Demo 6

Demo 6 is a speed test, which can be explicitly analysed using this execution protocol. While running the speed test the LaTeX output is deactivated and the root finding process iterated a large number of times to better measure the average execution time. The measurement results are saved as raw data in the file demo6speed.txt.
In this demo or speed test the following polynomials with a single root at $\frac{1}{3}$ are used (Example 9 from Bartoň and Jüttler):

$$\begin{aligned} f_2 &:= (t - \tfrac{1}{3})(3 - t) \\ f_4 &:= (t - \tfrac{1}{3})(2 - t)(t + 5)^2 \\ f_8 &:= (t - \tfrac{1}{3})(2 - t)^3(t + 5)^4 \\ f_{16} &:= (t - \tfrac{1}{3})(2 - t)^5(t + 5)^{10} \end{aligned}$$

Contents

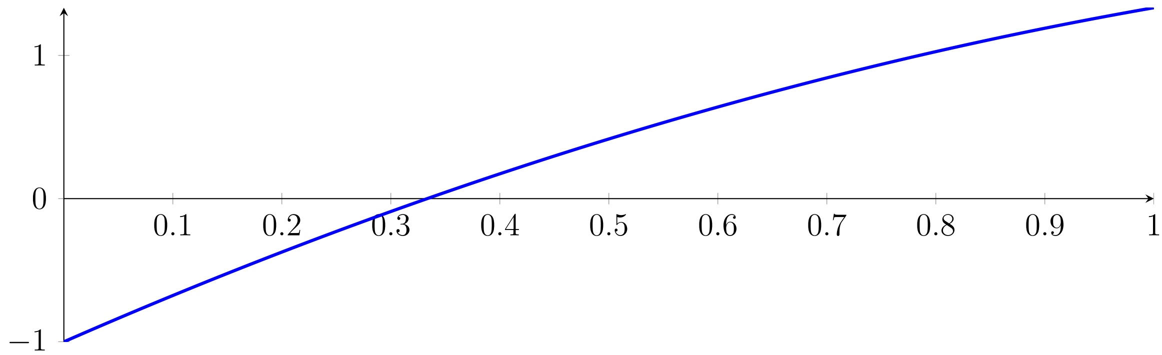
Part I
Numeric = double

1 Running BezClip on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

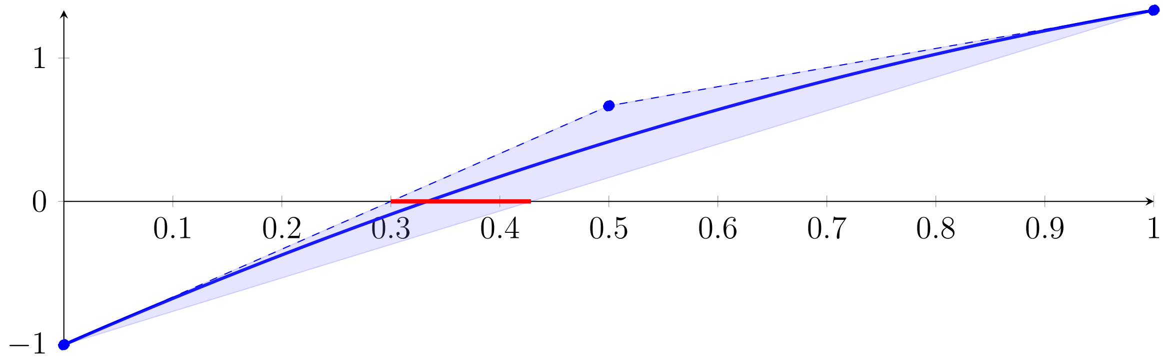
$$p = -1X^2 + 3.33333X - 1$$



1.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

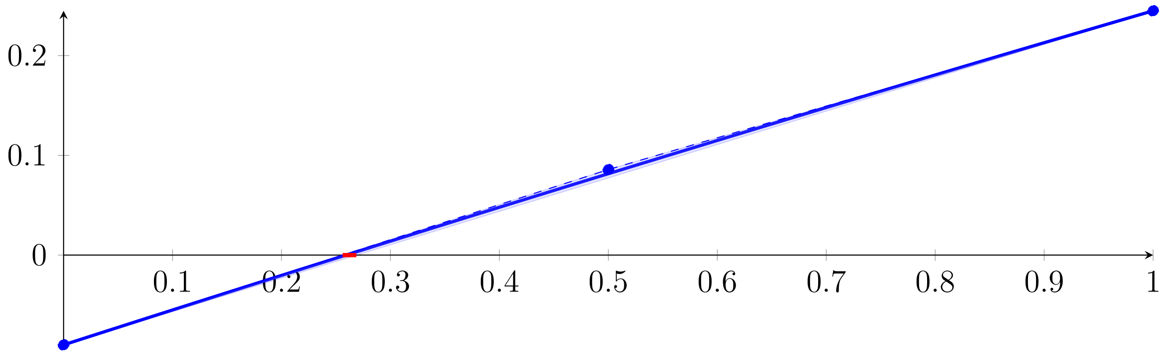
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

1.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -0.0165306X^2 + 0.351429X - 0.09$$
$$= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

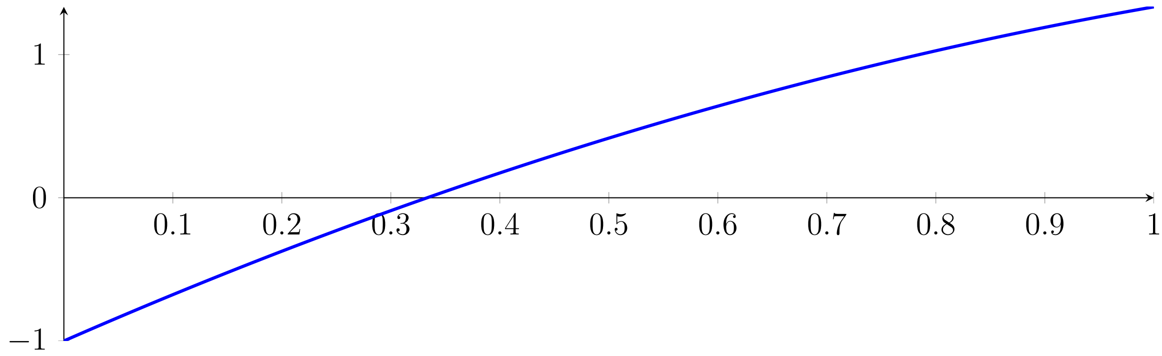
1.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Found root in interval $[0.332927, 0.334552]$ at recursion depth 3!

1.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.332927, 0.334552]$$

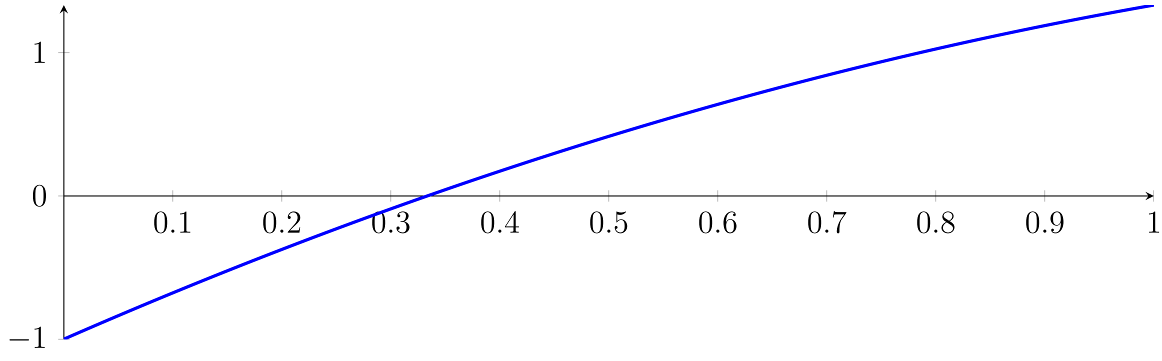
with precision $\varepsilon = 0.01$.

2 Running QuadClip on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0,1]$:

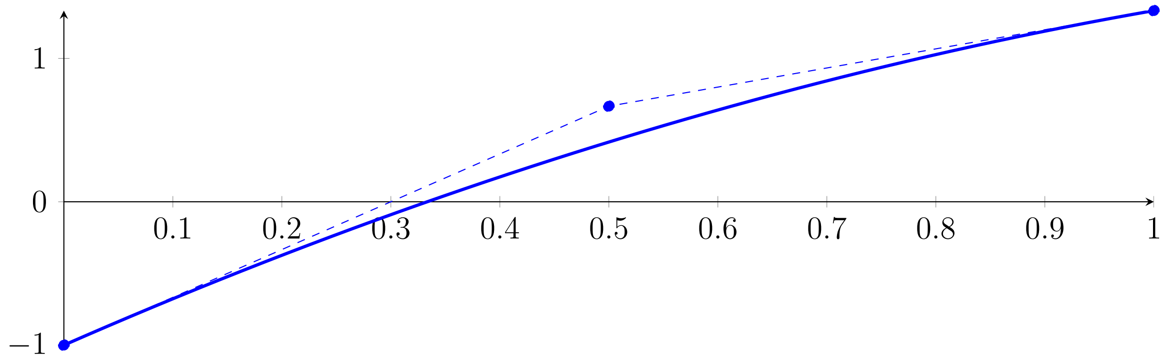
$$p = -1X^2 + 3.33333X - 1$$



2.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

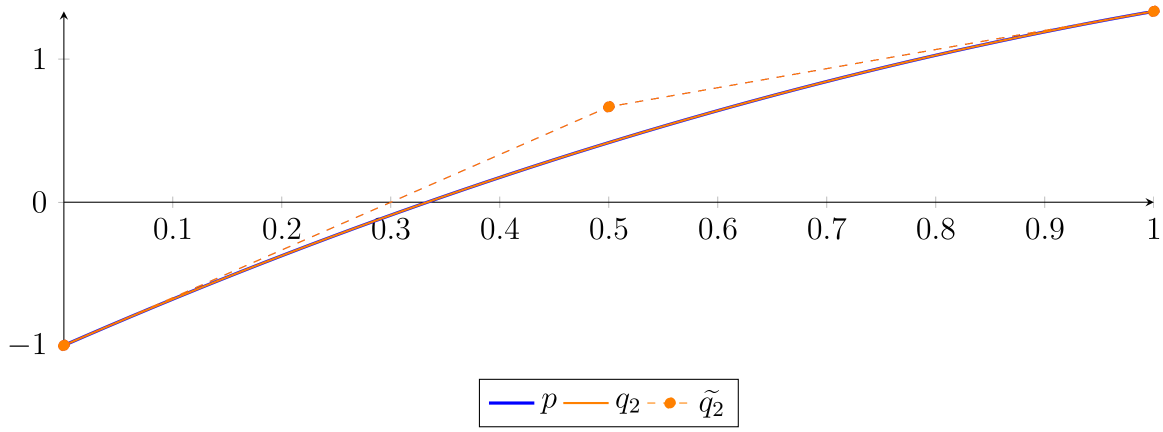
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.
Bounding polynomials M and m:

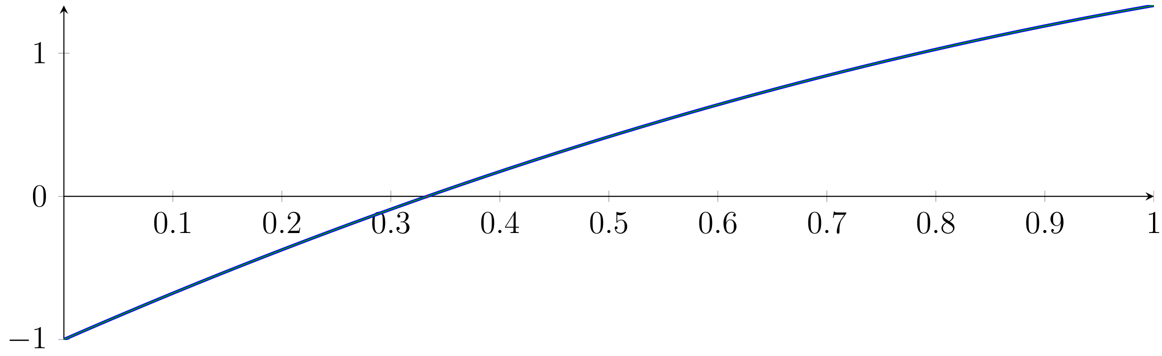
$$M = -1X^2 + 3.33333X - 1$$

$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $4.44089 \cdot 10^{-16}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

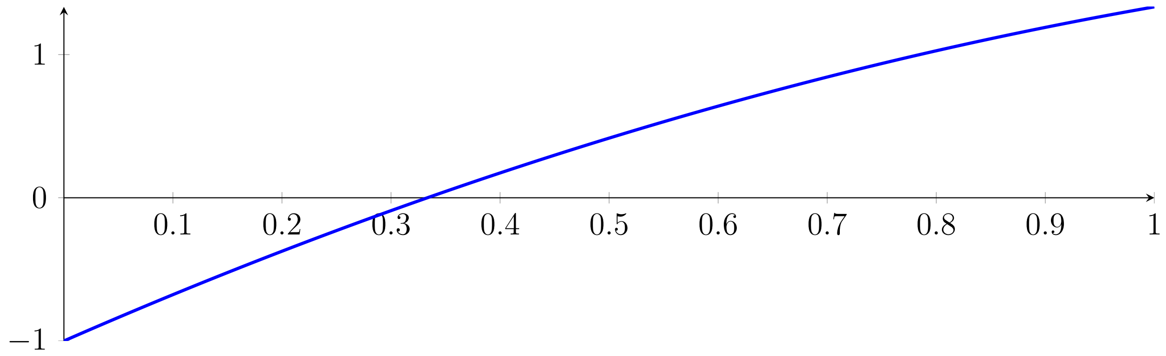
2.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

2.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

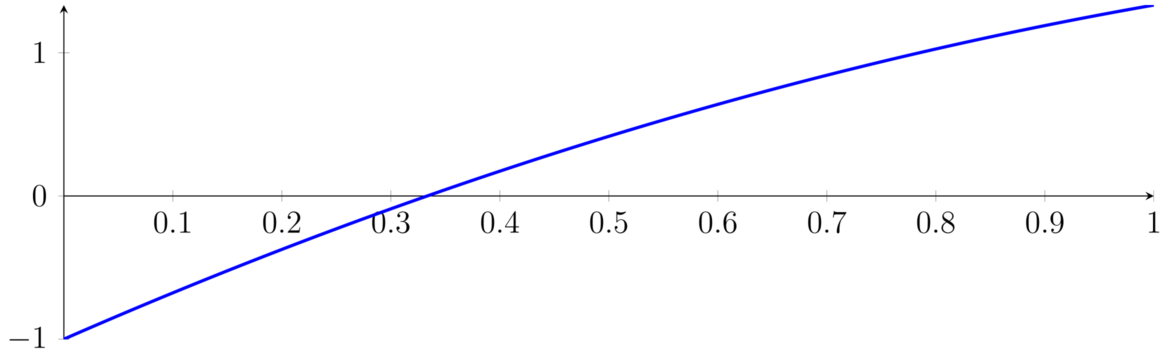
with precision $\varepsilon = 0.01$.

3 Running CubeClip on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

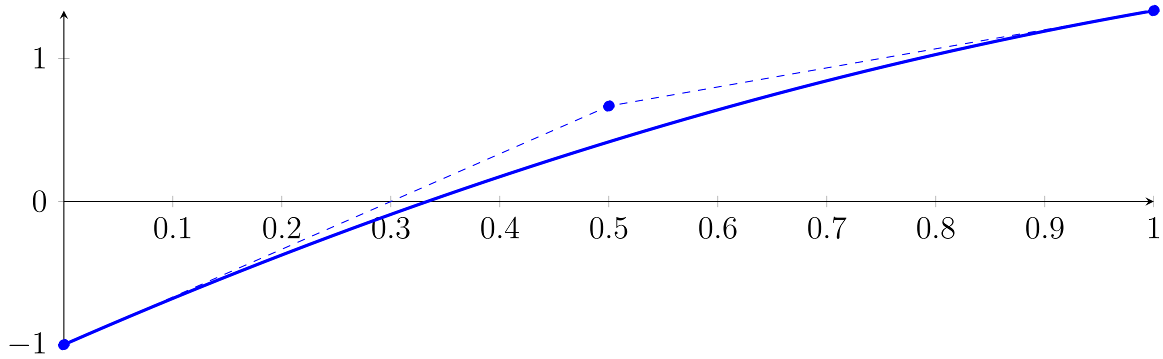
$$p = -1X^2 + 3.33333X - 1$$



3.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

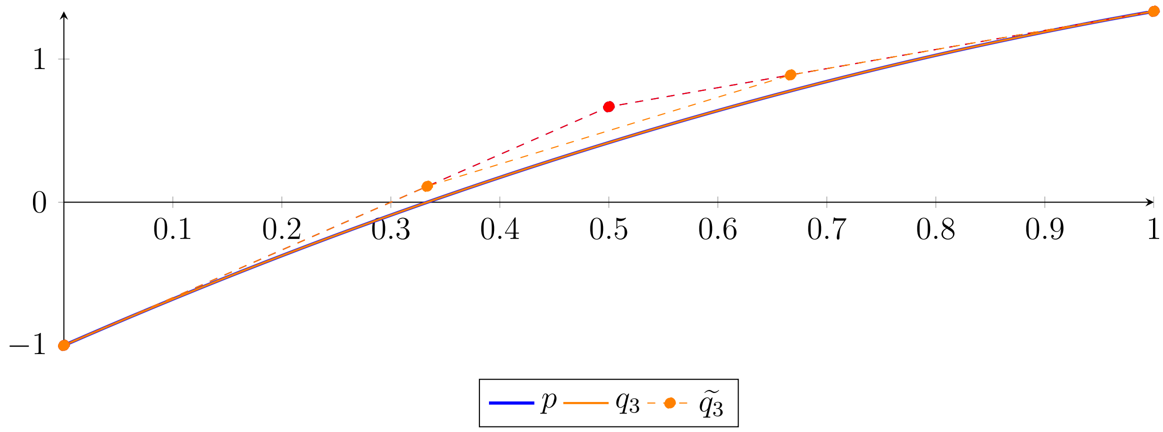
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.90958 \cdot 10^{-14}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134{\cdot}10^{-16}$.
Bounding polynomials M and m:

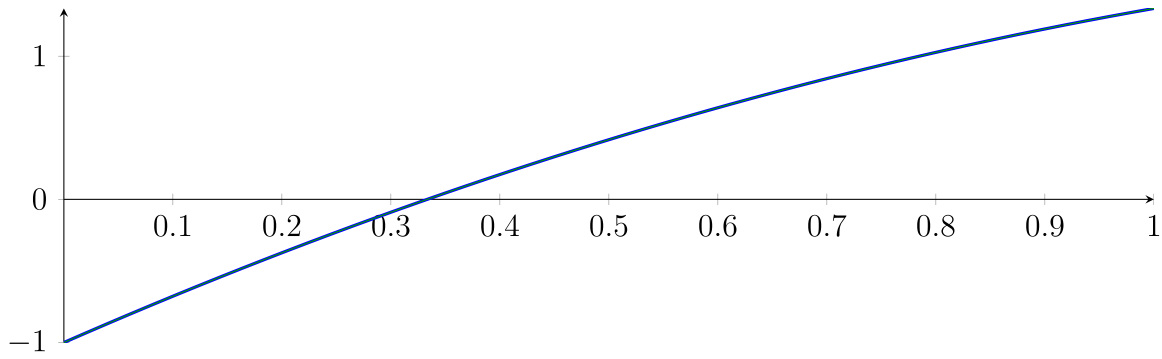
$$M = -1.88738{\cdot}10^{-14}X^3 - 1X^2 + 3.33333X - 1$$

$$m = -1.88738{\cdot}10^{-14}X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{-5.17655{\cdot}10^{13}\} \qquad N(m) = \{-5.17655{\cdot}10^{13}\}$$

Intersection intervals:

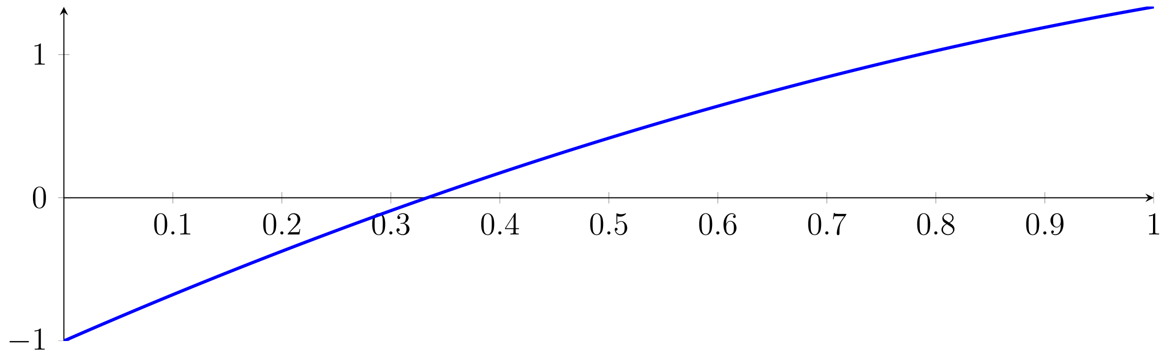


No intersection intervals with the x axis.

3.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

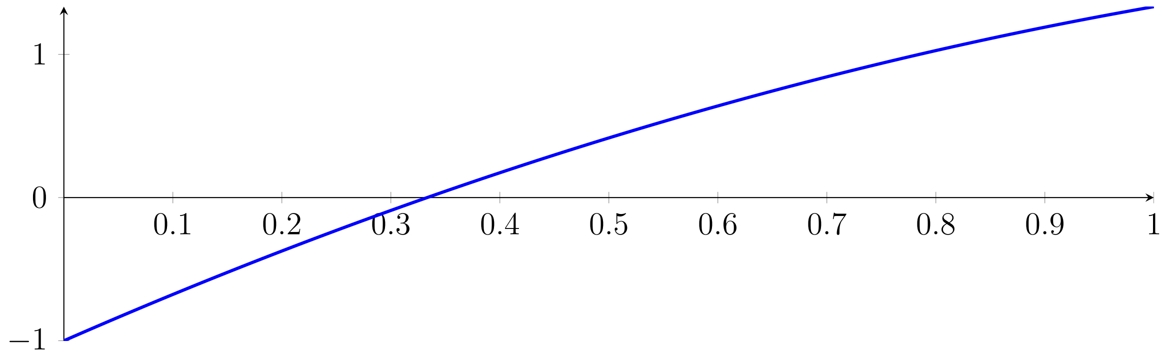
with precision $\varepsilon = 0.01$.

4 Running `BezClip` on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

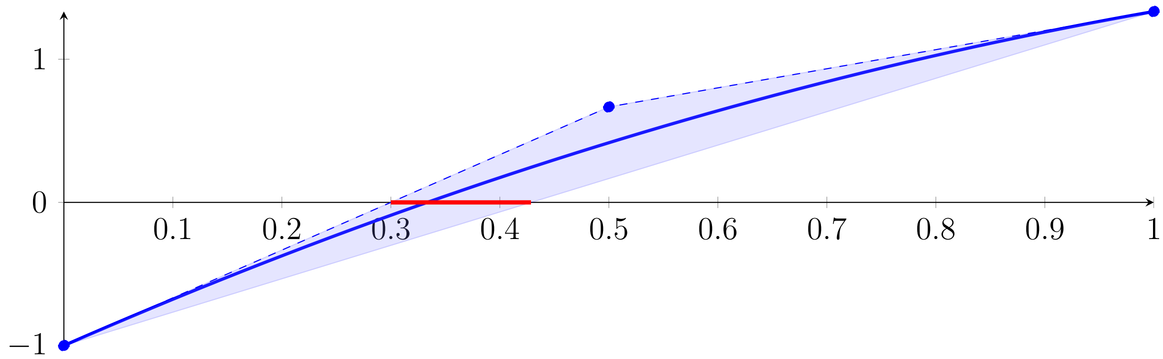
$$p = -1X^2 + 3.33333X - 1$$



4.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

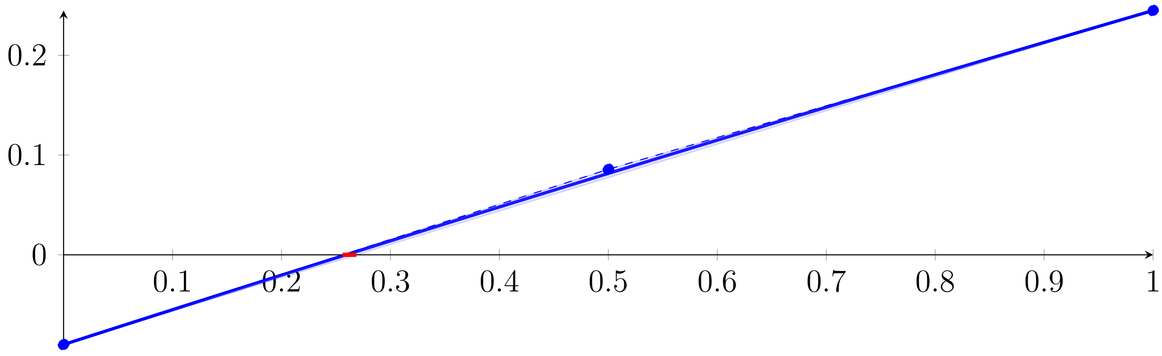
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

4.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

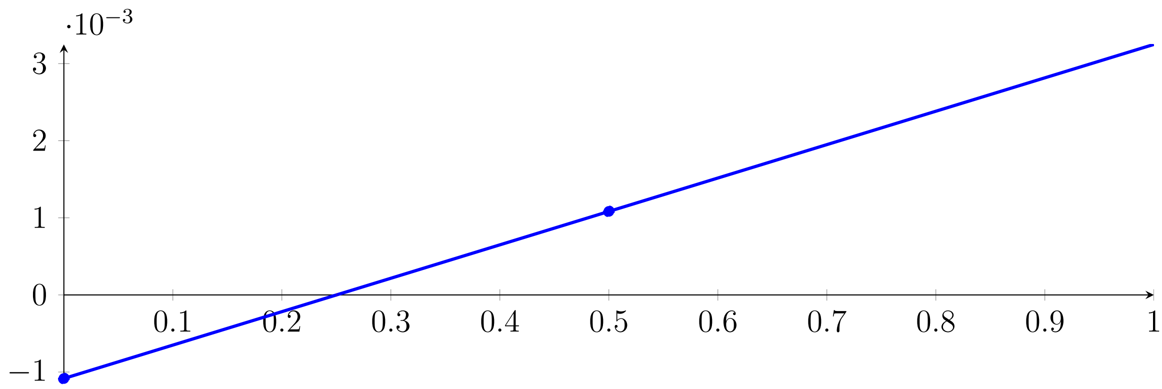
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

4.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

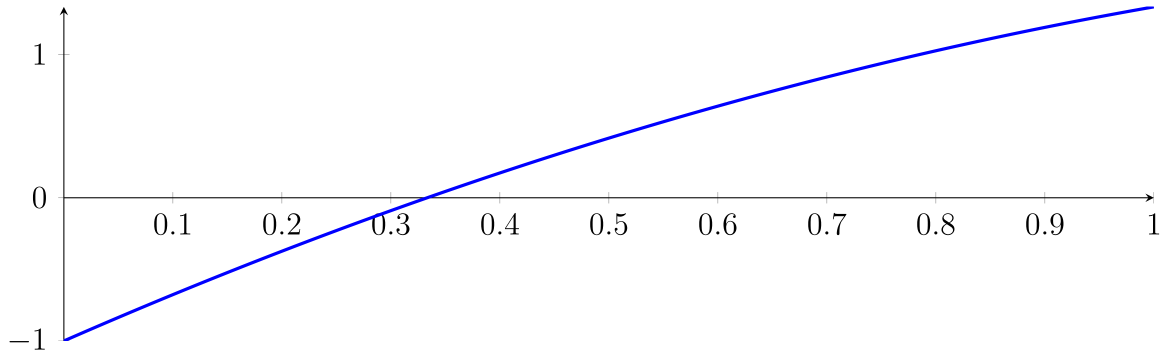
4.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Found root in interval $[0.333333, 0.333334]$ at recursion depth 4!

4.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333334]$$

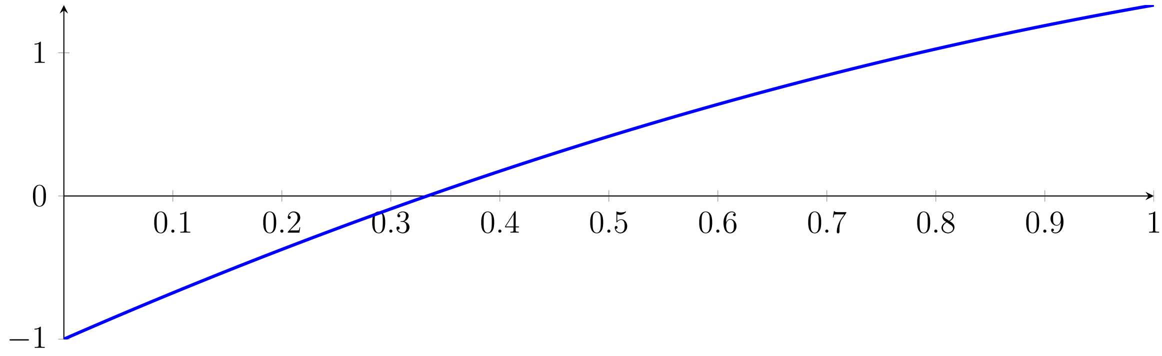
with precision $\varepsilon = 0.0001$.

5 Running QuadClip on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0,1]$:

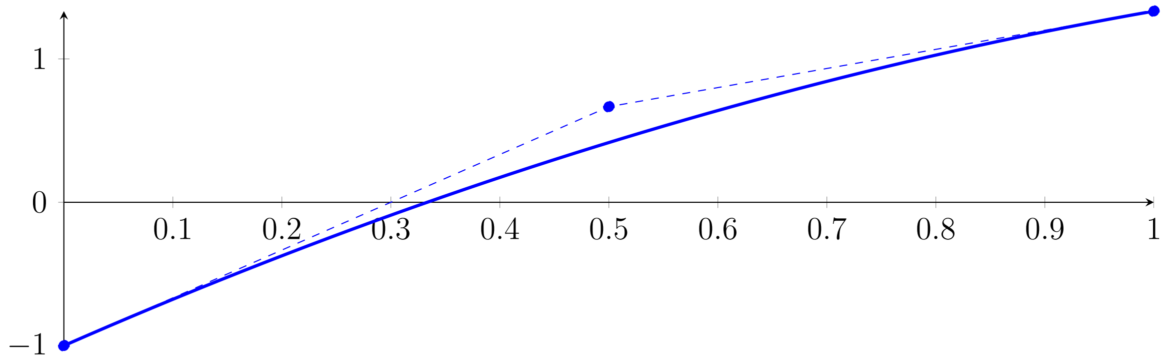
$$p = -1X^2 + 3.33333X - 1$$



5.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

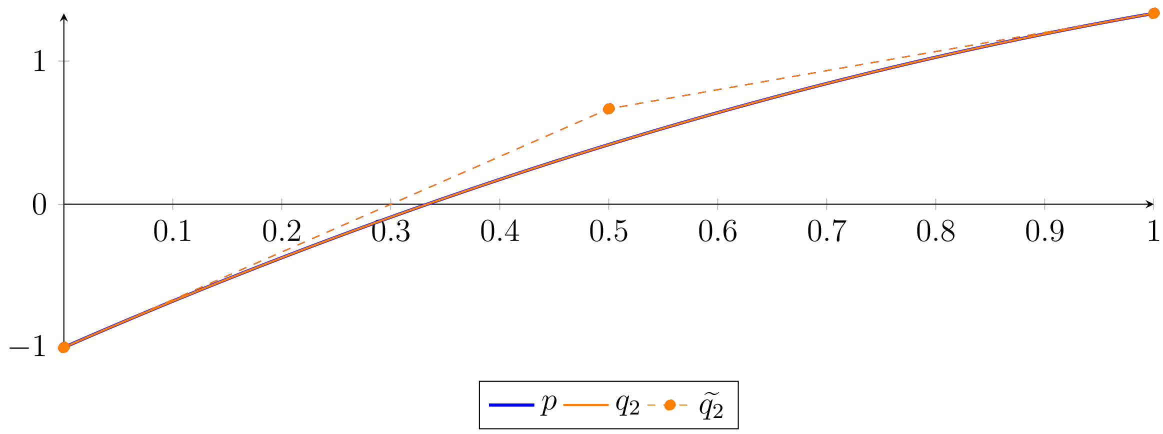
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.
Bounding polynomials M and m:

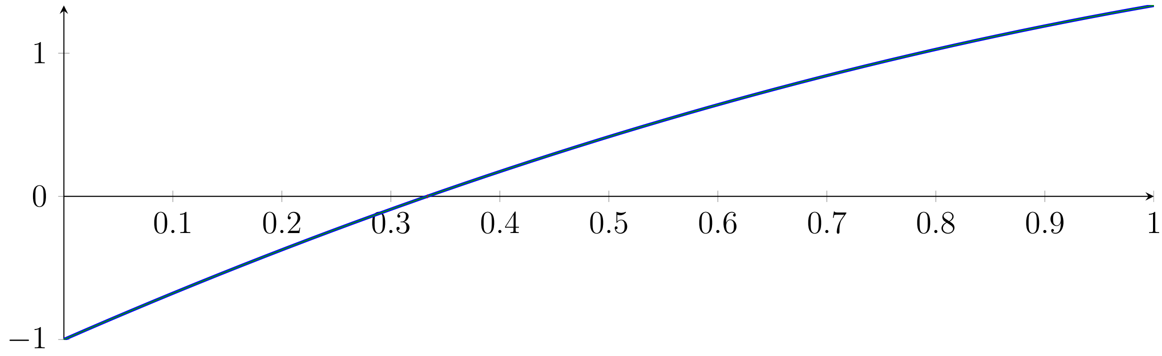
$$M = -1X^2 + 3.33333X - 1$$

$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $4.44089 \cdot 10^{-16}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

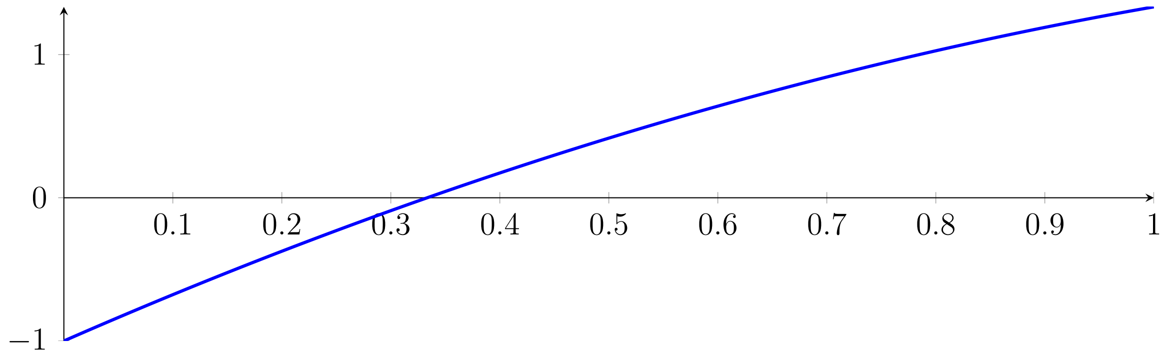
5.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

5.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

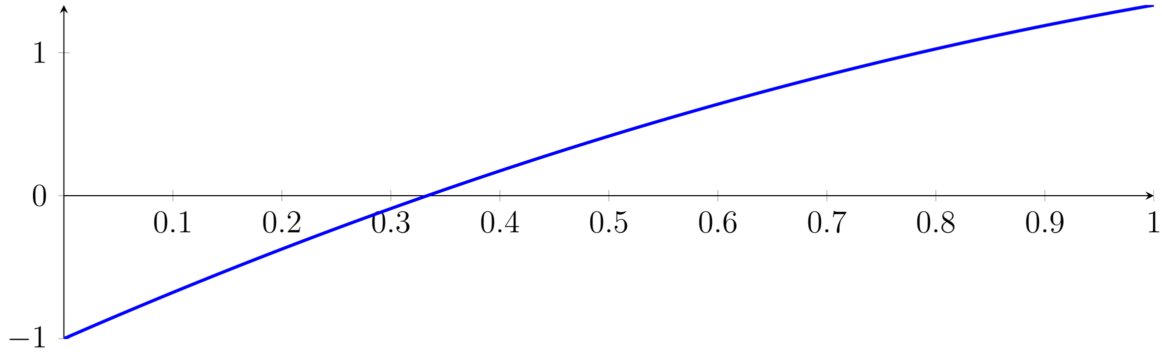
with precision $\varepsilon = 0.0001$.

6 Running CubeClip on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0,1]$:

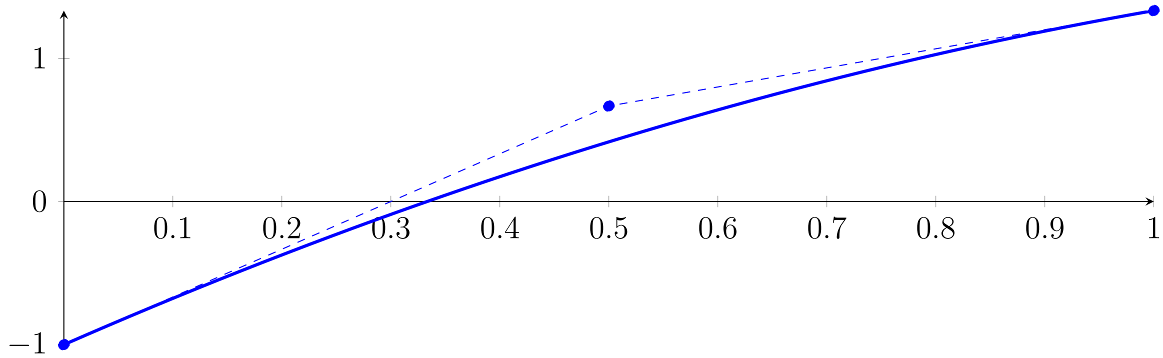
$$p = -1X^2 + 3.33333X - 1$$



6.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

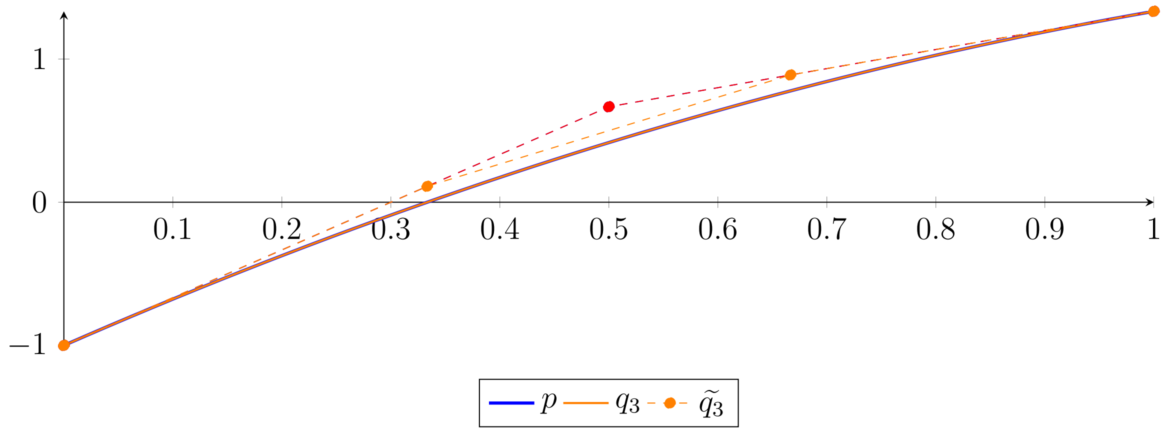
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.90958 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.
Bounding polynomials M and m:

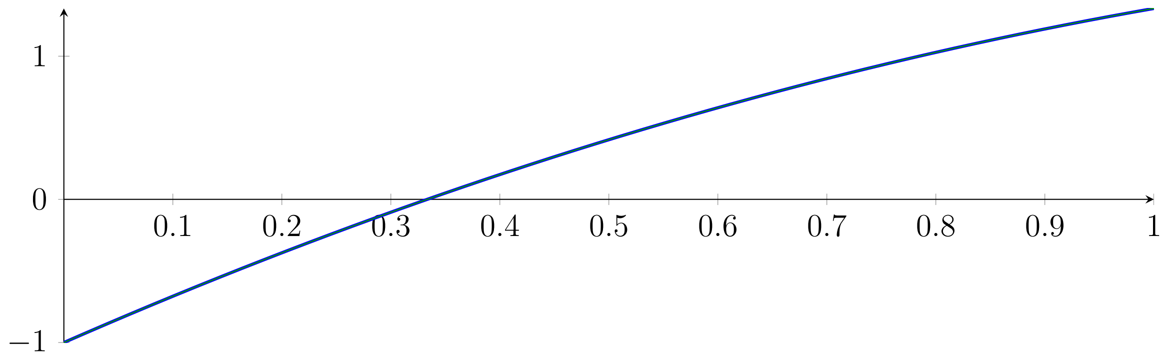
$$M = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

$$m = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{-5.17655 \cdot 10^{13}\} \qquad N(m) = \{-5.17655 \cdot 10^{13}\}$$

Intersection intervals:

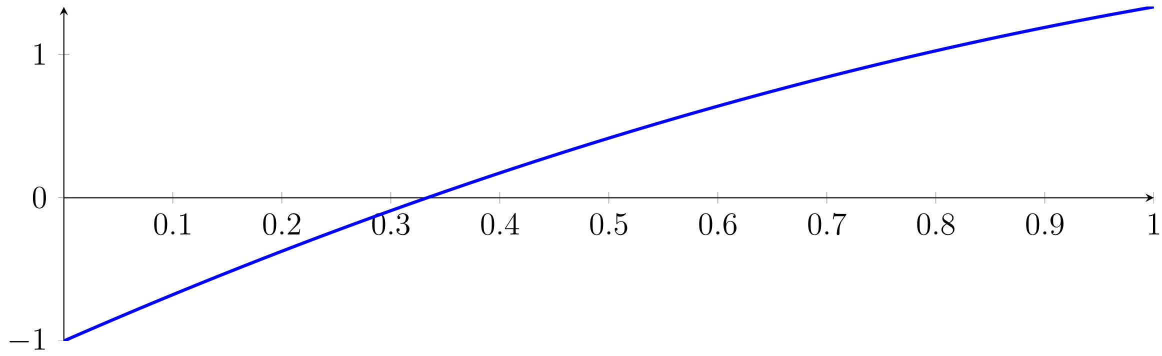


No intersection intervals with the x axis.

6.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

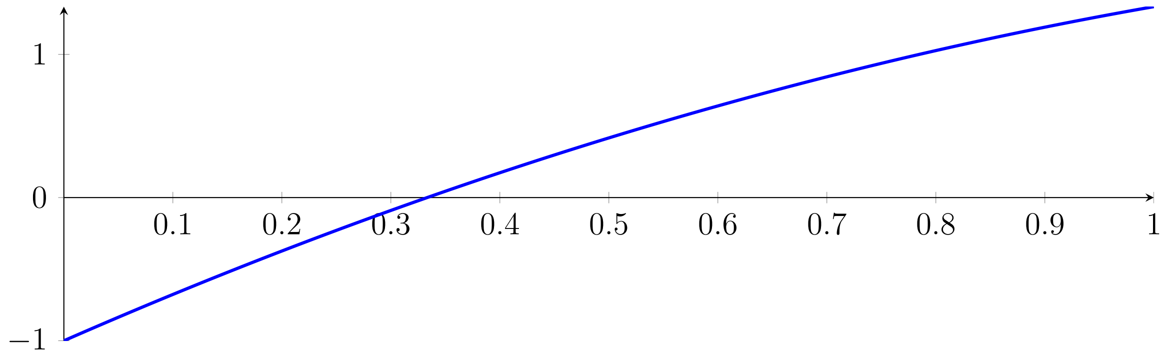
with precision $\varepsilon = 0.0001$.

7 Running BezClip on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

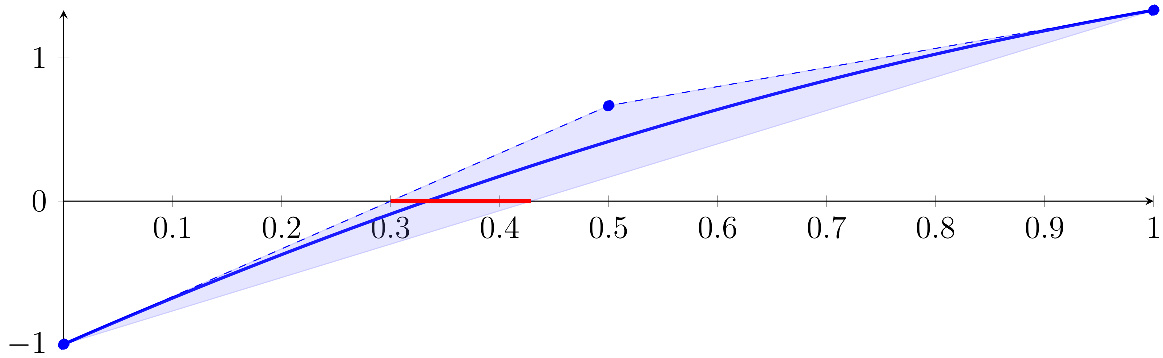
$$p = -1X^2 + 3.33333X - 1$$



7.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

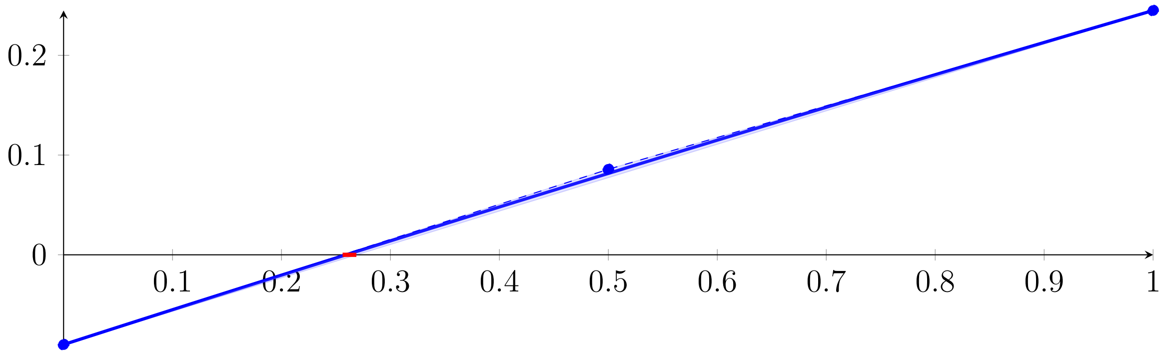
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

7.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

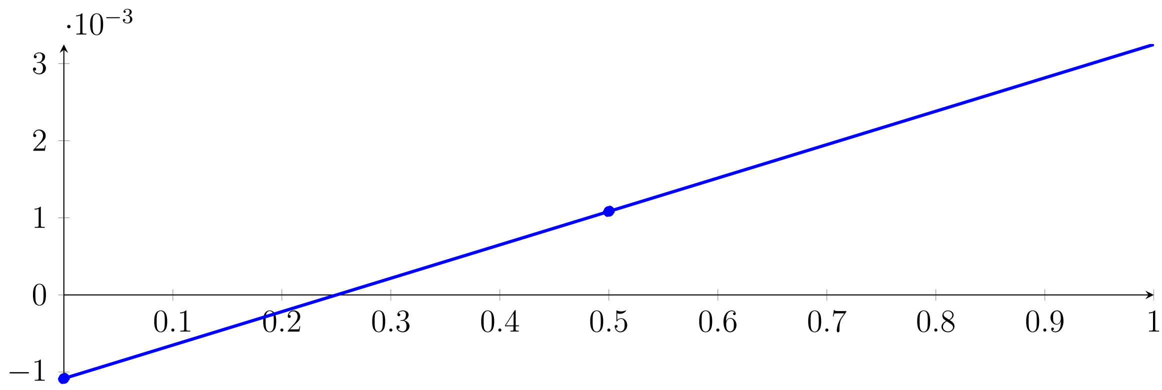
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

7.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

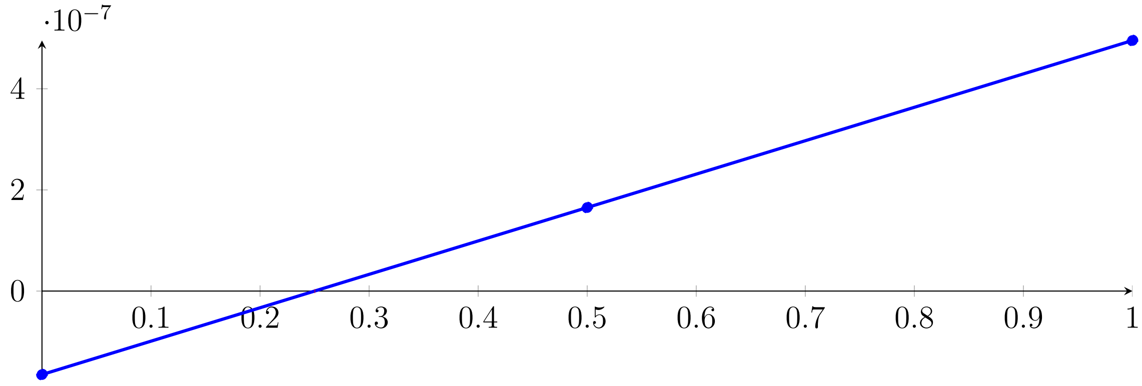
$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

7.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07}$$
$$= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $2.32306 \cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

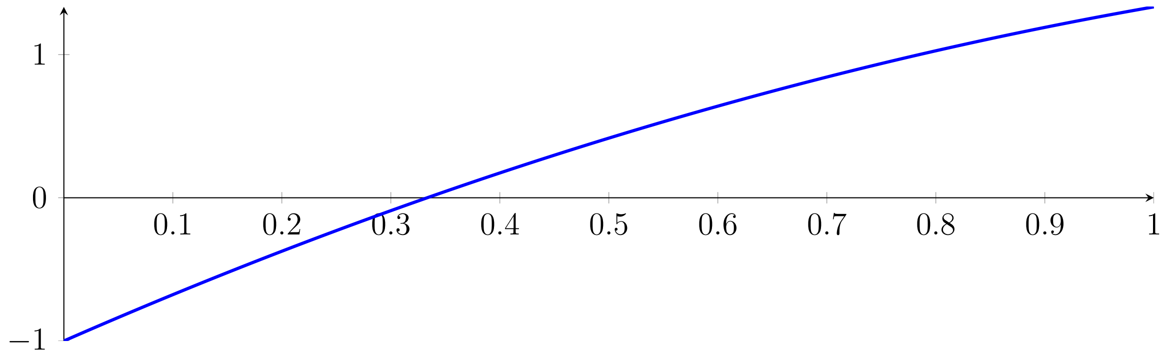
7.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

7.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

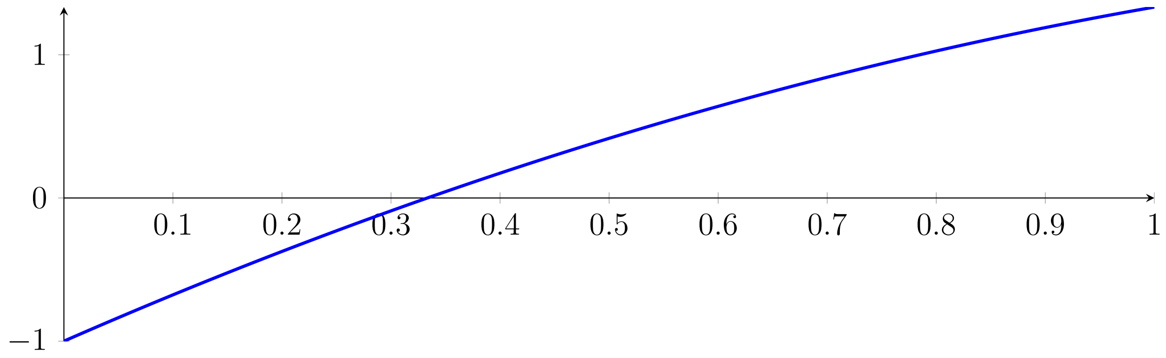
with precision $\varepsilon = 1 \cdot 10^{-08}$.

8 Running QuadClip on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

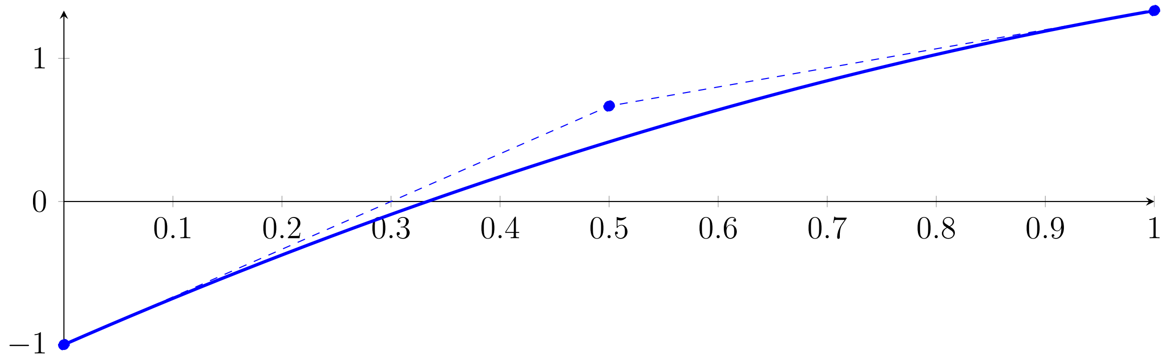
$$p = -1X^2 + 3.33333X - 1$$



8.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

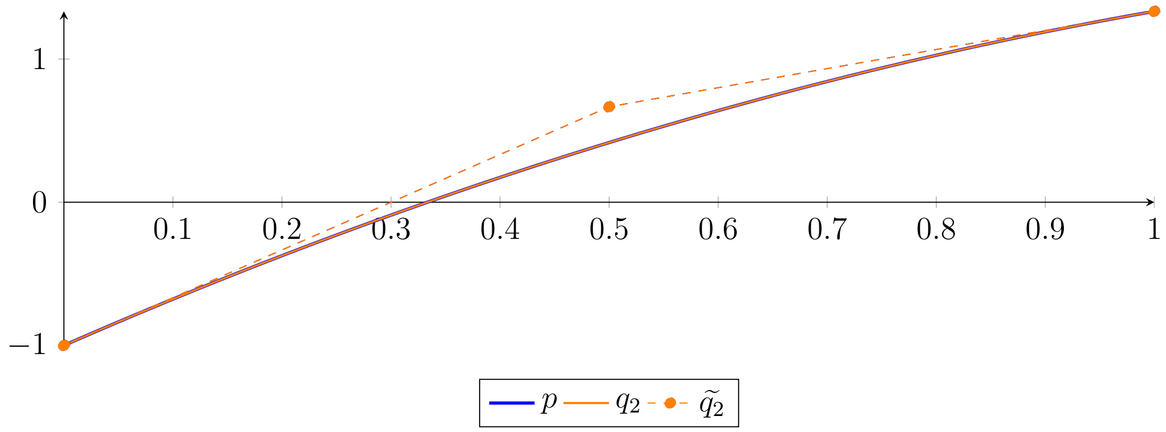
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



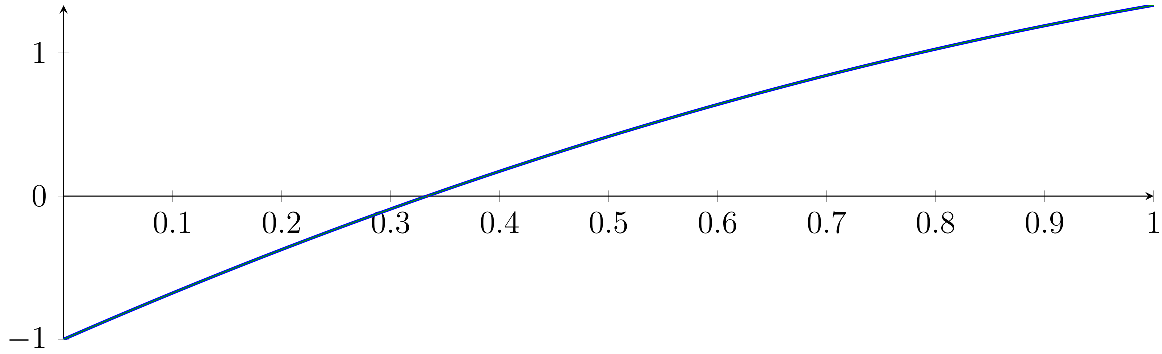
The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.
Bounding polynomials M and m:

$$M = -1X^2 + 3.33333X - 1$$
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $4.44089 \cdot 10^{-16}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

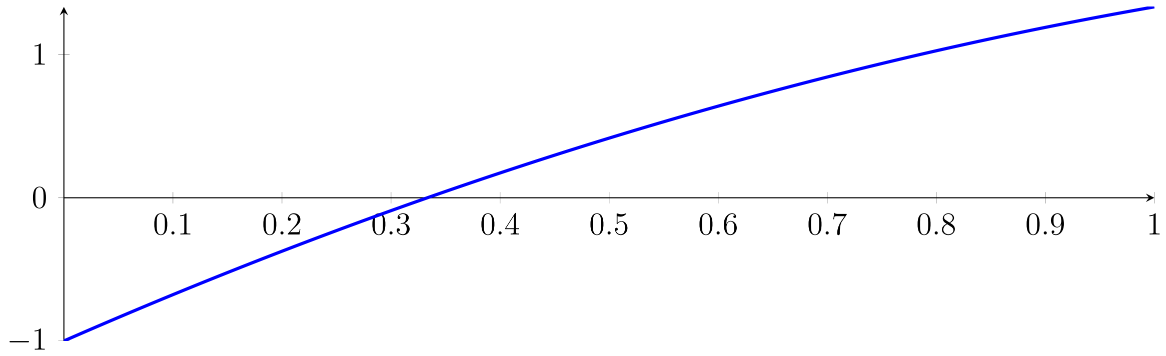
8.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

8.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

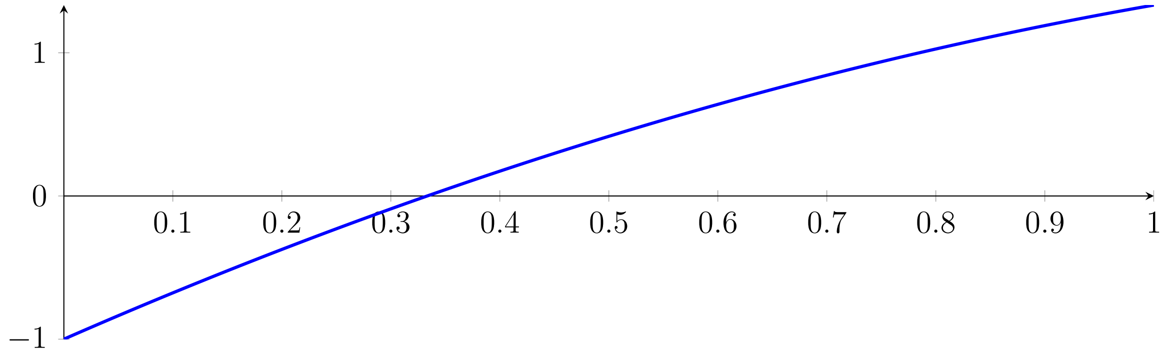
with precision $\varepsilon = 1 \cdot 10^{-08}$.

9 Running `CubeClip` on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

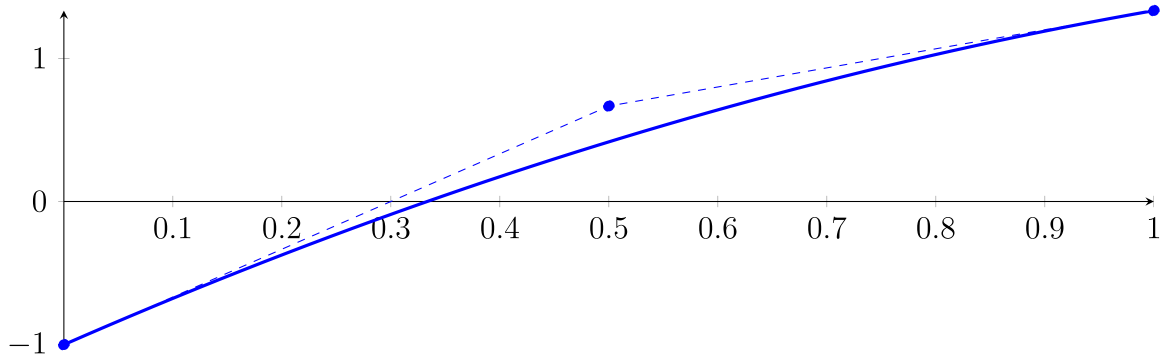
$$p = -1X^2 + 3.33333X - 1$$



9.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

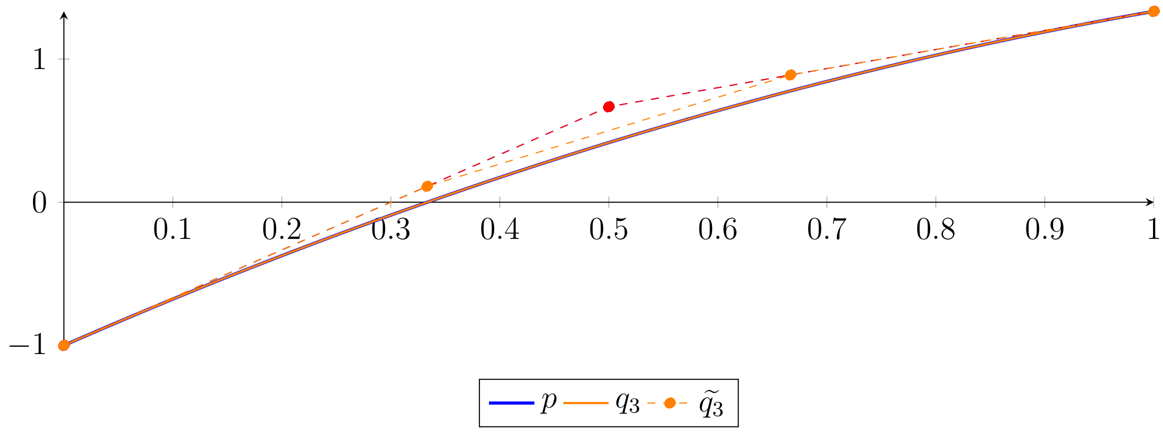
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.90958 \cdot 10^{-14}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134{\cdot}10^{-16}$.
Bounding polynomials M and m:

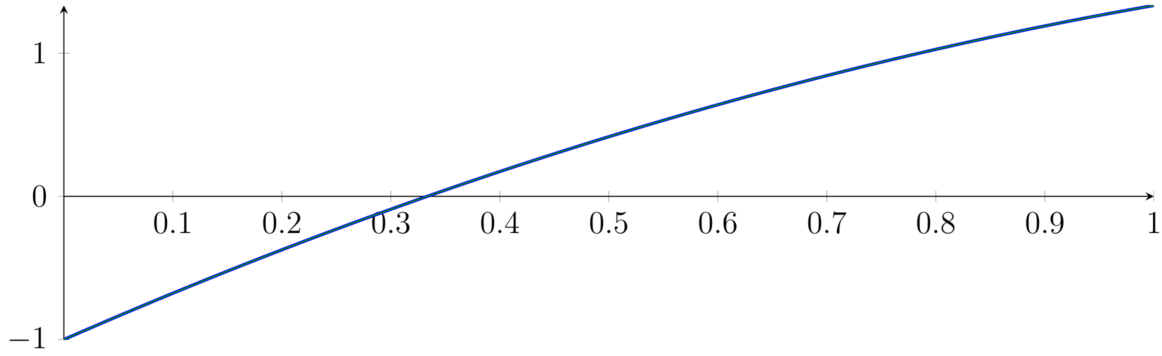
$$M = -1.88738{\cdot}10^{-14}X^3 - 1X^2 + 3.33333X - 1$$

$$m = -1.88738{\cdot}10^{-14}X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{-5.17655{\cdot}10^{13}\} \qquad N(m) = \{-5.17655{\cdot}10^{13}\}$$

Intersection intervals:

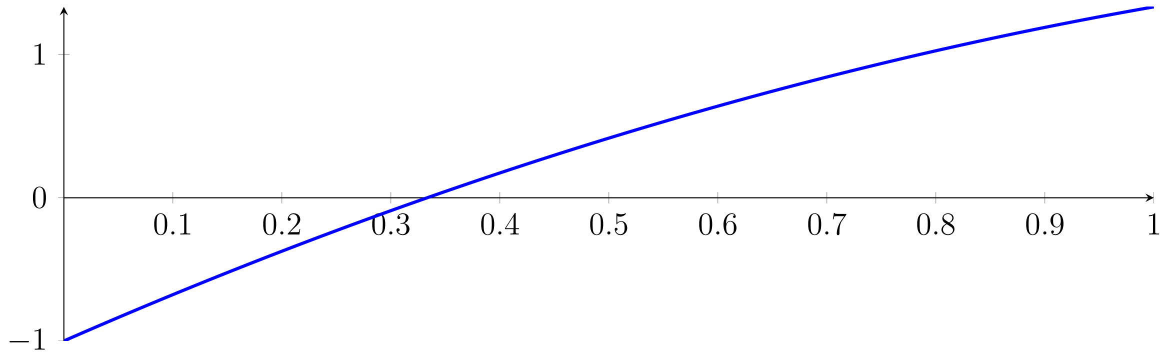


No intersection intervals with the x axis.

9.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

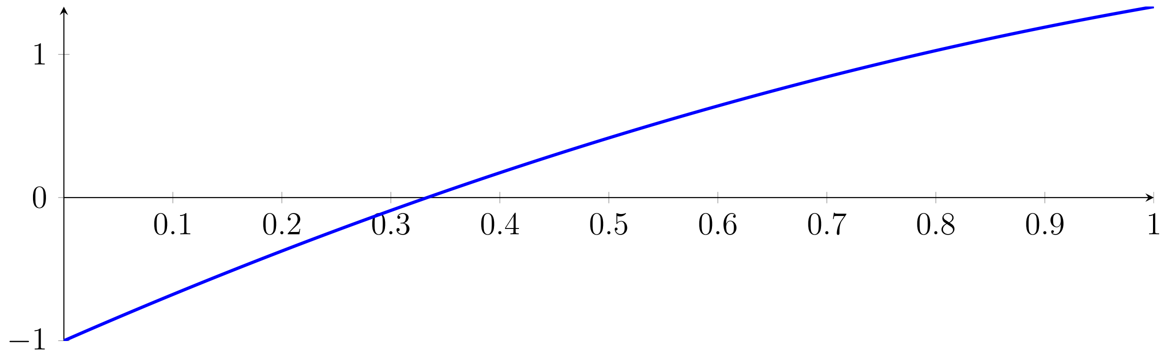
with precision $\varepsilon = 1 \cdot 10^{-08}$.

10 Running `BezClip` on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

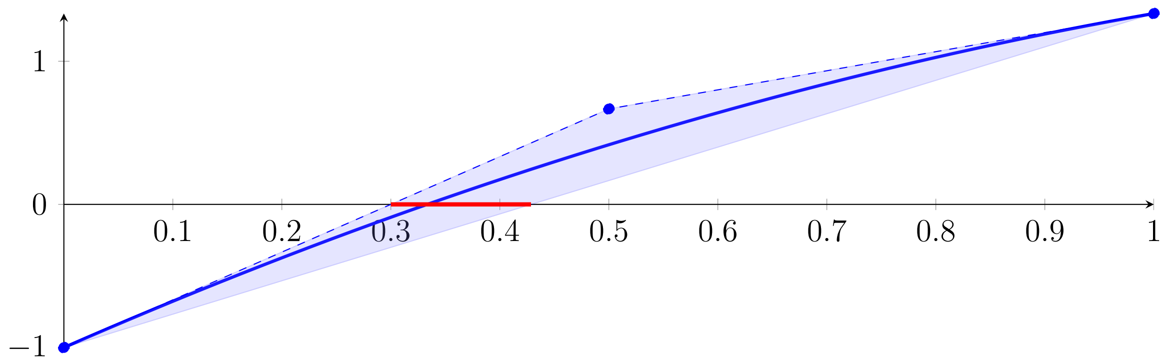
$$p = -1X^2 + 3.33333X - 1$$



10.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

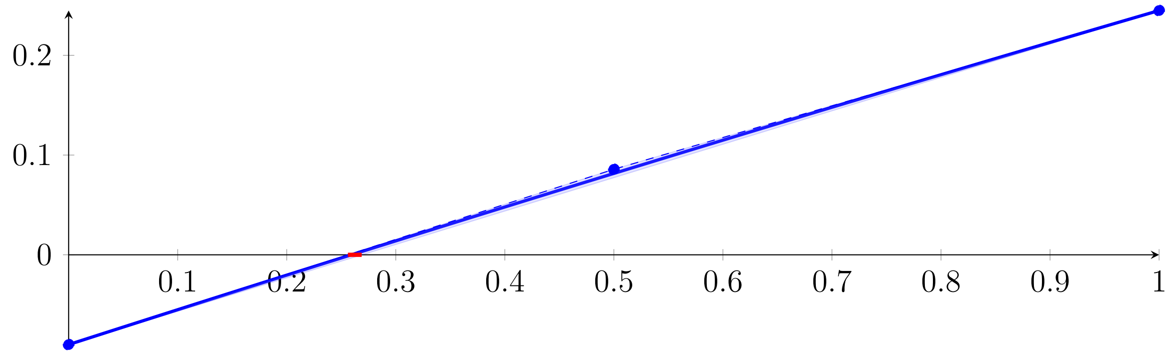
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

10.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

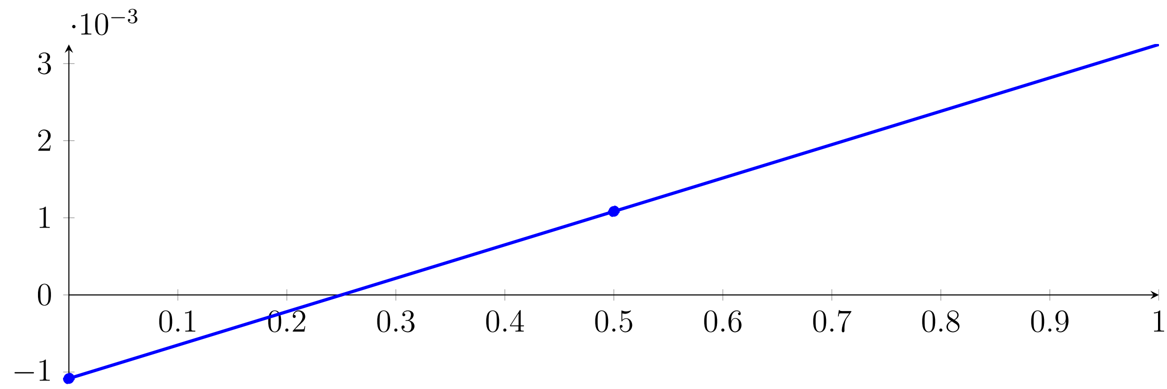
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

10.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -2.64151 \cdot 10^{-06} X^2 + 0.00433538X - 0.00108418$$
$$= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

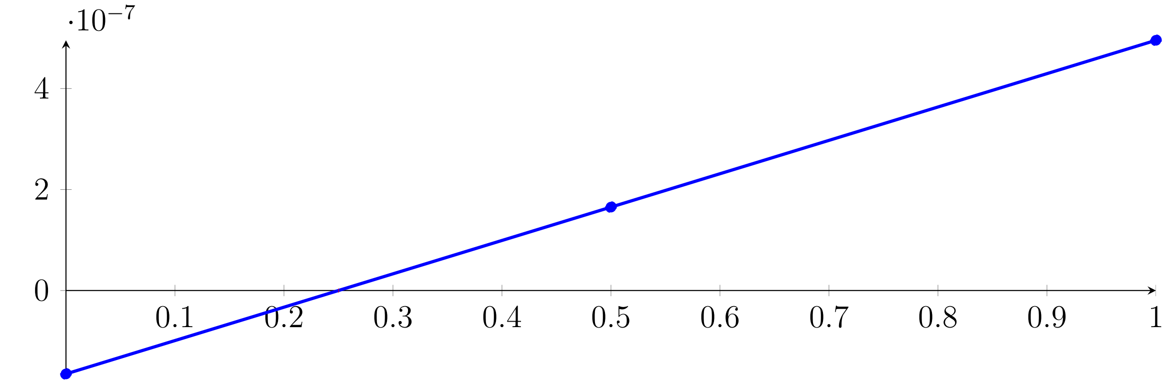
$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

10.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07}$$
$$= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

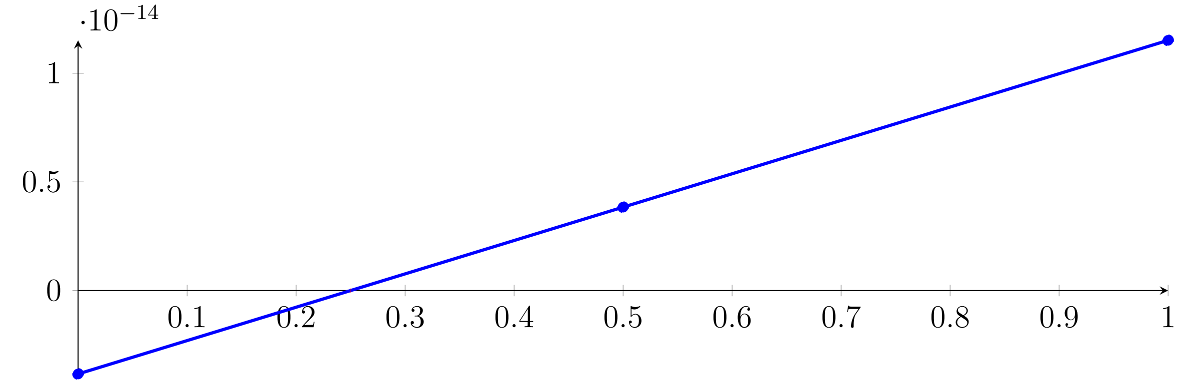
$$[0.25, 0.25]$$

Longest intersection interval: $2.32306 \cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

10.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.31322 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15}$$
$$= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.55112 \cdot 10^{-16}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

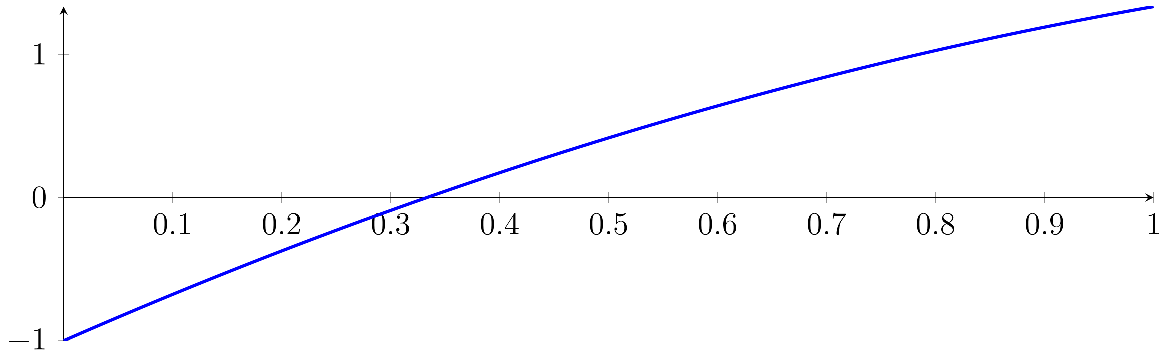
10.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

10.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

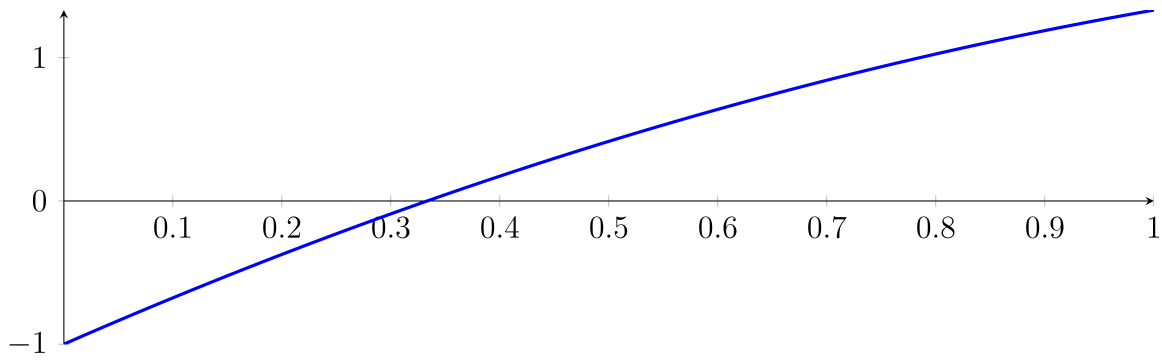
with precision $\varepsilon = 1 \cdot 10^{-16}$.

11 Running `QuadClip` on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called `QuadClip` with input polynomial on interval $[0, 1]$:

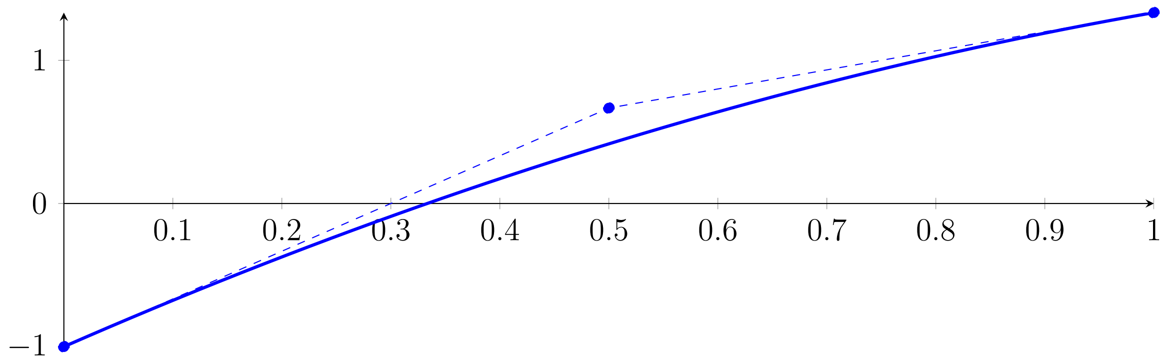
$$p = -1X^2 + 3.33333X - 1$$



11.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

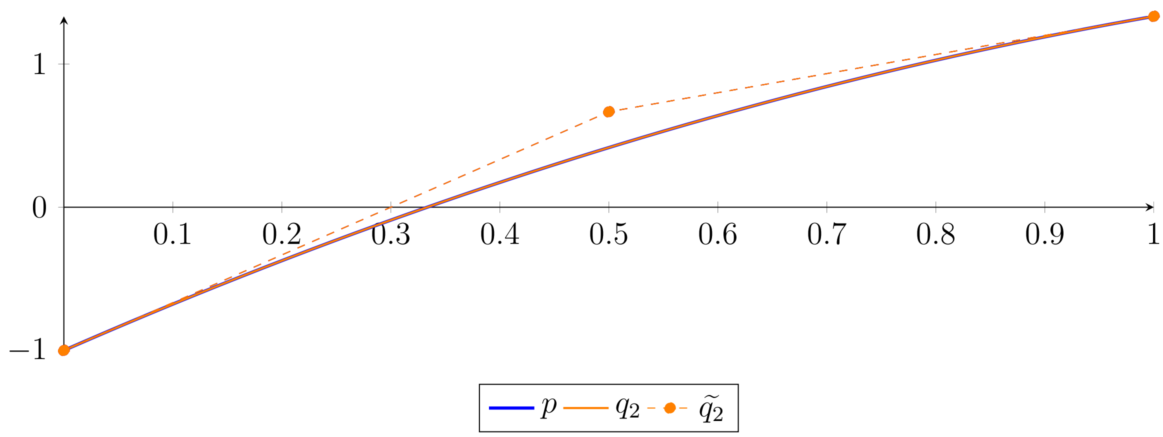
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



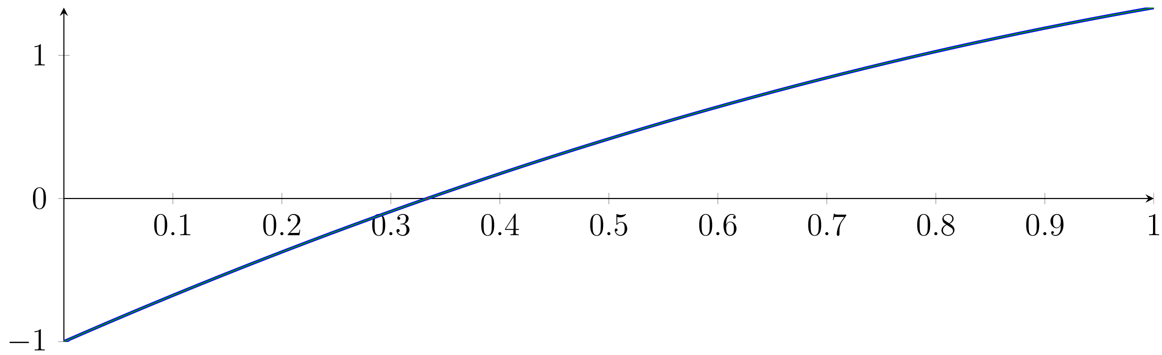
The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.
Bounding polynomials M and m:

$$M = -1X^2 + 3.33333X - 1$$
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



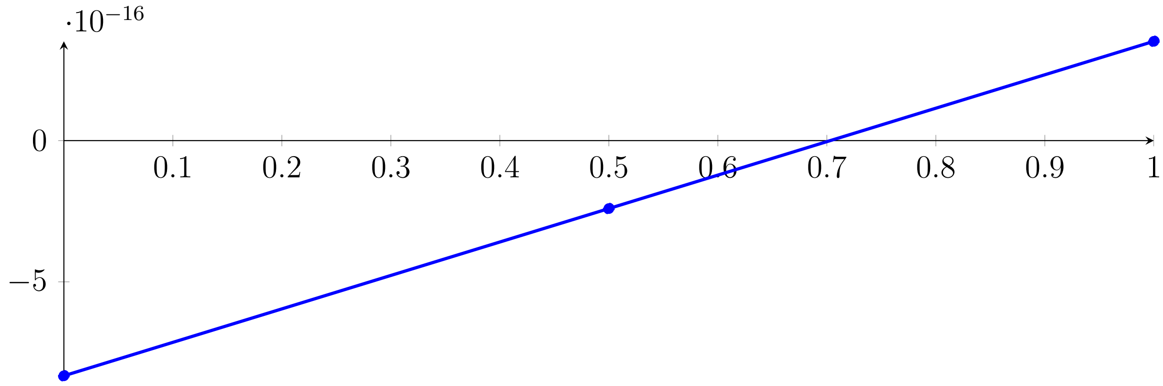
$$[0.333333, 0.333333]$$

Longest intersection interval: $4.44089 \cdot 10^{-16}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

11.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

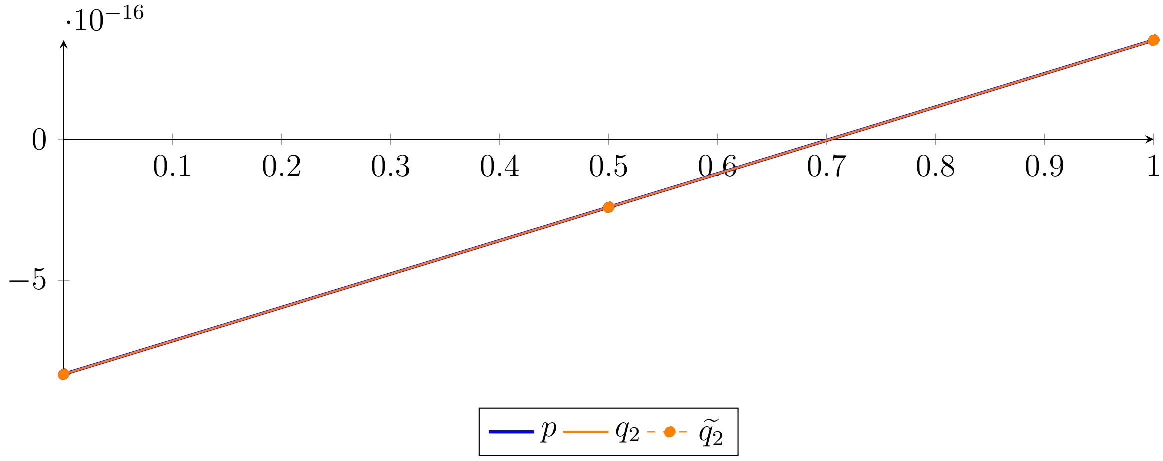
$$p = -1.97215 \cdot 10^{-31} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$
$$= -8.32667 \cdot 10^{-16} B_{0,2}(X) - 2.40548 \cdot 10^{-16} B_{1,2}(X) + 3.51571 \cdot 10^{-16} B_{2,2}(X)$$



Degree reduction and raising:

$$q_2 = 1.18329 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$
$$= -8.32667 \cdot 10^{-16} B_{0,2} - 2.40548 \cdot 10^{-16} B_{1,2} + 3.51571 \cdot 10^{-16} B_{2,2}$$

$$\tilde{q}_2 = 2.5638 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$
$$= -8.32667 \cdot 10^{-16} B_{0,2} - 2.40548 \cdot 10^{-16} B_{1,2} + 3.51571 \cdot 10^{-16} B_{2,2}$$



The maximum difference of the Bézier coefficients is $\delta = 1.2326 \cdot 10^{-30}$.

Bounding polynomials M and m:

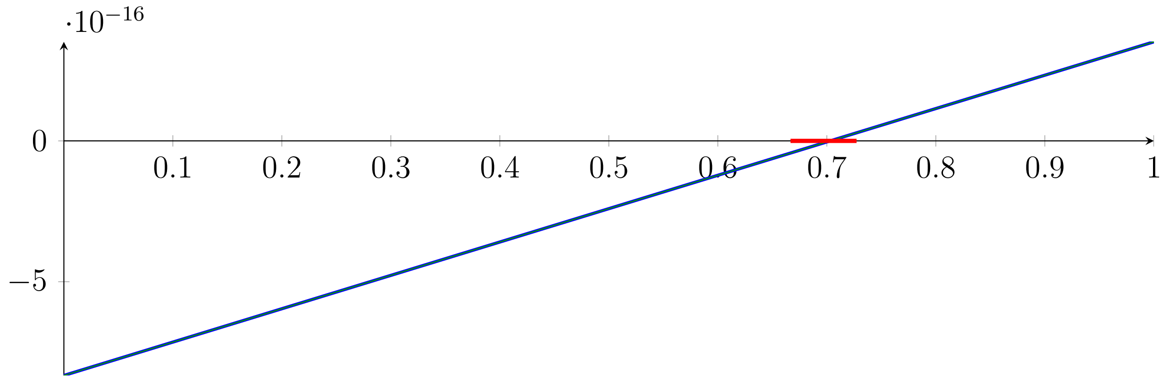
$$M = 1.08468 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$

$$m = 1.18329 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$

Root of M and m:

$$N(M) = \{-1.09178 \cdot 10^{15}, 0.727273\} \qquad N(m) = \{-1.0008 \cdot 10^{15}, 0.666667\}$$

Intersection intervals:



$$[0.666667, 0.727273]$$

Longest intersection interval: 0.0606061

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

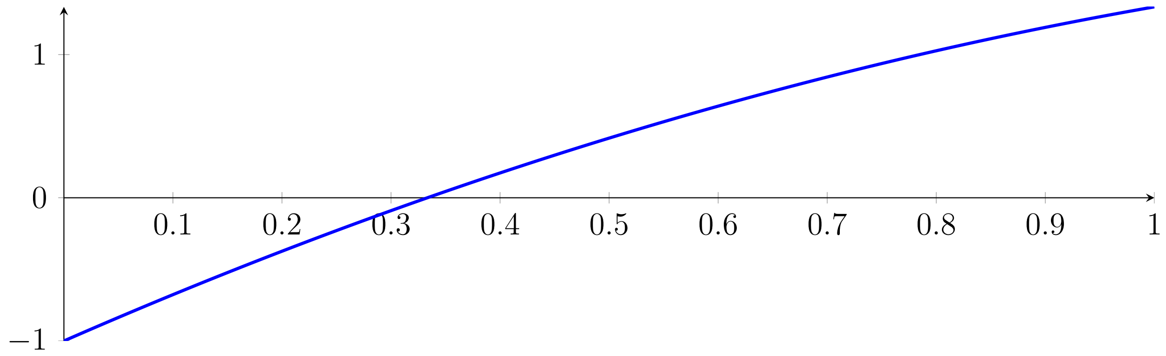
11.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

11.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

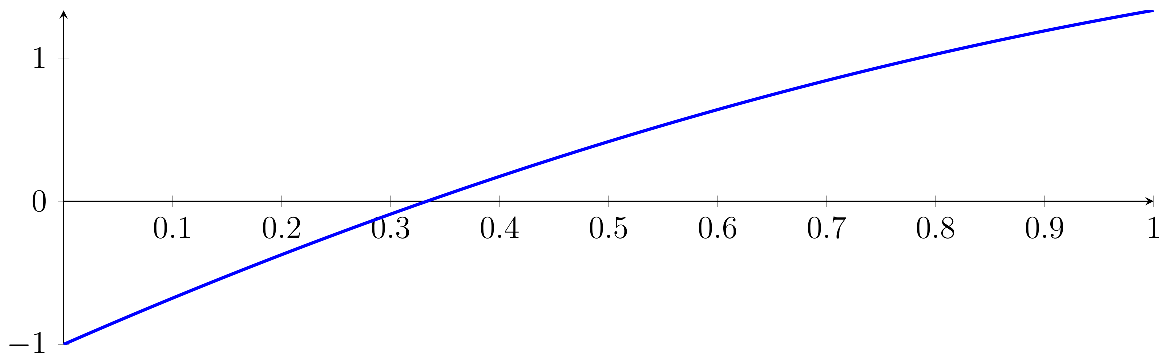
with precision $\varepsilon = 1 \cdot 10^{-16}$.

12 Running CubeClip on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

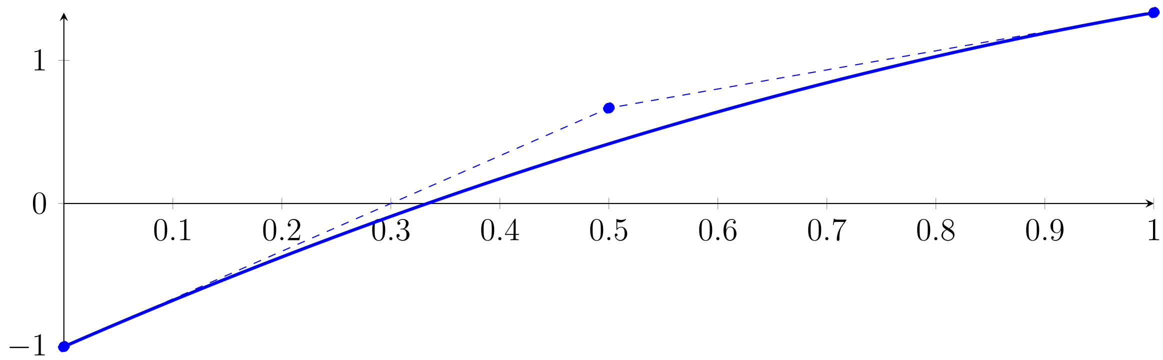
$$p = -1X^2 + 3.33333X - 1$$



12.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

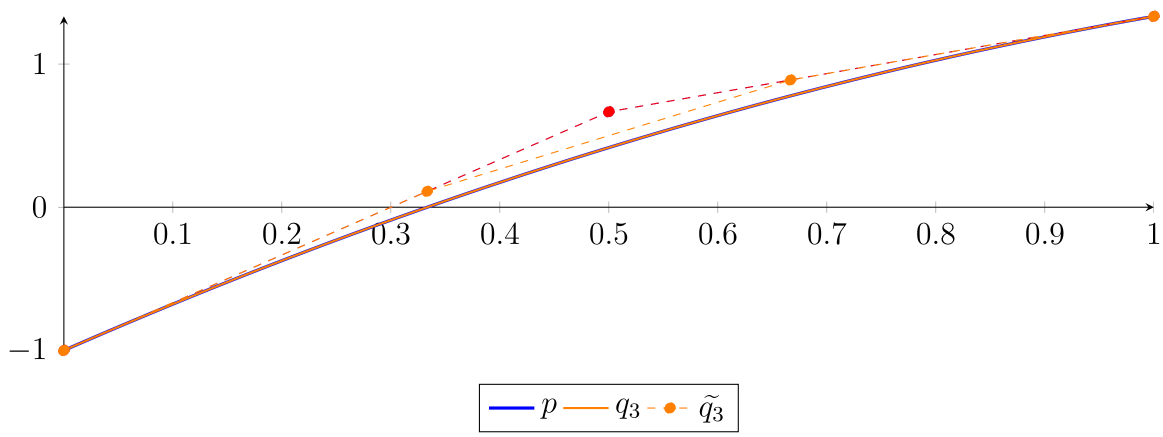
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.90958 \cdot 10^{-14}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134{\cdot}10^{-16}$.
Bounding polynomials M and m:

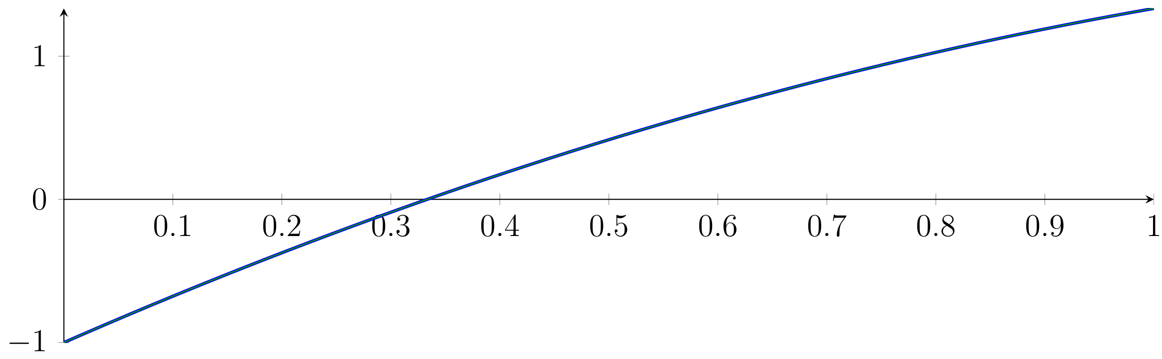
$$M = -1.88738{\cdot}10^{-14}X^3 - 1X^2 + 3.33333X - 1$$

$$m = -1.88738{\cdot}10^{-14}X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{-5.17655{\cdot}10^{13}\} \qquad N(m) = \{-5.17655{\cdot}10^{13}\}$$

Intersection intervals:

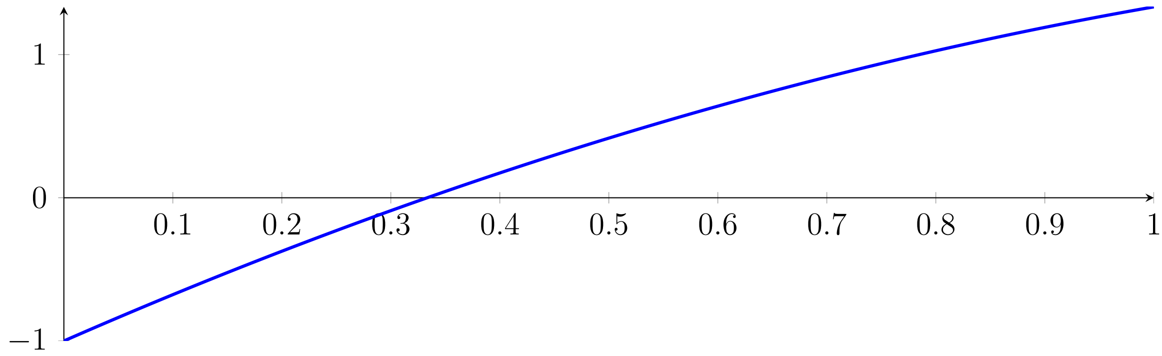


No intersection intervals with the x axis.

12.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

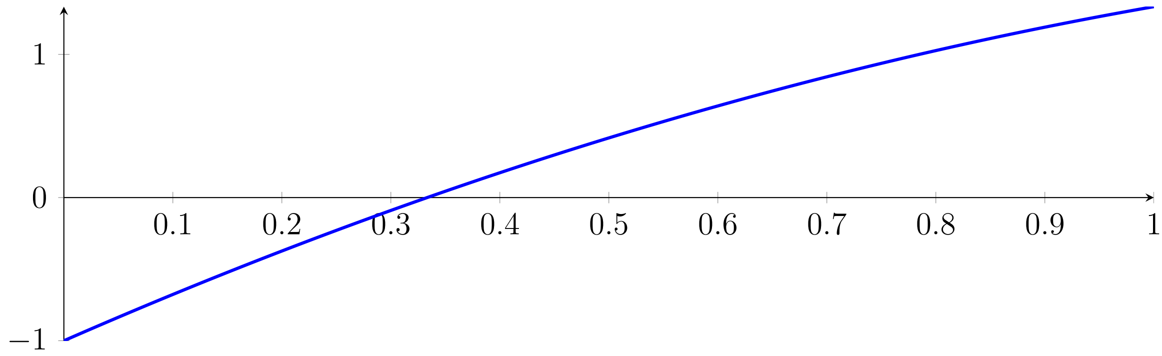
with precision $\varepsilon = 1{\cdot}10^{-16}$.

13 Running `BezClip` on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

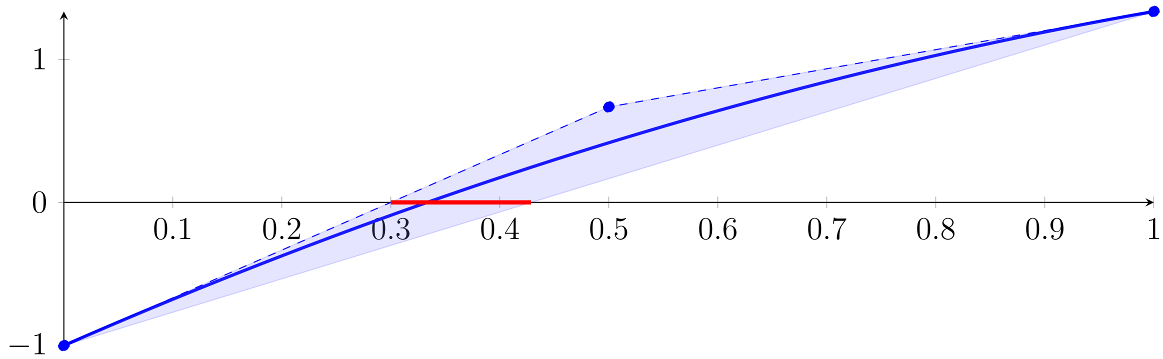
$$p = -1X^2 + 3.33333X - 1$$



13.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

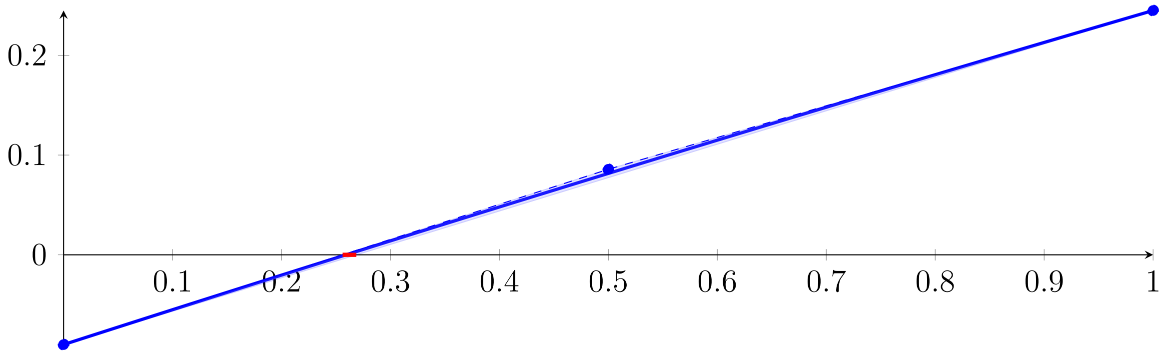
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

13.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

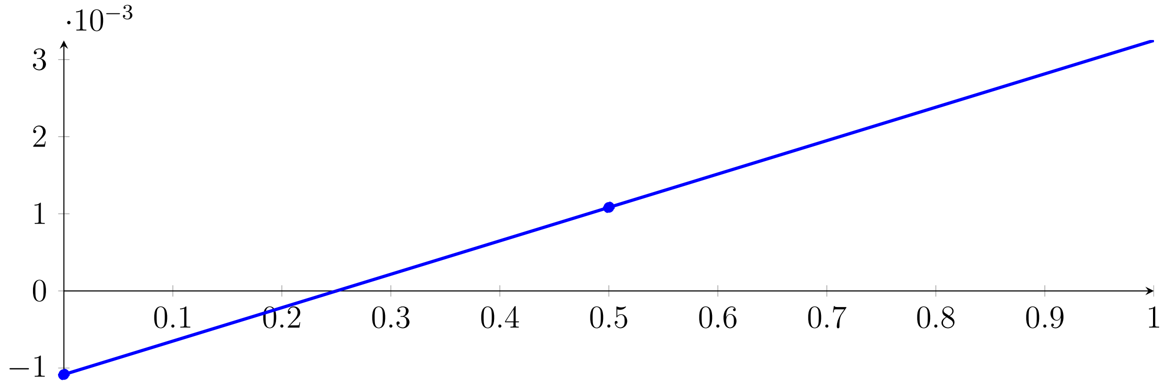
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

13.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

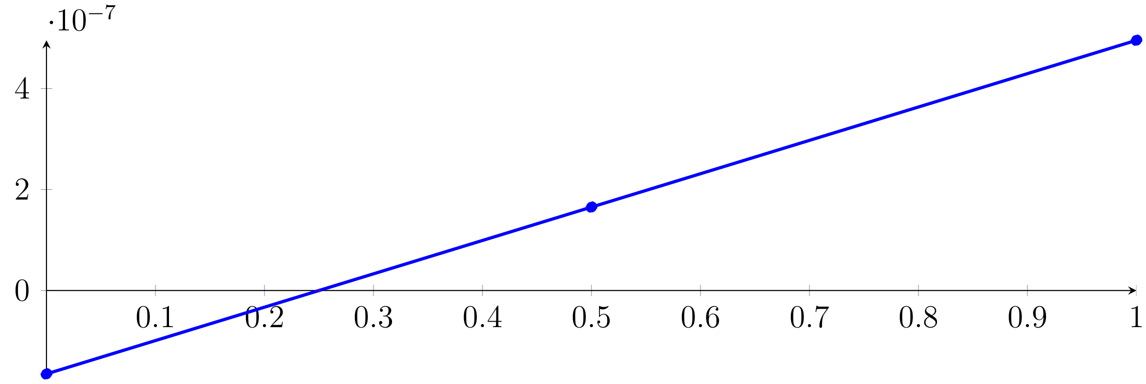
$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

13.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07} \\ &= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

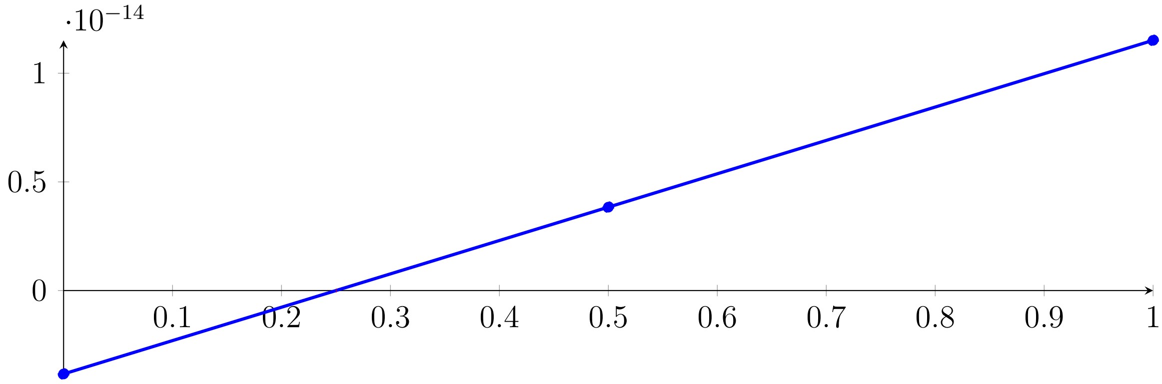
$$[0.25, 0.25]$$

Longest intersection interval: $2.32306 \cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

13.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -3.31322 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15} \\ &= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.55112 \cdot 10^{-16}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

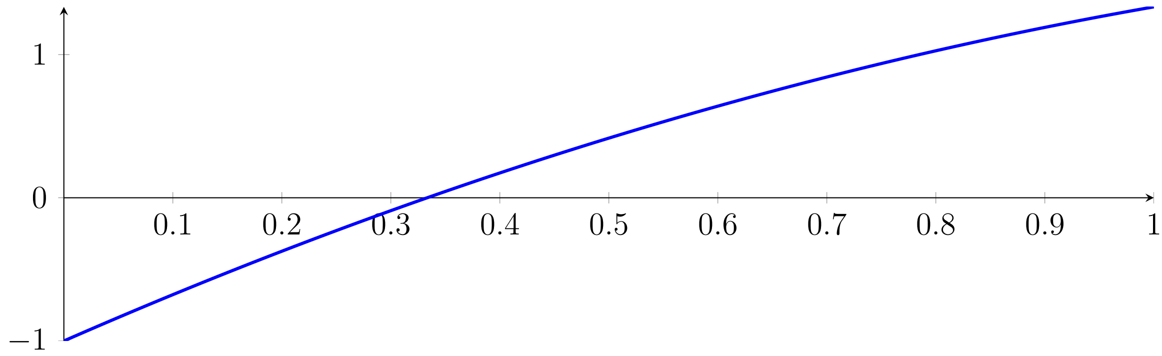
13.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

13.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

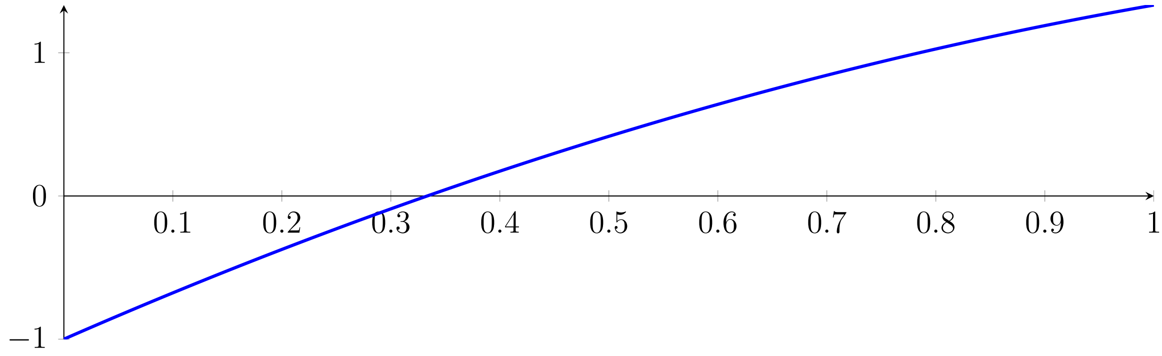
with precision $\varepsilon = 1 \cdot 10^{-32}$.

14 Running QuadClip on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

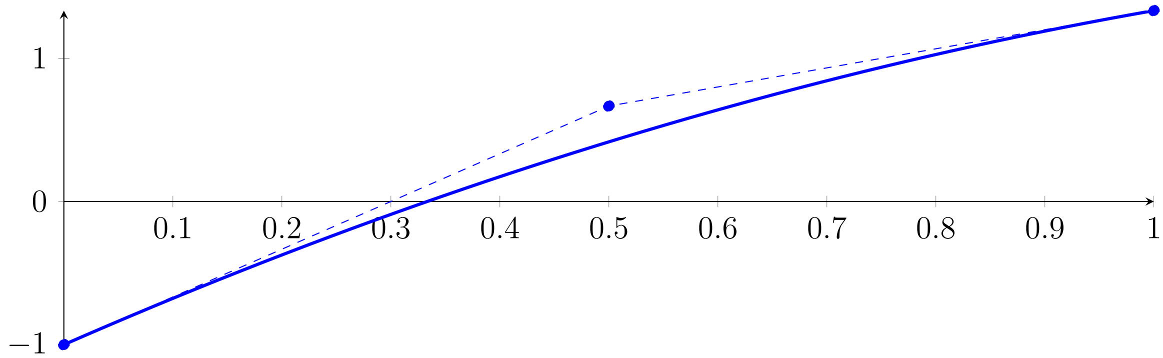
$$p = -1X^2 + 3.33333X - 1$$



14.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

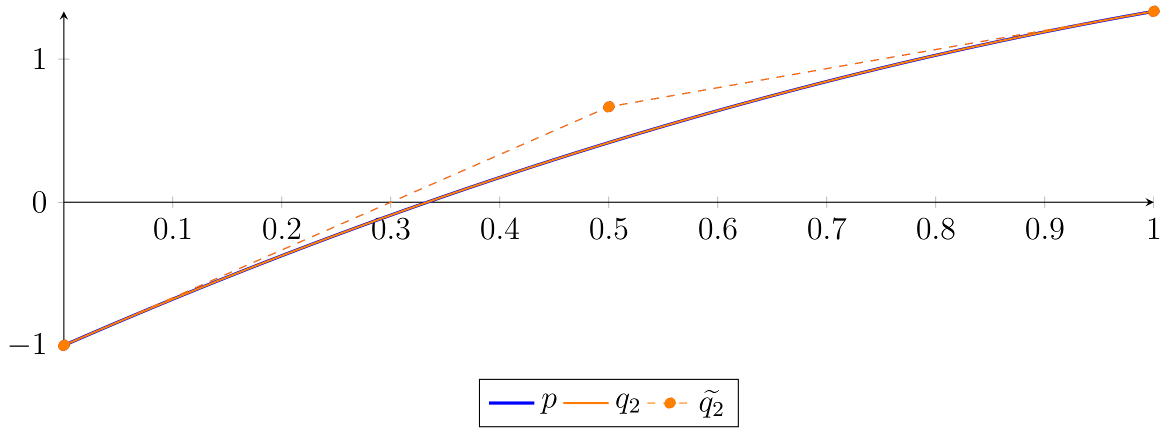
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.
Bounding polynomials M and m:

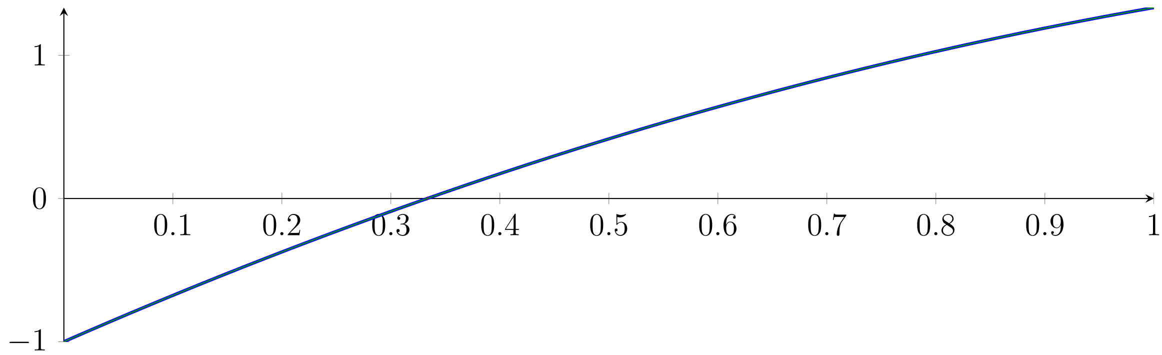
$$M = -1X^2 + 3.33333X - 1$$

$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



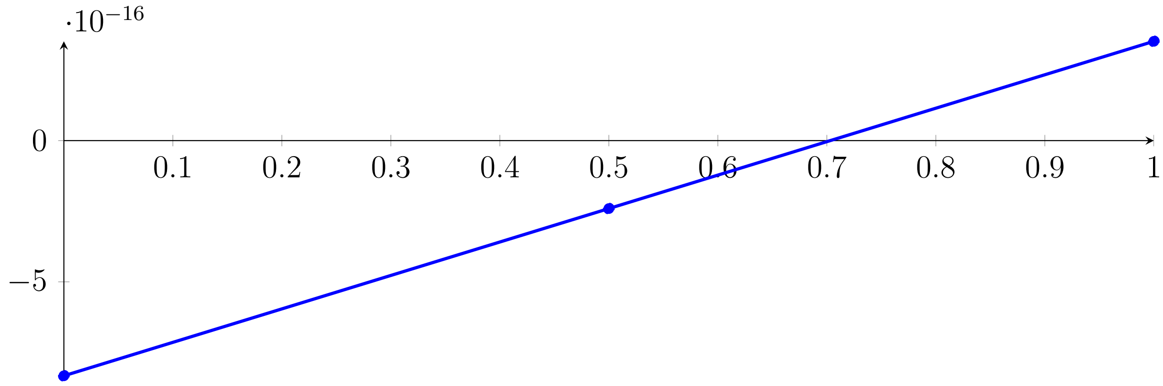
$$[0.333333, 0.333333]$$

Longest intersection interval: $4.44089 \cdot 10^{-16}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

14.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

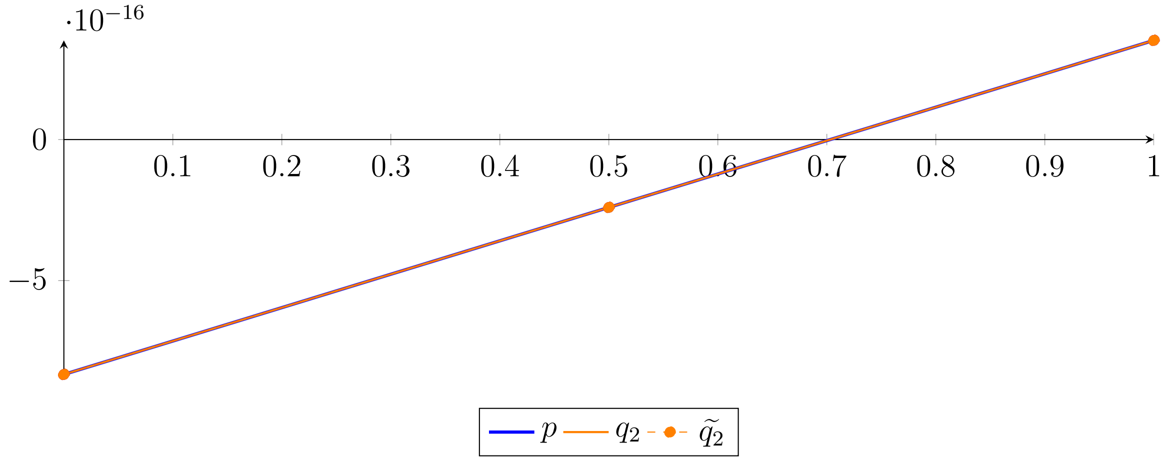
$$\begin{aligned} p &= -1.97215 \cdot 10^{-31} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2}(X) - 2.40548 \cdot 10^{-16} B_{1,2}(X) + 3.51571 \cdot 10^{-16} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.18329 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2} - 2.40548 \cdot 10^{-16} B_{1,2} + 3.51571 \cdot 10^{-16} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 2.5638 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2} - 2.40548 \cdot 10^{-16} B_{1,2} + 3.51571 \cdot 10^{-16} B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.2326 \cdot 10^{-30}$.

Bounding polynomials M and m:

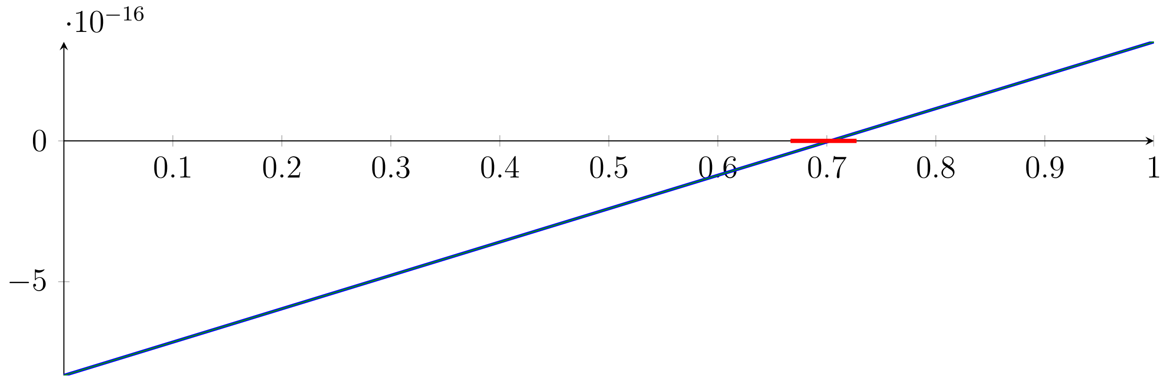
$$M = 1.08468 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$

$$m = 1.18329 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$

Root of M and m:

$$N(M) = \{-1.09178 \cdot 10^{15}, 0.727273\} \qquad N(m) = \{-1.0008 \cdot 10^{15}, 0.666667\}$$

Intersection intervals:



$$[0.666667, 0.727273]$$

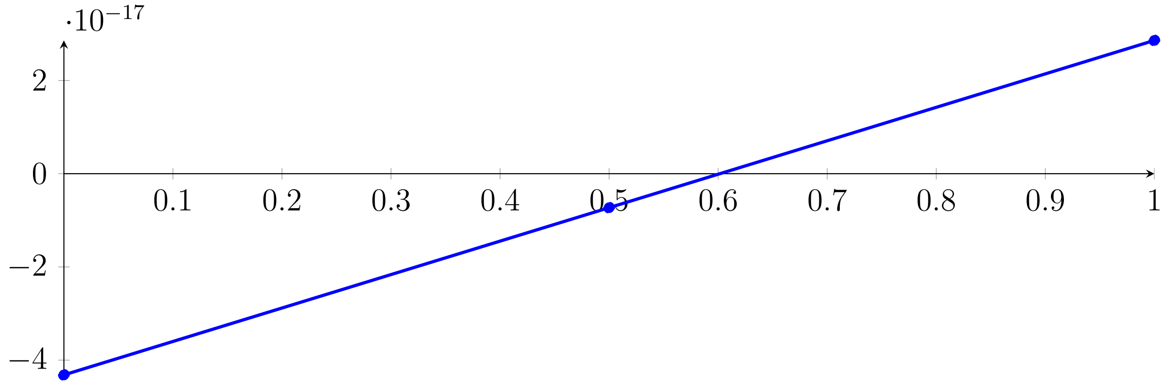
Longest intersection interval: 0.0606061

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

14.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

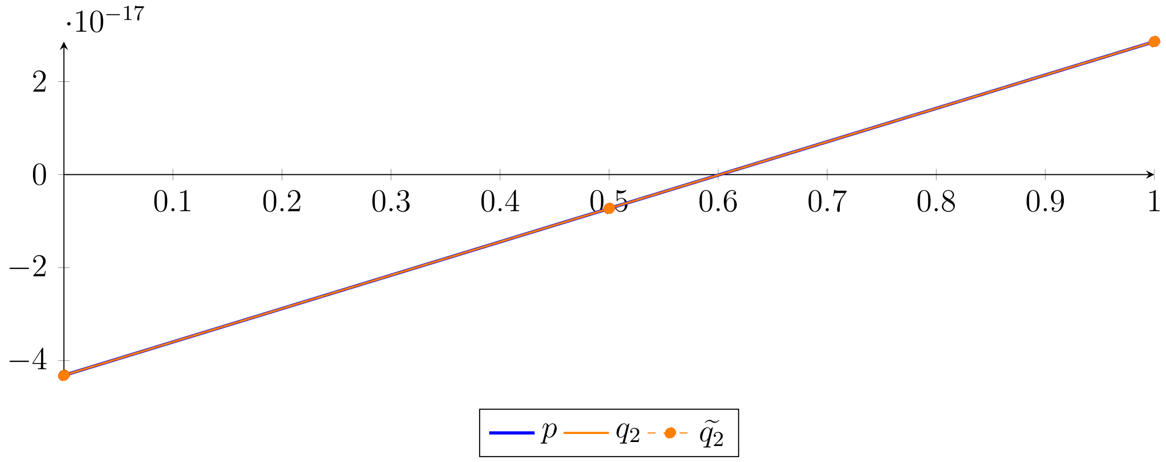
$$\begin{aligned} p &= 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17} \\ &= -4.31753 \cdot 10^{-17} B_{0,2}(X) - 7.28934 \cdot 10^{-18} B_{1,2}(X) + 2.85967 \cdot 10^{-17} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 5.54668{\cdot}10^{-32}X^2 + 7.1772{\cdot}10^{-17}X - 4.31753{\cdot}10^{-17} \\ &= -4.31753{\cdot}10^{-17}B_{0,2} - 7.28934{\cdot}10^{-18}B_{1,2} + 2.85967{\cdot}10^{-17}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 1.10934{\cdot}10^{-31}X^2 + 7.1772{\cdot}10^{-17}X - 4.31753{\cdot}10^{-17} \\ &= -4.31753{\cdot}10^{-17}B_{0,2} - 7.28934{\cdot}10^{-18}B_{1,2} + 2.85967{\cdot}10^{-17}B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.93038{\cdot}10^{-32}$.

Bounding polynomials M and m:

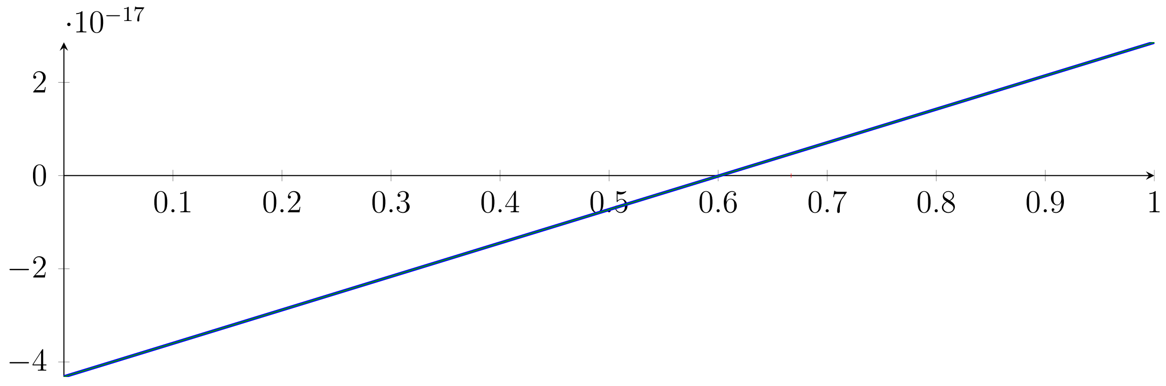
$$M = 5.54668{\cdot}10^{-32}X^2 + 7.1772{\cdot}10^{-17}X - 4.31753{\cdot}10^{-17}$$

$$m = 5.54668{\cdot}10^{-32}X^2 + 7.1772{\cdot}10^{-17}X - 4.31753{\cdot}10^{-17}$$

Root of M and m:

$$N(M) = \{-1.29396{\cdot}10^{15}, 0.666667\} \qquad N(m) = \{-1.29396{\cdot}10^{15}, 0.666667\}$$

Intersection intervals:



$$[0.666667, 0.666667]$$

Longest intersection interval: 0
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

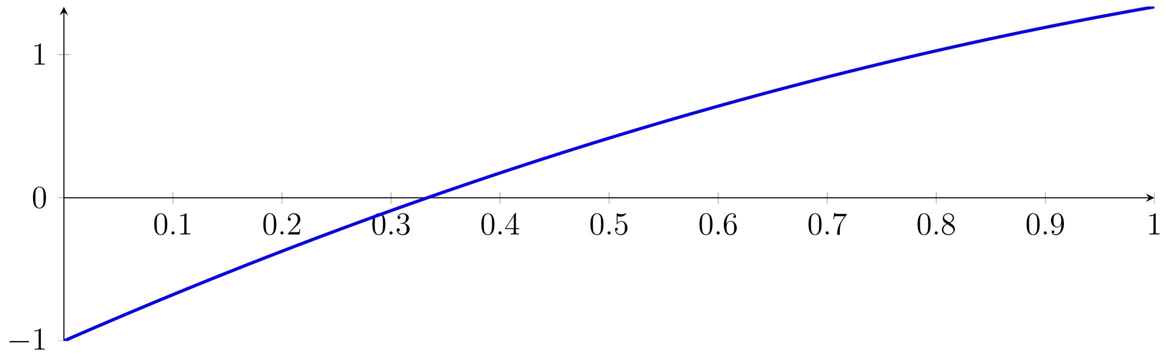
14.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

14.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

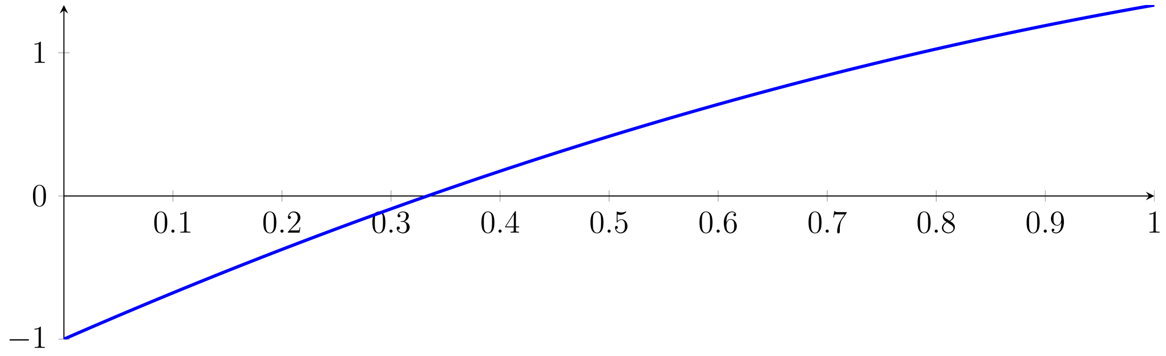
with precision $\varepsilon = 1 \cdot 10^{-32}$.

15 Running CubeClip on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

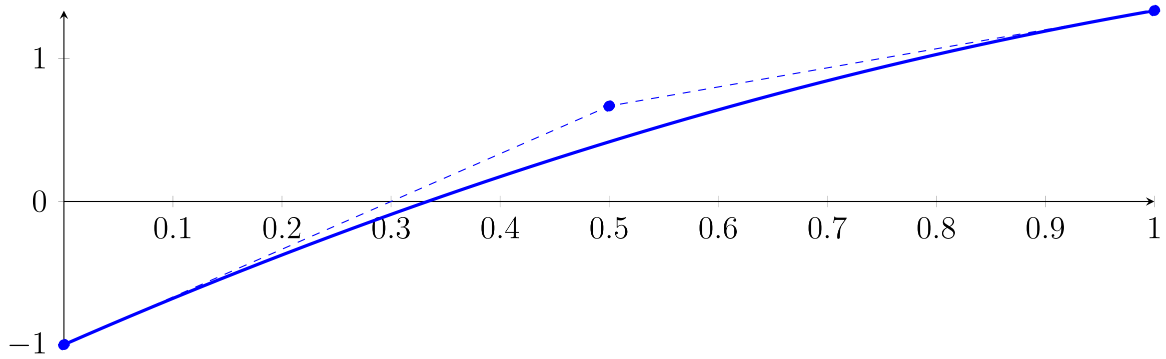
$$p = -1X^2 + 3.33333X - 1$$



15.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

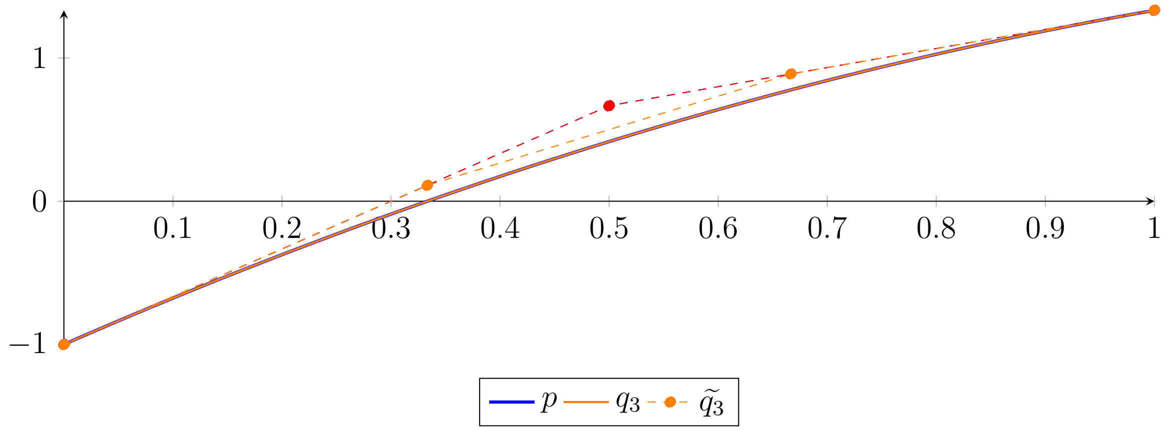
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.90958 \cdot 10^{-14}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.
Bounding polynomials M and m:

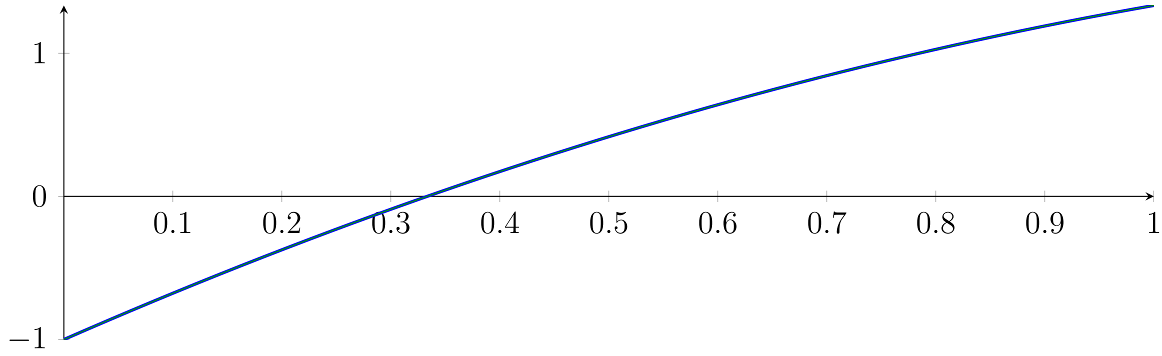
$$M = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

$$m = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{-5.17655 \cdot 10^{13}\} \qquad N(m) = \{-5.17655 \cdot 10^{13}\}$$

Intersection intervals:

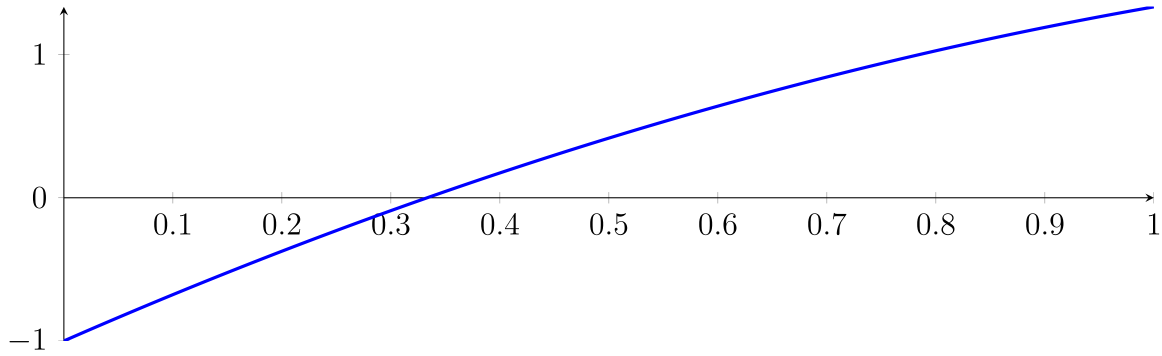


No intersection intervals with the x axis.

15.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

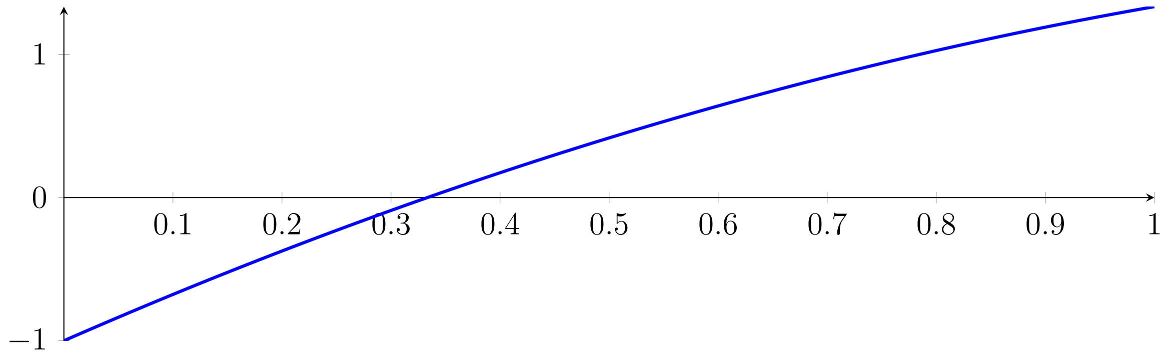
with precision $\varepsilon = 1 \cdot 10^{-32}$.

16 Running `BezClip` on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

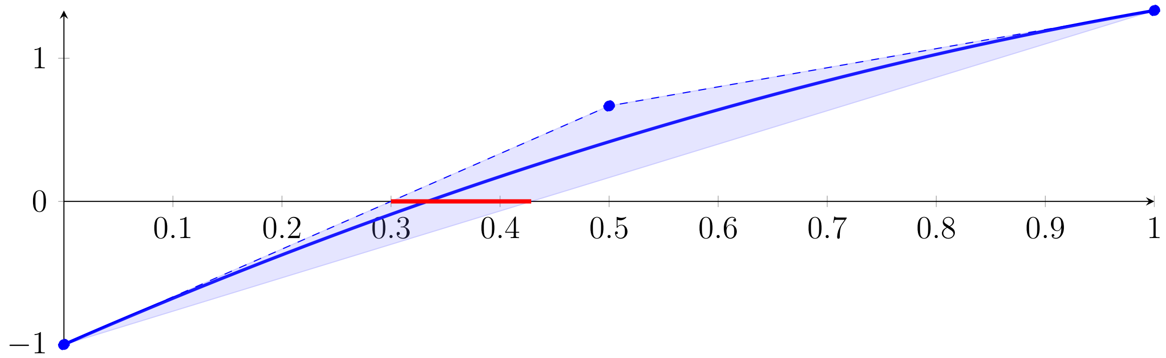
$$p = -1X^2 + 3.33333X - 1$$



16.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

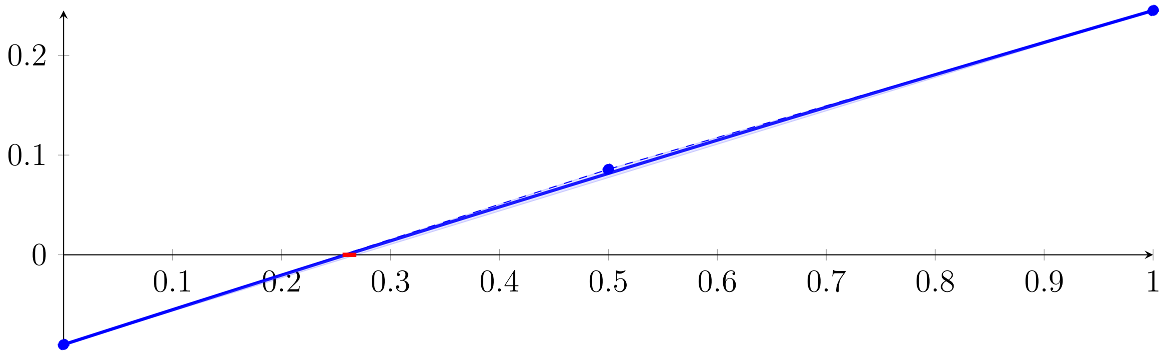
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

16.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

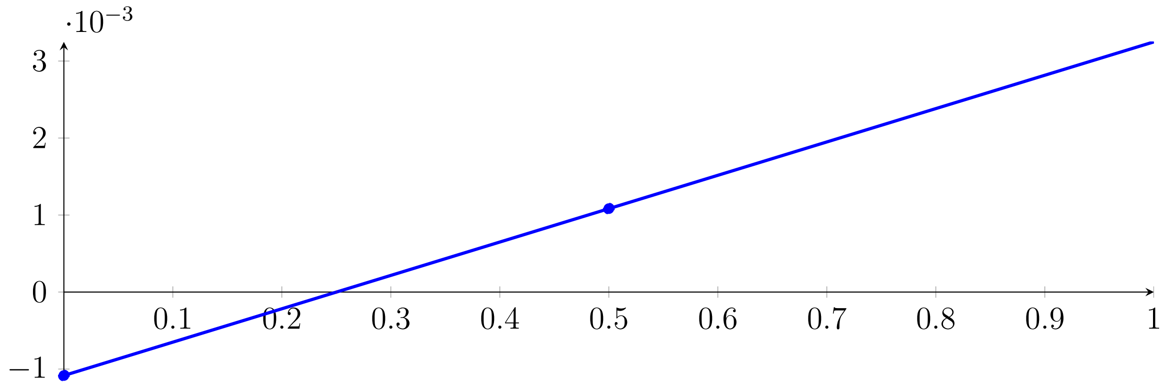
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

16.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

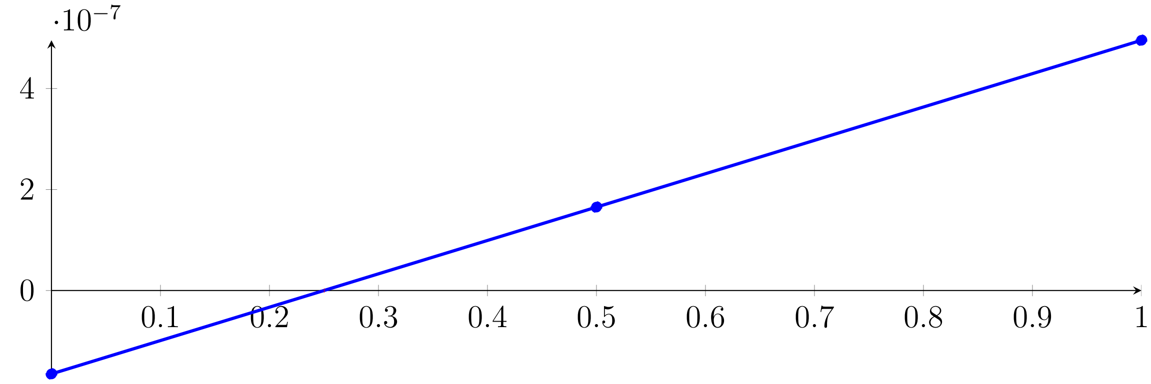
$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

16.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07}$$
$$= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

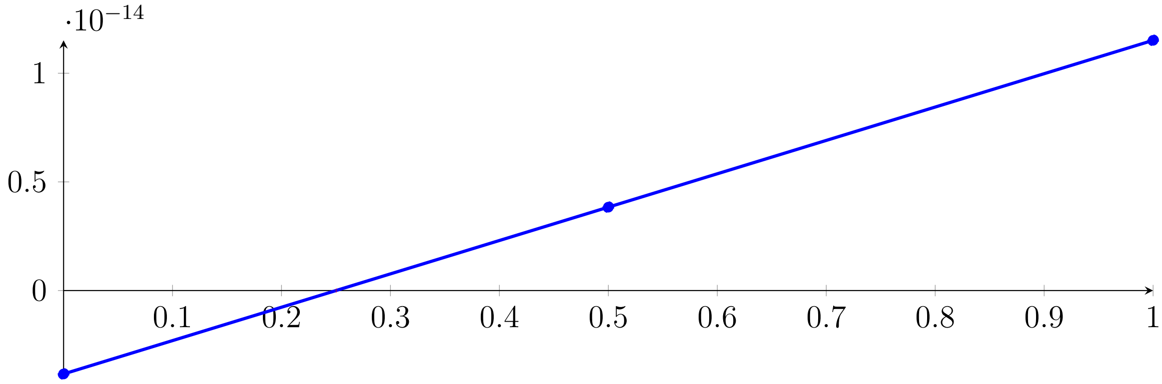
$$[0.25, 0.25]$$

Longest intersection interval: $2.32306 \cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

16.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.31322 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15}$$
$$= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.55112 \cdot 10^{-16}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

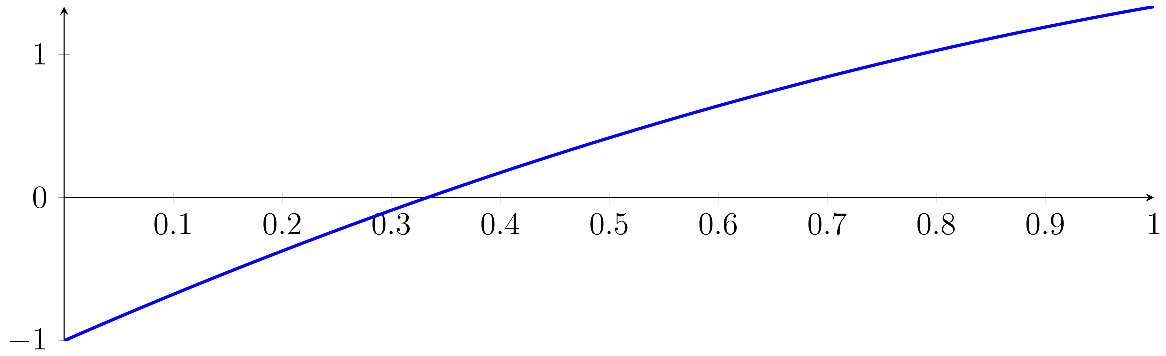
16.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

16.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

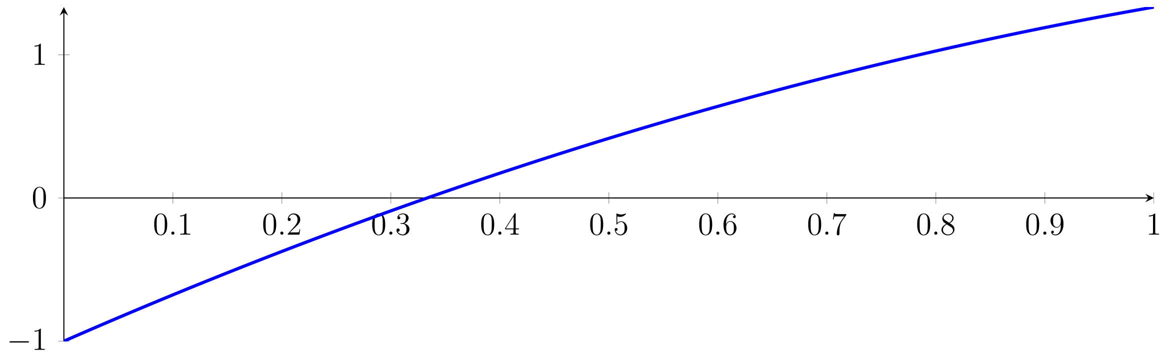
with precision $\varepsilon = 1 \cdot 10^{-64}$.

17 Running QuadClip on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

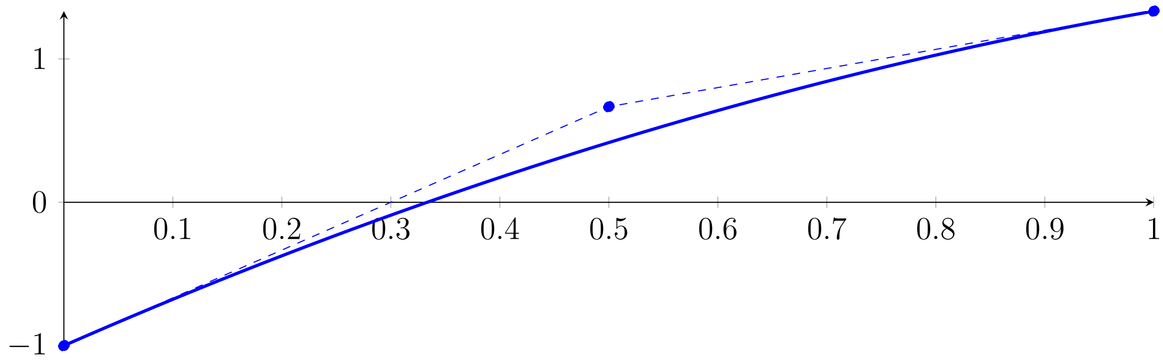
$$p = -1X^2 + 3.33333X - 1$$



17.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

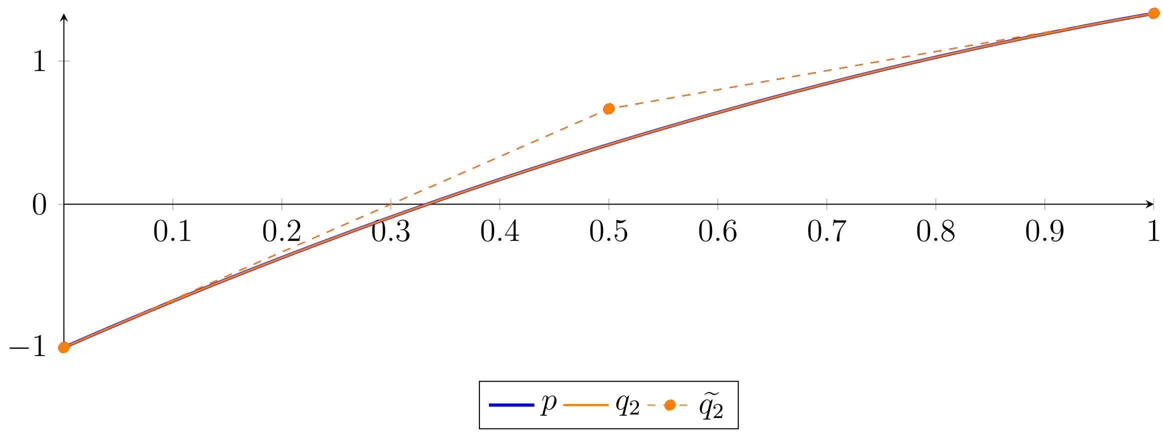
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.
Bounding polynomials M and m:

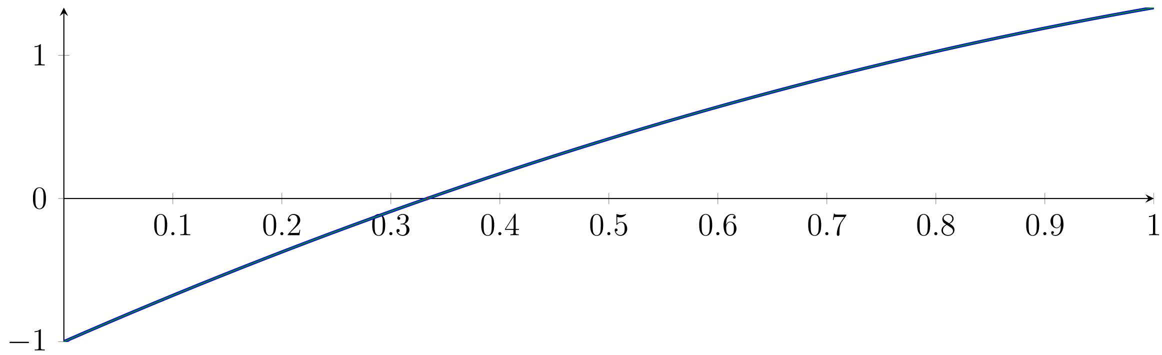
$$M = -1X^2 + 3.33333X - 1$$

$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



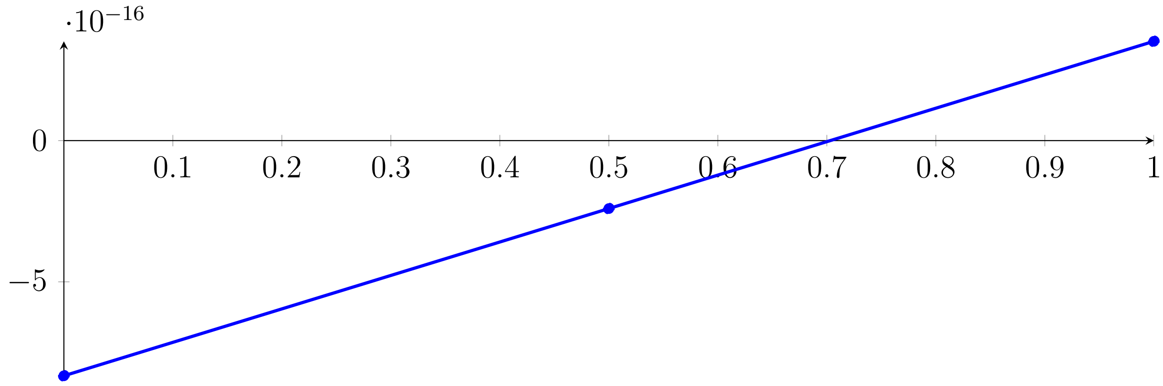
$$[0.333333, 0.333333]$$

Longest intersection interval: $4.44089 \cdot 10^{-16}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

17.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

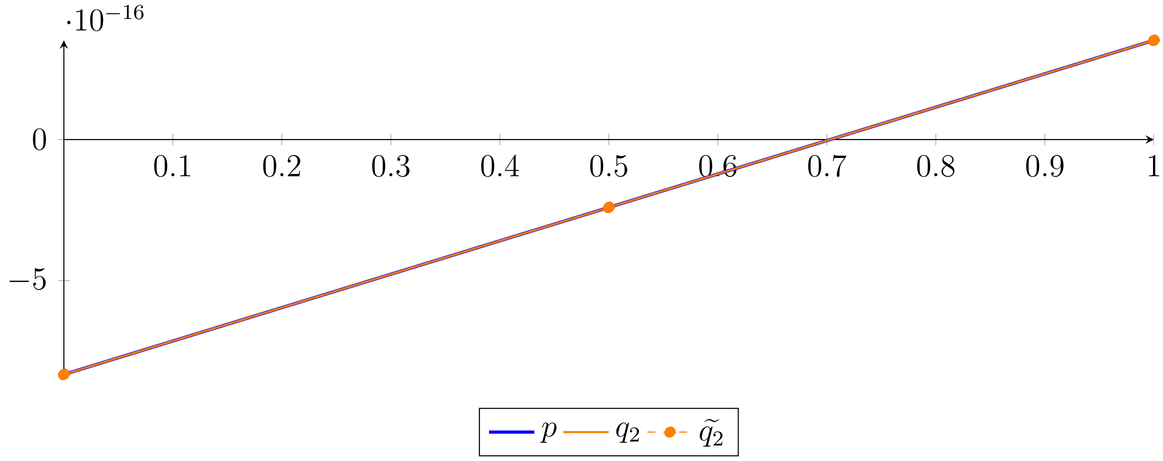
$$\begin{aligned} p &= -1.97215 \cdot 10^{-31} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2}(X) - 2.40548 \cdot 10^{-16} B_{1,2}(X) + 3.51571 \cdot 10^{-16} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.18329 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2} - 2.40548 \cdot 10^{-16} B_{1,2} + 3.51571 \cdot 10^{-16} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 2.5638 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2} - 2.40548 \cdot 10^{-16} B_{1,2} + 3.51571 \cdot 10^{-16} B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.2326 \cdot 10^{-30}$.

Bounding polynomials M and m:

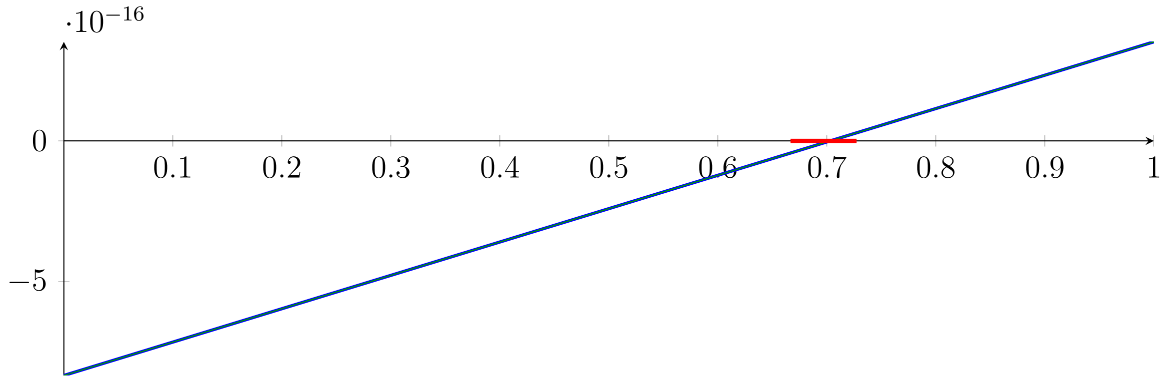
$$M = 1.08468 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$

$$m = 1.18329 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$

Root of M and m:

$$N(M) = \{-1.09178 \cdot 10^{15}, 0.727273\} \qquad N(m) = \{-1.0008 \cdot 10^{15}, 0.666667\}$$

Intersection intervals:



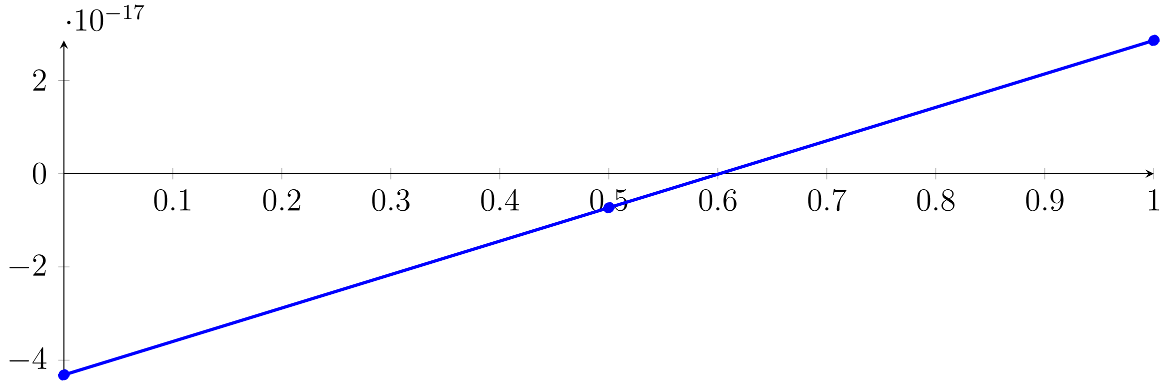
$$[0.666667, 0.727273]$$

Longest intersection interval: 0.0606061
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

17.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

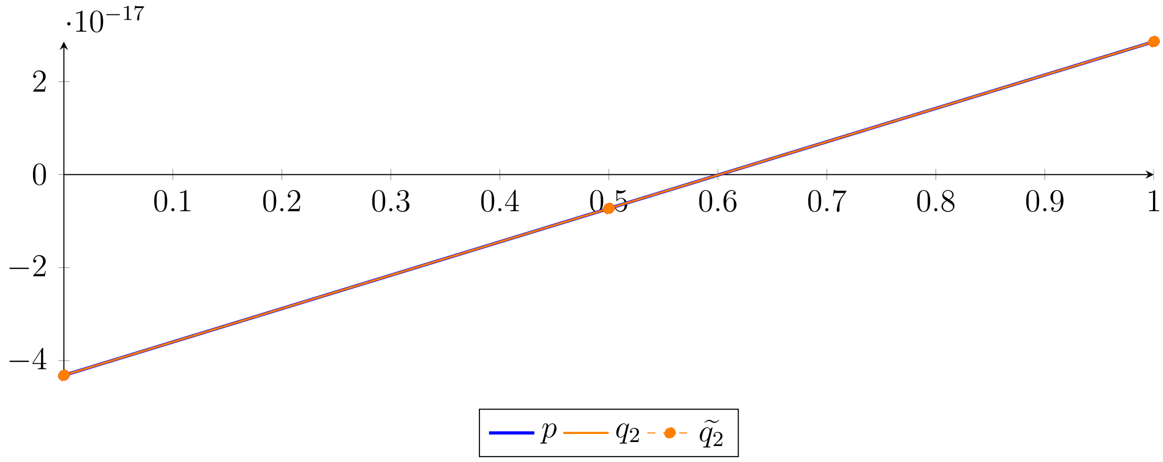
$$\begin{aligned} p &= 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17} \\ &= -4.31753 \cdot 10^{-17} B_{0,2}(X) - 7.28934 \cdot 10^{-18} B_{1,2}(X) + 2.85967 \cdot 10^{-17} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = 5.54668 \cdot 10^{-32} X^2 + 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17}$$
$$= -4.31753 \cdot 10^{-17} B_{0,2} - 7.28934 \cdot 10^{-18} B_{1,2} + 2.85967 \cdot 10^{-17} B_{2,2}$$

$$\widetilde{q_2} = 1.10934 \cdot 10^{-31} X^2 + 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17}$$
$$= -4.31753 \cdot 10^{-17} B_{0,2} - 7.28934 \cdot 10^{-18} B_{1,2} + 2.85967 \cdot 10^{-17} B_{2,2}$$



The maximum difference of the Bézier coefficients is $\delta = 4.93038 \cdot 10^{-32}$.

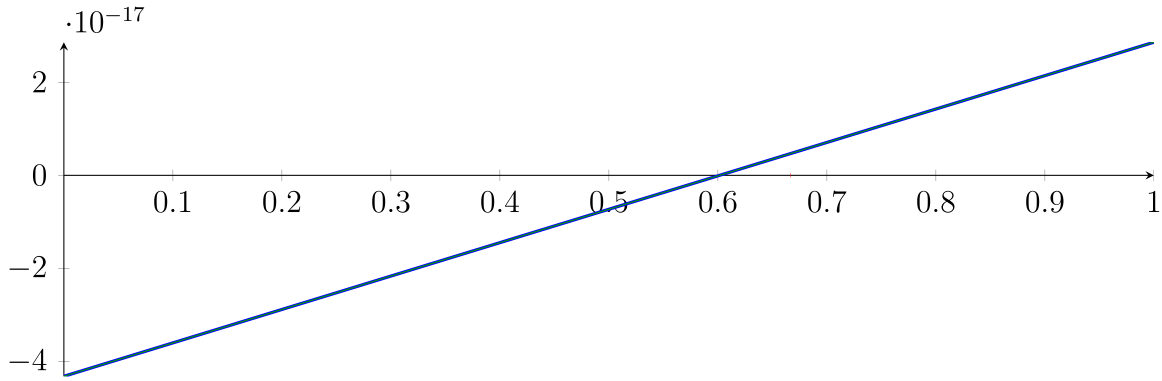
Bounding polynomials M and m:

$$M = 5.54668 \cdot 10^{-32} X^2 + 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17}$$
$$m = 5.54668 \cdot 10^{-32} X^2 + 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17}$$

Root of M and m:

$$N(M) = \{-1.29396 \cdot 10^{15}, 0.666667\} \qquad N(m) = \{-1.29396 \cdot 10^{15}, 0.666667\}$$

Intersection intervals:



$$[0.666667, 0.666667]$$

Longest intersection interval: 0
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

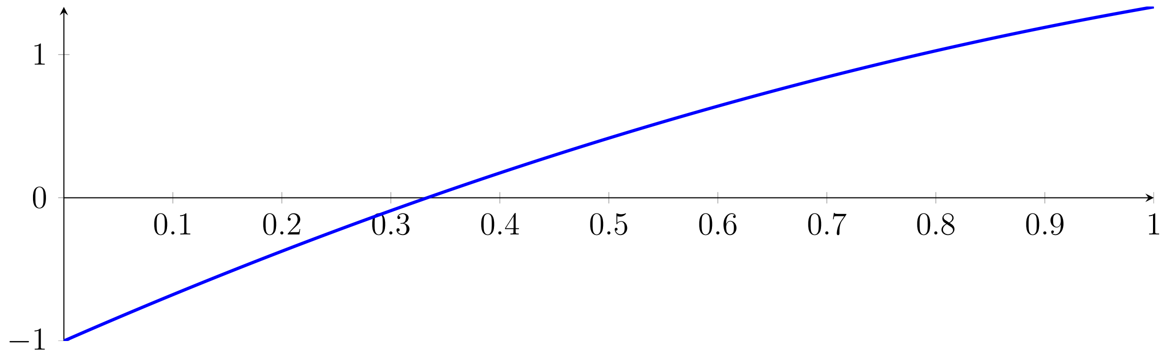
17.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

17.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

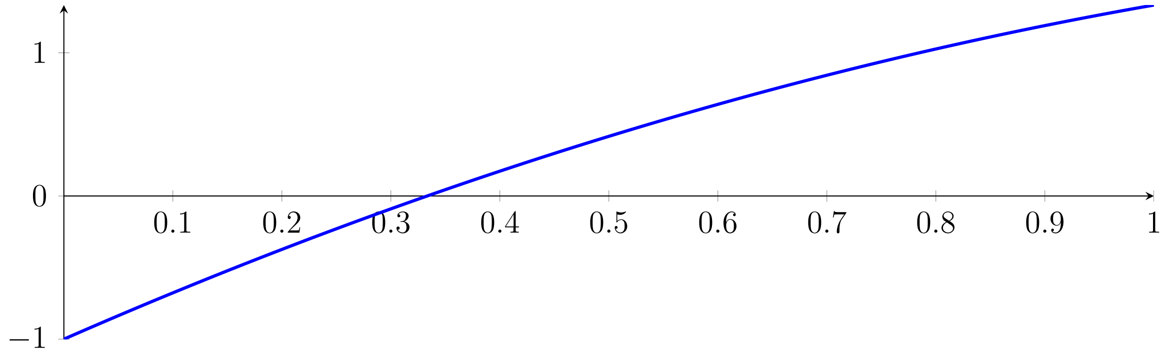
with precision $\varepsilon = 1 \cdot 10^{-64}$.

18 Running CubeClip on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

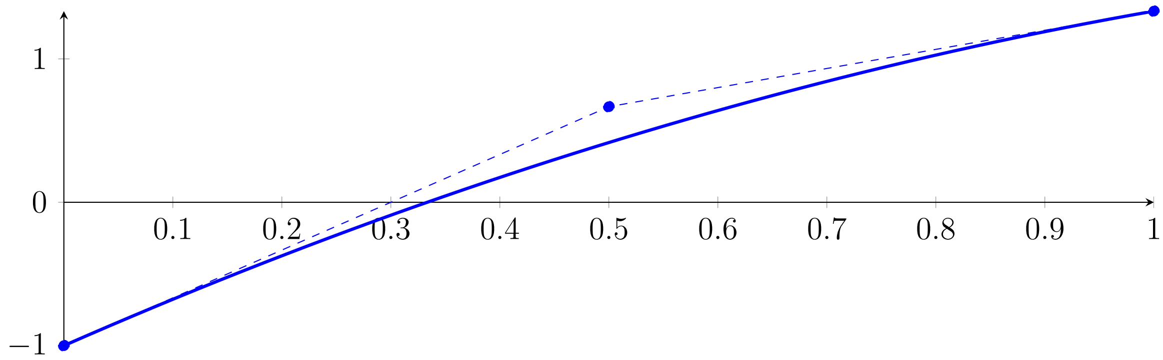
$$p = -1X^2 + 3.33333X - 1$$



18.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

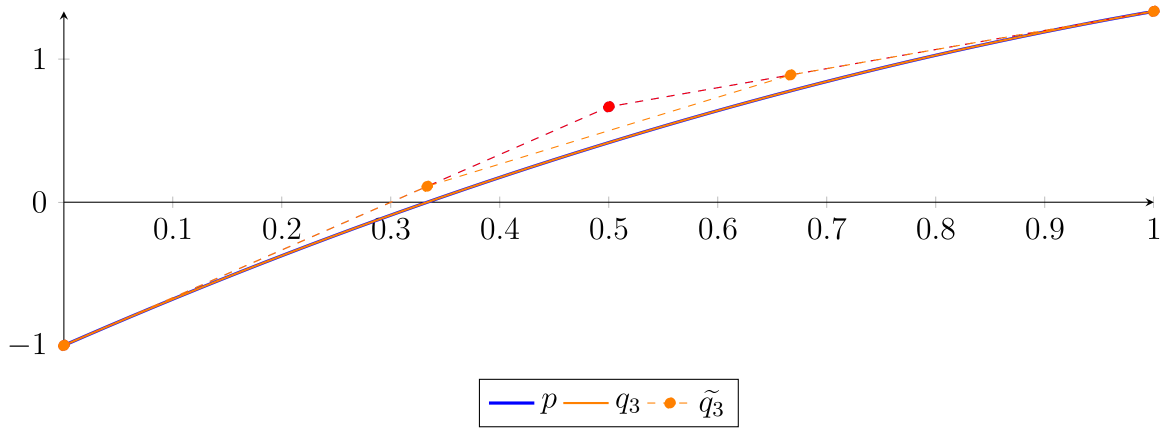
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.90958 \cdot 10^{-14}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.
Bounding polynomials M and m:

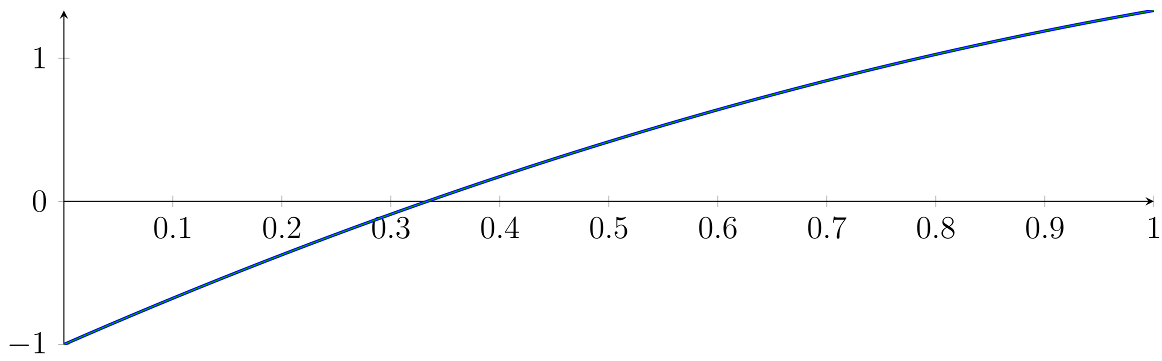
$$M = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

$$m = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{-5.17655 \cdot 10^{13}\} \qquad N(m) = \{-5.17655 \cdot 10^{13}\}$$

Intersection intervals:

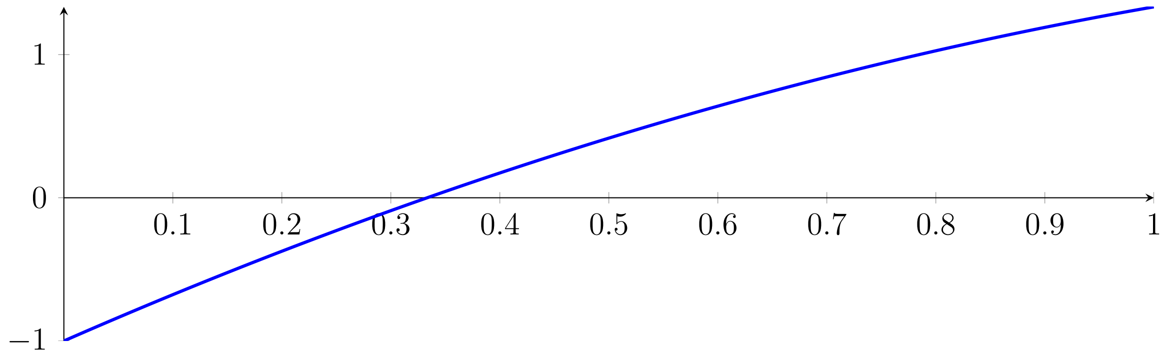


No intersection intervals with the x axis.

18.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

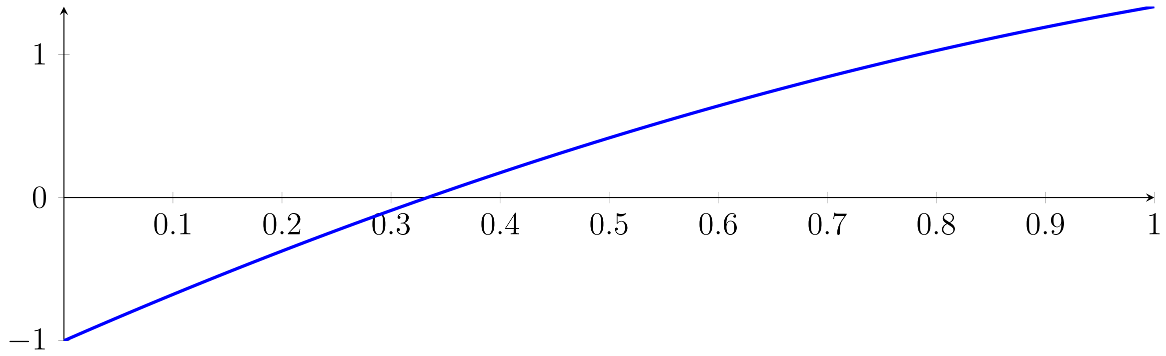
with precision $\varepsilon = 1 \cdot 10^{-64}$.

19 Running `BezClip` on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called `BezClip` with input polynomial on interval $[0,1]$:

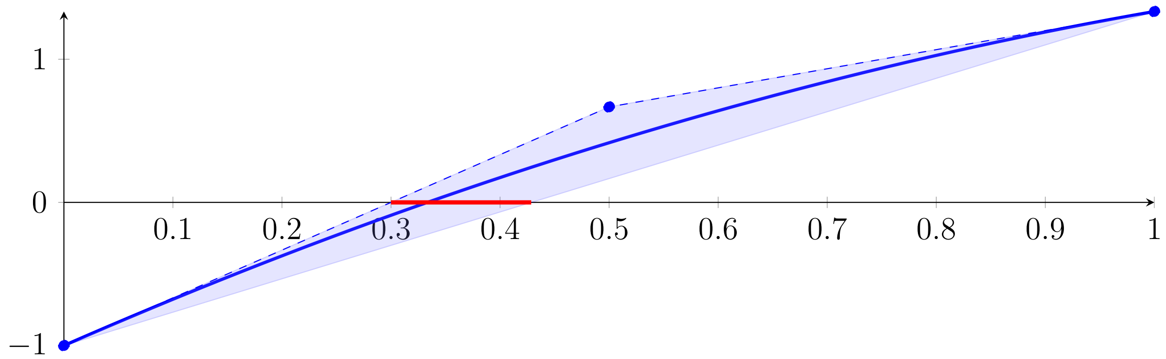
$$p = -1X^2 + 3.33333X - 1$$



19.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

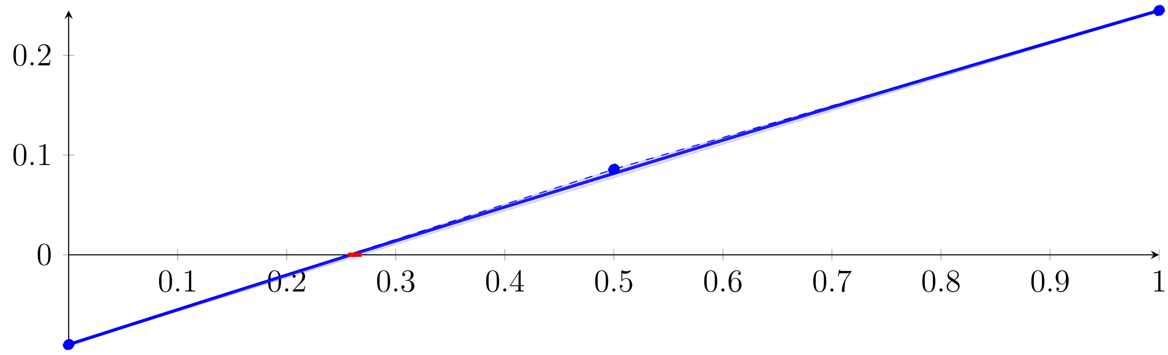
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\implies$ Selective recursion: interval 1: $[0.3, 0.428571]$,

19.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

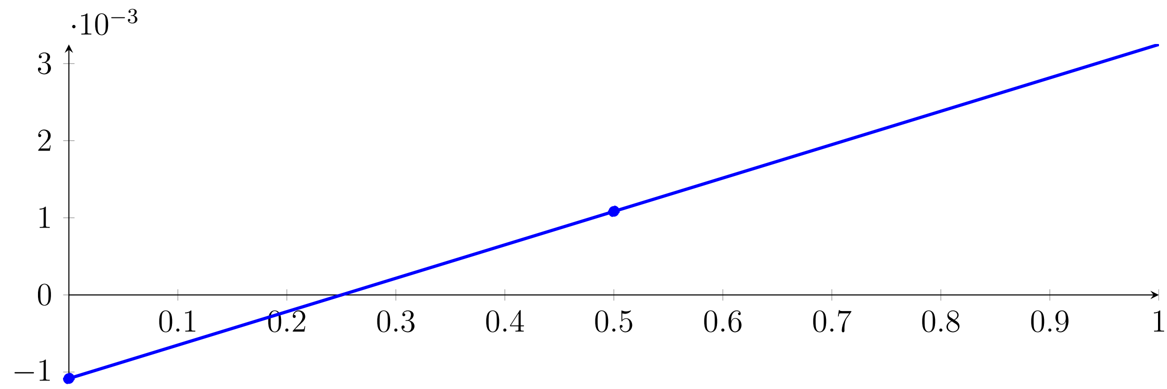
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

19.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

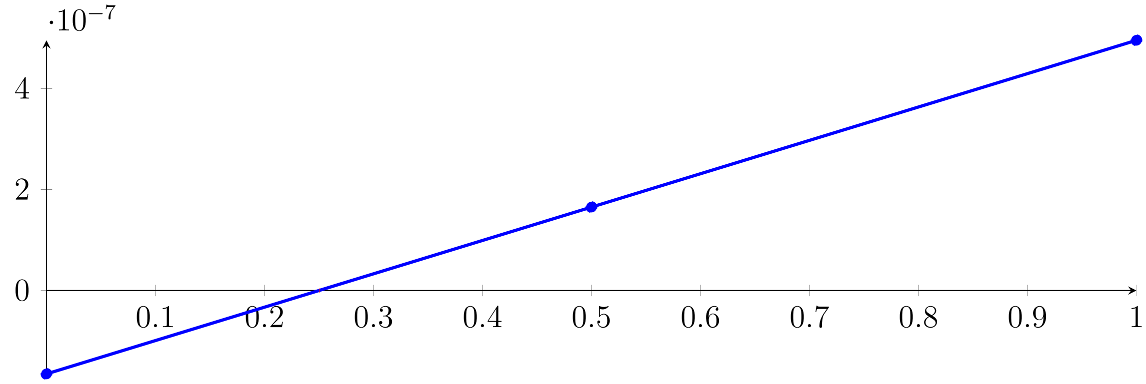
$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

19.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07}$$
$$= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

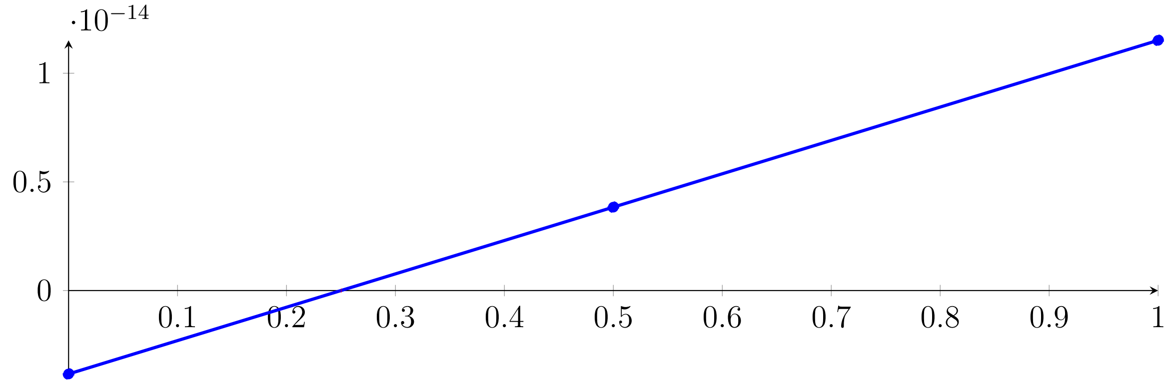
Longest intersection interval: $2.32306 \cdot 10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

19.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.31322 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15}$$
$$= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.55112 \cdot 10^{-16}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

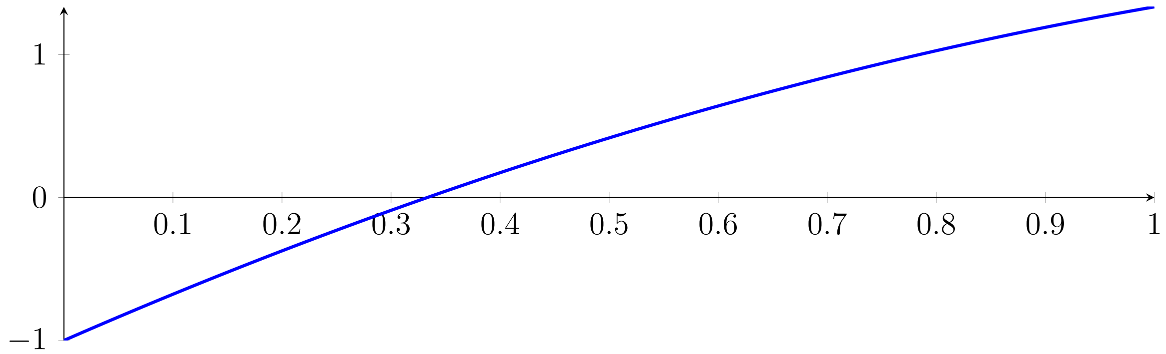
19.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

19.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

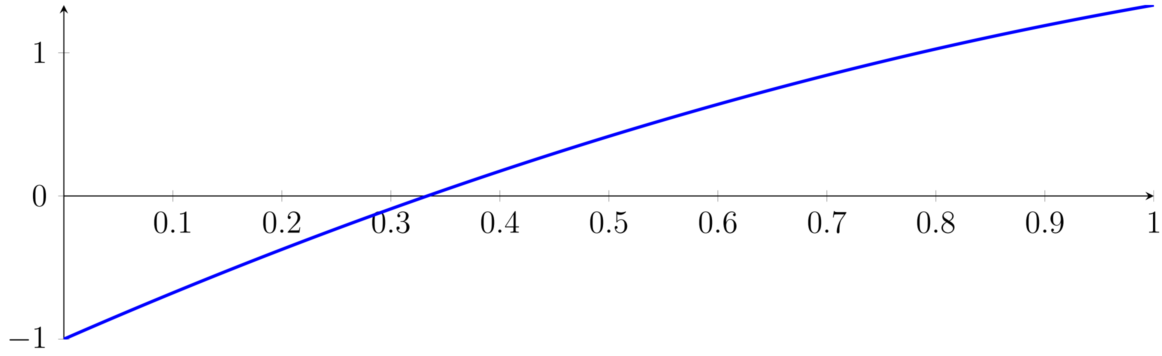
with precision $\varepsilon = 1 \cdot 10^{-128}$.

20 Running `QuadClip` on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called `QuadClip` with input polynomial on interval $[0, 1]$:

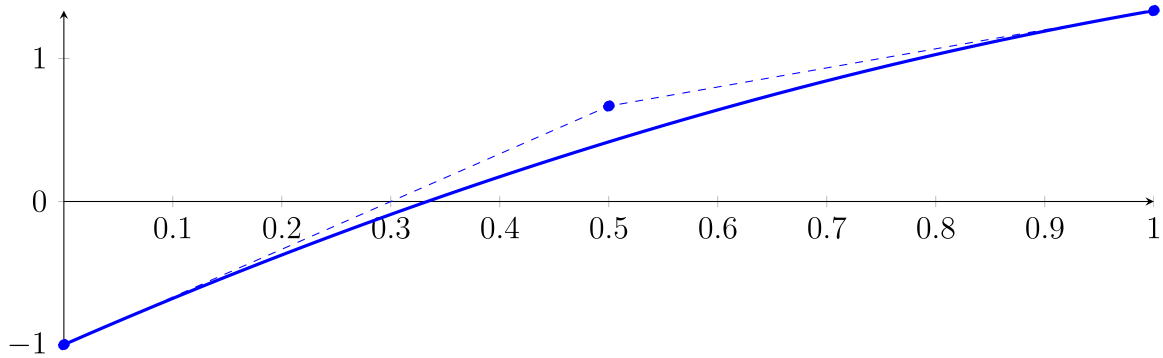
$$p = -1X^2 + 3.33333X - 1$$



20.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

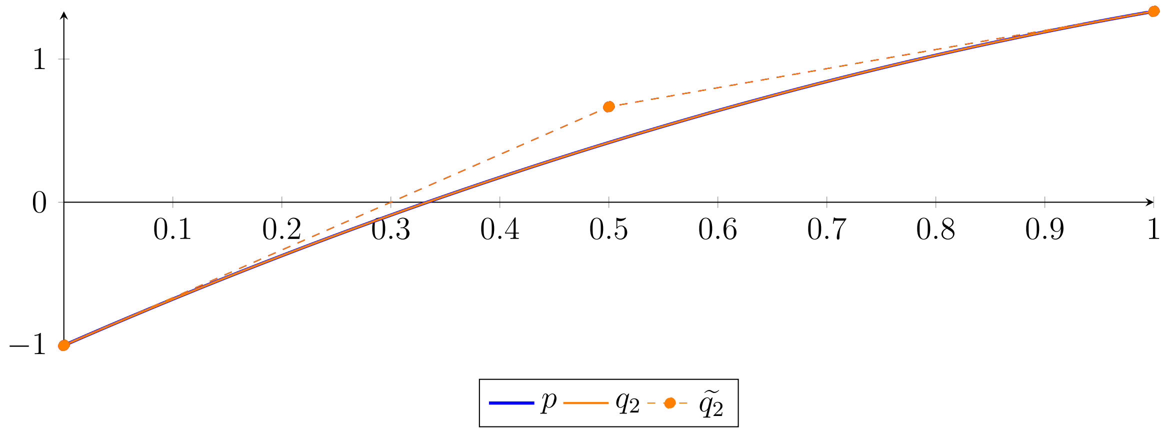
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134{\cdot}10^{-16}$.

Bounding polynomials M and m:

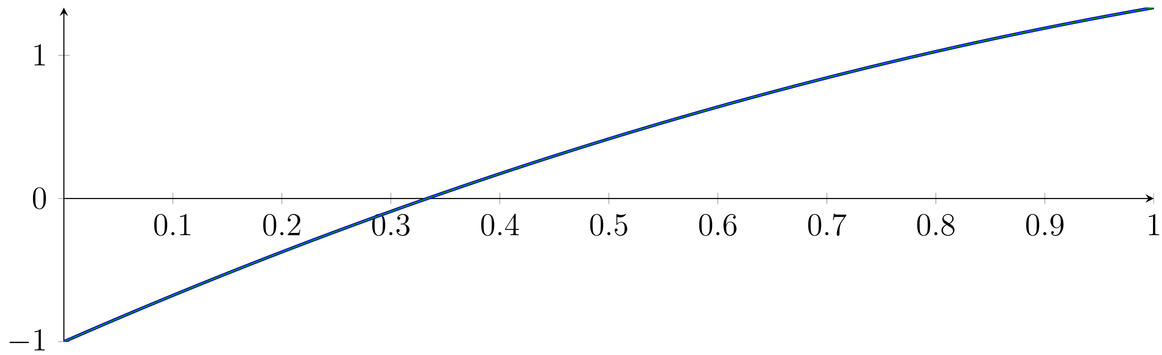
$$M = -1X^2 + 3.33333X - 1$$

$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

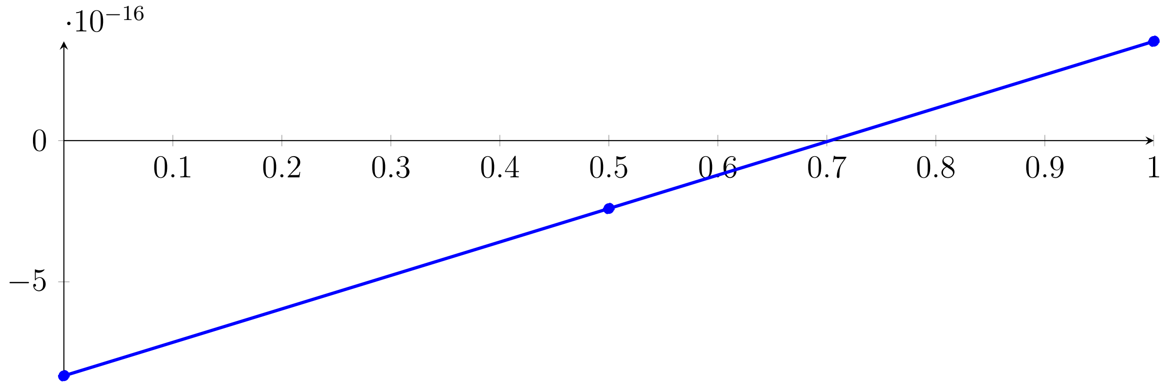
Longest intersection interval: $4.44089{\cdot}10^{-16}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

20.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

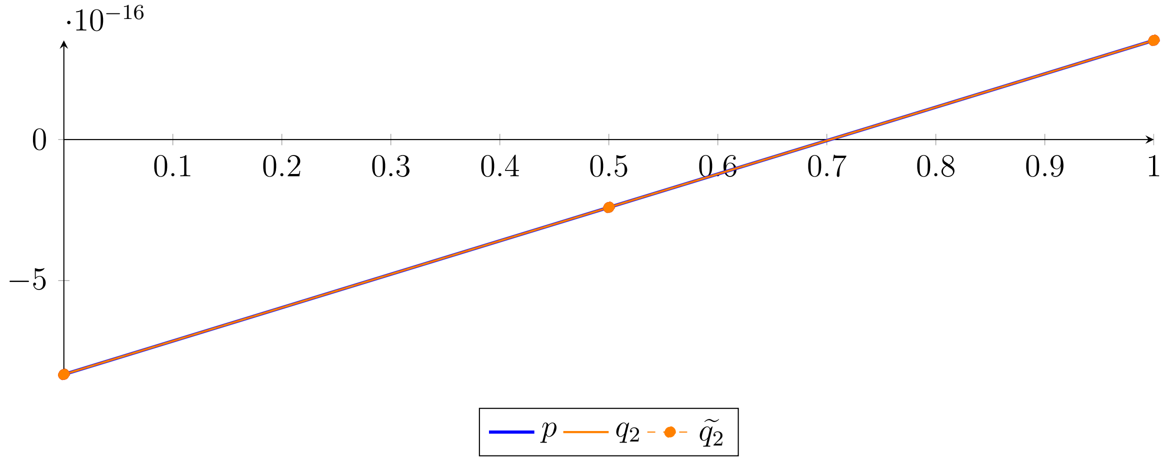
$$\begin{aligned} p &= -1.97215{\cdot}10^{-31}X^2 + 1.18424{\cdot}10^{-15}X - 8.32667{\cdot}10^{-16} \\ &= -8.32667{\cdot}10^{-16}B_{0,2}(X) - 2.40548{\cdot}10^{-16}B_{1,2}(X) + 3.51571{\cdot}10^{-16}B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.18329{\cdot}10^{-30}X^2 + 1.18424{\cdot}10^{-15}X - 8.32667{\cdot}10^{-16} \\ &= -8.32667{\cdot}10^{-16}B_{0,2} - 2.40548{\cdot}10^{-16}B_{1,2} + 3.51571{\cdot}10^{-16}B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 2.5638{\cdot}10^{-30}X^2 + 1.18424{\cdot}10^{-15}X - 8.32667{\cdot}10^{-16} \\ &= -8.32667{\cdot}10^{-16}B_{0,2} - 2.40548{\cdot}10^{-16}B_{1,2} + 3.51571{\cdot}10^{-16}B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.2326 \cdot 10^{-30}$.

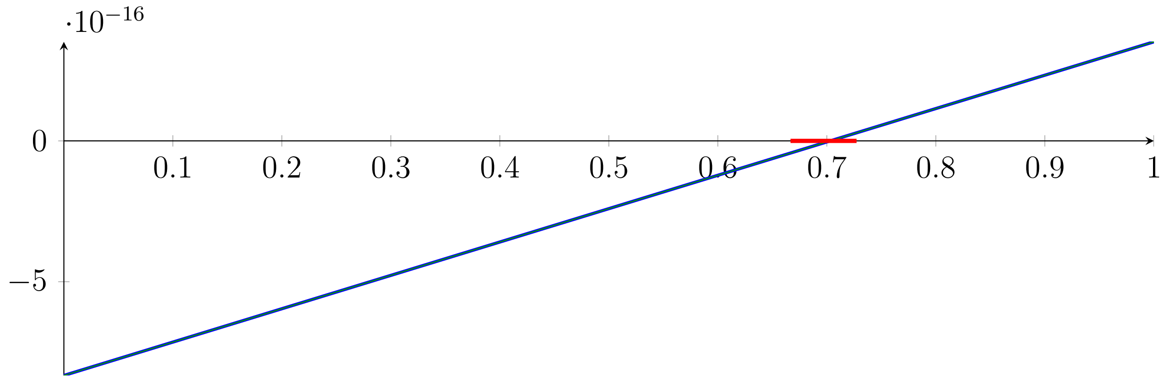
Bounding polynomials M and m:

$$M = 1.08468 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$
$$m = 1.18329 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$

Root of M and m:

$$N(M) = \{-1.09178 \cdot 10^{15}, 0.727273\} \qquad N(m) = \{-1.0008 \cdot 10^{15}, 0.666667\}$$

Intersection intervals:



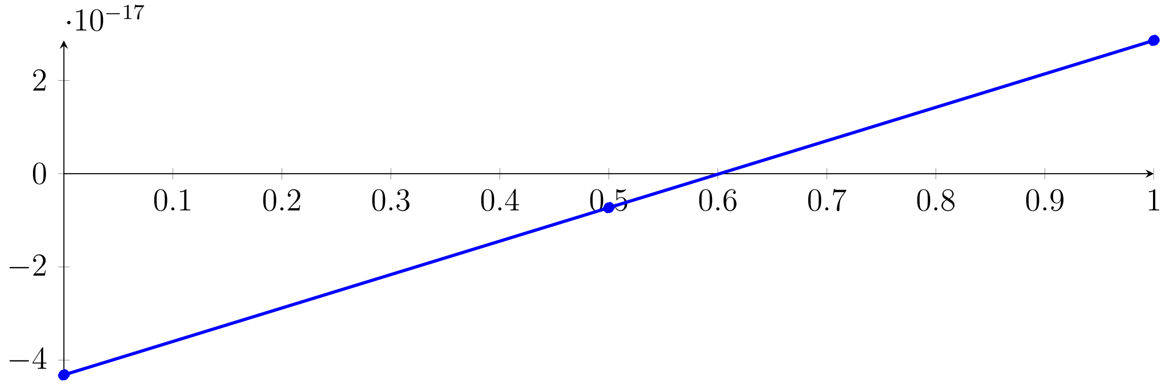
$$[0.666667, 0.727273]$$

Longest intersection interval: 0.0606061
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

20.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

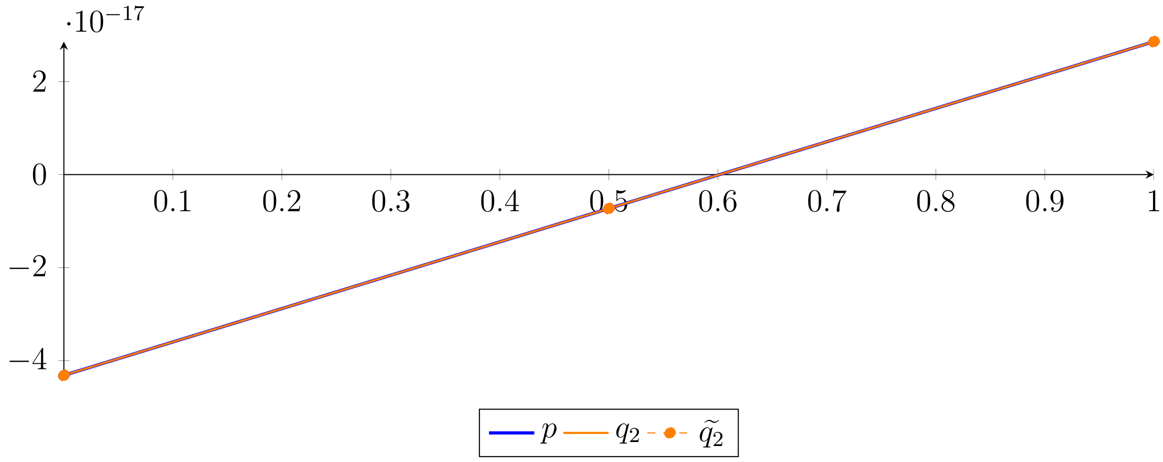
$$\begin{aligned} p &= 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17} \\ &= -4.31753 \cdot 10^{-17} B_{0,2}(X) - 7.28934 \cdot 10^{-18} B_{1,2}(X) + 2.85967 \cdot 10^{-17} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 5.54668{\cdot}10^{-32}X^2 + 7.1772{\cdot}10^{-17}X - 4.31753{\cdot}10^{-17} \\ &= -4.31753{\cdot}10^{-17}B_{0,2} - 7.28934{\cdot}10^{-18}B_{1,2} + 2.85967{\cdot}10^{-17}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 1.10934{\cdot}10^{-31}X^2 + 7.1772{\cdot}10^{-17}X - 4.31753{\cdot}10^{-17} \\ &= -4.31753{\cdot}10^{-17}B_{0,2} - 7.28934{\cdot}10^{-18}B_{1,2} + 2.85967{\cdot}10^{-17}B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.93038{\cdot}10^{-32}$.

Bounding polynomials M and m:

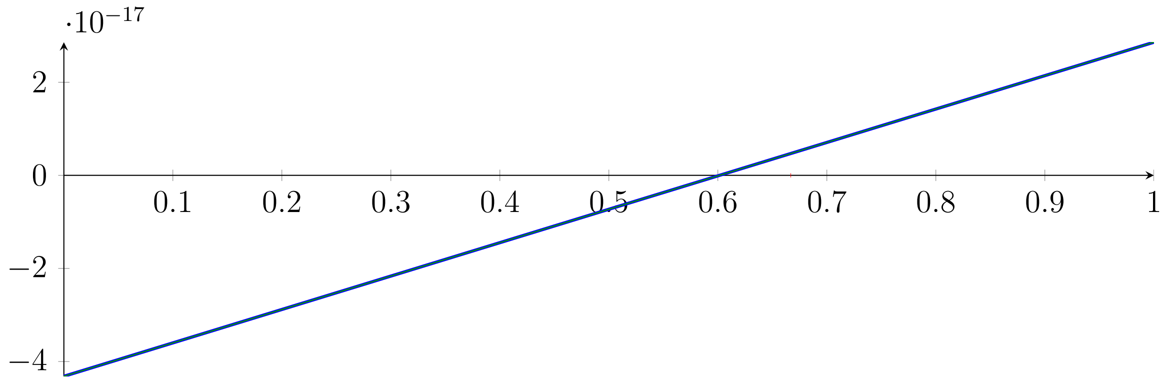
$$M = 5.54668{\cdot}10^{-32}X^2 + 7.1772{\cdot}10^{-17}X - 4.31753{\cdot}10^{-17}$$

$$m = 5.54668{\cdot}10^{-32}X^2 + 7.1772{\cdot}10^{-17}X - 4.31753{\cdot}10^{-17}$$

Root of M and m:

$$N(M) = \{-1.29396{\cdot}10^{15}, 0.666667\} \qquad N(m) = \{-1.29396{\cdot}10^{15}, 0.666667\}$$

Intersection intervals:



$$[0.666667, 0.666667]$$

Longest intersection interval: 0
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

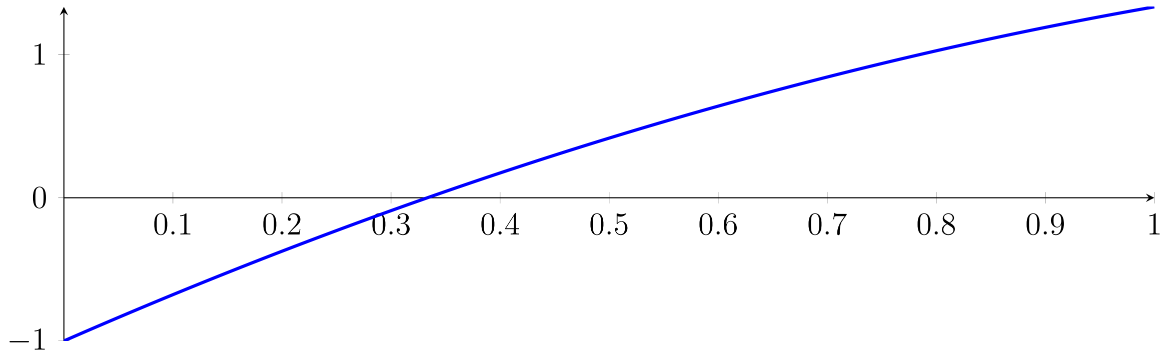
20.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

20.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

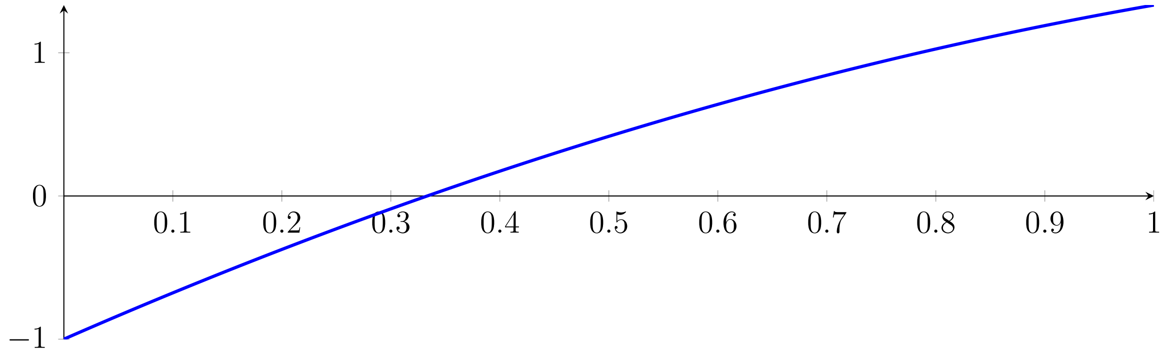
with precision $\varepsilon = 1 \cdot 10^{-128}$.

21 Running CubeClip on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0,1]$:

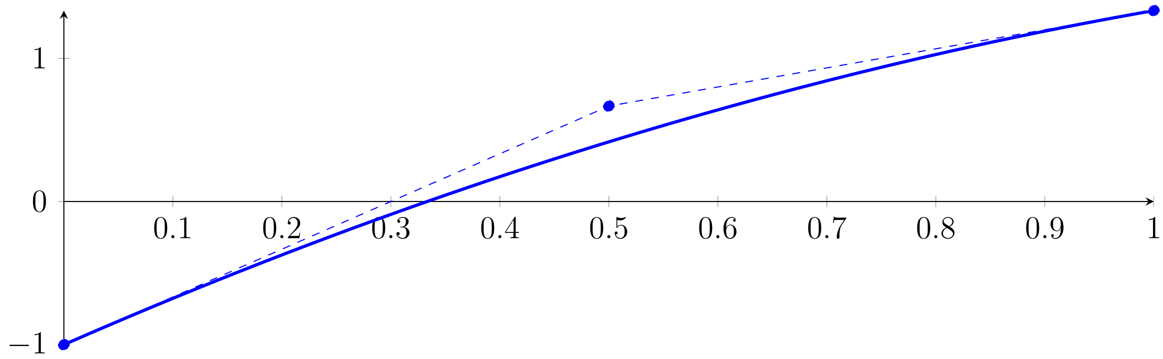
$$p = -1X^2 + 3.33333X - 1$$



21.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

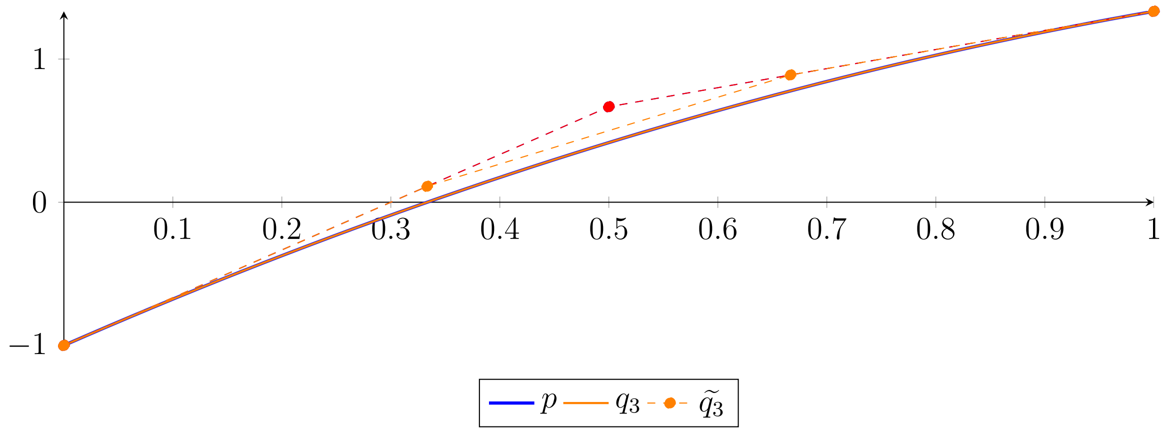
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.90958{\cdot}10^{-14}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.
Bounding polynomials M and m:

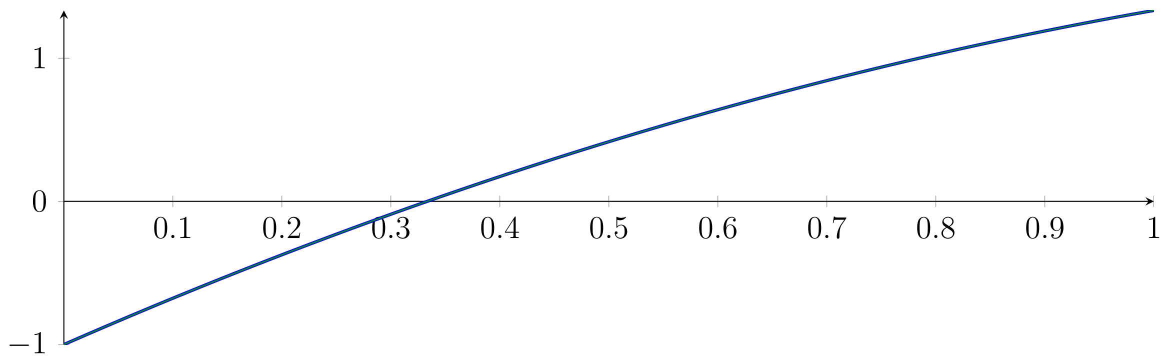
$$M = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

$$m = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{-5.17655 \cdot 10^{13}\} \qquad N(m) = \{-5.17655 \cdot 10^{13}\}$$

Intersection intervals:

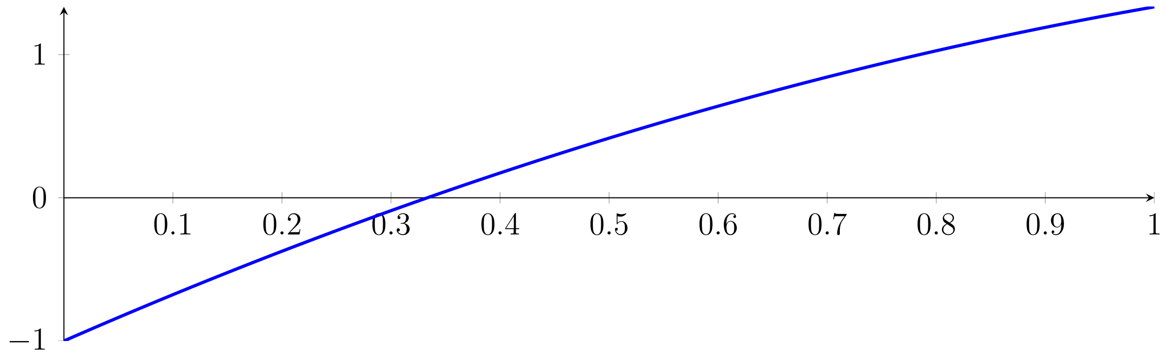


No intersection intervals with the x axis.

21.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

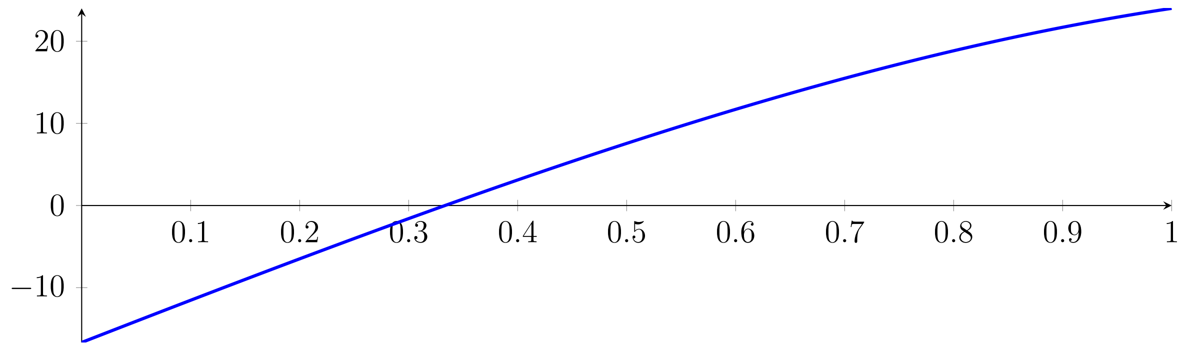
with precision $\varepsilon = 1 \cdot 10^{-128}$.

22 Running BezClip on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

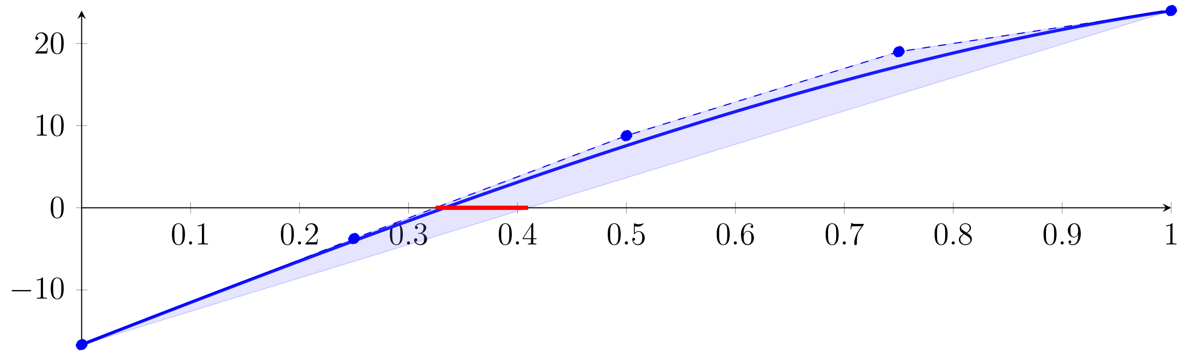
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



22.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

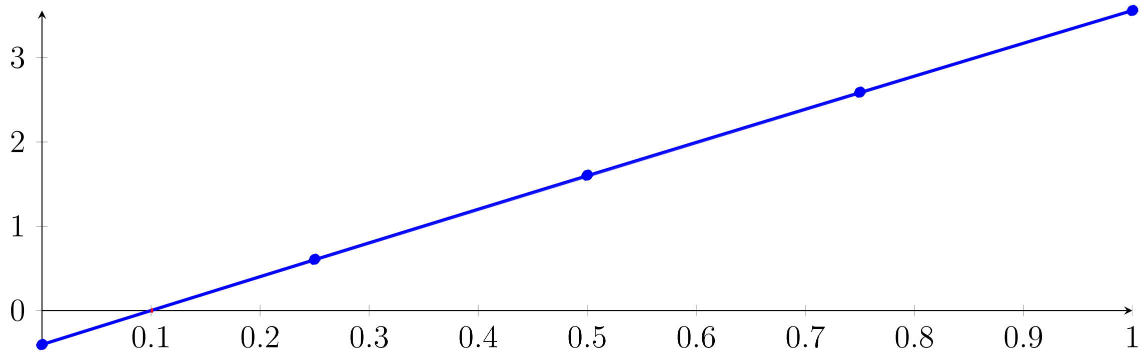
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

22.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

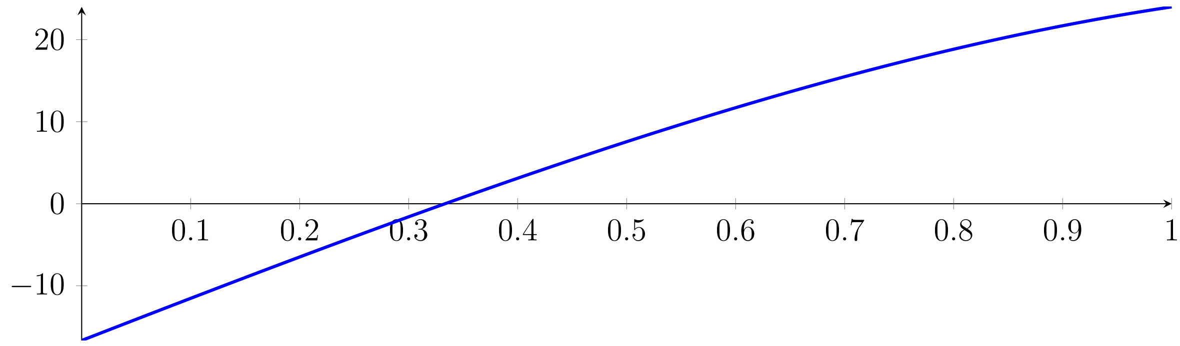
22.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Found root in interval $[0.333317, 0.333491]$ at recursion depth 3!

22.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333317, 0.333491]$$

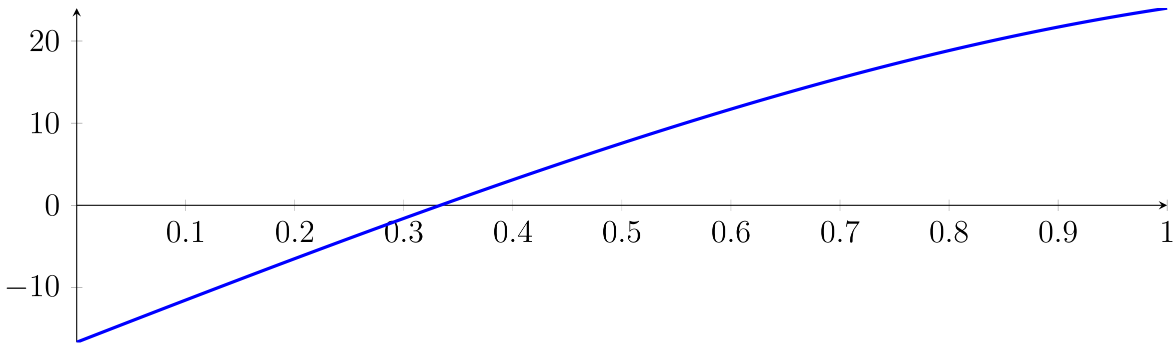
with precision $\varepsilon = 0.01$.

23 Running QuadClip on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$**:**

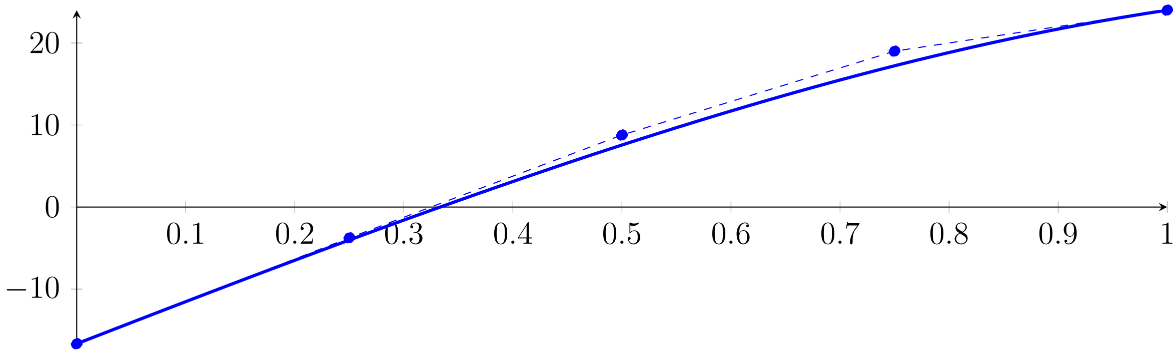
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



23.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

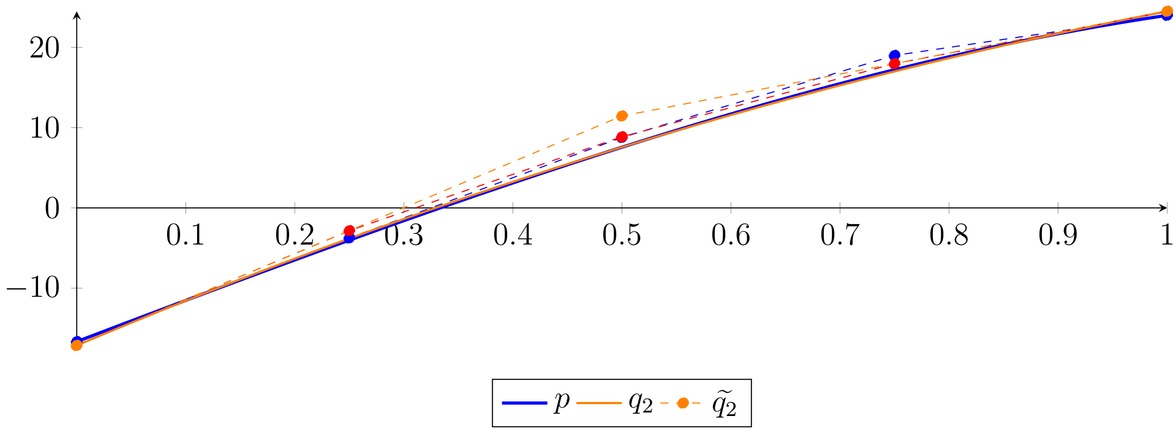
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -2.67519 \cdot 10^{-12}X^4 + 5.32907 \cdot 10^{-12}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



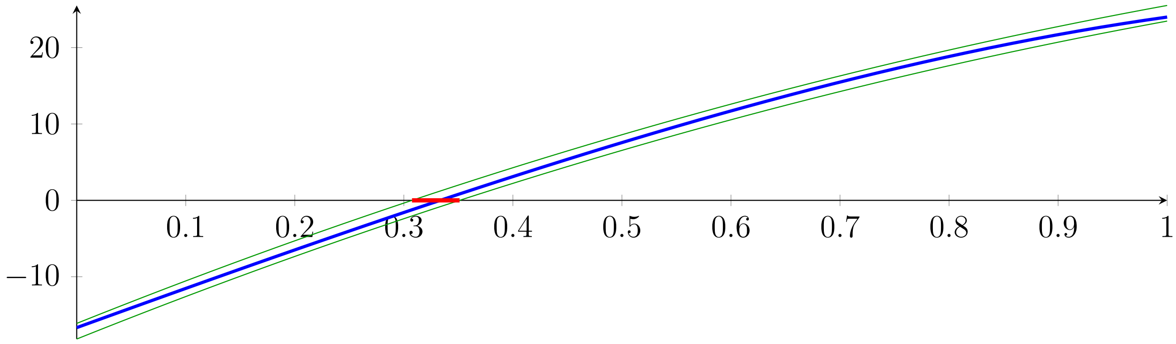
The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

$$M = -15.5476X^2 + 57.181X - 16.1119$$
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



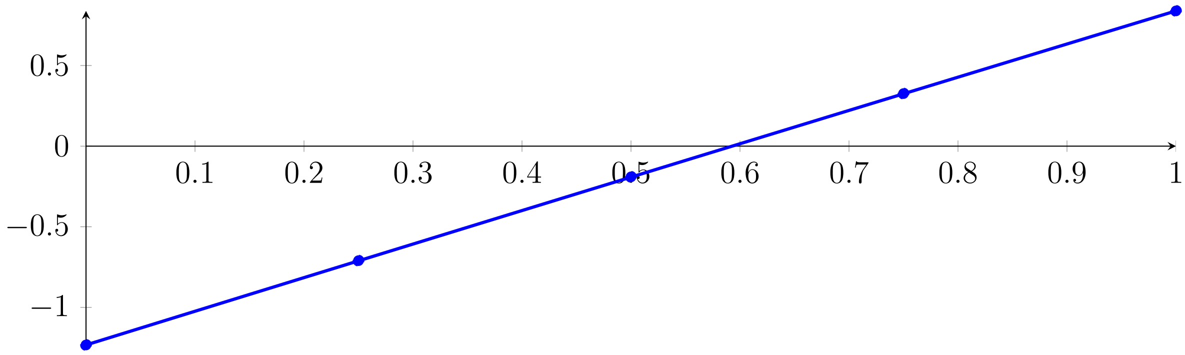
$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

23.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

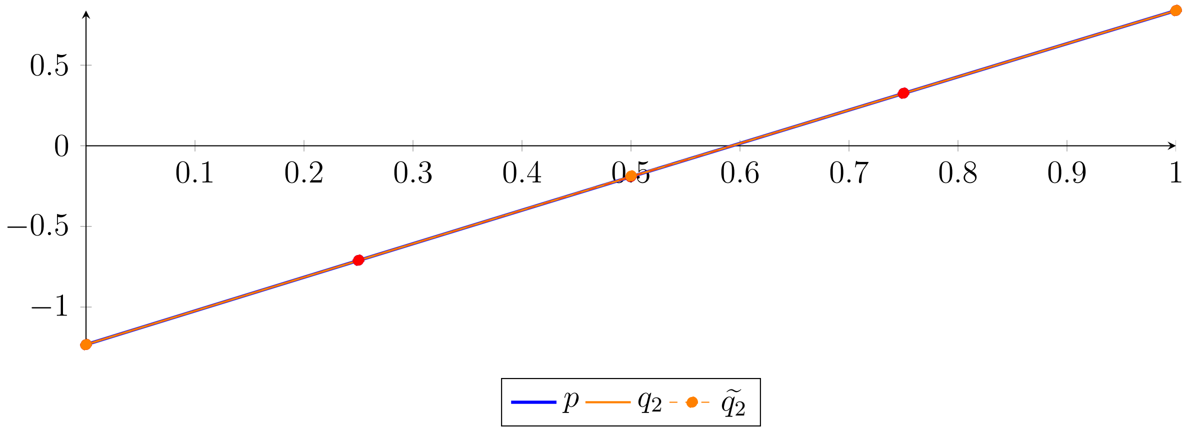
$$p = -3.62044 \cdot 10^{-06}X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278$$
$$= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -0.020089X^2 + 2.09166X - 1.23281$$
$$= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2}$$

$$\widetilde{q}_2 = 2.43583 \cdot 10^{-13}X^4 - 4.3876 \cdot 10^{-13}X^3 - 0.020089X^2 + 2.09166X - 1.23281$$
$$= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.
Bounding polynomials M and m:

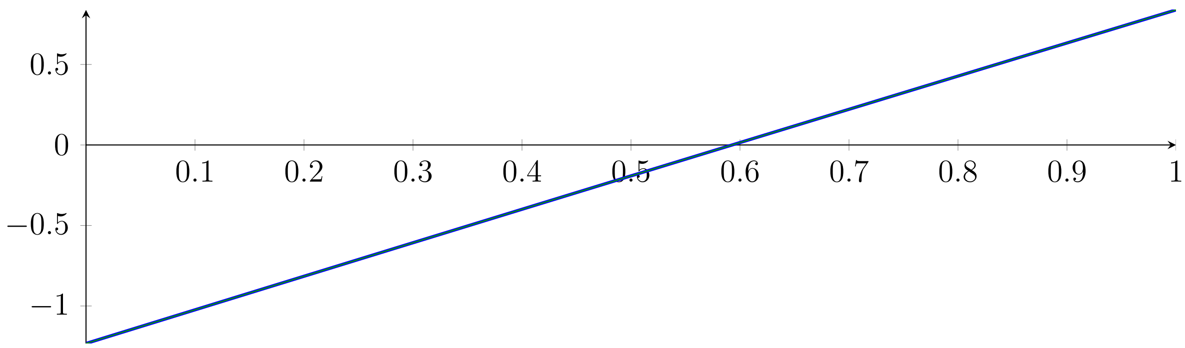
$$M = -0.020089X^2 + 2.09166X - 1.23274$$

$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

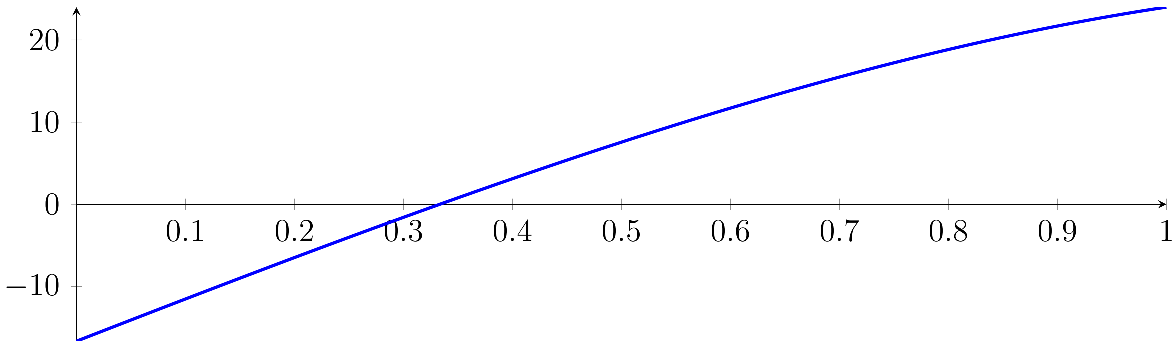
23.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Found root in interval $[0.333332, 0.333335]$ at recursion depth 3!

23.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333332, 0.333335]$$

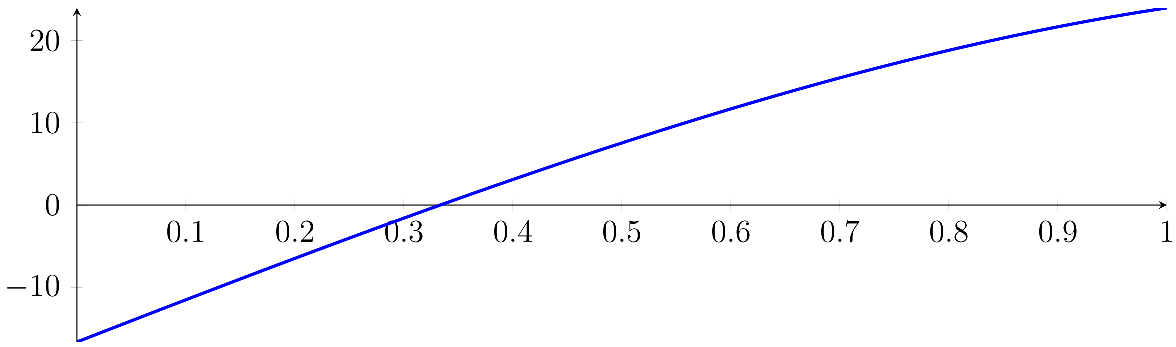
with precision $\varepsilon = 0.01$.

24 Running `CubeClip` on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

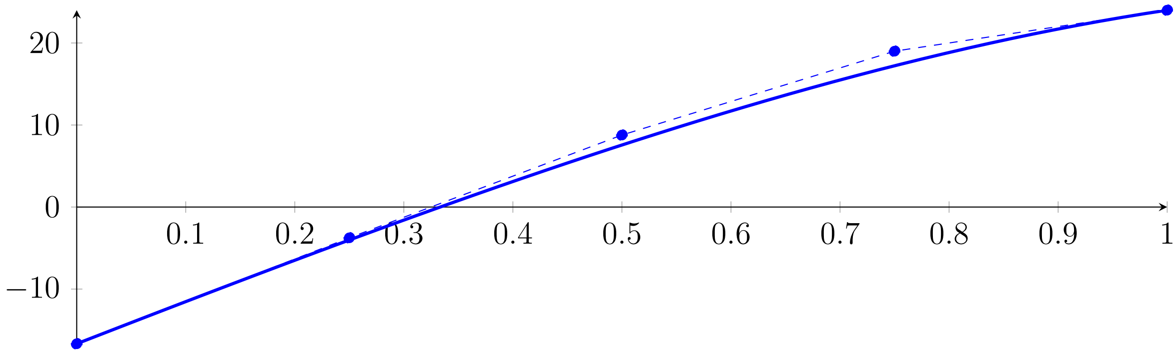
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



24.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

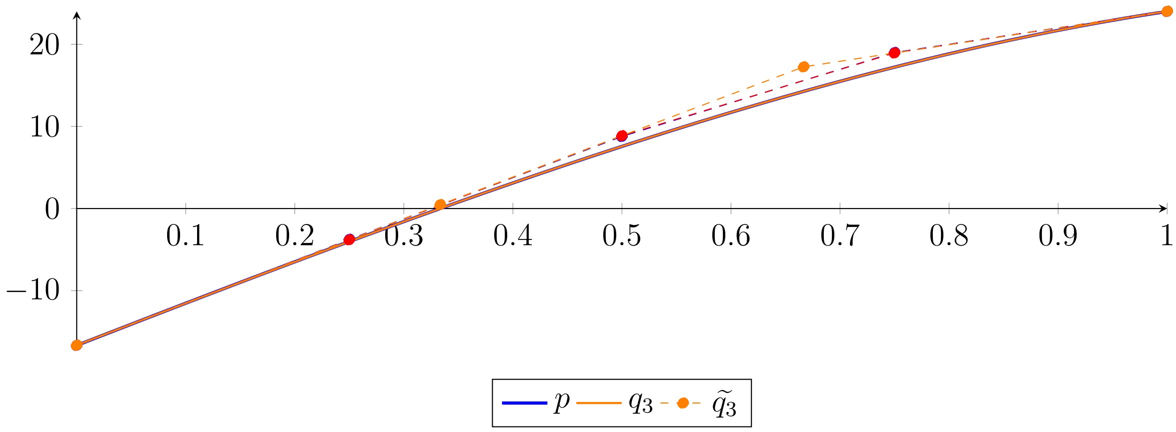
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -2.28084 \cdot 10^{-12}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



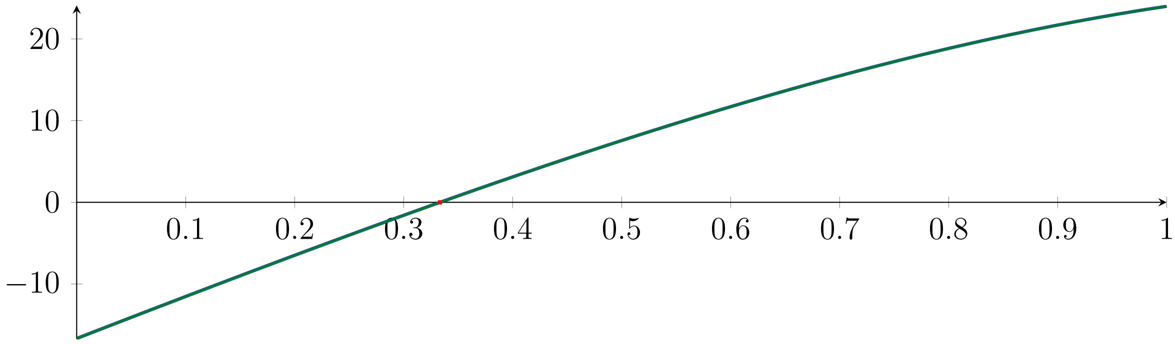
The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

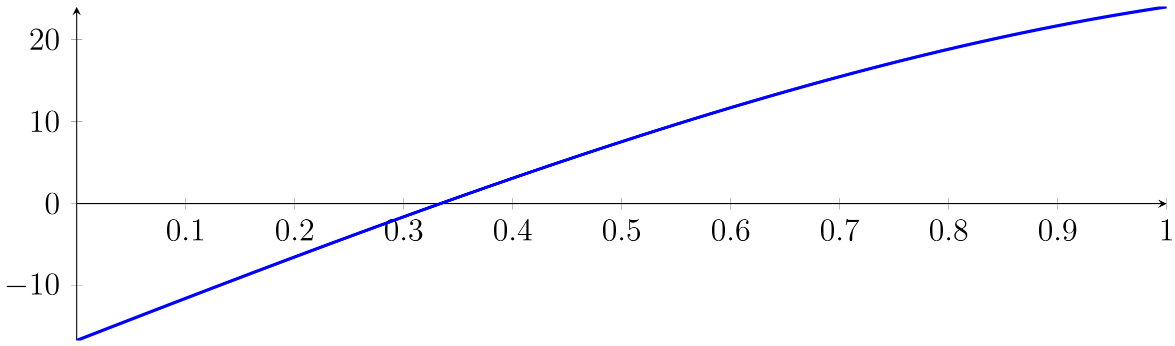
24.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Found root in interval $[0.331524, 0.335136]$ at recursion depth 2!

24.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.331524, 0.335136]$$

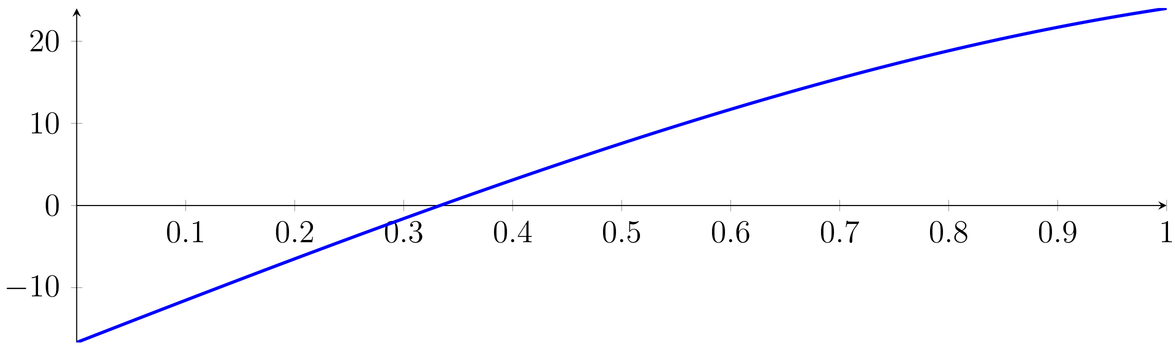
with precision $\varepsilon = 0.01$.

25 Running BezClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$**:**

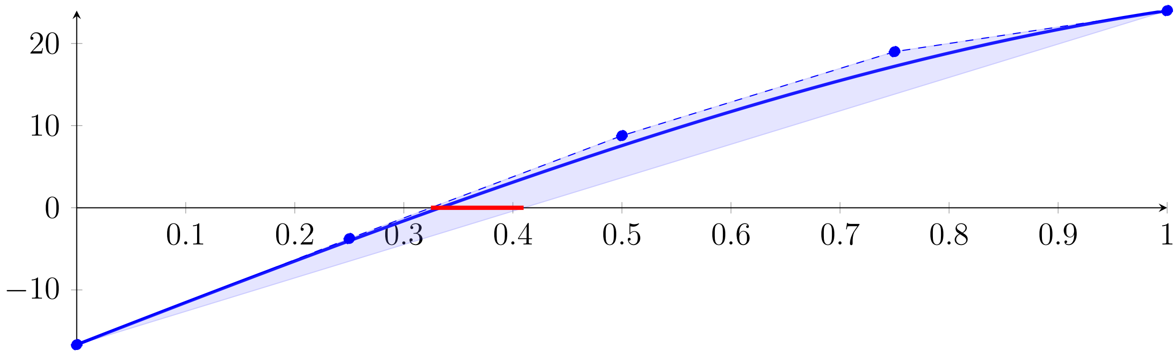
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



25.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x **axis:**

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x **axis:**

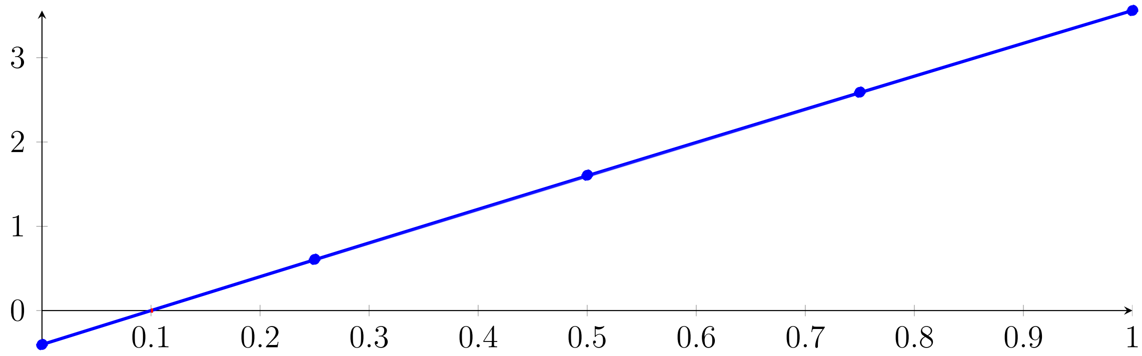
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

25.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

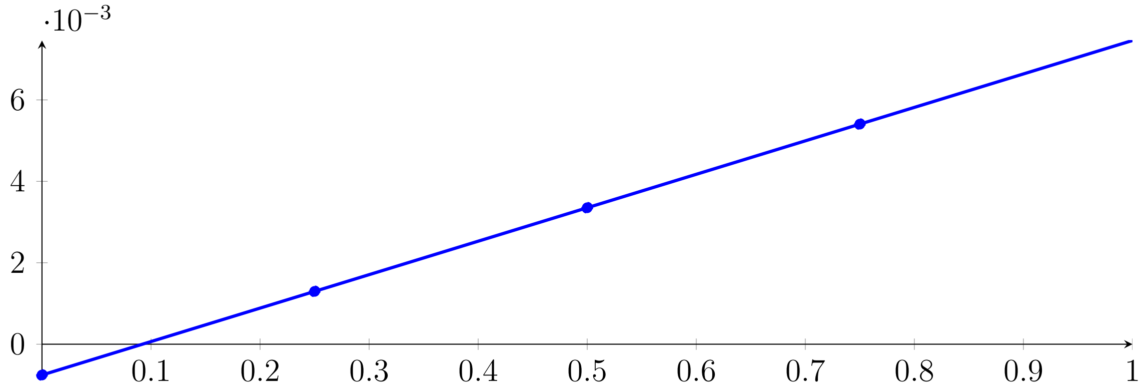
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

25.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.06393 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

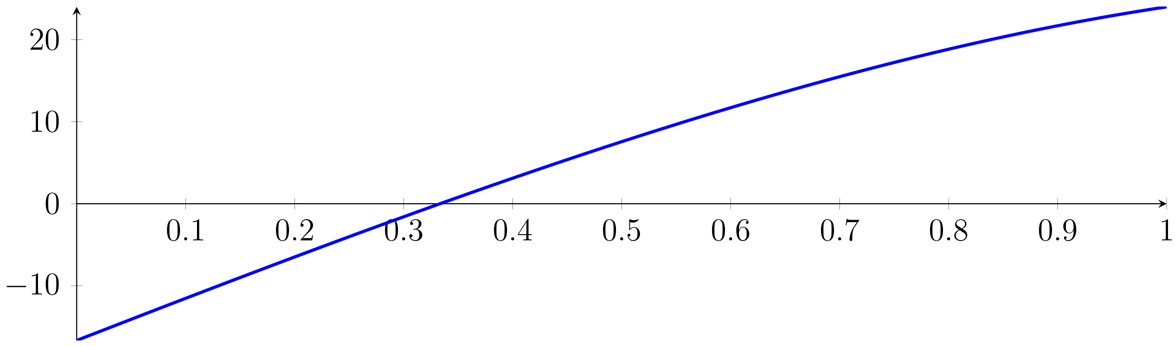
25.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

25.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

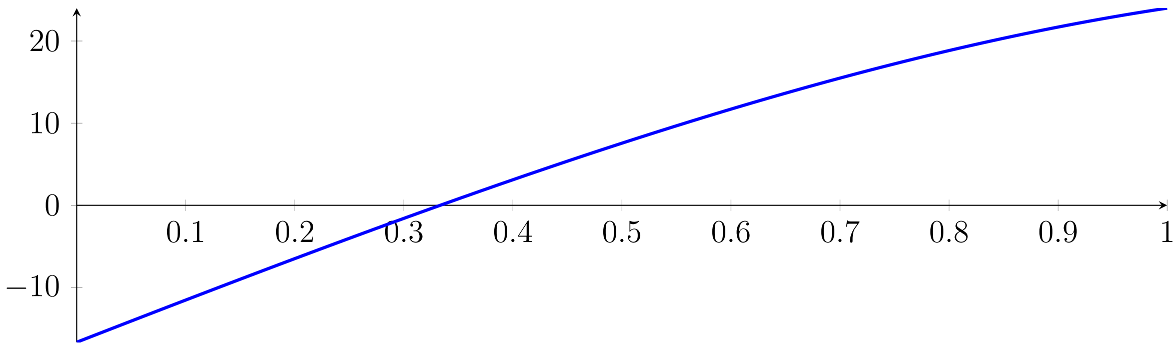
with precision $\varepsilon = 0.0001$.

26 Running QuadClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

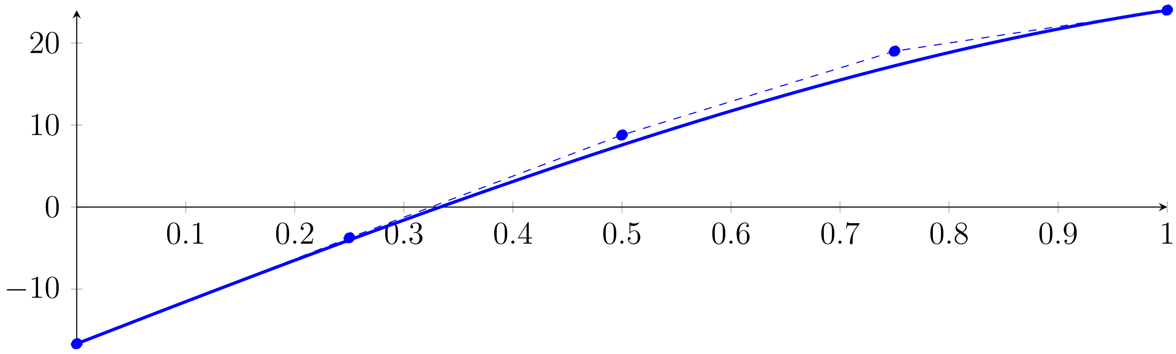
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



26.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

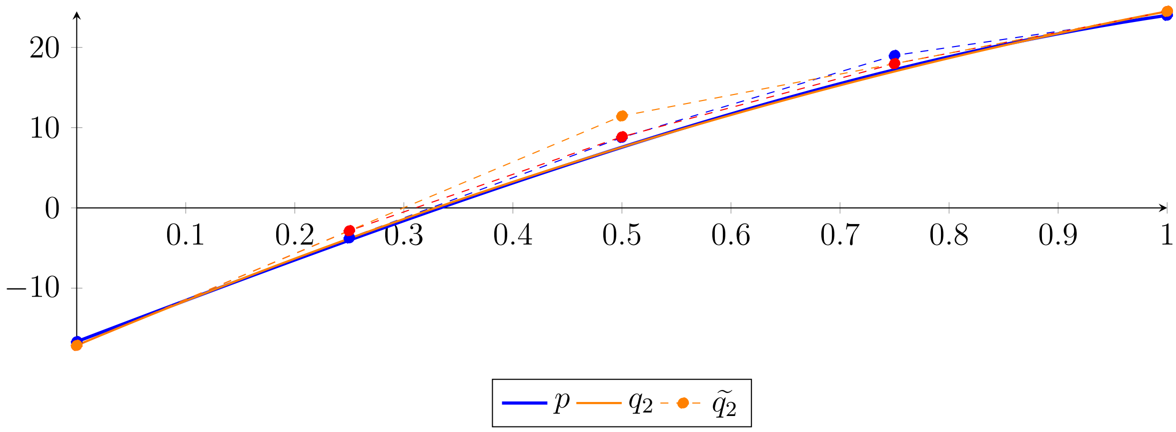
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -2.67519 \cdot 10^{-12}X^4 + 5.32907 \cdot 10^{-12}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

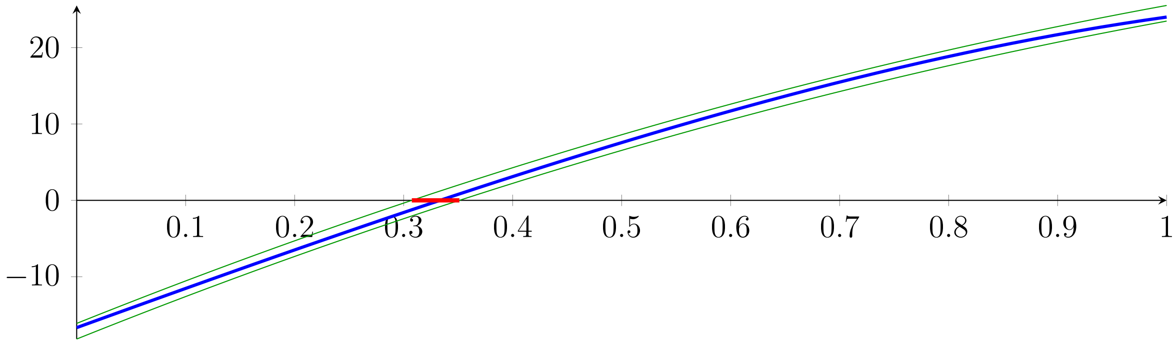
$$M = -15.5476X^2 + 57.181X - 16.1119$$

$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



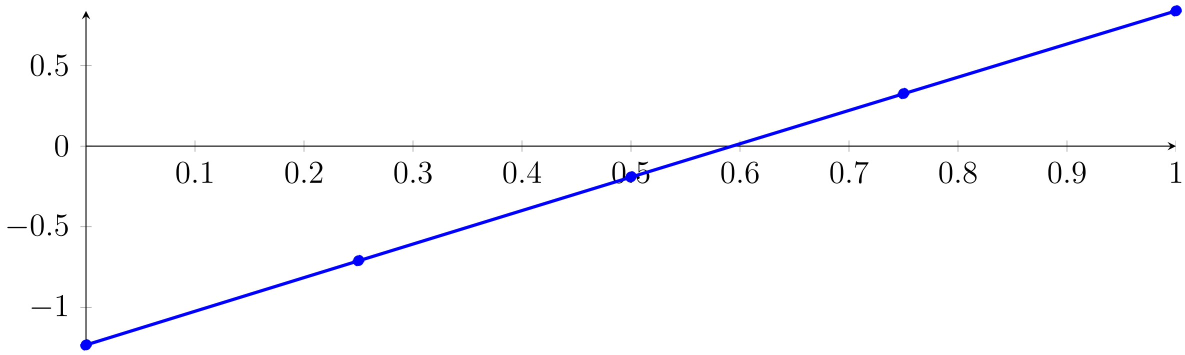
$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

26.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

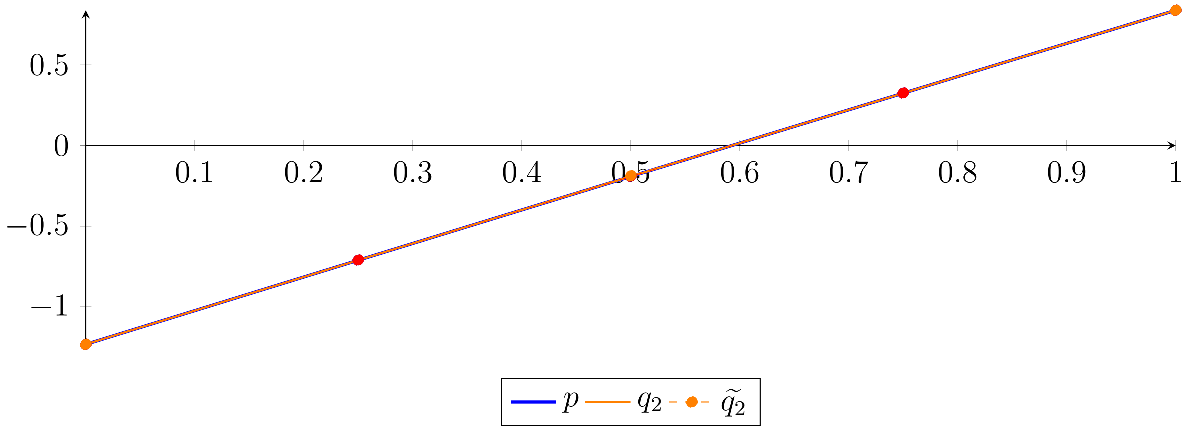
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 2.43583 \cdot 10^{-13} X^4 - 4.3876 \cdot 10^{-13} X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.
Bounding polynomials M and m:

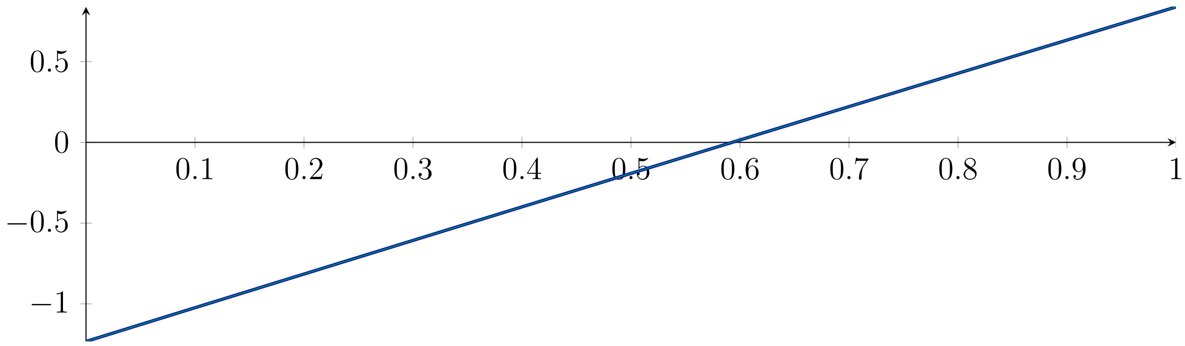
$$M = -0.020089X^2 + 2.09166X - 1.23274$$

$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$[0.592734, 0.592807]$

Longest intersection interval: $7.23183 \cdot 10^{-05}$
$\implies$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

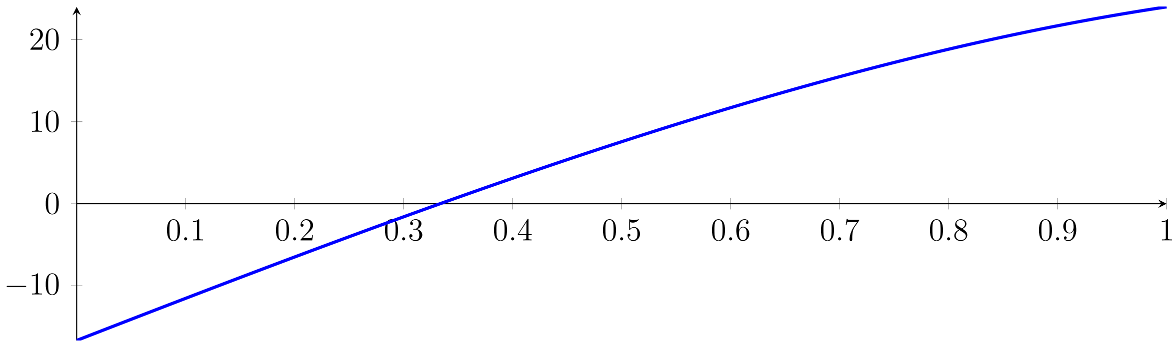
26.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Found root in interval $[0.333332, 0.333335]$ at recursion depth 3!

26.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333332, 0.333335]$$

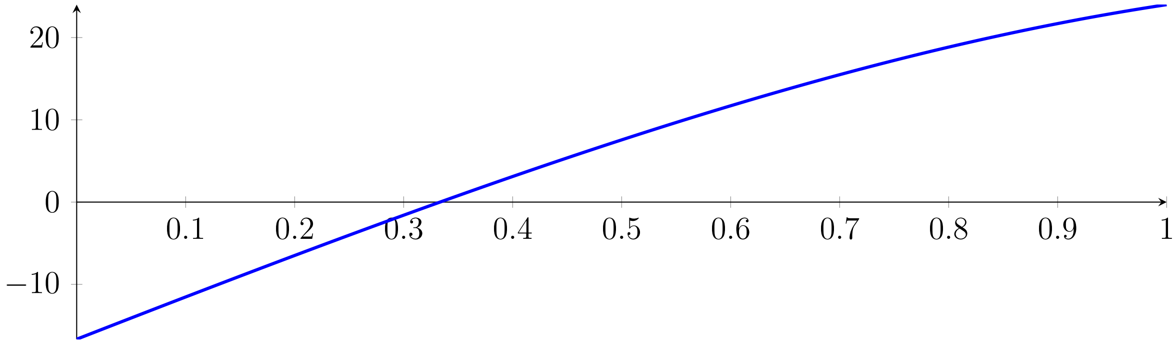
with precision $\varepsilon = 0.0001$.

27 Running `CubeClip` on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called `CubeClip` with input polynomial on interval $[0,1]$:

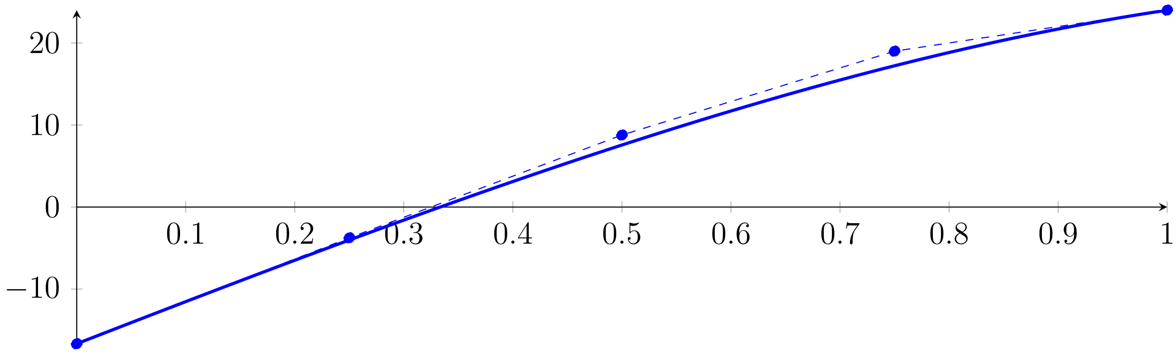
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



27.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

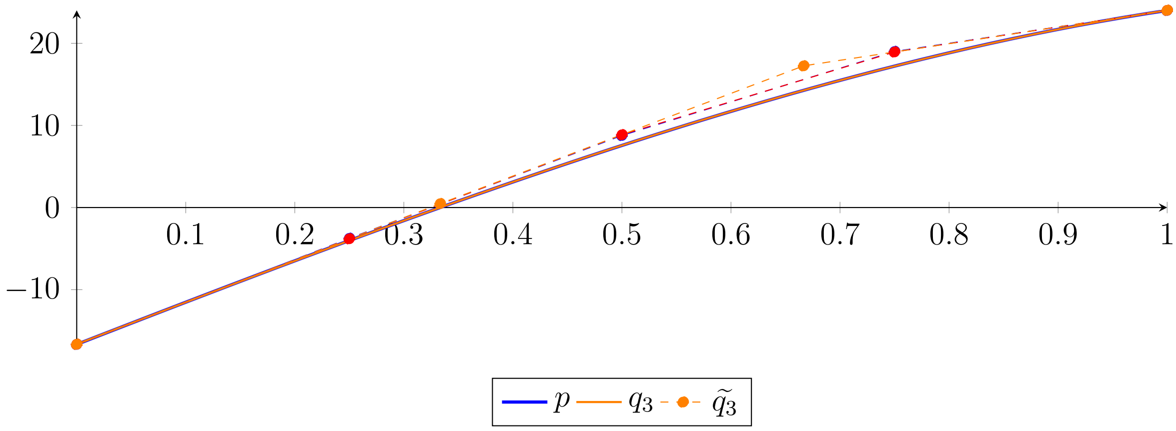
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -2.28084 \cdot 10^{-12}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



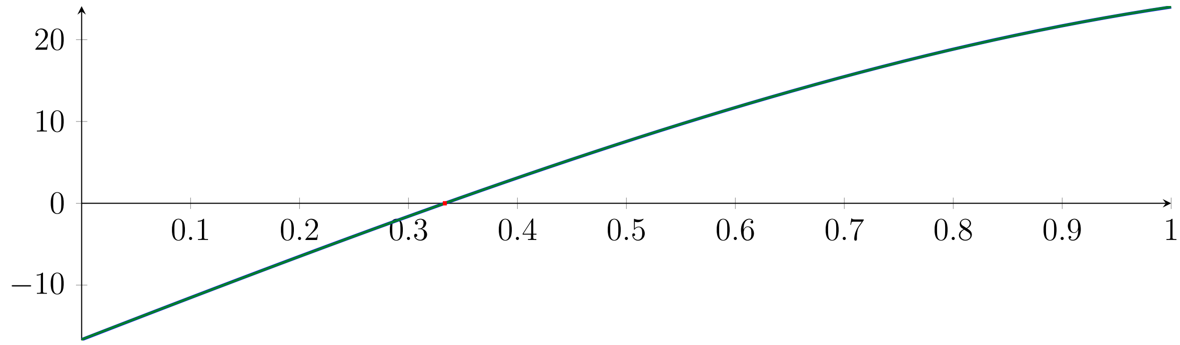
The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



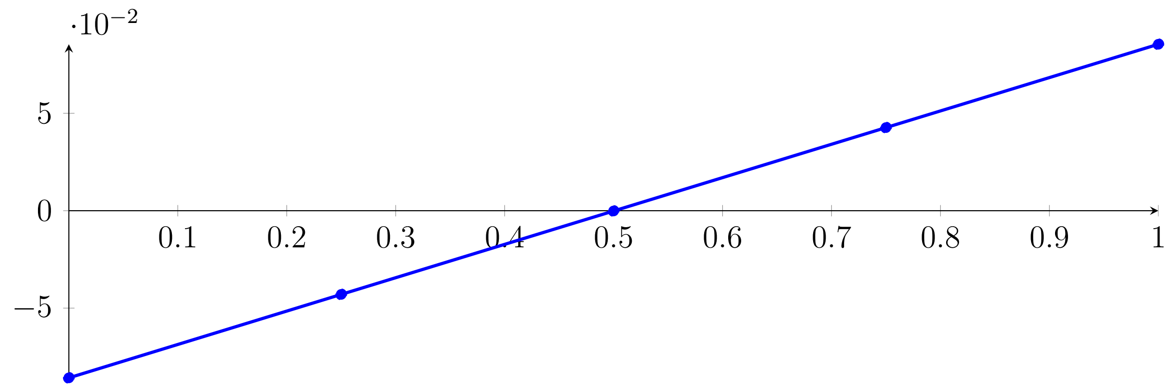
$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

27.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

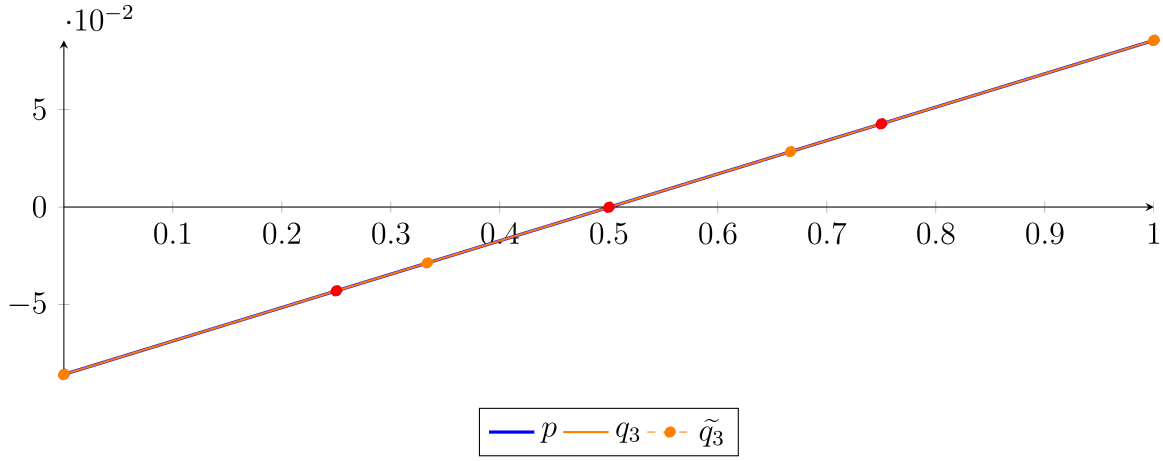
$$\begin{aligned} p &= -1.70219\cdot 10^{-10}X^4 - 4.23789\cdot 10^{-07}X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4}(X) - 0.0429507B_{1,4}(X) - 0.000129666B_{2,4}(X) \\ &\quad + 0.0426682B_{3,4}(X) + 0.0854427B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413\cdot 10^{-07}X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,3} - 0.0286693B_{1,3} + 0.02841B_{2,3} + 0.0854427B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 1.2032\cdot 10^{-14}X^4 - 4.2413\cdot 10^{-07}X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4} - 0.0429507B_{1,4} - 0.000129666B_{2,4} + 0.0426682B_{3,4} + 0.0854427B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45913 \cdot 10^{-11}$.
Bounding polynomials M and m:

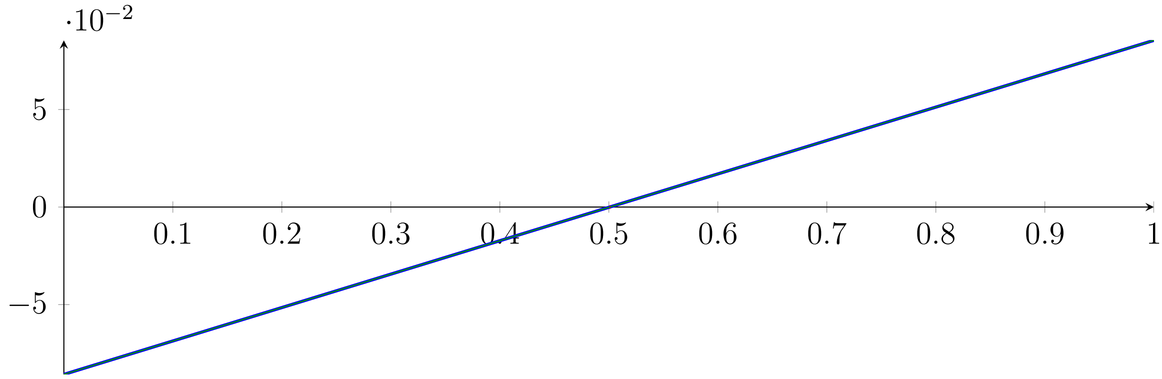
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

Longest intersection interval: $1.70047 \cdot 10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

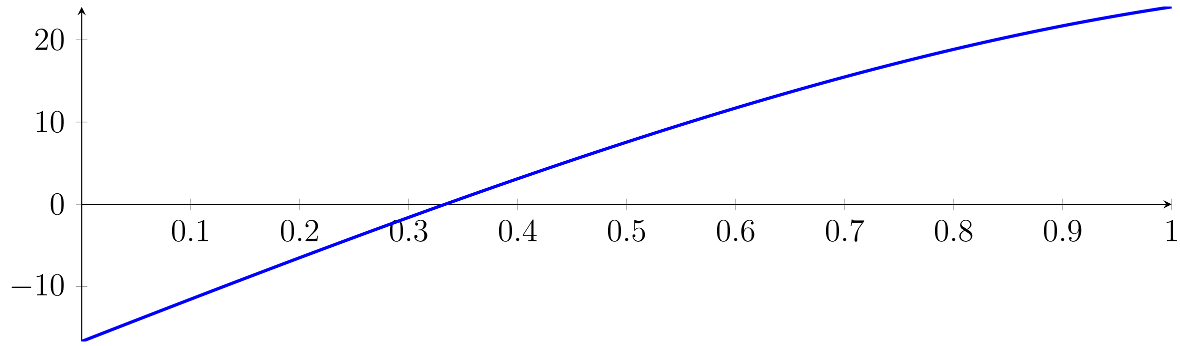
27.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

27.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

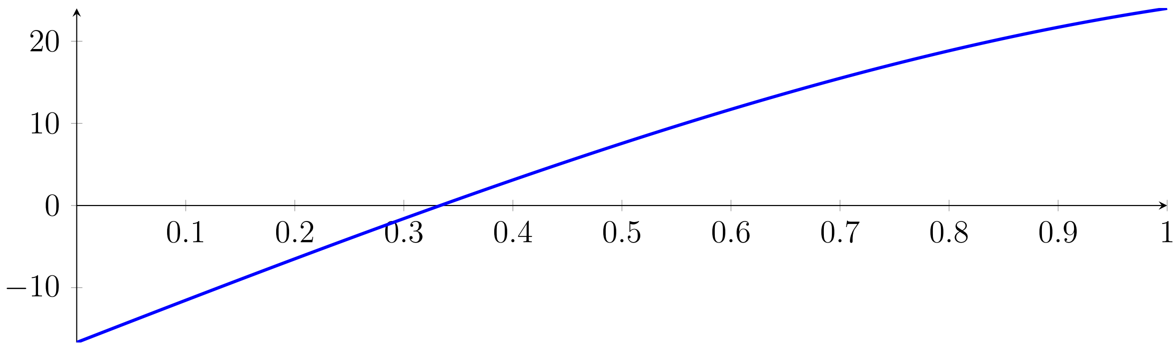
with precision $\varepsilon = 0.0001$.

28 Running `BezClip` on f_4 with epsilon 8

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

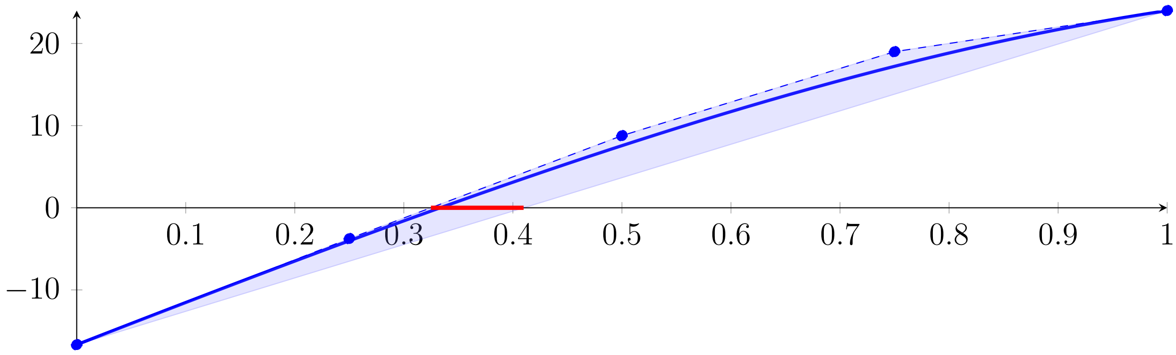
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



28.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

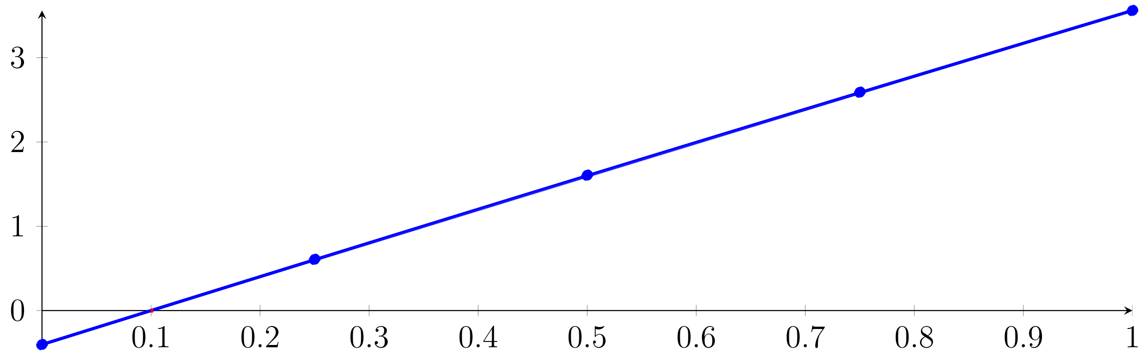
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

28.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

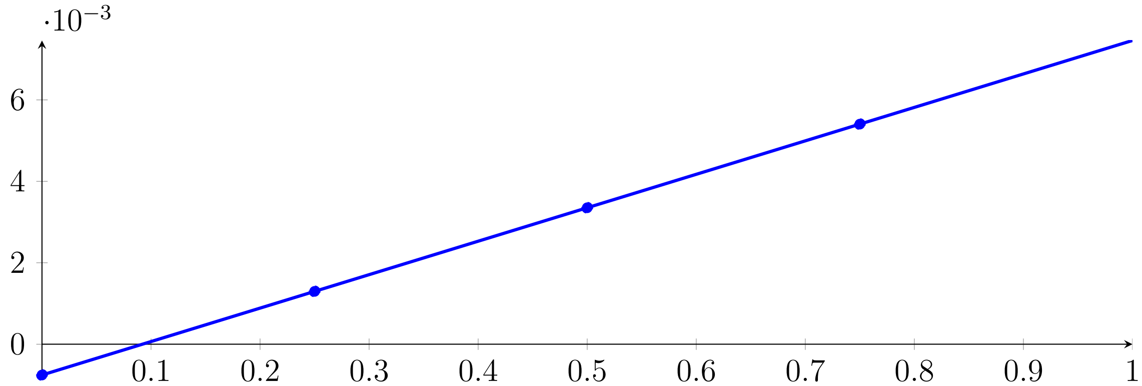
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

28.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.06393 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

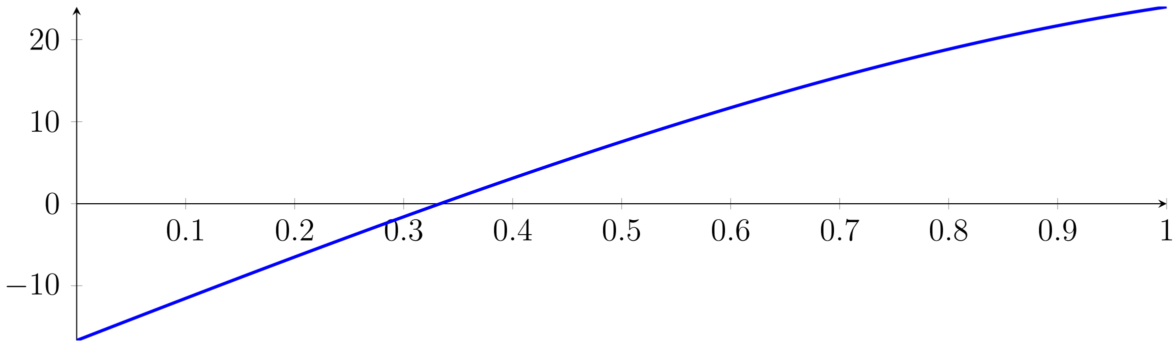
28.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

28.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

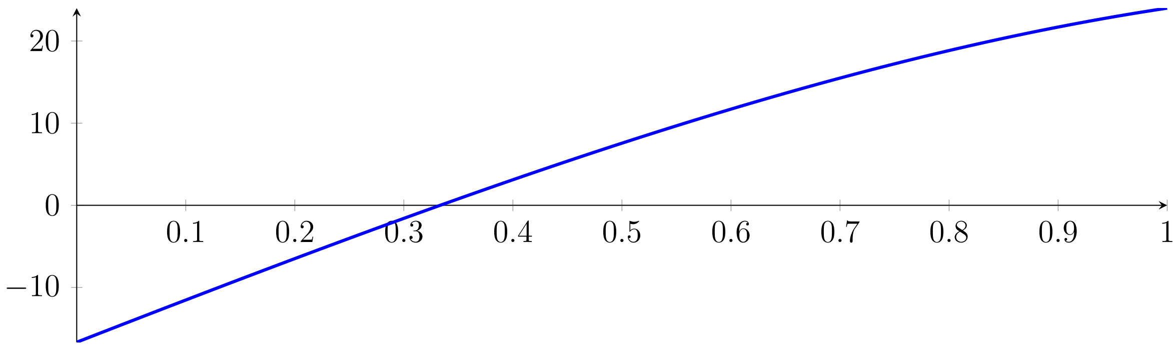
with precision $\varepsilon = 1 \cdot 10^{-08}$.

29 Running QuadClip on f_4 with epsilon 8

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0,1]$:

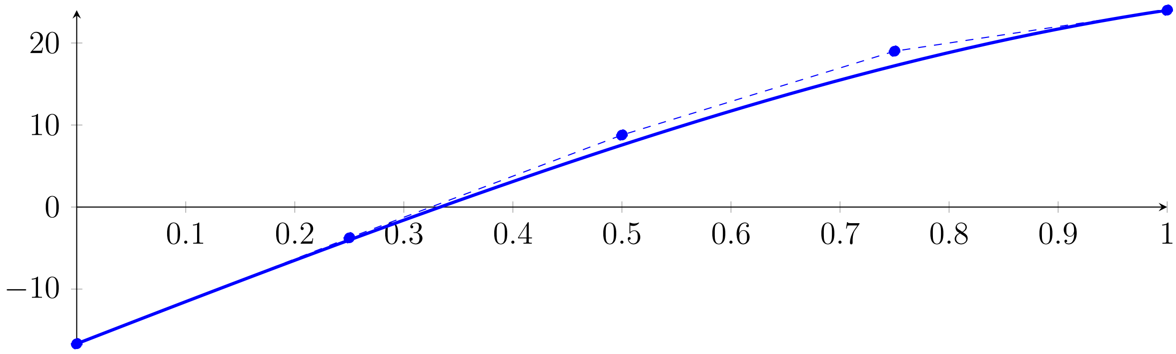
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



29.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

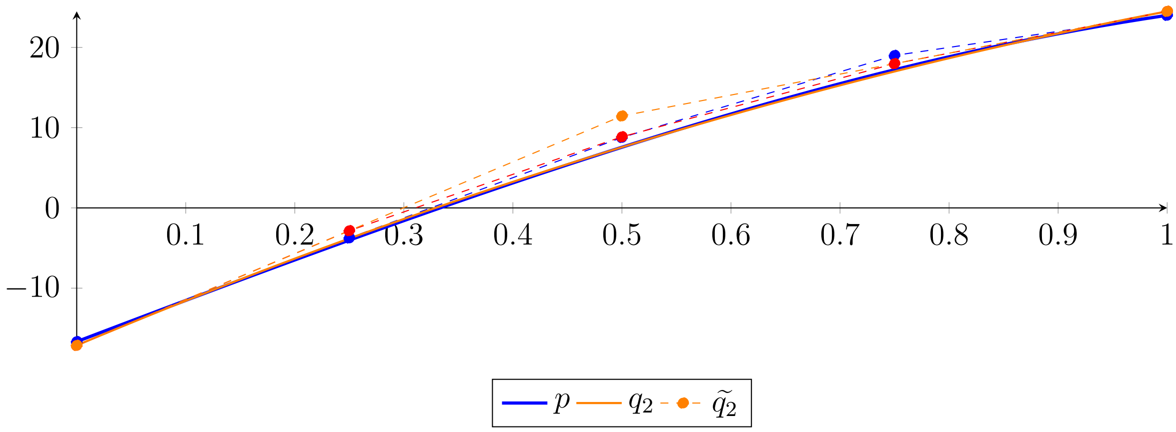
$$\begin{aligned}p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667\\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -15.5476X^2 + 57.181X - 17.1357\\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_2 &= -2.67519\cdot10^{-12}X^4 + 5.32907\cdot10^{-12}X^3 - 15.5476X^2 + 57.181X - 17.1357\\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

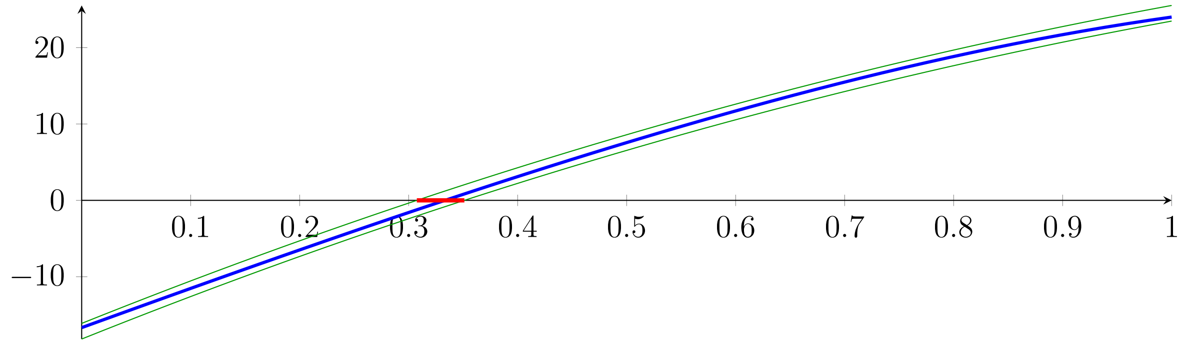
$$M = -15.5476X^2 + 57.181X - 16.1119$$

$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



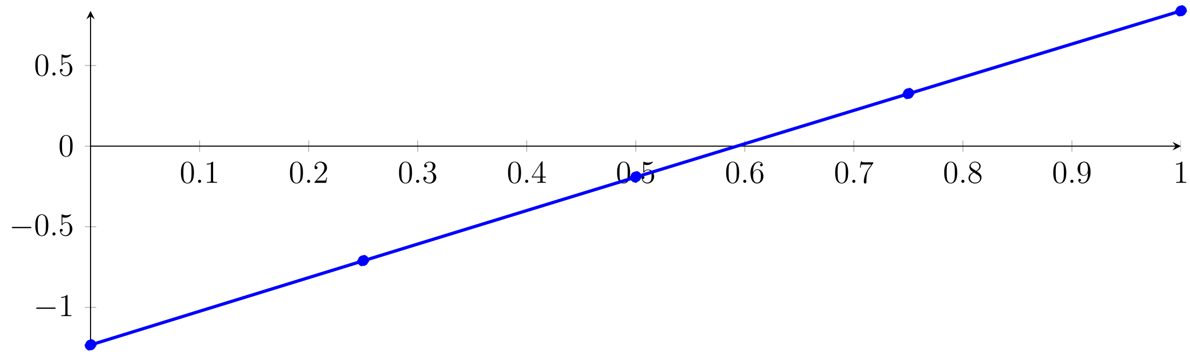
$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

29.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

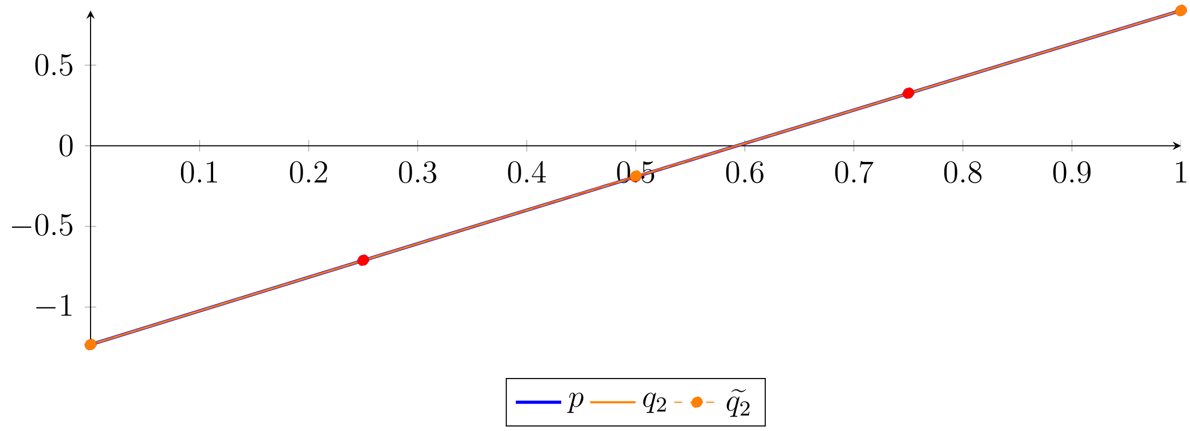
$$p = -3.62044 \cdot 10^{-06}X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278$$
$$= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -0.020089X^2 + 2.09166X - 1.23281$$
$$= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2}$$

$$\widetilde{q}_2 = 2.43583 \cdot 10^{-13}X^4 - 4.3876 \cdot 10^{-13}X^3 - 0.020089X^2 + 2.09166X - 1.23281$$
$$= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

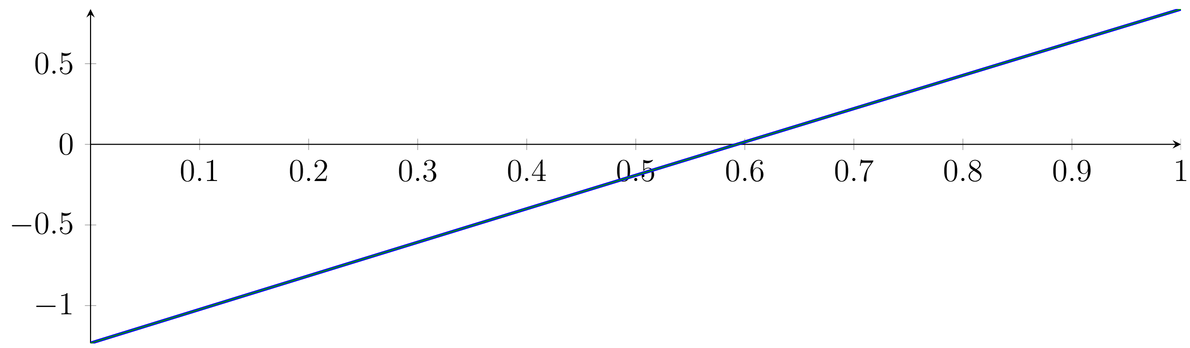
Bounding polynomials M and m:

$$M = -0.020089X^2 + 2.09166X - 1.23274$$
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

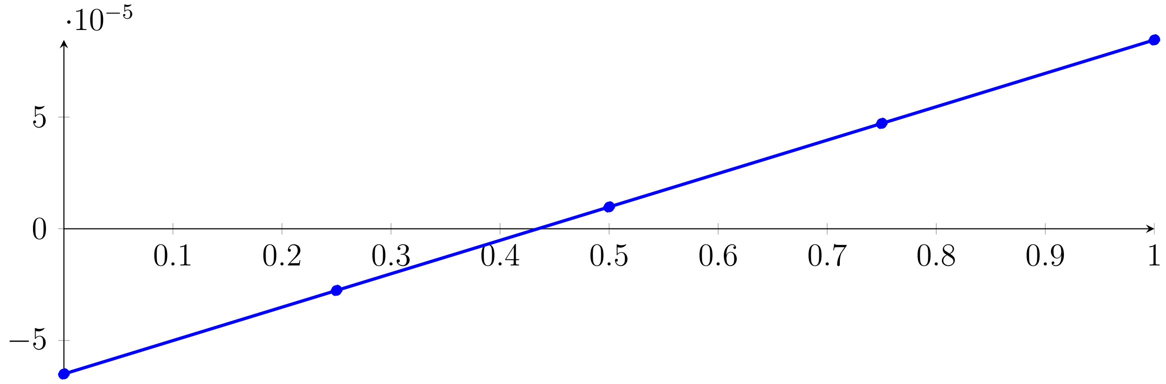
Longest intersection interval: $7.23183 \cdot 10^{-05}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

29.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

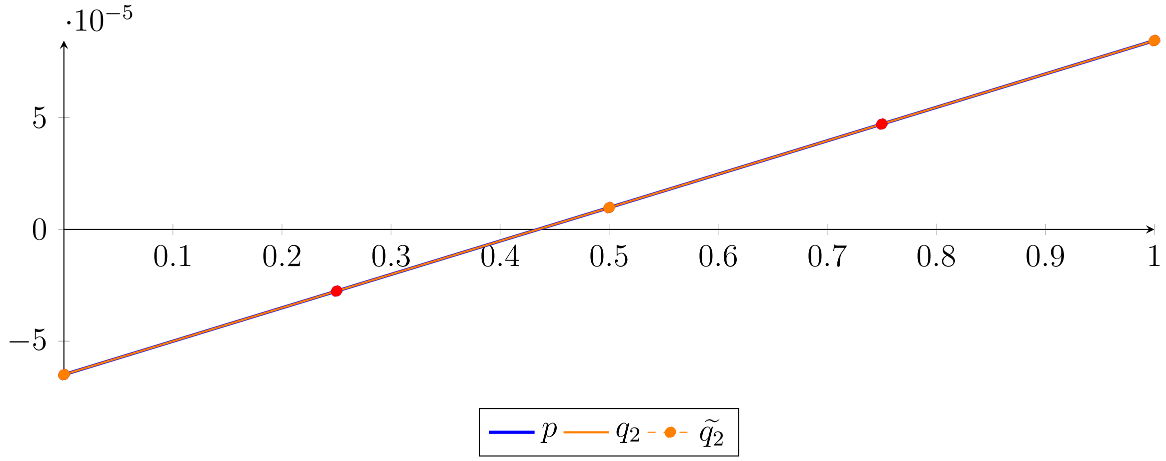
$$p = -2.71051 \cdot 10^{-20} X^4 - 2.82489 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777$$
$$\cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.06147{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$
$$= -6.50069{\cdot}10^{-05}B_{0,2} + 9.76779{\cdot}10^{-06}B_{1,2} + 8.45424{\cdot}10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = 6.72205{\cdot}10^{-18}X^4 - 1.21431{\cdot}10^{-17}X^3 - 1.06147{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$
$$= -6.50069{\cdot}10^{-05}B_{0,4} - 2.76196{\cdot}10^{-05}B_{1,4} + 9.76777{\cdot}10^{-06}B_{2,4} + 4.71551{\cdot}10^{-05}B_{3,4} + 8.45424{\cdot}10^{-05}B_{4,4}$$



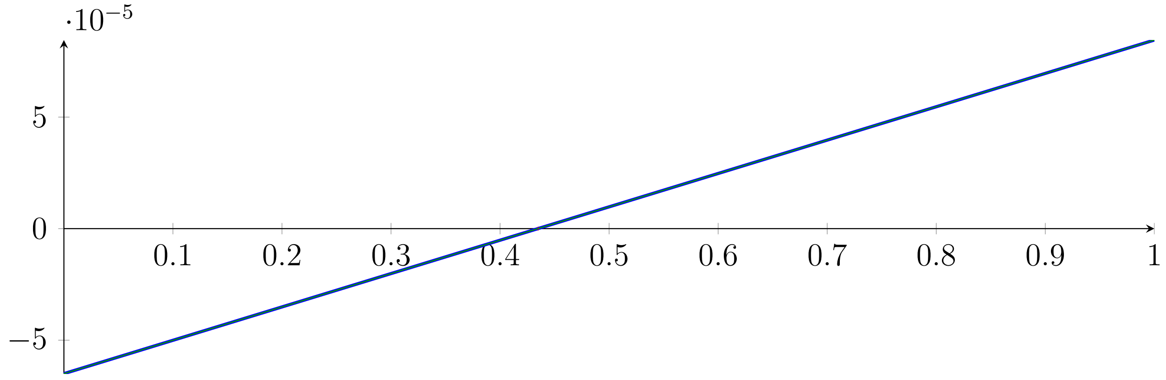
The maximum difference of the Bézier coefficients is $\delta = 2.88601{\cdot}10^{-17}$.
Bounding polynomials M and m:

$$M = -1.06147{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$
$$m = -1.06147{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{0.434685, 1.4089{\cdot}10^{6}\} \qquad N(m) = \{0.434685, 1.4089{\cdot}10^{6}\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

Longest intersection interval: $1.27678{\cdot}10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

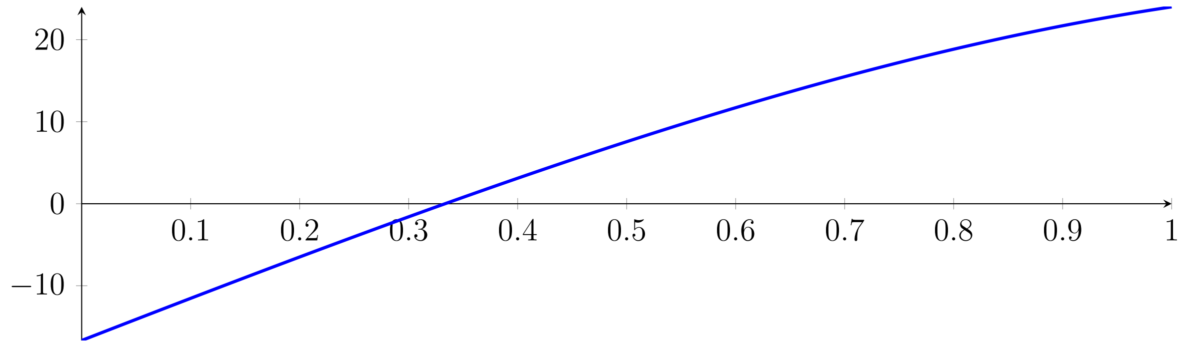
29.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

29.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

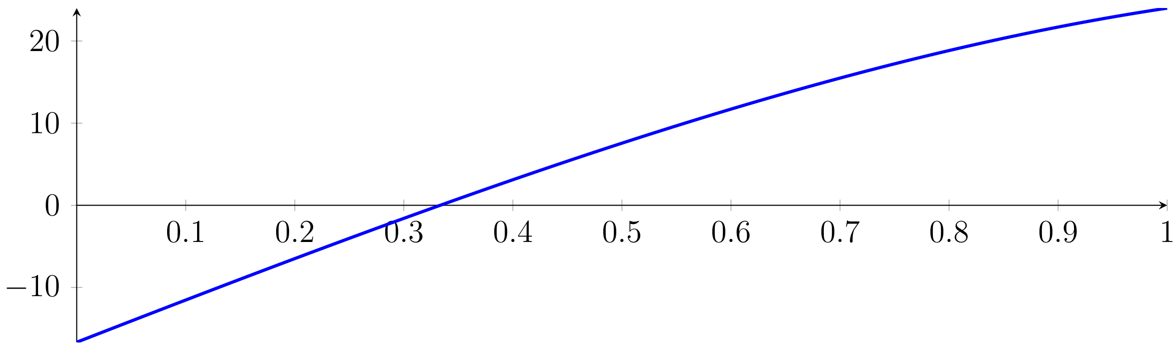
with precision $\varepsilon = 1 \cdot 10^{-08}$.

30 Running CubeClip on f_4 with epsilon 8

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0,1]$:

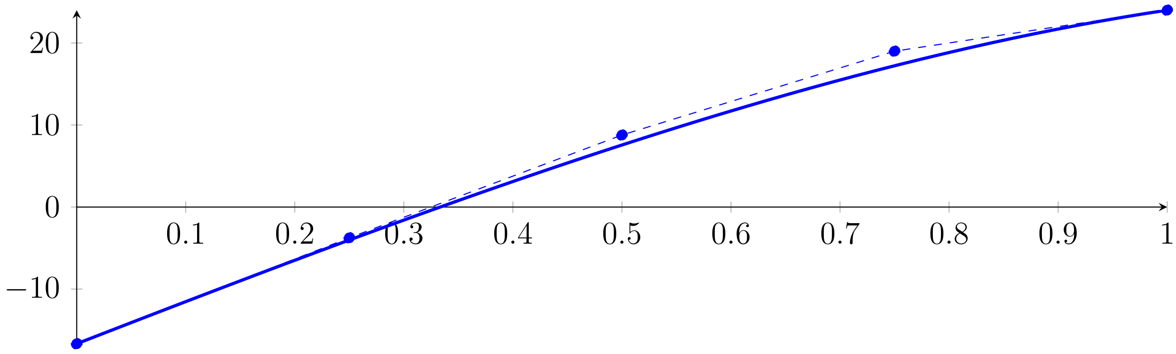
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



30.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

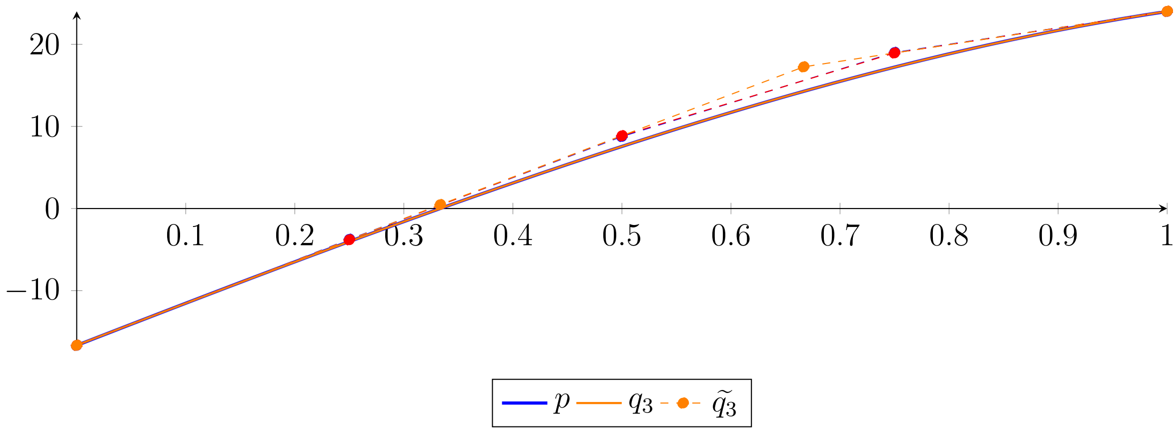
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -2.28084 \cdot 10^{-12}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



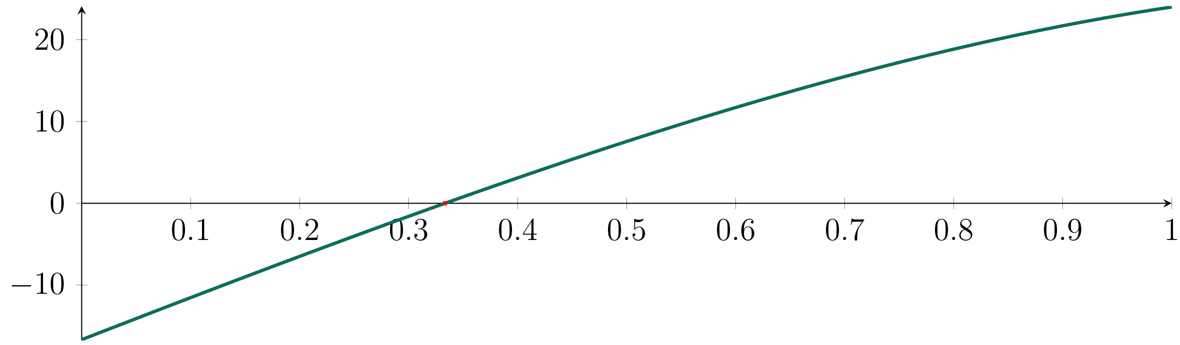
The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



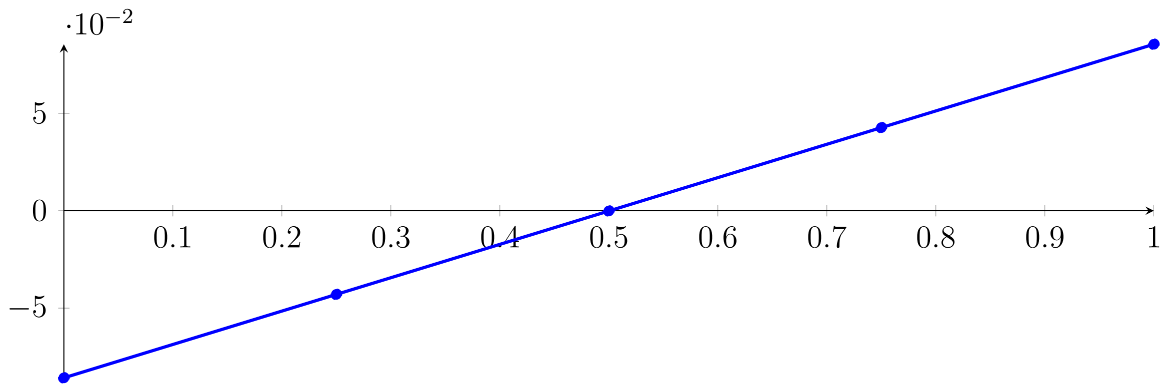
$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

30.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

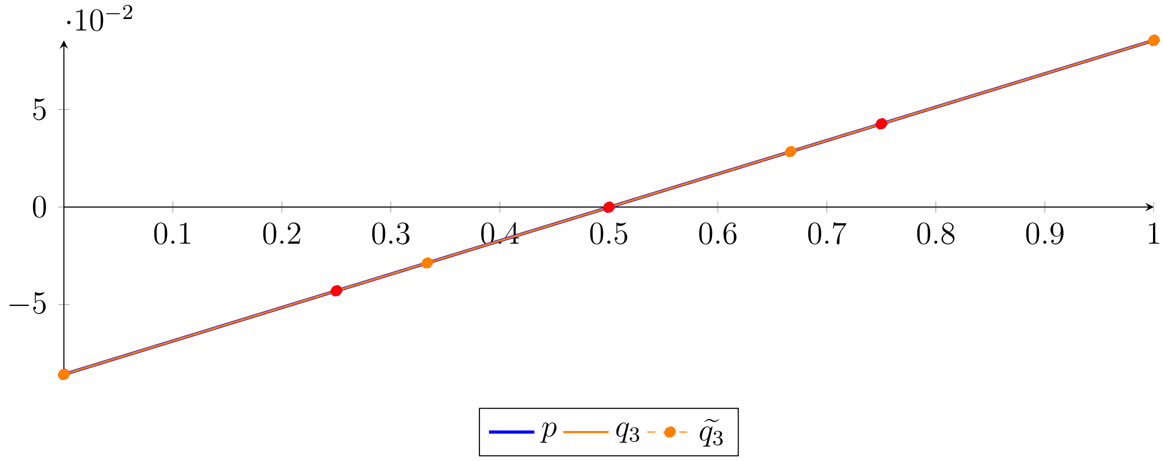
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4}(X) - 0.0429507B_{1,4}(X) - 0.000129666B_{2,4}(X) \\ &\quad + 0.0426682B_{3,4}(X) + 0.0854427B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,3} - 0.0286693B_{1,3} + 0.02841B_{2,3} + 0.0854427B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 1.2032 \cdot 10^{-14} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4} - 0.0429507B_{1,4} - 0.000129666B_{2,4} + 0.0426682B_{3,4} + 0.0854427B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45913 \cdot 10^{-11}$.

Bounding polynomials M and m:

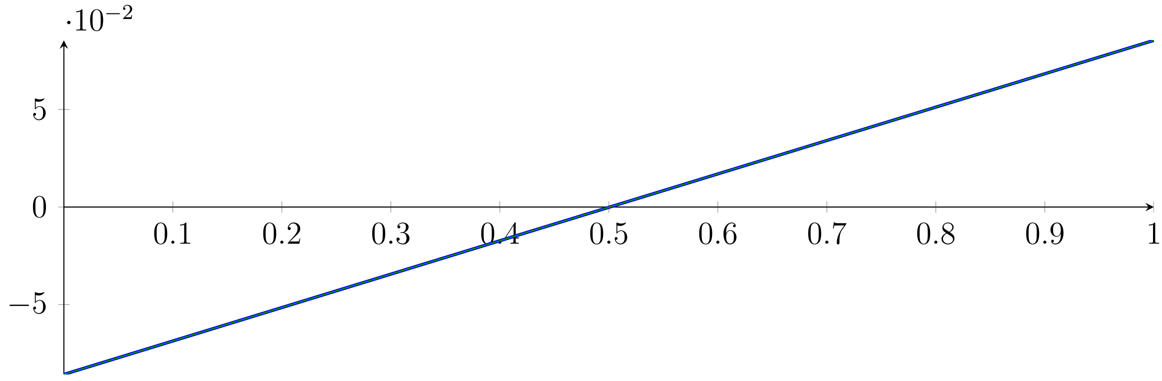
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

Longest intersection interval: $1.70047 \cdot 10^{-10}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

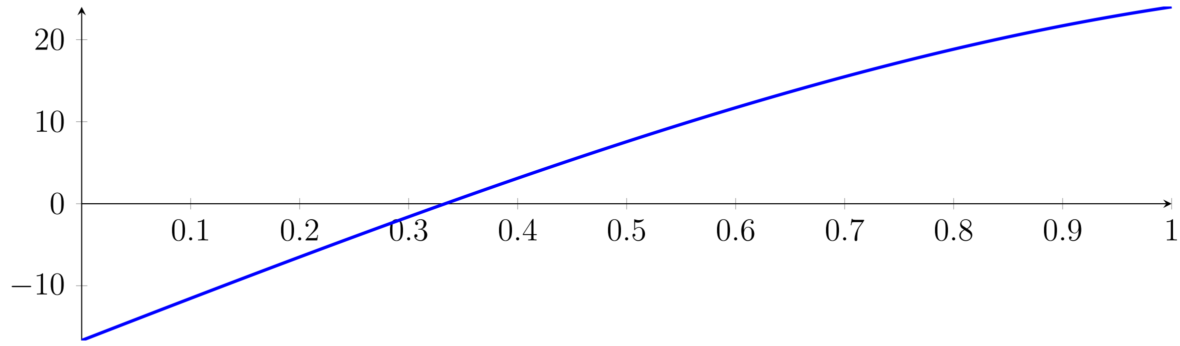
30.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

30.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

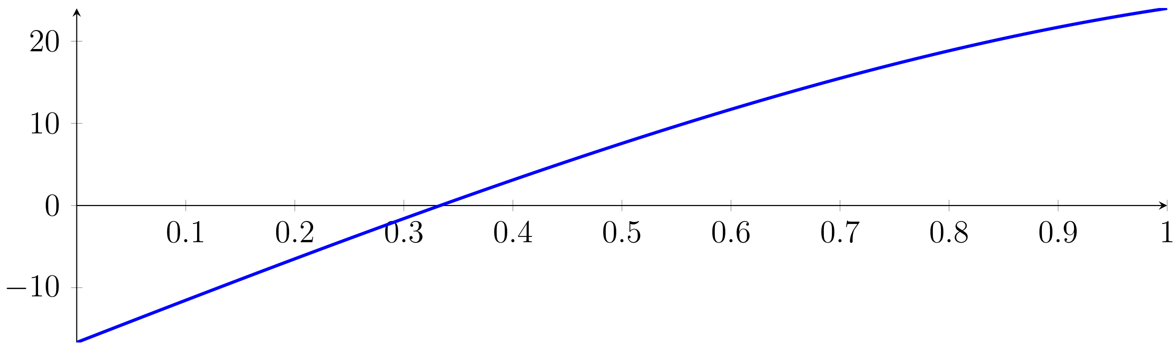
with precision $\varepsilon = 1 \cdot 10^{-08}$.

31 Running BezClip on f_4 with epsilon 16

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

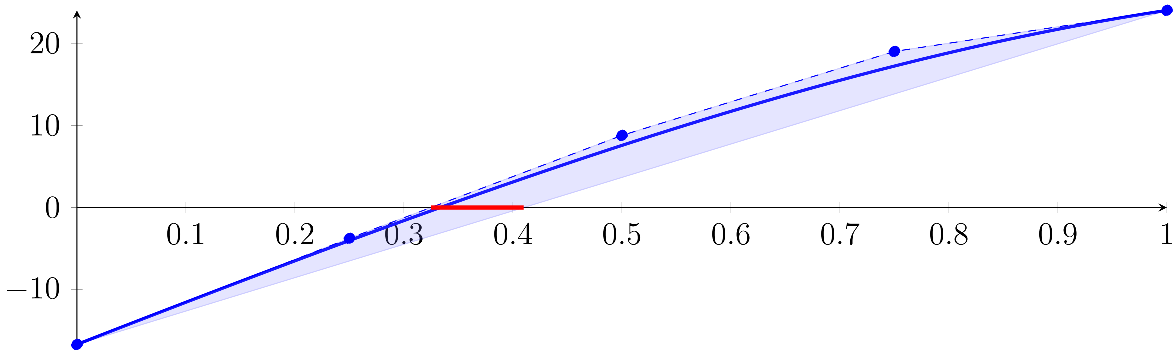
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



31.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

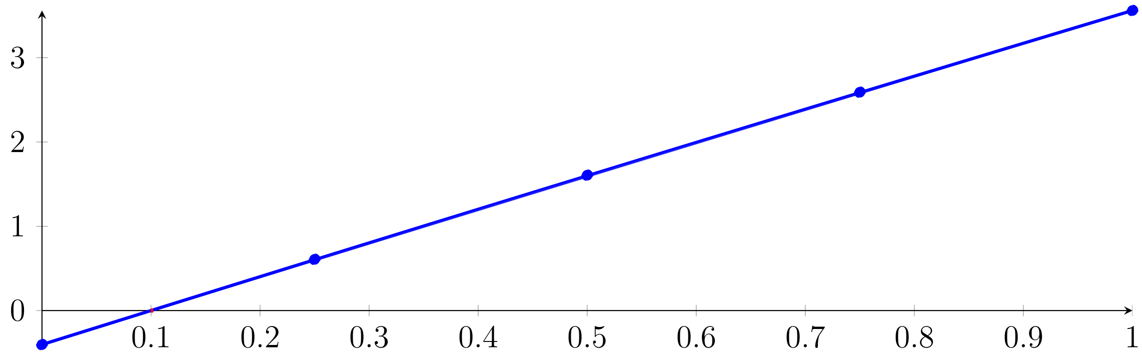
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

31.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

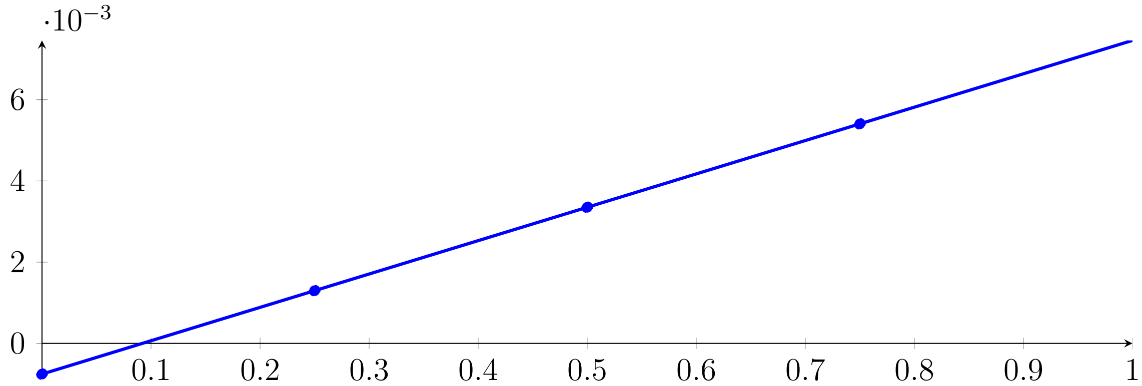
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

31.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.06393 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

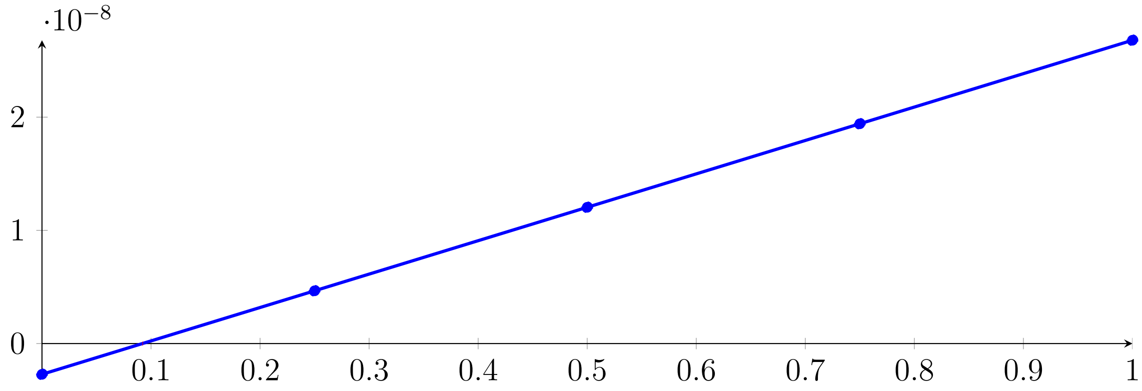
$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

31.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -9.92617 \cdot 10^{-24} X^4 + 6.61744 \cdot 10^{-24} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09}$$
$$= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373$$
$$\cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28974 \cdot 10^{-11}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

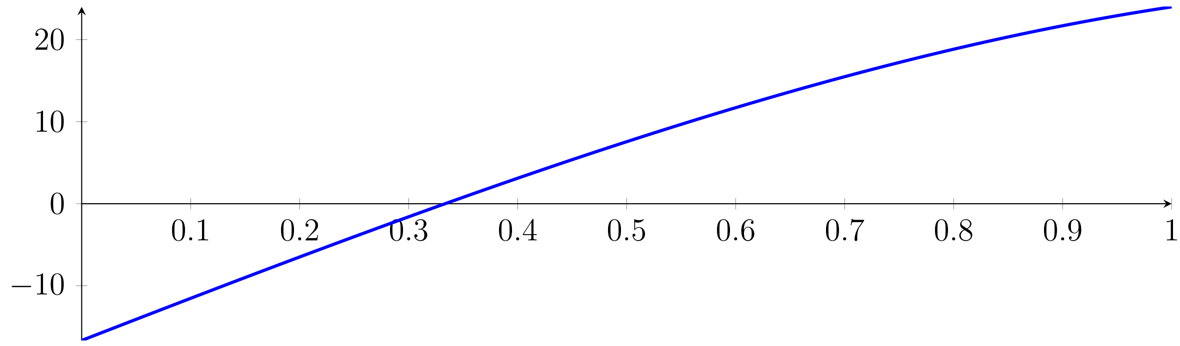
31.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

31.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

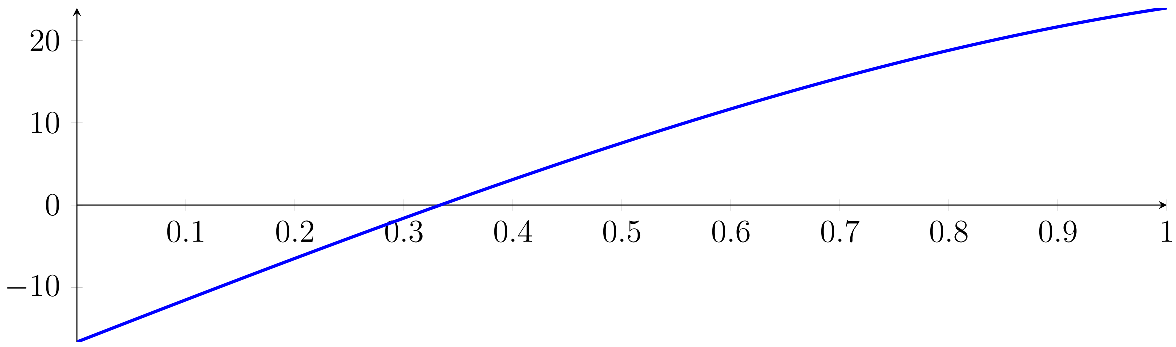
with precision $\varepsilon = 1 \cdot 10^{-16}$.

32 Running QuadClip on f_4 with epsilon 16

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

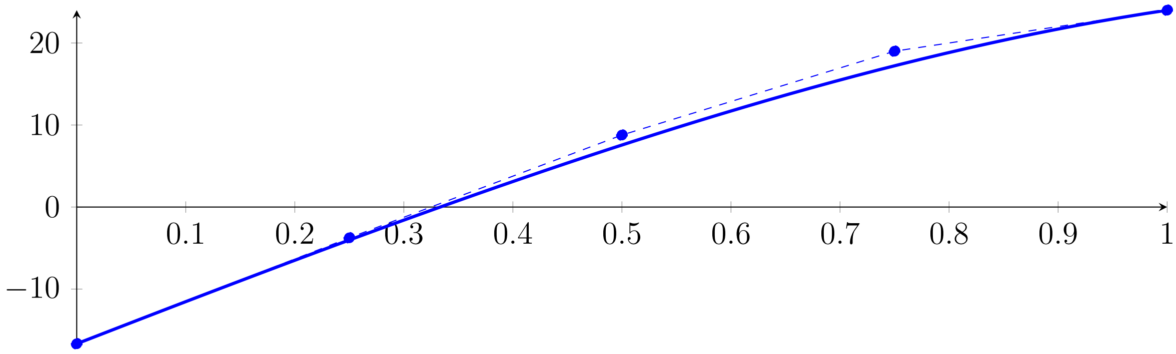
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



32.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

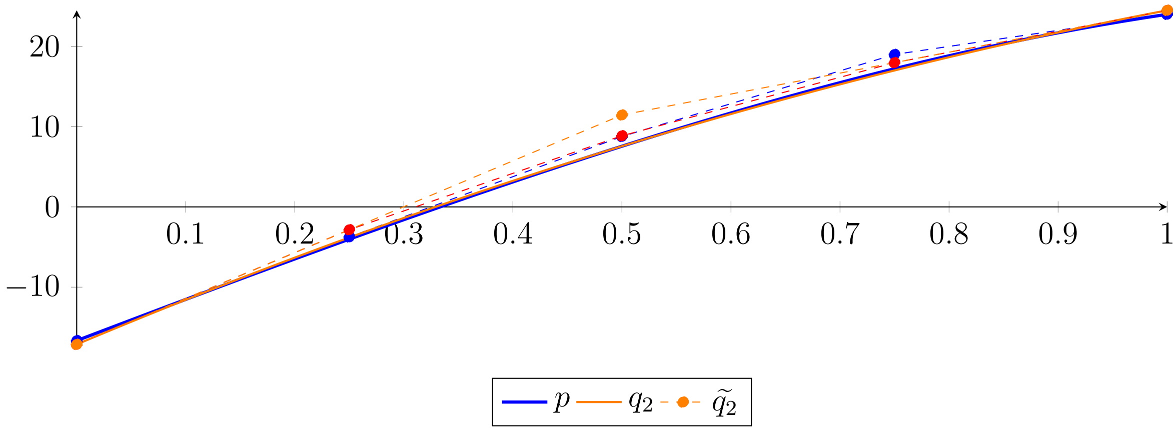
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -2.67519 \cdot 10^{-12}X^4 + 5.32907 \cdot 10^{-12}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

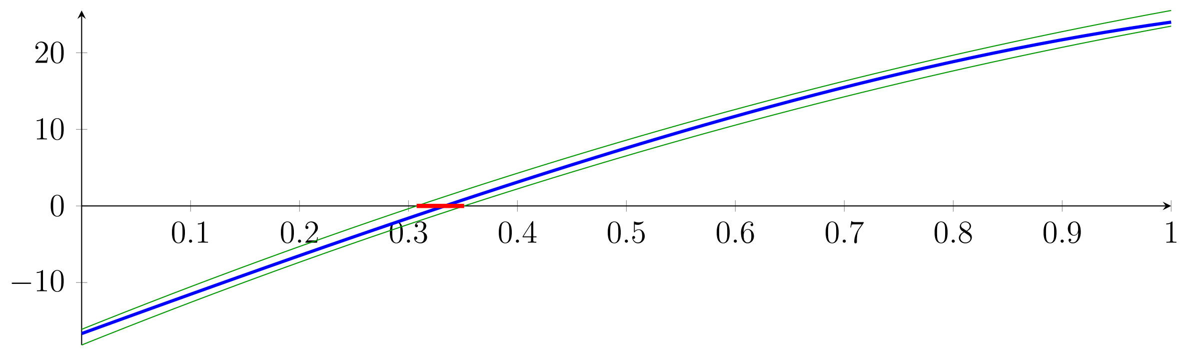
$$M = -15.5476X^2 + 57.181X - 16.1119$$

$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



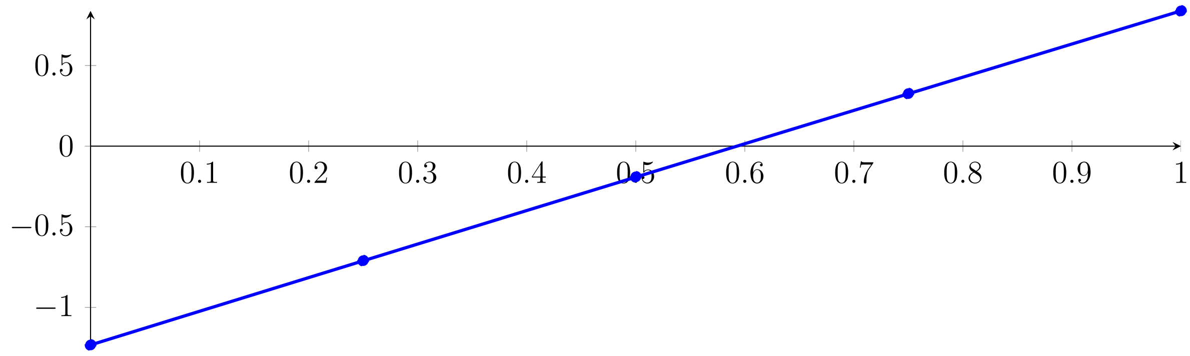
$[0.307477, 0.351097]$

Longest intersection interval: 0.0436205
$\implies$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

32.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

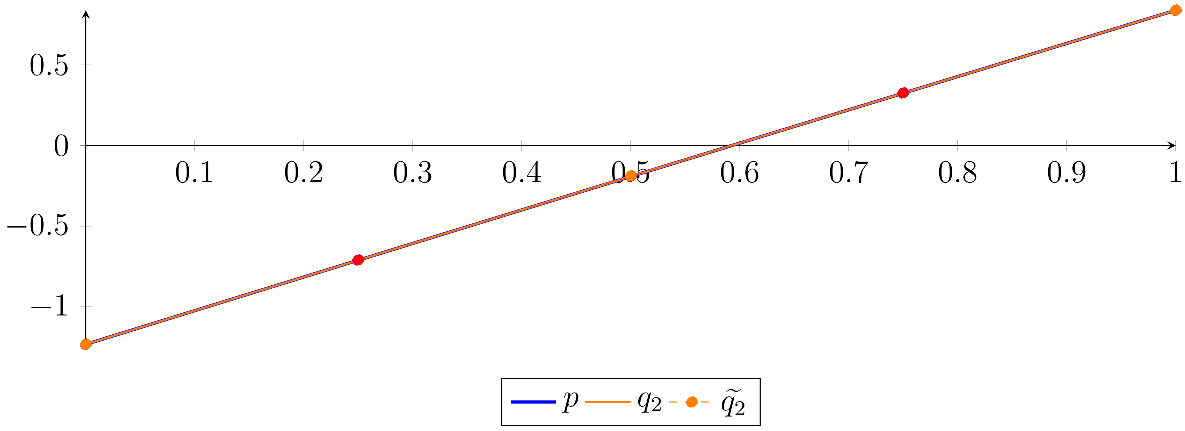
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06}X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 2.43583 \cdot 10^{-13}X^4 - 4.3876 \cdot 10^{-13}X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713{\cdot}10^{-05}$.

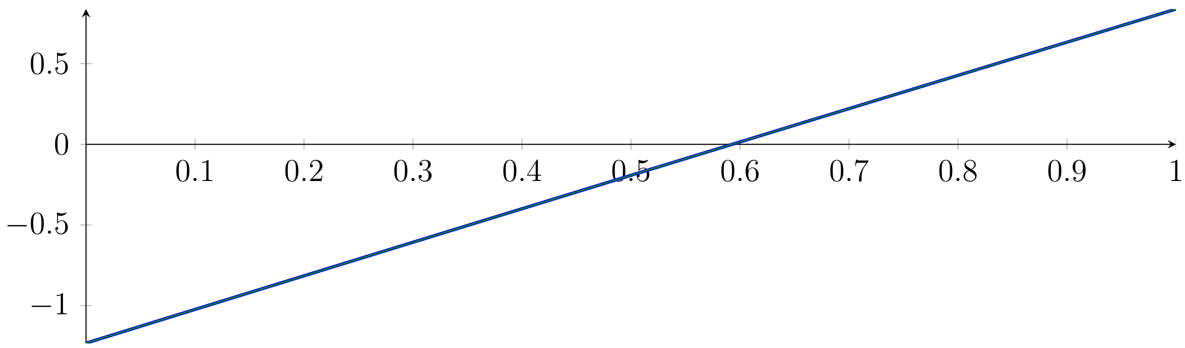
Bounding polynomials M and m:

$$M = -0.020089X^2 + 2.09166X - 1.23274$$
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

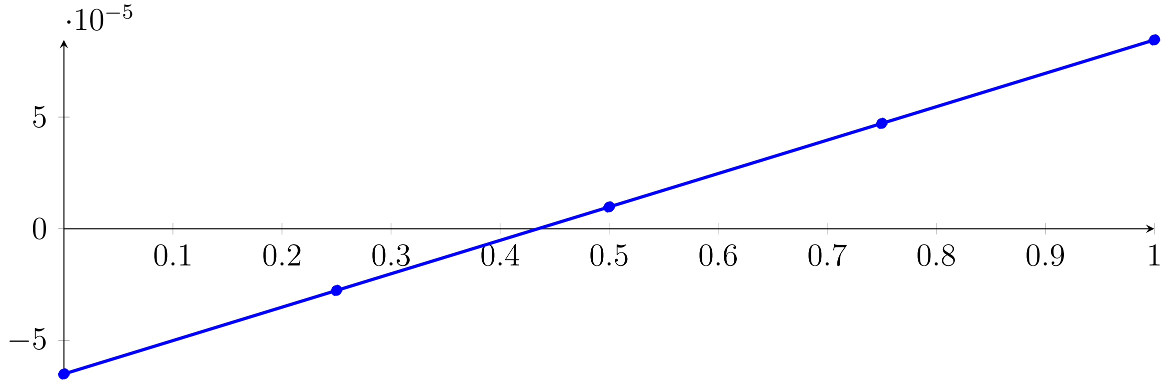
Longest intersection interval: $7.23183{\cdot}10^{-05}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

32.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

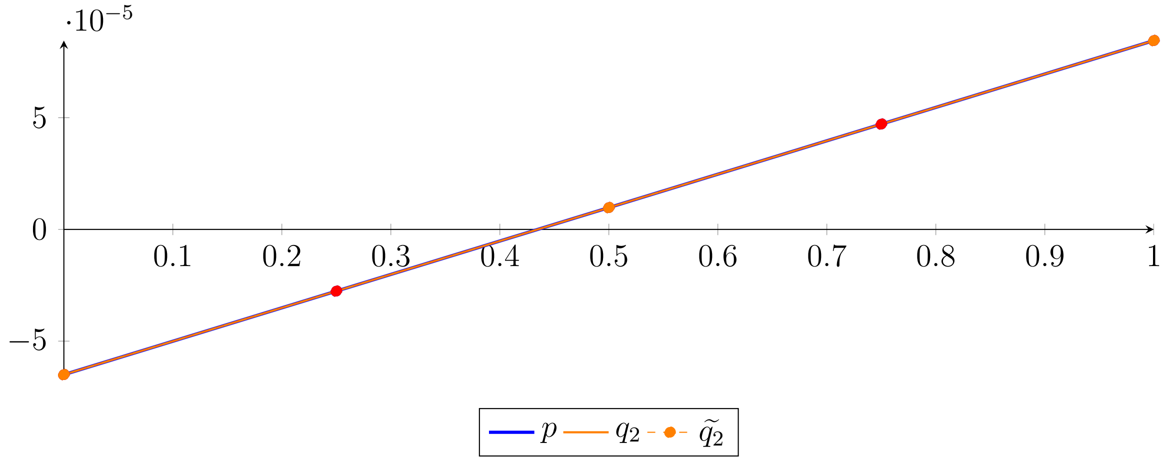
$$p = -2.71051{\cdot}10^{-20}X^4 - 2.82489{\cdot}10^{-16}X^3 - 1.06146{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$
$$= -6.50069{\cdot}10^{-05}B_{0,4}(X) - 2.76196{\cdot}10^{-05}B_{1,4}(X) + 9.76777 {\cdot}10^{-06}B_{2,4}(X) + 4.71551{\cdot}10^{-05}B_{3,4}(X) + 8.45424{\cdot}10^{-05}B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.06147\cdot 10^{-10}X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$
$$= -6.50069\cdot 10^{-05}B_{0,2} + 9.76779\cdot 10^{-06}B_{1,2} + 8.45424\cdot 10^{-05}B_{2,2}$$

$$\widetilde{q_2} = 6.72205\cdot 10^{-18}X^4 - 1.21431\cdot 10^{-17}X^3 - 1.06147\cdot 10^{-10}X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$
$$= -6.50069\cdot 10^{-05}B_{0,4} - 2.76196\cdot 10^{-05}B_{1,4} + 9.76777\cdot 10^{-06}B_{2,4} + 4.71551\cdot 10^{-05}B_{3,4} + 8.45424\cdot 10^{-05}B_{4,4}$$



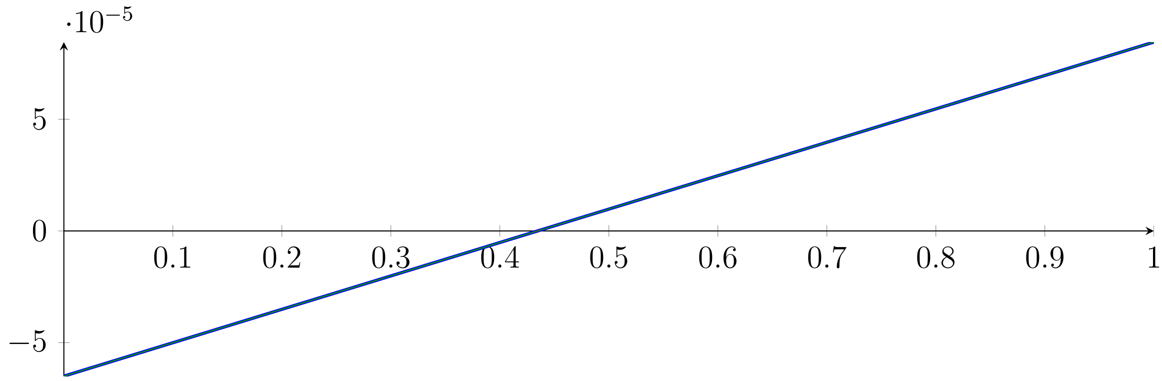
The maximum difference of the Bézier coefficients is $\delta = 2.88601\cdot 10^{-17}$.
Bounding polynomials M and m:

$$M = -1.06147\cdot 10^{-10}X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$
$$m = -1.06147\cdot 10^{-10}X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.434685, 1.4089\cdot 10^{6}\} \qquad N(m) = \{0.434685, 1.4089\cdot 10^{6}\}$$

Intersection intervals:



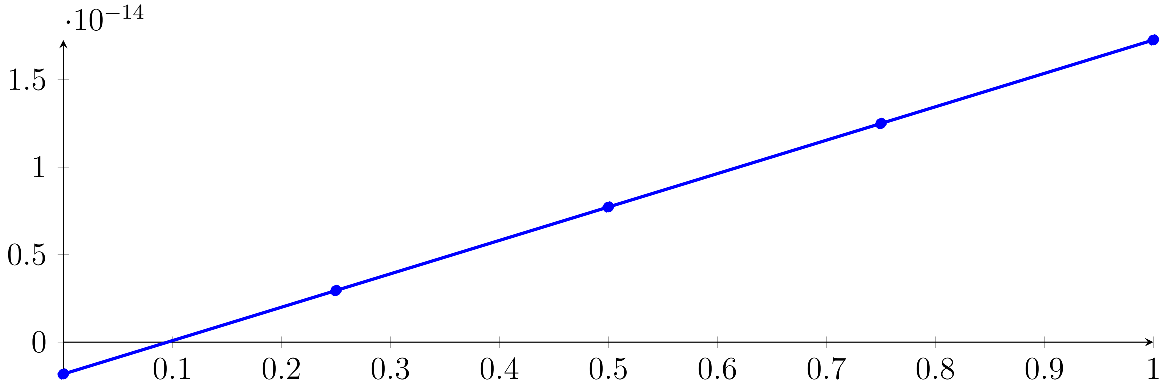
$$[0.434685, 0.434685]$$

Longest intersection interval: $1.27678\cdot 10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

32.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

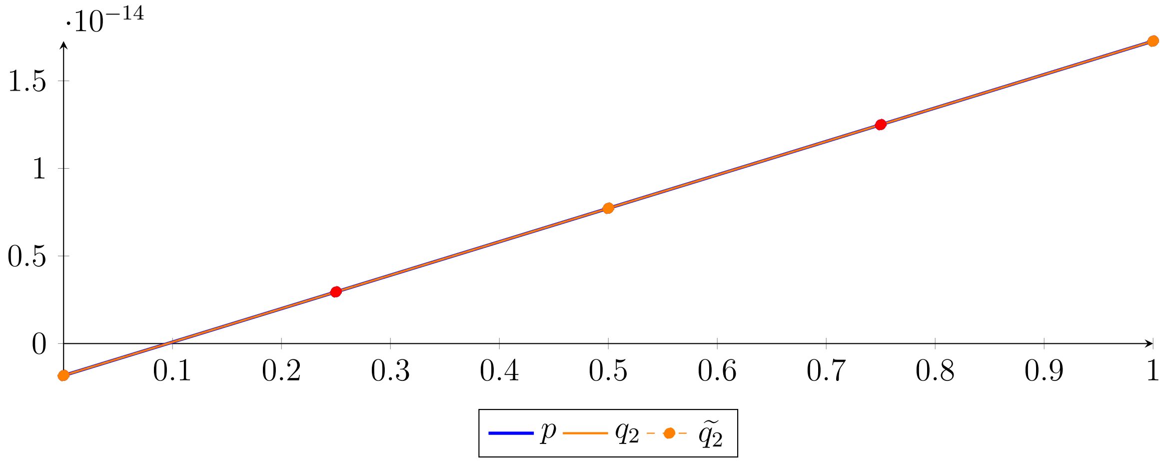
$$p = -1.41995\cdot10^{-29}X^4 + 6.31089\cdot10^{-30}X^3 + 4.73317\cdot10^{-30}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$
$$= -1.8241\cdot10^{-15}B_{0,4}(X) + 2.94943\cdot10^{-15}B_{1,4}(X) + 7.72295$$
$$\cdot10^{-15}B_{2,4}(X) + 1.24965\cdot10^{-14}B_{3,4}(X) + 1.727\cdot10^{-14}B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -2.68213\cdot10^{-29}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$
$$= -1.8241\cdot10^{-15}B_{0,2} + 7.72295\cdot10^{-15}B_{1,2} + 1.727\cdot10^{-14}B_{2,2}$$

$$\widetilde{q}_2 = -2.18041\cdot10^{-27}X^4 + 3.9443\cdot10^{-27}X^3 - 2.24352\cdot10^{-27}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$
$$= -1.8241\cdot10^{-15}B_{0,4} + 2.94943\cdot10^{-15}B_{1,4} + 7.72295\cdot10^{-15}B_{2,4} + 1.24965\cdot10^{-14}B_{3,4} + 1.727\cdot10^{-14}B_{4,4}$$



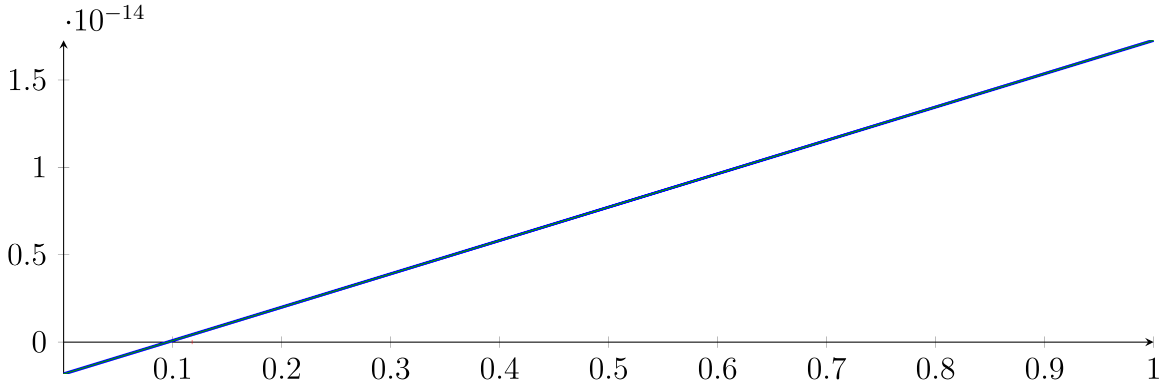
The maximum difference of the Bézier coefficients is $\delta = 1.73549\cdot10^{-28}$.
Bounding polynomials M and m:

$$M = -2.68213\cdot10^{-29}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$
$$m = -2.68213\cdot10^{-29}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$

Root of M and m:

$$N(M) = \{0.117647, 7.11901\cdot10^{14}\} \qquad N(m) = \{0.117647, 7.11901\cdot10^{14}\}$$

Intersection intervals:



$[0.117647, 0.117647]$

Longest intersection interval: 0
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

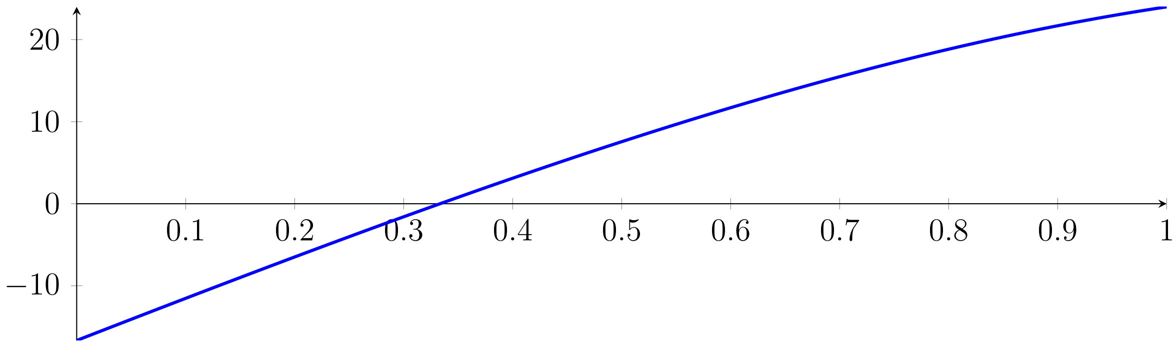
32.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

32.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

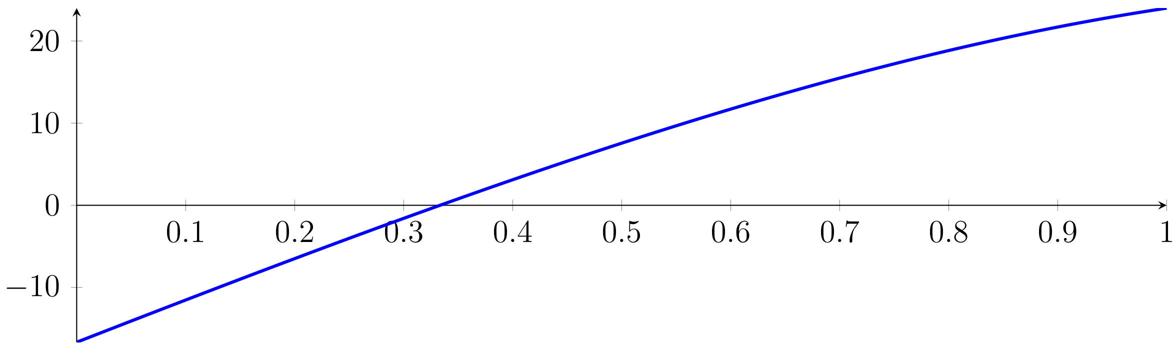
with precision $\varepsilon = 1 \cdot 10^{-16}$.

33 Running CubeClip on f_4 with epsilon 16

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

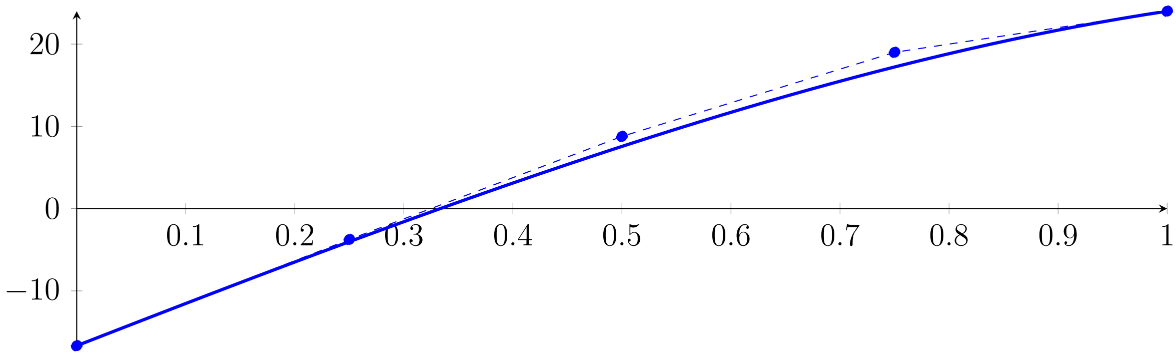
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



33.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

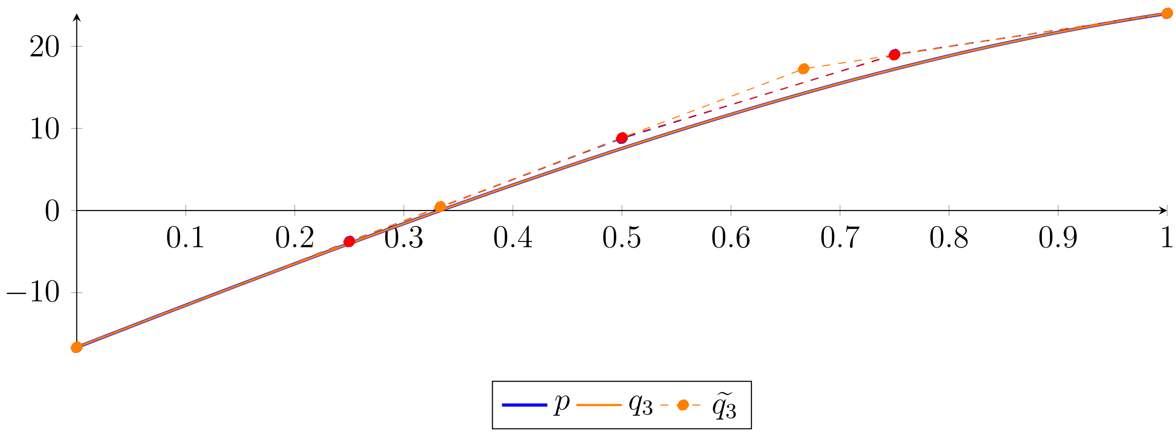
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -2.28084 \cdot 10^{-12}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



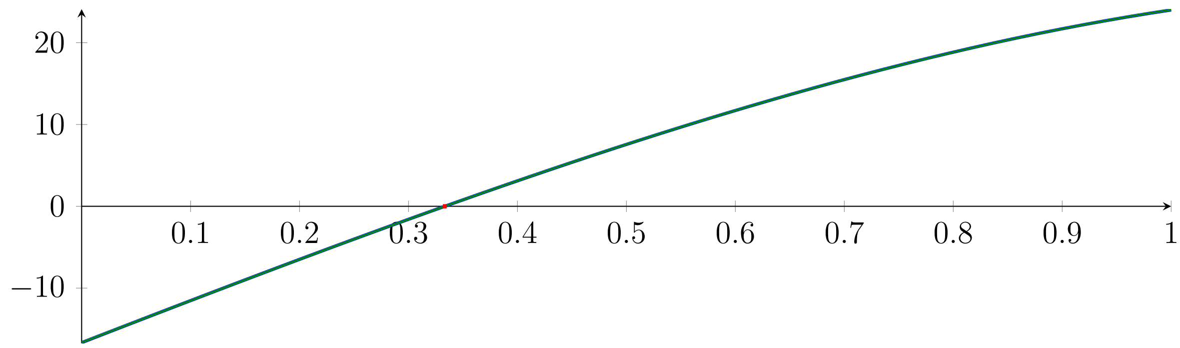
The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



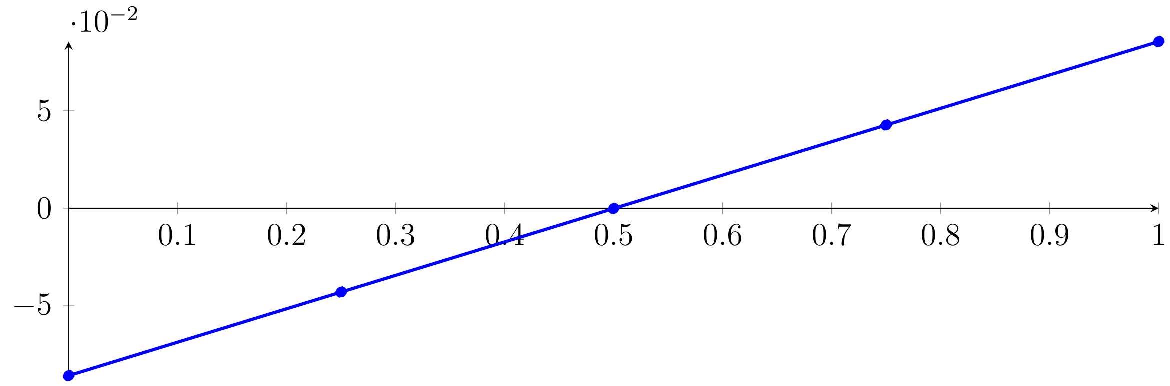
$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

33.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

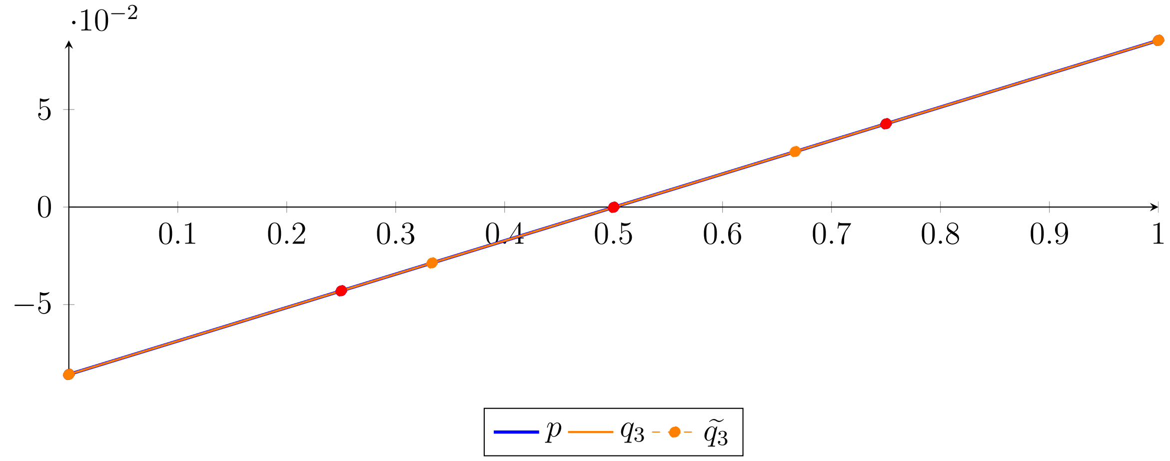
$$\begin{aligned}p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4}(X) - 0.0429507B_{1,4}(X) - 0.000129666B_{2,4}(X) \\ &\quad + 0.0426682B_{3,4}(X) + 0.0854427B_{4,4}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,3} - 0.0286693B_{1,3} + 0.02841B_{2,3} + 0.0854427B_{3,3}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= 1.2032 \cdot 10^{-14} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4} - 0.0429507B_{1,4} - 0.000129666B_{2,4} + 0.0426682B_{3,4} + 0.0854427B_{4,4}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45913 \cdot 10^{-11}$.
Bounding polynomials M and m:

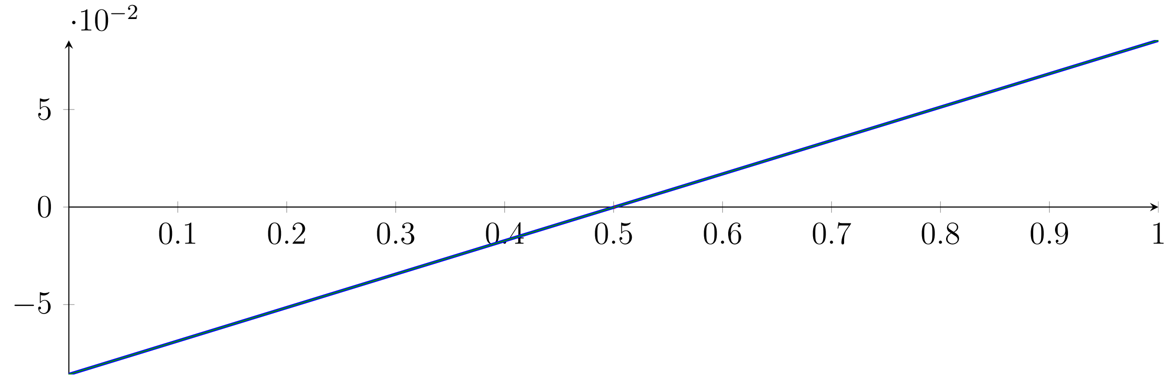
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



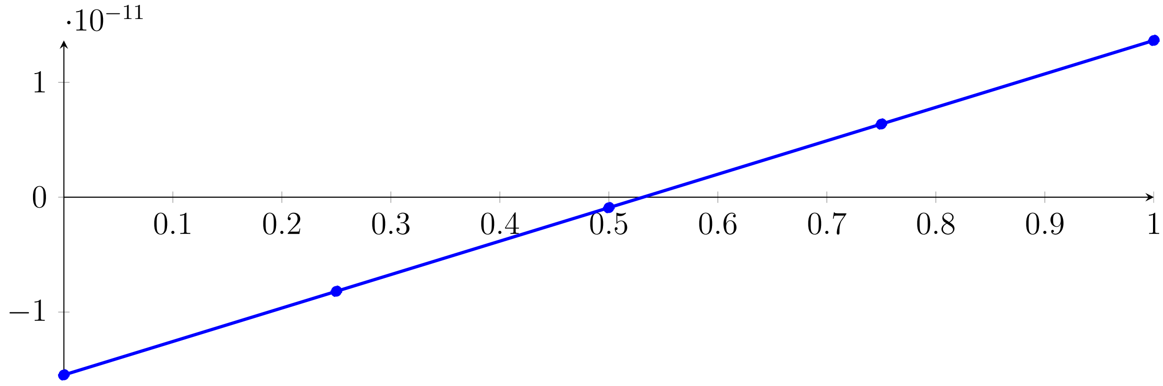
$$[0.500825, 0.500825]$$

Longest intersection interval: $1.70047 \cdot 10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

33.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

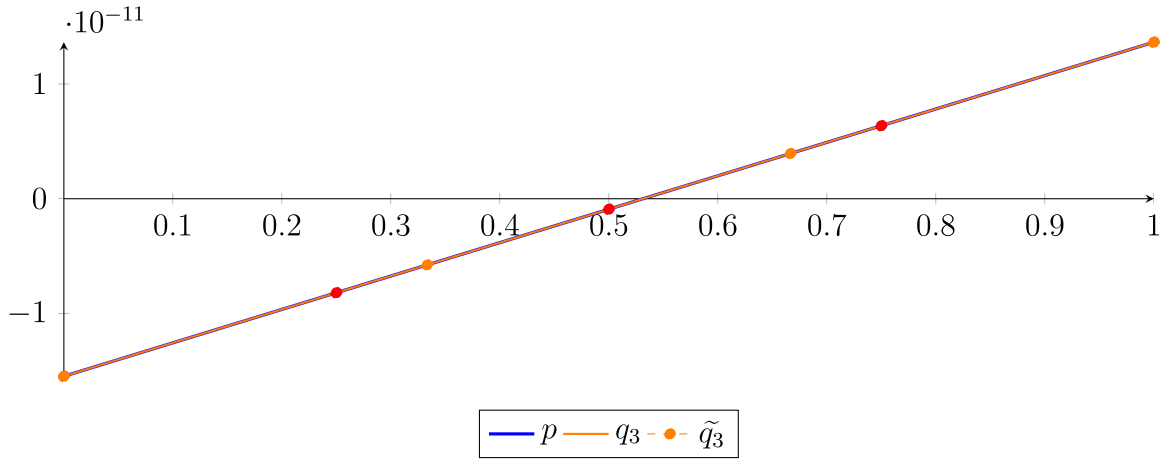
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.01312 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\ &= -1.5473 \cdot 10^{-11} B_{0,4}(X) - 8.19335 \cdot 10^{-12} B_{1,4}(X) - 9.13745 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36586 \cdot 10^{-12} B_{3,4}(X) + 1.36455 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -6.78547\cdot 10^{-25}X^3 - 2.95652\cdot 10^{-24}X^2 + 2.91184\cdot 10^{-11}X - 1.5473\cdot 10^{-11}$$
$$= -1.5473\cdot 10^{-11}B_{0,3} - 5.76681\cdot 10^{-12}B_{1,3} + 3.93932\cdot 10^{-12}B_{2,3} + 1.36455\cdot 10^{-11}B_{3,3}$$
$$\widetilde{q}_3 = 2.83697\cdot 10^{-24}X^4 - 6.85009\cdot 10^{-24}X^3 + 1.39587\cdot 10^{-24}X^2 + 2.91184\cdot 10^{-11}X - 1.5473\cdot 10^{-11}$$
$$= -1.5473\cdot 10^{-11}B_{0,4} - 8.19335\cdot 10^{-12}B_{1,4} - 9.13745\cdot 10^{-13}B_{2,4} + 6.36586\cdot 10^{-12}B_{3,4} + 1.36455\cdot 10^{-11}B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.84343\cdot 10^{-25}$.

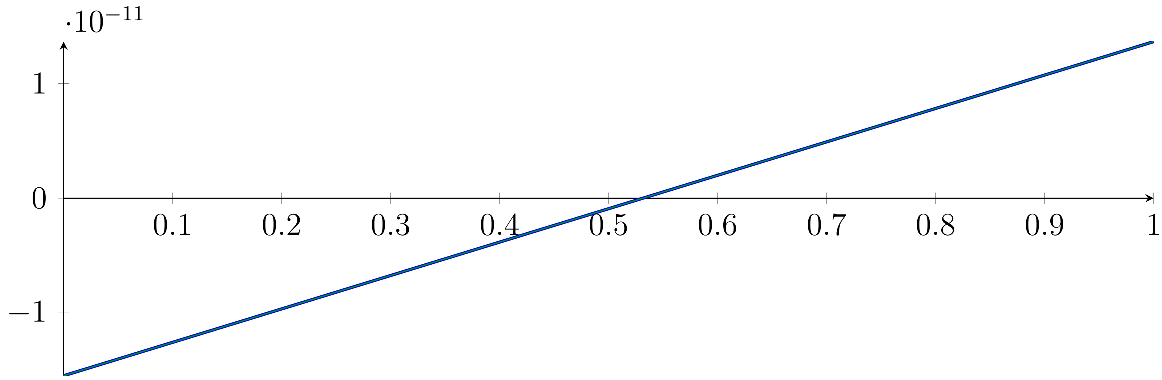
Bounding polynomials M and m:

$$M = -6.78547\cdot 10^{-25}X^3 - 2.95652\cdot 10^{-24}X^2 + 2.91184\cdot 10^{-11}X - 1.5473\cdot 10^{-11}$$
$$m = -6.78547\cdot 10^{-25}X^3 - 2.95652\cdot 10^{-24}X^2 + 2.91184\cdot 10^{-11}X - 1.5473\cdot 10^{-11}$$

Root of M and m:

$$N(M) = \{-6.51982\cdot 10^6, 0.53138, 6.51982\cdot 10^6\} \quad N(m) = \{-6.51982\cdot 10^6, 0.53138, 6.51982\cdot 10^6\}$$

Intersection intervals:



$[0.53138, 0.53138]$

Longest intersection interval: 0

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

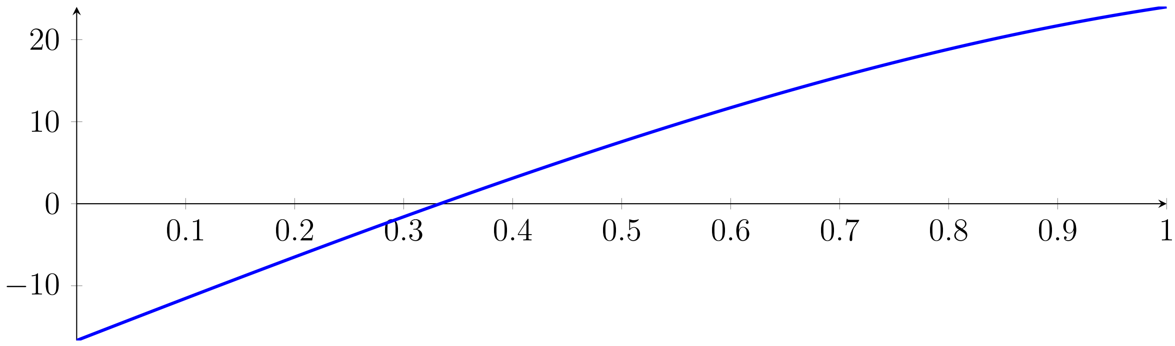
33.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

33.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

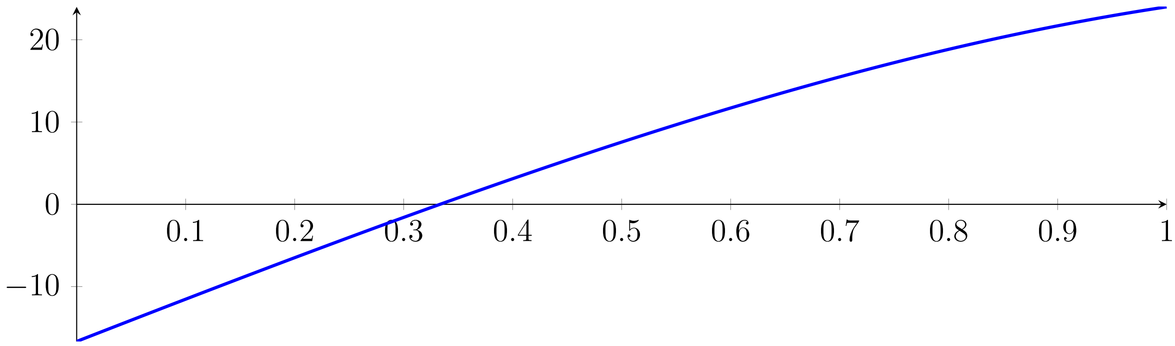
with precision $\varepsilon = 1 \cdot 10^{-16}$.

34 Running BezClip on f_4 with epsilon 32

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

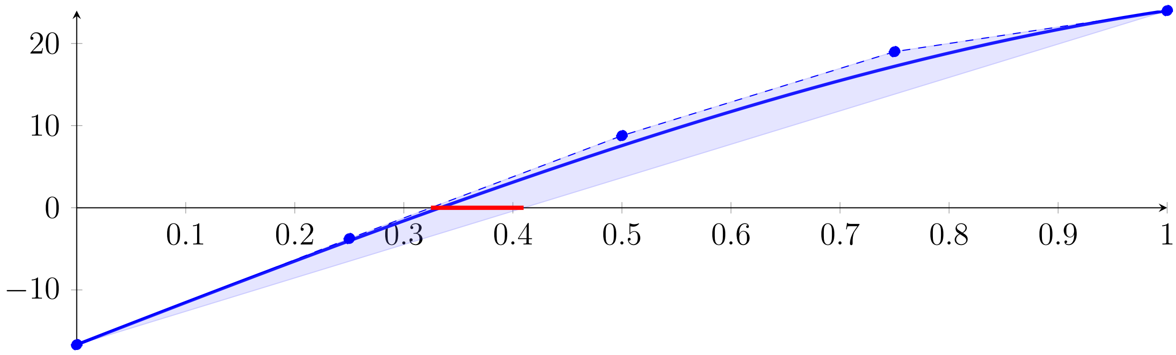
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



34.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

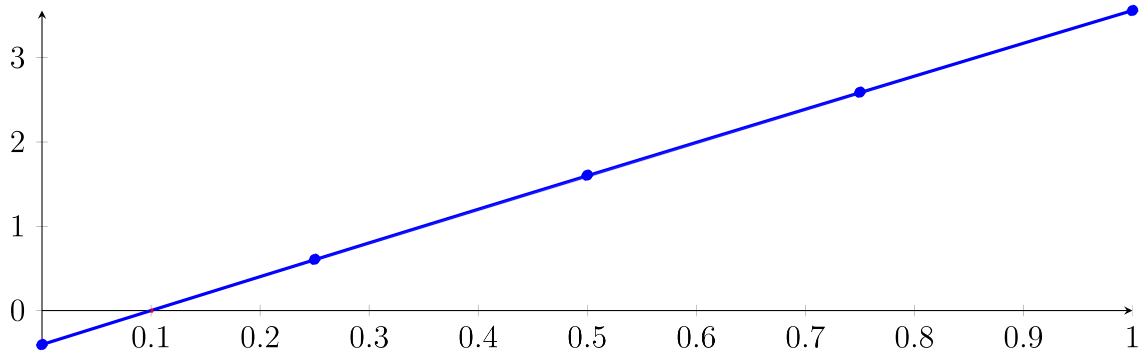
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

34.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

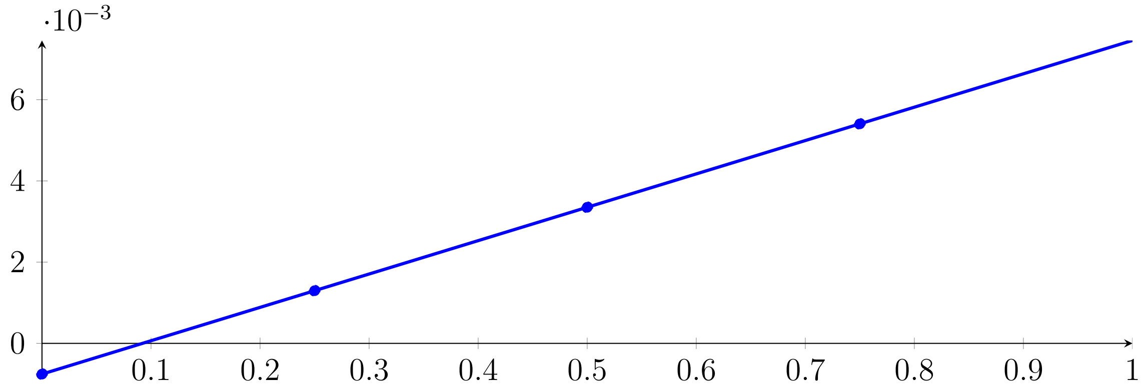
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

34.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.06393 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

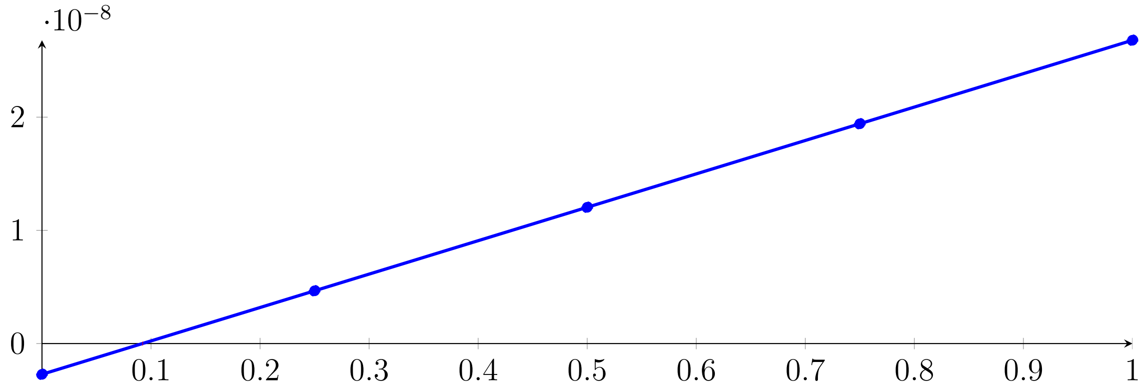
$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

34.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -9.92617 \cdot 10^{-24} X^4 + 6.61744 \cdot 10^{-24} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09}$$
$$= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28974 \cdot 10^{-11}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

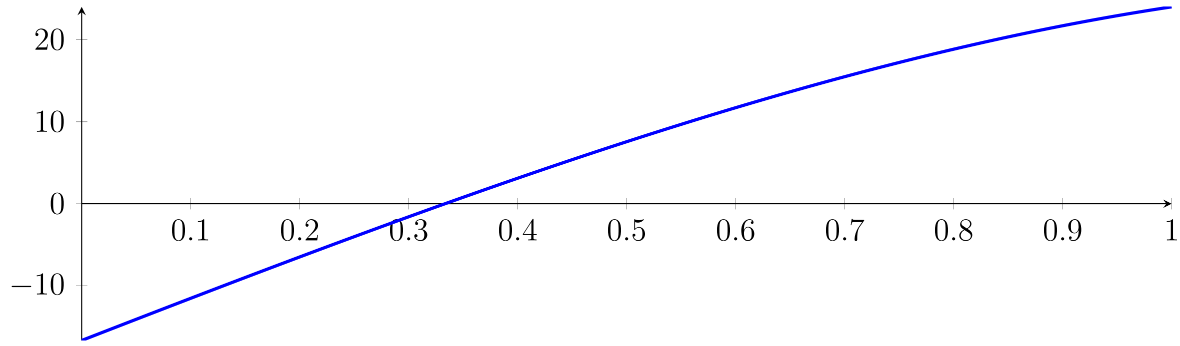
34.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

34.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

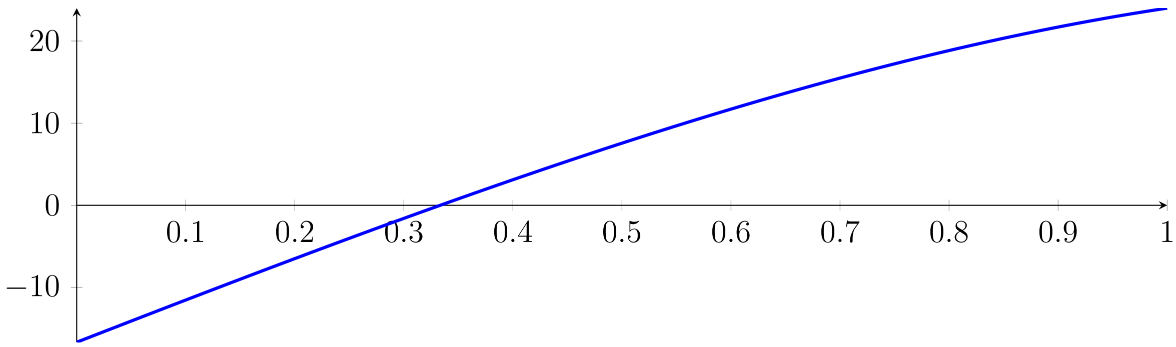
with precision $\varepsilon = 1 \cdot 10^{-32}$.

35 Running QuadClip on f_4 with epsilon 32

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

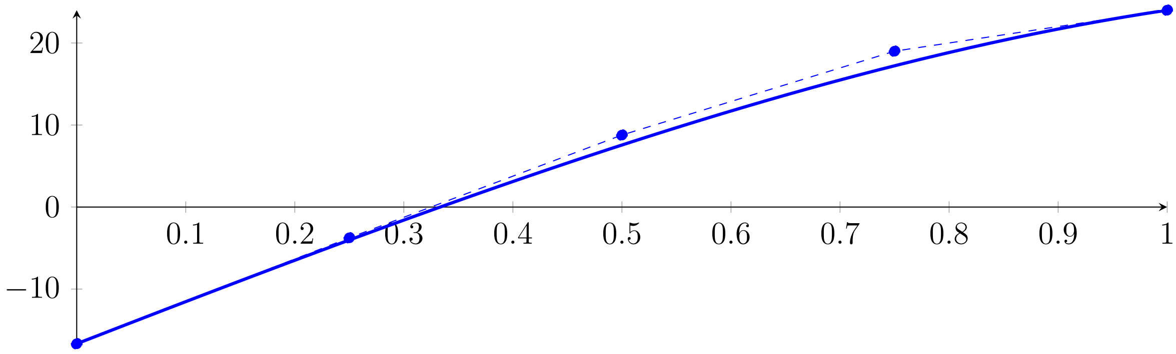
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



35.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

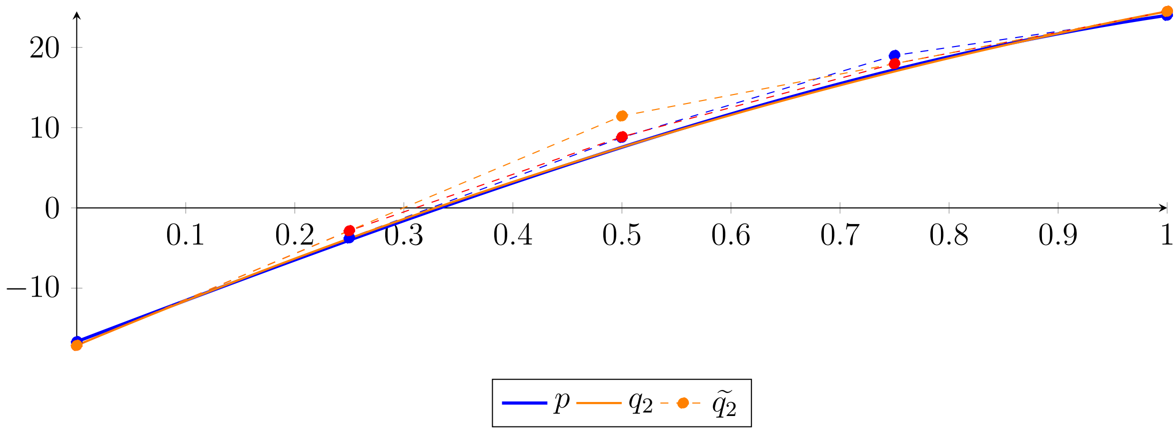
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -2.67519 \cdot 10^{-12}X^4 + 5.32907 \cdot 10^{-12}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

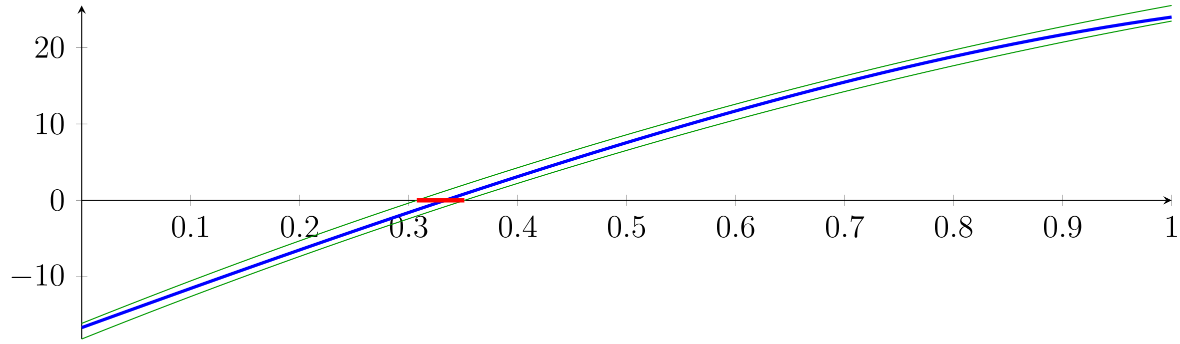
$$M = -15.5476X^2 + 57.181X - 16.1119$$

$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



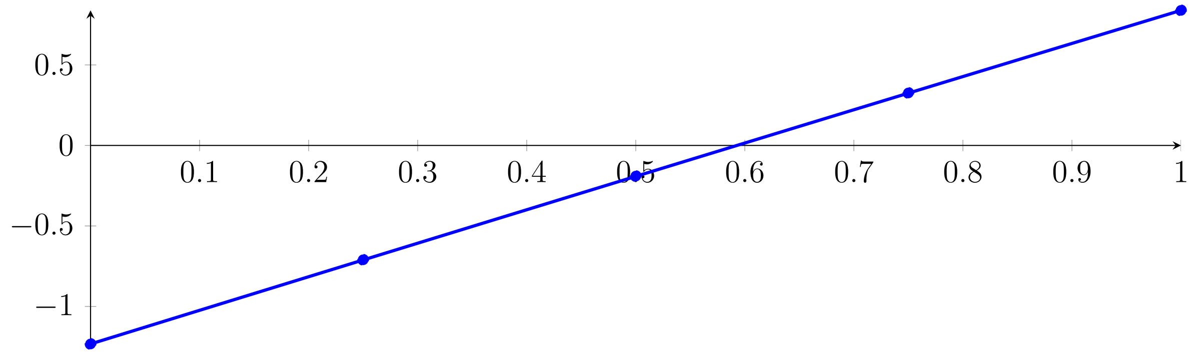
$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

35.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

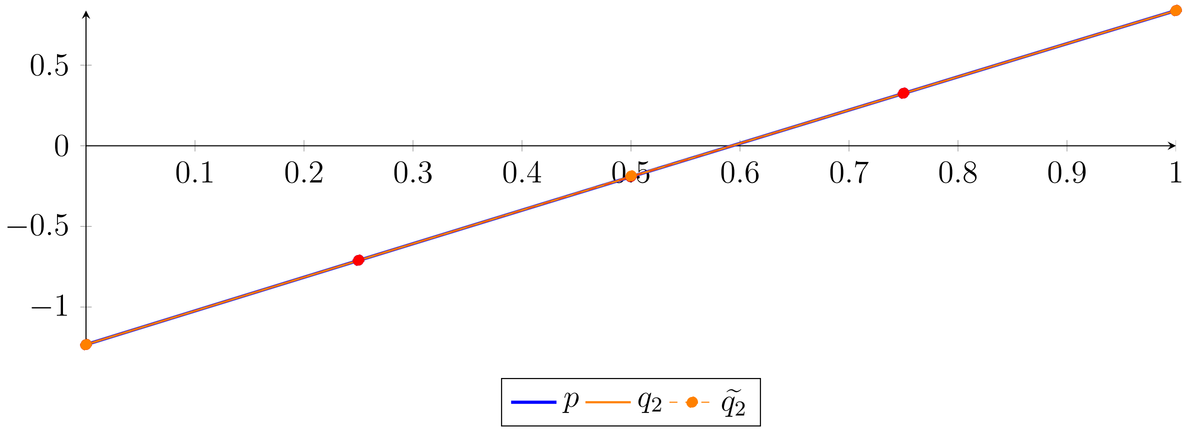
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06}X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 2.43583 \cdot 10^{-13}X^4 - 4.3876 \cdot 10^{-13}X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



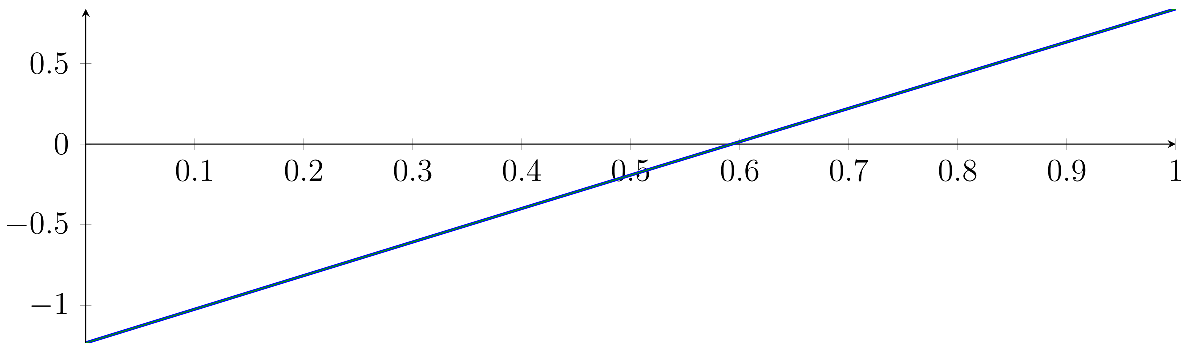
The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.
Bounding polynomials M and m:

$$M = -0.020089X^2 + 2.09166X - 1.23274$$
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



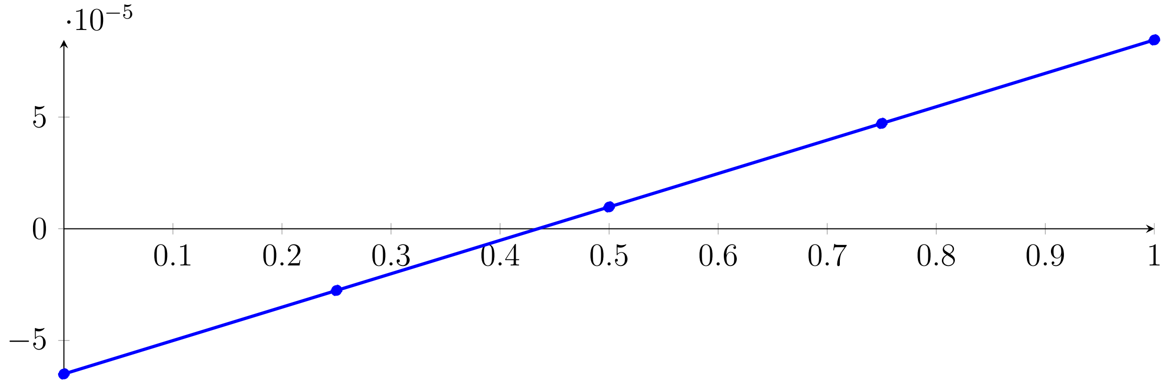
$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

35.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

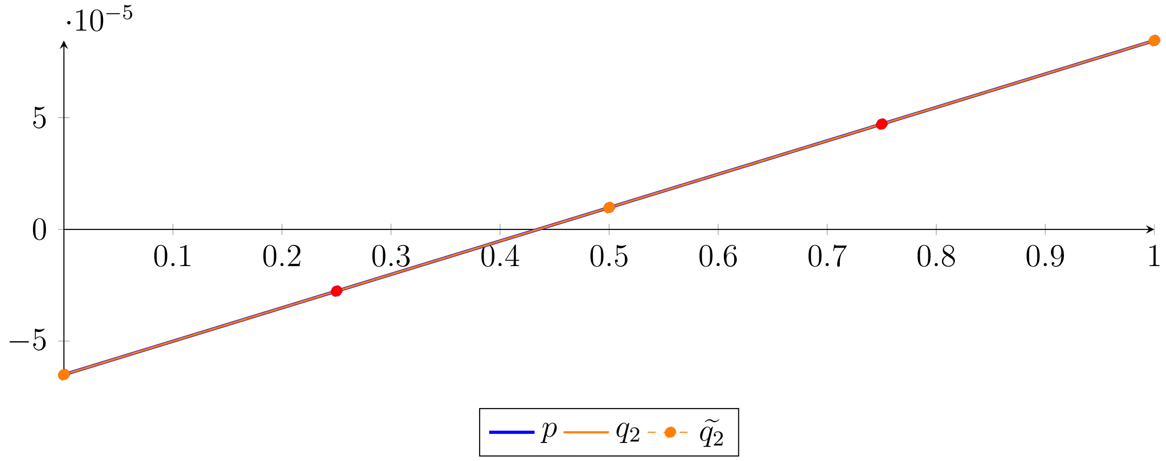
$$p = -2.71051 \cdot 10^{-20}X^4 - 2.82489 \cdot 10^{-16}X^3 - 1.06146 \cdot 10^{-10}X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05}B_{0,4}(X) - 2.76196 \cdot 10^{-05}B_{1,4}(X) + 9.76777$$
$$\cdot 10^{-06}B_{2,4}(X) + 4.71551 \cdot 10^{-05}B_{3,4}(X) + 8.45424 \cdot 10^{-05}B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.06147\cdot 10^{-10}X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$
$$= -6.50069\cdot 10^{-05}B_{0,2} + 9.76779\cdot 10^{-06}B_{1,2} + 8.45424\cdot 10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = 6.72205\cdot 10^{-18}X^4 - 1.21431\cdot 10^{-17}X^3 - 1.06147\cdot 10^{-10}X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$
$$= -6.50069\cdot 10^{-05}B_{0,4} - 2.76196\cdot 10^{-05}B_{1,4} + 9.76777\cdot 10^{-06}B_{2,4} + 4.71551\cdot 10^{-05}B_{3,4} + 8.45424\cdot 10^{-05}B_{4,4}$$



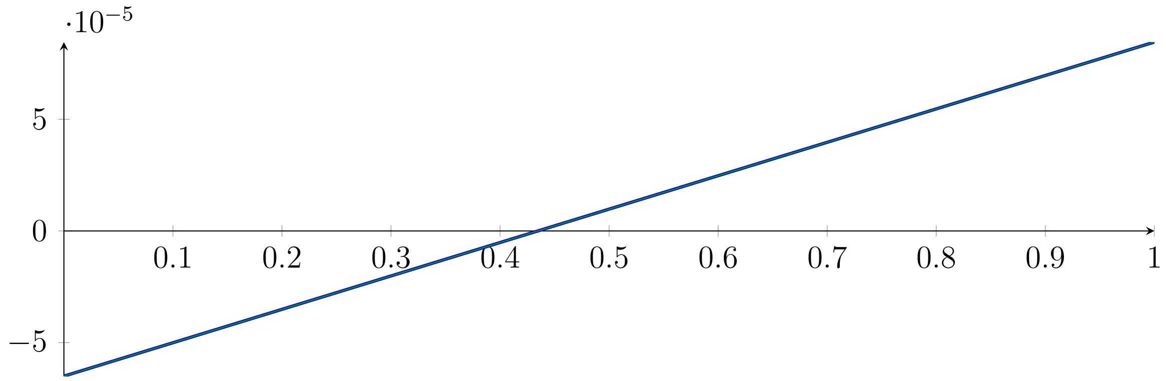
The maximum difference of the Bézier coefficients is $\delta = 2.88601\cdot 10^{-17}$.
Bounding polynomials M and m:

$$M = -1.06147\cdot 10^{-10}X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$
$$m = -1.06147\cdot 10^{-10}X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.434685, 1.4089\cdot 10^{6}\} \qquad N(m) = \{0.434685, 1.4089\cdot 10^{6}\}$$

Intersection intervals:



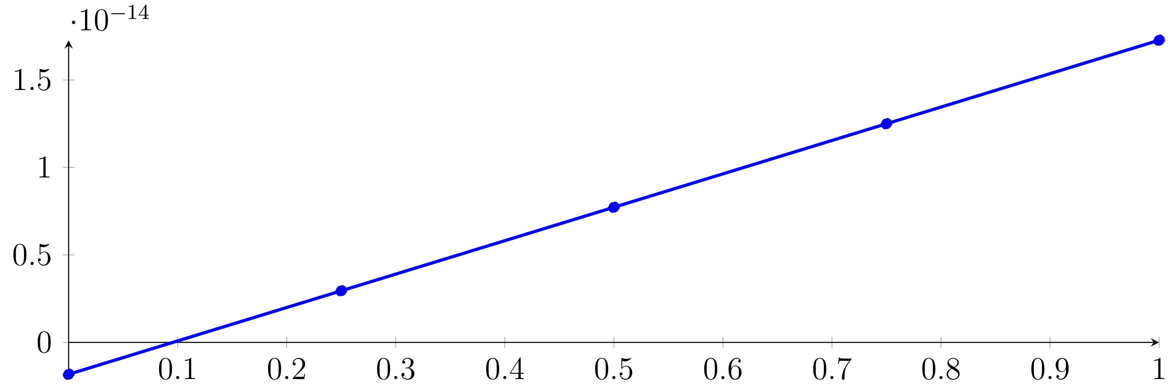
$$[0.434685, 0.434685]$$

Longest intersection interval: $1.27678\cdot 10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

35.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

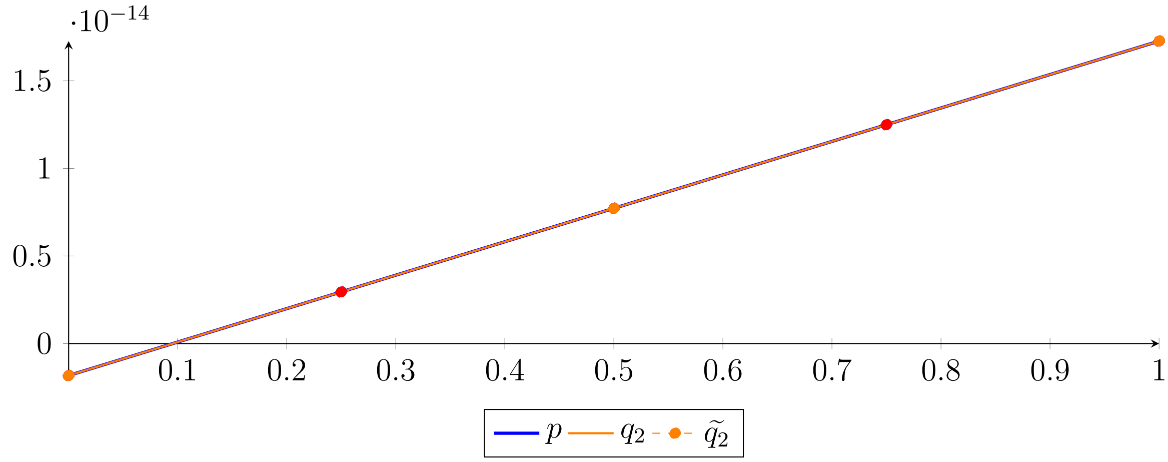
$$p = -1.41995\cdot10^{-29}X^4 + 6.31089\cdot10^{-30}X^3 + 4.73317\cdot10^{-30}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$
$$= -1.8241\cdot10^{-15}B_{0,4}(X) + 2.94943\cdot10^{-15}B_{1,4}(X) + 7.72295$$
$$\cdot10^{-15}B_{2,4}(X) + 1.24965\cdot10^{-14}B_{3,4}(X) + 1.727\cdot10^{-14}B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -2.68213\cdot10^{-29}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$
$$= -1.8241\cdot10^{-15}B_{0,2} + 7.72295\cdot10^{-15}B_{1,2} + 1.727\cdot10^{-14}B_{2,2}$$

$$\widetilde{q}_2 = -2.18041\cdot10^{-27}X^4 + 3.9443\cdot10^{-27}X^3 - 2.24352\cdot10^{-27}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$
$$= -1.8241\cdot10^{-15}B_{0,4} + 2.94943\cdot10^{-15}B_{1,4} + 7.72295\cdot10^{-15}B_{2,4} + 1.24965\cdot10^{-14}B_{3,4} + 1.727\cdot10^{-14}B_{4,4}$$



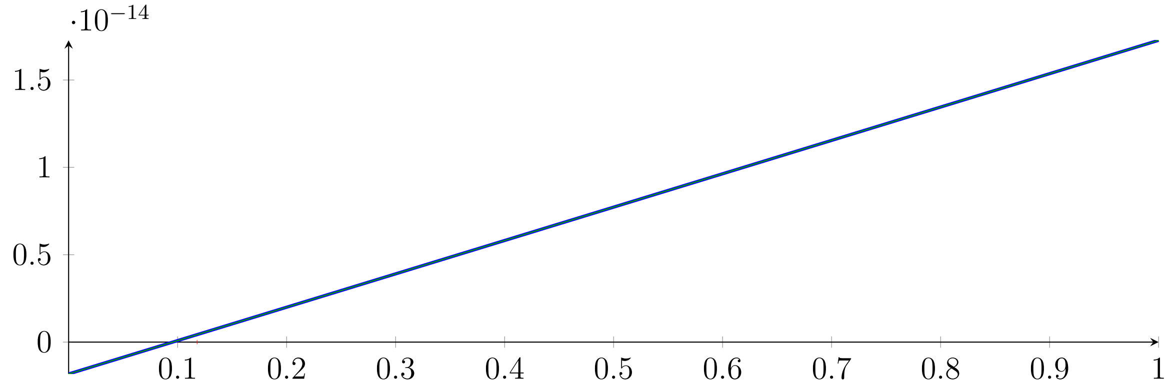
The maximum difference of the Bézier coefficients is $\delta = 1.73549\cdot10^{-28}$.
Bounding polynomials M and m:

$$M = -2.68213\cdot10^{-29}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$
$$m = -2.68213\cdot10^{-29}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$

Root of M and m:

$$N(M) = \{0.117647, 7.11901\cdot10^{14}\} \qquad N(m) = \{0.117647, 7.11901\cdot10^{14}\}$$

Intersection intervals:



$[0.117647, 0.117647]$

Longest intersection interval: 0

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

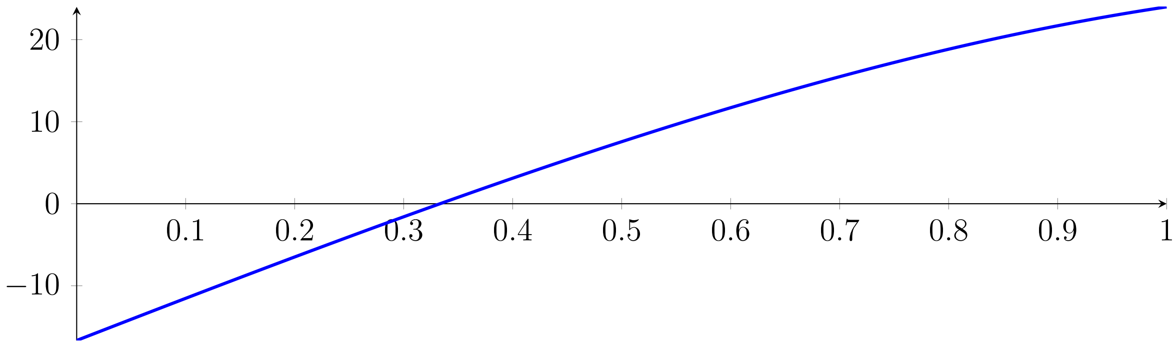
35.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

35.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-32}$.

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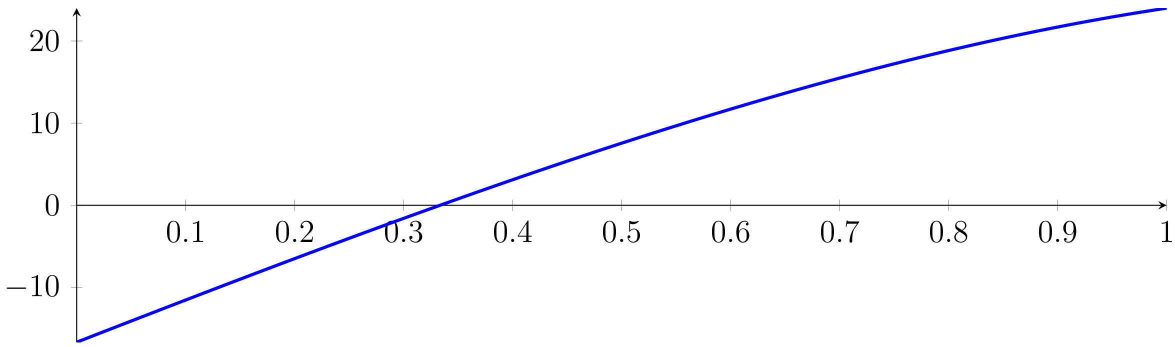
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

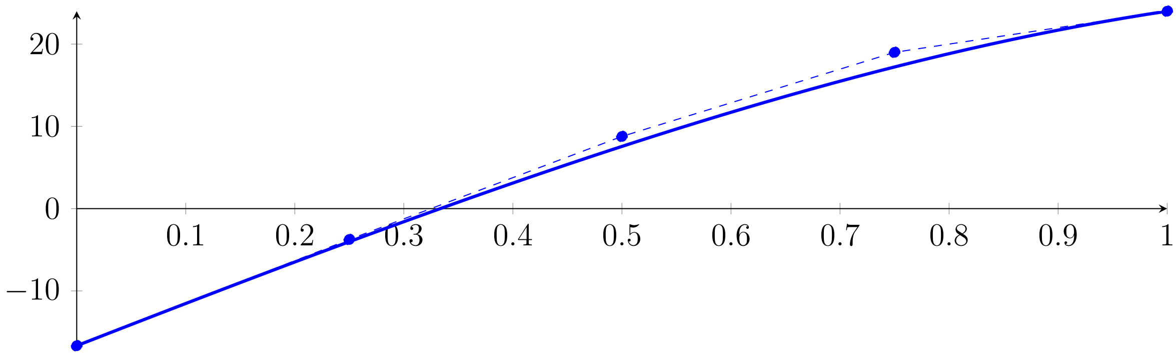
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



36.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

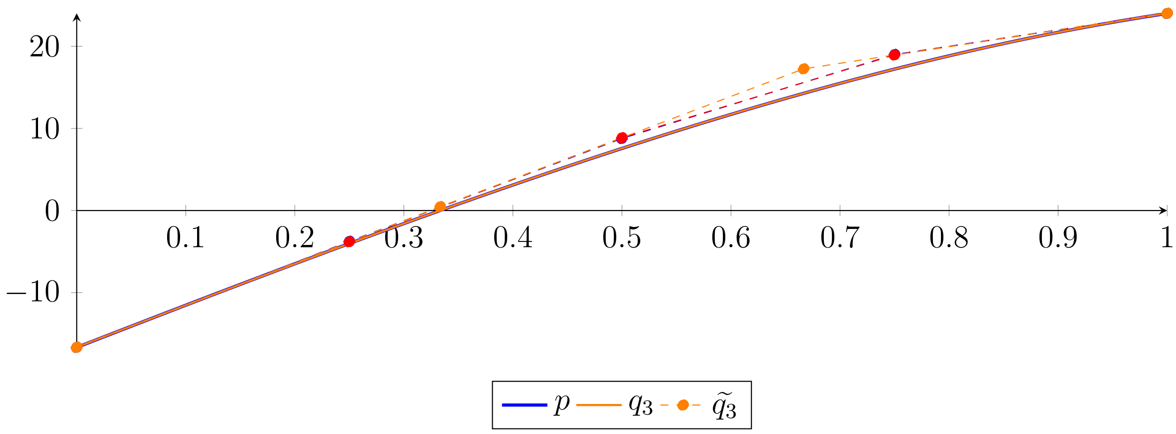
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -2.28084 \cdot 10^{-12}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



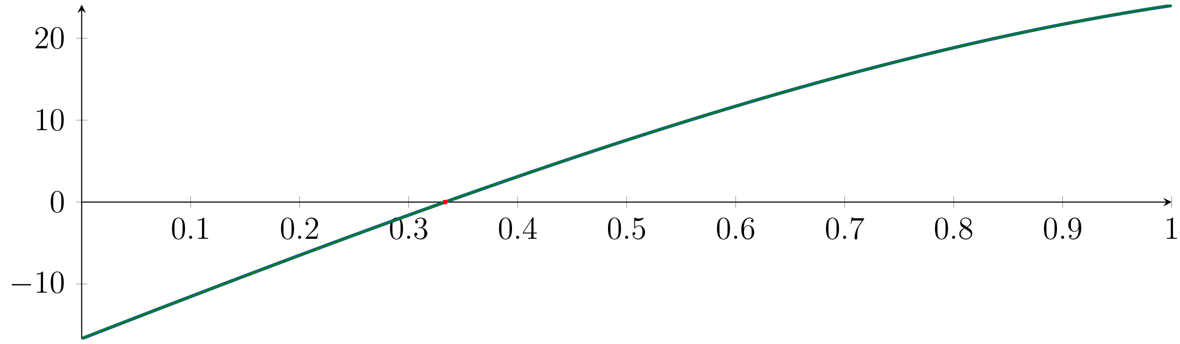
The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



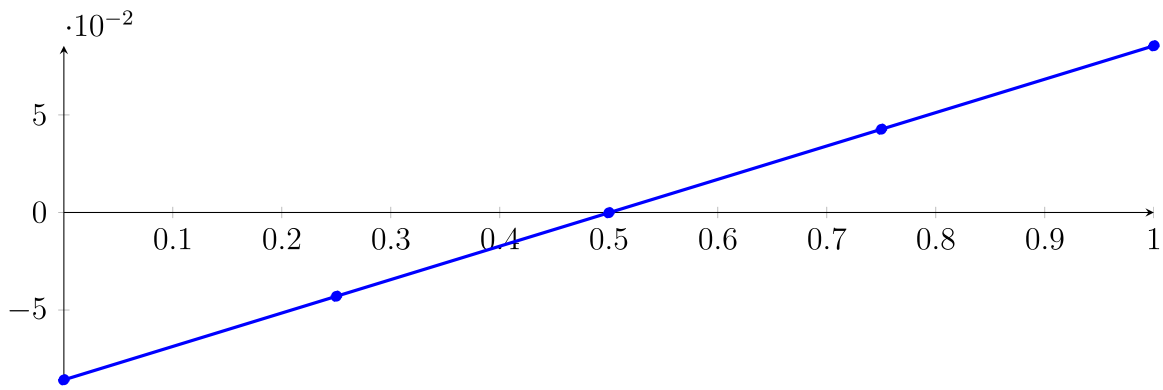
$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

36.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

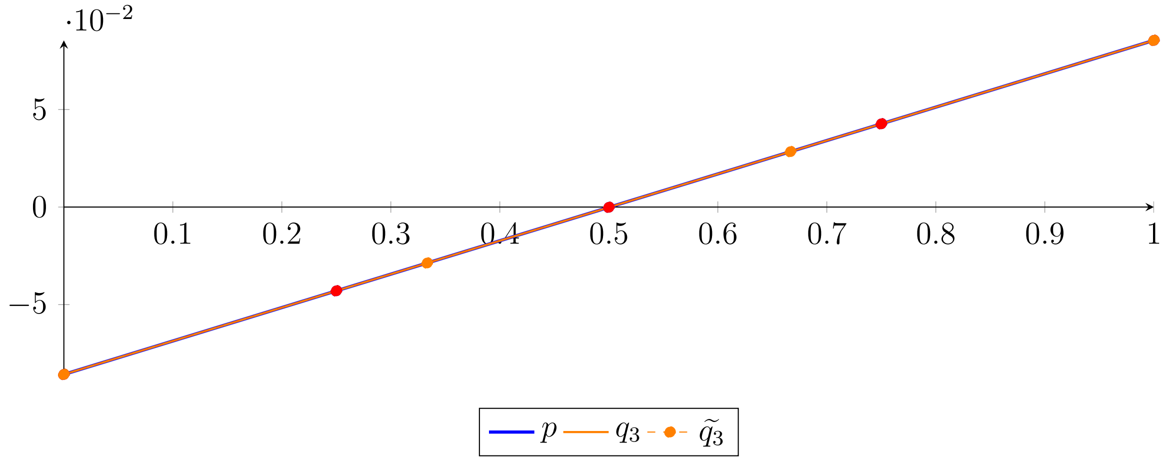
$$\begin{aligned} p &= -1.70219{\cdot}10^{-10}X^4 - 4.23789{\cdot}10^{-07}X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4}(X) - 0.0429507B_{1,4}(X) - 0.000129666B_{2,4}(X) \\ &\quad + 0.0426682B_{3,4}(X) + 0.0854427B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413{\cdot}10^{-07}X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,3} - 0.0286693B_{1,3} + 0.02841B_{2,3} + 0.0854427B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 1.2032{\cdot}10^{-14}X^4 - 4.2413{\cdot}10^{-07}X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4} - 0.0429507B_{1,4} - 0.000129666B_{2,4} + 0.0426682B_{3,4} + 0.0854427B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45913{\cdot}10^{-11}$.
Bounding polynomials M and m:

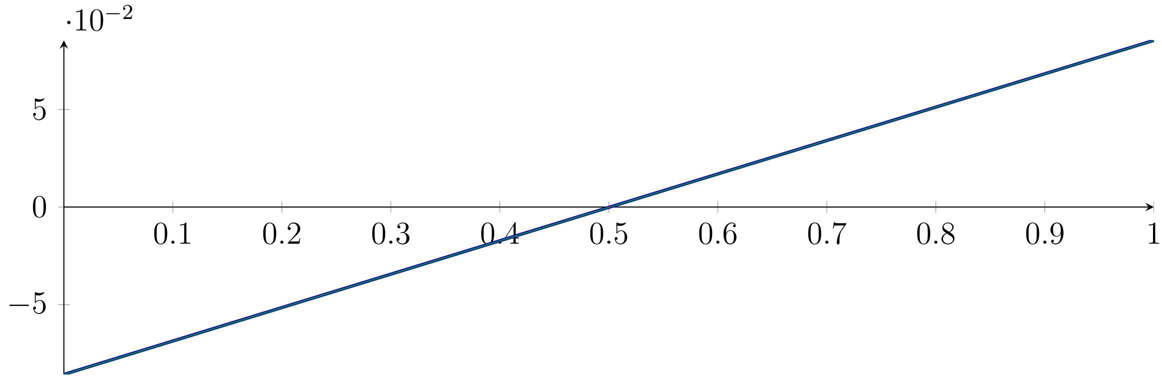
$$M = -4.2413{\cdot}10^{-07}X^3 - 0.000138529X^2 + 0.171376X - 0.0857948$$

$$m = -4.2413{\cdot}10^{-07}X^3 - 0.000138529X^2 + 0.171376X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



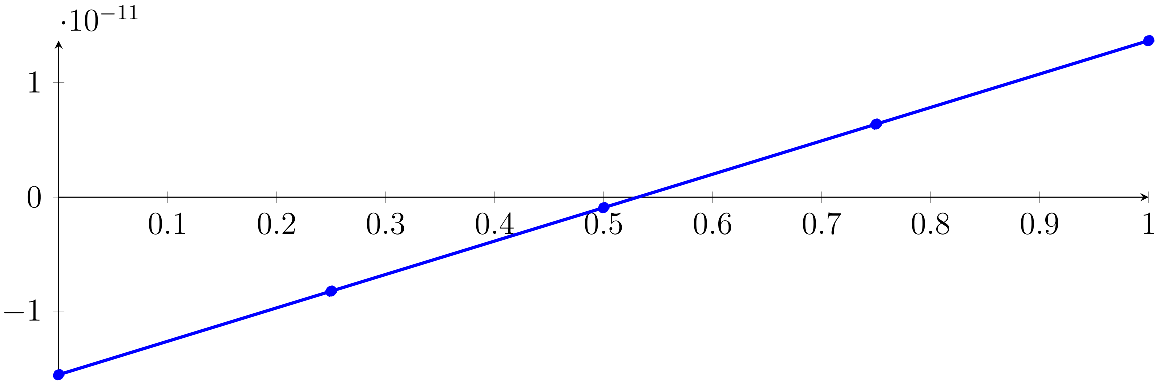
$$[0.500825, 0.500825]$$

Longest intersection interval: $1.70047{\cdot}10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

36.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

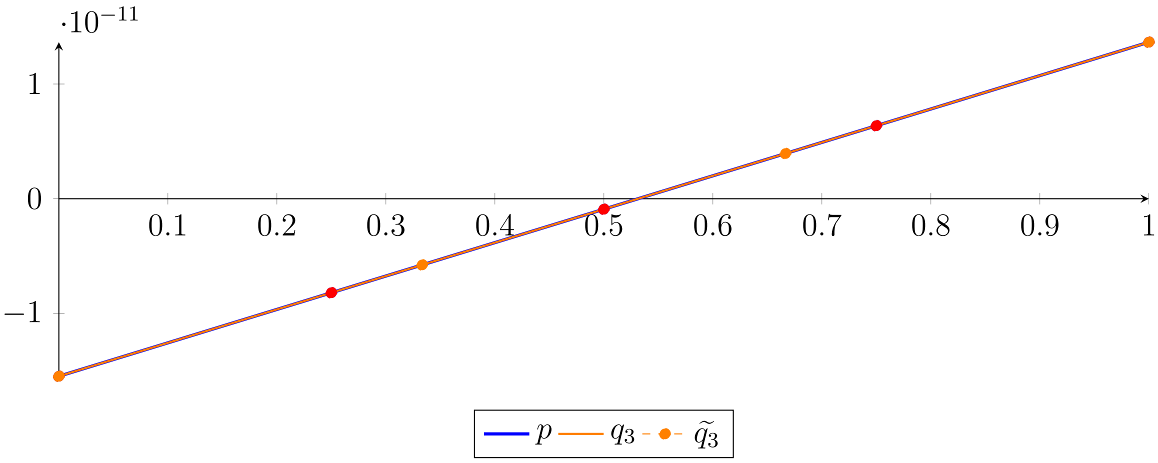
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.01312{\cdot}10^{-24}X^2 + 2.91184{\cdot}10^{-11}X - 1.5473{\cdot}10^{-11} \\ &= -1.5473{\cdot}10^{-11}B_{0,4}(X) - 8.19335{\cdot}10^{-12}B_{1,4}(X) - 9.13745 \\ &\quad {\cdot}10^{-13}B_{2,4}(X) + 6.36586{\cdot}10^{-12}B_{3,4}(X) + 1.36455{\cdot}10^{-11}B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -6.78547\cdot 10^{-25}X^3 - 2.95652\cdot 10^{-24}X^2 + 2.91184\cdot 10^{-11}X - 1.5473\cdot 10^{-11}$$
$$= -1.5473\cdot 10^{-11}B_{0,3} - 5.76681\cdot 10^{-12}B_{1,3} + 3.93932\cdot 10^{-12}B_{2,3} + 1.36455\cdot 10^{-11}B_{3,3}$$
$$\widetilde{q}_3 = 2.83697\cdot 10^{-24}X^4 - 6.85009\cdot 10^{-24}X^3 + 1.39587\cdot 10^{-24}X^2 + 2.91184\cdot 10^{-11}X - 1.5473\cdot 10^{-11}$$
$$= -1.5473\cdot 10^{-11}B_{0,4} - 8.19335\cdot 10^{-12}B_{1,4} - 9.13745\cdot 10^{-13}B_{2,4} + 6.36586\cdot 10^{-12}B_{3,4} + 1.36455\cdot 10^{-11}B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.84343\cdot 10^{-25}$.

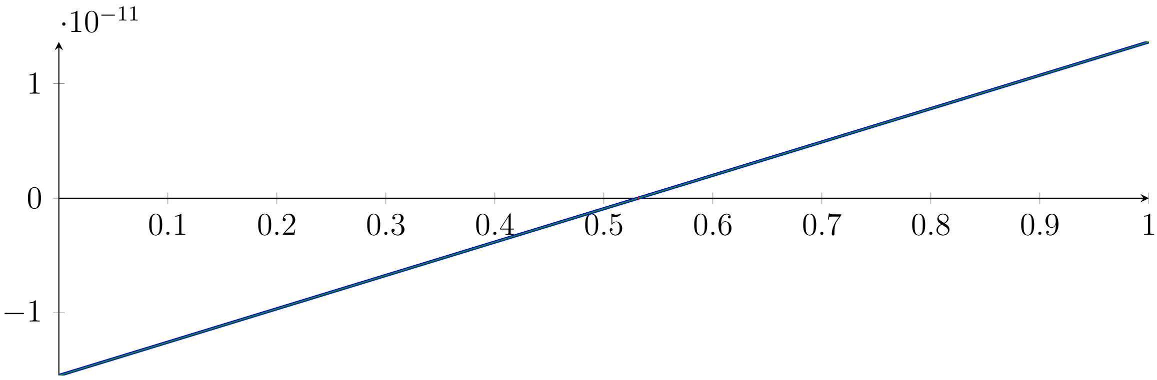
Bounding polynomials M and m:

$$M = -6.78547\cdot 10^{-25}X^3 - 2.95652\cdot 10^{-24}X^2 + 2.91184\cdot 10^{-11}X - 1.5473\cdot 10^{-11}$$
$$m = -6.78547\cdot 10^{-25}X^3 - 2.95652\cdot 10^{-24}X^2 + 2.91184\cdot 10^{-11}X - 1.5473\cdot 10^{-11}$$

Root of M and m:

$$N(M) = \{-6.51982\cdot 10^{6}, 0.53138, 6.51982\cdot 10^{6}\} \quad N(m) = \{-6.51982\cdot 10^{6}, 0.53138, 6.51982\cdot 10^{6}\}$$

Intersection intervals:



$$[0.53138, 0.53138]$$

Longest intersection interval: 0

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

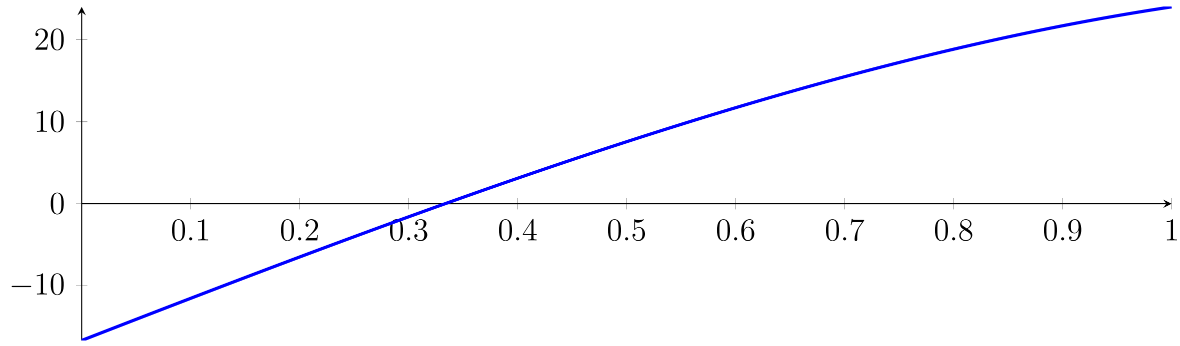
36.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

36.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

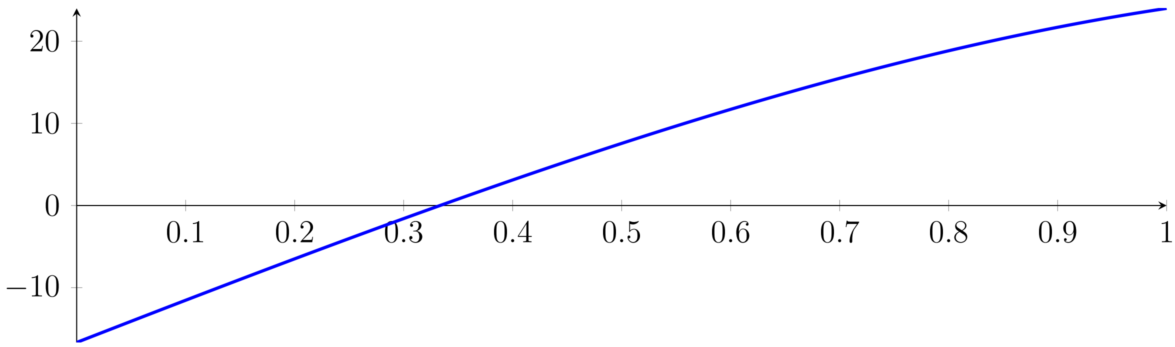
with precision $\varepsilon = 1 \cdot 10^{-32}$.

37 Running `BezClip` on f_4 with epsilon 64

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

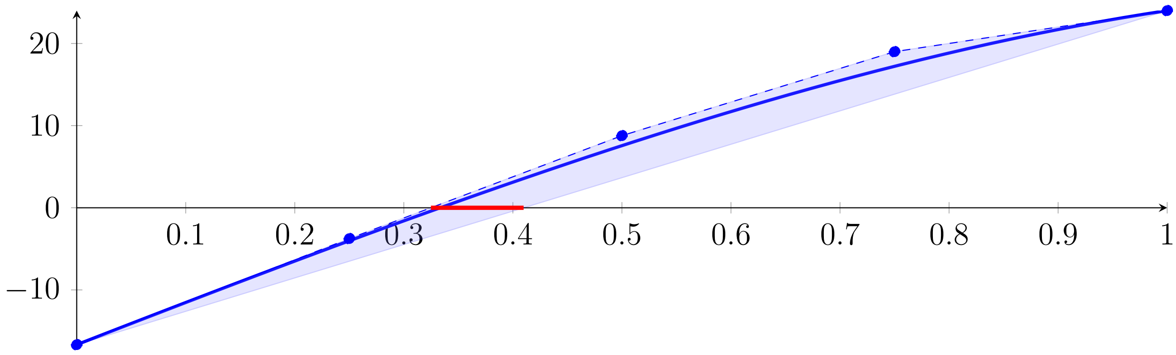
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



37.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

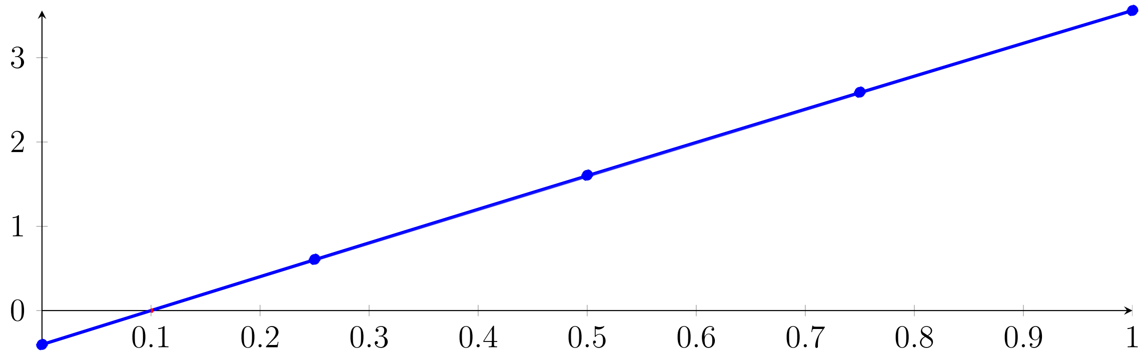
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

37.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

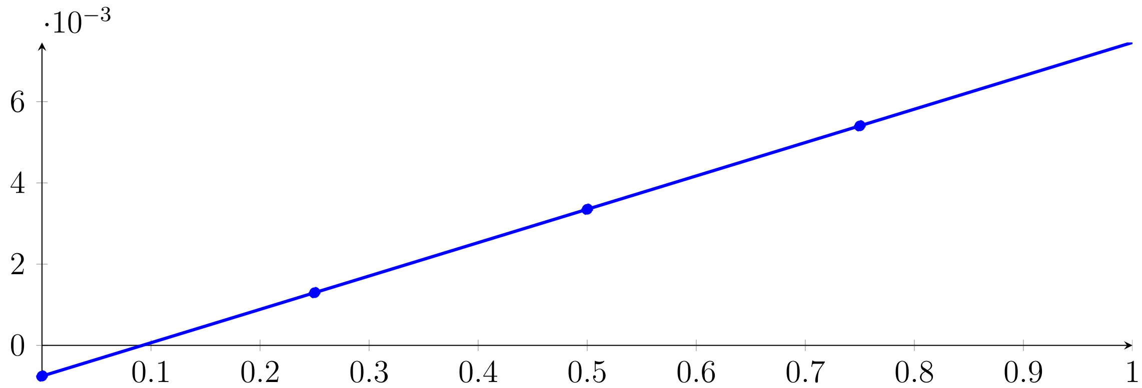
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

37.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.06393 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

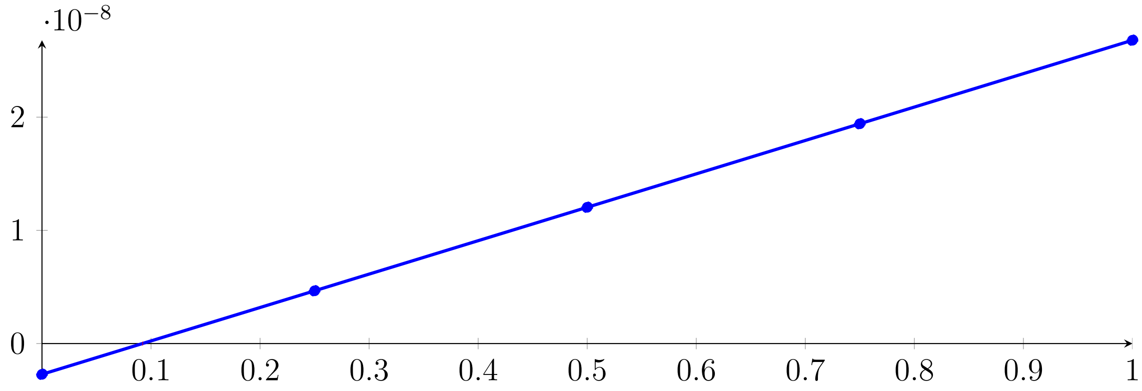
$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

37.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -9.92617 \cdot 10^{-24} X^4 + 6.61744 \cdot 10^{-24} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09}$$
$$= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28974 \cdot 10^{-11}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

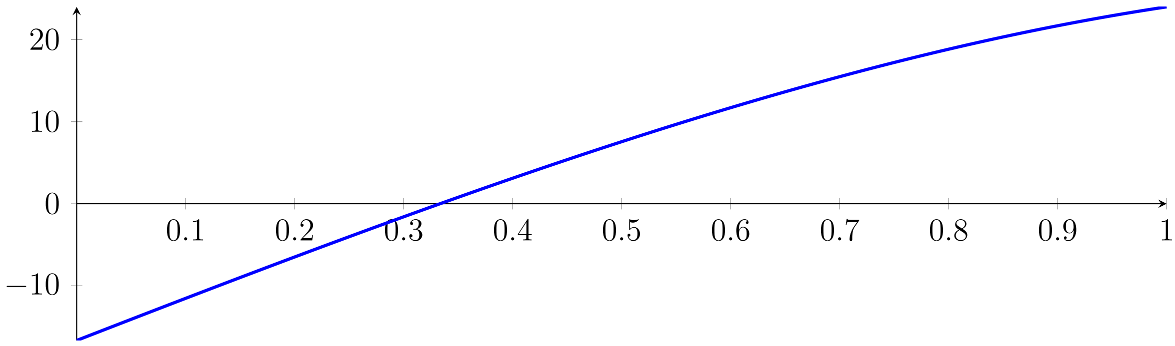
37.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

37.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

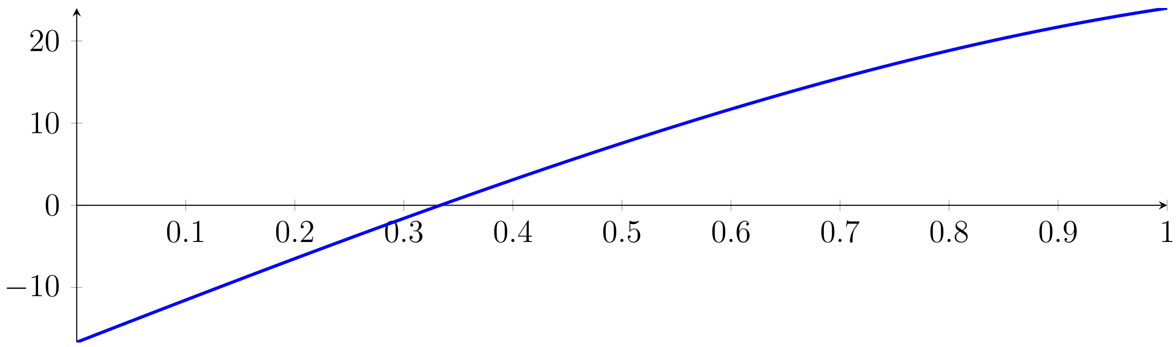
with precision $\varepsilon = 1 \cdot 10^{-64}$.

38 Running QuadClip on f_4 with epsilon 64

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

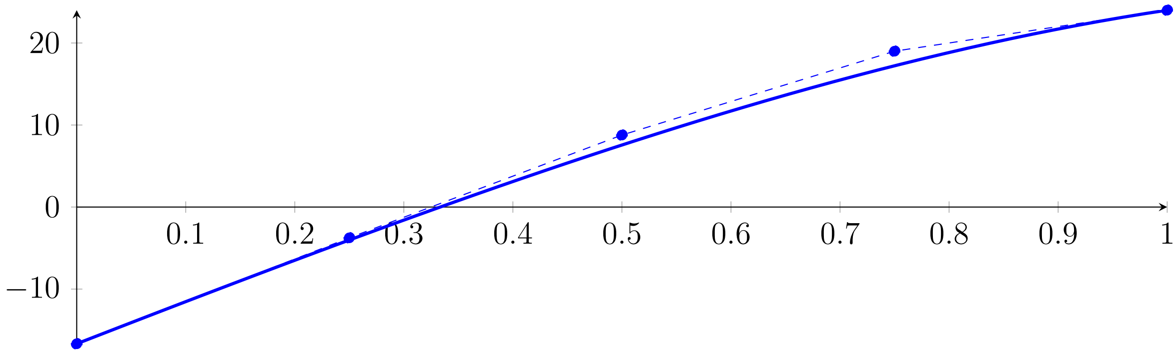
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



38.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

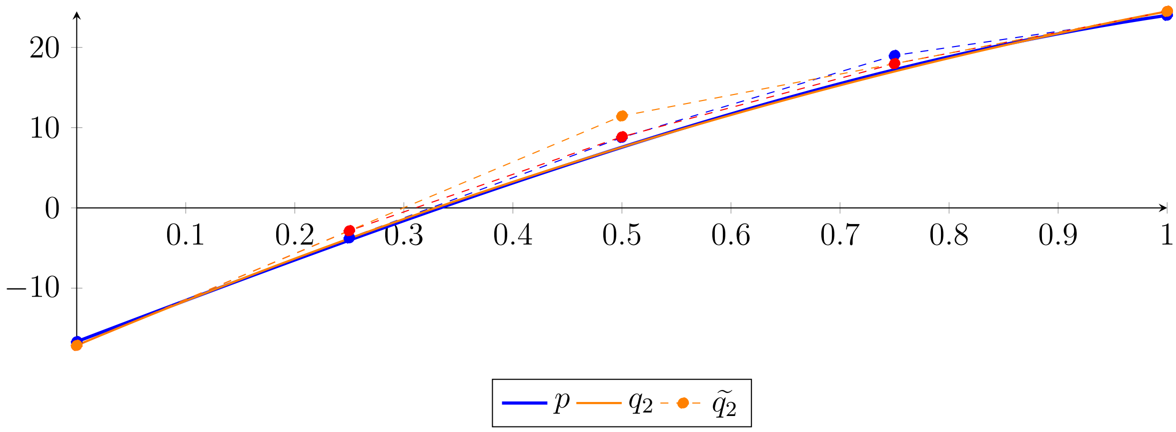
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -2.67519 \cdot 10^{-12}X^4 + 5.32907 \cdot 10^{-12}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

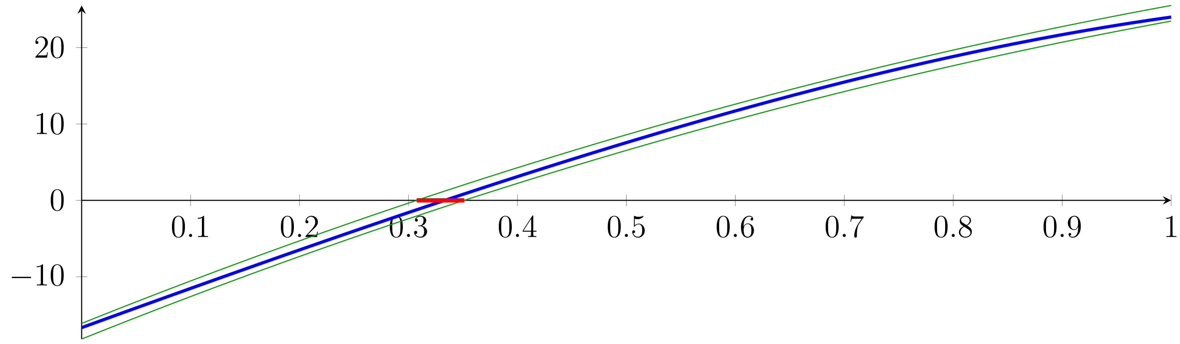
$$M = -15.5476X^2 + 57.181X - 16.1119$$

$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



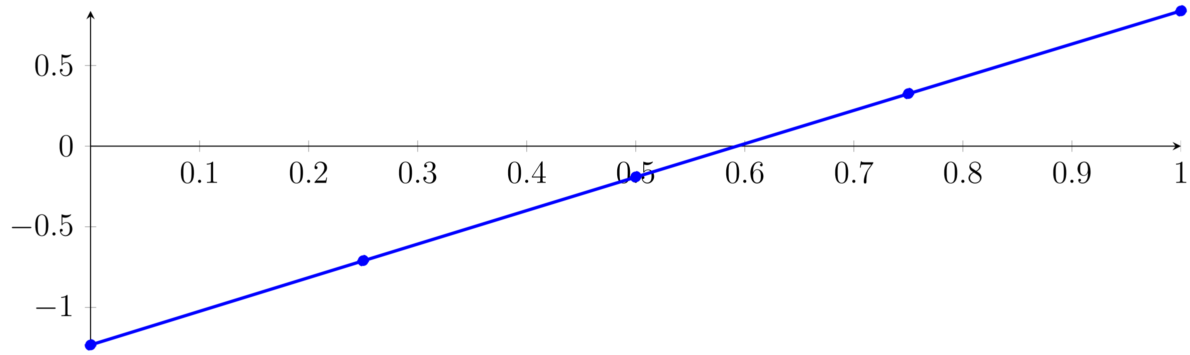
$[0.307477, 0.351097]$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

38.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

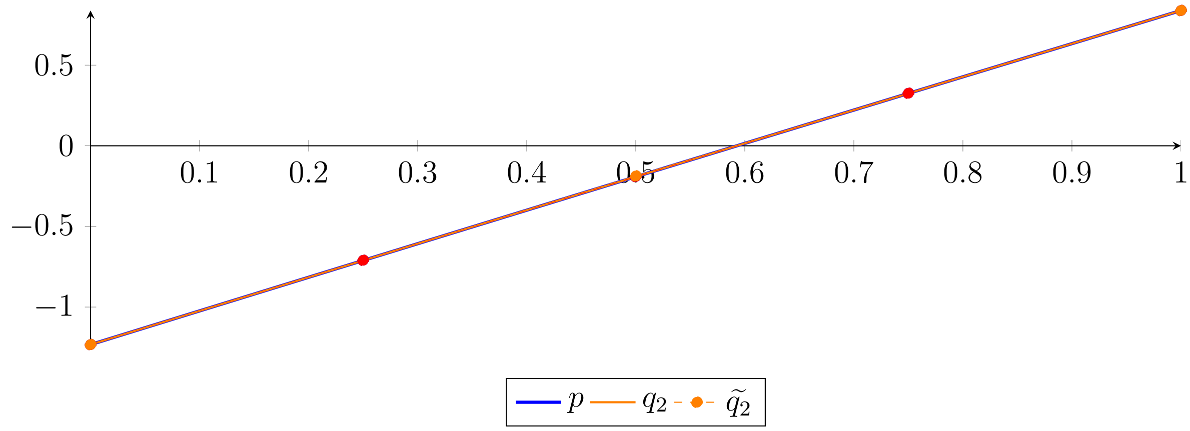
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 2.43583 \cdot 10^{-13} X^4 - 4.3876 \cdot 10^{-13} X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



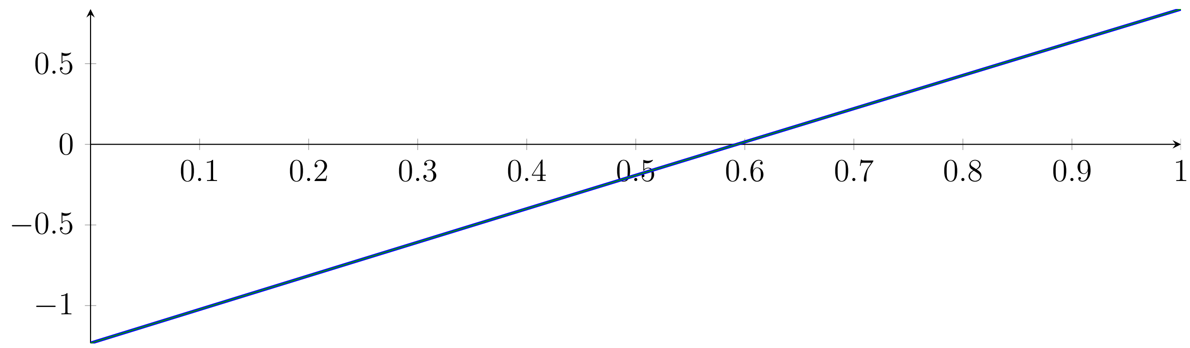
The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.
Bounding polynomials M and m:

$$M = -0.020089X^2 + 2.09166X - 1.23274$$
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



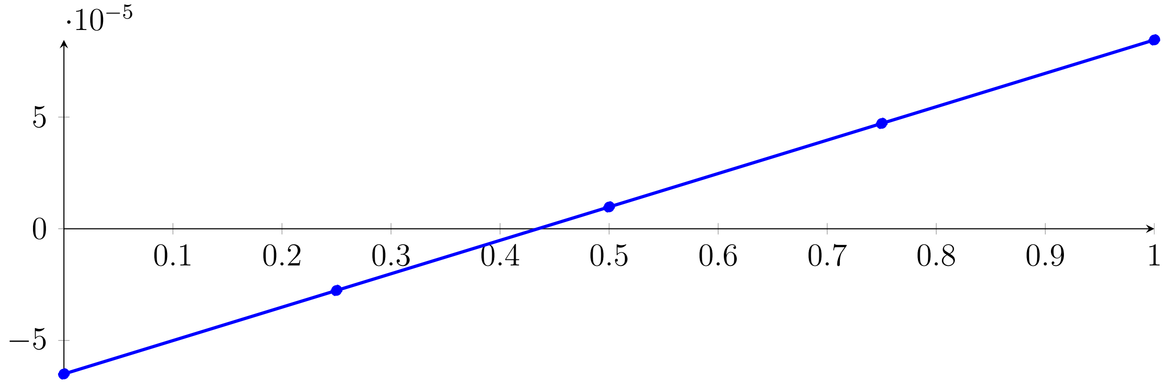
$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

38.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

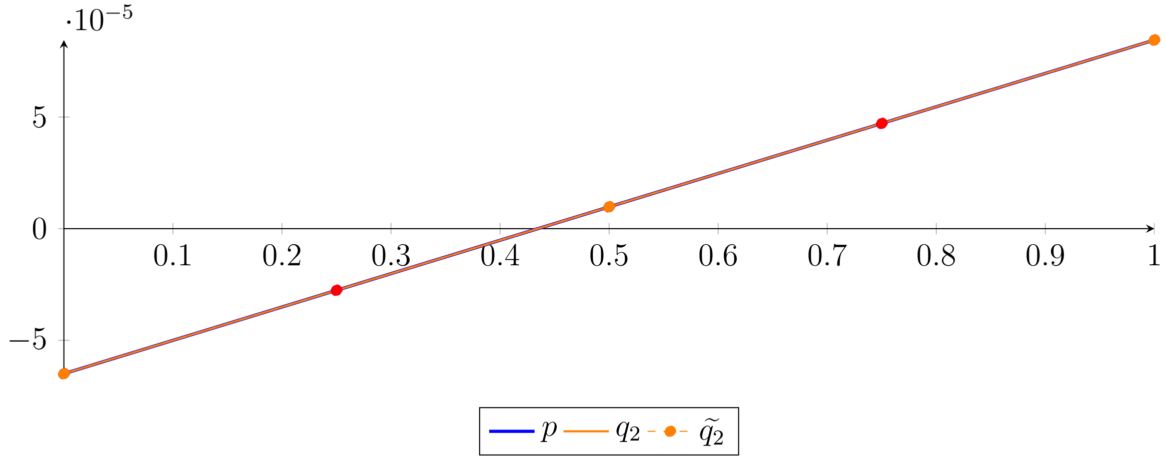
$$p = -2.71051 \cdot 10^{-20} X^4 - 2.82489 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777$$
$$\cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.06147\cdot10^{-10}X^2 + 0.000149549X - 6.50069\cdot10^{-05}$$
$$= -6.50069\cdot10^{-05}B_{0,2} + 9.76779\cdot10^{-06}B_{1,2} + 8.45424\cdot10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = 6.72205\cdot10^{-18}X^4 - 1.21431\cdot10^{-17}X^3 - 1.06147\cdot10^{-10}X^2 + 0.000149549X - 6.50069\cdot10^{-05}$$
$$= -6.50069\cdot10^{-05}B_{0,4} - 2.76196\cdot10^{-05}B_{1,4} + 9.76777\cdot10^{-06}B_{2,4} + 4.71551\cdot10^{-05}B_{3,4} + 8.45424\cdot10^{-05}B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.88601\cdot10^{-17}$.

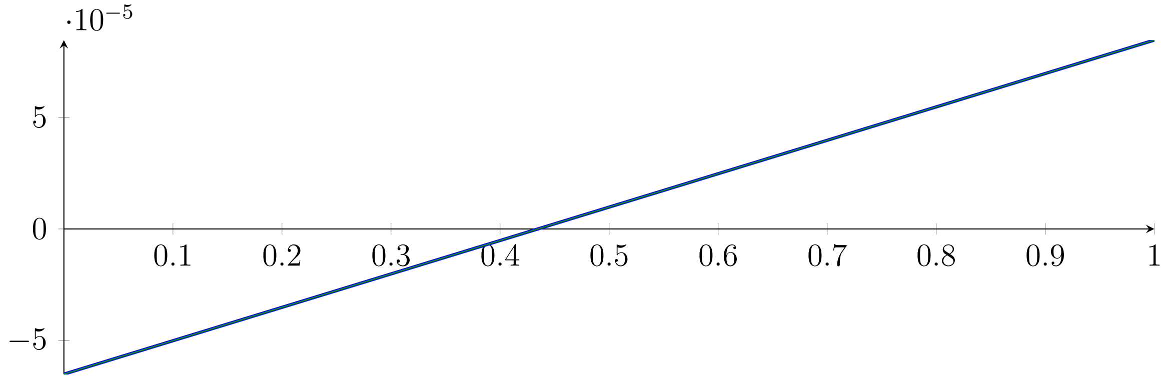
Bounding polynomials M and m:

$$M = -1.06147\cdot10^{-10}X^2 + 0.000149549X - 6.50069\cdot10^{-05}$$
$$m = -1.06147\cdot10^{-10}X^2 + 0.000149549X - 6.50069\cdot10^{-05}$$

Root of M and m:

$$N(M) = \{0.434685, 1.4089\cdot10^{6}\} \qquad N(m) = \{0.434685, 1.4089\cdot10^{6}\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

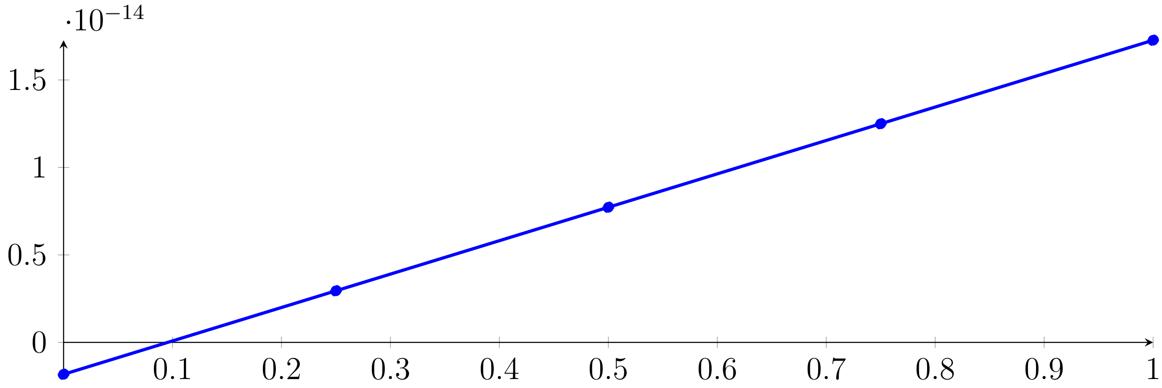
Longest intersection interval: $1.27678\cdot10^{-10}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

38.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

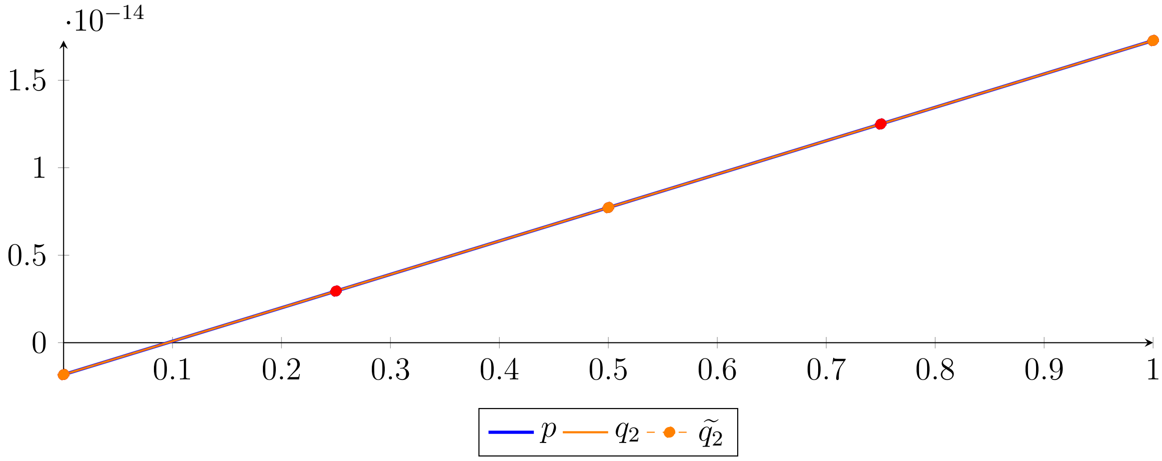
$$p = -1.41995\cdot10^{-29}X^4 + 6.31089\cdot10^{-30}X^3 + 4.73317\cdot10^{-30}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$
$$= -1.8241\cdot10^{-15}B_{0,4}(X) + 2.94943\cdot10^{-15}B_{1,4}(X) + 7.72295\cdot10^{-15}B_{2,4}(X) + 1.24965\cdot10^{-14}B_{3,4}(X) + 1.727\cdot10^{-14}B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -2.68213\cdot10^{-29}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$
$$= -1.8241\cdot10^{-15}B_{0,2} + 7.72295\cdot10^{-15}B_{1,2} + 1.727\cdot10^{-14}B_{2,2}$$

$$\widetilde{q}_2 = -2.18041\cdot10^{-27}X^4 + 3.9443\cdot10^{-27}X^3 - 2.24352\cdot10^{-27}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$
$$= -1.8241\cdot10^{-15}B_{0,4} + 2.94943\cdot10^{-15}B_{1,4} + 7.72295\cdot10^{-15}B_{2,4} + 1.24965\cdot10^{-14}B_{3,4} + 1.727\cdot10^{-14}B_{4,4}$$



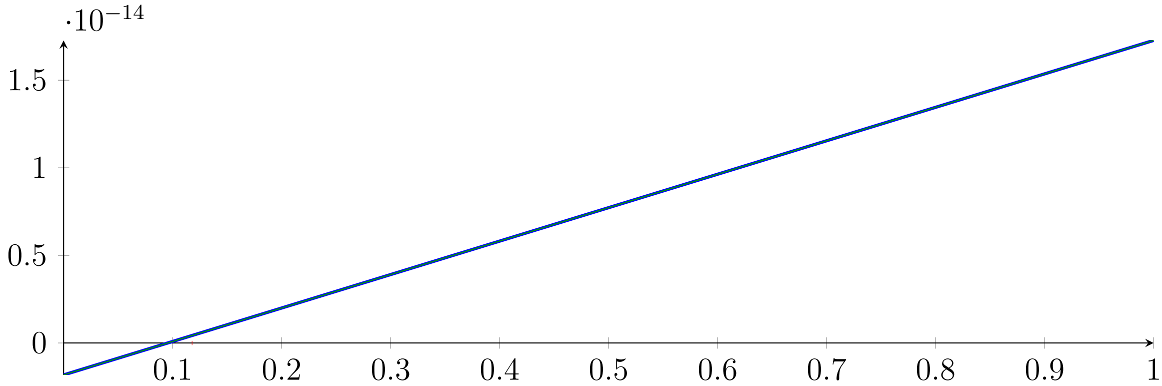
The maximum difference of the Bézier coefficients is $\delta = 1.73549\cdot10^{-28}$.
Bounding polynomials M and m:

$$M = -2.68213\cdot10^{-29}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$
$$m = -2.68213\cdot10^{-29}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$

Root of M and m:

$$N(M) = \{0.117647, 7.11901\cdot10^{14}\} \qquad N(m) = \{0.117647, 7.11901\cdot10^{14}\}$$

Intersection intervals:



$[0.117647, 0.117647]$

Longest intersection interval: 0
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

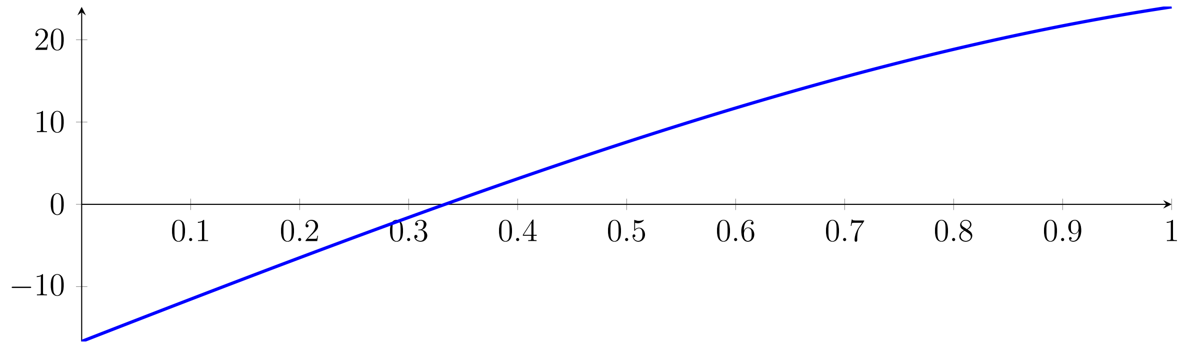
38.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

38.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

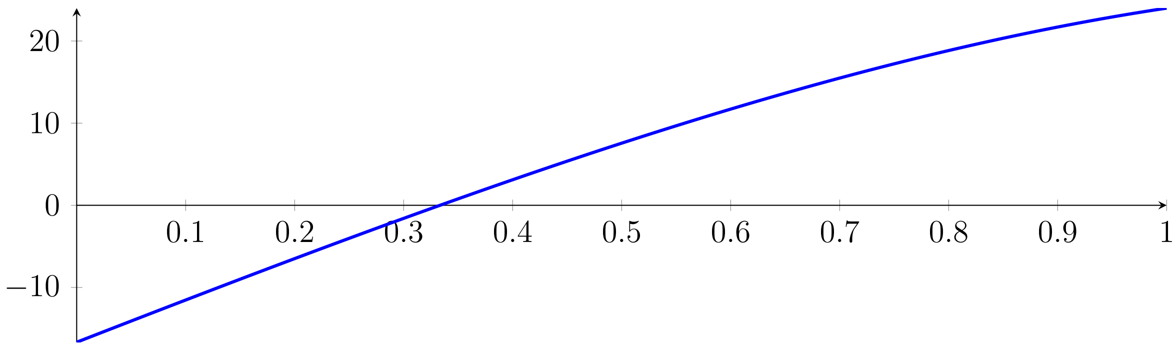
with precision $\varepsilon = 1{\cdot}10^{-64}$.

39 Running CubeClip on f_4 with epsilon 64

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0,1]$:

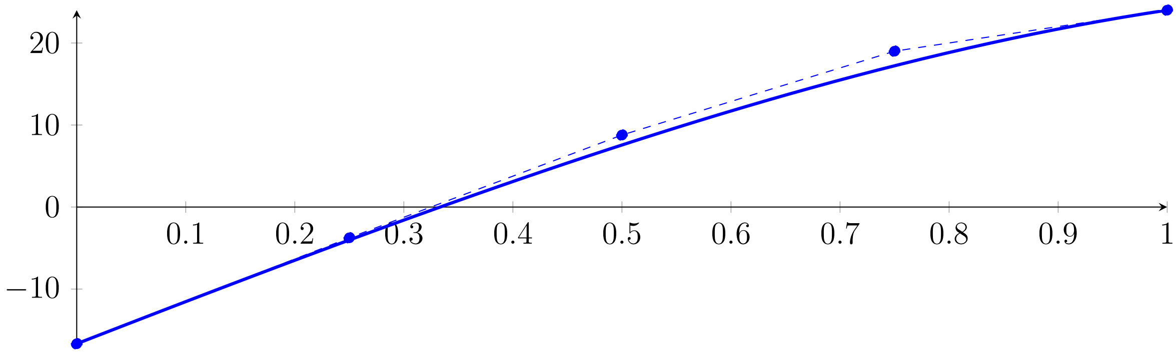
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



39.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

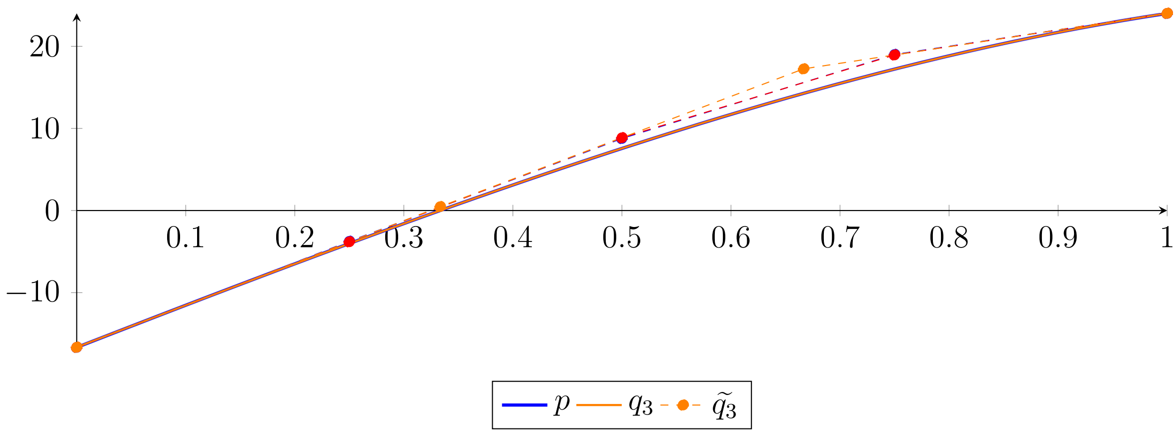
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -2.28084 \cdot 10^{-12}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



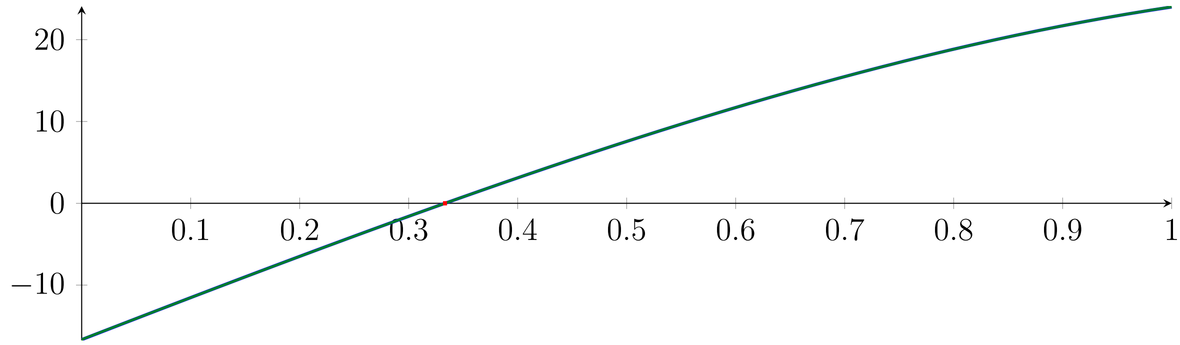
The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



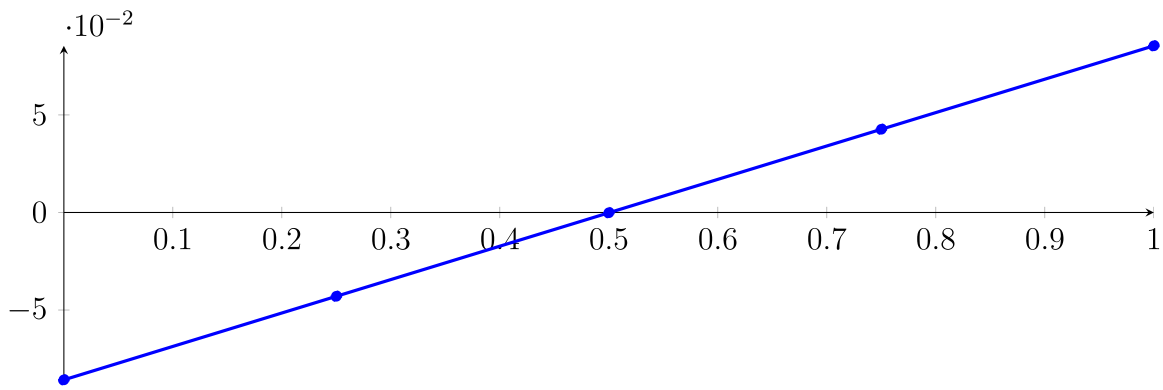
$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

39.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

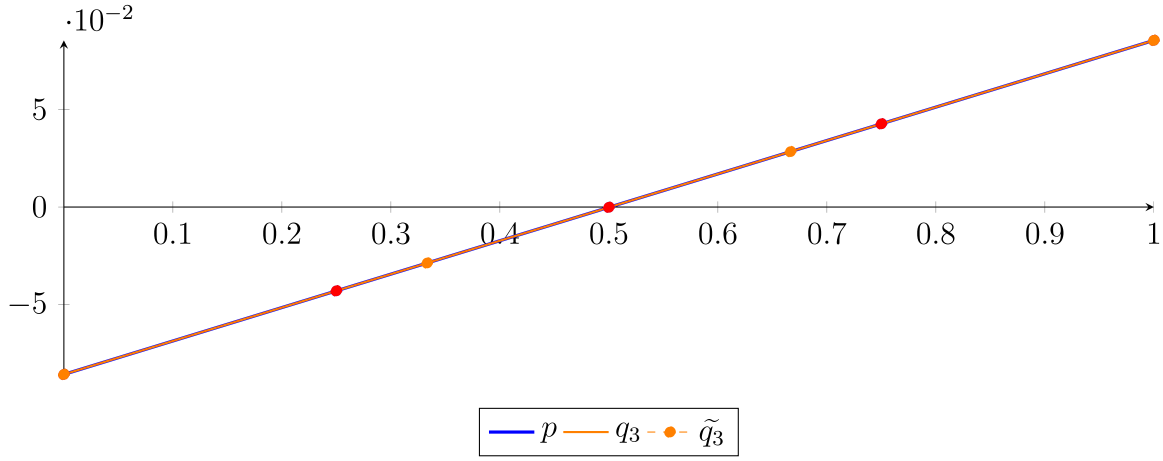
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4}(X) - 0.0429507B_{1,4}(X) - 0.000129666B_{2,4}(X) \\ &\quad + 0.0426682B_{3,4}(X) + 0.0854427B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,3} - 0.0286693B_{1,3} + 0.02841B_{2,3} + 0.0854427B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 1.2032 \cdot 10^{-14} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4} - 0.0429507B_{1,4} - 0.000129666B_{2,4} + 0.0426682B_{3,4} + 0.0854427B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45913{\cdot}10^{-11}$.

Bounding polynomials M and m:

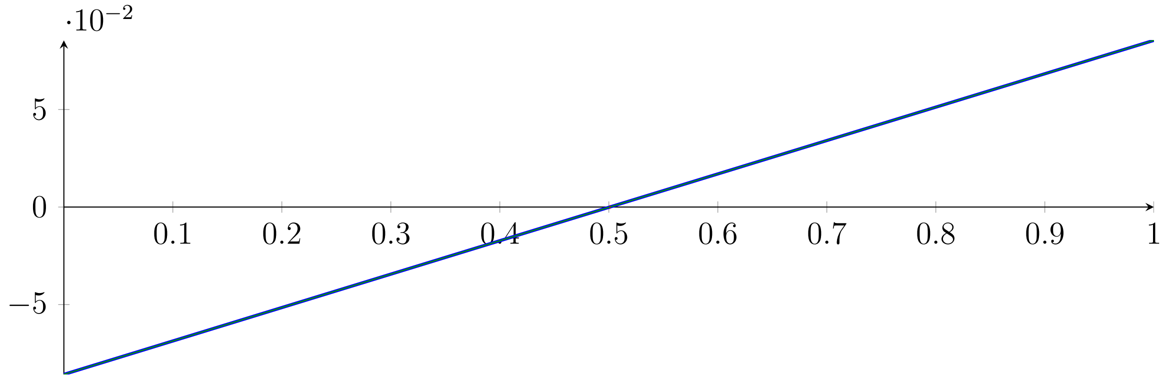
$$M = -4.2413{\cdot}10^{-07}X^3 - 0.000138529X^2 + 0.171376X - 0.0857948$$

$$m = -4.2413{\cdot}10^{-07}X^3 - 0.000138529X^2 + 0.171376X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

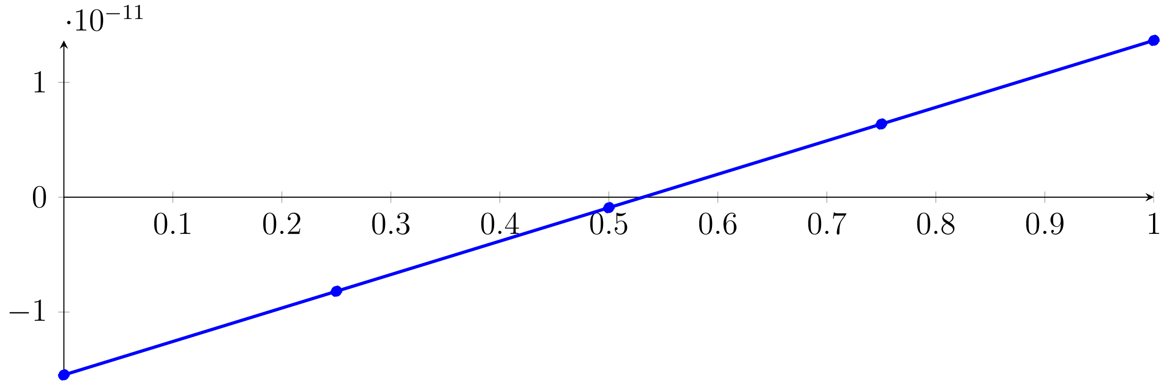
Longest intersection interval: $1.70047{\cdot}10^{-10}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

39.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

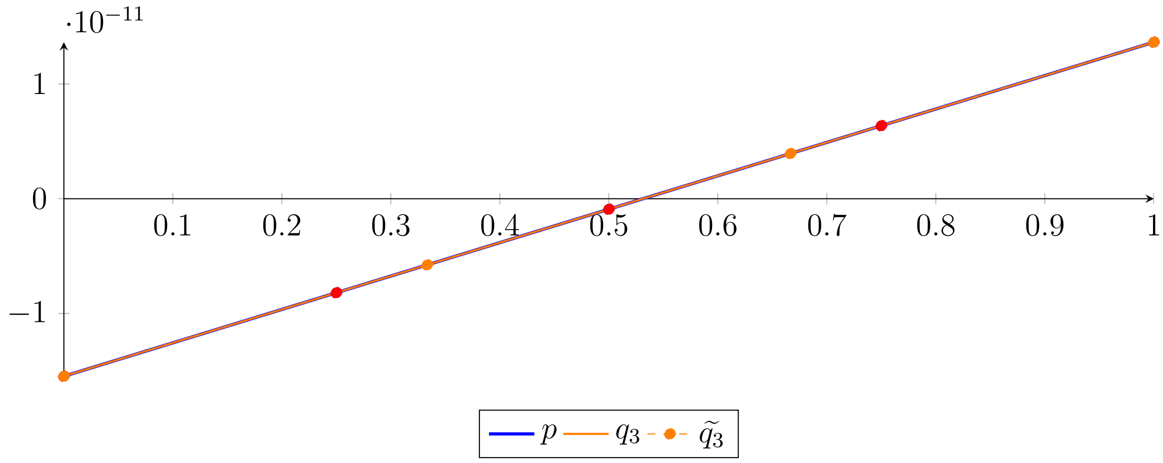
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.01312{\cdot}10^{-24}X^2 + 2.91184{\cdot}10^{-11}X - 1.5473{\cdot}10^{-11} \\ &= -1.5473{\cdot}10^{-11}B_{0,4}(X) - 8.19335{\cdot}10^{-12}B_{1,4}(X) - 9.13745 \\ &\quad {\cdot}10^{-13}B_{2,4}(X) + 6.36586{\cdot}10^{-12}B_{3,4}(X) + 1.36455{\cdot}10^{-11}B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -6.78547\cdot10^{-25}X^3 - 2.95652\cdot10^{-24}X^2 + 2.91184\cdot10^{-11}X - 1.5473\cdot10^{-11}$$
$$= -1.5473\cdot10^{-11}B_{0,3} - 5.76681\cdot10^{-12}B_{1,3} + 3.93932\cdot10^{-12}B_{2,3} + 1.36455\cdot10^{-11}B_{3,3}$$
$$\widetilde{q}_3 = 2.83697\cdot10^{-24}X^4 - 6.85009\cdot10^{-24}X^3 + 1.39587\cdot10^{-24}X^2 + 2.91184\cdot10^{-11}X - 1.5473\cdot10^{-11}$$
$$= -1.5473\cdot10^{-11}B_{0,4} - 8.19335\cdot10^{-12}B_{1,4} - 9.13745\cdot10^{-13}B_{2,4} + 6.36586\cdot10^{-12}B_{3,4} + 1.36455\cdot10^{-11}B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.84343\cdot10^{-25}$.

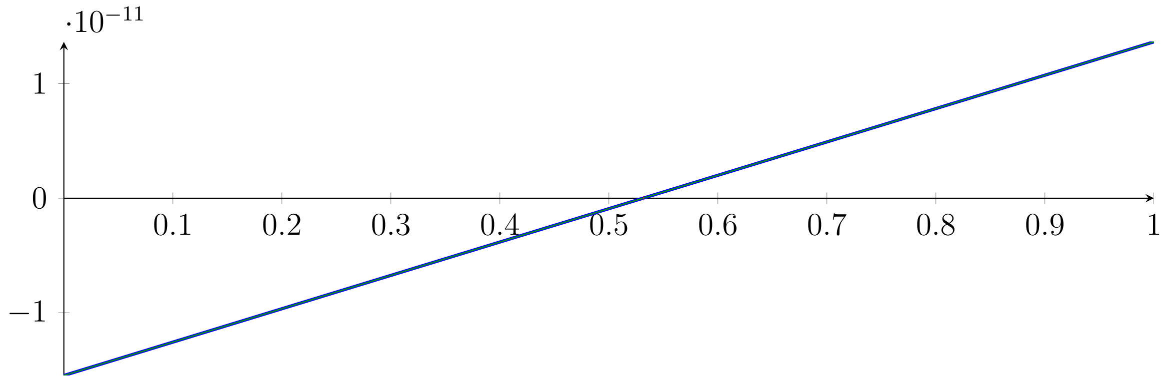
Bounding polynomials M and m:

$$M = -6.78547\cdot10^{-25}X^3 - 2.95652\cdot10^{-24}X^2 + 2.91184\cdot10^{-11}X - 1.5473\cdot10^{-11}$$
$$m = -6.78547\cdot10^{-25}X^3 - 2.95652\cdot10^{-24}X^2 + 2.91184\cdot10^{-11}X - 1.5473\cdot10^{-11}$$

Root of M and m:

$$N(M) = \{-6.51982\cdot10^6, 0.53138, 6.51982\cdot10^6\} \quad N(m) = \{-6.51982\cdot10^6, 0.53138, 6.51982\cdot10^6\}$$

Intersection intervals:



$[0.53138, 0.53138]$

Longest intersection interval: 0

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

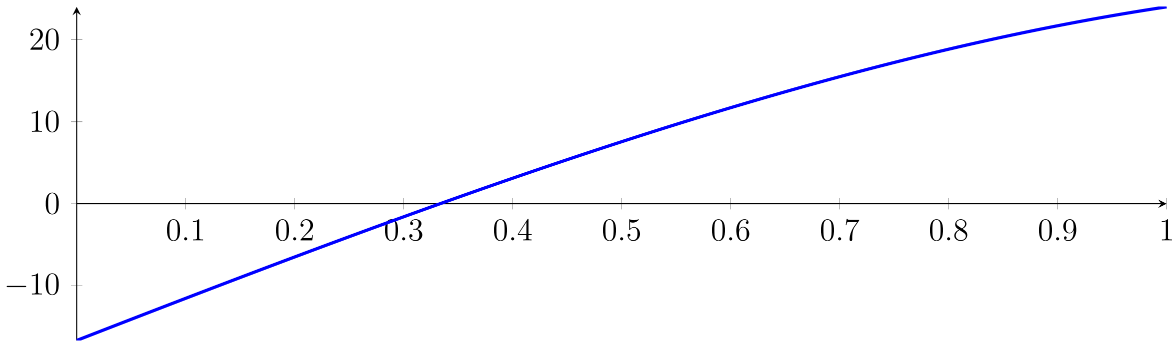
39.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

39.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

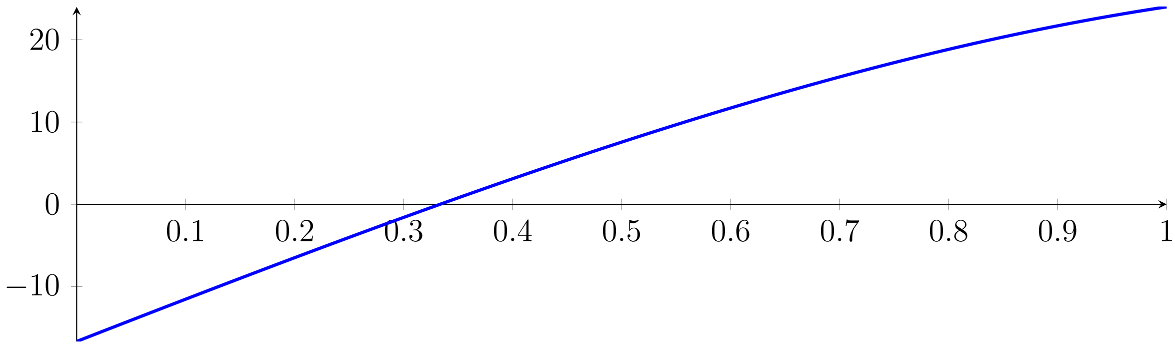
with precision $\varepsilon = 1 \cdot 10^{-64}$.

40 Running BezClip on f_4 with epsilon 128

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

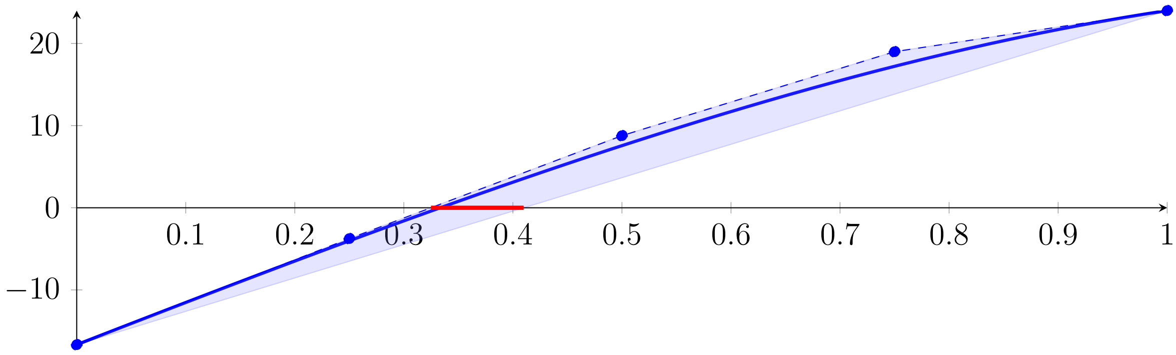
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



40.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

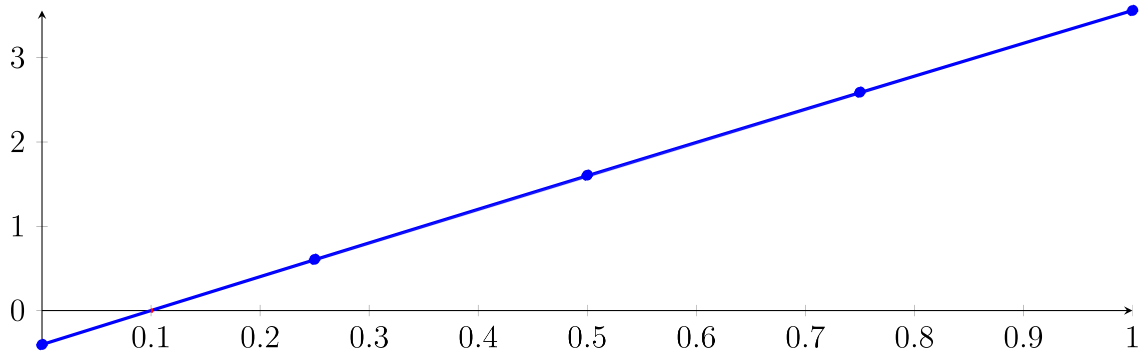
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

40.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

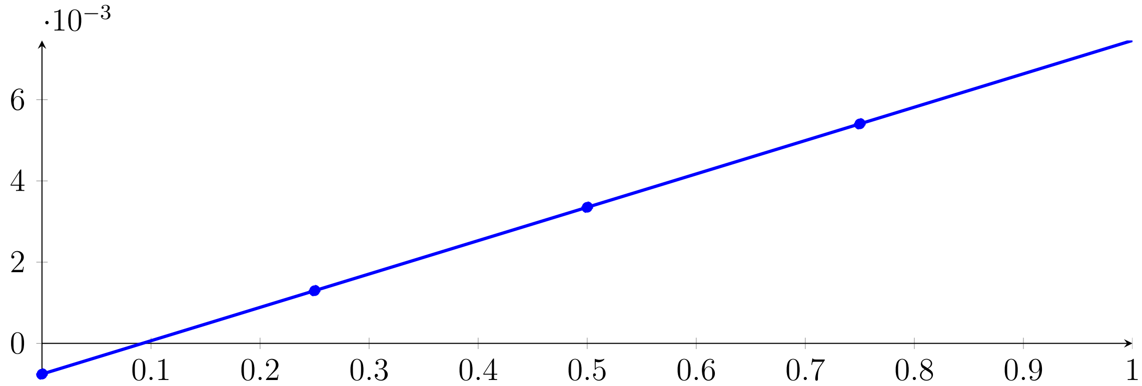
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

40.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.06393 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

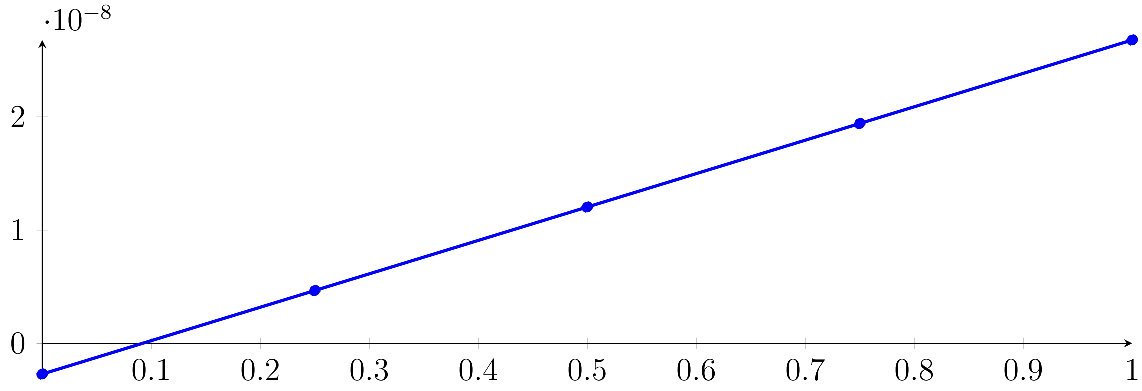
$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

40.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -9.92617 \cdot 10^{-24} X^4 + 6.61744 \cdot 10^{-24} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09}$$
$$= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28974 \cdot 10^{-11}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

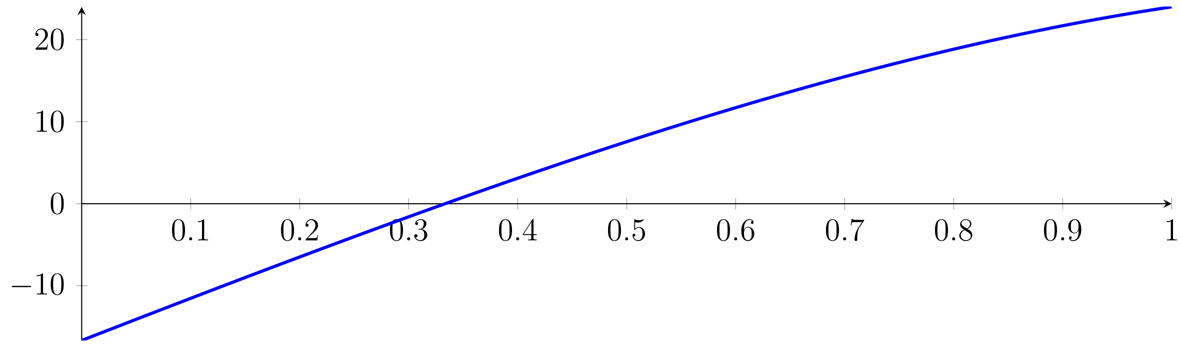
40.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

40.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

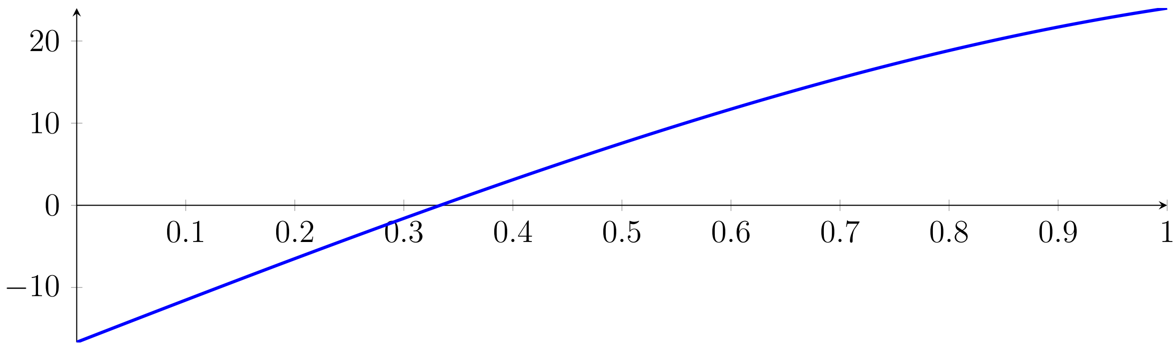
with precision $\varepsilon = 1 \cdot 10^{-128}$.

41 Running QuadClip on f_4 with epsilon 128

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

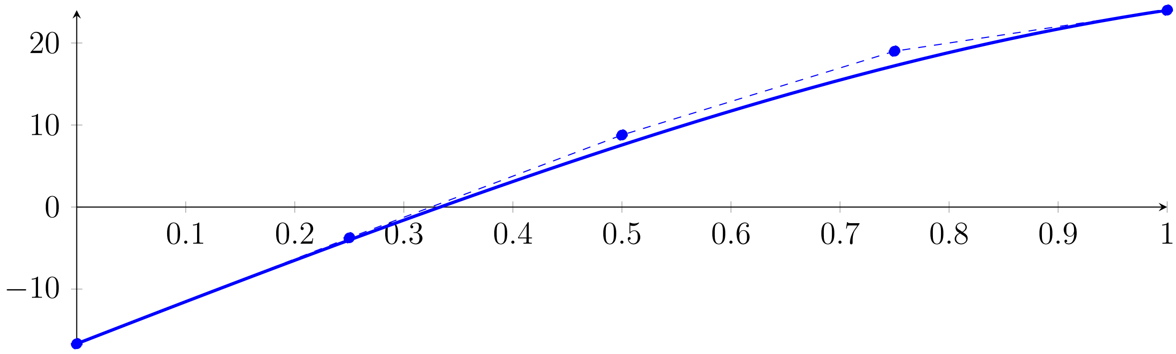
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



41.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

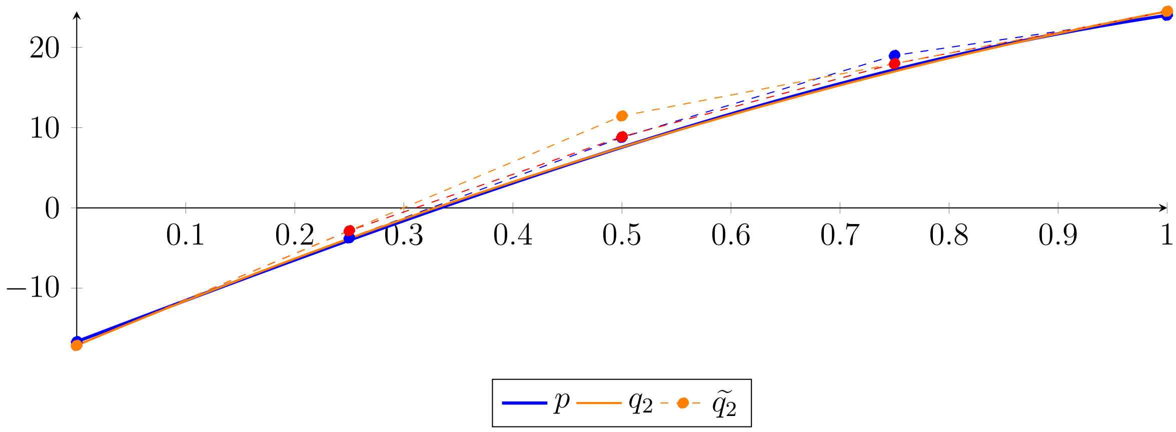
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$
$$= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -15.5476X^2 + 57.181X - 17.1357$$
$$= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2}$$

$$\widetilde{q_2} = -2.67519 \cdot 10^{-12}X^4 + 5.32907 \cdot 10^{-12}X^3 - 15.5476X^2 + 57.181X - 17.1357$$
$$= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4}$$



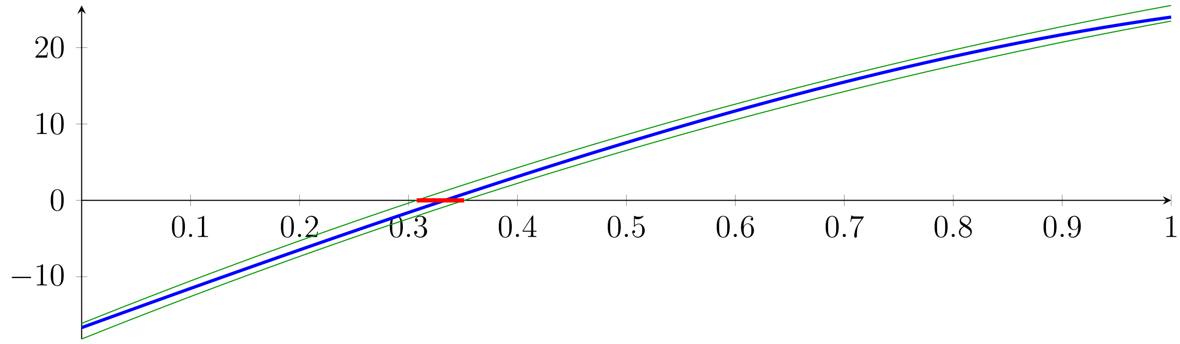
The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

$$M = -15.5476X^2 + 57.181X - 16.1119$$
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



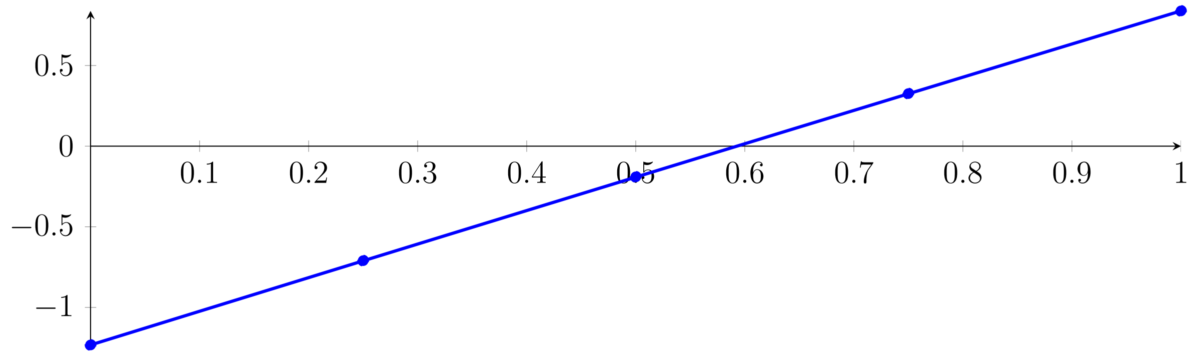
$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

41.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

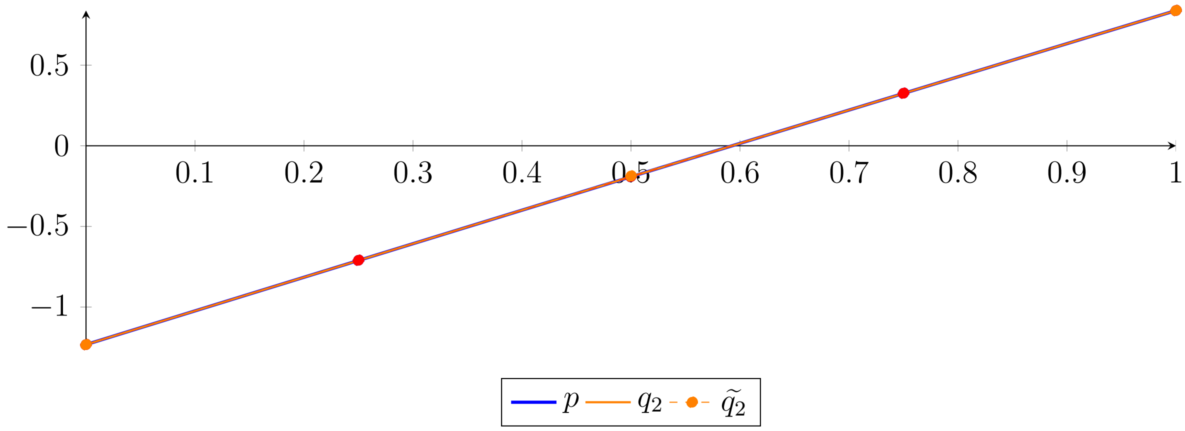
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 2.43583 \cdot 10^{-13} X^4 - 4.3876 \cdot 10^{-13} X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



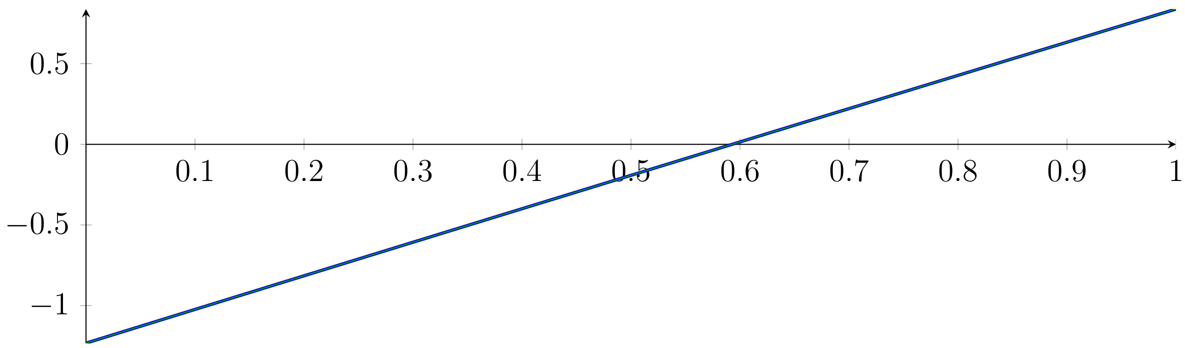
The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.
Bounding polynomials M and m:

$$M = -0.020089X^2 + 2.09166X - 1.23274$$
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



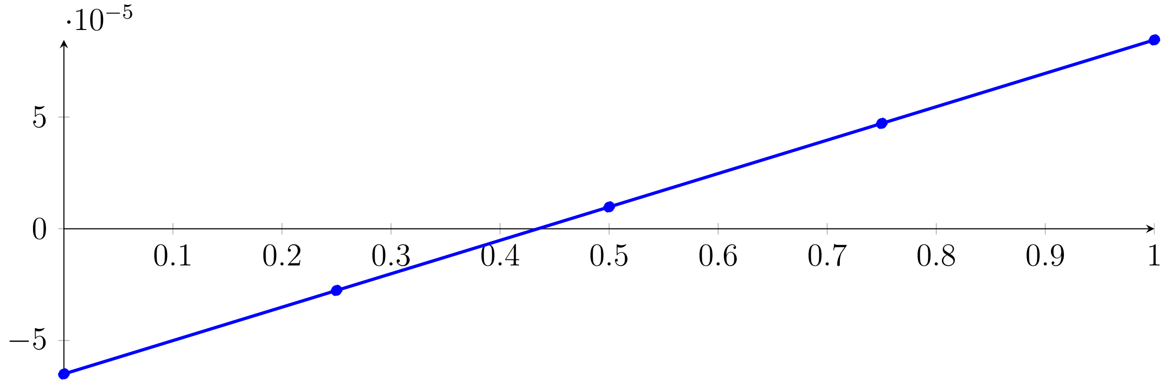
$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

41.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

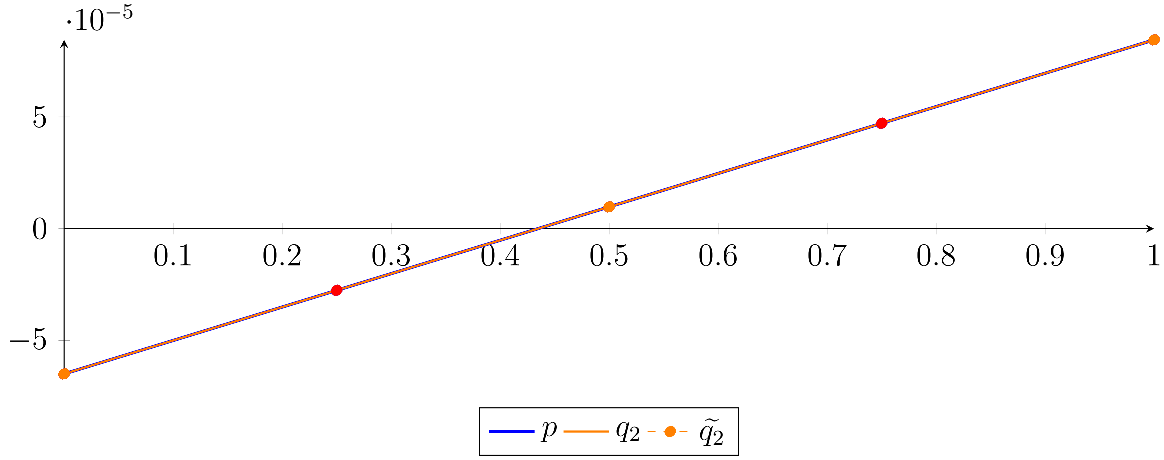
$$p = -2.71051 \cdot 10^{-20} X^4 - 2.82489 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.06147\cdot 10^{-10} X^2 + 0.000149549X - 6.50069\cdot 10^{-05} \\ &= -6.50069\cdot 10^{-05} B_{0,2} + 9.76779\cdot 10^{-06} B_{1,2} + 8.45424\cdot 10^{-05} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 6.72205\cdot 10^{-18} X^4 - 1.21431\cdot 10^{-17} X^3 - 1.06147\cdot 10^{-10} X^2 + 0.000149549X - 6.50069\cdot 10^{-05} \\ &= -6.50069\cdot 10^{-05} B_{0,4} - 2.76196\cdot 10^{-05} B_{1,4} + 9.76777\cdot 10^{-06} B_{2,4} + 4.71551\cdot 10^{-05} B_{3,4} + 8.45424\cdot 10^{-05} B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.88601\cdot 10^{-17}$.

Bounding polynomials M and m:

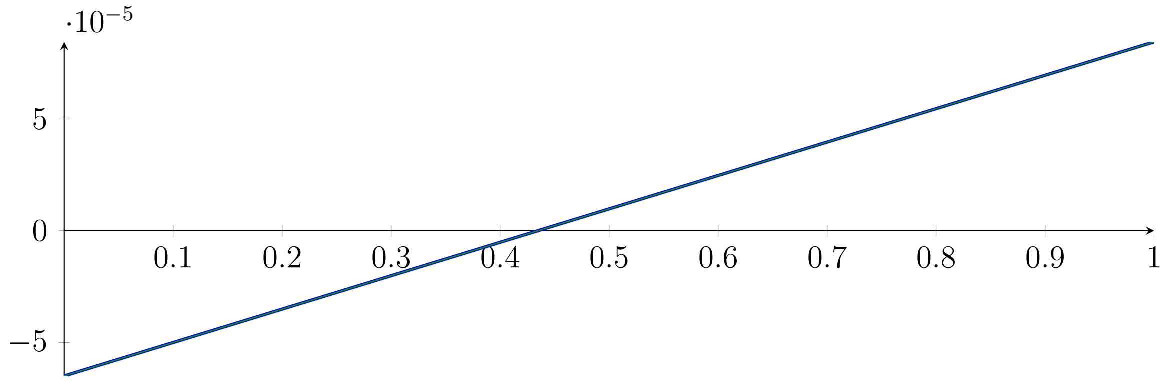
$$M = -1.06147\cdot 10^{-10} X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$

$$m = -1.06147\cdot 10^{-10} X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.434685, 1.4089\cdot 10^{6}\} \qquad N(m) = \{0.434685, 1.4089\cdot 10^{6}\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

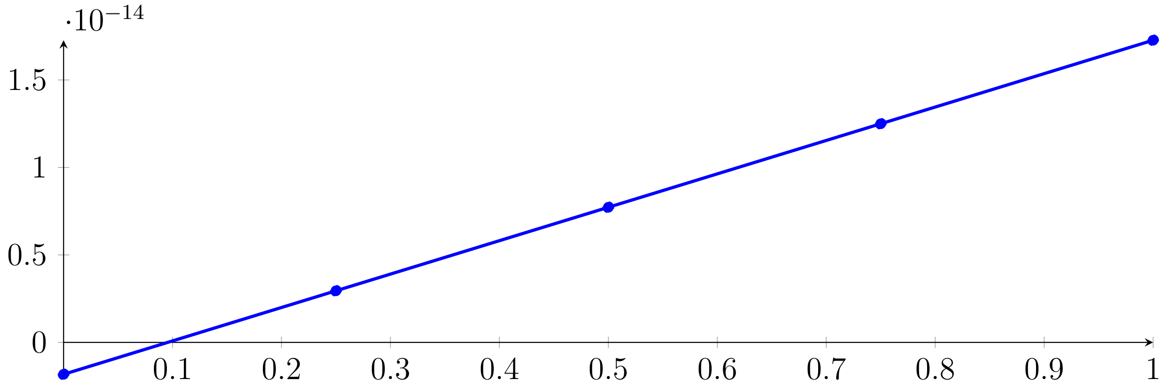
Longest intersection interval: $1.27678\cdot 10^{-10}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

41.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

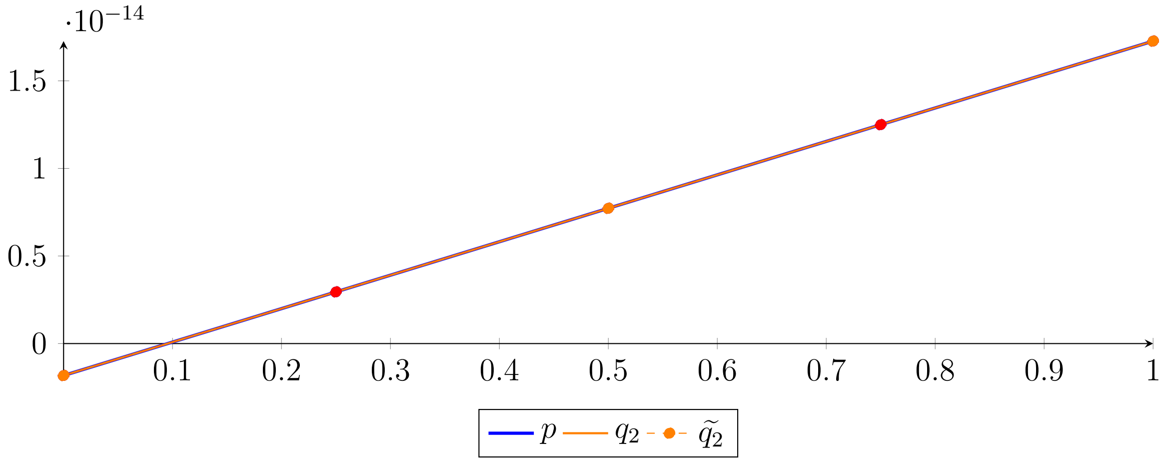
$$p = -1.41995\cdot10^{-29}X^4 + 6.31089\cdot10^{-30}X^3 + 4.73317\cdot10^{-30}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$
$$= -1.8241\cdot10^{-15}B_{0,4}(X) + 2.94943\cdot10^{-15}B_{1,4}(X) + 7.72295$$
$$\cdot10^{-15}B_{2,4}(X) + 1.24965\cdot10^{-14}B_{3,4}(X) + 1.727\cdot10^{-14}B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -2.68213\cdot10^{-29}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$
$$= -1.8241\cdot10^{-15}B_{0,2} + 7.72295\cdot10^{-15}B_{1,2} + 1.727\cdot10^{-14}B_{2,2}$$

$$\widetilde{q}_2 = -2.18041\cdot10^{-27}X^4 + 3.9443\cdot10^{-27}X^3 - 2.24352\cdot10^{-27}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$
$$= -1.8241\cdot10^{-15}B_{0,4} + 2.94943\cdot10^{-15}B_{1,4} + 7.72295\cdot10^{-15}B_{2,4} + 1.24965\cdot10^{-14}B_{3,4} + 1.727\cdot10^{-14}B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 1.73549\cdot10^{-28}$.

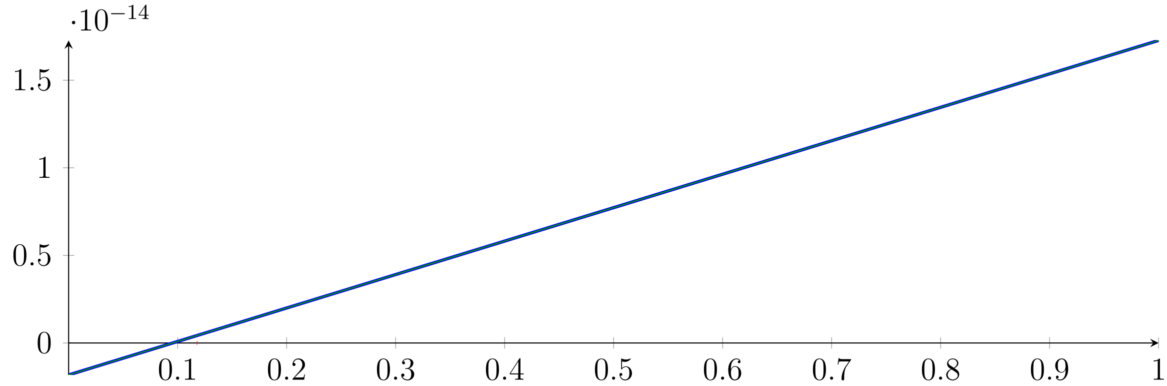
Bounding polynomials M and m:

$$M = -2.68213\cdot10^{-29}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$
$$m = -2.68213\cdot10^{-29}X^2 + 1.90941\cdot10^{-14}X - 1.8241\cdot10^{-15}$$

Root of M and m:

$$N(M) = \{0.117647, 7.11901\cdot10^{14}\} \qquad N(m) = \{0.117647, 7.11901\cdot10^{14}\}$$

Intersection intervals:



$[0.117647, 0.117647]$

Longest intersection interval: 0
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

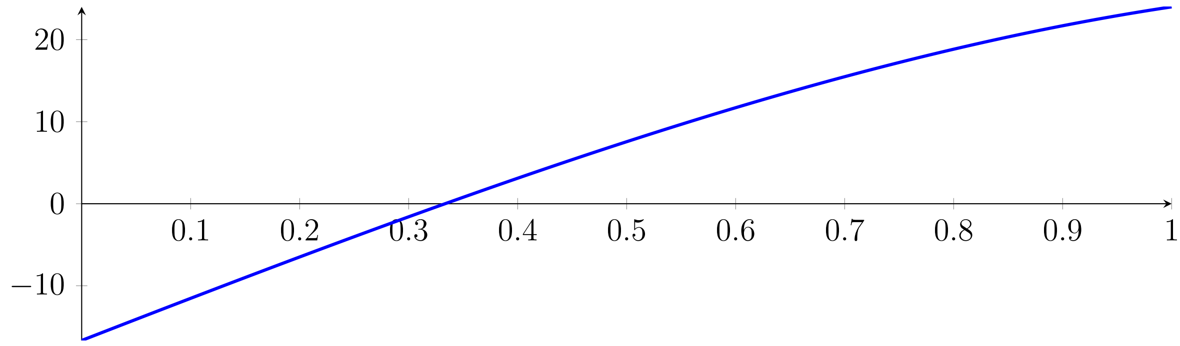
41.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

41.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

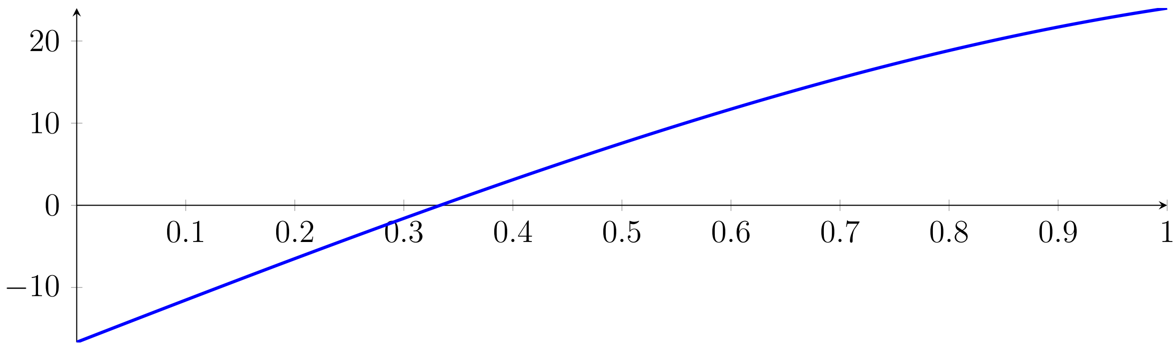
with precision $\varepsilon = 1 \cdot 10^{-128}$.

42 Running CubeClip on f_4 with epsilon 128

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

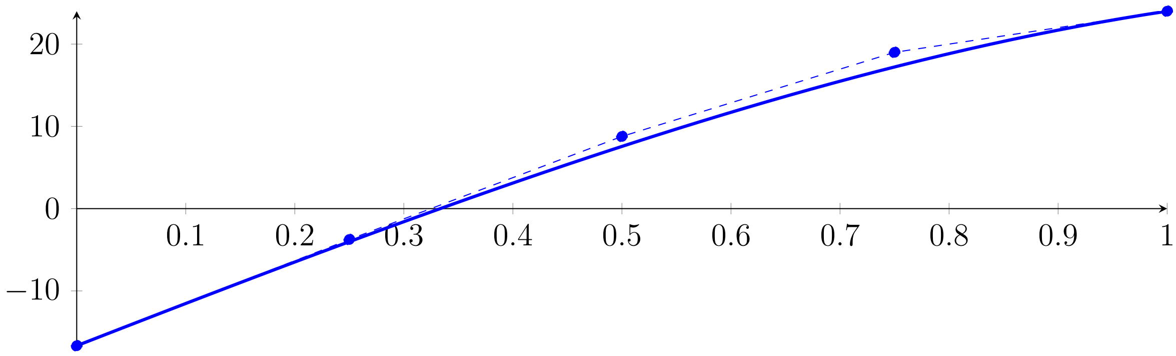
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



42.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

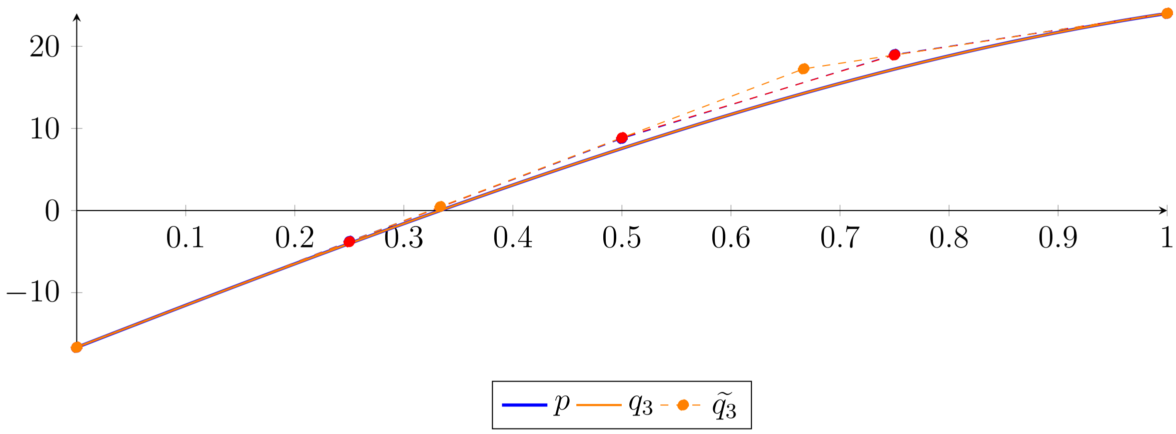
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$
$$= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X)$$



Degree reduction and raising:

$$q_3 = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524$$
$$= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3}$$

$$\widetilde{q_3} = -2.28084 \cdot 10^{-12}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524$$
$$= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m:

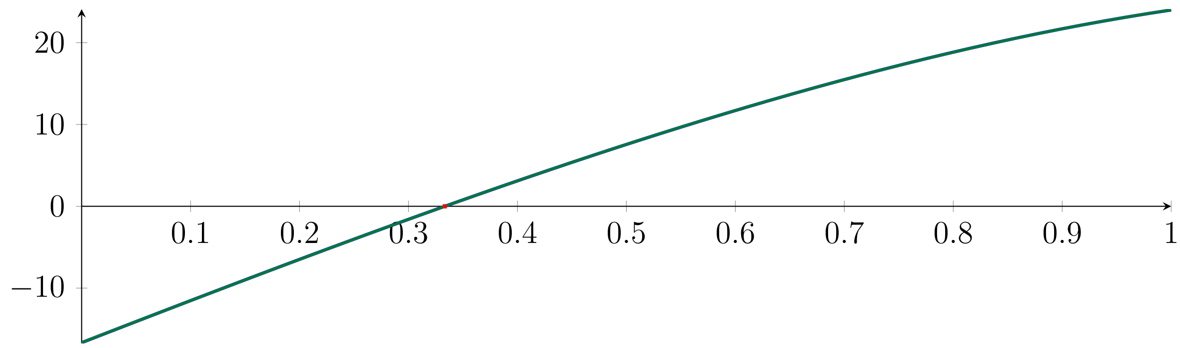
$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

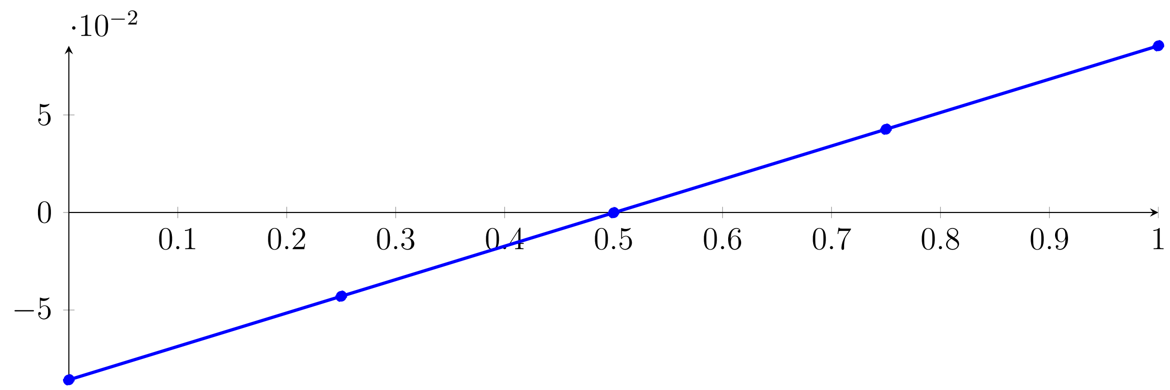
Longest intersection interval: 0.00361204

$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

42.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

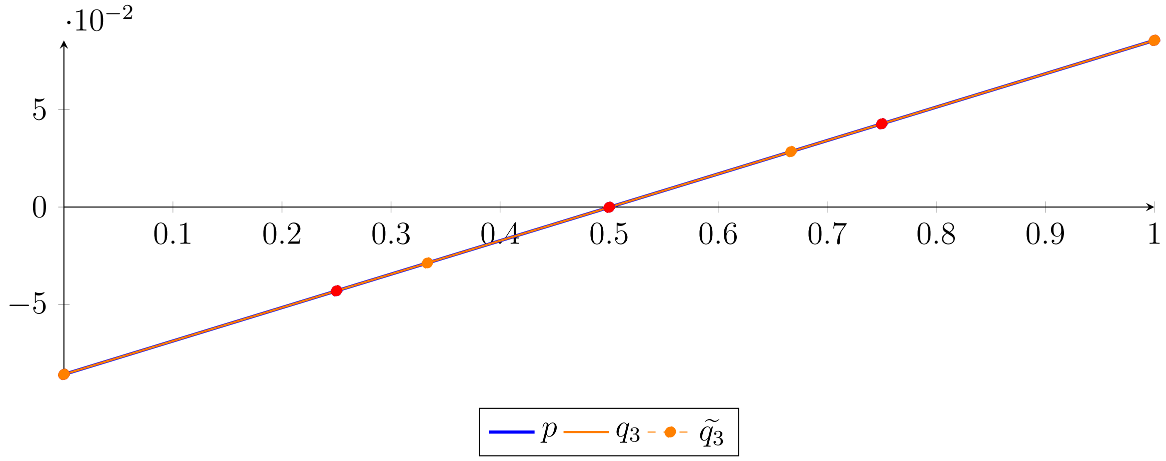
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4}(X) - 0.0429507B_{1,4}(X) - 0.000129666B_{2,4}(X) \\ &\quad + 0.0426682B_{3,4}(X) + 0.0854427B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,3} - 0.0286693B_{1,3} + 0.02841B_{2,3} + 0.0854427B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 1.2032 \cdot 10^{-14} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4} - 0.0429507B_{1,4} - 0.000129666B_{2,4} + 0.0426682B_{3,4} + 0.0854427B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45913 \cdot 10^{-11}$.
Bounding polynomials M and m:

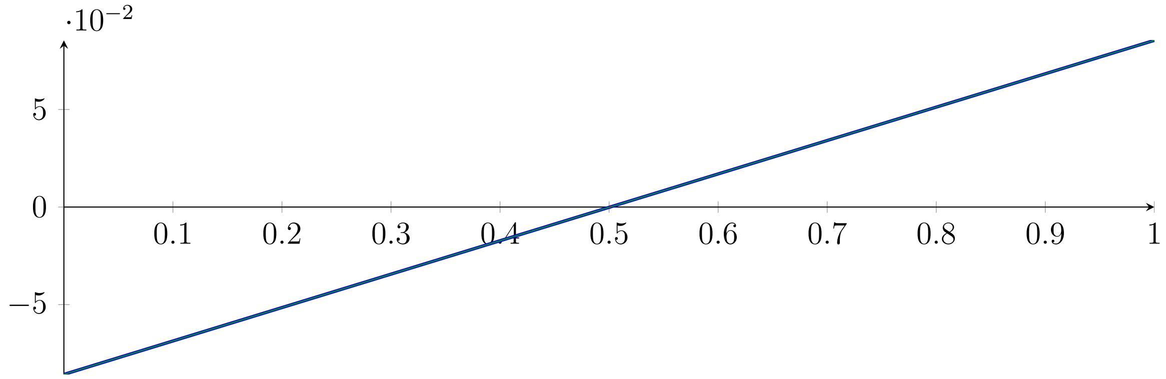
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



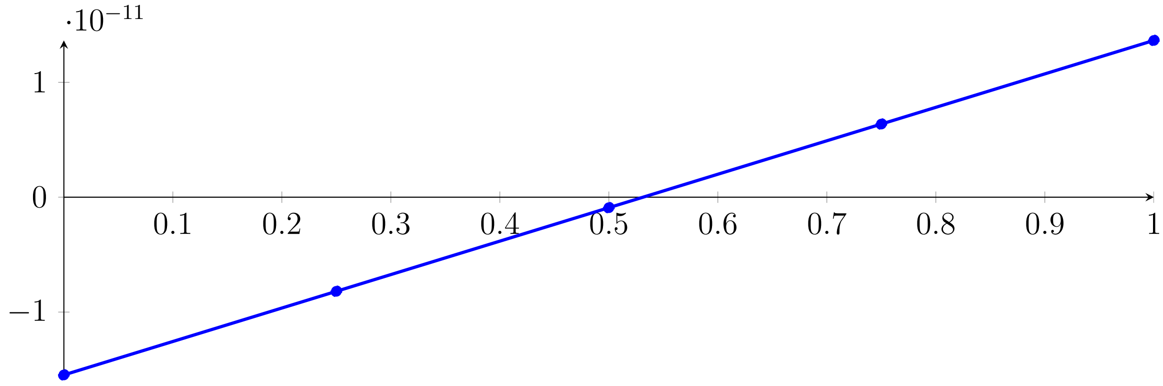
$$[0.500825, 0.500825]$$

Longest intersection interval: $1.70047 \cdot 10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

42.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

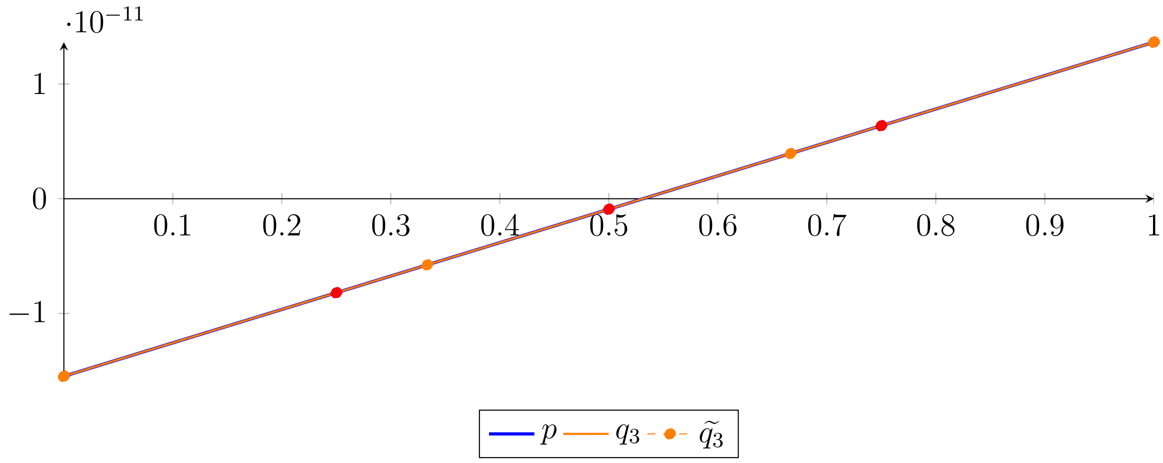
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.01312 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\ &= -1.5473 \cdot 10^{-11} B_{0,4}(X) - 8.19335 \cdot 10^{-12} B_{1,4}(X) - 9.13745 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36586 \cdot 10^{-12} B_{3,4}(X) + 1.36455 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -6.78547\cdot10^{-25}X^3 - 2.95652\cdot10^{-24}X^2 + 2.91184\cdot10^{-11}X - 1.5473\cdot10^{-11}$$
$$= -1.5473\cdot10^{-11}B_{0,3} - 5.76681\cdot10^{-12}B_{1,3} + 3.93932\cdot10^{-12}B_{2,3} + 1.36455\cdot10^{-11}B_{3,3}$$
$$\widetilde{q}_3 = 2.83697\cdot10^{-24}X^4 - 6.85009\cdot10^{-24}X^3 + 1.39587\cdot10^{-24}X^2 + 2.91184\cdot10^{-11}X - 1.5473\cdot10^{-11}$$
$$= -1.5473\cdot10^{-11}B_{0,4} - 8.19335\cdot10^{-12}B_{1,4} - 9.13745\cdot10^{-13}B_{2,4} + 6.36586\cdot10^{-12}B_{3,4} + 1.36455\cdot10^{-11}B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.84343\cdot10^{-25}$.

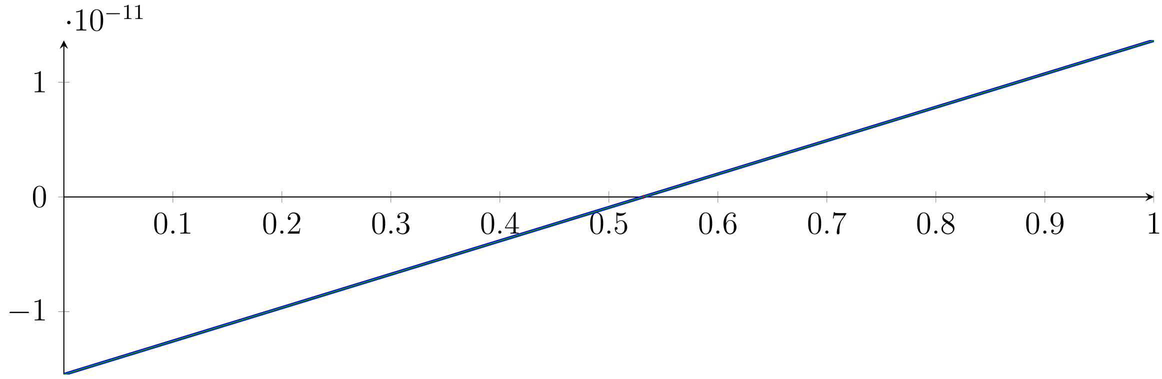
Bounding polynomials M and m:

$$M = -6.78547\cdot10^{-25}X^3 - 2.95652\cdot10^{-24}X^2 + 2.91184\cdot10^{-11}X - 1.5473\cdot10^{-11}$$
$$m = -6.78547\cdot10^{-25}X^3 - 2.95652\cdot10^{-24}X^2 + 2.91184\cdot10^{-11}X - 1.5473\cdot10^{-11}$$

Root of M and m:

$$N(M) = \{-6.51982\cdot10^6, 0.53138, 6.51982\cdot10^6\} \quad N(m) = \{-6.51982\cdot10^6, 0.53138, 6.51982\cdot10^6\}$$

Intersection intervals:



$[0.53138, 0.53138]$

Longest intersection interval: 0

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

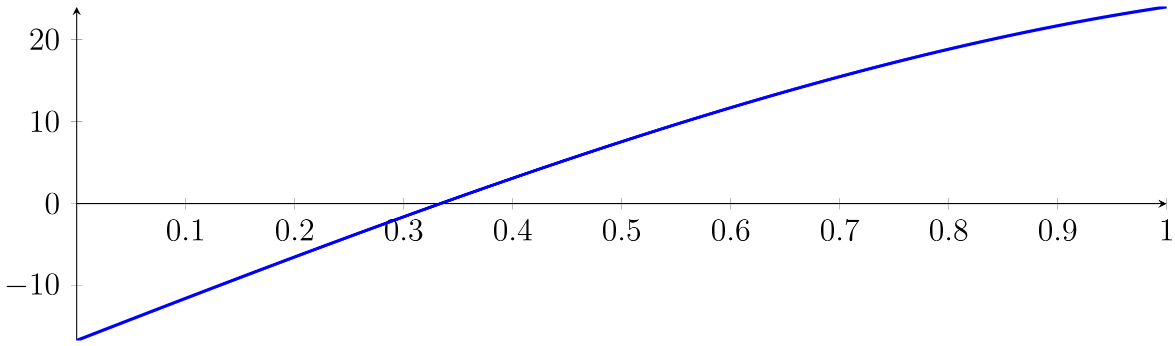
42.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

42.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

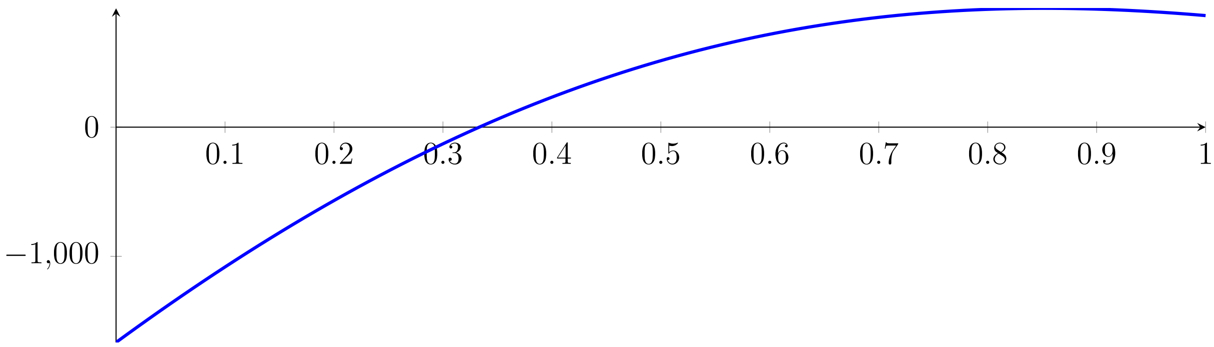
with precision $\varepsilon = 1 \cdot 10^{-128}$.

43 Running BezClip on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

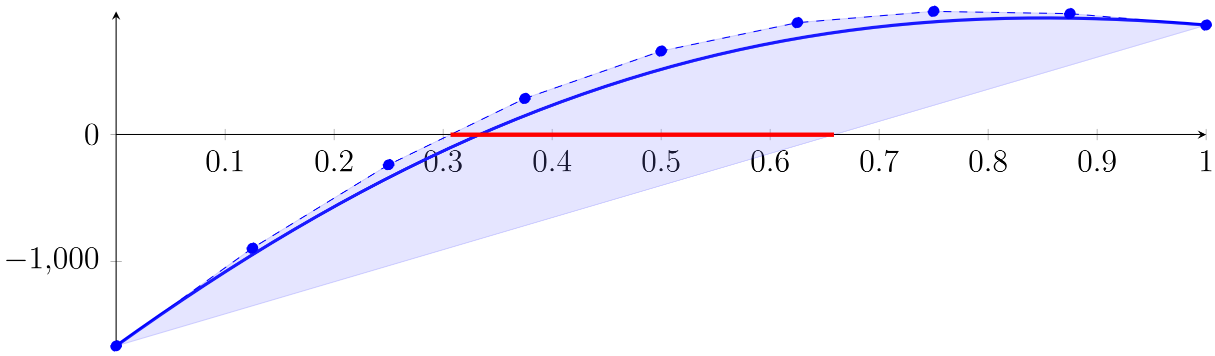
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



43.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

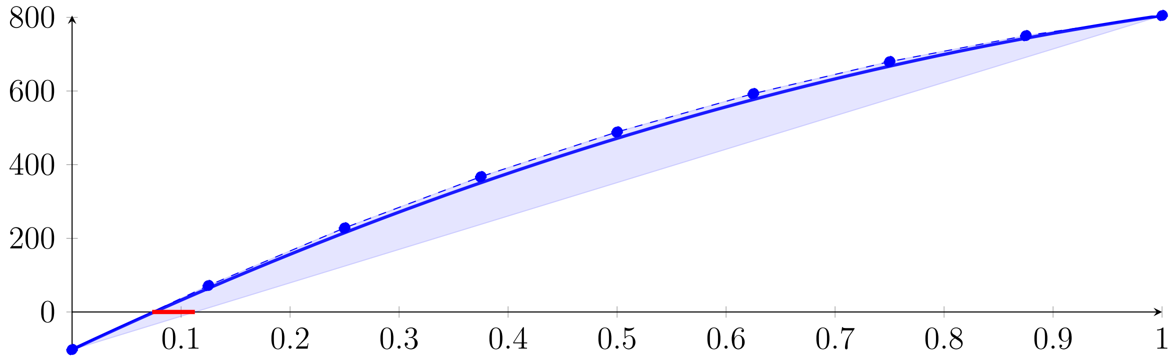
$$[0.306796, 0.658588]$$

Longest intersection interval: 0.351792
$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

43.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\ &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\ &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\ &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

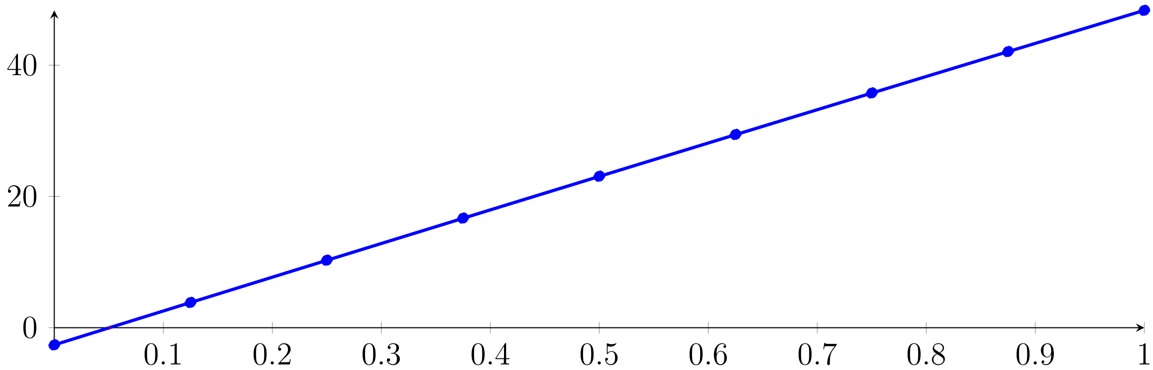
Longest intersection interval: 0.0391855

$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

43.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.41789 \cdot 10^{-13}X^8 - 1.26477 \cdot 10^{-12}X^7 - 5.05786 \cdot 10^{-10}X^6 + 3.66765 \cdot 10^{-08}X^5 \\ &\quad + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\ &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\ &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

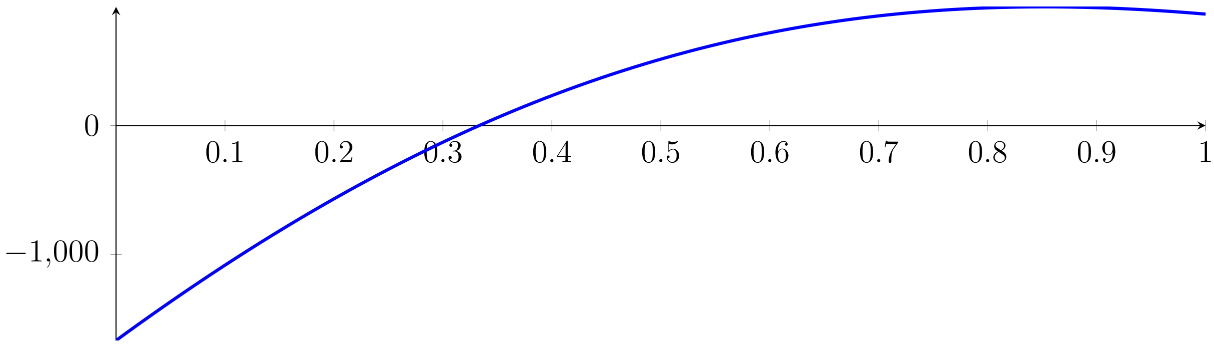
43.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Found root in interval $[0.333333, 0.333343]$ at recursion depth 4!

43.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333343]$$

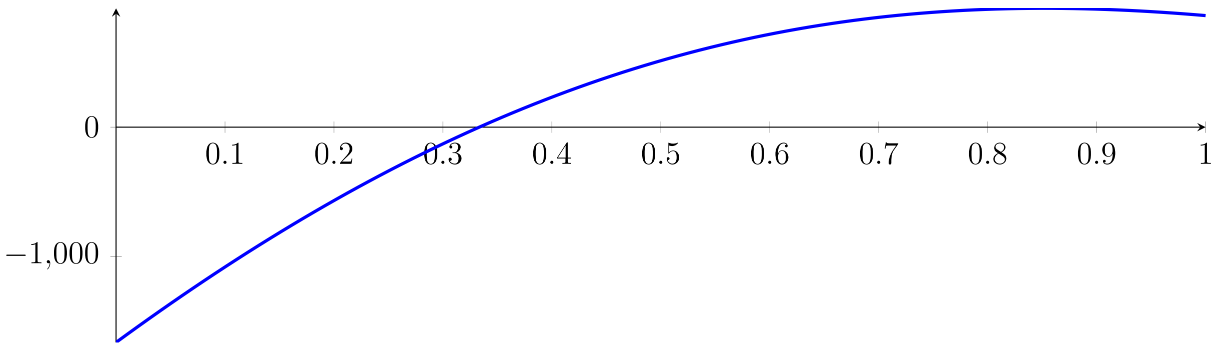
with precision $\varepsilon = 0.01$.

44 Running QuadClip on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

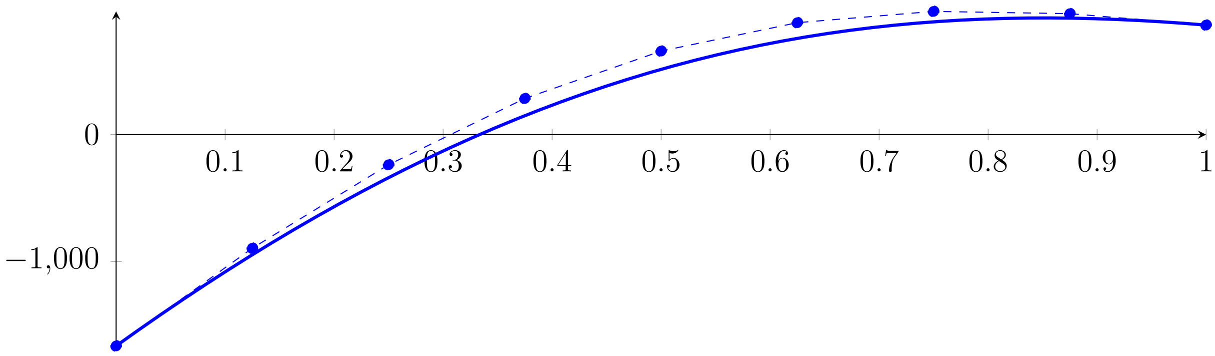
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



44.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

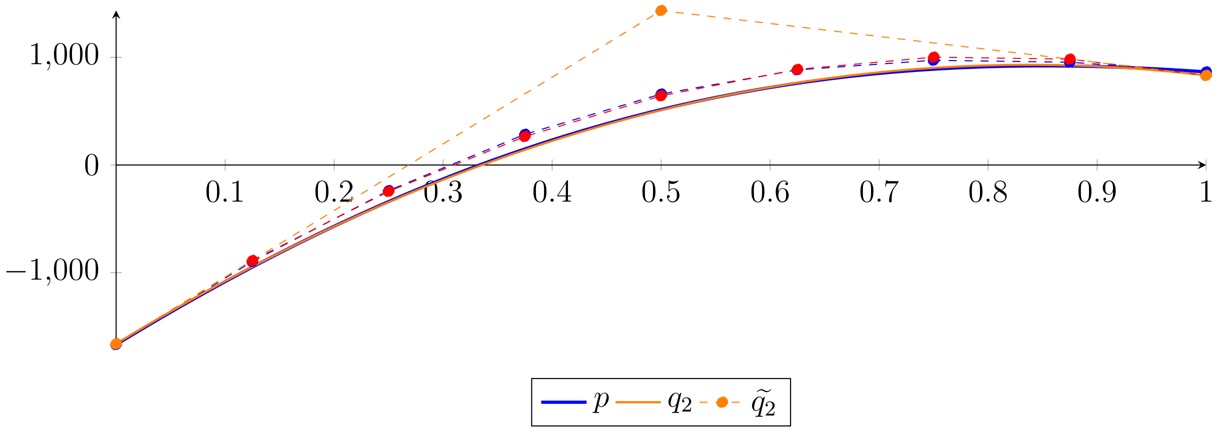
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -3.91297 \cdot 10^{-06}X^8 + 1.58774 \cdot 10^{-05}X^7 - 2.63335 \cdot 10^{-05}X^6 + 2.29285 \cdot 10^{-05}X^5 \\ &\quad - 1.11931 \cdot 10^{-05}X^4 + 3.0313 \cdot 10^{-06}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

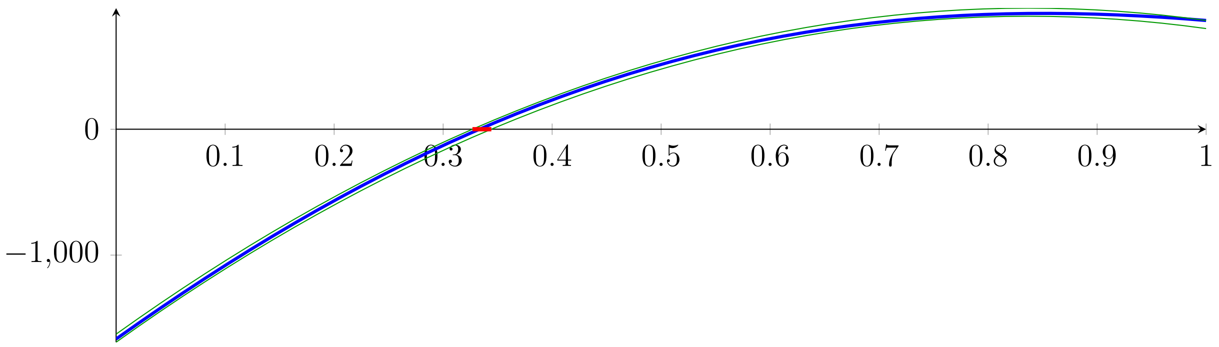
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

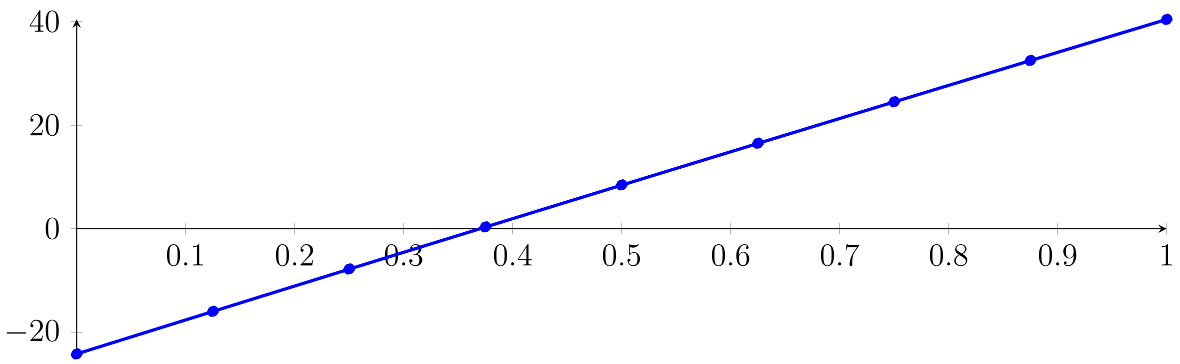
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

44.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

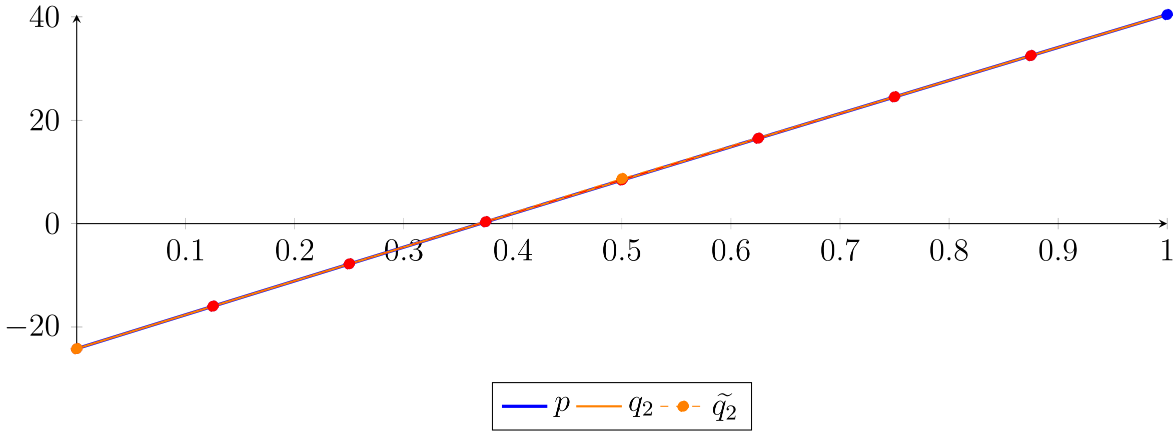
$$\begin{aligned} p = & -5.40012\cdot 10^{-13}X^8 - 7.38964\cdot 10^{-12}X^7 - 1.94416\cdot 10^{-09}X^6 + 1.19265\cdot 10^{-07}X^5 \\ & + 8.12335\cdot 10^{-05}X^4 - 0.0012733X^3 - 1.18084X^2 + 65.8155X - 24.1945 \\ = & -24.1945B_{0,8}(X) - 15.9676B_{1,8}(X) - 7.78282B_{2,8}(X) + 0.35975B_{3,8}(X) \\ & + 8.4601B_{4,8}(X) + 16.5182B_{5,8}(X) + 24.5341B_{6,8}(X) + 32.5077B_{7,8}(X) + 40.4389B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 1.00159 \cdot 10^{-08}X^8 - 3.3372 \cdot 10^{-08}X^7 + 4.23875 \cdot 10^{-08}X^6 - 2.49721 \cdot 10^{-08}X^5 \\ &\quad + 6.08793 \cdot 10^{-09}X^4 + 1.46429 \cdot 10^{-10}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m:

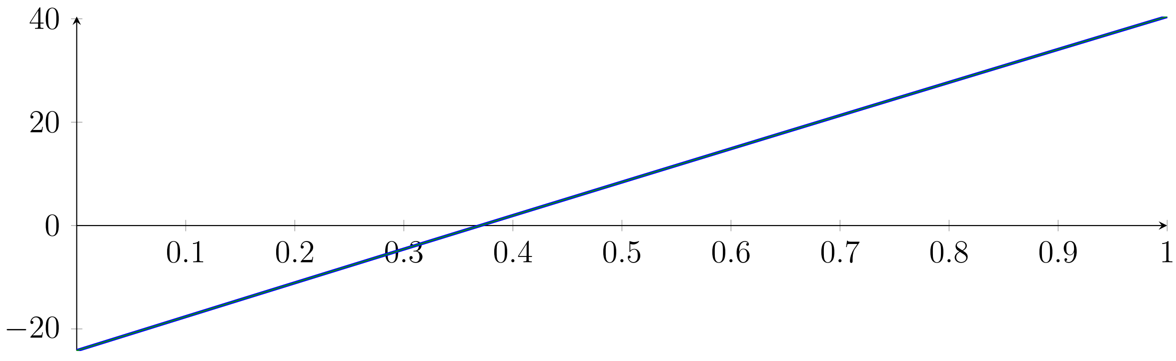
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$N(M) = \{0.370068, 55.2832\}$ $\qquad$ $N(m) = \{0.37007, 55.2832\}$

Intersection intervals:



$[0.370068, 0.37007]$

Longest intersection interval: $1.74588 \cdot 10^{-06}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

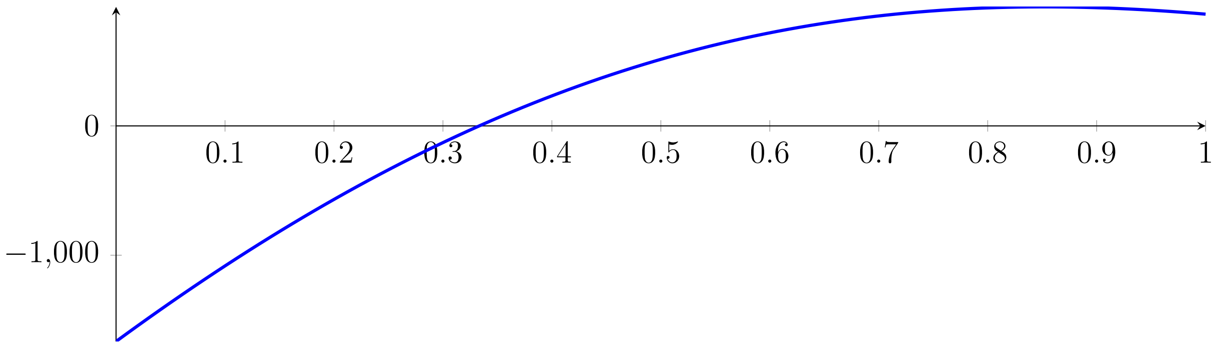
44.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

44.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



Result: Root Intervals

$$[0.333333, 0.333333]$$

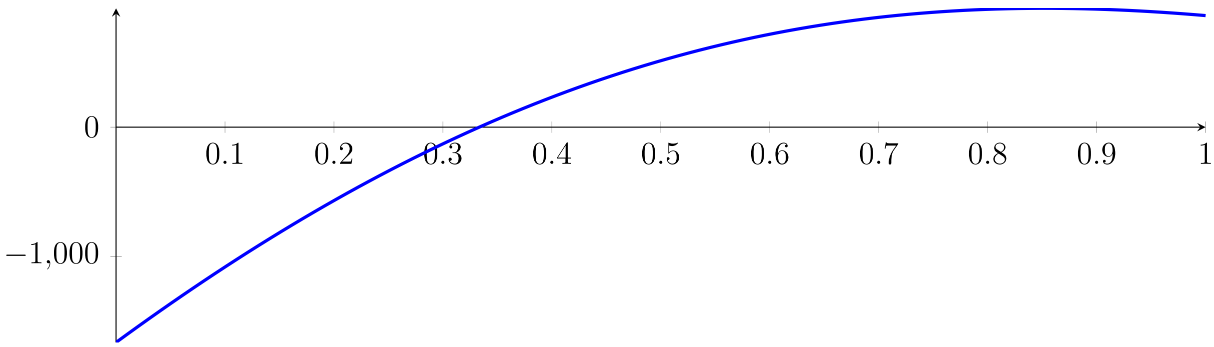
with precision $\varepsilon = 0.01$.

45 Running CubeClip on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

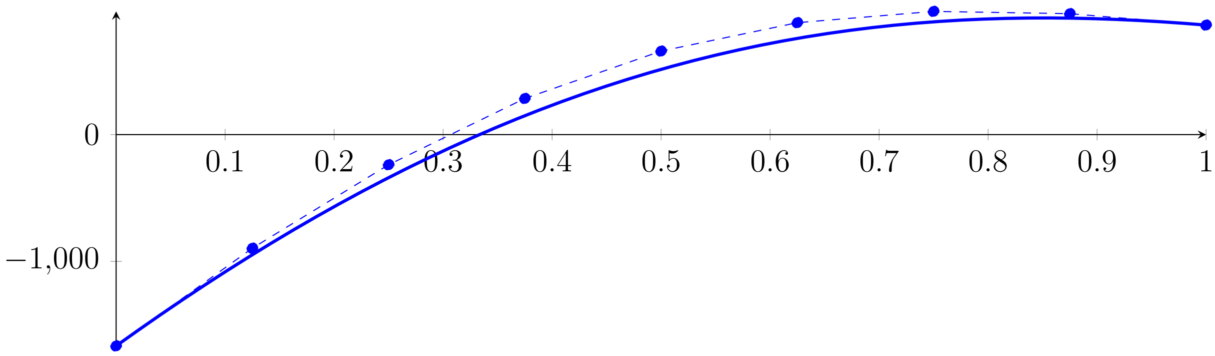
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



45.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

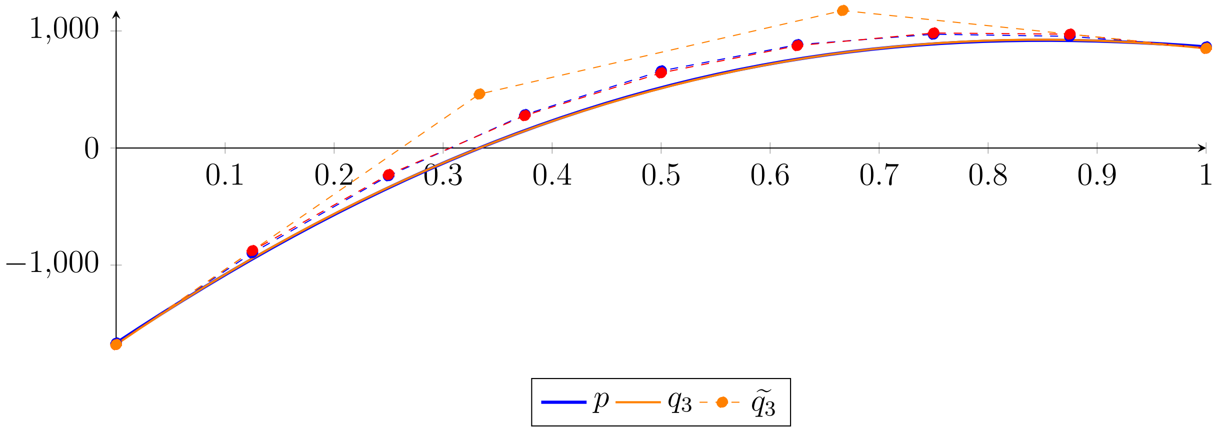
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2.38117 \cdot 10^{-06}X^8 - 9.49957 \cdot 10^{-06}X^7 + 1.54757 \cdot 10^{-05}X^6 - 1.32609 \cdot 10^{-05}X^5 \\ &\quad + 6.41819 \cdot 10^{-06}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

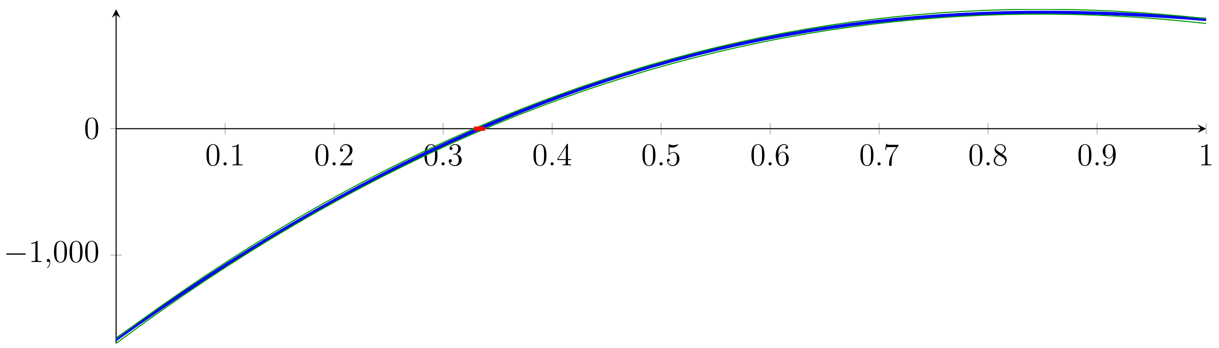
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$[0.328258, 0.338551]$

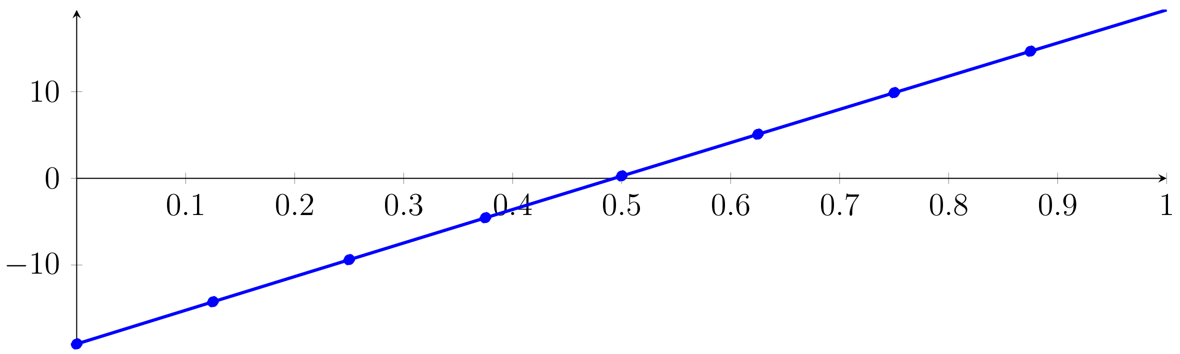
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

45.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

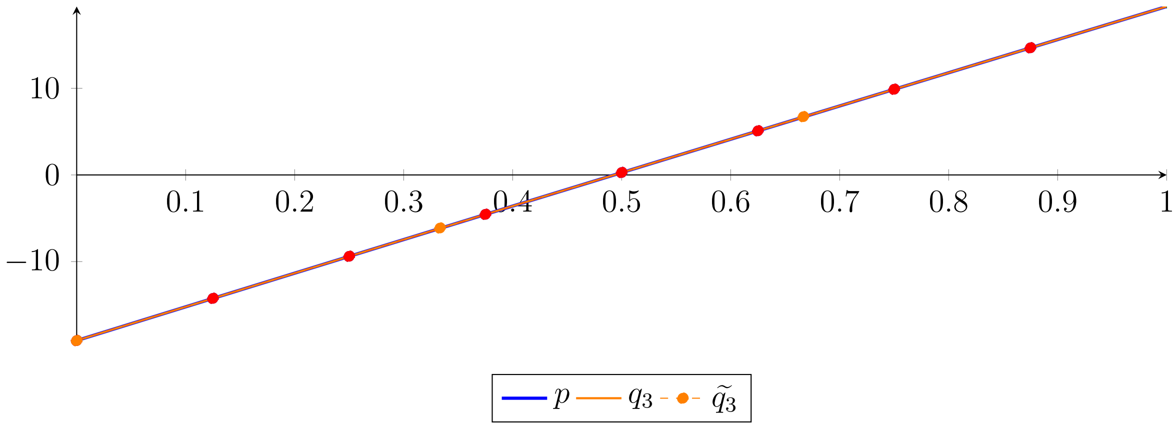
$$\begin{aligned} p &= -1.27898{\cdot}10^{-13}X^8 - 1.13687{\cdot}10^{-13}X^7 - 8.23661{\cdot}10^{-11}X^6 + 8.72882{\cdot}10^{-09}X^5 \\ &\quad + 1.00963{\cdot}10^{-05}X^4 - 0.000261161X^3 - 0.416284X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8}(X) - 14.2419B_{1,8}(X) - 9.38619B_{2,8}(X) - 4.54539B_{3,8}(X) \\ &\quad + 0.280537B_{4,8}(X) + 5.09158B_{5,8}(X) + 9.88774B_{6,8}(X) + 14.669B_{7,8}(X) + 19.4354B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2.3353\cdot10^{-08}X^8 - 9.31856\cdot10^{-08}X^7 + 1.51861\cdot10^{-07}X^6 - 1.30228\cdot10^{-07}X^5 \\ &\quad + 6.31618\cdot10^{-08}X^4 - 0.000240962X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16887\cdot10^{-07}$.
Bounding polynomials M and m:

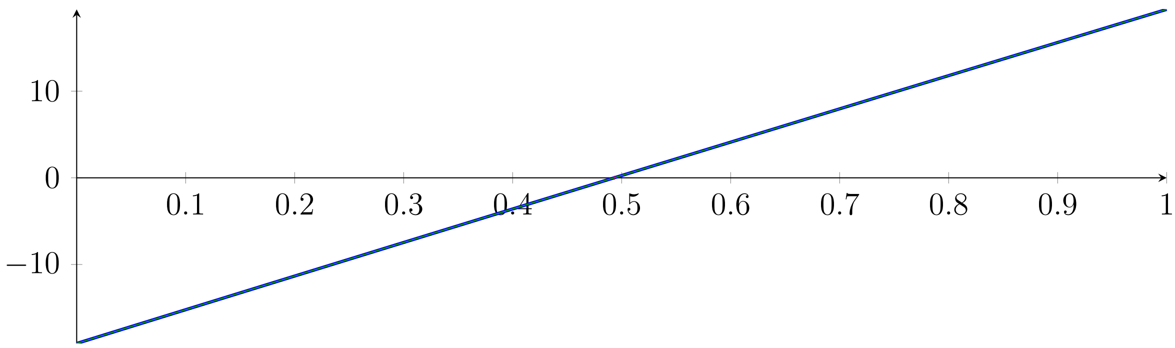
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.12517\cdot10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

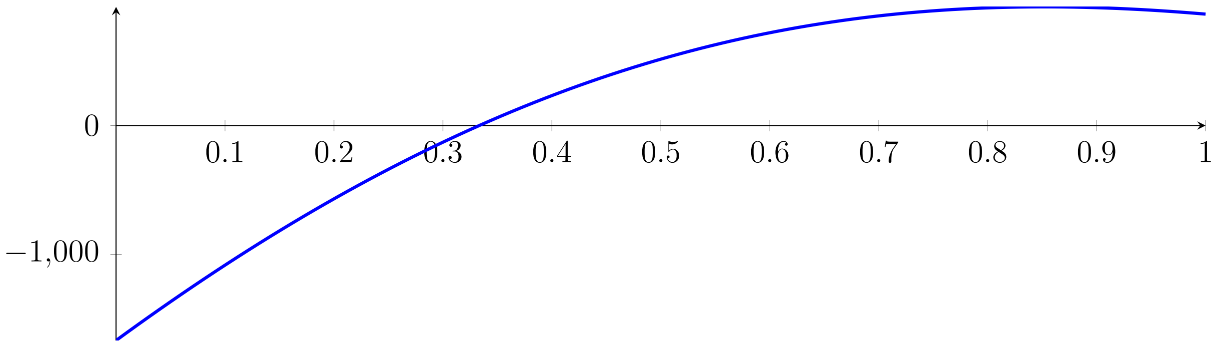
45.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

45.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

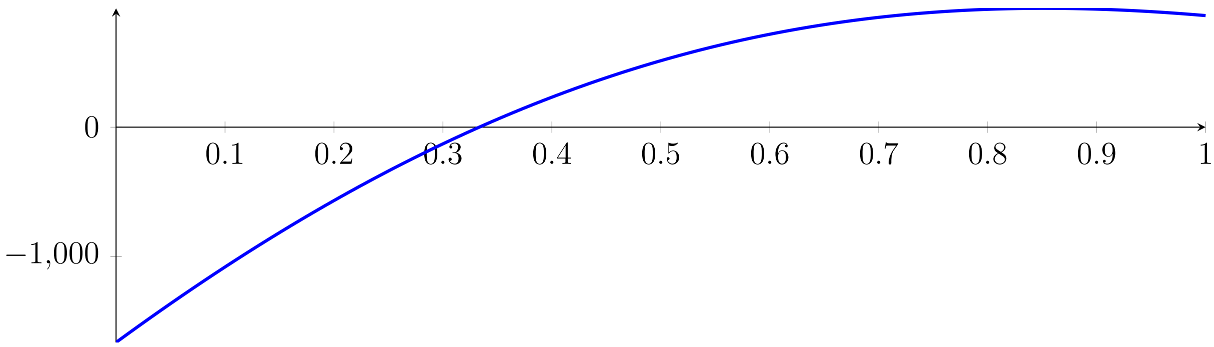
with precision $\varepsilon = 0.01$.

46 Running BezClip on f_8 with epsilon 4

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

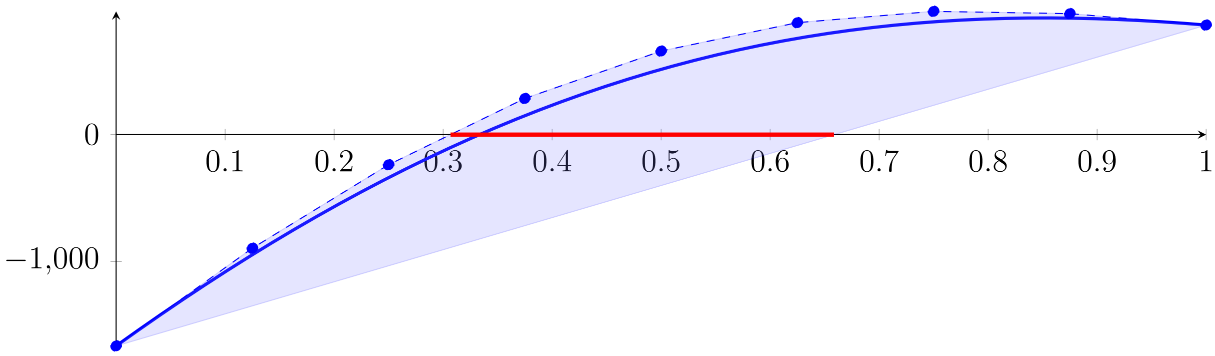
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



46.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

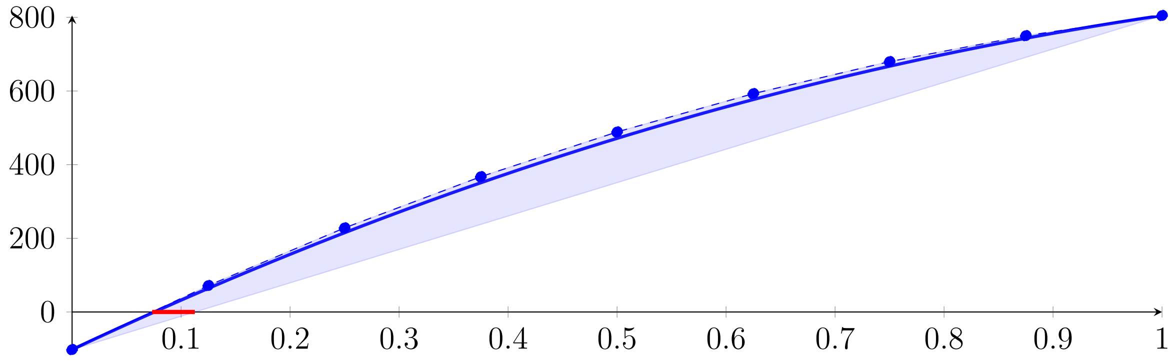
Longest intersection interval: 0.351792

$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

46.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5\\ &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17\\ &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X)\\ &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

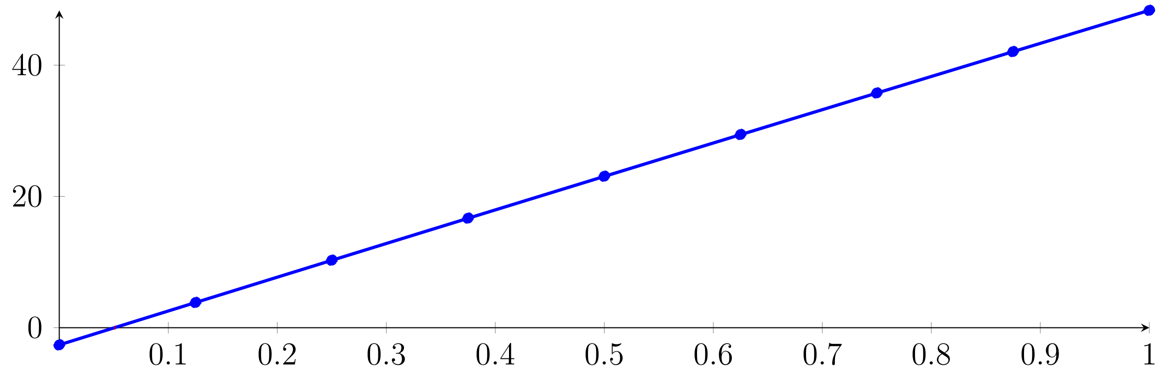
Longest intersection interval: 0.0391855

$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

46.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -5.41789\cdot 10^{-13}X^8 - 1.26477\cdot 10^{-12}X^7 - 5.05786\cdot 10^{-10}X^6 + 3.66765\cdot 10^{-08}X^5\\ &\quad + 3.25466\cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683\\ &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X)\\ &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

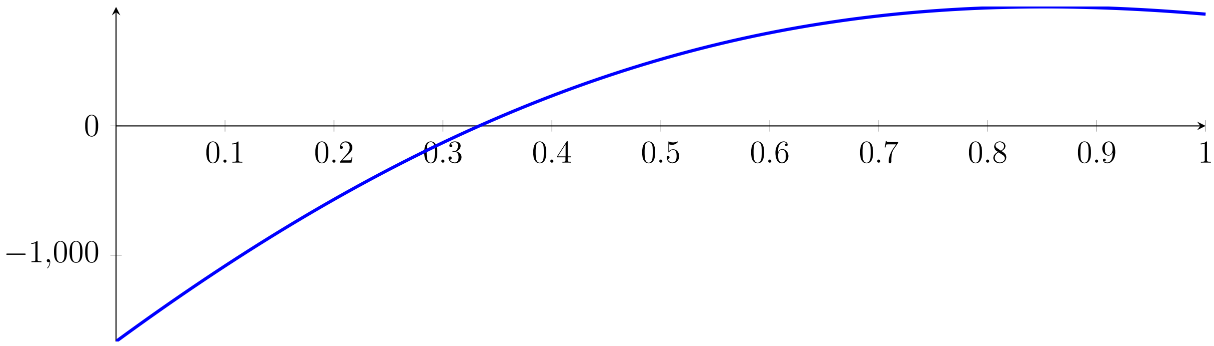
46.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Found root in interval $[0.333333, 0.333343]$ at recursion depth 4!

46.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333343]$$

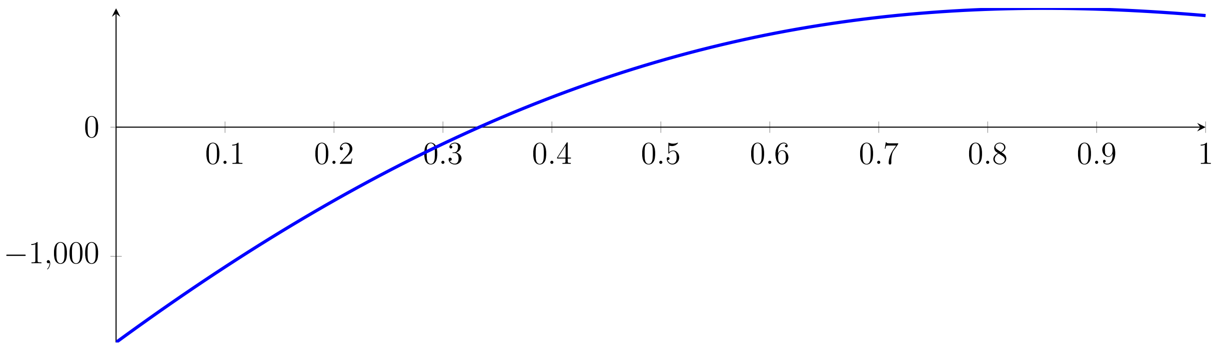
with precision $\varepsilon = 0.0001$.

47 Running QuadClip on f_8 with epsilon 4

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

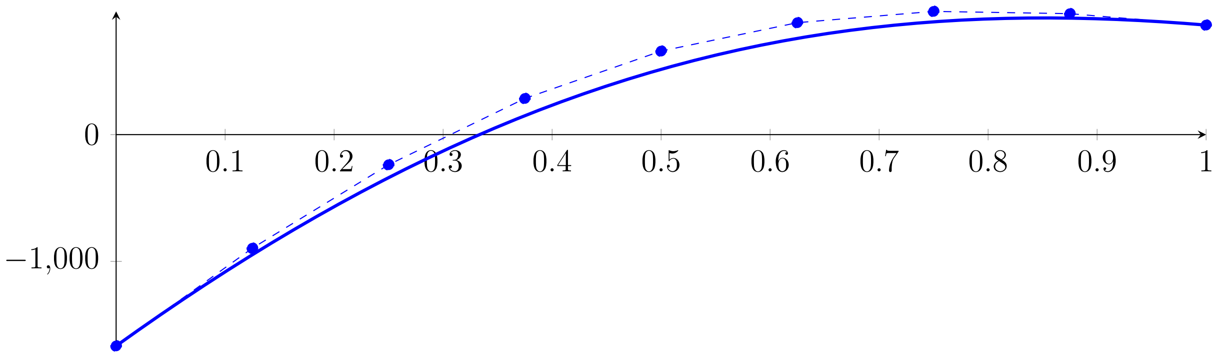
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



47.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

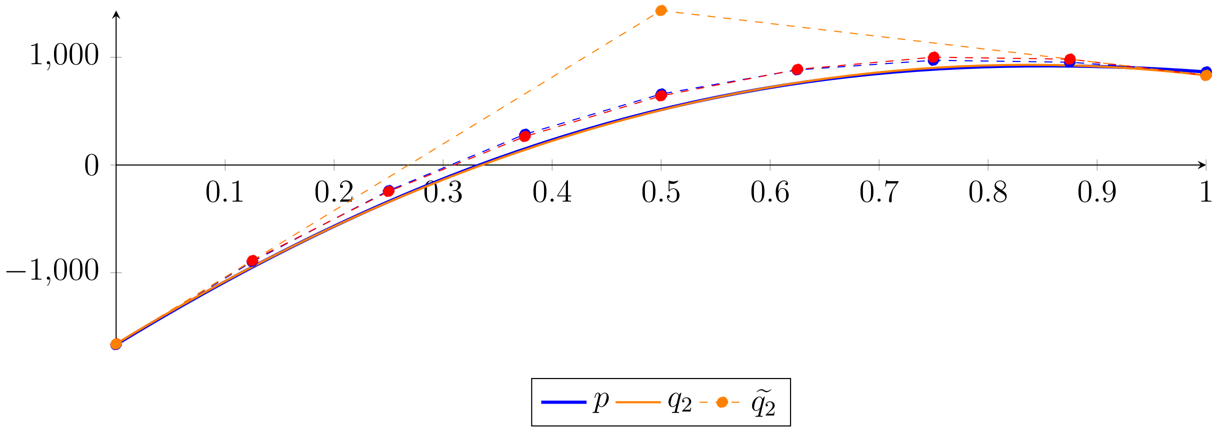
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -3.91297 \cdot 10^{-06}X^8 + 1.58774 \cdot 10^{-05}X^7 - 2.63335 \cdot 10^{-05}X^6 + 2.29285 \cdot 10^{-05}X^5 \\ &\quad - 1.11931 \cdot 10^{-05}X^4 + 3.0313 \cdot 10^{-06}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

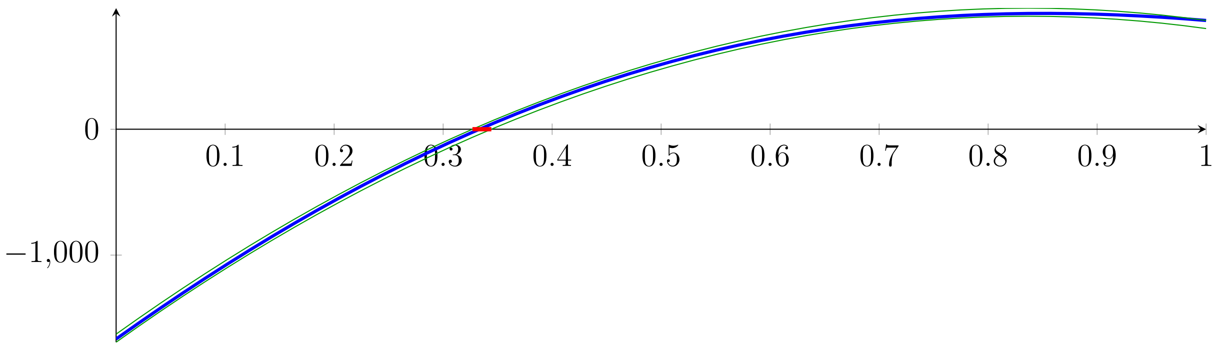
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

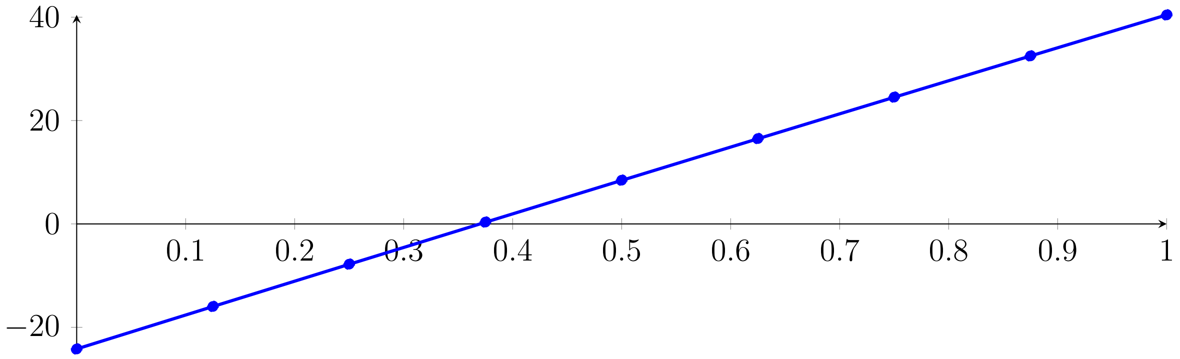
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

47.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

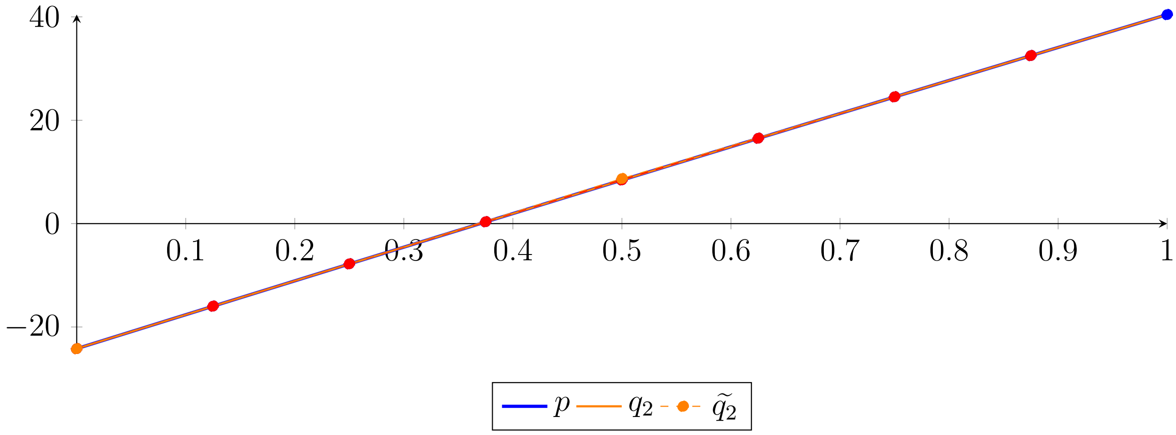
$$\begin{aligned} p = &-5.40012\cdot10^{-13}X^8 - 7.38964\cdot10^{-12}X^7 - 1.94416\cdot10^{-09}X^6 + 1.19265\cdot10^{-07}X^5 \\ &+ 8.12335\cdot10^{-05}X^4 - 0.0012733X^3 - 1.18084X^2 + 65.8155X - 24.1945 \\ = &-24.1945B_{0,8}(X) - 15.9676B_{1,8}(X) - 7.78282B_{2,8}(X) + 0.35975B_{3,8}(X) \\ &+ 8.4601B_{4,8}(X) + 16.5182B_{5,8}(X) + 24.5341B_{6,8}(X) + 32.5077B_{7,8}(X) + 40.4389B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 1.00159{\cdot}10^{-08}X^8 - 3.3372{\cdot}10^{-08}X^7 + 4.23875{\cdot}10^{-08}X^6 - 2.49721{\cdot}10^{-08}X^5 \\ &\quad + 6.08793{\cdot}10^{-09}X^4 + 1.46429{\cdot}10^{-10}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894{\cdot}10^{-05}$.

Bounding polynomials M and m:

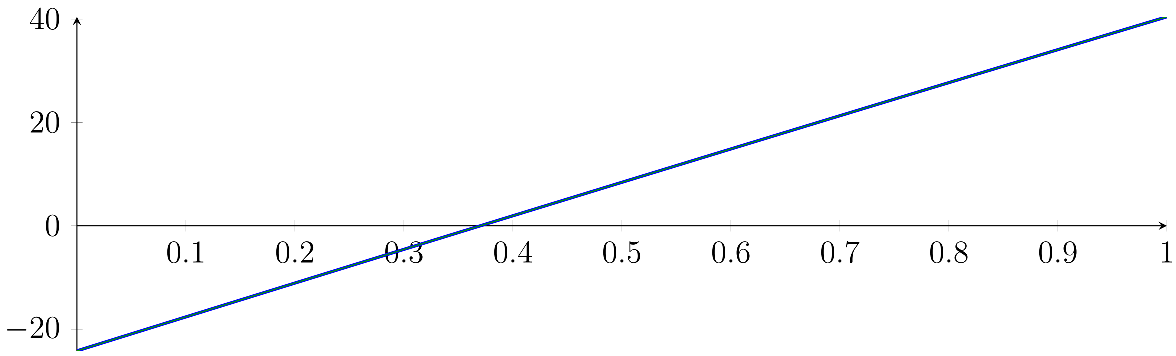
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588{\cdot}10^{-06}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

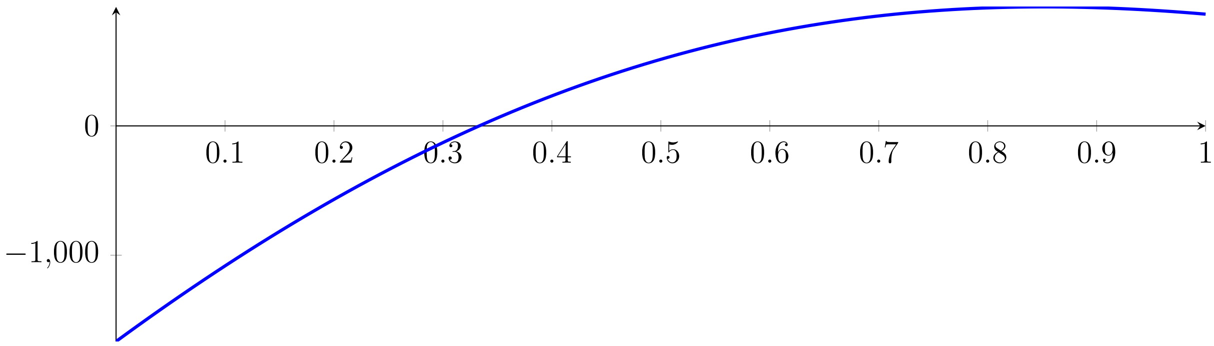
47.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

47.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

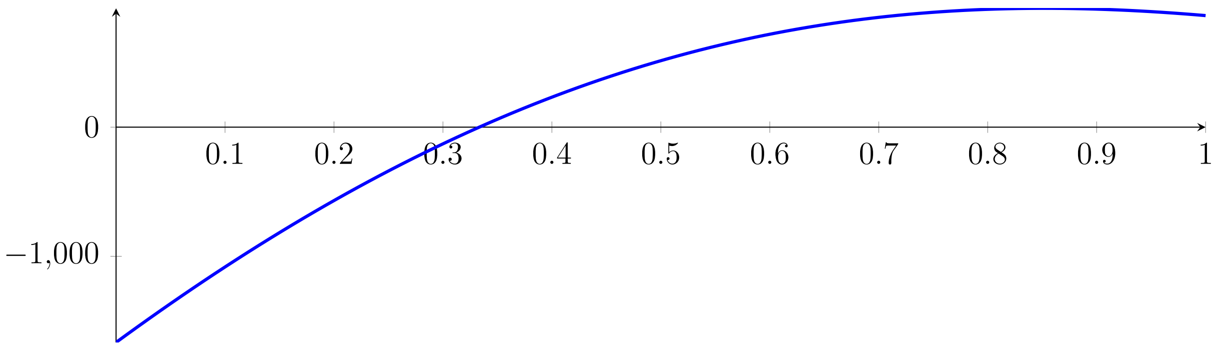
with precision $\varepsilon = 0.0001$.

48 Running `CubeClip` on f_8 with epsilon 4

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

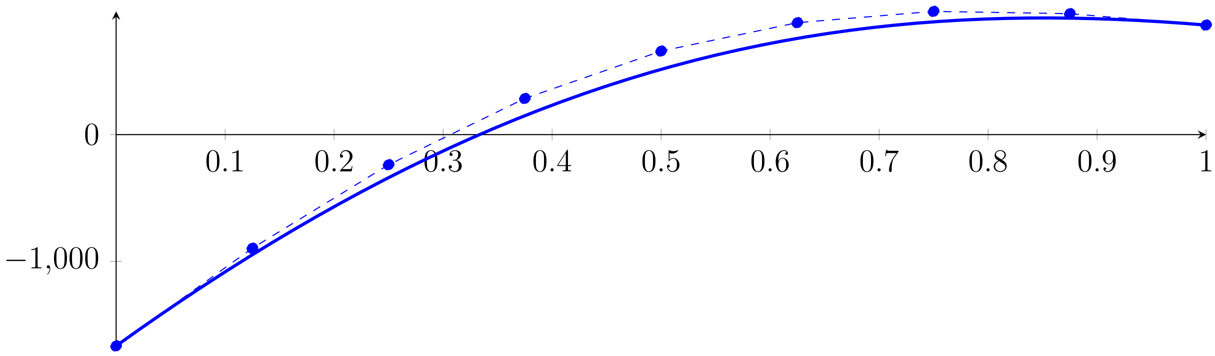
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



48.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

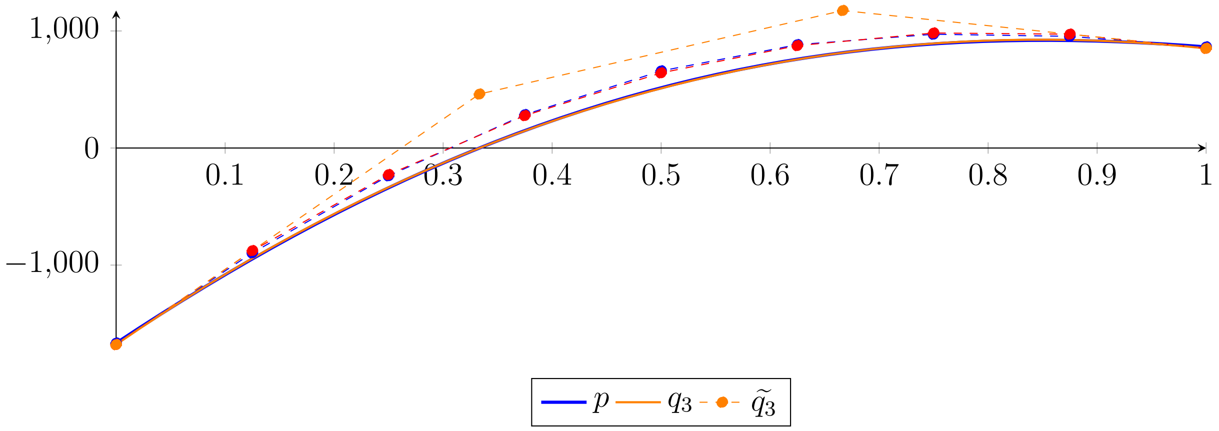
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 2.38117 \cdot 10^{-06}X^8 - 9.49957 \cdot 10^{-06}X^7 + 1.54757 \cdot 10^{-05}X^6 - 1.32609 \cdot 10^{-05}X^5 \\ &\quad + 6.41819 \cdot 10^{-06}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

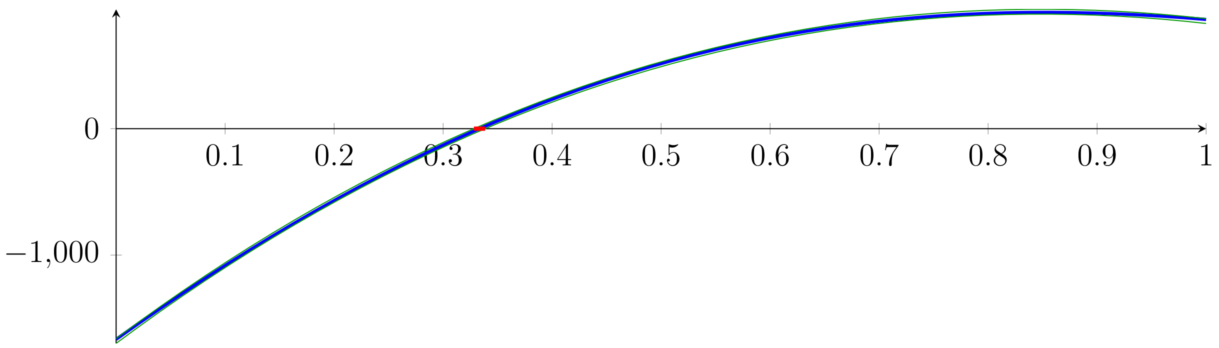
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



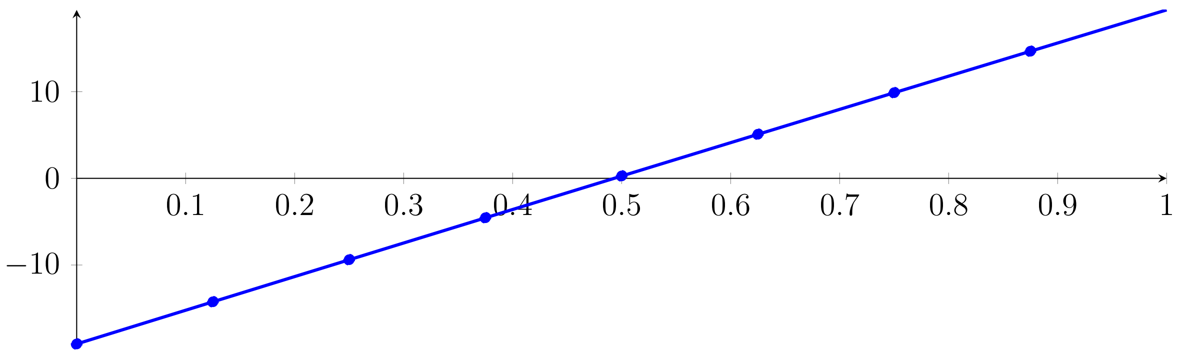
$[0.328258, 0.338551]$

Longest intersection interval: 0.0102926
$\implies$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

48.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

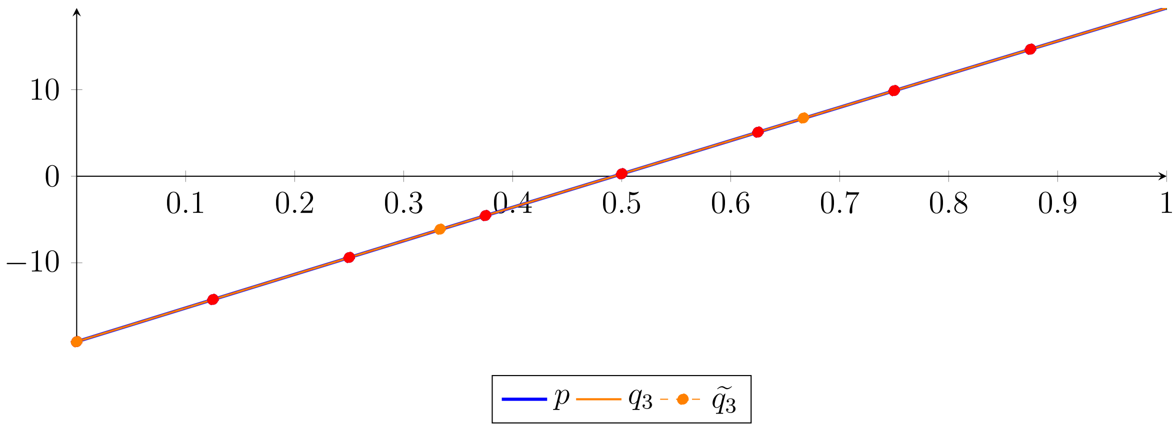
$$\begin{aligned} p = &-1.27898\cdot10^{-13}X^8 - 1.13687\cdot10^{-13}X^7 - 8.23661\cdot10^{-11}X^6 + 8.72882\cdot10^{-09}X^5 \\ &+ 1.00963\cdot10^{-05}X^4 - 0.000261161X^3 - 0.416284X^2 + 38.9643X - 19.1124 \\ = &-19.1124B_{0,8}(X) - 14.2419B_{1,8}(X) - 9.38619B_{2,8}(X) - 4.54539B_{3,8}(X) \\ &+ 0.280537B_{4,8}(X) + 5.09158B_{5,8}(X) + 9.88774B_{6,8}(X) + 14.669B_{7,8}(X) + 19.4354B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 2.3353{\cdot}10^{-08}X^8 - 9.31856{\cdot}10^{-08}X^7 + 1.51861{\cdot}10^{-07}X^6 - 1.30228{\cdot}10^{-07}X^5 \\ &\quad + 6.31618{\cdot}10^{-08}X^4 - 0.000240962X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16887{\cdot}10^{-07}$.

Bounding polynomials M and m:

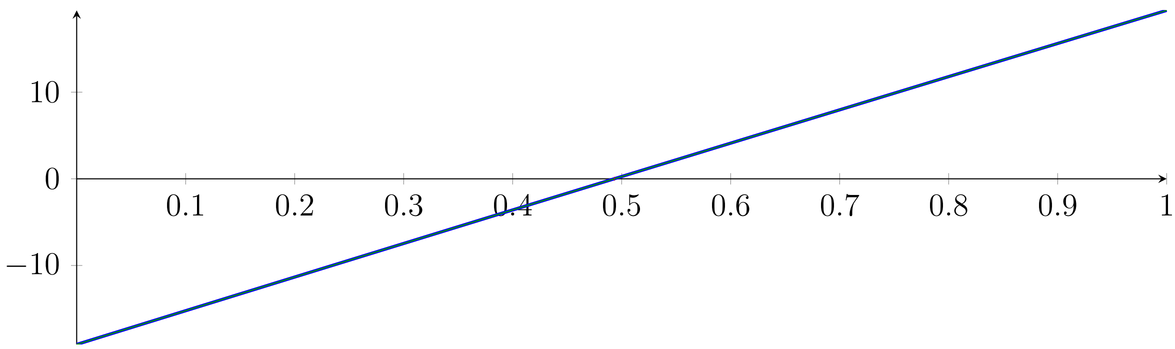
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$[0.493109, 0.493109]$

Longest intersection interval: $1.12517{\cdot}10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

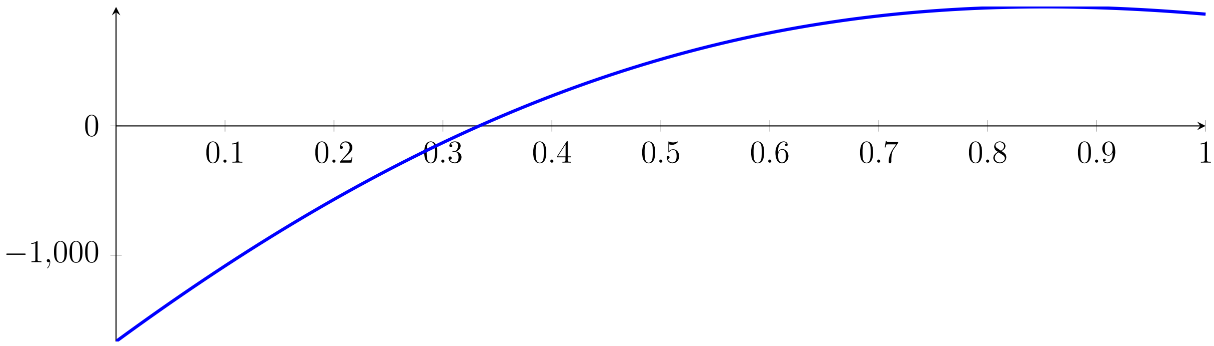
48.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

48.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

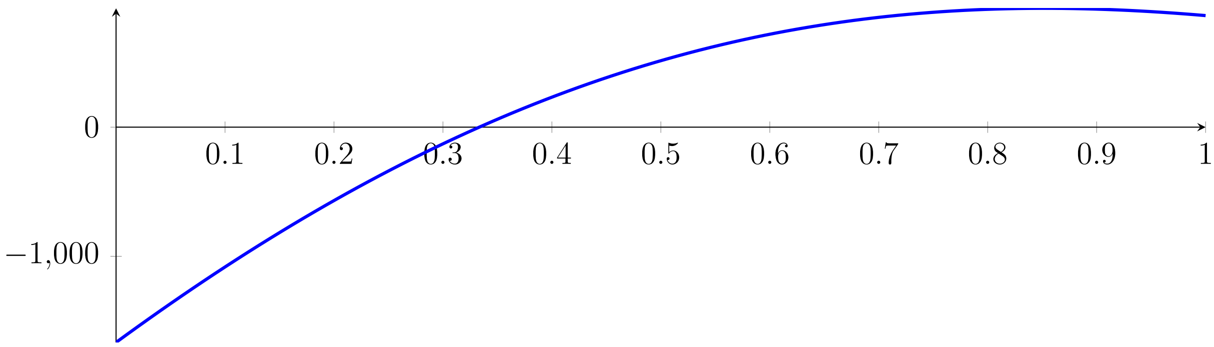
with precision $\varepsilon = 0.0001$.

49 Running BezClip on f_8 with epsilon 8

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

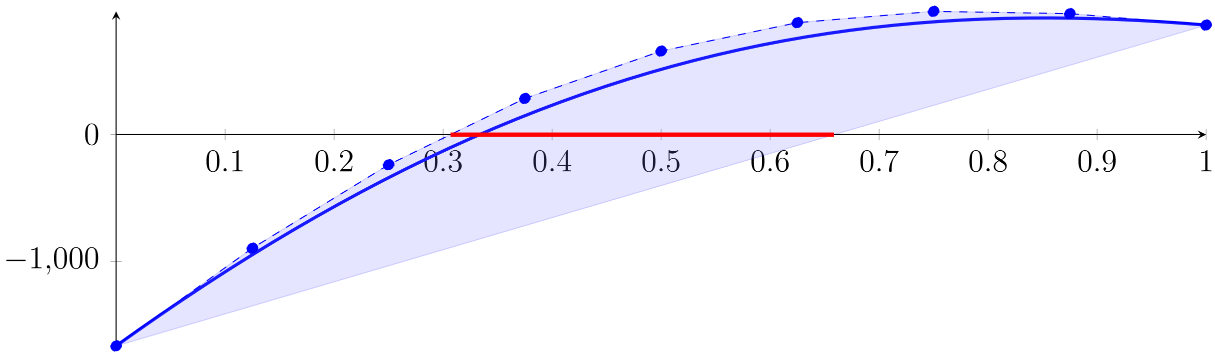
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



49.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$
$$= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X)$$
$$+ 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

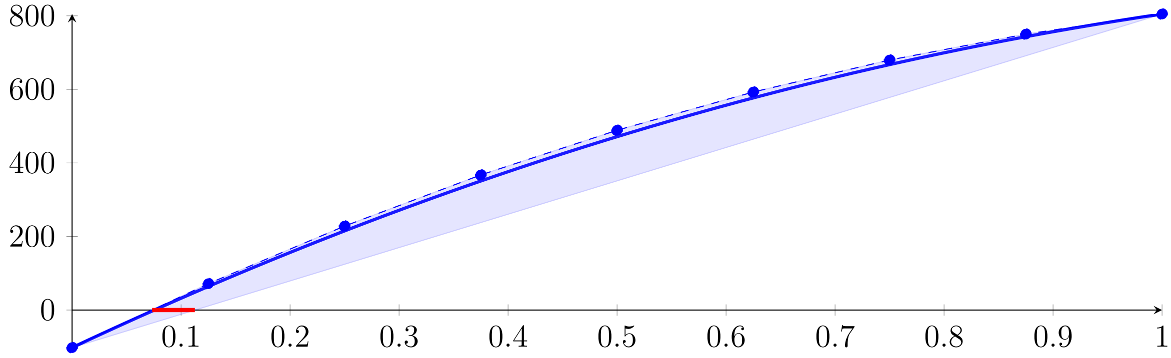
$$[0.306796, 0.658588]$$

Longest intersection interval: 0.351792
$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

49.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p = &-0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5\\ &+ 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17\\ = &-102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X)\\ &+ 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

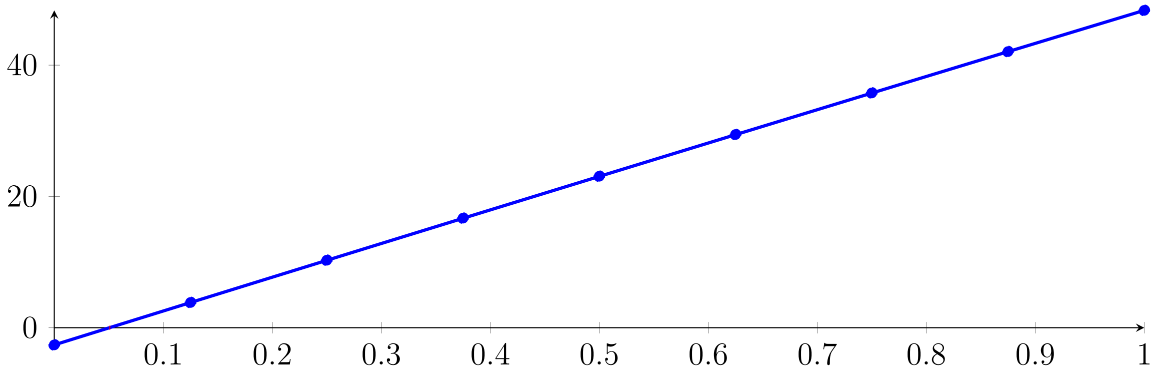
Longest intersection interval: 0.0391855

$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

49.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p = &-5.41789 \cdot 10^{-13}X^8 - 1.26477 \cdot 10^{-12}X^7 - 5.05786 \cdot 10^{-10}X^6 + 3.66765 \cdot 10^{-08}X^5\\ &+ 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683\\ = &-2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X)\\ &+ 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

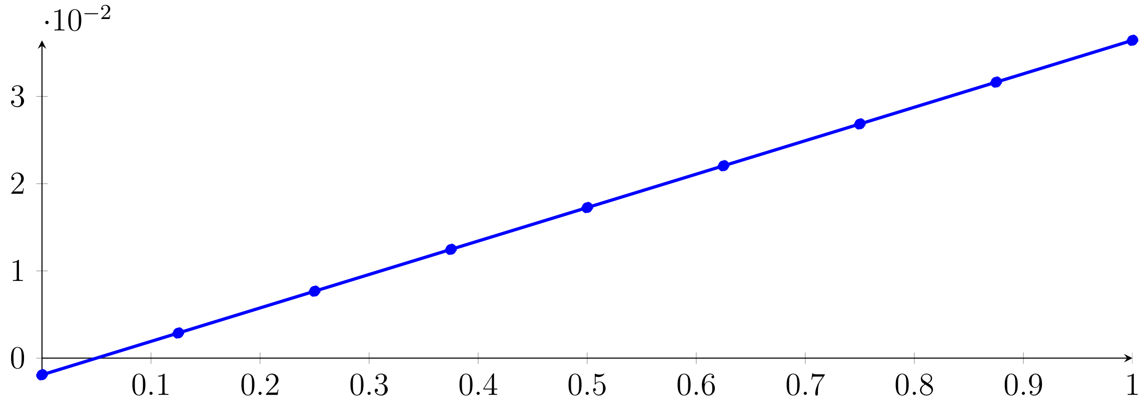
Longest intersection interval: 0.000742589

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

49.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -7.11237{\cdot}10^{-16}X^8 + 5.27356{\cdot}10^{-16}X^7 - 7.38298{\cdot}10^{-15}X^6 + 1.06859{\cdot}10^{-15}X^5 \\ & - 1.09288{\cdot}10^{-15}X^4 - 2.37227{\cdot}10^{-13}X^3 - 4.12146{\cdot}10^{-07}X^2 + 0.0383444X - 0.00191378 \\ = & -0.00191378B_{0,8}(X) + 0.00287927B_{1,8}(X) + 0.00767231B_{2,8}(X) \\ & + 0.0124653B_{3,8}(X) + 0.0172583B_{4,8}(X) + 0.0220513B_{5,8}(X) \\ & + 0.0268443B_{6,8}(X) + 0.0316373B_{7,8}(X) + 0.0364302B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

Longest intersection interval: $5.36469{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

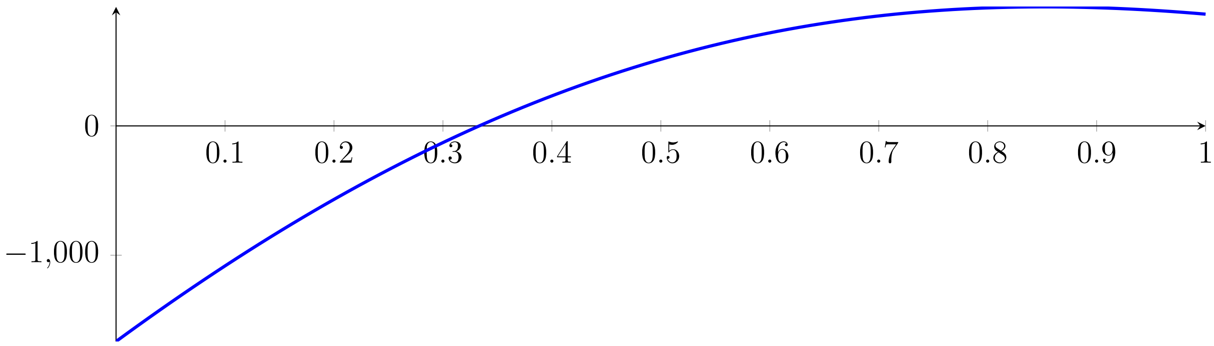
49.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

49.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

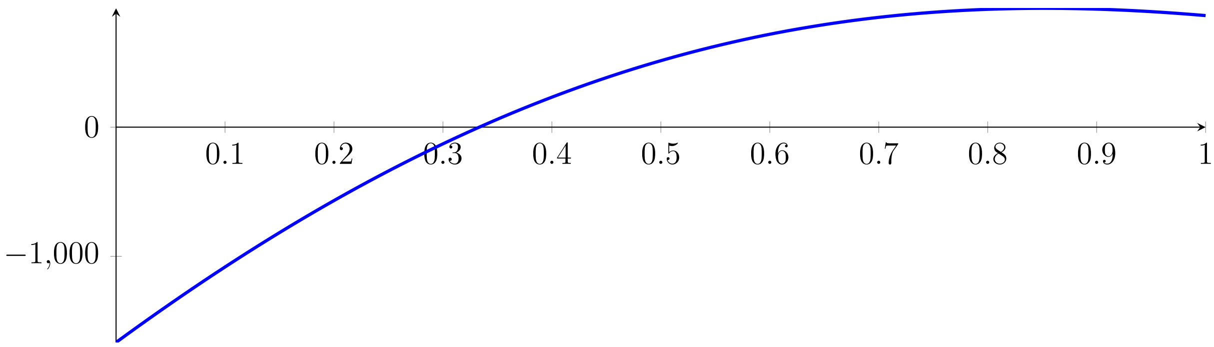
with precision $\varepsilon = 1 \cdot 10^{-08}$.

50 Running QuadClip on f_8 with epsilon 8

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

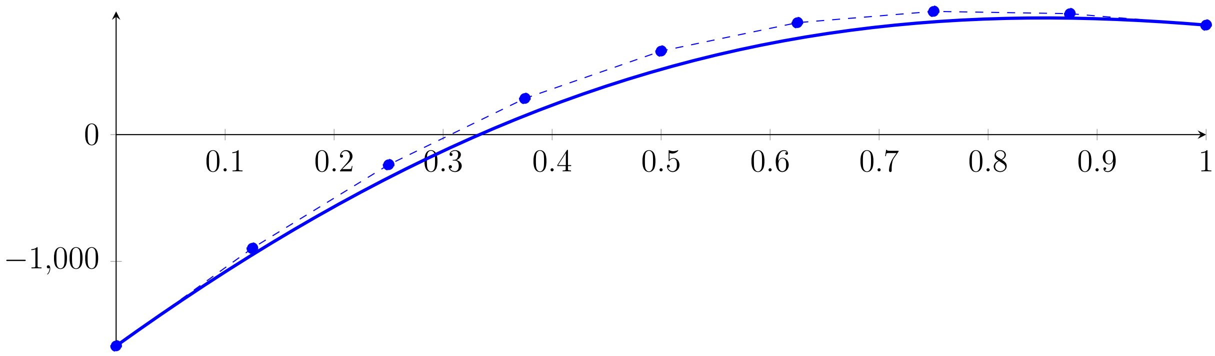
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



50.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

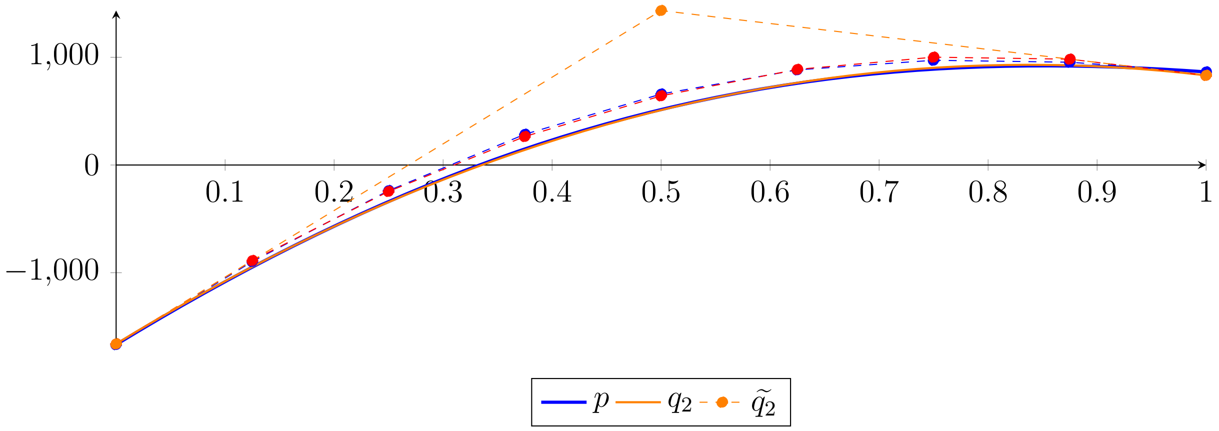
$$\begin{aligned}p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67\\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X)\\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -3695.78X^2 + 6187.64X - 1660\\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2}\end{aligned}$$

$$\begin{aligned}\tilde{q}_2 &= -3.91297 \cdot 10^{-06}X^8 + 1.58774 \cdot 10^{-05}X^7 - 2.63335 \cdot 10^{-05}X^6 + 2.29285 \cdot 10^{-05}X^5\\ &\quad - 1.11931 \cdot 10^{-05}X^4 + 3.0313 \cdot 10^{-06}X^3 - 3695.78X^2 + 6187.64X - 1660\\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8}\\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

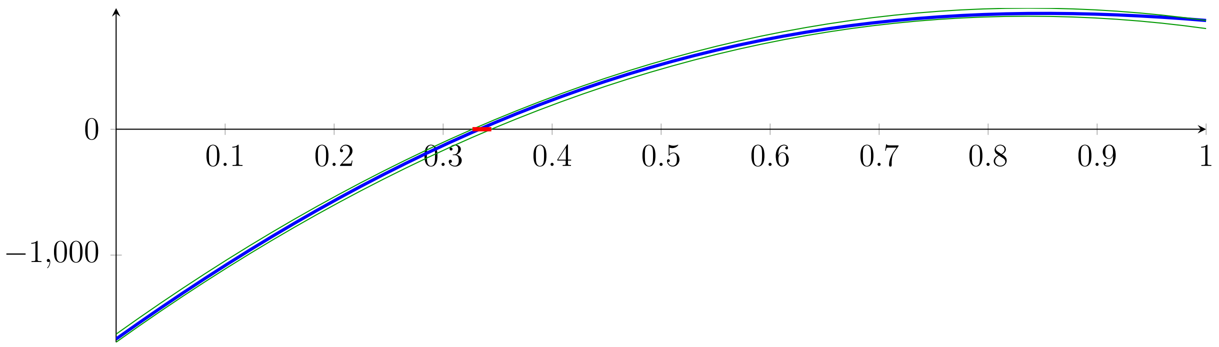
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

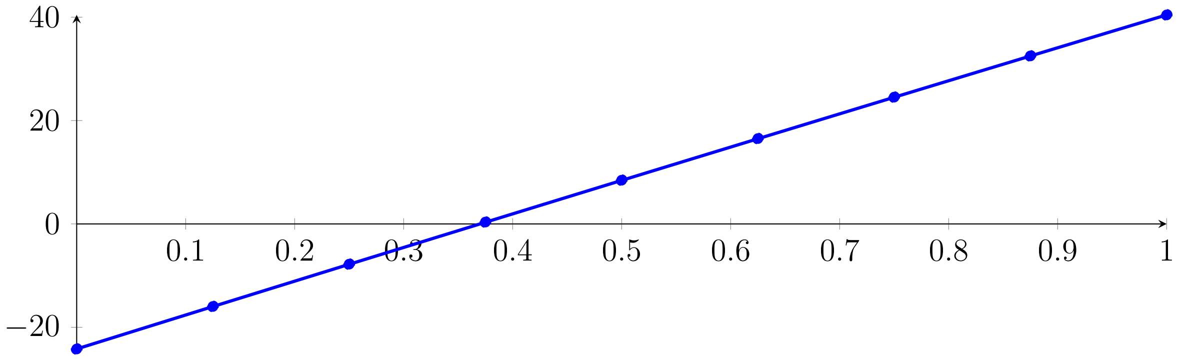
Longest intersection interval: 0.0173372

$\implies$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

50.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

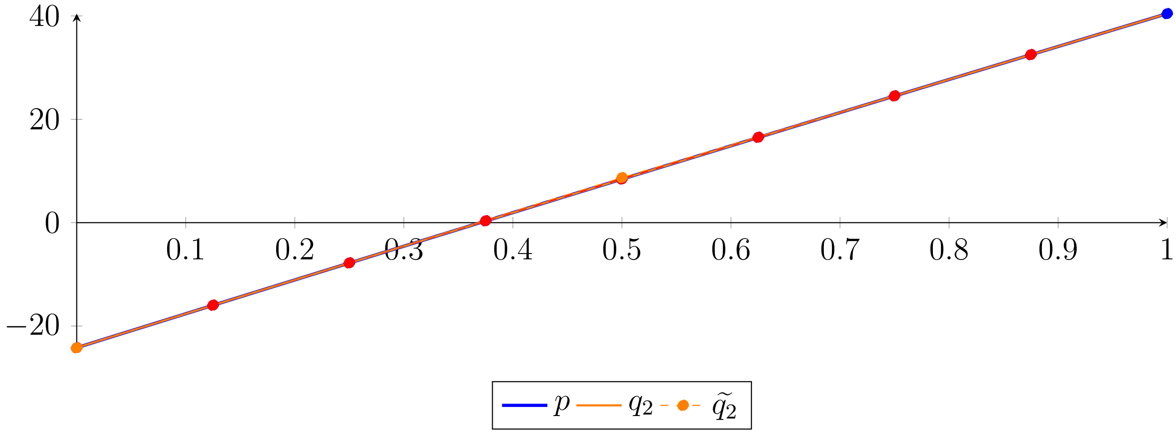
$$\begin{aligned} p = &-5.40012\cdot10^{-13}X^8 - 7.38964\cdot10^{-12}X^7 - 1.94416\cdot10^{-09}X^6 + 1.19265\cdot10^{-07}X^5 \\ &+ 8.12335\cdot10^{-05}X^4 - 0.0012733X^3 - 1.18084X^2 + 65.8155X - 24.1945 \\ = &-24.1945B_{0,8}(X) - 15.9676B_{1,8}(X) - 7.78282B_{2,8}(X) + 0.35975B_{3,8}(X) \\ &+ 8.4601B_{4,8}(X) + 16.5182B_{5,8}(X) + 24.5341B_{6,8}(X) + 32.5077B_{7,8}(X) + 40.4389B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 1.00159\cdot 10^{-08}X^8 - 3.3372\cdot 10^{-08}X^7 + 4.23875\cdot 10^{-08}X^6 - 2.49721\cdot 10^{-08}X^5 \\ &\quad + 6.08793\cdot 10^{-09}X^4 + 1.46429\cdot 10^{-10}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894\cdot 10^{-05}$.
Bounding polynomials M and m:

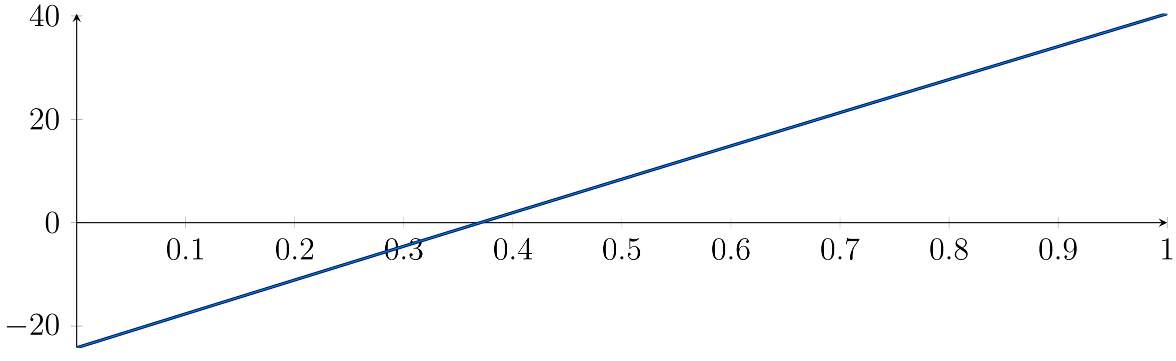
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



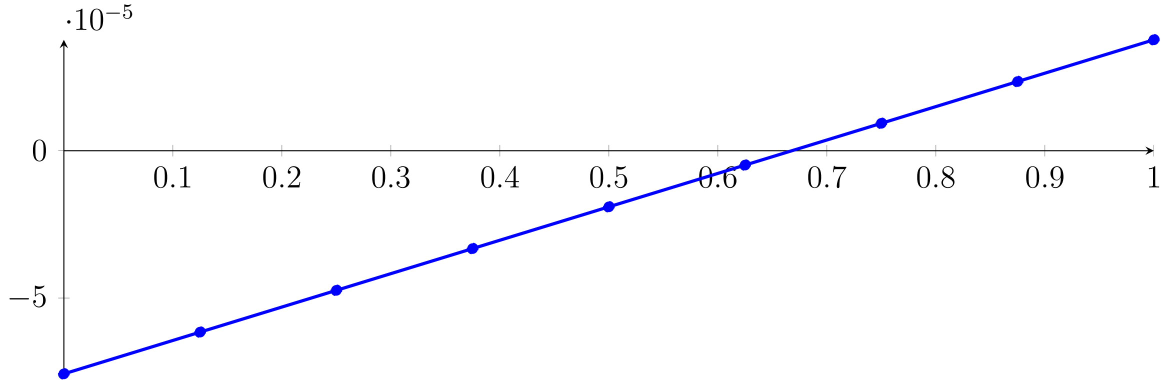
$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588\cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

50.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

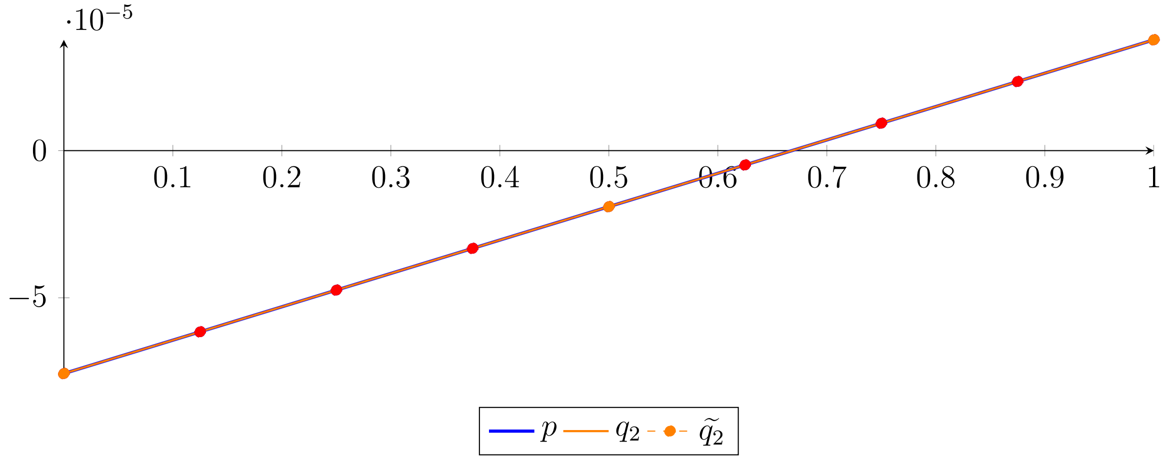
$$\begin{aligned}p = &-2.1684\cdot 10^{-19}X^8 - 4.33681\cdot 10^{-19}X^7 + 2.12504\cdot 10^{-17}X^6 - 1.51788\cdot 10^{-18}X^5 \\ &+ 7.58942\cdot 10^{-18}X^4 - 3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05} \\ = &-7.5732\cdot 10^{-05}B_{0,8}(X) - 6.15596\cdot 10^{-05}B_{1,8}(X) - 4.73873\cdot 10^{-05}B_{2,8}(X) \\ &- 3.32149\cdot 10^{-05}B_{3,8}(X) - 1.90425\cdot 10^{-05}B_{4,8}(X) - 4.87016\cdot 10^{-06}B_{5,8}(X) \\ &+ 9.3022\cdot 10^{-06}B_{6,8}(X) + 2.34746\cdot 10^{-05}B_{7,8}(X) + 3.76469\cdot 10^{-05}B_{8,8}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05} \\ &= -7.5732\cdot 10^{-05}B_{0,2} - 1.90425\cdot 10^{-05}B_{1,2} + 3.76469\cdot 10^{-05}B_{2,2}\end{aligned}$$

$$\begin{aligned}\tilde{q}_2 &= 5.62292\cdot 10^{-14}X^8 - 2.22643\cdot 10^{-13}X^7 + 3.60043\cdot 10^{-13}X^6 - 3.05846\cdot 10^{-13}X^5 + 1.46182 \\ &\cdot 10^{-13}X^4 - 3.90612\cdot 10^{-14}X^3 - 3.59793\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05} \\ &= -7.5732\cdot 10^{-05}B_{0,8} - 6.15596\cdot 10^{-05}B_{1,8} - 4.73873\cdot 10^{-05}B_{2,8} - 3.32149\cdot 10^{-05}B_{3,8} - 1.90425 \\ &\cdot 10^{-05}B_{4,8} - 4.87016\cdot 10^{-06}B_{5,8} + 9.3022\cdot 10^{-06}B_{6,8} + 2.34746\cdot 10^{-05}B_{7,8} + 3.76469\cdot 10^{-05}B_{8,8}\end{aligned}$$



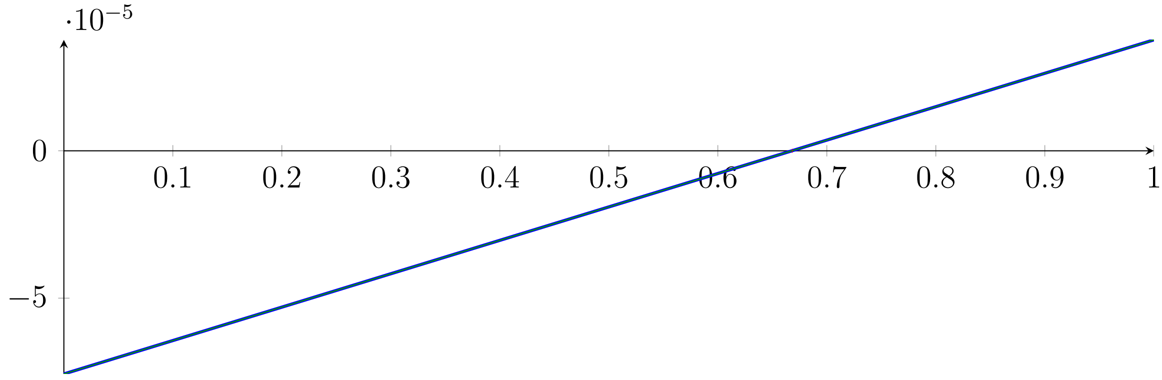
The maximum difference of the Bézier coefficients is $\delta = 2.98887\cdot 10^{-16}$.
Bounding polynomials M and m:

$$M = -3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05}$$
$$m = -3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.667955, 3.14644\cdot 10^{7}\} \qquad N(m) = \{0.667955, 3.14644\cdot 10^{7}\}$$

Intersection intervals:



$[0.667955, 0.667955]$

Longest intersection interval: $1.88052 \cdot 10^{-09}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

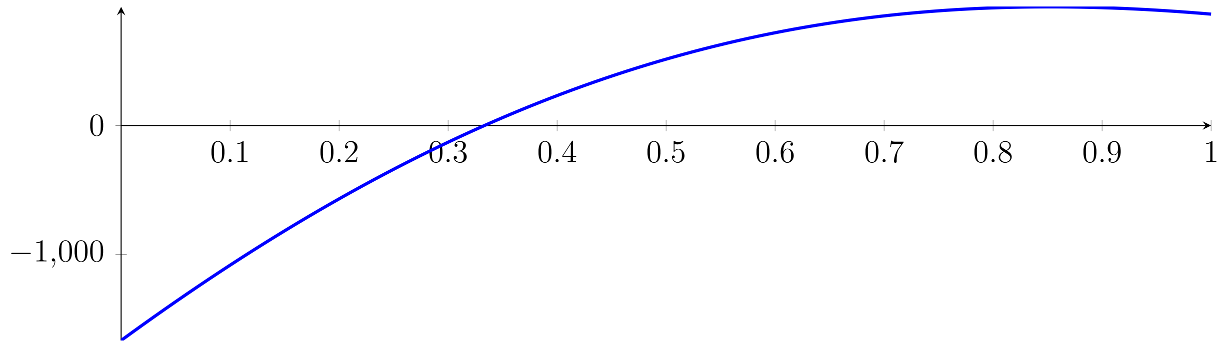
50.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

50.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

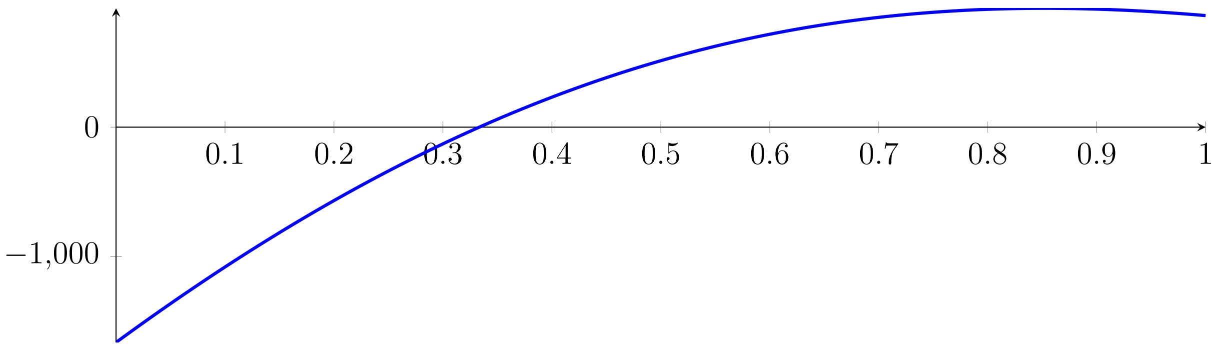
with precision $\varepsilon = 1 \cdot 10^{-08}$.

51 Running CubeClip on f_8 with epsilon 8

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

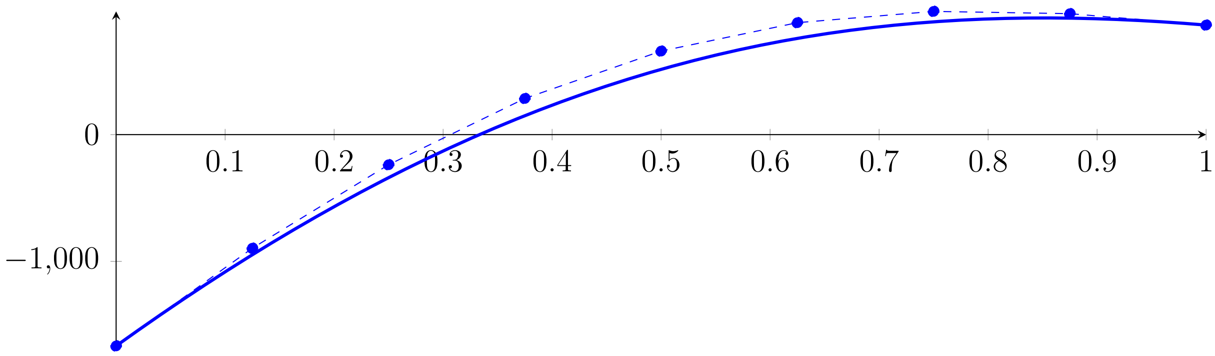
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



51.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

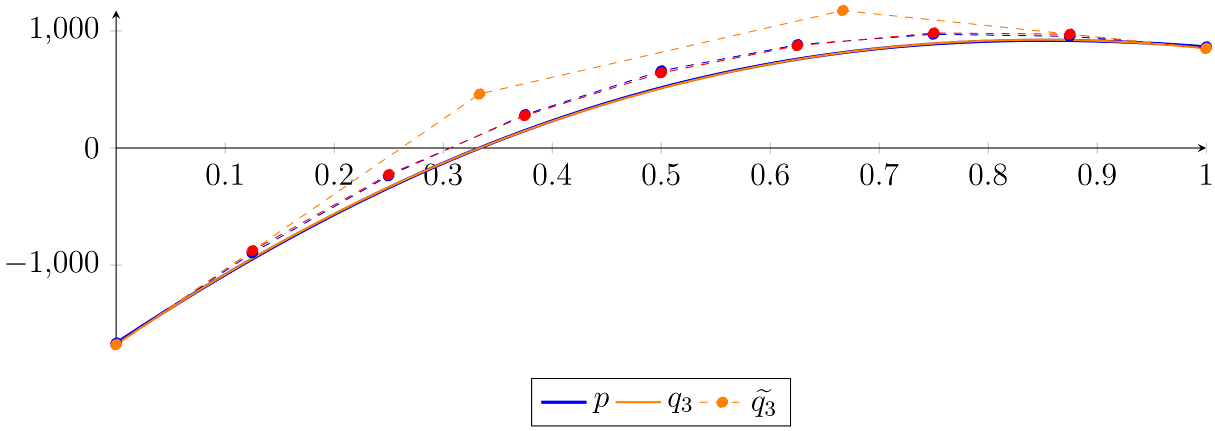
$$\begin{aligned}p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67\\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X)\\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47\\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3}\end{aligned}$$

$$\begin{aligned}\tilde{q}_3 &= 2.38117 \cdot 10^{-06}X^8 - 9.49957 \cdot 10^{-06}X^7 + 1.54757 \cdot 10^{-05}X^6 - 1.32609 \cdot 10^{-05}X^5\\ &\quad + 6.41819 \cdot 10^{-06}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47\\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8}\\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

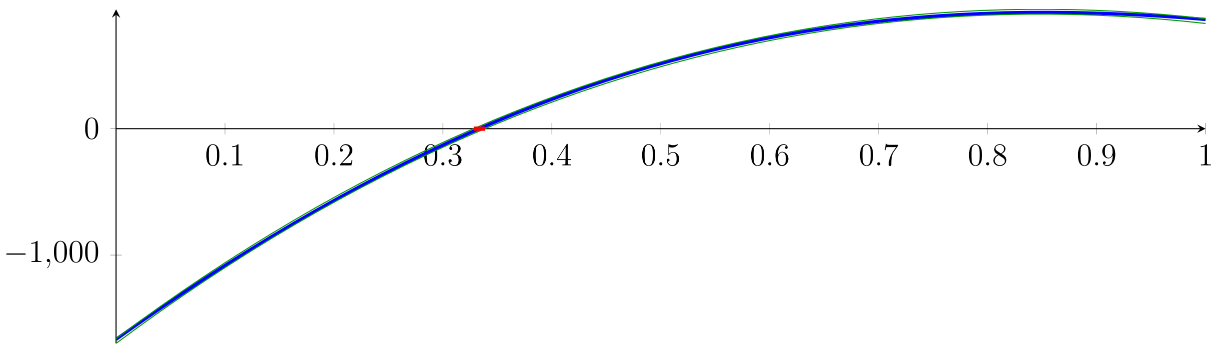
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$[0.328258, 0.338551]$

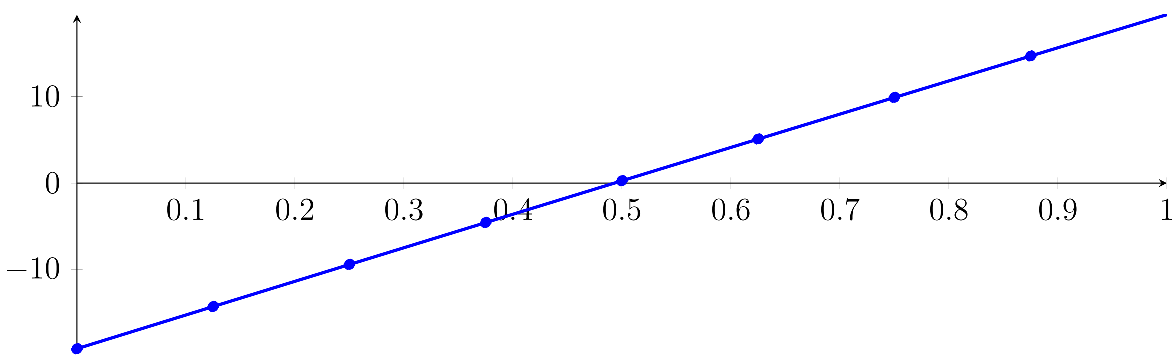
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

51.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

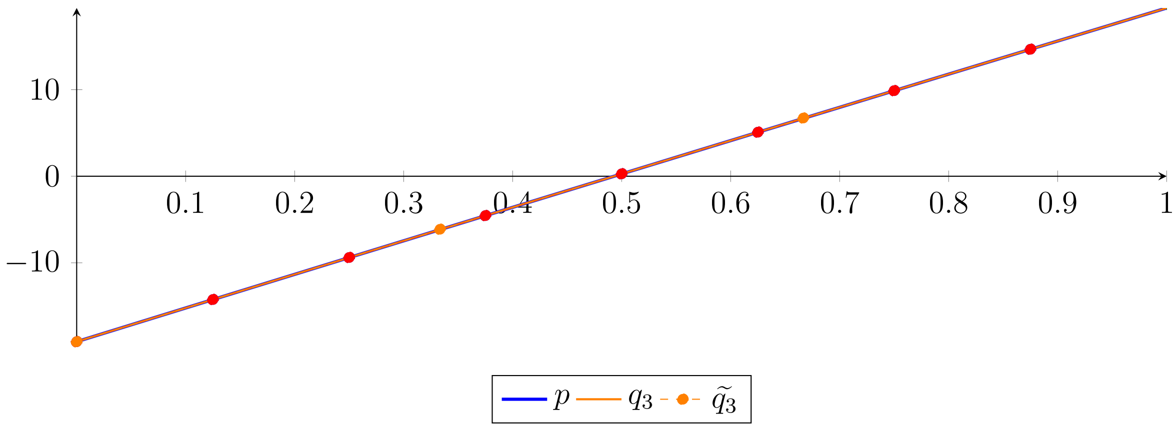
$$p = -1.27898\cdot10^{-13}X^8 - 1.13687\cdot10^{-13}X^7 - 8.23661\cdot10^{-11}X^6 + 8.72882\cdot10^{-09}X^5 + 1.00963\cdot10^{-05}X^4 - 0.000261161X^3 - 0.416284X^2 + 38.9643X - 19.1124$$
$$= -19.1124B_{0,8}(X) - 14.2419B_{1,8}(X) - 9.38619B_{2,8}(X) - 4.54539B_{3,8}(X) + 0.280537B_{4,8}(X) + 5.09158B_{5,8}(X) + 9.88774B_{6,8}(X) + 14.669B_{7,8}(X) + 19.4354B_{8,8}(X)$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 2.3353\cdot 10^{-08}X^8 - 9.31856\cdot 10^{-08}X^7 + 1.51861\cdot 10^{-07}X^6 - 1.30228\cdot 10^{-07}X^5 \\ &\quad + 6.31618\cdot 10^{-08}X^4 - 0.000240962X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16887\cdot 10^{-07}$.

Bounding polynomials M and m:

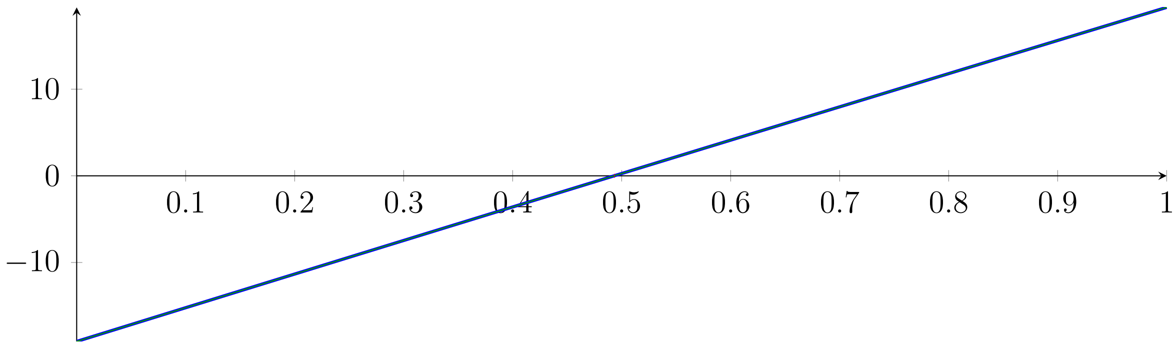
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.12517\cdot 10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

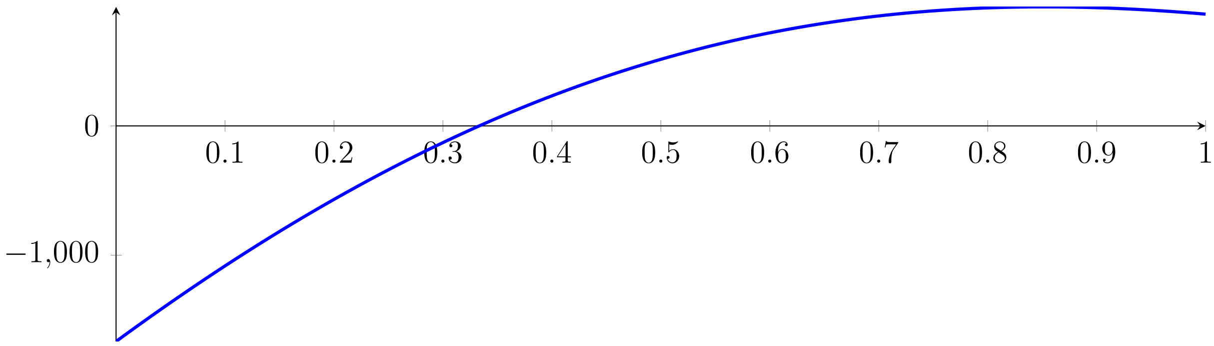
51.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

51.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

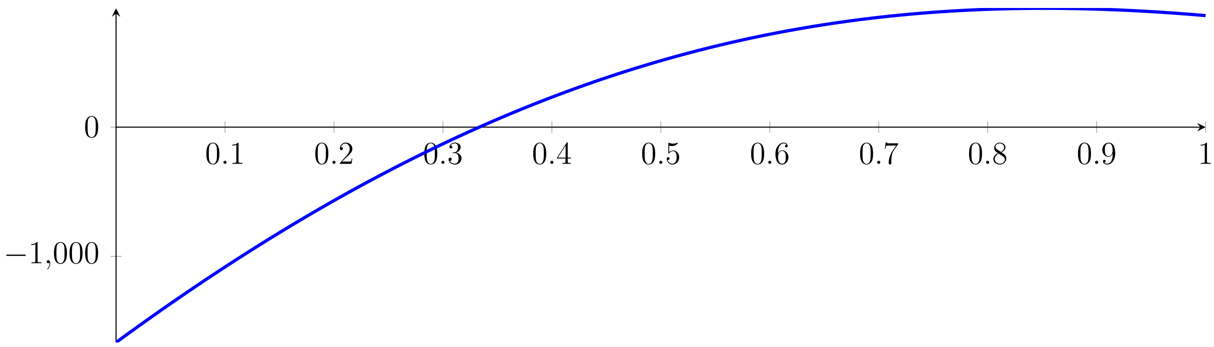
with precision $\varepsilon = 1 \cdot 10^{-08}$.

52 Running `BezClip` on f_8 with epsilon 16

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called `BezClip` with input polynomial on interval $[0,1]$:

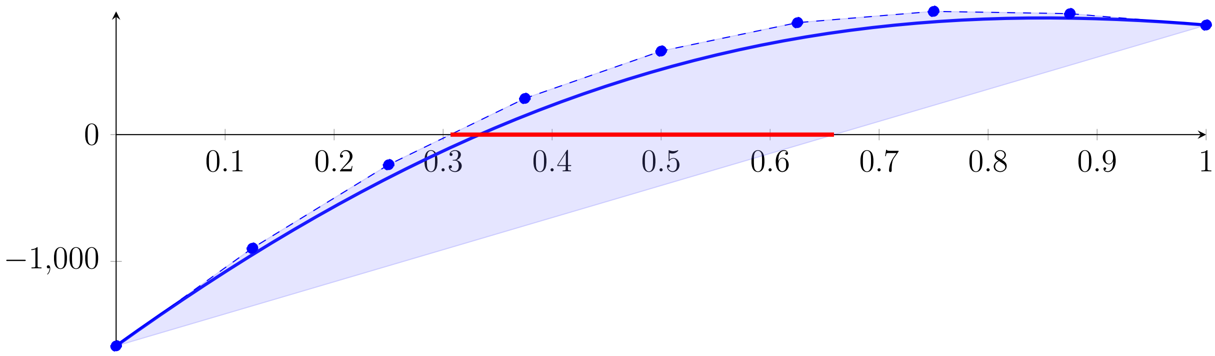
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



52.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

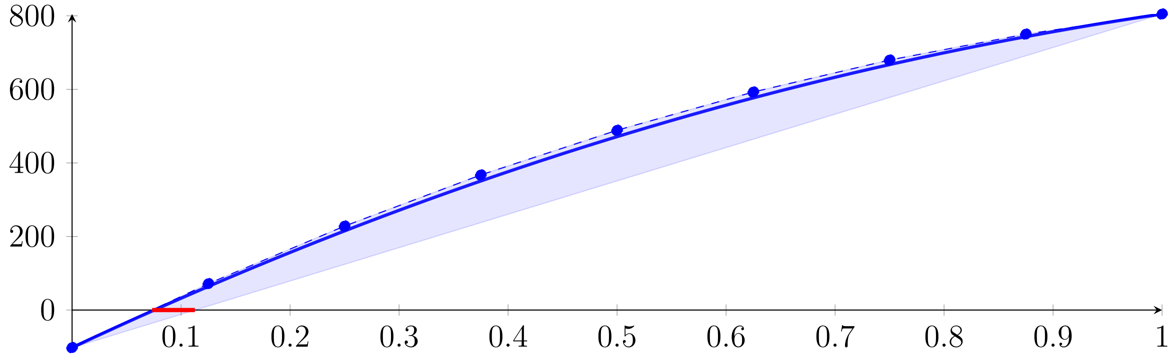
$$[0.306796, 0.658588]$$

Longest intersection interval: 0.351792
$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

52.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\ &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\ &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\ &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

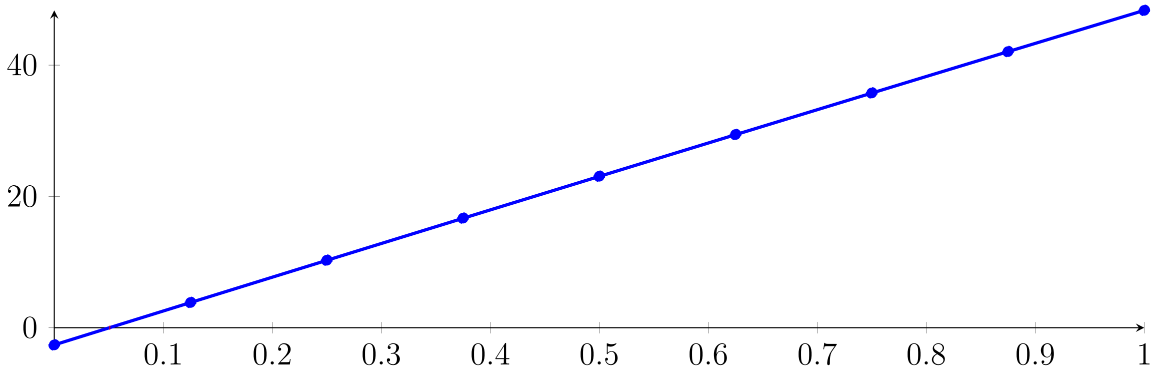
$$[0.0734515, 0.112637]$$

Longest intersection interval: 0.0391855
$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

52.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -5.41789 \cdot 10^{-13}X^8 - 1.26477 \cdot 10^{-12}X^7 - 5.05786 \cdot 10^{-10}X^6 + 3.66765 \cdot 10^{-08}X^5 \\ &\quad + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\ &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\ &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

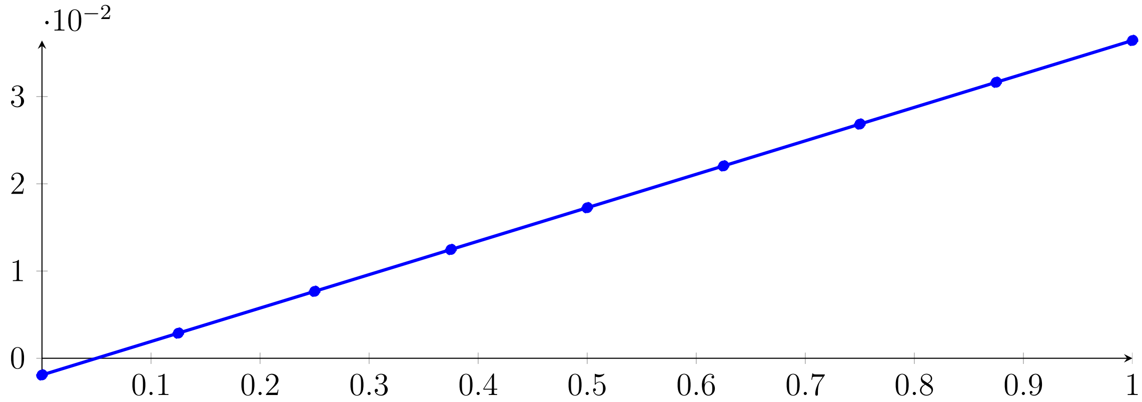
$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

52.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-7.11237\cdot10^{-16}X^8 + 5.27356\cdot10^{-16}X^7 - 7.38298\cdot10^{-15}X^6 + 1.06859\cdot10^{-15}X^5 \\ &- 1.09288\cdot10^{-15}X^4 - 2.37227\cdot10^{-13}X^3 - 4.12146\cdot10^{-07}X^2 + 0.0383444X - 0.00191378 \\ = &-0.00191378B_{0,8}(X) + 0.00287927B_{1,8}(X) + 0.00767231B_{2,8}(X) \\ &+ 0.0124653B_{3,8}(X) + 0.0172583B_{4,8}(X) + 0.0220513B_{5,8}(X) \\ &+ 0.0268443B_{6,8}(X) + 0.0316373B_{7,8}(X) + 0.0364302B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

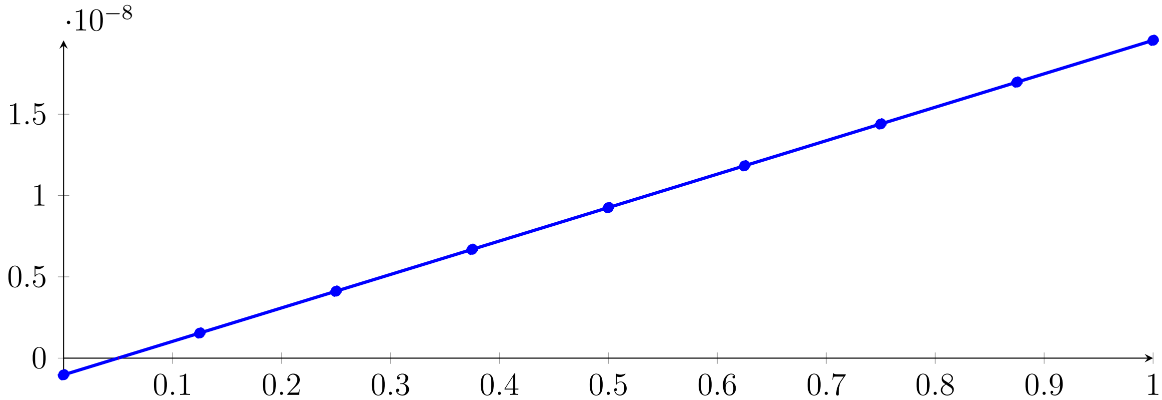
Longest intersection interval: $5.36469\cdot10^{-07}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

52.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-3.09366\cdot10^{-22}X^8 + 4.49986\cdot10^{-22}X^7 - 4.03002\cdot10^{-21}X^6 + 3.70577\cdot10^{-22}X^5 - 3.47416 \\ &\cdot10^{-22}X^4 + 9.26442\cdot10^{-23}X^3 - 1.18608\cdot10^{-19}X^2 + 2.05706\cdot10^{-08}X - 1.02667\cdot10^{-09} \\ = &-1.02667\cdot10^{-09}B_{0,8}(X) + 1.54465\cdot10^{-09}B_{1,8}(X) + 4.11597\cdot10^{-09}B_{2,8}(X) \\ &+ 6.68729\cdot10^{-09}B_{3,8}(X) + 9.25862\cdot10^{-09}B_{4,8}(X) + 1.18299\cdot10^{-08}B_{5,8}(X) \\ &+ 1.44013\cdot10^{-08}B_{6,8}(X) + 1.69726\cdot10^{-08}B_{7,8}(X) + 1.95439\cdot10^{-08}B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87728 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

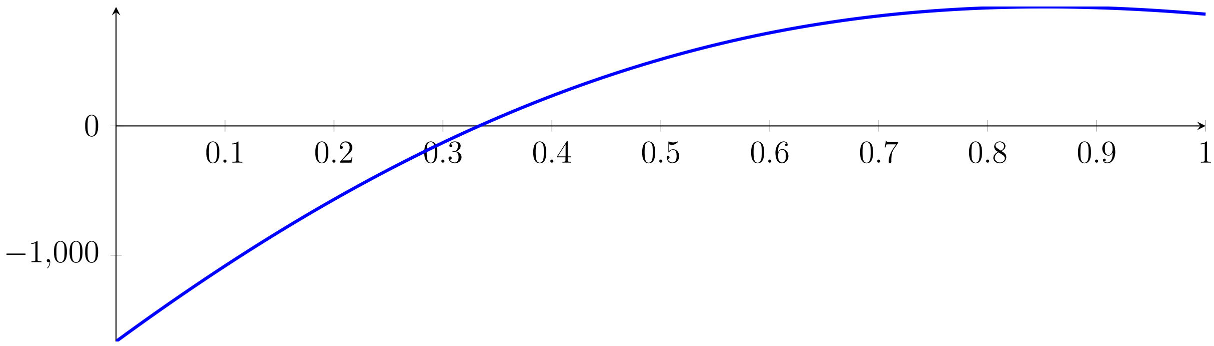
52.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

52.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-16}$.

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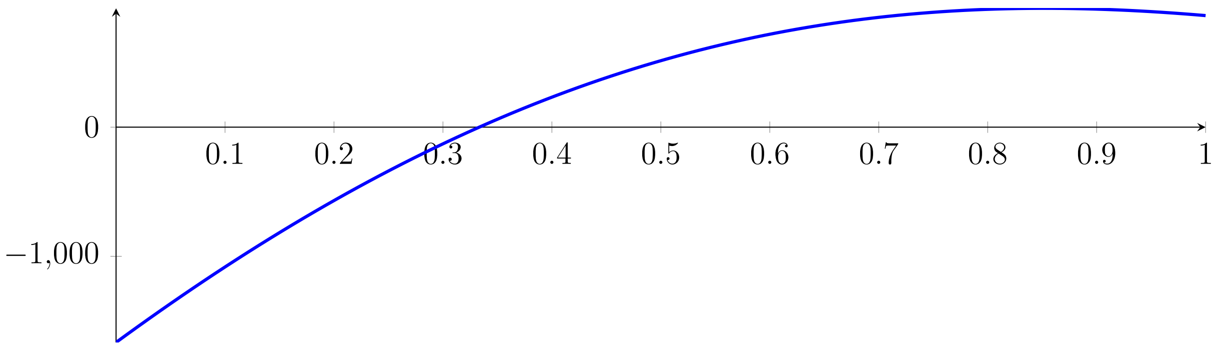
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

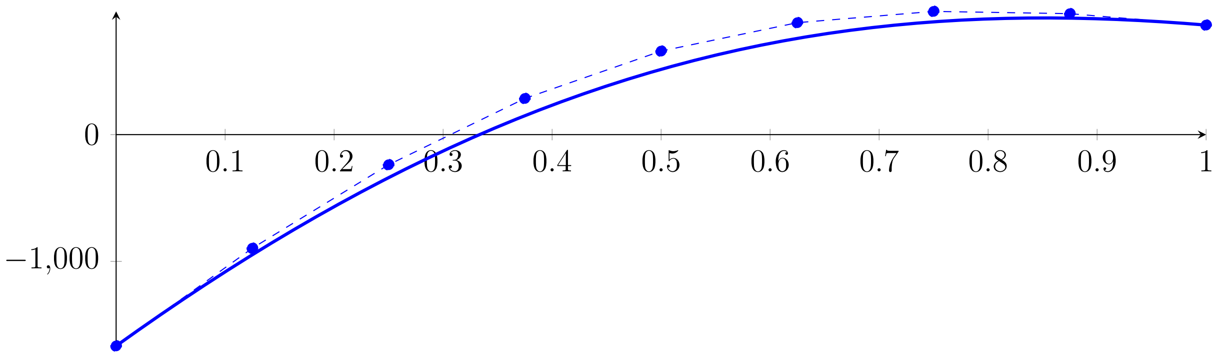
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



53.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

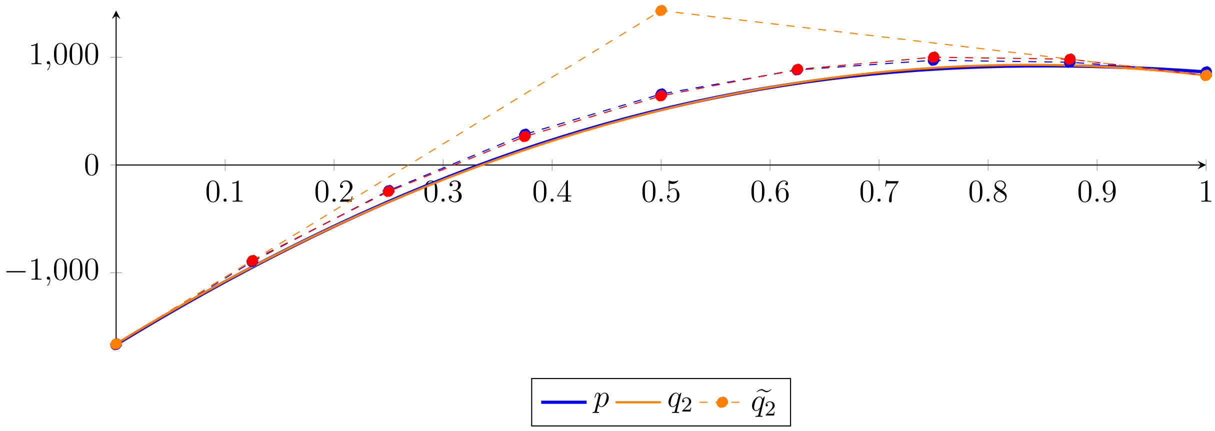
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -3.91297 \cdot 10^{-06}X^8 + 1.58774 \cdot 10^{-05}X^7 - 2.63335 \cdot 10^{-05}X^6 + 2.29285 \cdot 10^{-05}X^5 \\ &\quad - 1.11931 \cdot 10^{-05}X^4 + 3.0313 \cdot 10^{-06}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

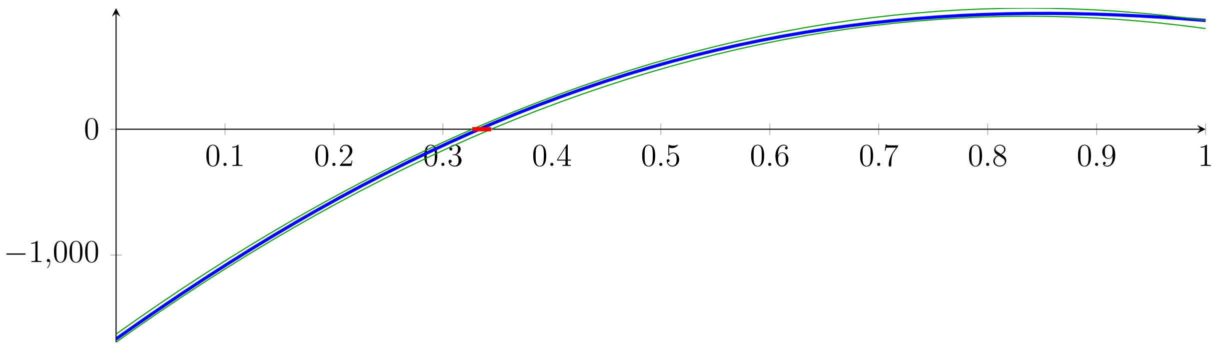
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

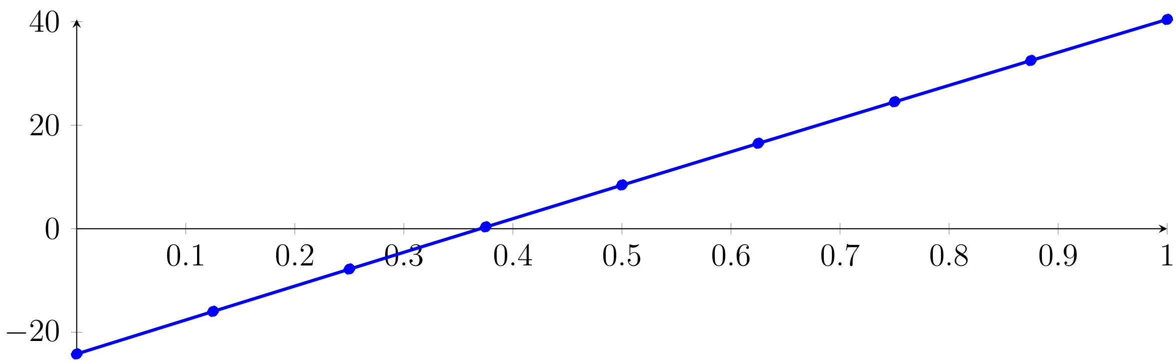
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

53.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

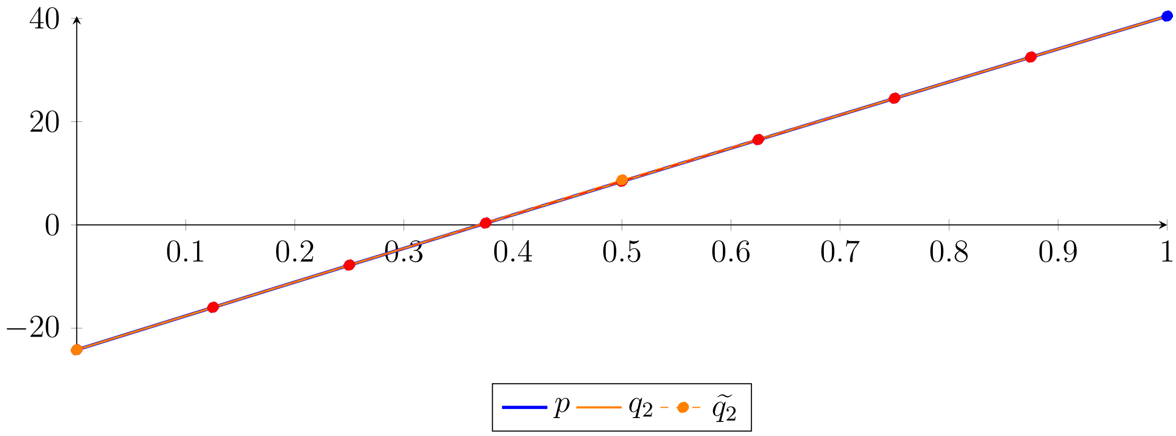
$$\begin{aligned} p = &-5.40012\cdot 10^{-13}X^8 - 7.38964\cdot 10^{-12}X^7 - 1.94416\cdot 10^{-09}X^6 + 1.19265\cdot 10^{-07}X^5 \\ &+ 8.12335\cdot 10^{-05}X^4 - 0.0012733X^3 - 1.18084X^2 + 65.8155X - 24.1945 \\ = &-24.1945B_{0,8}(X) - 15.9676B_{1,8}(X) - 7.78282B_{2,8}(X) + 0.35975B_{3,8}(X) \\ &+ 8.4601B_{4,8}(X) + 16.5182B_{5,8}(X) + 24.5341B_{6,8}(X) + 32.5077B_{7,8}(X) + 40.4389B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 1.00159{\cdot}10^{-08}X^8 - 3.3372{\cdot}10^{-08}X^7 + 4.23875{\cdot}10^{-08}X^6 - 2.49721{\cdot}10^{-08}X^5 \\ &\quad + 6.08793{\cdot}10^{-09}X^4 + 1.46429{\cdot}10^{-10}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894{\cdot}10^{-05}$.
Bounding polynomials M and m:

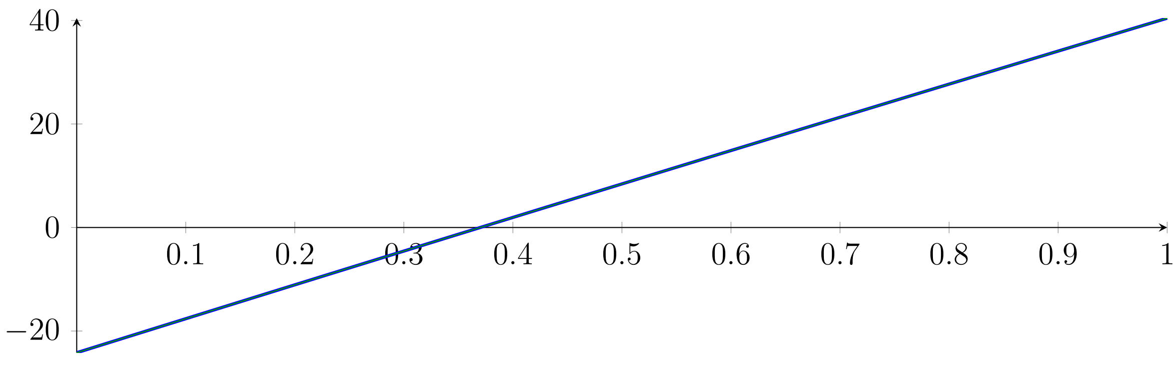
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



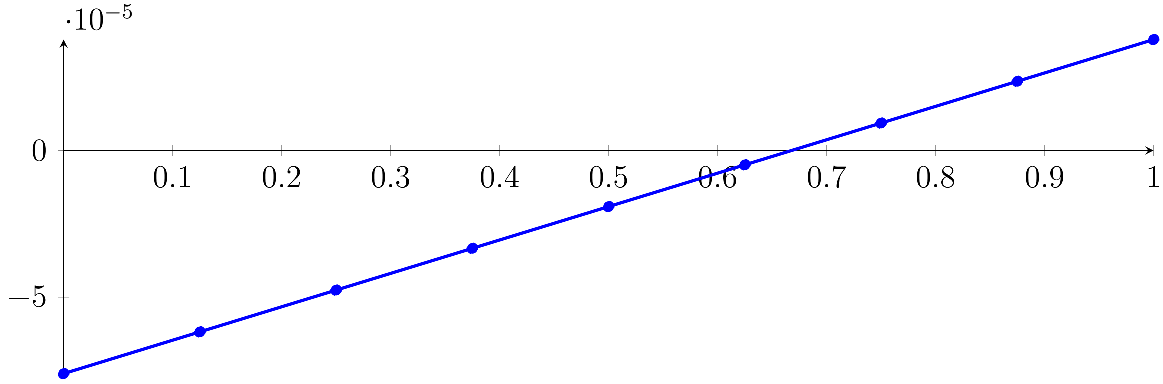
$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588{\cdot}10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

53.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

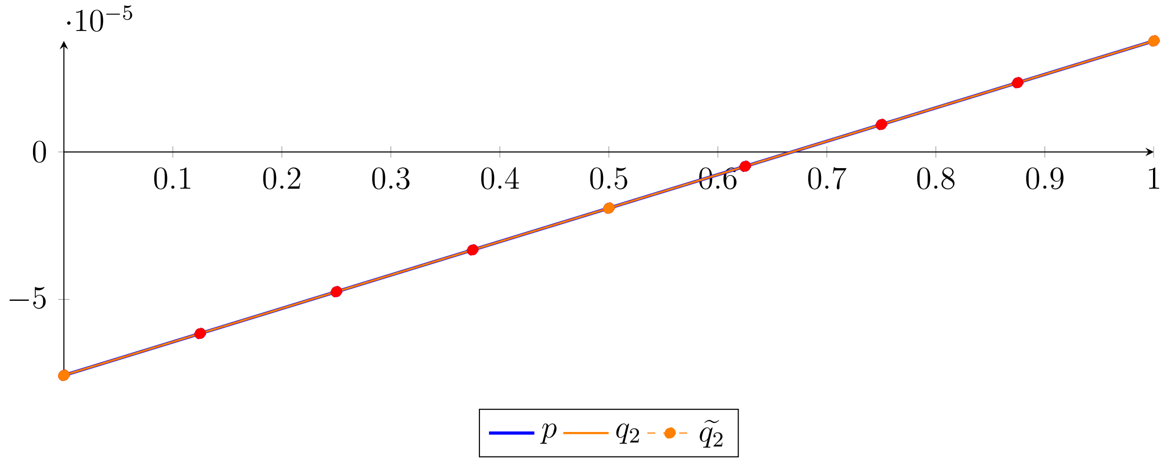
$$p = -2.1684\cdot10^{-19}X^8 - 4.33681\cdot10^{-19}X^7 + 2.12504\cdot10^{-17}X^6 - 1.51788\cdot10^{-18}X^5 + 7.58942\cdot10^{-18}X^4 - 3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$= -7.5732\cdot10^{-05}B_{0,8}(X) - 6.15596\cdot10^{-05}B_{1,8}(X) - 4.73873\cdot10^{-05}B_{2,8}(X) - 3.32149\cdot10^{-05}B_{3,8}(X) - 1.90425\cdot10^{-05}B_{4,8}(X) - 4.87016\cdot10^{-06}B_{5,8}(X) + 9.3022\cdot10^{-06}B_{6,8}(X) + 2.34746\cdot10^{-05}B_{7,8}(X) + 3.76469\cdot10^{-05}B_{8,8}(X)$$



Degree reduction and raising:

$$q_2 = -3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$= -7.5732\cdot10^{-05}B_{0,2} - 1.90425\cdot10^{-05}B_{1,2} + 3.76469\cdot10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = 5.62292\cdot10^{-14}X^8 - 2.22643\cdot10^{-13}X^7 + 3.60043\cdot10^{-13}X^6 - 3.05846\cdot10^{-13}X^5 + 1.46182\cdot10^{-13}X^4 - 3.90612\cdot10^{-14}X^3 - 3.59793\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$= -7.5732\cdot10^{-05}B_{0,8} - 6.15596\cdot10^{-05}B_{1,8} - 4.73873\cdot10^{-05}B_{2,8} - 3.32149\cdot10^{-05}B_{3,8} - 1.90425\cdot10^{-05}B_{4,8} - 4.87016\cdot10^{-06}B_{5,8} + 9.3022\cdot10^{-06}B_{6,8} + 2.34746\cdot10^{-05}B_{7,8} + 3.76469\cdot10^{-05}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.98887\cdot10^{-16}$.

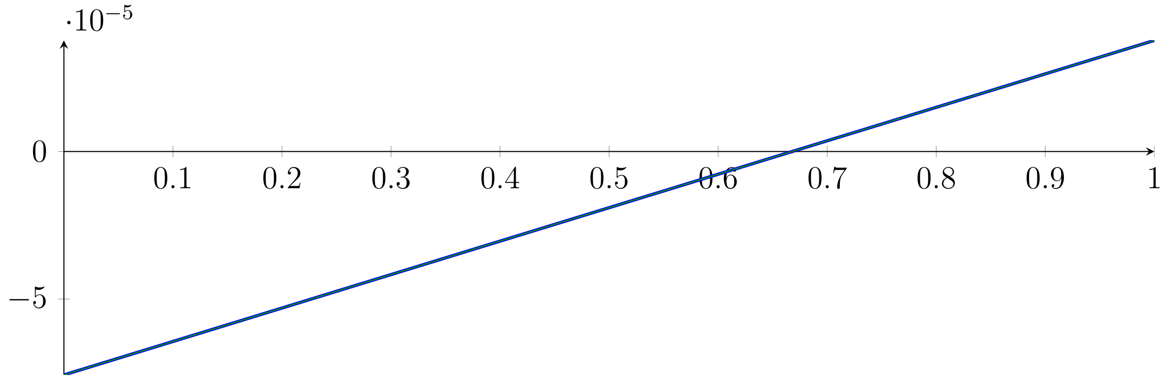
Bounding polynomials M and m:

$$M = -3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$m = -3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$

Root of M and m:

$N(M) = \{0.667955, 3.14644\cdot10^{7}\}$ $\quad$ $N(m) = \{0.667955, 3.14644\cdot10^{7}\}$

Intersection intervals:



$[0.667955, 0.667955]$

Longest intersection interval: $1.88052 \cdot 10^{-09}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

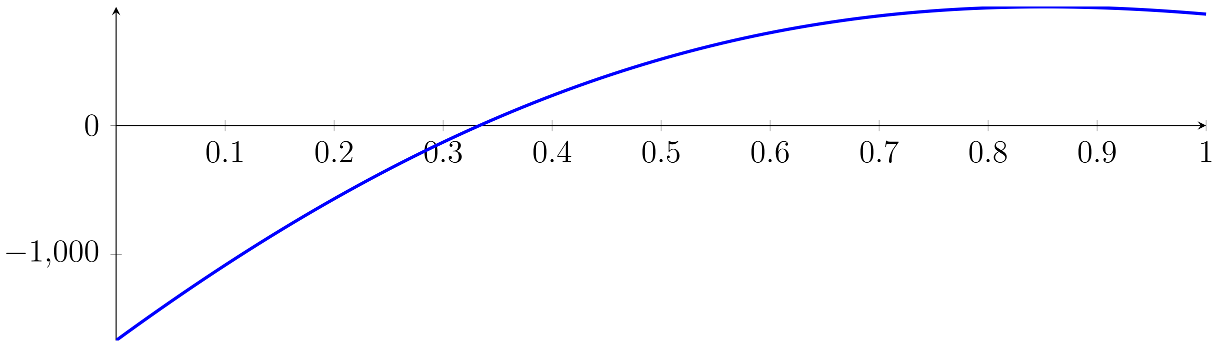
53.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 4!
p(0) = 4.52469e-14 - p(1) 2.58458e-13

53.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

with precision $\varepsilon = 1 \cdot 10^{-16}$.

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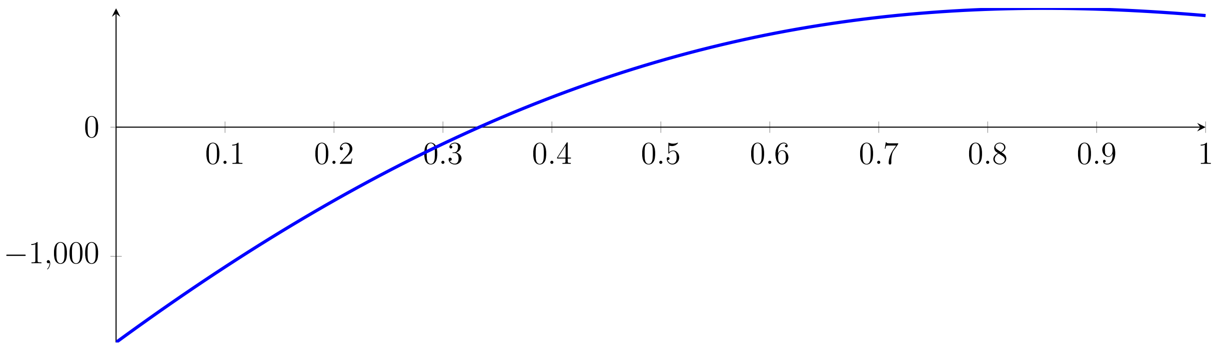
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

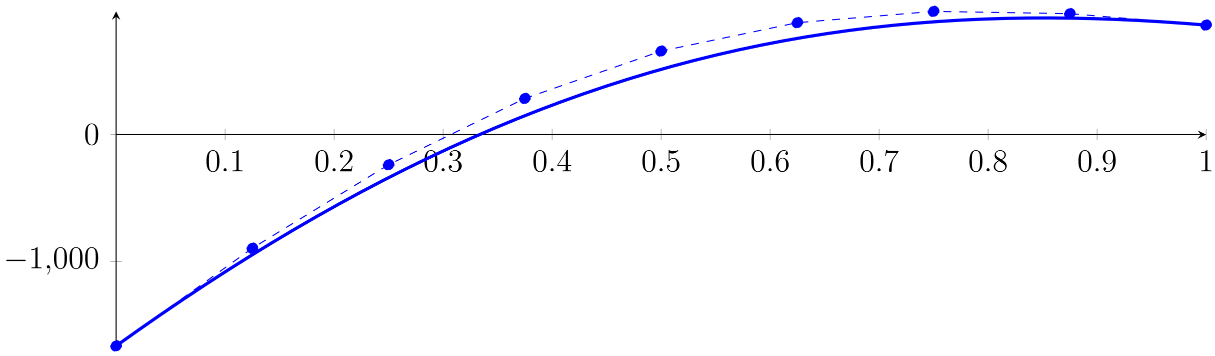
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



54.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

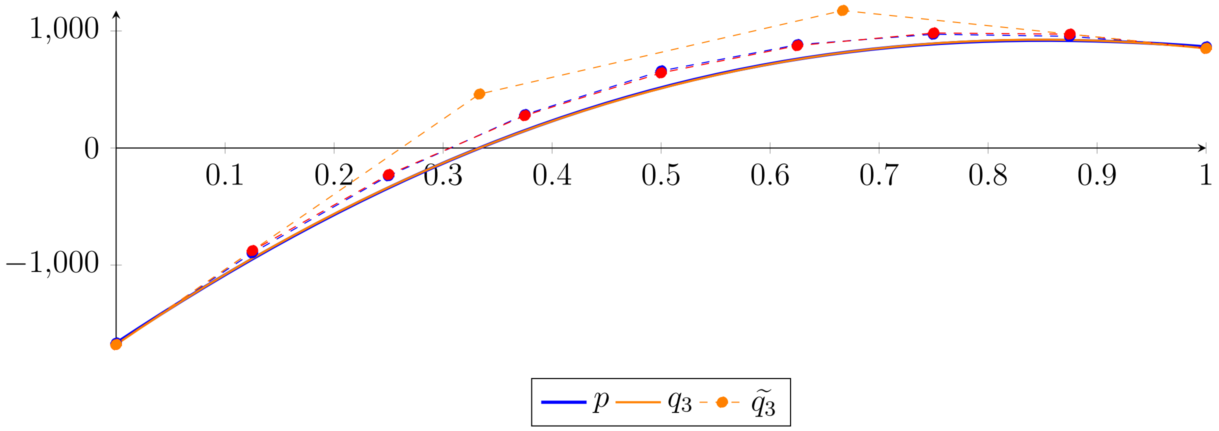
$$\begin{aligned}p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67\\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X)\\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47\\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= 2.38117{\cdot}10^{-06}X^8 - 9.49957{\cdot}10^{-06}X^7 + 1.54757{\cdot}10^{-05}X^6 - 1.32609{\cdot}10^{-05}X^5\\ &\quad + 6.41819{\cdot}10^{-06}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47\\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8}\\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

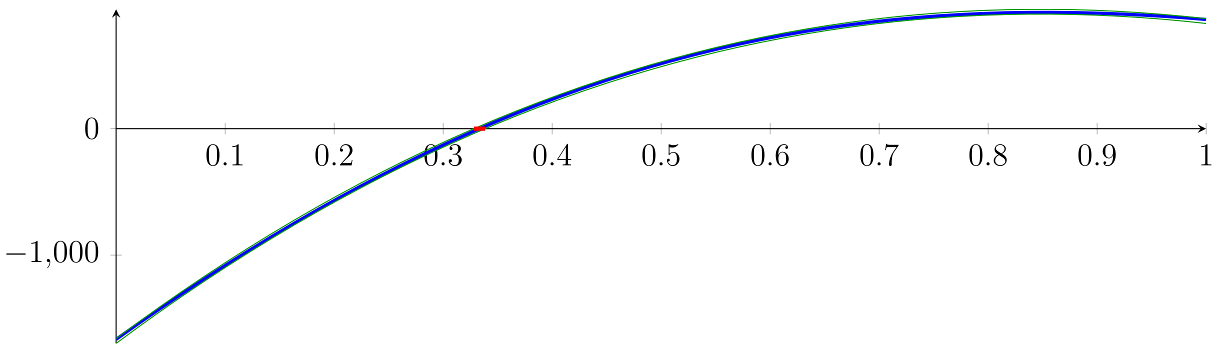
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$[0.328258, 0.338551]$

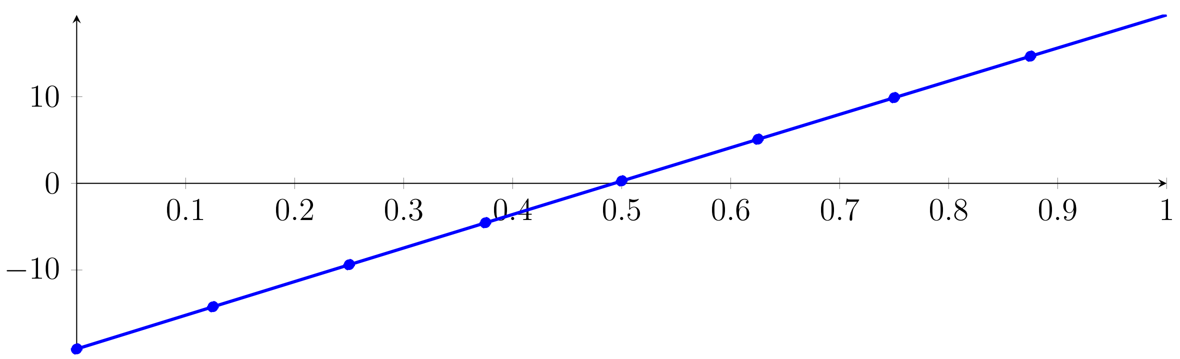
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

54.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

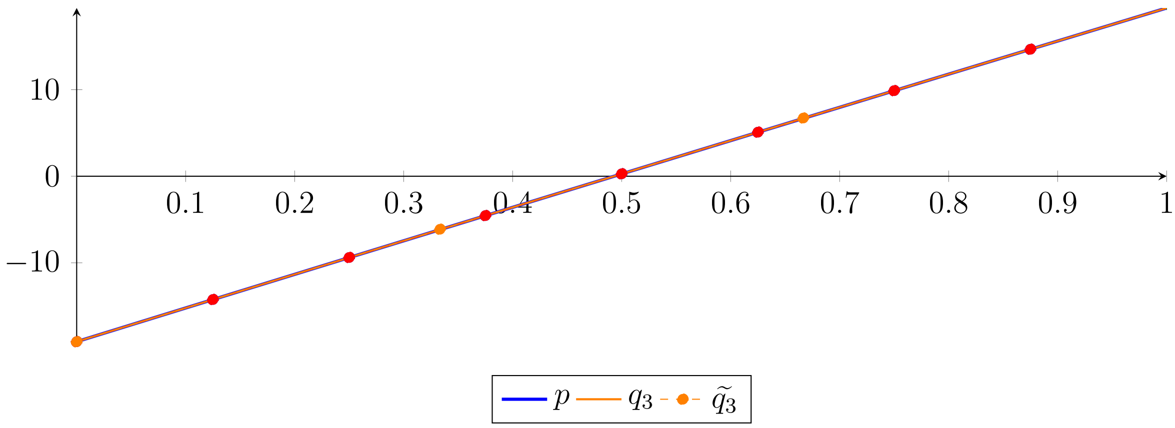
$$\begin{aligned} p &= -1.27898\cdot10^{-13}X^8 - 1.13687\cdot10^{-13}X^7 - 8.23661\cdot10^{-11}X^6 + 8.72882\cdot10^{-09}X^5 \\ &\quad + 1.00963\cdot10^{-05}X^4 - 0.000261161X^3 - 0.416284X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8}(X) - 14.2419B_{1,8}(X) - 9.38619B_{2,8}(X) - 4.54539B_{3,8}(X) \\ &\quad + 0.280537B_{4,8}(X) + 5.09158B_{5,8}(X) + 9.88774B_{6,8}(X) + 14.669B_{7,8}(X) + 19.4354B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2.3353\cdot 10^{-08}X^8 - 9.31856\cdot 10^{-08}X^7 + 1.51861\cdot 10^{-07}X^6 - 1.30228\cdot 10^{-07}X^5 \\ &\quad + 6.31618\cdot 10^{-08}X^4 - 0.000240962X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16887\cdot 10^{-07}$.

Bounding polynomials M and m:

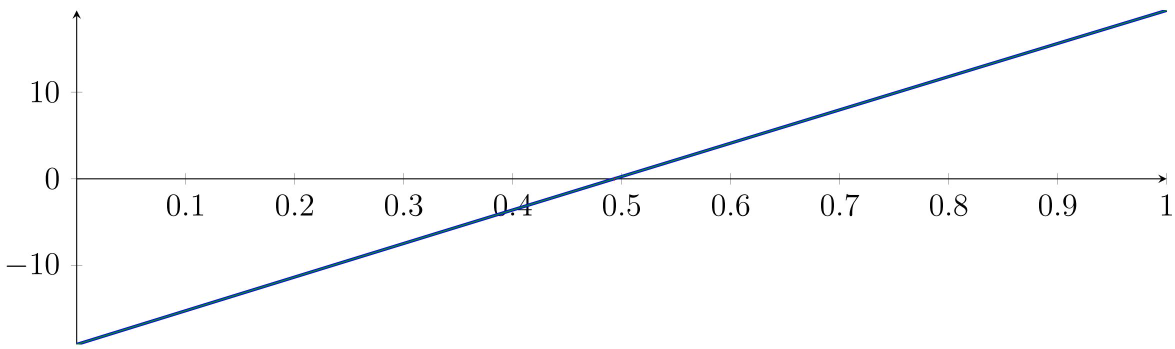
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

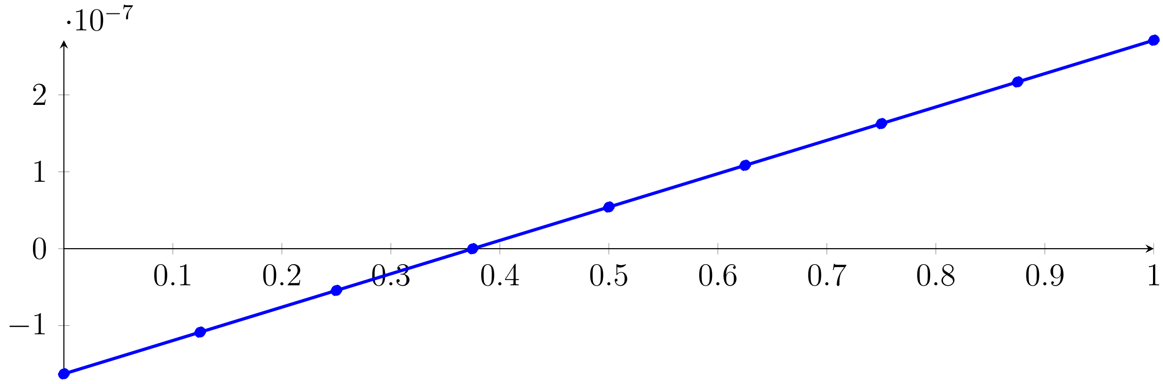
Longest intersection interval: $1.12517\cdot 10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

54.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

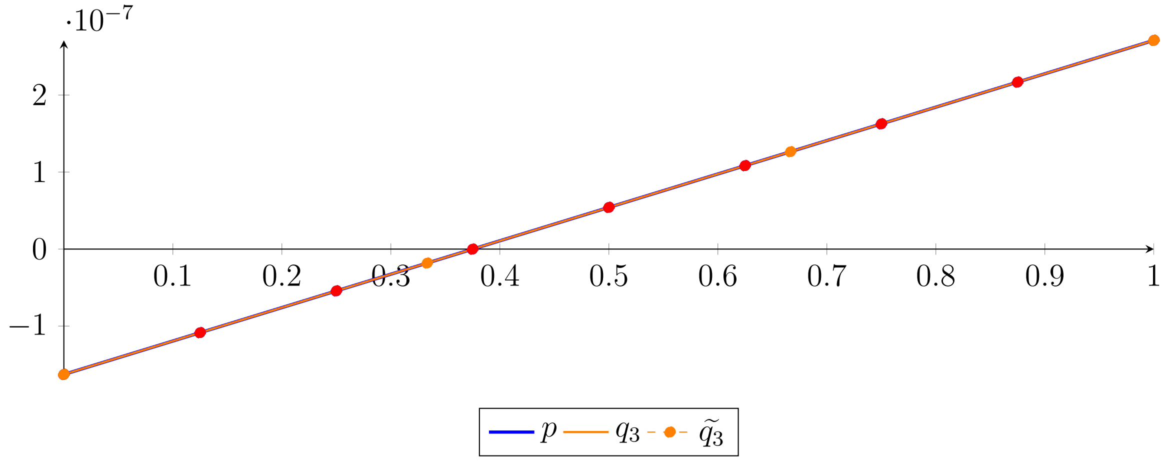
$$p = -3.81165{\cdot}10^{-21}X^8 + 6.77626{\cdot}10^{-21}X^7 + 2.96462{\cdot}10^{-21}X^6 + 5.92923{\cdot}10^{-21}X^5 + 1.11173 {\cdot}10^{-20}X^4 + 1.48231{\cdot}10^{-21}X^3 - 5.27494{\cdot}10^{-17}X^2 + 4.33793{\cdot}10^{-07}X - 1.6278{\cdot}10^{-07}$$
$$= -1.6278{\cdot}10^{-07}B_{0,8}(X) - 1.08555{\cdot}10^{-07}B_{1,8}(X) - 5.43313{\cdot}10^{-08}B_{2,8}(X) - 1.07093{\cdot}10^{-10}B_{3,8}(X) + 5.41171{\cdot}10^{-08}B_{4,8}(X) + 1.08341{\cdot}10^{-07}B_{5,8}(X) + 1.62565{\cdot}10^{-07}B_{6,8}(X) + 2.1679{\cdot}10^{-07}B_{7,8}(X) + 2.71014{\cdot}10^{-07}B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = -1.90582{\cdot}10^{-21}X^3 - 5.2746{\cdot}10^{-17}X^2 + 4.33793{\cdot}10^{-07}X - 1.6278{\cdot}10^{-07}$$
$$= -1.6278{\cdot}10^{-07}B_{0,3} - 1.81818{\cdot}10^{-08}B_{1,3} + 1.26416{\cdot}10^{-07}B_{2,3} + 2.71014{\cdot}10^{-07}B_{3,3}$$

$$\widetilde{q}_3 = 1.50585{\cdot}10^{-16}X^8 - 5.82707{\cdot}10^{-16}X^7 + 9.15943{\cdot}10^{-16}X^6 - 7.54824{\cdot}10^{-16}X^5 + 3.52096 {\cdot}10^{-16}X^4 - 9.31289{\cdot}10^{-17}X^3 - 3.98474{\cdot}10^{-17}X^2 + 4.33793{\cdot}10^{-07}X - 1.6278{\cdot}10^{-07}$$
$$= -1.6278{\cdot}10^{-07}B_{0,8} - 1.08555{\cdot}10^{-07}B_{1,8} - 5.43313{\cdot}10^{-08}B_{2,8} - 1.07093{\cdot}10^{-10}B_{3,8} + 5.41171 {\cdot}10^{-08}B_{4,8} + 1.08341{\cdot}10^{-07}B_{5,8} + 1.62565{\cdot}10^{-07}B_{6,8} + 2.1679{\cdot}10^{-07}B_{7,8} + 2.71014{\cdot}10^{-07}B_{8,8}$$



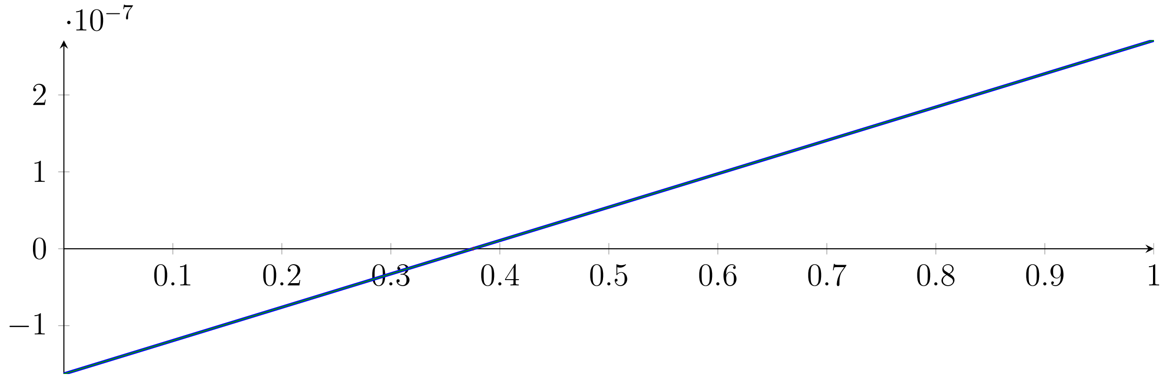
The maximum difference of the Bézier coefficients is $\delta = 8.66435{\cdot}10^{-19}$.
Bounding polynomials M and m:

$$M = -1.85288{\cdot}10^{-21}X^3 - 5.2746{\cdot}10^{-17}X^2 + 4.33793{\cdot}10^{-07}X - 1.6278{\cdot}10^{-07}$$
$$m = -1.90582{\cdot}10^{-21}X^3 - 5.2746{\cdot}10^{-17}X^2 + 4.33793{\cdot}10^{-07}X - 1.6278{\cdot}10^{-07}$$

Root of M and m:

$$N(M) = \{-1.49969{\cdot}10^{7}, 0.375247, 1.49696{\cdot}10^{7}\} \quad N(m) = \{-1.46018{\cdot}10^{7}, 0.375247, 1.45759{\cdot}10^{7}\}$$

Intersection intervals:



$[0.375247, 0.375247]$

Longest intersection interval: $7.69251 \cdot 10^{-09}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

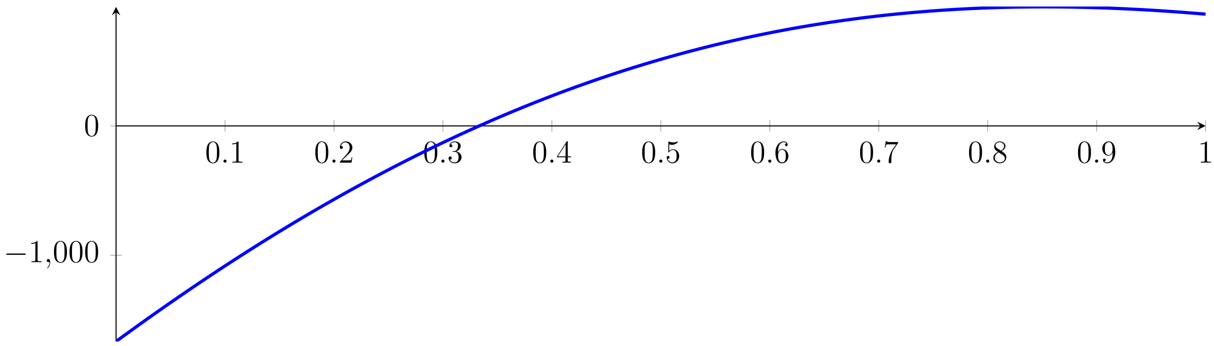
54.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

54.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-16}$.

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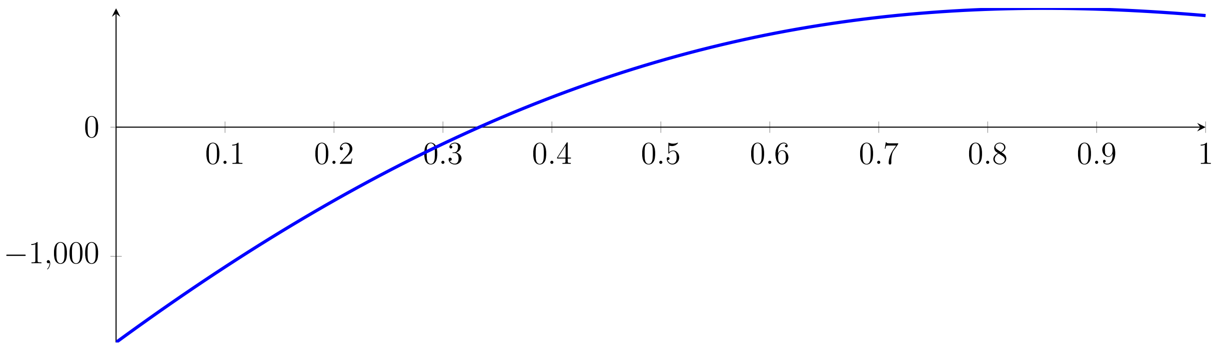
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

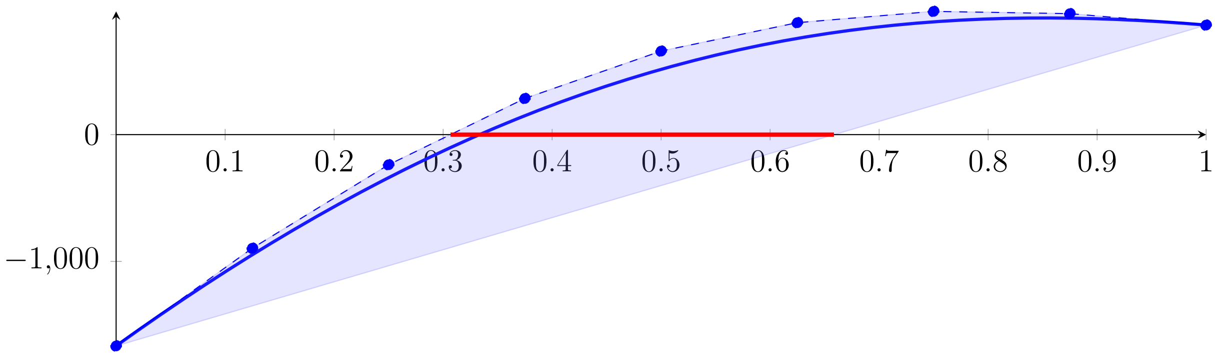
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



55.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

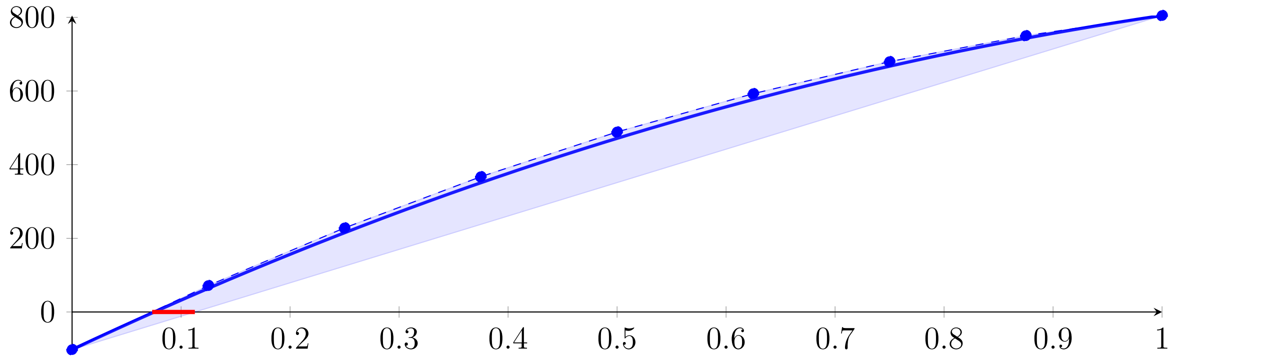
$$[0.306796, 0.658588]$$

Longest intersection interval: 0.351792
$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

55.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\ &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\ &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\ &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

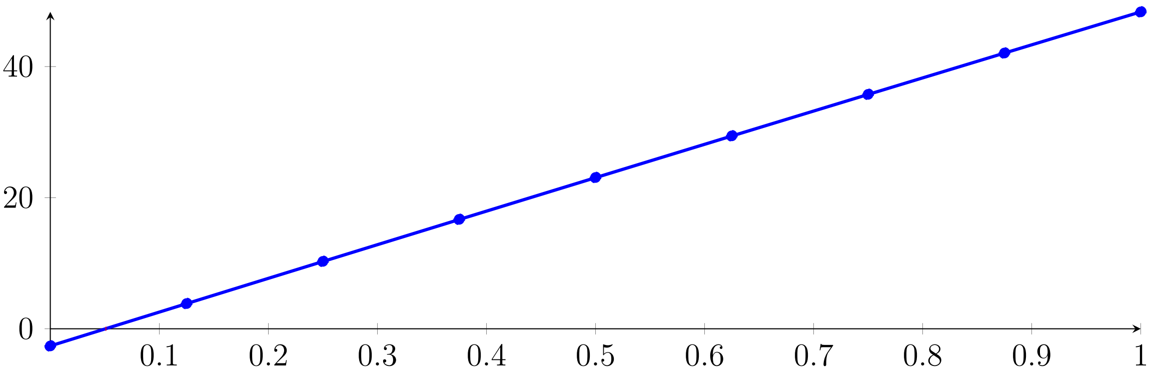
Longest intersection interval: 0.0391855

$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

55.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.41789 \cdot 10^{-13}X^8 - 1.26477 \cdot 10^{-12}X^7 - 5.05786 \cdot 10^{-10}X^6 + 3.66765 \cdot 10^{-08}X^5 \\ &\quad + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\ &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\ &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

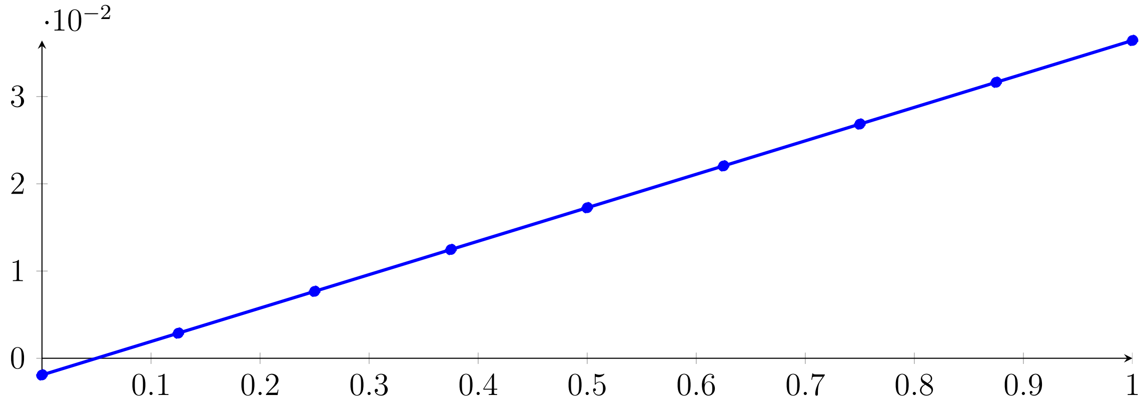
Longest intersection interval: 0.000742589

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

55.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -7.11237\cdot 10^{-16}X^8 + 5.27356\cdot 10^{-16}X^7 - 7.38298\cdot 10^{-15}X^6 + 1.06859\cdot 10^{-15}X^5 \\ &\quad - 1.09288\cdot 10^{-15}X^4 - 2.37227\cdot 10^{-13}X^3 - 4.12146\cdot 10^{-07}X^2 + 0.0383444X - 0.00191378 \\ &= -0.00191378B_{0,8}(X) + 0.00287927B_{1,8}(X) + 0.00767231B_{2,8}(X) \\ &\quad + 0.0124653B_{3,8}(X) + 0.0172583B_{4,8}(X) + 0.0220513B_{5,8}(X) \\ &\quad + 0.0268443B_{6,8}(X) + 0.0316373B_{7,8}(X) + 0.0364302B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

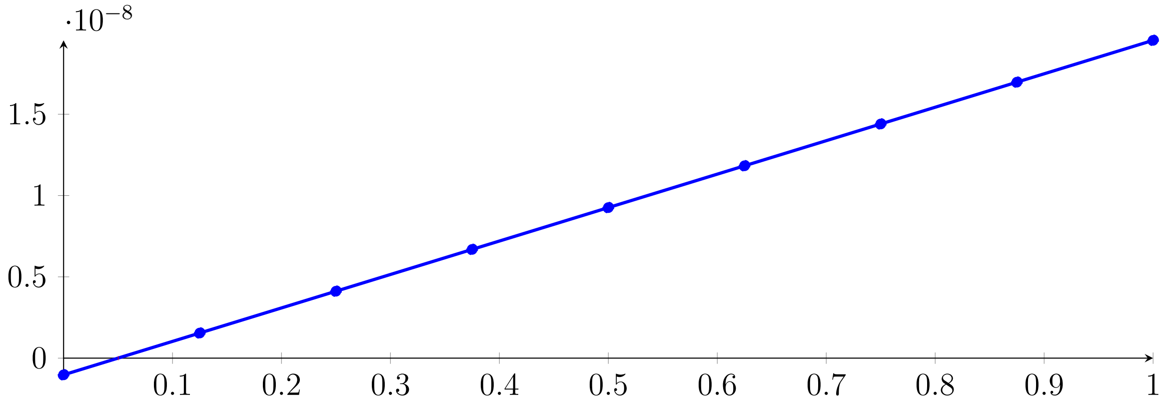
$$[0.0499103, 0.0499109]$$

Longest intersection interval: $5.36469\cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

55.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -3.09366\cdot 10^{-22}X^8 + 4.49986\cdot 10^{-22}X^7 - 4.03002\cdot 10^{-21}X^6 + 3.70577\cdot 10^{-22}X^5 - 3.47416 \\ &\quad \cdot 10^{-22}X^4 + 9.26442\cdot 10^{-23}X^3 - 1.18608\cdot 10^{-19}X^2 + 2.05706\cdot 10^{-08}X - 1.02667\cdot 10^{-09} \\ &= -1.02667\cdot 10^{-09}B_{0,8}(X) + 1.54465\cdot 10^{-09}B_{1,8}(X) + 4.11597\cdot 10^{-09}B_{2,8}(X) \\ &\quad + 6.68729\cdot 10^{-09}B_{3,8}(X) + 9.25862\cdot 10^{-09}B_{4,8}(X) + 1.18299\cdot 10^{-08}B_{5,8}(X) \\ &\quad + 1.44013\cdot 10^{-08}B_{6,8}(X) + 1.69726\cdot 10^{-08}B_{7,8}(X) + 1.95439\cdot 10^{-08}B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87728 \cdot 10^{-13}$
$\implies$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

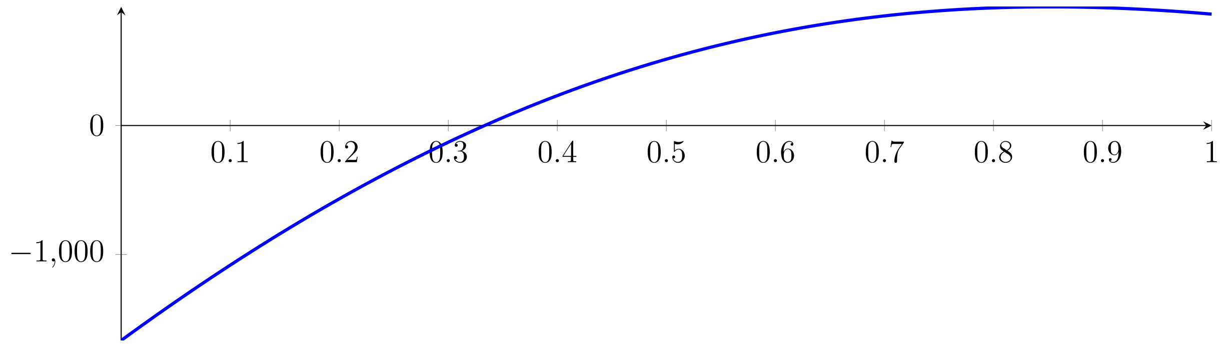
55.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

55.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-32}$.

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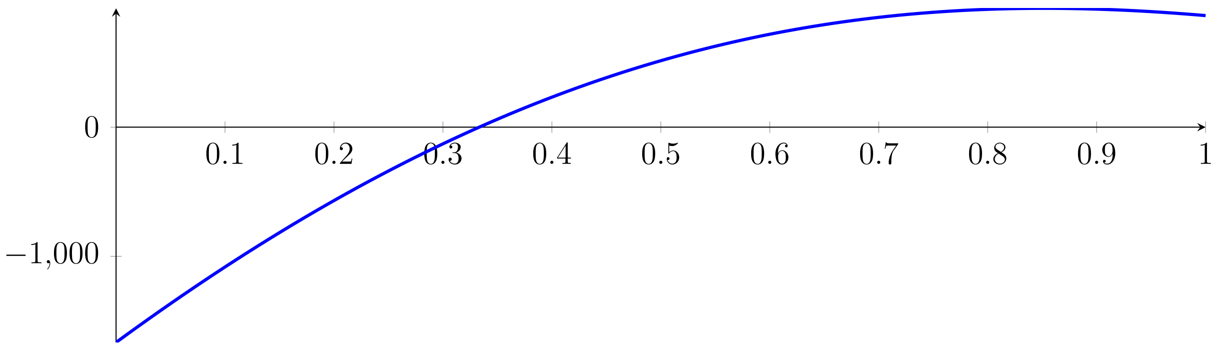
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

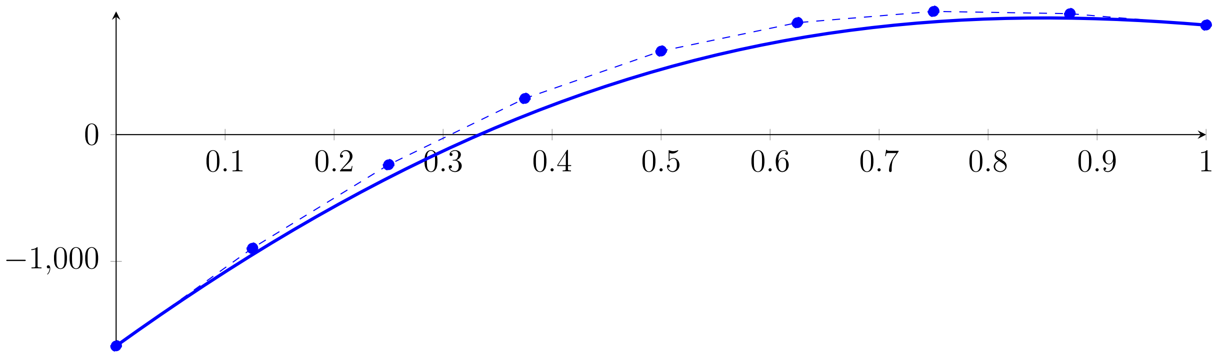
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



56.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

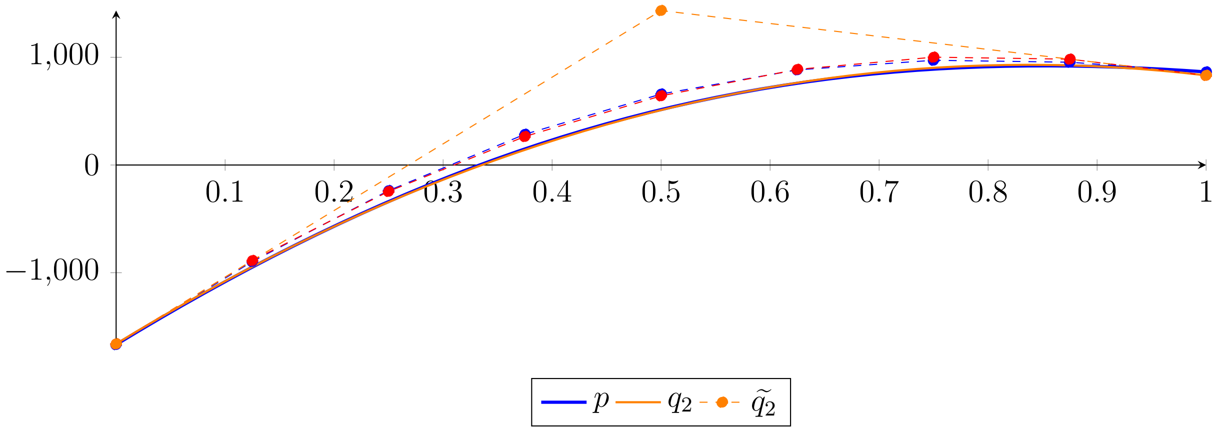
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -3.91297 \cdot 10^{-06}X^8 + 1.58774 \cdot 10^{-05}X^7 - 2.63335 \cdot 10^{-05}X^6 + 2.29285 \cdot 10^{-05}X^5 \\ &\quad - 1.11931 \cdot 10^{-05}X^4 + 3.0313 \cdot 10^{-06}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

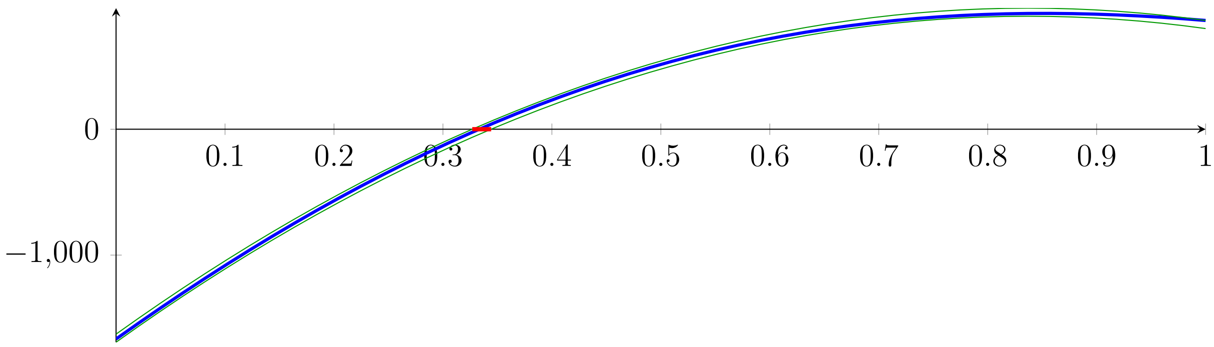
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

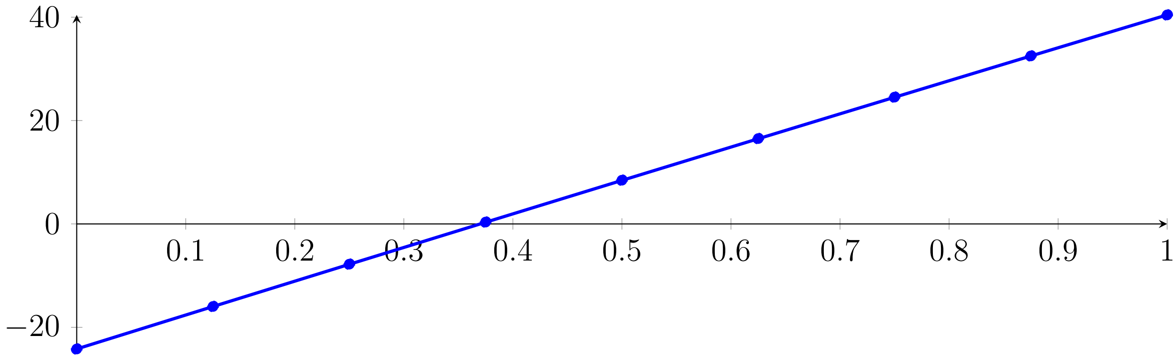
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

56.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

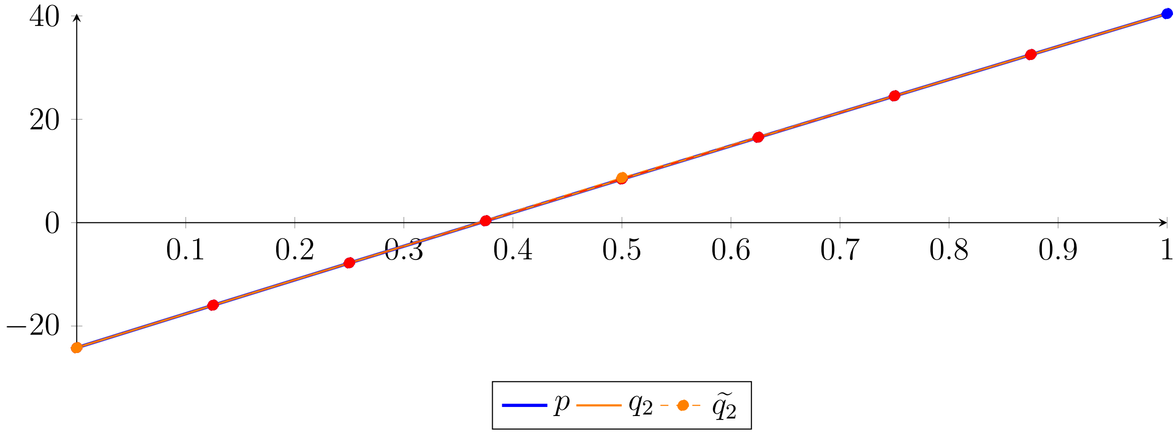
$$\begin{aligned} p = & -5.40012\cdot10^{-13}X^8 - 7.38964\cdot10^{-12}X^7 - 1.94416\cdot10^{-09}X^6 + 1.19265\cdot10^{-07}X^5 \\ & + 8.12335\cdot10^{-05}X^4 - 0.0012733X^3 - 1.18084X^2 + 65.8155X - 24.1945 \\ = & -24.1945B_{0,8}(X) - 15.9676B_{1,8}(X) - 7.78282B_{2,8}(X) + 0.35975B_{3,8}(X) \\ & + 8.4601B_{4,8}(X) + 16.5182B_{5,8}(X) + 24.5341B_{6,8}(X) + 32.5077B_{7,8}(X) + 40.4389B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 1.00159{\cdot}10^{-08}X^8 - 3.3372{\cdot}10^{-08}X^7 + 4.23875{\cdot}10^{-08}X^6 - 2.49721{\cdot}10^{-08}X^5 \\ &\quad + 6.08793{\cdot}10^{-09}X^4 + 1.46429{\cdot}10^{-10}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894{\cdot}10^{-05}$.
Bounding polynomials M and m:

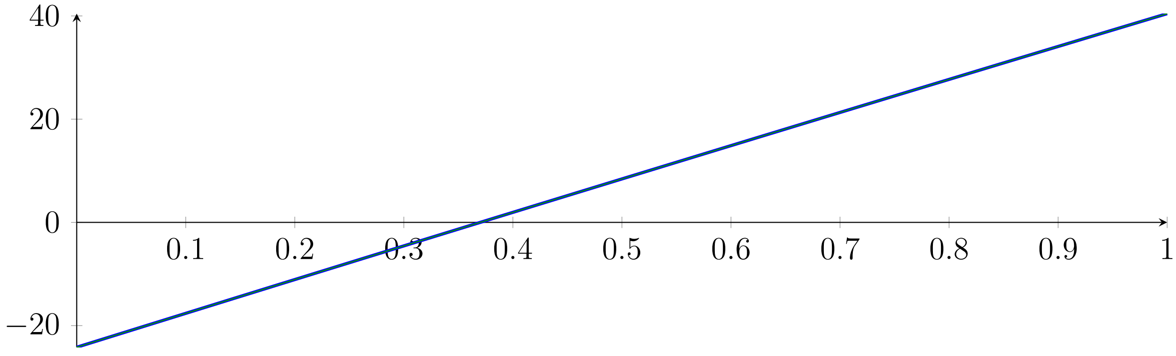
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



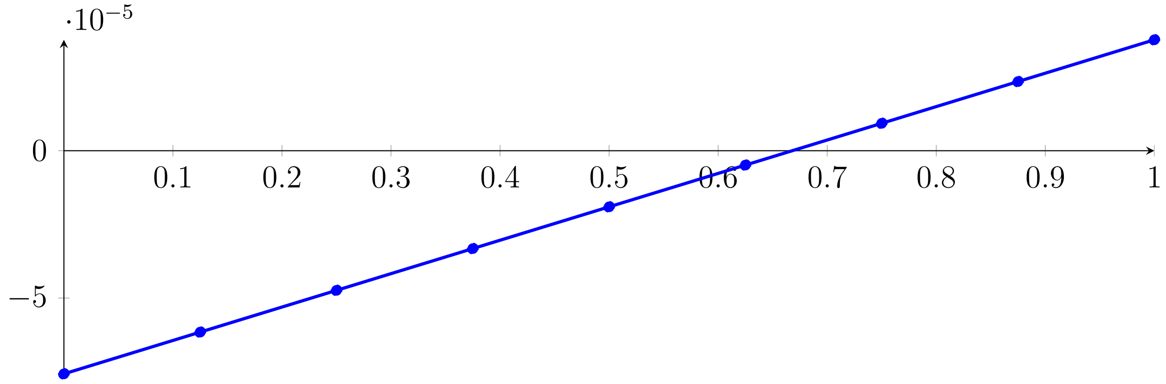
$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588{\cdot}10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

56.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

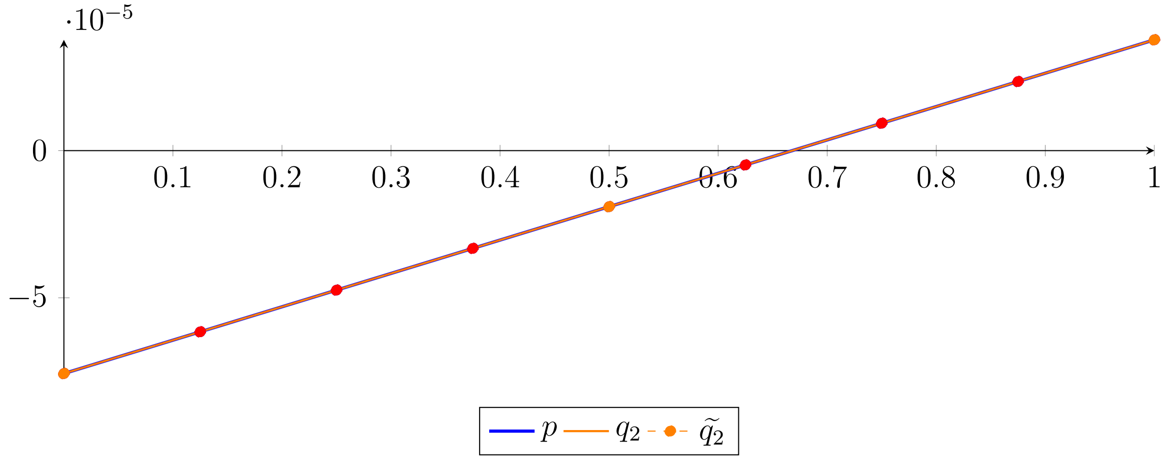
$$\begin{aligned} p = &-2.1684\cdot 10^{-19}X^8 - 4.33681\cdot 10^{-19}X^7 + 2.12504\cdot 10^{-17}X^6 - 1.51788\cdot 10^{-18}X^5 \\ &+ 7.58942\cdot 10^{-18}X^4 - 3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05} \\ = &-7.5732\cdot 10^{-05}B_{0,8}(X) - 6.15596\cdot 10^{-05}B_{1,8}(X) - 4.73873\cdot 10^{-05}B_{2,8}(X) \\ &- 3.32149\cdot 10^{-05}B_{3,8}(X) - 1.90425\cdot 10^{-05}B_{4,8}(X) - 4.87016\cdot 10^{-06}B_{5,8}(X) \\ &+ 9.3022\cdot 10^{-06}B_{6,8}(X) + 2.34746\cdot 10^{-05}B_{7,8}(X) + 3.76469\cdot 10^{-05}B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05} \\ &= -7.5732\cdot 10^{-05}B_{0,2} - 1.90425\cdot 10^{-05}B_{1,2} + 3.76469\cdot 10^{-05}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 5.62292\cdot 10^{-14}X^8 - 2.22643\cdot 10^{-13}X^7 + 3.60043\cdot 10^{-13}X^6 - 3.05846\cdot 10^{-13}X^5 + 1.46182 \\ &\cdot 10^{-13}X^4 - 3.90612\cdot 10^{-14}X^3 - 3.59793\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05} \\ &= -7.5732\cdot 10^{-05}B_{0,8} - 6.15596\cdot 10^{-05}B_{1,8} - 4.73873\cdot 10^{-05}B_{2,8} - 3.32149\cdot 10^{-05}B_{3,8} - 1.90425 \\ &\cdot 10^{-05}B_{4,8} - 4.87016\cdot 10^{-06}B_{5,8} + 9.3022\cdot 10^{-06}B_{6,8} + 2.34746\cdot 10^{-05}B_{7,8} + 3.76469\cdot 10^{-05}B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.98887\cdot 10^{-16}$.

Bounding polynomials M and m:

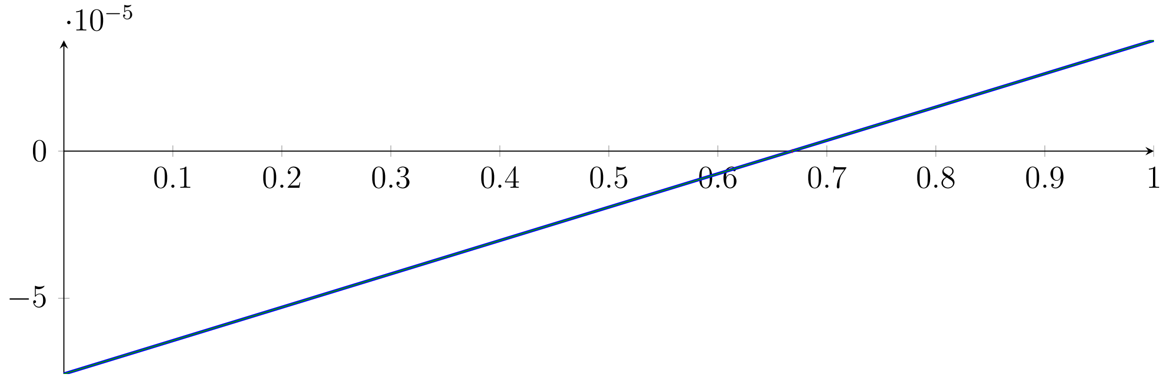
$$M = -3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05}$$

$$m = -3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.667955, 3.14644\cdot 10^{7}\} \qquad N(m) = \{0.667955, 3.14644\cdot 10^{7}\}$$

Intersection intervals:



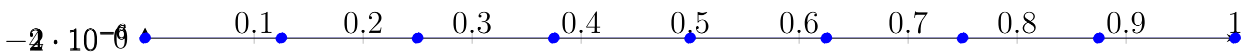
$[0.667955, 0.667955]$

Longest intersection interval: $1.88052 \cdot 10^{-09}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

56.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

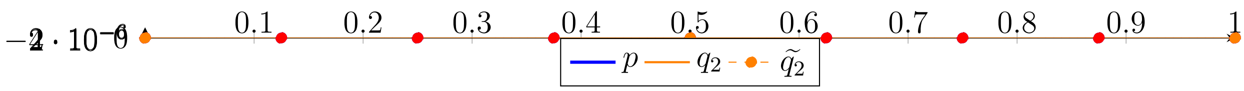
$$\begin{aligned}p &= -1.31266 \cdot 10^{-27} X^8 - 6.92683 \cdot 10^{-26} X^6 - 8.48183 \cdot 10^{-27} X^5 - 1.06023 \\ &\quad \cdot 10^{-26} X^4 + 1.41364 \cdot 10^{-27} X^3 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\ &= 4.52469 \cdot 10^{-14} B_{0,8}(X) + 7.18983 \cdot 10^{-14} B_{1,8}(X) + 9.85497 \cdot 10^{-14} B_{2,8}(X) \\ &\quad + 1.25201 \cdot 10^{-13} B_{3,8}(X) + 1.51852 \cdot 10^{-13} B_{4,8}(X) + 1.78504 \cdot 10^{-13} B_{5,8}(X) \\ &\quad + 2.05155 \cdot 10^{-13} B_{6,8}(X) + 2.31807 \cdot 10^{-13} B_{7,8}(X) + 2.58458 \cdot 10^{-13} B_{8,8}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= 2.44862 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\ &= 4.52469 \cdot 10^{-14} B_{0,2} + 1.51852 \cdot 10^{-13} B_{1,2} + 2.58458 \cdot 10^{-13} B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_2 &= -9.59093 \cdot 10^{-23} X^8 + 4.40717 \cdot 10^{-22} X^7 - 8.25115 \cdot 10^{-22} X^6 + 8.08254 \cdot 10^{-22} X^5 - 4.43405 \\ &\quad \cdot 10^{-22} X^4 + 1.3589 \cdot 10^{-22} X^3 - 2.18259 \cdot 10^{-23} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\ &= 4.52469 \cdot 10^{-14} B_{0,8} + 7.18983 \cdot 10^{-14} B_{1,8} + 9.85497 \cdot 10^{-14} B_{2,8} + 1.25201 \cdot 10^{-13} B_{3,8} + 1.51852 \\ &\quad \cdot 10^{-13} B_{4,8} + 1.78504 \cdot 10^{-13} B_{5,8} + 2.05155 \cdot 10^{-13} B_{6,8} + 2.31807 \cdot 10^{-13} B_{7,8} + 2.58458 \cdot 10^{-13} B_{8,8}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.40606 \cdot 10^{-25}$.
Bounding polynomials M and m:

$$M = 2.42338 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14}$$
$$m = 2.47387 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14}$$

Root of M and m:

$$N(M) = \{-8.79809 \cdot 10^{13}, -0.213542\} \qquad N(m) = \{-8.61853 \cdot 10^{13}, -0.214286\}$$

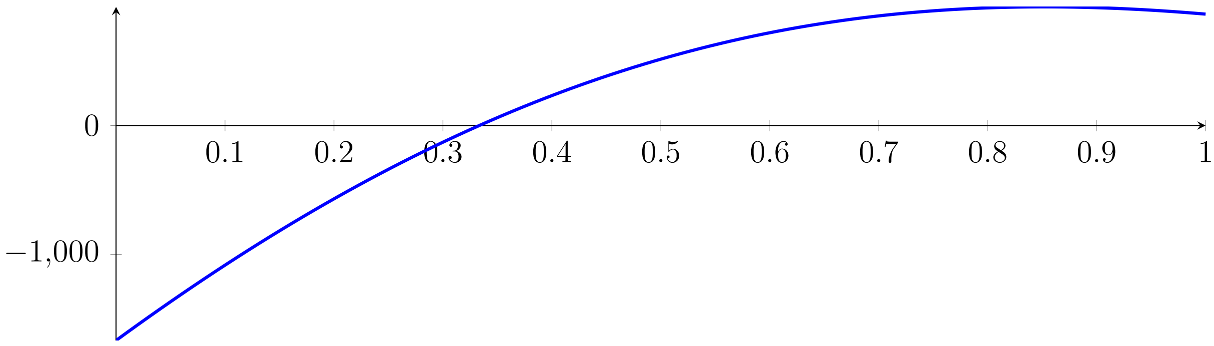
Intersection intervals:

No intersection intervals with the x axis.

56.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0,1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

with precision $\varepsilon = 1 \cdot 10^{-32}$.

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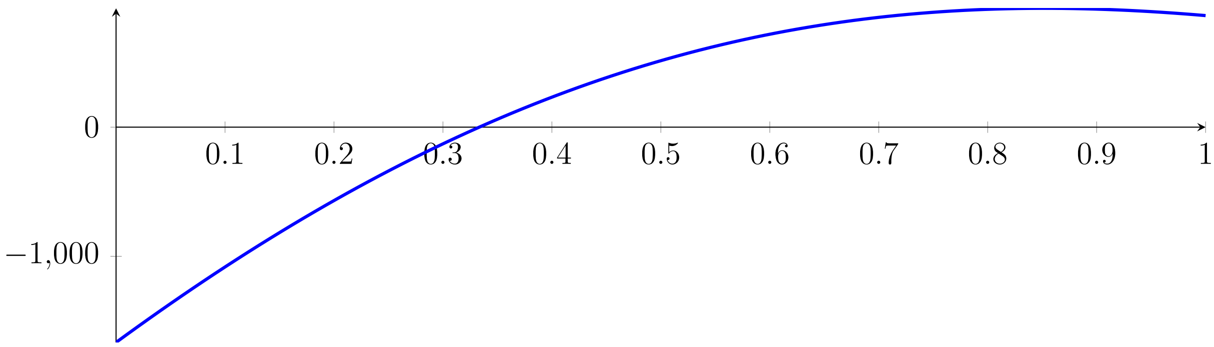
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

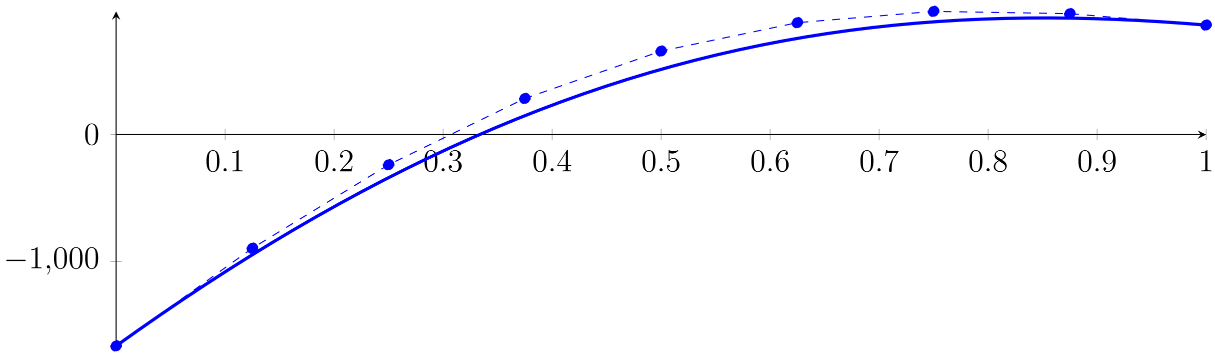
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



57.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

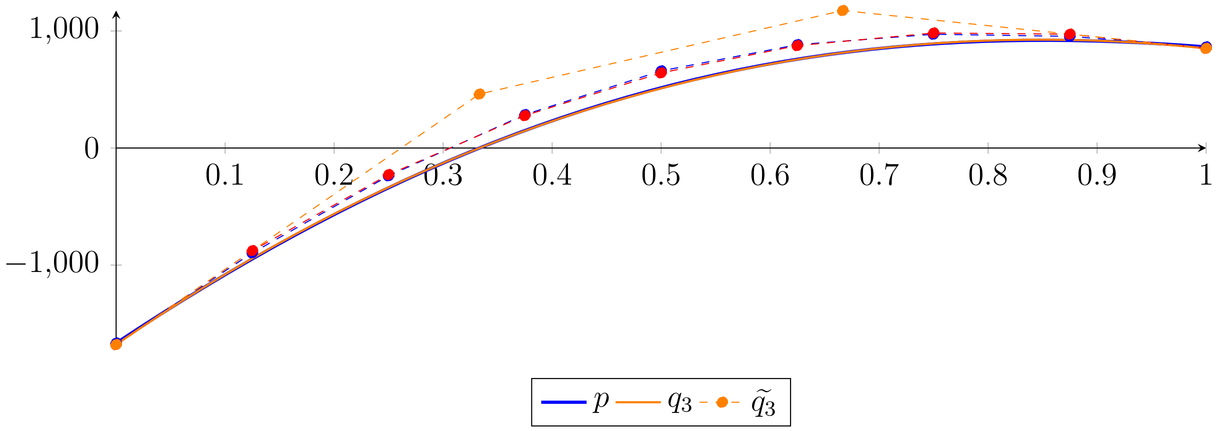
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 2.38117 \cdot 10^{-06}X^8 - 9.49957 \cdot 10^{-06}X^7 + 1.54757 \cdot 10^{-05}X^6 - 1.32609 \cdot 10^{-05}X^5 \\ &\quad + 6.41819 \cdot 10^{-06}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

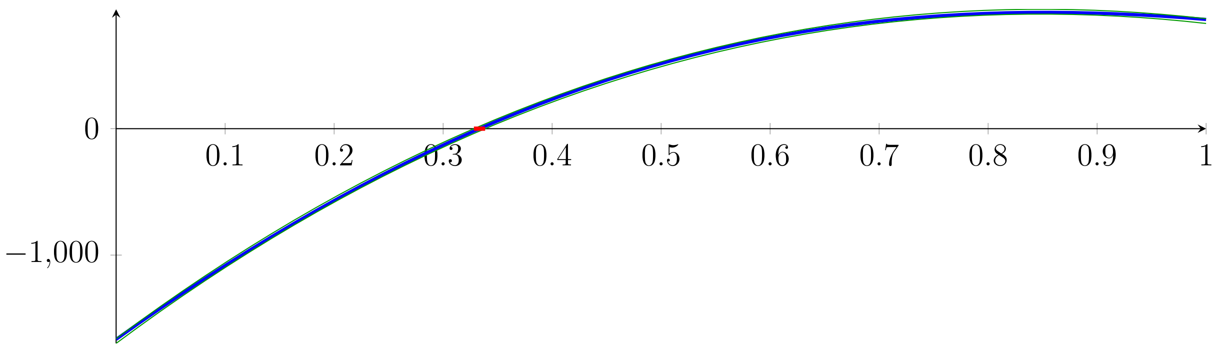
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

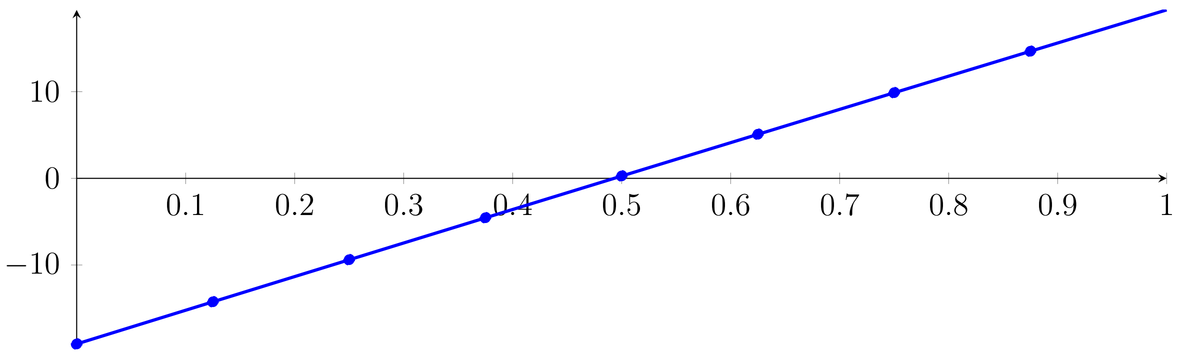
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

57.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

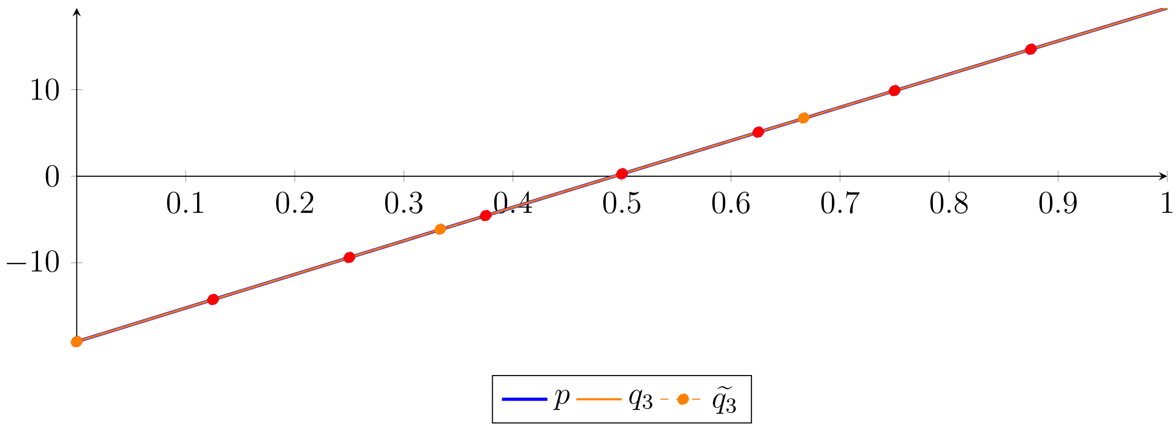
$$\begin{aligned} p = &-1.27898\cdot10^{-13}X^8 - 1.13687\cdot10^{-13}X^7 - 8.23661\cdot10^{-11}X^6 + 8.72882\cdot10^{-09}X^5 \\ &+ 1.00963\cdot10^{-05}X^4 - 0.000261161X^3 - 0.416284X^2 + 38.9643X - 19.1124 \\ = &-19.1124B_{0,8}(X) - 14.2419B_{1,8}(X) - 9.38619B_{2,8}(X) - 4.54539B_{3,8}(X) \\ &+ 0.280537B_{4,8}(X) + 5.09158B_{5,8}(X) + 9.88774B_{6,8}(X) + 14.669B_{7,8}(X) + 19.4354B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124\\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}\end{aligned}$$

$$\begin{aligned}\widetilde{q_3} &= 2.3353\cdot10^{-08}X^8 - 9.31856\cdot10^{-08}X^7 + 1.51861\cdot10^{-07}X^6 - 1.30228\cdot10^{-07}X^5\\ &\quad + 6.31618\cdot10^{-08}X^4 - 0.000240962X^3 - 0.416297X^2 + 38.9643X - 19.1124\\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}\\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16887\cdot10^{-07}$.

Bounding polynomials M and m:

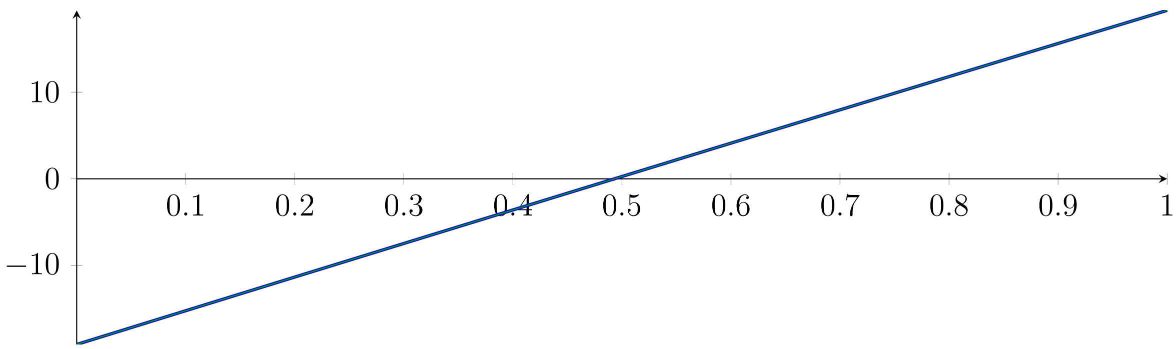
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

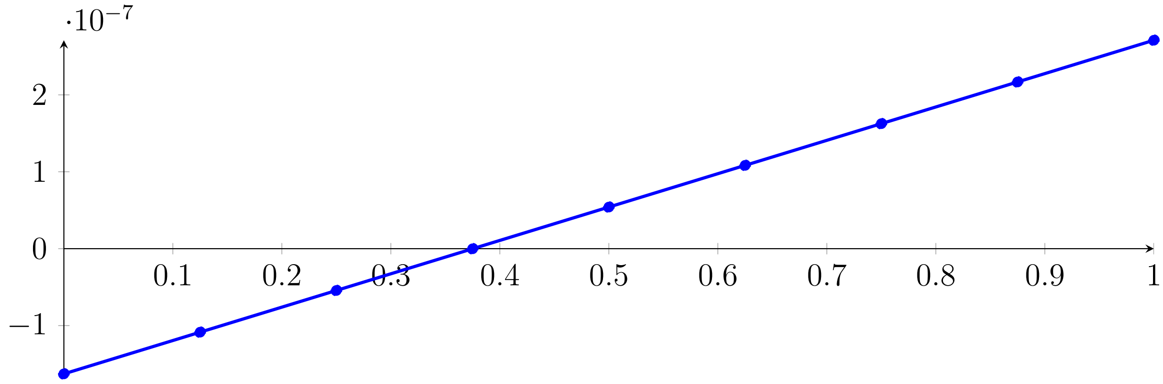
Longest intersection interval: $1.12517\cdot10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

57.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

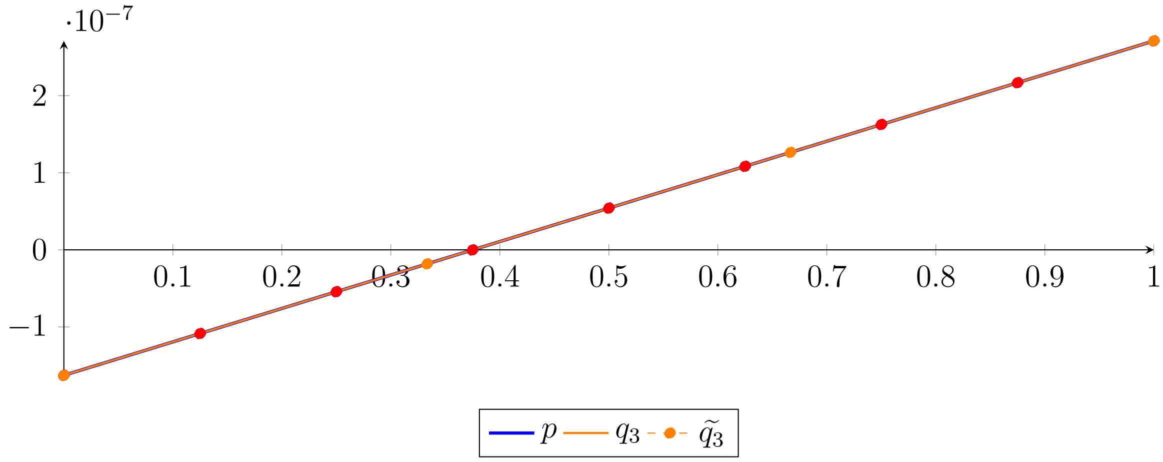
$$p = -3.81165\cdot10^{-21}X^8 + 6.77626\cdot10^{-21}X^7 + 2.96462\cdot10^{-21}X^6 + 5.92923\cdot10^{-21}X^5 + 1.11173\cdot10^{-20}X^4 + 1.48231\cdot10^{-21}X^3 - 5.27494\cdot10^{-17}X^2 + 4.33793\cdot10^{-07}X - 1.6278\cdot10^{-07}$$
$$= -1.6278\cdot10^{-07}B_{0,8}(X) - 1.08555\cdot10^{-07}B_{1,8}(X) - 5.43313\cdot10^{-08}B_{2,8}(X) - 1.07093\cdot10^{-10}B_{3,8}(X) + 5.41171\cdot10^{-08}B_{4,8}(X) + 1.08341\cdot10^{-07}B_{5,8}(X) + 1.62565\cdot10^{-07}B_{6,8}(X) + 2.1679\cdot10^{-07}B_{7,8}(X) + 2.71014\cdot10^{-07}B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = -1.90582\cdot10^{-21}X^3 - 5.2746\cdot10^{-17}X^2 + 4.33793\cdot10^{-07}X - 1.6278\cdot10^{-07}$$
$$= -1.6278\cdot10^{-07}B_{0,3} - 1.81818\cdot10^{-08}B_{1,3} + 1.26416\cdot10^{-07}B_{2,3} + 2.71014\cdot10^{-07}B_{3,3}$$

$$\tilde{q}_3 = 1.50585\cdot10^{-16}X^8 - 5.82707\cdot10^{-16}X^7 + 9.15943\cdot10^{-16}X^6 - 7.54824\cdot10^{-16}X^5 + 3.52096\cdot10^{-16}X^4 - 9.31289\cdot10^{-17}X^3 - 3.98474\cdot10^{-17}X^2 + 4.33793\cdot10^{-07}X - 1.6278\cdot10^{-07}$$
$$= -1.6278\cdot10^{-07}B_{0,8} - 1.08555\cdot10^{-07}B_{1,8} - 5.43313\cdot10^{-08}B_{2,8} - 1.07093\cdot10^{-10}B_{3,8} + 5.41171\cdot10^{-08}B_{4,8} + 1.08341\cdot10^{-07}B_{5,8} + 1.62565\cdot10^{-07}B_{6,8} + 2.1679\cdot10^{-07}B_{7,8} + 2.71014\cdot10^{-07}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 8.66435\cdot10^{-19}$.

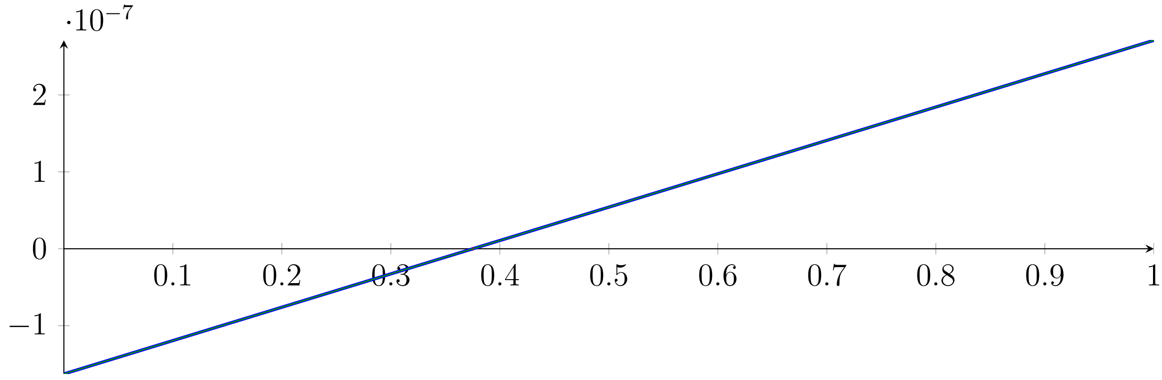
Bounding polynomials M and m:

$$M = -1.85288\cdot10^{-21}X^3 - 5.2746\cdot10^{-17}X^2 + 4.33793\cdot10^{-07}X - 1.6278\cdot10^{-07}$$
$$m = -1.90582\cdot10^{-21}X^3 - 5.2746\cdot10^{-17}X^2 + 4.33793\cdot10^{-07}X - 1.6278\cdot10^{-07}$$

Root of M and m:

$$N(M) = \{-1.49969\cdot10^{7}, 0.375247, 1.49696\cdot10^{7}\} \quad N(m) = \{-1.46018\cdot10^{7}, 0.375247, 1.45759\cdot10^{7}\}$$

Intersection intervals:



$[0.375247, 0.375247]$

Longest intersection interval: $7.69251 \cdot 10^{-09}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

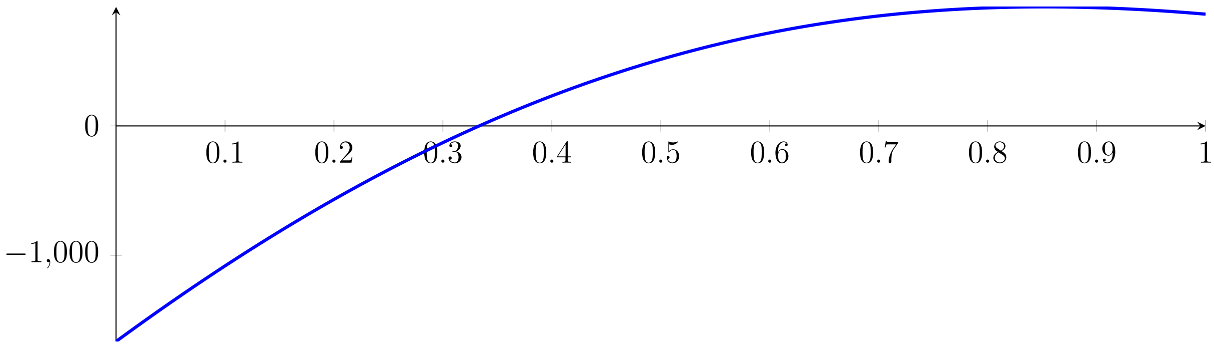
57.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

57.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-32}$.

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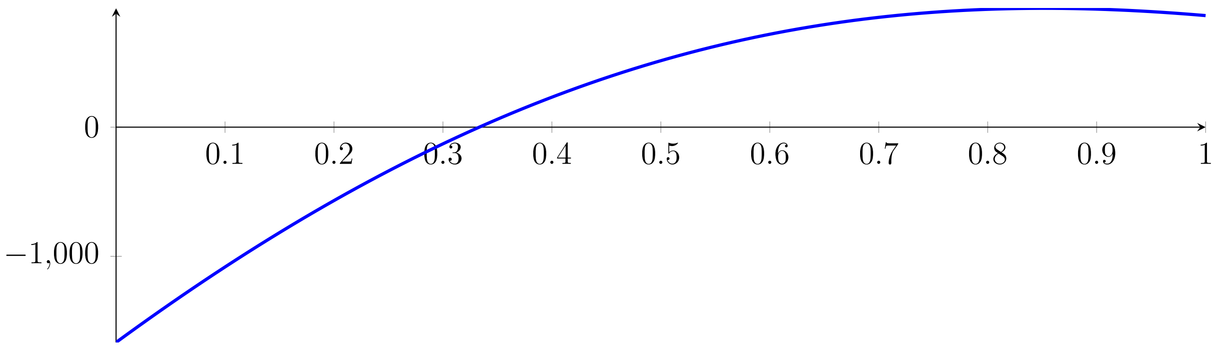
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

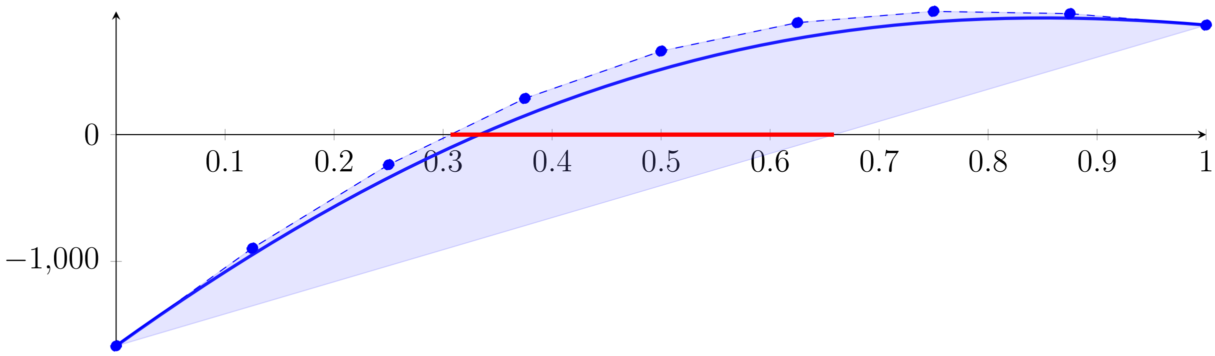
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



58.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

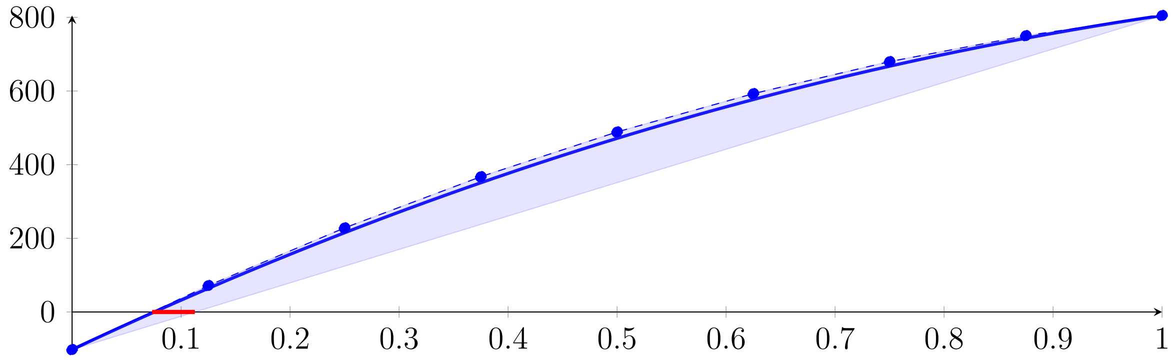
Longest intersection interval: 0.351792

$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

58.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\ & + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\ = & -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\ & + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

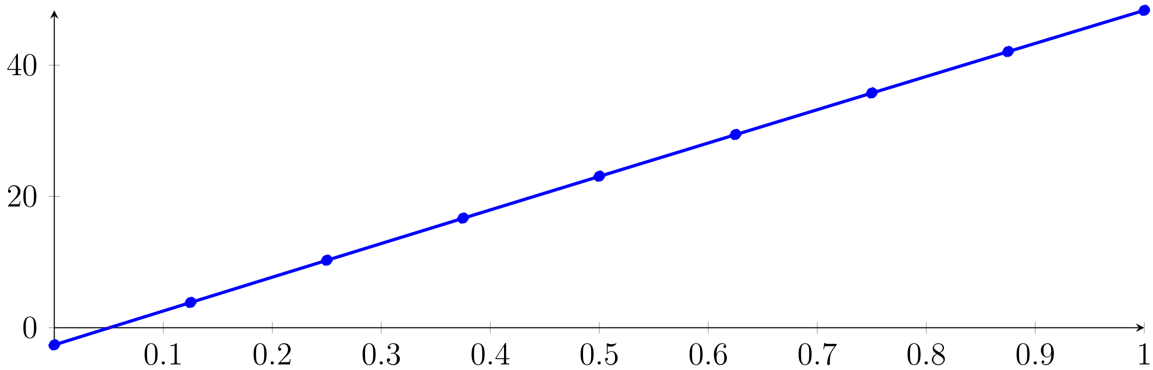
Longest intersection interval: 0.0391855

$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

58.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -5.41789 \cdot 10^{-13}X^8 - 1.26477 \cdot 10^{-12}X^7 - 5.05786 \cdot 10^{-10}X^6 + 3.66765 \cdot 10^{-08}X^5 \\ & + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\ = & -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\ & + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

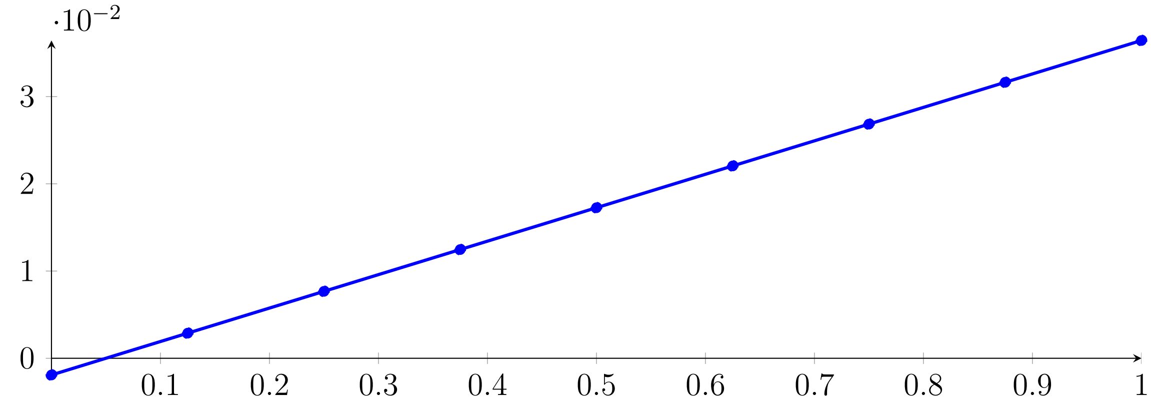
Longest intersection interval: 0.000742589

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

58.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -7.11237\cdot10^{-16}X^8 + 5.27356\cdot10^{-16}X^7 - 7.38298\cdot10^{-15}X^6 + 1.06859\cdot10^{-15}X^5 \\ &\quad - 1.09288\cdot10^{-15}X^4 - 2.37227\cdot10^{-13}X^3 - 4.12146\cdot10^{-07}X^2 + 0.0383444X - 0.00191378 \\ &= -0.00191378B_{0,8}(X) + 0.00287927B_{1,8}(X) + 0.00767231B_{2,8}(X) \\ &\quad + 0.0124653B_{3,8}(X) + 0.0172583B_{4,8}(X) + 0.0220513B_{5,8}(X) \\ &\quad + 0.0268443B_{6,8}(X) + 0.0316373B_{7,8}(X) + 0.0364302B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

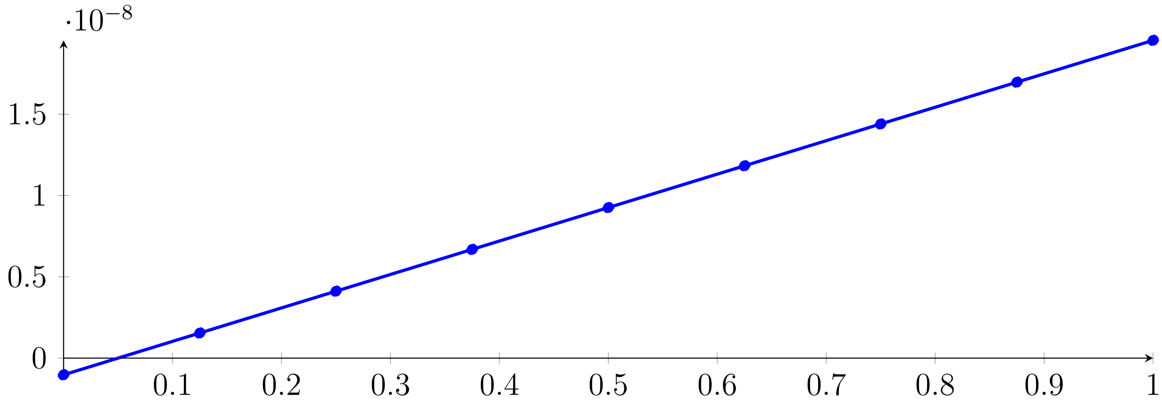
$$[0.0499103, 0.0499109]$$

Longest intersection interval: $5.36469\cdot10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

58.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -3.09366\cdot10^{-22}X^8 + 4.49986\cdot10^{-22}X^7 - 4.03002\cdot10^{-21}X^6 + 3.70577\cdot10^{-22}X^5 - 3.47416 \\ &\quad \cdot10^{-22}X^4 + 9.26442\cdot10^{-23}X^3 - 1.18608\cdot10^{-19}X^2 + 2.05706\cdot10^{-08}X - 1.02667\cdot10^{-09} \\ &= -1.02667\cdot10^{-09}B_{0,8}(X) + 1.54465\cdot10^{-09}B_{1,8}(X) + 4.11597\cdot10^{-09}B_{2,8}(X) \\ &\quad + 6.68729\cdot10^{-09}B_{3,8}(X) + 9.25862\cdot10^{-09}B_{4,8}(X) + 1.18299\cdot10^{-08}B_{5,8}(X) \\ &\quad + 1.44013\cdot10^{-08}B_{6,8}(X) + 1.69726\cdot10^{-08}B_{7,8}(X) + 1.95439\cdot10^{-08}B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87728 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

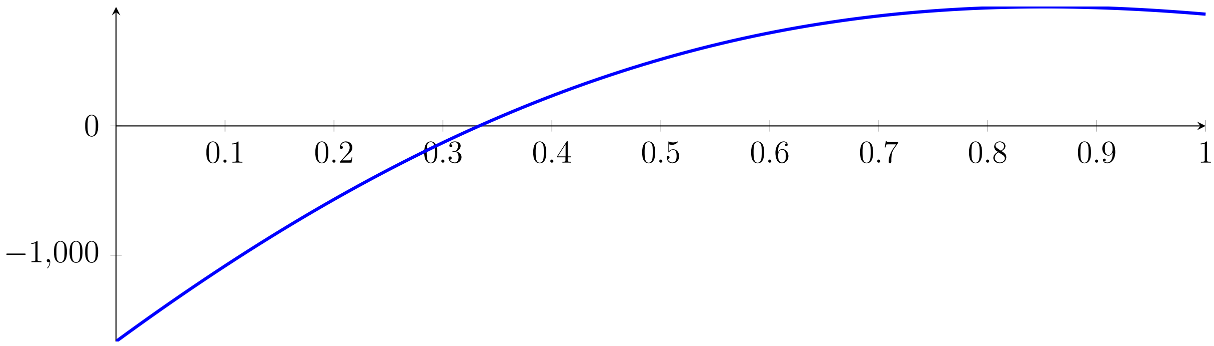
58.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

58.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-64}$.

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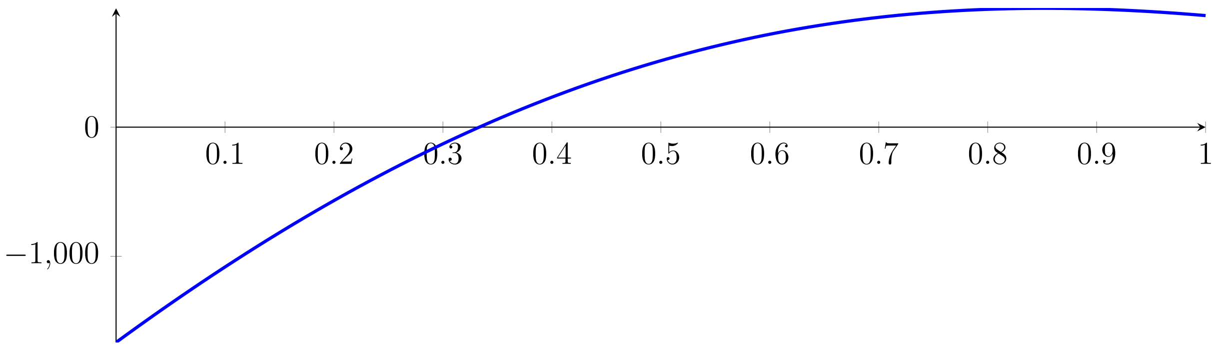
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0,1]$:

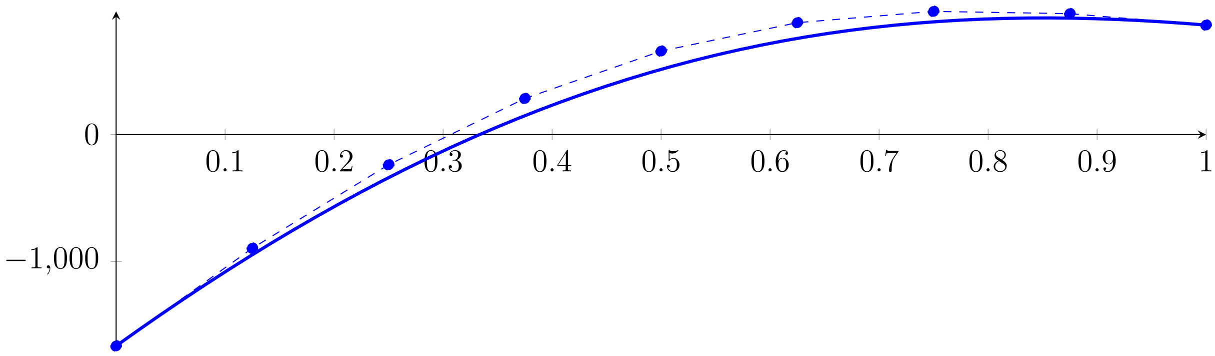
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



59.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

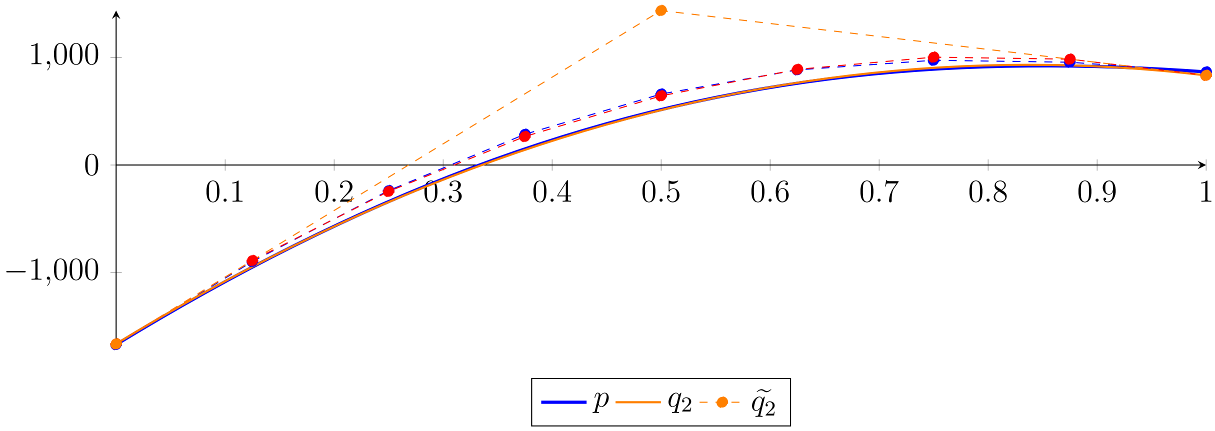
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -3.91297 \cdot 10^{-06}X^8 + 1.58774 \cdot 10^{-05}X^7 - 2.63335 \cdot 10^{-05}X^6 + 2.29285 \cdot 10^{-05}X^5 \\ &\quad - 1.11931 \cdot 10^{-05}X^4 + 3.0313 \cdot 10^{-06}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

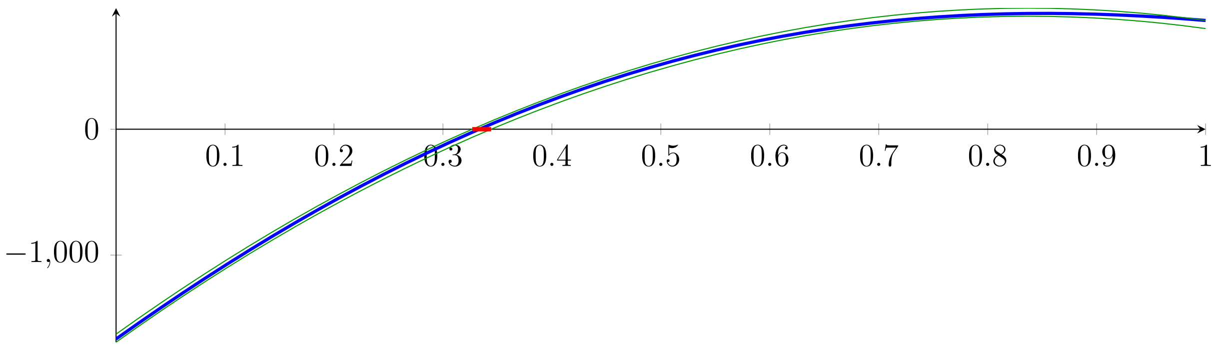
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

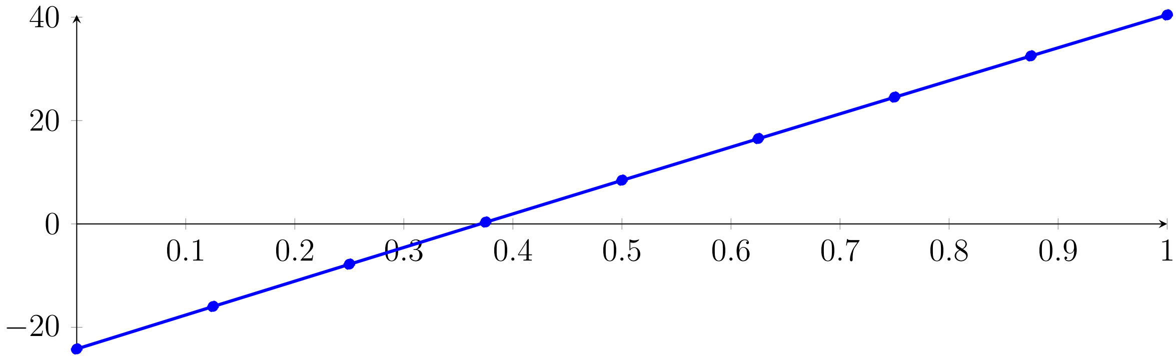
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

59.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

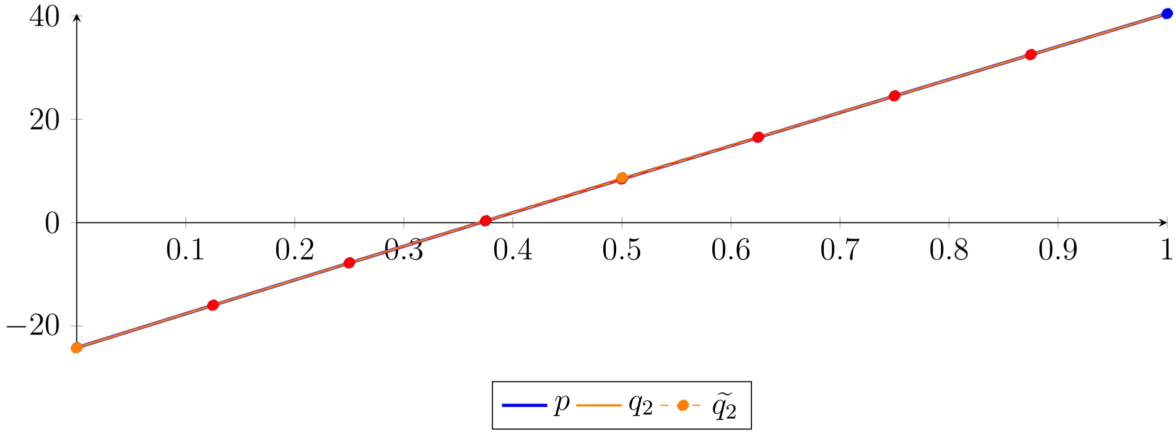
$$\begin{aligned} p = & -5.40012\cdot10^{-13}X^8 - 7.38964\cdot10^{-12}X^7 - 1.94416\cdot10^{-09}X^6 + 1.19265\cdot10^{-07}X^5 \\ & + 8.12335\cdot10^{-05}X^4 - 0.0012733X^3 - 1.18084X^2 + 65.8155X - 24.1945 \\ = & -24.1945B_{0,8}(X) - 15.9676B_{1,8}(X) - 7.78282B_{2,8}(X) + 0.35975B_{3,8}(X) \\ & + 8.4601B_{4,8}(X) + 16.5182B_{5,8}(X) + 24.5341B_{6,8}(X) + 32.5077B_{7,8}(X) + 40.4389B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 1.00159 \cdot 10^{-08}X^8 - 3.3372 \cdot 10^{-08}X^7 + 4.23875 \cdot 10^{-08}X^6 - 2.49721 \cdot 10^{-08}X^5 \\ &\quad + 6.08793 \cdot 10^{-09}X^4 + 1.46429 \cdot 10^{-10}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.
Bounding polynomials M and m:

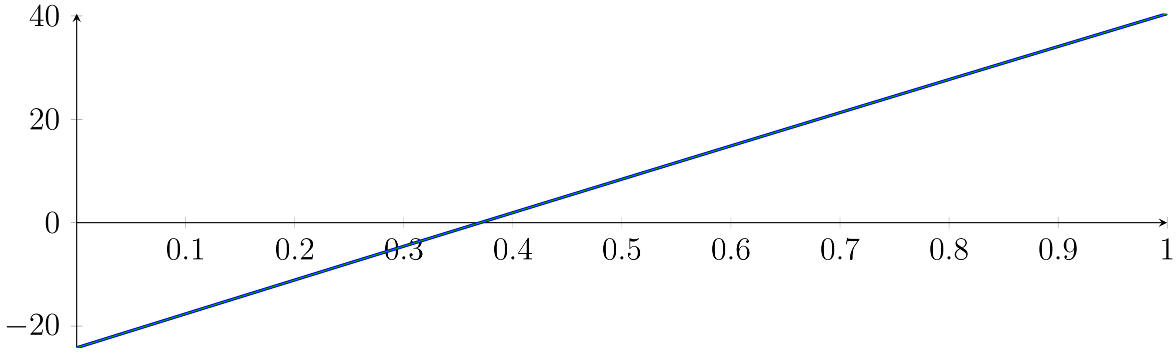
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



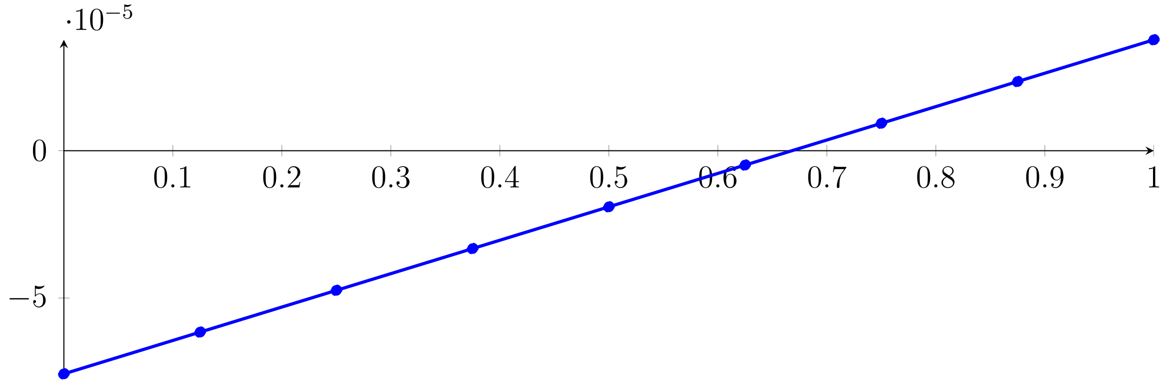
$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

59.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

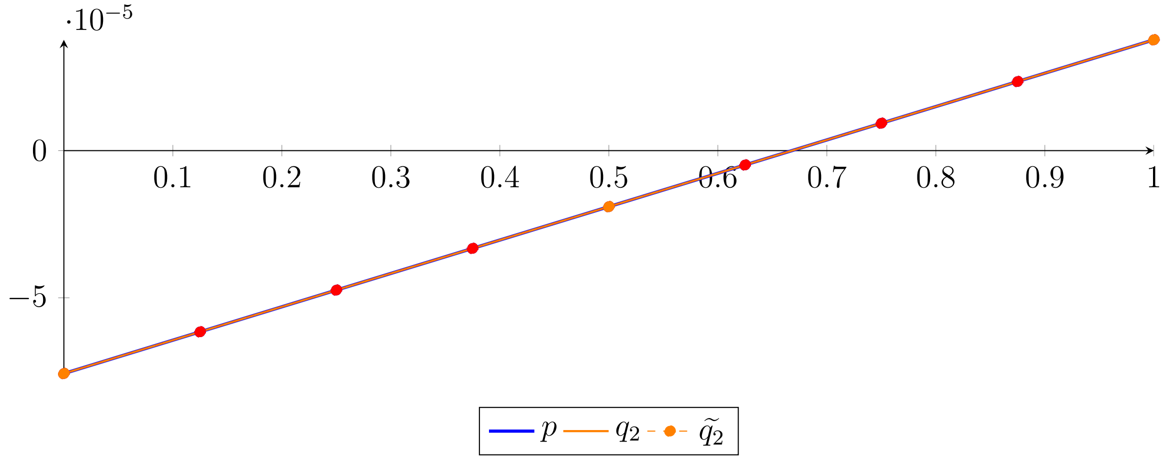
$$p = -2.1684\cdot10^{-19}X^8 - 4.33681\cdot10^{-19}X^7 + 2.12504\cdot10^{-17}X^6 - 1.51788\cdot10^{-18}X^5 + 7.58942\cdot10^{-18}X^4 - 3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$= -7.5732\cdot10^{-05}B_{0,8}(X) - 6.15596\cdot10^{-05}B_{1,8}(X) - 4.73873\cdot10^{-05}B_{2,8}(X) - 3.32149\cdot10^{-05}B_{3,8}(X) - 1.90425\cdot10^{-05}B_{4,8}(X) - 4.87016\cdot10^{-06}B_{5,8}(X) + 9.3022\cdot10^{-06}B_{6,8}(X) + 2.34746\cdot10^{-05}B_{7,8}(X) + 3.76469\cdot10^{-05}B_{8,8}(X)$$



Degree reduction and raising:

$$q_2 = -3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$= -7.5732\cdot10^{-05}B_{0,2} - 1.90425\cdot10^{-05}B_{1,2} + 3.76469\cdot10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = 5.62292\cdot10^{-14}X^8 - 2.22643\cdot10^{-13}X^7 + 3.60043\cdot10^{-13}X^6 - 3.05846\cdot10^{-13}X^5 + 1.46182\cdot10^{-13}X^4 - 3.90612\cdot10^{-14}X^3 - 3.59793\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$= -7.5732\cdot10^{-05}B_{0,8} - 6.15596\cdot10^{-05}B_{1,8} - 4.73873\cdot10^{-05}B_{2,8} - 3.32149\cdot10^{-05}B_{3,8} - 1.90425\cdot10^{-05}B_{4,8} - 4.87016\cdot10^{-06}B_{5,8} + 9.3022\cdot10^{-06}B_{6,8} + 2.34746\cdot10^{-05}B_{7,8} + 3.76469\cdot10^{-05}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.98887\cdot10^{-16}$.

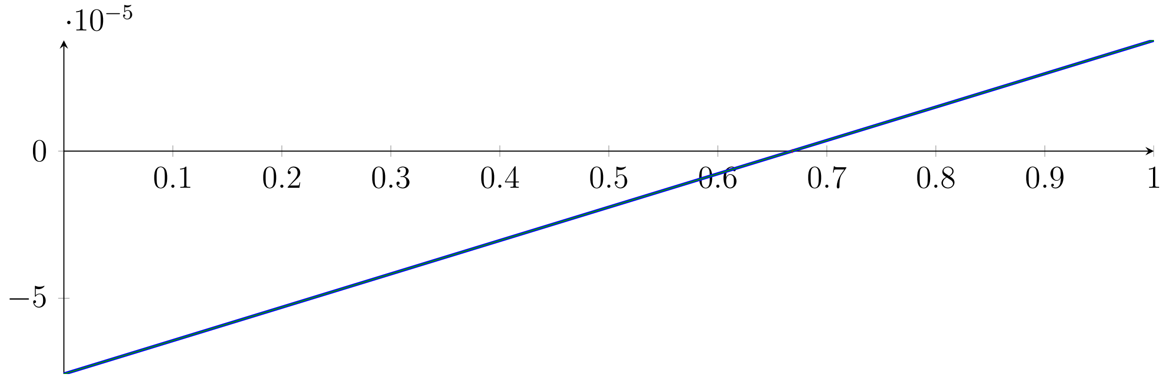
Bounding polynomials M and m:

$$M = -3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$m = -3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$

Root of M and m:

$$N(M) = \{0.667955, 3.14644\cdot10^{7}\} \qquad N(m) = \{0.667955, 3.14644\cdot10^{7}\}$$

Intersection intervals:



$[0.667955, 0.667955]$

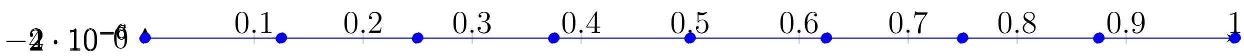
Longest intersection interval: $1.88052 \cdot 10^{-09}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

59.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

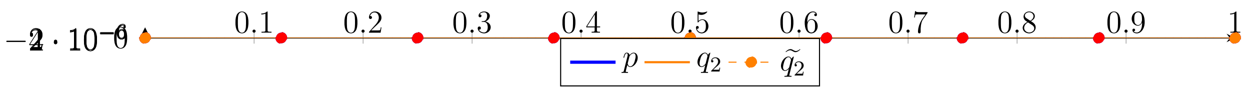
$$p = -1.31266 \cdot 10^{-27} X^8 - 6.92683 \cdot 10^{-26} X^6 - 8.48183 \cdot 10^{-27} X^5 - 1.06023 \cdot 10^{-26} X^4 + 1.41364 \cdot 10^{-27} X^3 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14}$$
$$= 4.52469 \cdot 10^{-14} B_{0,8}(X) + 7.18983 \cdot 10^{-14} B_{1,8}(X) + 9.85497 \cdot 10^{-14} B_{2,8}(X) + 1.25201 \cdot 10^{-13} B_{3,8}(X) + 1.51852 \cdot 10^{-13} B_{4,8}(X) + 1.78504 \cdot 10^{-13} B_{5,8}(X) + 2.05155 \cdot 10^{-13} B_{6,8}(X) + 2.31807 \cdot 10^{-13} B_{7,8}(X) + 2.58458 \cdot 10^{-13} B_{8,8}(X)$$



Degree reduction and raising:

$$q_2 = 2.44862 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14}$$
$$= 4.52469 \cdot 10^{-14} B_{0,2} + 1.51852 \cdot 10^{-13} B_{1,2} + 2.58458 \cdot 10^{-13} B_{2,2}$$

$$\widetilde{q}_2 = -9.59093 \cdot 10^{-23} X^8 + 4.40717 \cdot 10^{-22} X^7 - 8.25115 \cdot 10^{-22} X^6 + 8.08254 \cdot 10^{-22} X^5 - 4.43405 \cdot 10^{-22} X^4 + 1.3589 \cdot 10^{-22} X^3 - 2.18259 \cdot 10^{-23} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14}$$
$$= 4.52469 \cdot 10^{-14} B_{0,8} + 7.18983 \cdot 10^{-14} B_{1,8} + 9.85497 \cdot 10^{-14} B_{2,8} + 1.25201 \cdot 10^{-13} B_{3,8} + 1.51852 \cdot 10^{-13} B_{4,8} + 1.78504 \cdot 10^{-13} B_{5,8} + 2.05155 \cdot 10^{-13} B_{6,8} + 2.31807 \cdot 10^{-13} B_{7,8} + 2.58458 \cdot 10^{-13} B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 6.40606 \cdot 10^{-25}$.

Bounding polynomials M and m:

$$M = 2.42338 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14}$$
$$m = 2.47387 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14}$$

Root of M and m:

$$N(M) = \{-8.79809 \cdot 10^{13}, -0.213542\} \qquad N(m) = \{-8.61853 \cdot 10^{13}, -0.214286\}$$

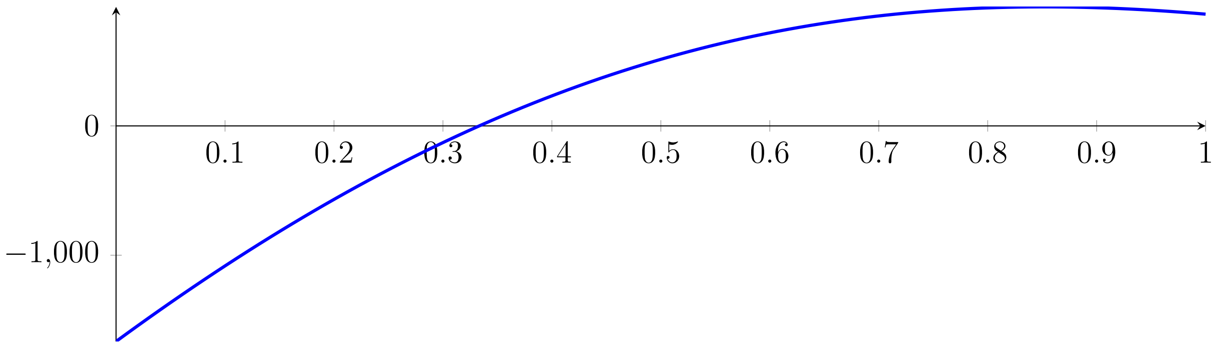
Intersection intervals:

No intersection intervals with the x axis.

59.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

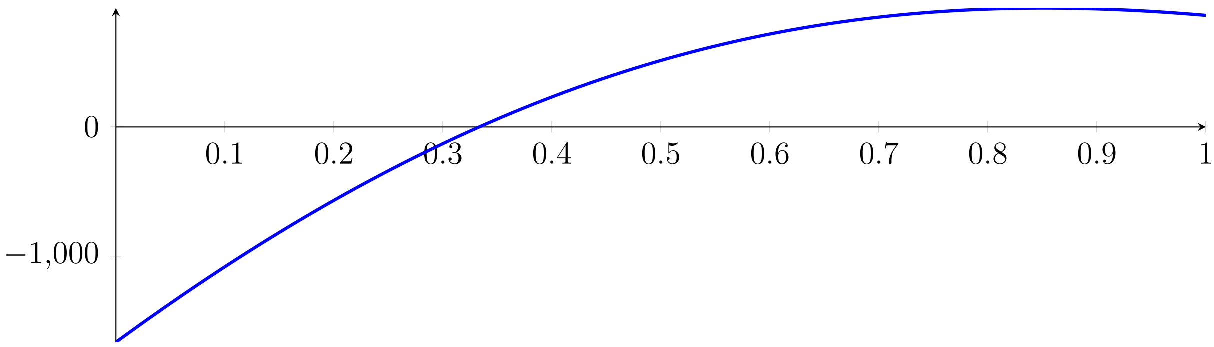
with precision $\varepsilon = 1 \cdot 10^{-64}$.

60 Running `CubeClip` on f_8 with epsilon 64

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

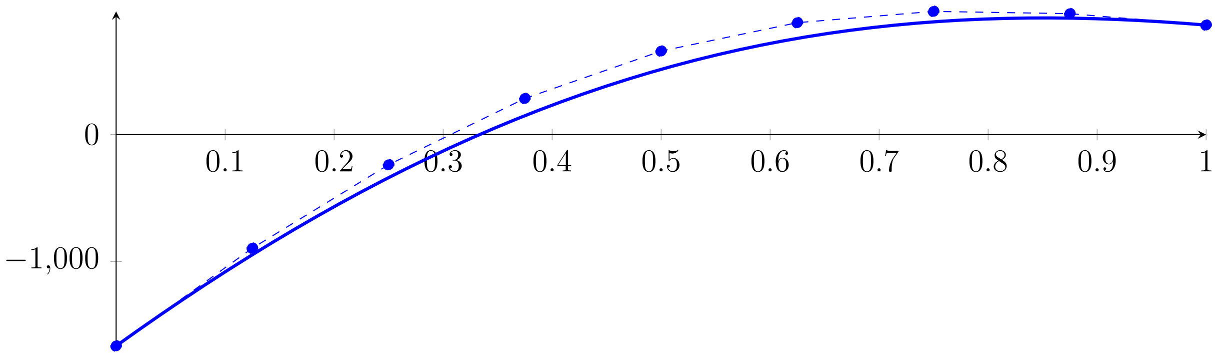
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



60.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

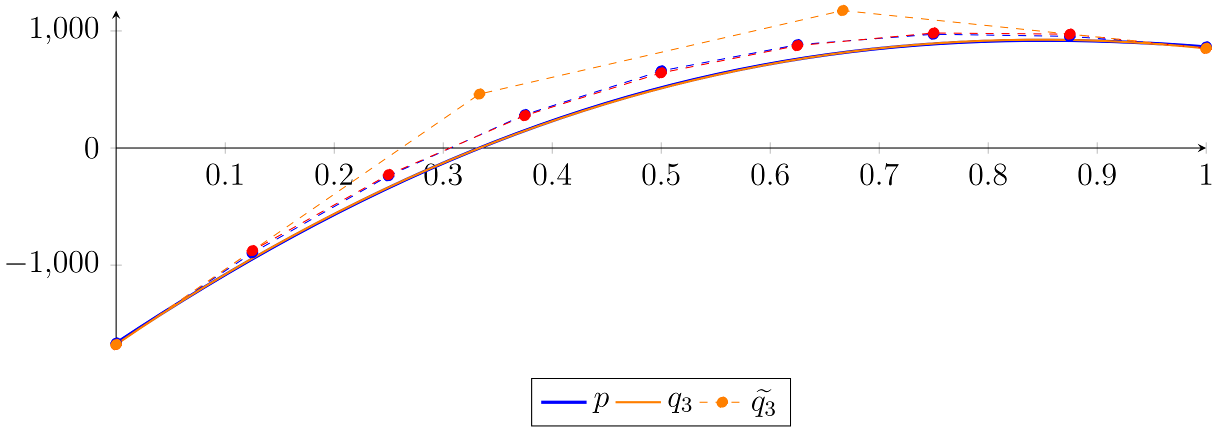
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 2.38117 \cdot 10^{-06}X^8 - 9.49957 \cdot 10^{-06}X^7 + 1.54757 \cdot 10^{-05}X^6 - 1.32609 \cdot 10^{-05}X^5 \\ &\quad + 6.41819 \cdot 10^{-06}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

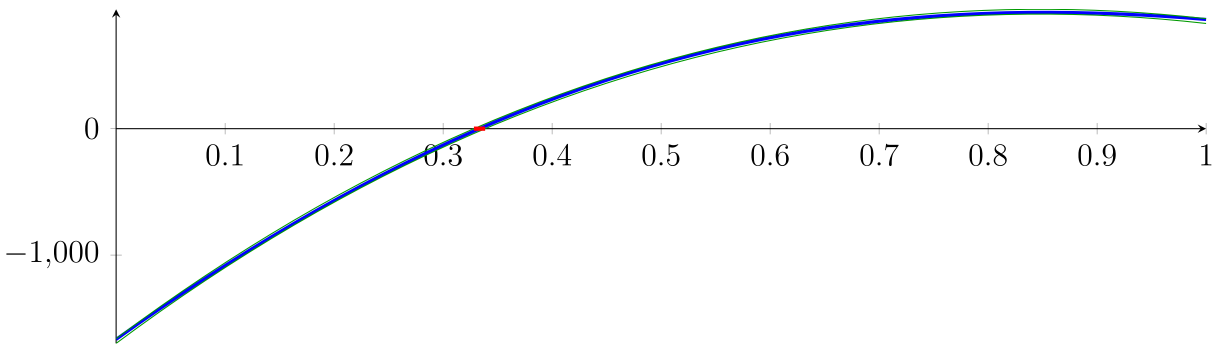
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$[0.328258, 0.338551]$

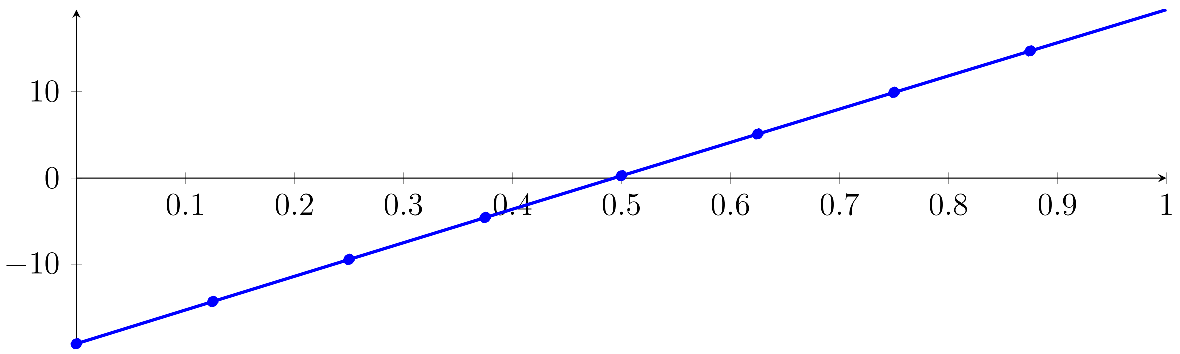
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

60.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

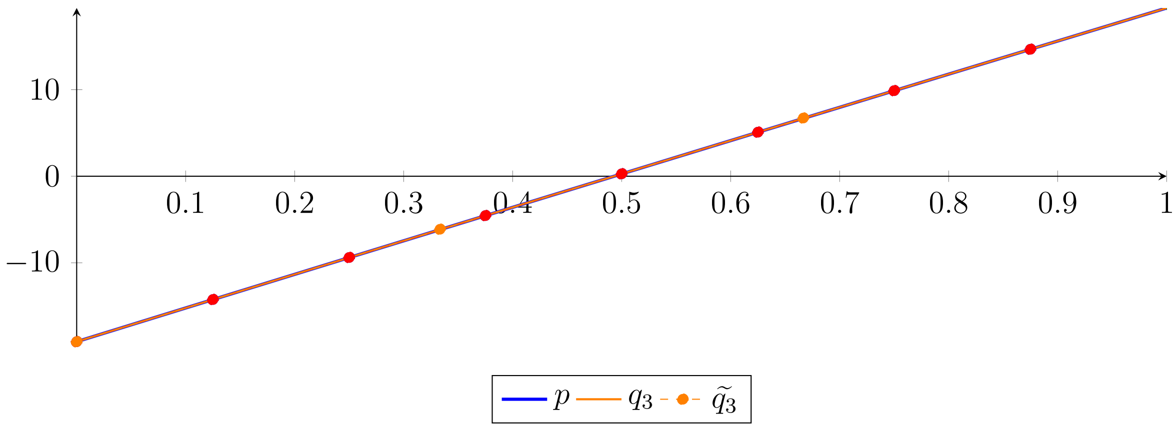
$$\begin{aligned} p = &-1.27898\cdot 10^{-13}X^8 - 1.13687\cdot 10^{-13}X^7 - 8.23661\cdot 10^{-11}X^6 + 8.72882\cdot 10^{-09}X^5 \\ &+ 1.00963\cdot 10^{-05}X^4 - 0.000261161X^3 - 0.416284X^2 + 38.9643X - 19.1124 \\ = &-19.1124B_{0,8}(X) - 14.2419B_{1,8}(X) - 9.38619B_{2,8}(X) - 4.54539B_{3,8}(X) \\ &+ 0.280537B_{4,8}(X) + 5.09158B_{5,8}(X) + 9.88774B_{6,8}(X) + 14.669B_{7,8}(X) + 19.4354B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= 2.3353 \cdot 10^{-08}X^8 - 9.31856 \cdot 10^{-08}X^7 + 1.51861 \cdot 10^{-07}X^6 - 1.30228 \cdot 10^{-07}X^5 \\ &\quad + 6.31618 \cdot 10^{-08}X^4 - 0.000240962X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16887 \cdot 10^{-07}$.

Bounding polynomials M and m:

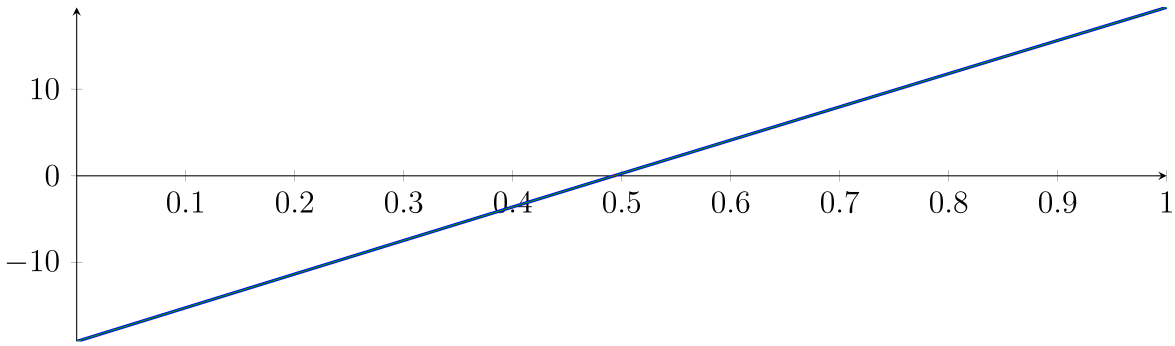
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

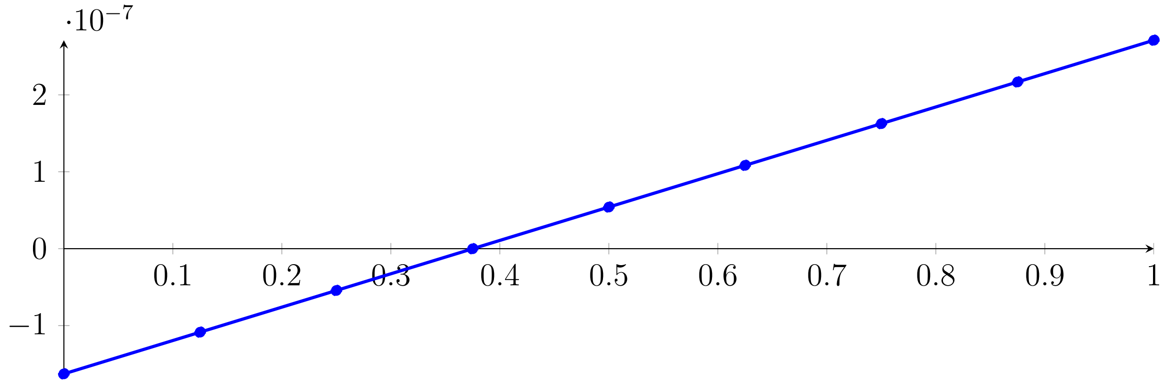
Longest intersection interval: $1.12517 \cdot 10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

60.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

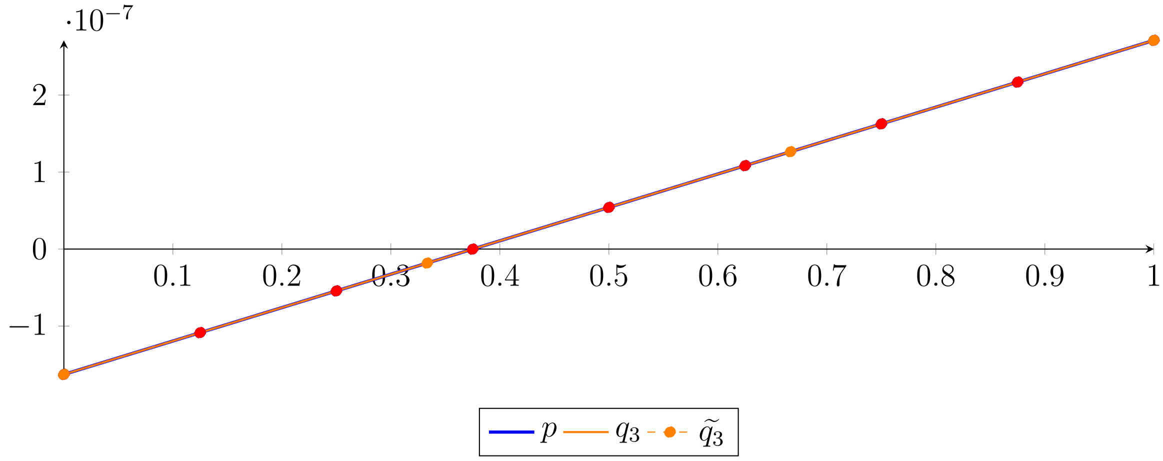
$$p = -3.81165\cdot10^{-21}X^8 + 6.77626\cdot10^{-21}X^7 + 2.96462\cdot10^{-21}X^6 + 5.92923\cdot10^{-21}X^5 + 1.11173\cdot10^{-20}X^4 + 1.48231\cdot10^{-21}X^3 - 5.27494\cdot10^{-17}X^2 + 4.33793\cdot10^{-07}X - 1.6278\cdot10^{-07}$$
$$= -1.6278\cdot10^{-07}B_{0,8}(X) - 1.08555\cdot10^{-07}B_{1,8}(X) - 5.43313\cdot10^{-08}B_{2,8}(X) - 1.07093\cdot10^{-10}B_{3,8}(X) + 5.41171\cdot10^{-08}B_{4,8}(X) + 1.08341\cdot10^{-07}B_{5,8}(X) + 1.62565\cdot10^{-07}B_{6,8}(X) + 2.1679\cdot10^{-07}B_{7,8}(X) + 2.71014\cdot10^{-07}B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = -1.90582\cdot10^{-21}X^3 - 5.2746\cdot10^{-17}X^2 + 4.33793\cdot10^{-07}X - 1.6278\cdot10^{-07}$$
$$= -1.6278\cdot10^{-07}B_{0,3} - 1.81818\cdot10^{-08}B_{1,3} + 1.26416\cdot10^{-07}B_{2,3} + 2.71014\cdot10^{-07}B_{3,3}$$

$$\tilde{q}_3 = 1.50585\cdot10^{-16}X^8 - 5.82707\cdot10^{-16}X^7 + 9.15943\cdot10^{-16}X^6 - 7.54824\cdot10^{-16}X^5 + 3.52096\cdot10^{-16}X^4 - 9.31289\cdot10^{-17}X^3 - 3.98474\cdot10^{-17}X^2 + 4.33793\cdot10^{-07}X - 1.6278\cdot10^{-07}$$
$$= -1.6278\cdot10^{-07}B_{0,8} - 1.08555\cdot10^{-07}B_{1,8} - 5.43313\cdot10^{-08}B_{2,8} - 1.07093\cdot10^{-10}B_{3,8} + 5.41171\cdot10^{-08}B_{4,8} + 1.08341\cdot10^{-07}B_{5,8} + 1.62565\cdot10^{-07}B_{6,8} + 2.1679\cdot10^{-07}B_{7,8} + 2.71014\cdot10^{-07}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 8.66435\cdot10^{-19}$.

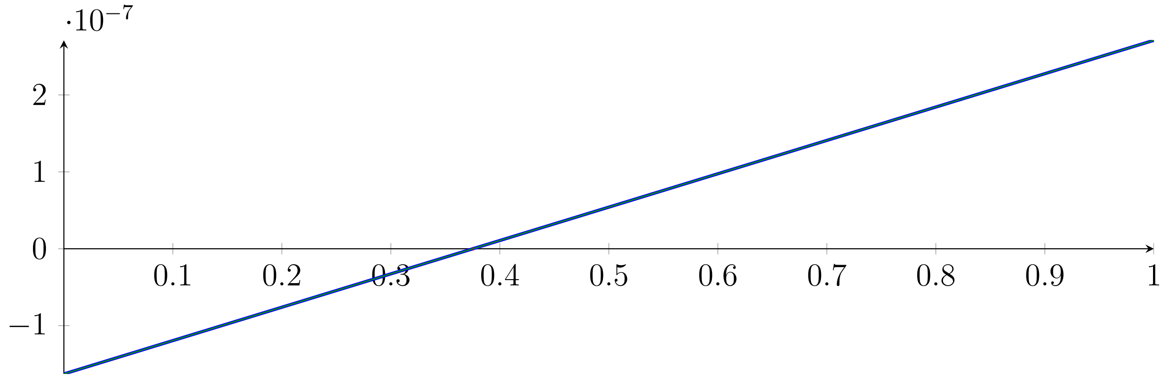
Bounding polynomials M and m:

$$M = -1.85288\cdot10^{-21}X^3 - 5.2746\cdot10^{-17}X^2 + 4.33793\cdot10^{-07}X - 1.6278\cdot10^{-07}$$
$$m = -1.90582\cdot10^{-21}X^3 - 5.2746\cdot10^{-17}X^2 + 4.33793\cdot10^{-07}X - 1.6278\cdot10^{-07}$$

Root of M and m:

$$N(M) = \{-1.49969\cdot10^{7}, 0.375247, 1.49696\cdot10^{7}\} \quad N(m) = \{-1.46018\cdot10^{7}, 0.375247, 1.45759\cdot10^{7}\}$$

Intersection intervals:



$[0.375247, 0.375247]$

Longest intersection interval: $7.69251 \cdot 10^{-09}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

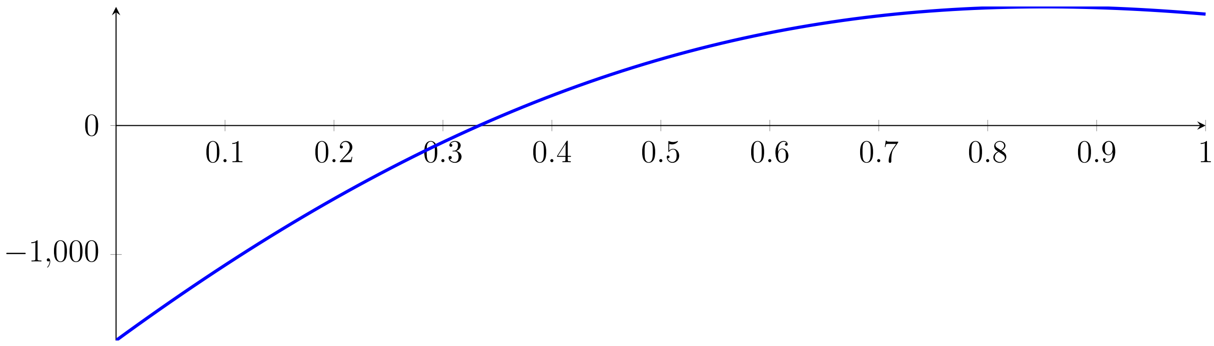
60.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

60.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-64}$.

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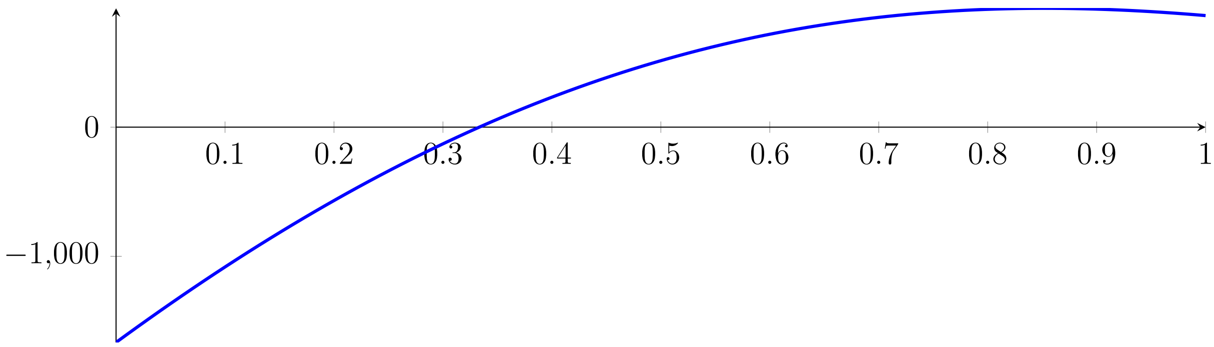
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

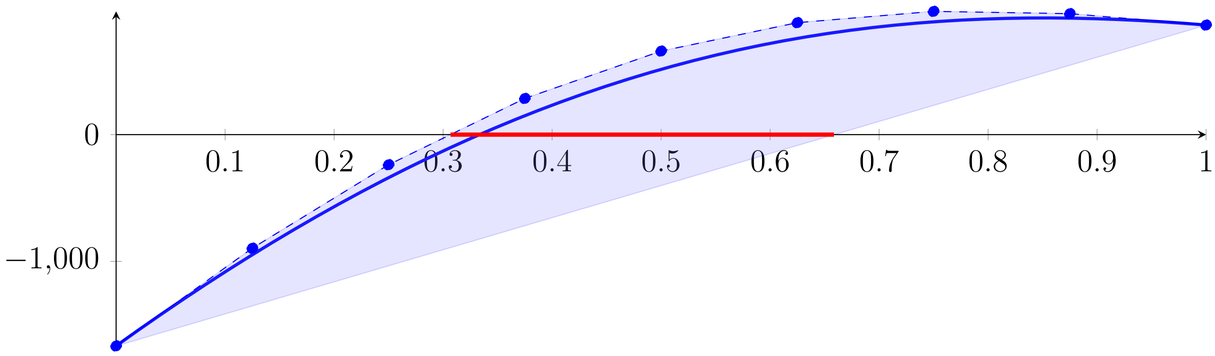
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



61.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

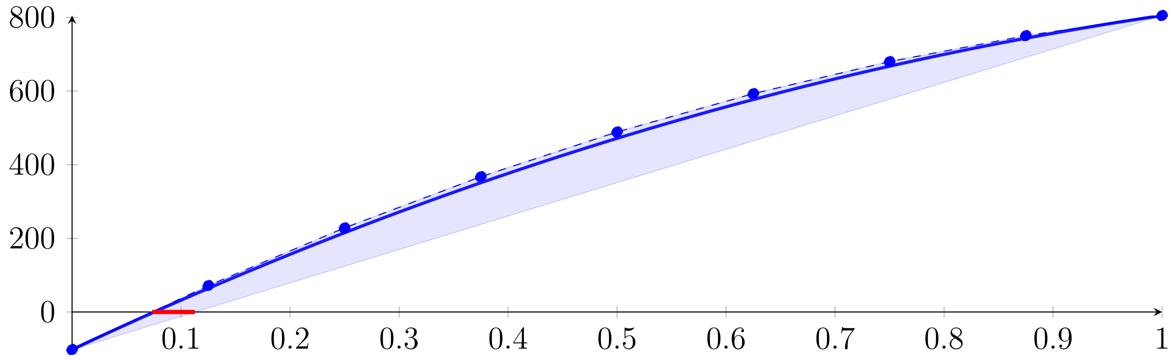
$$[0.306796, 0.658588]$$

Longest intersection interval: 0.351792
$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

61.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17$$
$$= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

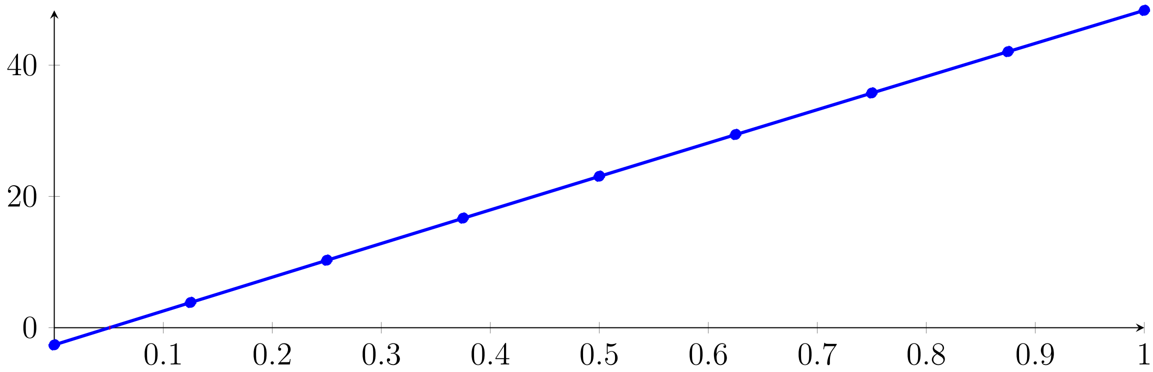
$$[0.0734515, 0.112637]$$

Longest intersection interval: 0.0391855
$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

61.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -5.41789 \cdot 10^{-13}X^8 - 1.26477 \cdot 10^{-12}X^7 - 5.05786 \cdot 10^{-10}X^6 + 3.66765 \cdot 10^{-08}X^5 + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683$$
$$= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

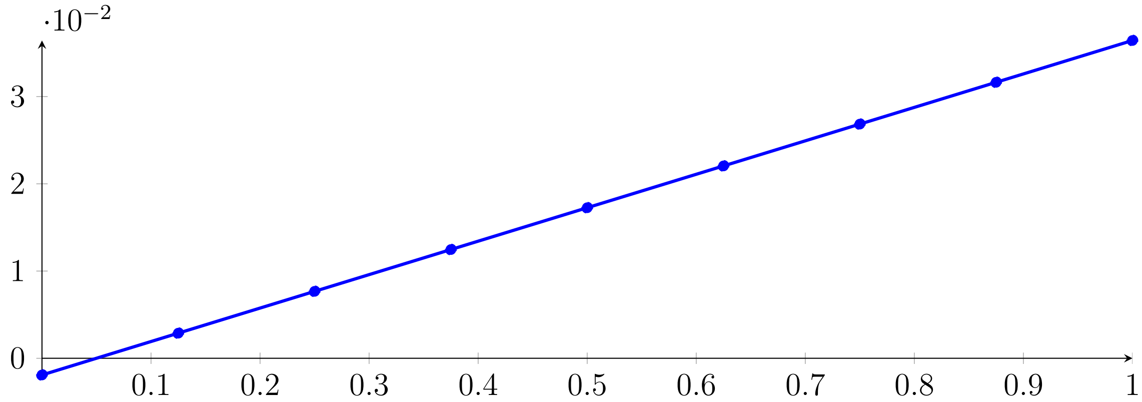
$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

61.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -7.11237\cdot 10^{-16}X^8 + 5.27356\cdot 10^{-16}X^7 - 7.38298\cdot 10^{-15}X^6 + 1.06859\cdot 10^{-15}X^5 \\ & - 1.09288\cdot 10^{-15}X^4 - 2.37227\cdot 10^{-13}X^3 - 4.12146\cdot 10^{-07}X^2 + 0.0383444X - 0.00191378 \\ = & -0.00191378B_{0,8}(X) + 0.00287927B_{1,8}(X) + 0.00767231B_{2,8}(X) \\ & + 0.0124653B_{3,8}(X) + 0.0172583B_{4,8}(X) + 0.0220513B_{5,8}(X) \\ & + 0.0268443B_{6,8}(X) + 0.0316373B_{7,8}(X) + 0.0364302B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

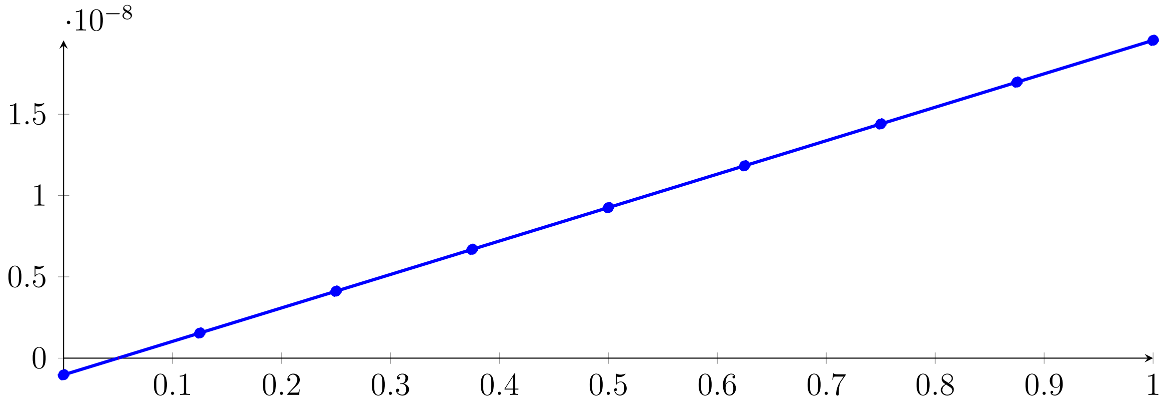
$$[0.0499103, 0.0499109]$$

Longest intersection interval: $5.36469\cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

61.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -3.09366\cdot 10^{-22}X^8 + 4.49986\cdot 10^{-22}X^7 - 4.03002\cdot 10^{-21}X^6 + 3.70577\cdot 10^{-22}X^5 - 3.47416 \\ & \cdot 10^{-22}X^4 + 9.26442\cdot 10^{-23}X^3 - 1.18608\cdot 10^{-19}X^2 + 2.05706\cdot 10^{-08}X - 1.02667\cdot 10^{-09} \\ = & -1.02667\cdot 10^{-09}B_{0,8}(X) + 1.54465\cdot 10^{-09}B_{1,8}(X) + 4.11597\cdot 10^{-09}B_{2,8}(X) \\ & + 6.68729\cdot 10^{-09}B_{3,8}(X) + 9.25862\cdot 10^{-09}B_{4,8}(X) + 1.18299\cdot 10^{-08}B_{5,8}(X) \\ & + 1.44013\cdot 10^{-08}B_{6,8}(X) + 1.69726\cdot 10^{-08}B_{7,8}(X) + 1.95439\cdot 10^{-08}B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87728 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

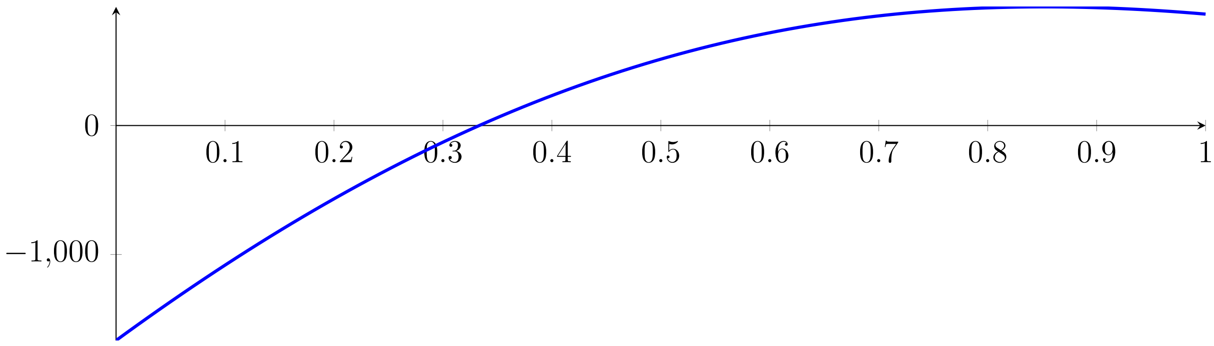
61.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

61.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-128}$.

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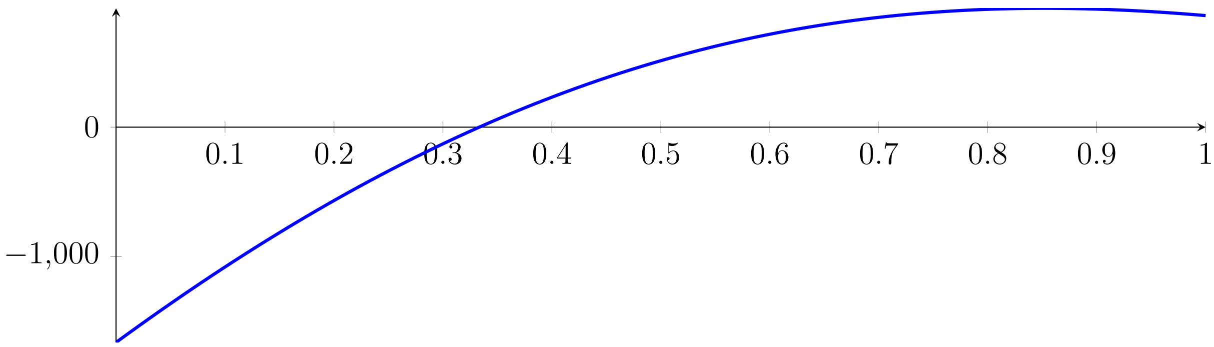
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

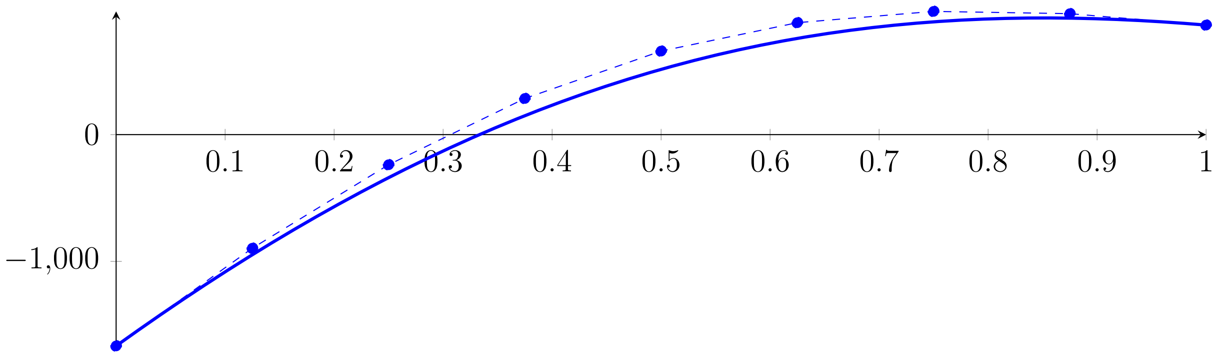
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



62.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

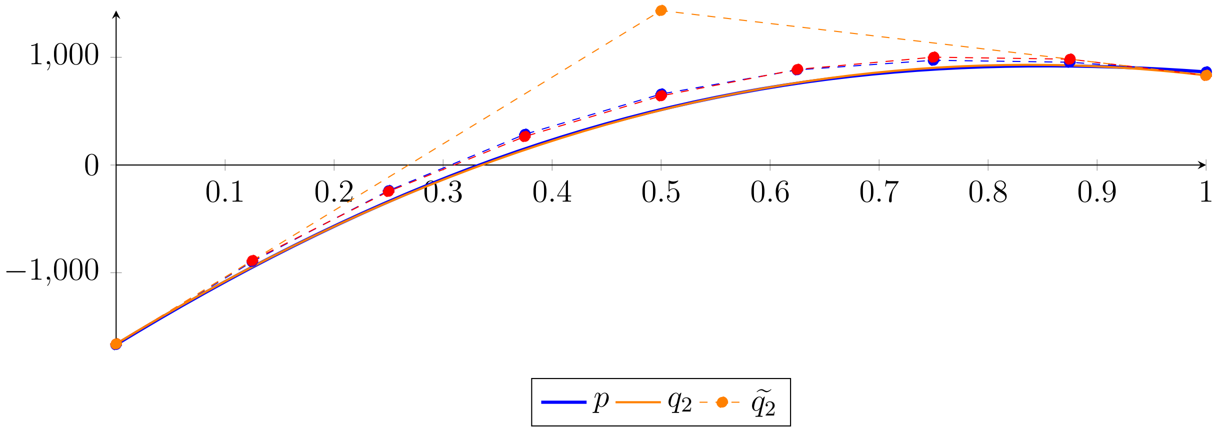
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -3.91297 \cdot 10^{-06}X^8 + 1.58774 \cdot 10^{-05}X^7 - 2.63335 \cdot 10^{-05}X^6 + 2.29285 \cdot 10^{-05}X^5 \\ &\quad - 1.11931 \cdot 10^{-05}X^4 + 3.0313 \cdot 10^{-06}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

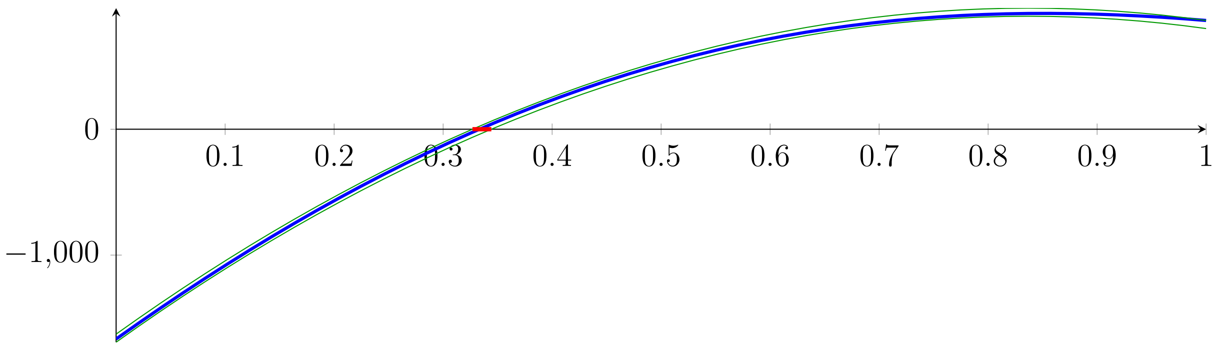
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

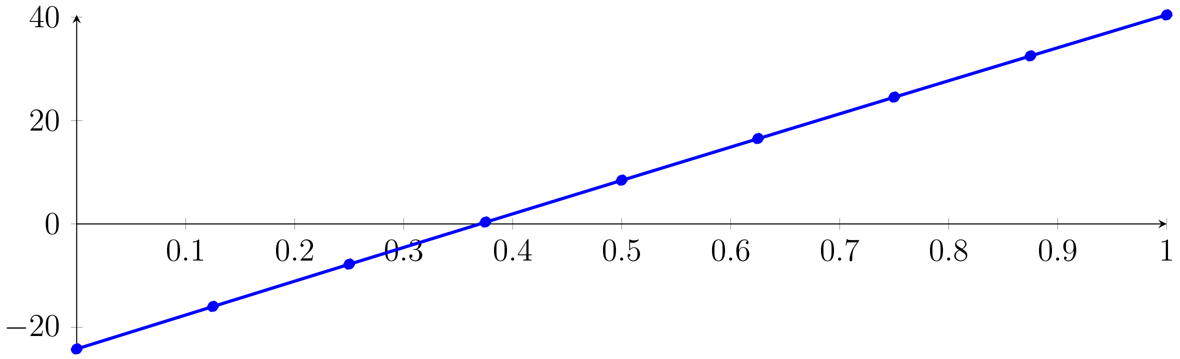
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

62.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

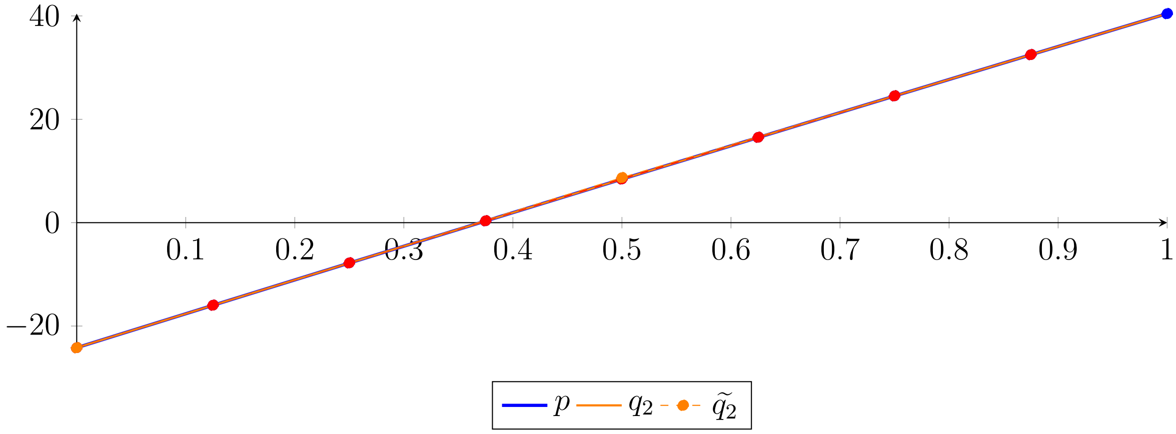
$$\begin{aligned}p = &-5.40012\cdot10^{-13}X^8 - 7.38964\cdot10^{-12}X^7 - 1.94416\cdot10^{-09}X^6 + 1.19265\cdot10^{-07}X^5\\ &+ 8.12335\cdot10^{-05}X^4 - 0.0012733X^3 - 1.18084X^2 + 65.8155X - 24.1945\\ = &-24.1945B_{0,8}(X) - 15.9676B_{1,8}(X) - 7.78282B_{2,8}(X) + 0.35975B_{3,8}(X)\\ &+ 8.4601B_{4,8}(X) + 16.5182B_{5,8}(X) + 24.5341B_{6,8}(X) + 32.5077B_{7,8}(X) + 40.4389B_{8,8}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 1.00159 \cdot 10^{-08}X^8 - 3.3372 \cdot 10^{-08}X^7 + 4.23875 \cdot 10^{-08}X^6 - 2.49721 \cdot 10^{-08}X^5 \\ &\quad + 6.08793 \cdot 10^{-09}X^4 + 1.46429 \cdot 10^{-10}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.
Bounding polynomials M and m:

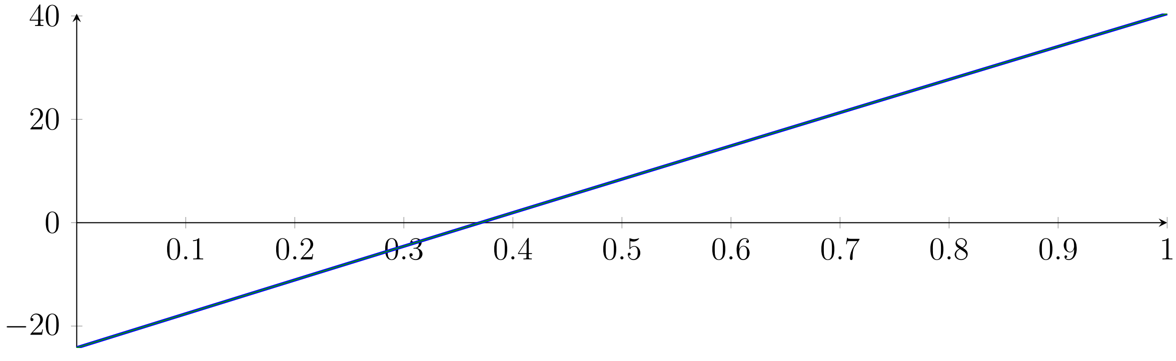
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



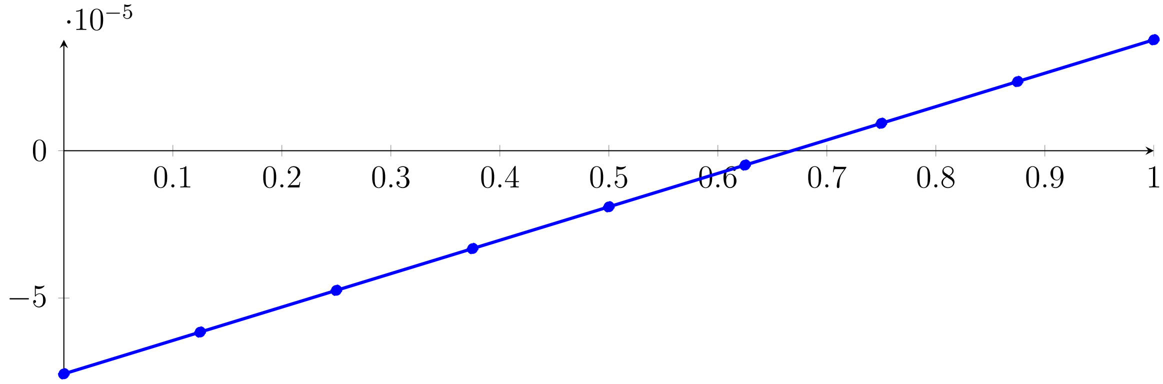
$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

62.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

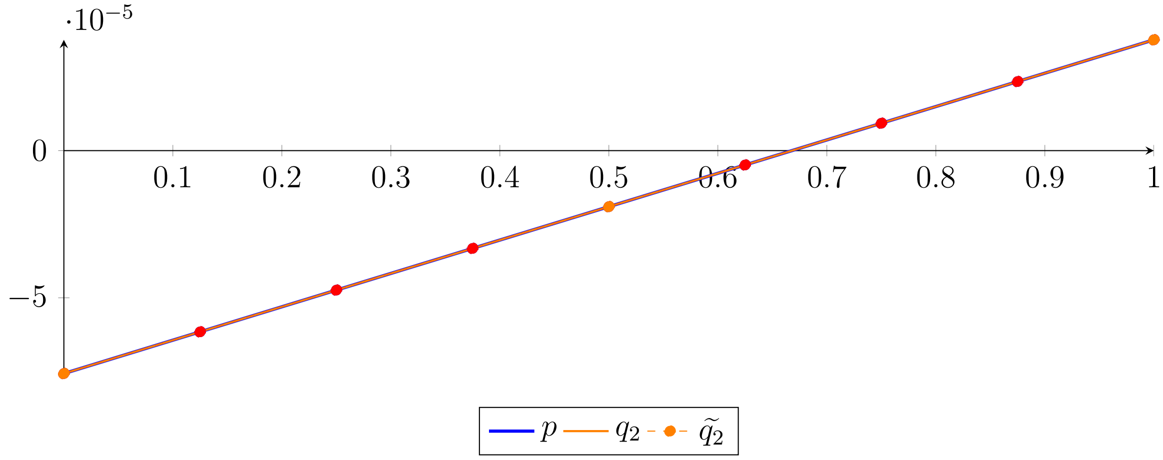
$$\begin{aligned}p = &-2.1684\cdot 10^{-19}X^8 - 4.33681\cdot 10^{-19}X^7 + 2.12504\cdot 10^{-17}X^6 - 1.51788\cdot 10^{-18}X^5 \\ &+ 7.58942\cdot 10^{-18}X^4 - 3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05} \\ = &-7.5732\cdot 10^{-05}B_{0,8}(X) - 6.15596\cdot 10^{-05}B_{1,8}(X) - 4.73873\cdot 10^{-05}B_{2,8}(X) \\ &- 3.32149\cdot 10^{-05}B_{3,8}(X) - 1.90425\cdot 10^{-05}B_{4,8}(X) - 4.87016\cdot 10^{-06}B_{5,8}(X) \\ &+ 9.3022\cdot 10^{-06}B_{6,8}(X) + 2.34746\cdot 10^{-05}B_{7,8}(X) + 3.76469\cdot 10^{-05}B_{8,8}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05} \\ &= -7.5732\cdot 10^{-05}B_{0,2} - 1.90425\cdot 10^{-05}B_{1,2} + 3.76469\cdot 10^{-05}B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_2 = &\,5.62292\cdot 10^{-14}X^8 - 2.22643\cdot 10^{-13}X^7 + 3.60043\cdot 10^{-13}X^6 - 3.05846\cdot 10^{-13}X^5 + 1.46182 \\ &\cdot 10^{-13}X^4 - 3.90612\cdot 10^{-14}X^3 - 3.59793\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05} \\ = &-7.5732\cdot 10^{-05}B_{0,8} - 6.15596\cdot 10^{-05}B_{1,8} - 4.73873\cdot 10^{-05}B_{2,8} - 3.32149\cdot 10^{-05}B_{3,8} - 1.90425 \\ &\cdot 10^{-05}B_{4,8} - 4.87016\cdot 10^{-06}B_{5,8} + 9.3022\cdot 10^{-06}B_{6,8} + 2.34746\cdot 10^{-05}B_{7,8} + 3.76469\cdot 10^{-05}B_{8,8}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.98887\cdot 10^{-16}$.

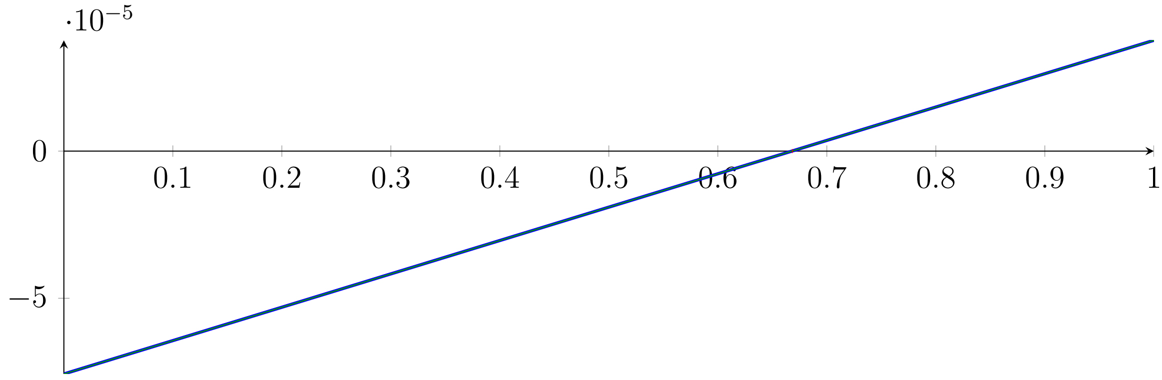
Bounding polynomials M and m:

$$M = -3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05}$$
$$m = -3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.667955, 3.14644\cdot 10^{7}\} \qquad N(m) = \{0.667955, 3.14644\cdot 10^{7}\}$$

Intersection intervals:



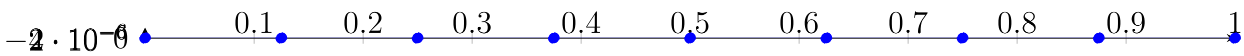
$[0.667955, 0.667955]$

Longest intersection interval: $1.88052 \cdot 10^{-09}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

62.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

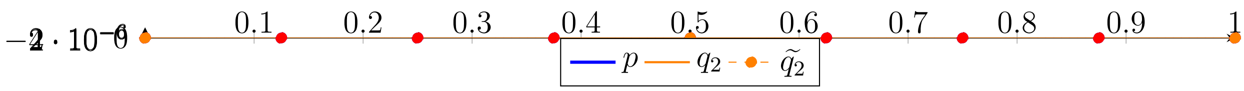
$$\begin{aligned} p = & -1.31266 \cdot 10^{-27} X^8 - 6.92683 \cdot 10^{-26} X^6 - 8.48183 \cdot 10^{-27} X^5 - 1.06023 \\ & \cdot 10^{-26} X^4 + 1.41364 \cdot 10^{-27} X^3 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\ = & \, 4.52469 \cdot 10^{-14} B_{0,8}(X) + 7.18983 \cdot 10^{-14} B_{1,8}(X) + 9.85497 \cdot 10^{-14} B_{2,8}(X) \\ & + 1.25201 \cdot 10^{-13} B_{3,8}(X) + 1.51852 \cdot 10^{-13} B_{4,8}(X) + 1.78504 \cdot 10^{-13} B_{5,8}(X) \\ & + 2.05155 \cdot 10^{-13} B_{6,8}(X) + 2.31807 \cdot 10^{-13} B_{7,8}(X) + 2.58458 \cdot 10^{-13} B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 = & \, 2.44862 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\ = & \, 4.52469 \cdot 10^{-14} B_{0,2} + 1.51852 \cdot 10^{-13} B_{1,2} + 2.58458 \cdot 10^{-13} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 = & -9.59093 \cdot 10^{-23} X^8 + 4.40717 \cdot 10^{-22} X^7 - 8.25115 \cdot 10^{-22} X^6 + 8.08254 \cdot 10^{-22} X^5 - 4.43405 \\ & \cdot 10^{-22} X^4 + 1.3589 \cdot 10^{-22} X^3 - 2.18259 \cdot 10^{-23} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\ = & \, 4.52469 \cdot 10^{-14} B_{0,8} + 7.18983 \cdot 10^{-14} B_{1,8} + 9.85497 \cdot 10^{-14} B_{2,8} + 1.25201 \cdot 10^{-13} B_{3,8} + 1.51852 \\ & \cdot 10^{-13} B_{4,8} + 1.78504 \cdot 10^{-13} B_{5,8} + 2.05155 \cdot 10^{-13} B_{6,8} + 2.31807 \cdot 10^{-13} B_{7,8} + 2.58458 \cdot 10^{-13} B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.40606 \cdot 10^{-25}$.
Bounding polynomials M and m:

$$M = 2.42338 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14}$$
$$m = 2.47387 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14}$$

Root of M and m:

$$N(M) = \{-8.79809 \cdot 10^{13}, -0.213542\} \qquad N(m) = \{-8.61853 \cdot 10^{13}, -0.214286\}$$

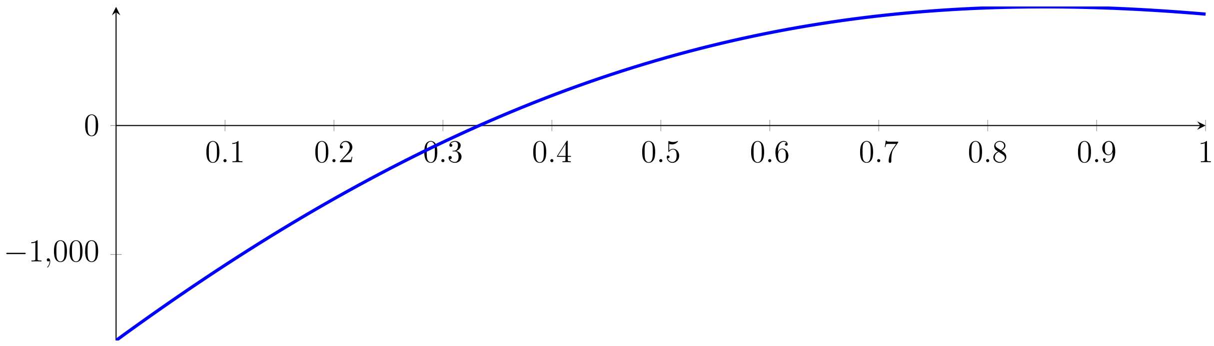
Intersection intervals:

No intersection intervals with the x axis.

62.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

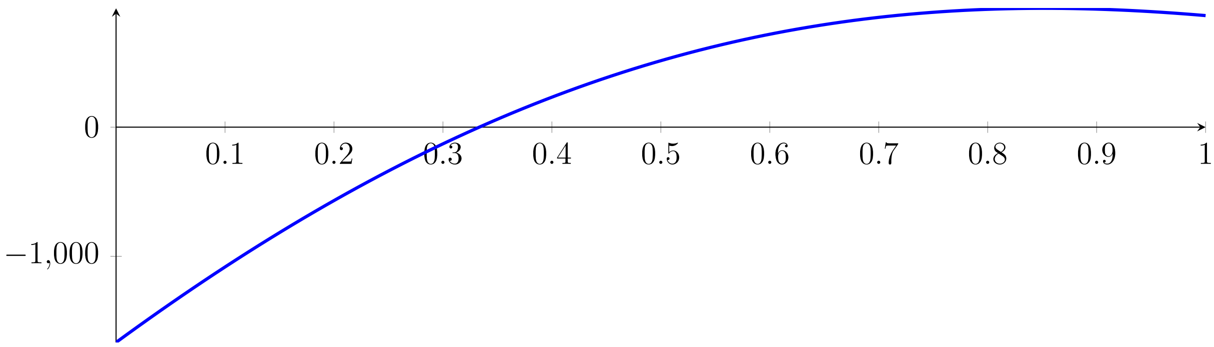
with precision $\varepsilon = 1 \cdot 10^{-128}$.

63 Running CubeClip on f_8 with epsilon 128

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

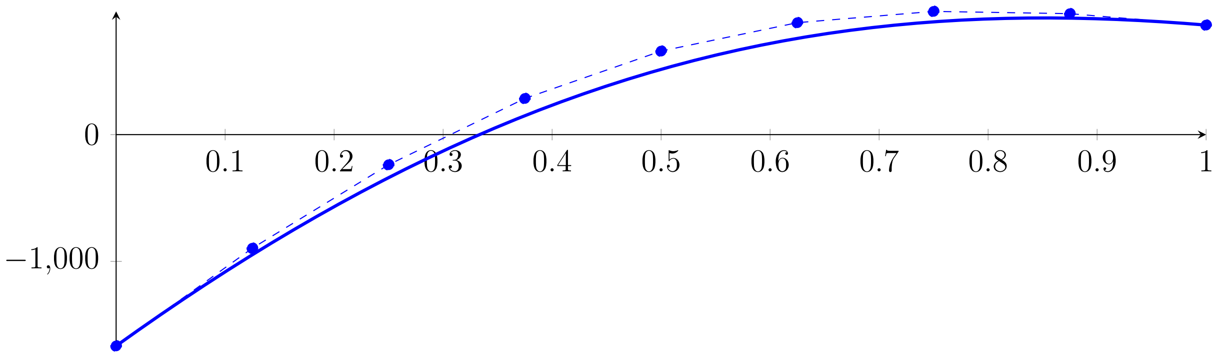
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



63.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

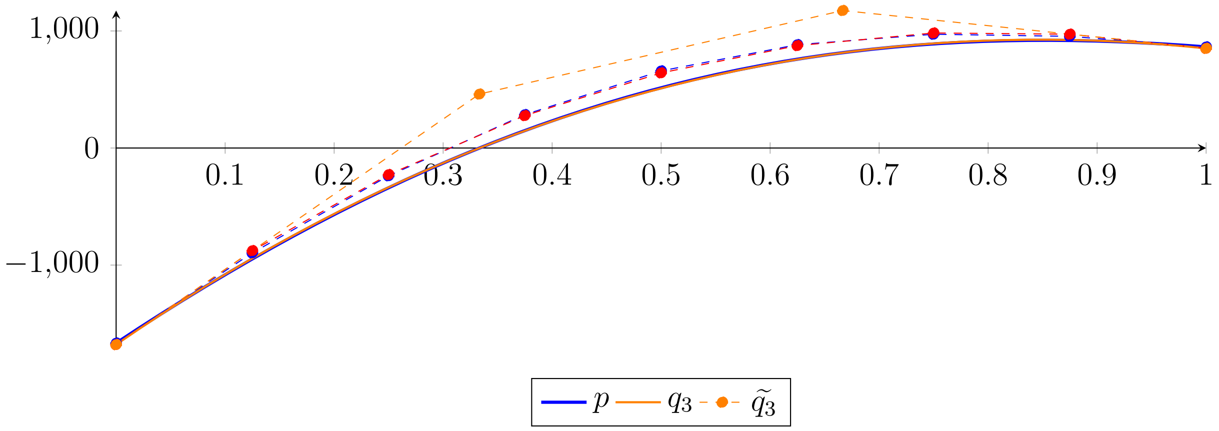
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 2.38117 \cdot 10^{-06}X^8 - 9.49957 \cdot 10^{-06}X^7 + 1.54757 \cdot 10^{-05}X^6 - 1.32609 \cdot 10^{-05}X^5 \\ &\quad + 6.41819 \cdot 10^{-06}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

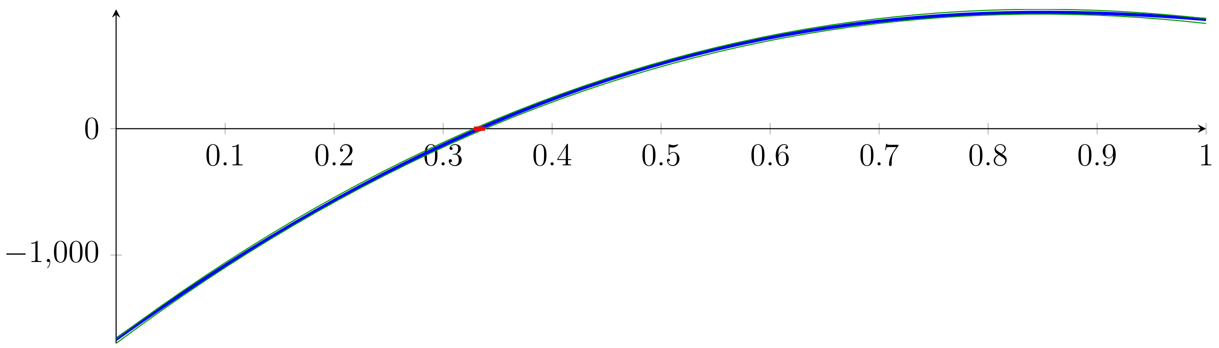
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$[0.328258, 0.338551]$

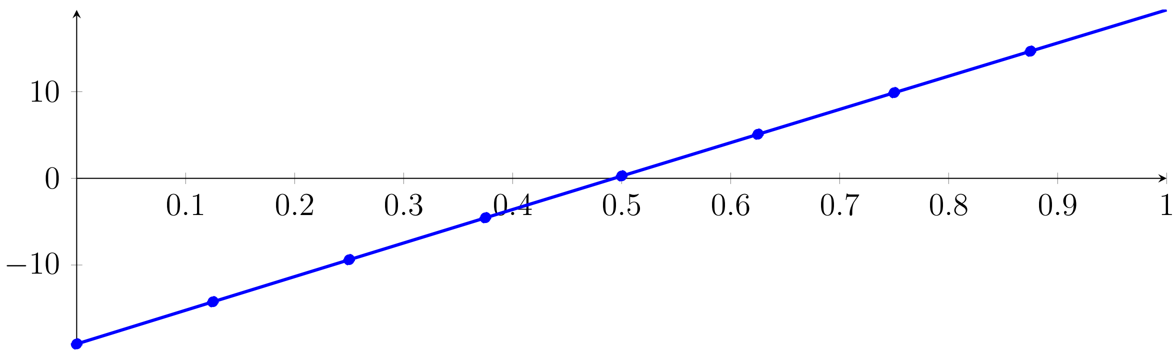
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

63.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

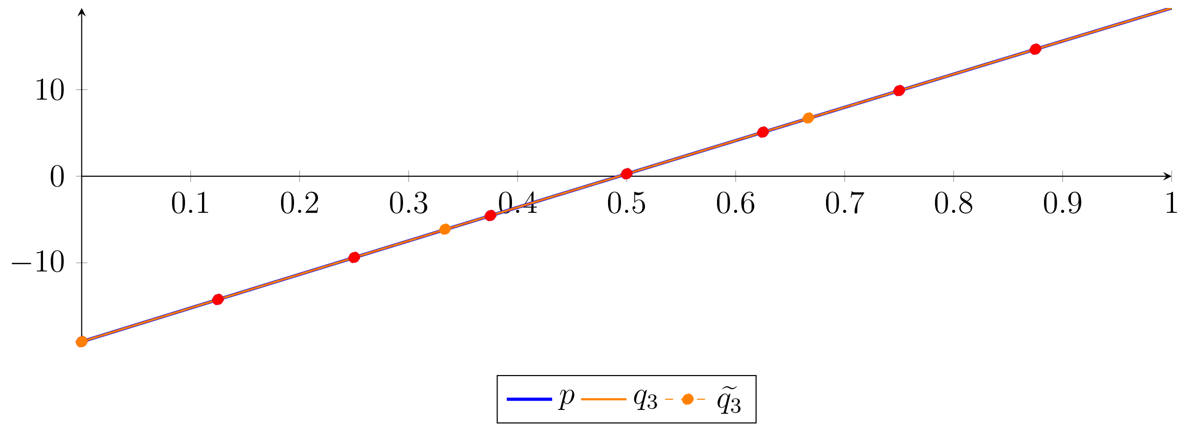
$$\begin{aligned} p = &-1.27898\cdot10^{-13}X^8 - 1.13687\cdot10^{-13}X^7 - 8.23661\cdot10^{-11}X^6 + 8.72882\cdot10^{-09}X^5 \\ &+ 1.00963\cdot10^{-05}X^4 - 0.000261161X^3 - 0.416284X^2 + 38.9643X - 19.1124 \\ = &-19.1124B_{0,8}(X) - 14.2419B_{1,8}(X) - 9.38619B_{2,8}(X) - 4.54539B_{3,8}(X) \\ &+ 0.280537B_{4,8}(X) + 5.09158B_{5,8}(X) + 9.88774B_{6,8}(X) + 14.669B_{7,8}(X) + 19.4354B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2.3353{\cdot}10^{-08}X^8 - 9.31856{\cdot}10^{-08}X^7 + 1.51861{\cdot}10^{-07}X^6 - 1.30228{\cdot}10^{-07}X^5 \\ &\quad + 6.31618{\cdot}10^{-08}X^4 - 0.000240962X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16887{\cdot}10^{-07}$.

Bounding polynomials M and m:

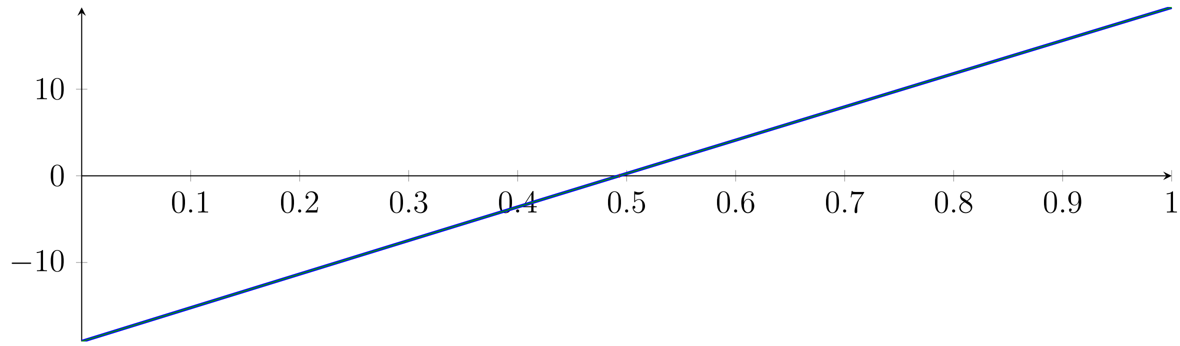
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

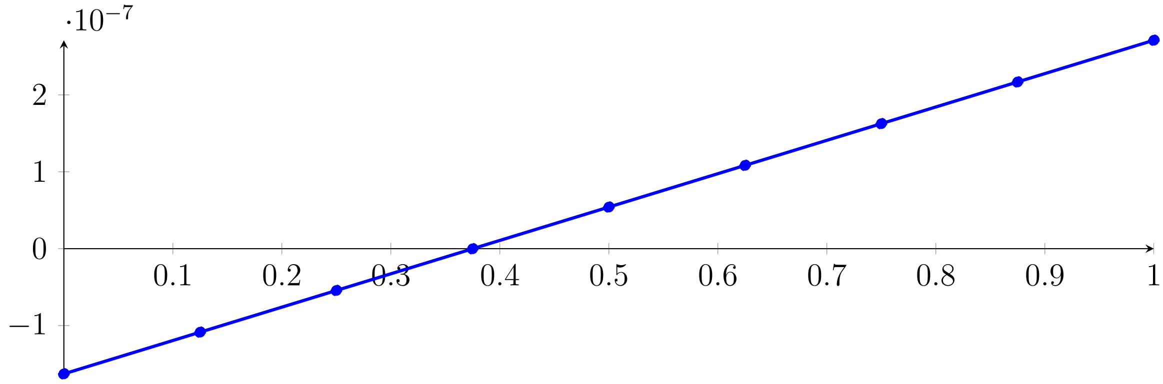
Longest intersection interval: $1.12517{\cdot}10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

63.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

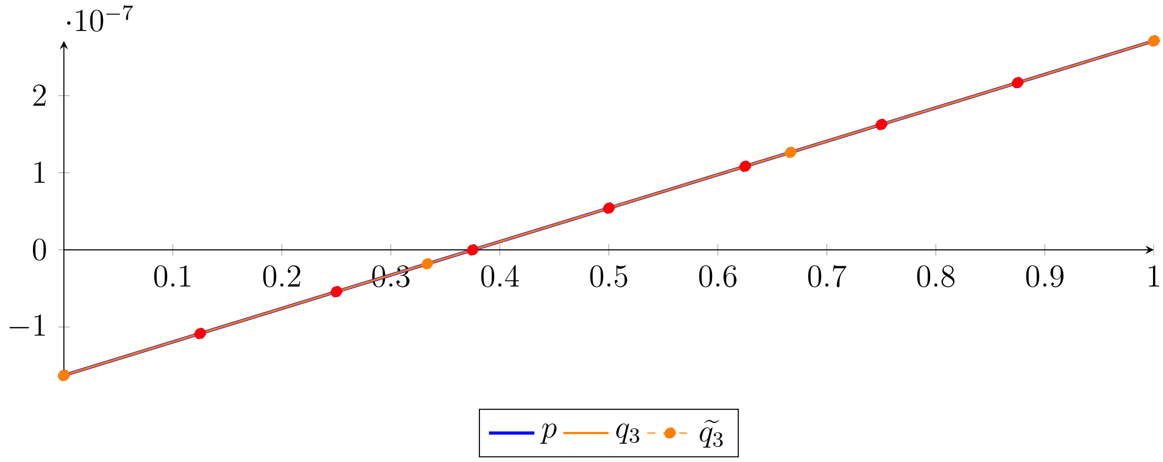
$$p = -3.81165\cdot10^{-21}X^8 + 6.77626\cdot10^{-21}X^7 + 2.96462\cdot10^{-21}X^6 + 5.92923\cdot10^{-21}X^5 + 1.11173\cdot10^{-20}X^4 + 1.48231\cdot10^{-21}X^3 - 5.27494\cdot10^{-17}X^2 + 4.33793\cdot10^{-07}X - 1.6278\cdot10^{-07}$$
$$= -1.6278\cdot10^{-07}B_{0,8}(X) - 1.08555\cdot10^{-07}B_{1,8}(X) - 5.43313\cdot10^{-08}B_{2,8}(X) - 1.07093\cdot10^{-10}B_{3,8}(X) + 5.41171\cdot10^{-08}B_{4,8}(X) + 1.08341\cdot10^{-07}B_{5,8}(X) + 1.62565\cdot10^{-07}B_{6,8}(X) + 2.1679\cdot10^{-07}B_{7,8}(X) + 2.71014\cdot10^{-07}B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = -1.90582\cdot10^{-21}X^3 - 5.2746\cdot10^{-17}X^2 + 4.33793\cdot10^{-07}X - 1.6278\cdot10^{-07}$$
$$= -1.6278\cdot10^{-07}B_{0,3} - 1.81818\cdot10^{-08}B_{1,3} + 1.26416\cdot10^{-07}B_{2,3} + 2.71014\cdot10^{-07}B_{3,3}$$

$$\widetilde{q}_3 = 1.50585\cdot10^{-16}X^8 - 5.82707\cdot10^{-16}X^7 + 9.15943\cdot10^{-16}X^6 - 7.54824\cdot10^{-16}X^5 + 3.52096\cdot10^{-16}X^4 - 9.31289\cdot10^{-17}X^3 - 3.98474\cdot10^{-17}X^2 + 4.33793\cdot10^{-07}X - 1.6278\cdot10^{-07}$$
$$= -1.6278\cdot10^{-07}B_{0,8} - 1.08555\cdot10^{-07}B_{1,8} - 5.43313\cdot10^{-08}B_{2,8} - 1.07093\cdot10^{-10}B_{3,8} + 5.41171\cdot10^{-08}B_{4,8} + 1.08341\cdot10^{-07}B_{5,8} + 1.62565\cdot10^{-07}B_{6,8} + 2.1679\cdot10^{-07}B_{7,8} + 2.71014\cdot10^{-07}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 8.66435\cdot10^{-19}$.

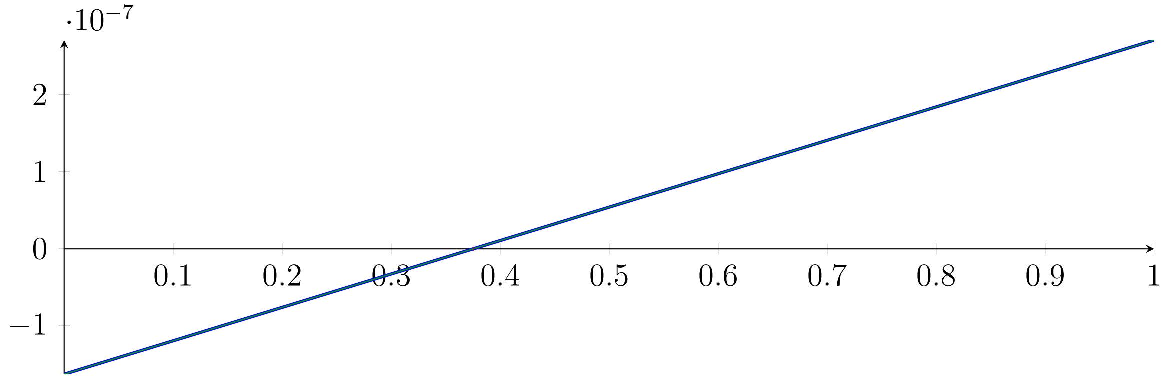
Bounding polynomials M and m:

$$M = -1.85288\cdot10^{-21}X^3 - 5.2746\cdot10^{-17}X^2 + 4.33793\cdot10^{-07}X - 1.6278\cdot10^{-07}$$
$$m = -1.90582\cdot10^{-21}X^3 - 5.2746\cdot10^{-17}X^2 + 4.33793\cdot10^{-07}X - 1.6278\cdot10^{-07}$$

Root of M and m:

$$N(M) = \{-1.49969\cdot10^7, 0.375247, 1.49696\cdot10^7\} \quad N(m) = \{-1.46018\cdot10^7, 0.375247, 1.45759\cdot10^7\}$$

Intersection intervals:



$[0.375247, 0.375247]$

Longest intersection interval: $7.69251 \cdot 10^{-09}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

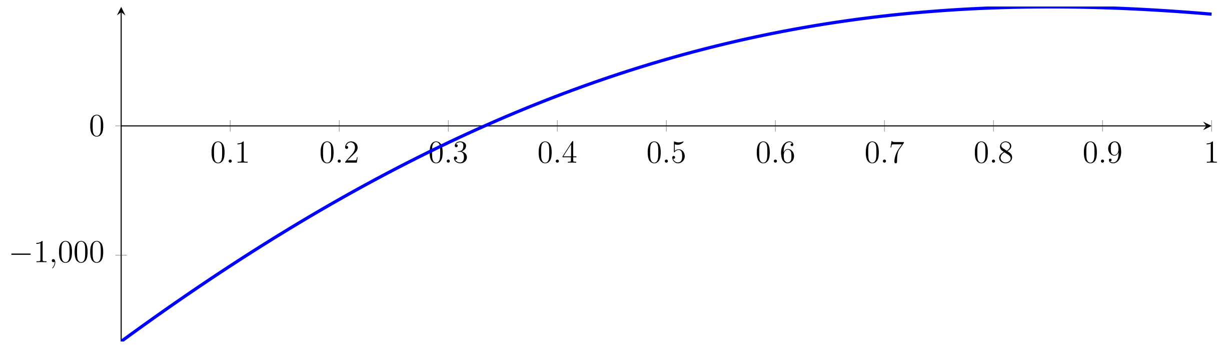
63.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

63.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

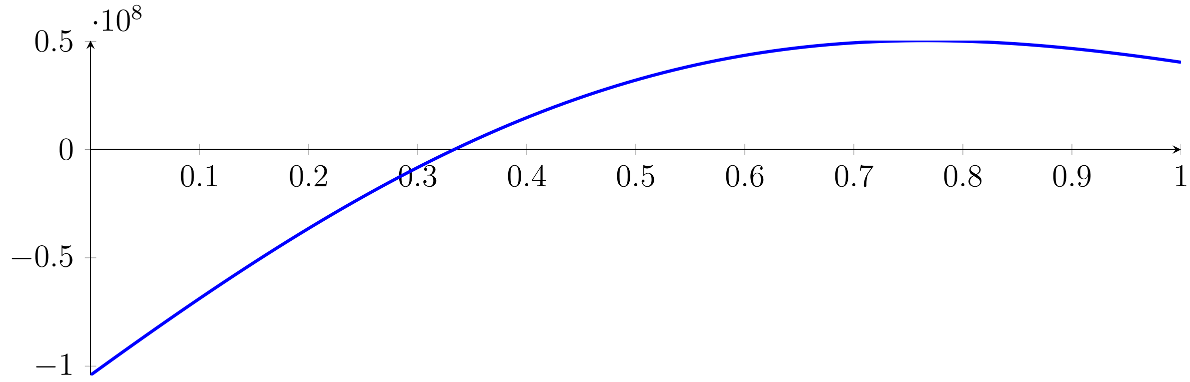
with precision $\varepsilon = 1 \cdot 10^{-128}$.

64 Running BezClip on f_{16} with epsilon 2

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called BezClip with input polynomial on interval $[0, 1]$:

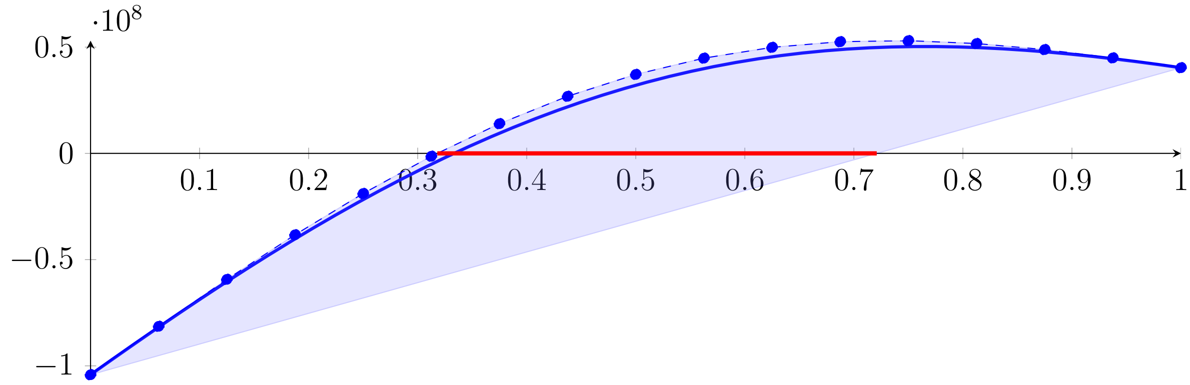
$$\begin{aligned}p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10}\\ &+ 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5}\\ &+ 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}\end{aligned}$$



64.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p = &-0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11}\\ &+ 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\\ &\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}\\ = &-1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203\\ &\cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X)\\ &+ 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901\\ &\cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X)\\ &+ 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

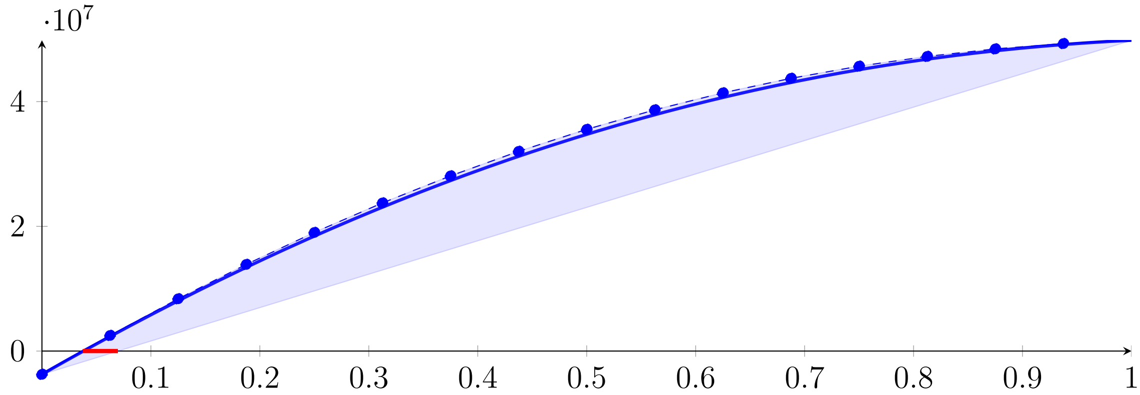
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

64.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p = &-0.00483322X^{16} + 0.0186809X^{15} - 0.0194312X^{14} - 0.0738695X^{13} - 1.11673X^{12}\\ &- 5.0471X^{11} + 36.3082X^{10} + 692.914X^{9} + 1886.96X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5}\\ &+ 3.91945\cdot 10^{6}X^{4} - 7.54561\cdot 10^{6}X^{3} - 4.29225\cdot 10^{7}X^{2} + 9.97982\cdot 10^{7}X - 3.73501\cdot 10^{6}\\ = &-3.73501\cdot 10^{6}B_{0,16}(X) + 2.50237\cdot 10^{6}B_{1,16}(X) + 8.38207\cdot 10^{6}B_{2,16}(X) + 1.38906\\ &\cdot 10^{7}B_{3,16}(X) + 1.90167\cdot 10^{7}B_{4,16}(X) + 2.37512\cdot 10^{7}B_{5,16}(X) + 2.80875\cdot 10^{7}B_{6,16}(X)\\ &+ 3.20213\cdot 10^{7}B_{7,16}(X) + 3.55508\cdot 10^{7}B_{8,16}(X) + 3.86768\cdot 10^{7}B_{9,16}(X) + 4.14025\\ &\cdot 10^{7}B_{10,16}(X) + 4.37336\cdot 10^{7}B_{11,16}(X) + 4.56783\cdot 10^{7}B_{12,16}(X) + 4.7247\cdot 10^{7}B_{13,16}(X)\\ &+ 4.84524\cdot 10^{7}B_{14,16}(X) + 4.93093\cdot 10^{7}B_{15,16}(X) + 4.98343\cdot 10^{7}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

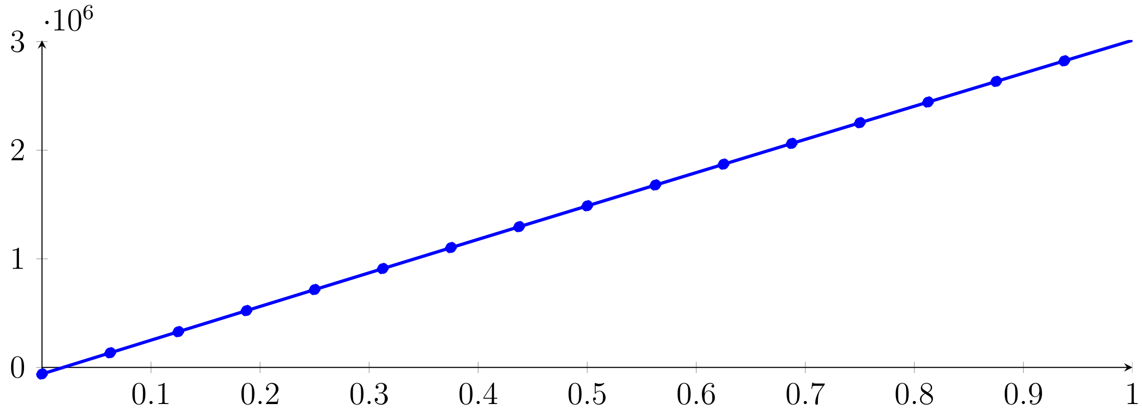
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

64.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p = &-0.000205056X^{16} + 0.000776167X^{15} - 0.000863333X^{14} - 0.000217499X^{13} - 0.00809276X^{12}\\ &- 5.09773\cdot 10^{-05}X^{11} - 0.00564923X^{10} - 0.000162811X^{9} - 0.000215376X^{8} - 3.32948\cdot 10^{-07}X^{7}\\ &- 0.000259866X^{6} + 0.0161077X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot 10^{6}X - 60508.5\\ = &-60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X)\\ &+ 910202B_{5,16}(X) + 1.1032\cdot 10^{6}B_{6,16}(X) + 1.29582\cdot 10^{6}B_{7,16}(X) + 1.48805\cdot 10^{6}B_{8,16}(X)\\ &+ 1.6799\cdot 10^{6}B_{9,16}(X) + 1.87137\cdot 10^{6}B_{10,16}(X) + 2.06245\cdot 10^{6}B_{11,16}(X) + 2.25315\cdot 10^{6}B_{12,16}(X)\\ &+ 2.44346\cdot 10^{6}B_{13,16}(X) + 2.63339\cdot 10^{6}B_{14,16}(X) + 2.82293\cdot 10^{6}B_{15,16}(X) + 3.01209\cdot 10^{6}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

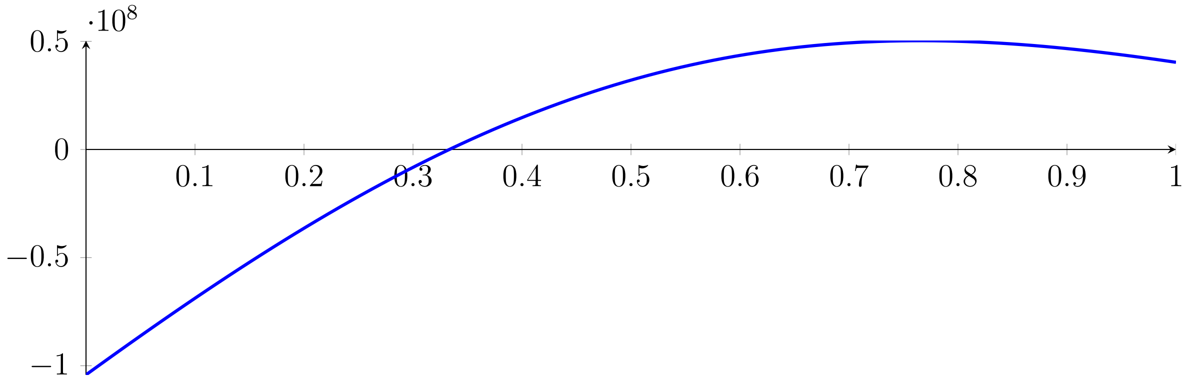
64.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Found root in interval $[0.333333, 0.333337]$ at recursion depth 4!

64.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333337]$$

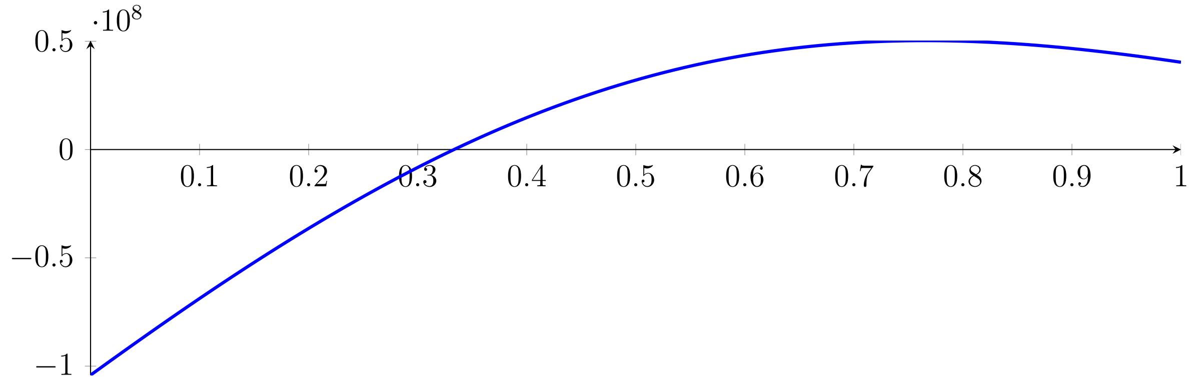
with precision $\varepsilon = 0.01$.

65 Running QuadClip on f_{16} with epsilon 2

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

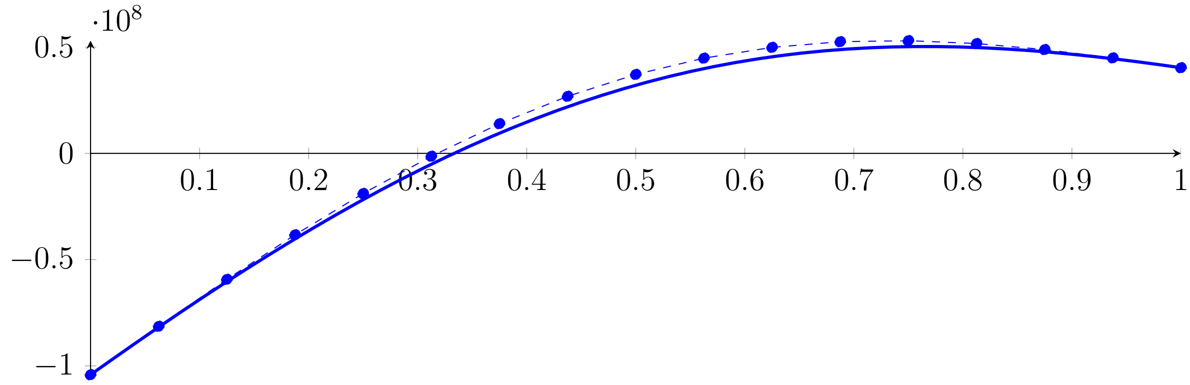
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



65.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

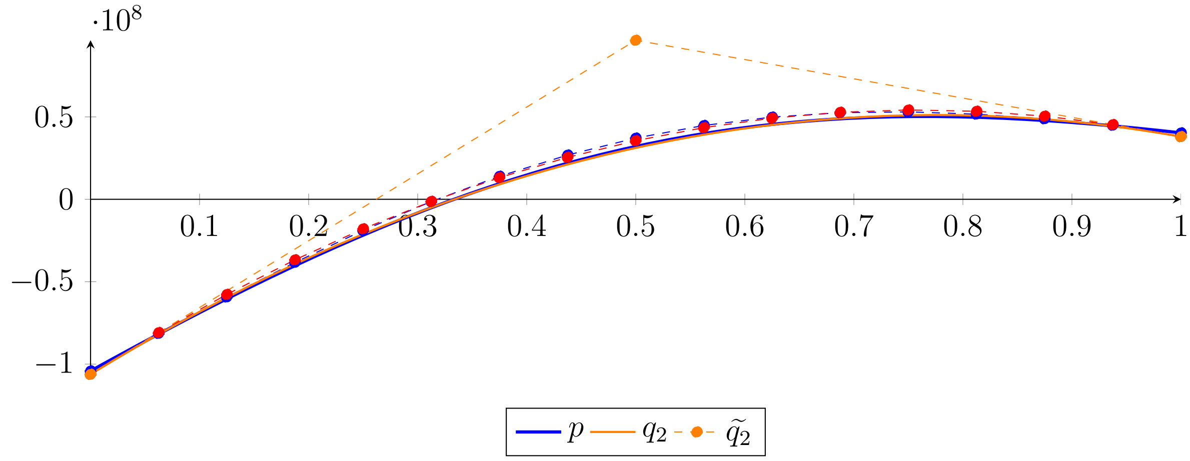
$$\begin{aligned} p &= -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\quad \cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ &= -1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.06197{\cdot}10^8$$
$$= -1.06197{\cdot}10^8 B_{0,2} + 9.65115{\cdot}10^7 B_{1,2} + 3.80384{\cdot}10^7 B_{2,2}$$

$$\widetilde{q}_2 = -1.41787{\cdot}10^6 X^{16} + 1.11761{\cdot}10^7 X^{15} - 3.98898{\cdot}10^7 X^{14} + 8.52437{\cdot}10^7 X^{13} - 1.21528{\cdot}10^8 X^{12} + 1.21946{\cdot}10^8 X^{11} - 8.86062{\cdot}10^7 X^{10} + 4.72904{\cdot}10^7 X^9 - 1.86355{\cdot}10^7 X^8 + 5.41059{\cdot}10^6 X^7 - 1.14487{\cdot}10^6 X^6 + 172240 X^5 - 17636.6 X^4 + 1159.74 X^3 - 2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.06197{\cdot}10^8$$
$$= -1.06197{\cdot}10^8 B_{0,16} - 8.08583{\cdot}10^7 B_{1,16} - 5.76963{\cdot}10^7 B_{2,16} - 3.67107{\cdot}10^7 B_{3,16} - 1.79017{\cdot}10^7 B_{4,16} - 1.26923{\cdot}10^6 B_{5,16} + 1.31867{\cdot}10^7 B_{6,16} + 2.54662{\cdot}10^7 B_{7,16} + 3.55691{\cdot}10^7 B_{8,16} + 4.34957{\cdot}10^7 B_{9,16} + 4.92455{\cdot}10^7 B_{10,16} + 5.2819{\cdot}10^7 B_{11,16} + 5.42159{\cdot}10^7 B_{12,16} + 5.34363{\cdot}10^7 B_{13,16} + 5.04802{\cdot}10^7 B_{14,16} + 4.53476{\cdot}10^7 B_{15,16} + 3.80384{\cdot}10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233{\cdot}10^6$.

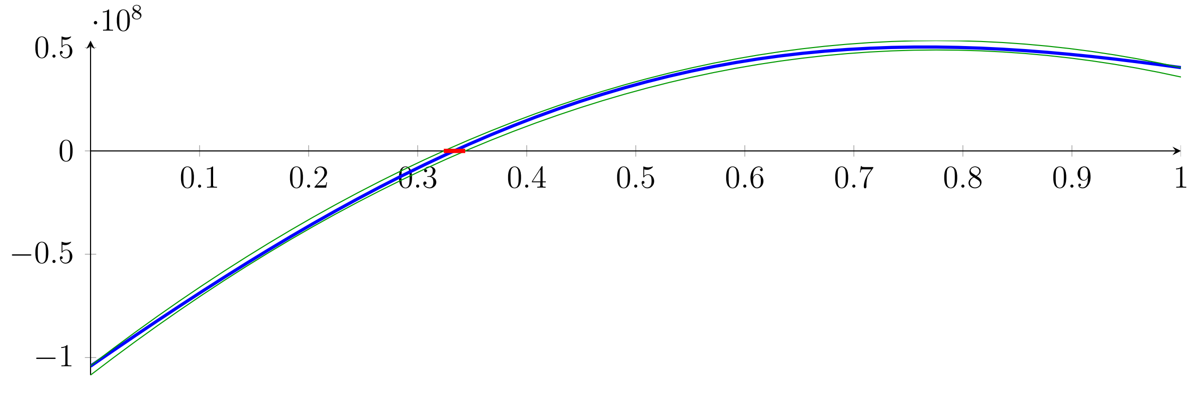
Bounding polynomials M and m:

$$M = -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.03924{\cdot}10^8$$
$$m = -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.08469{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

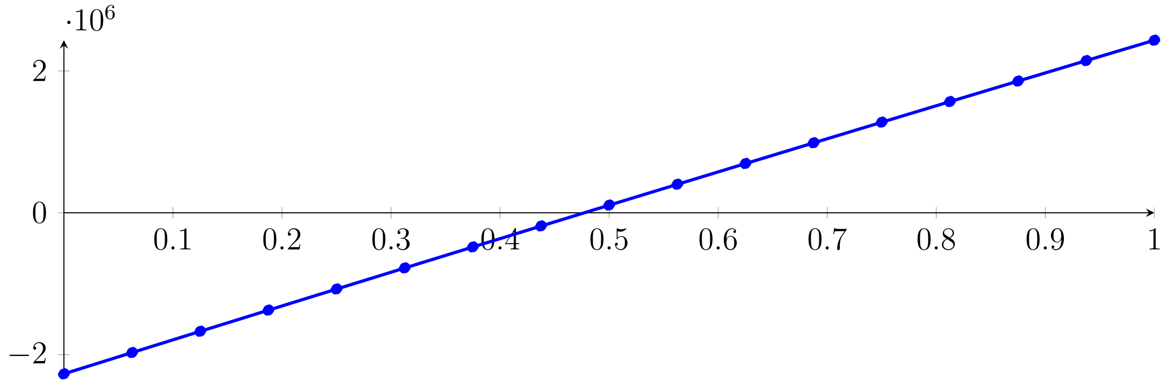
Longest intersection interval: 0.0196686

$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

65.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

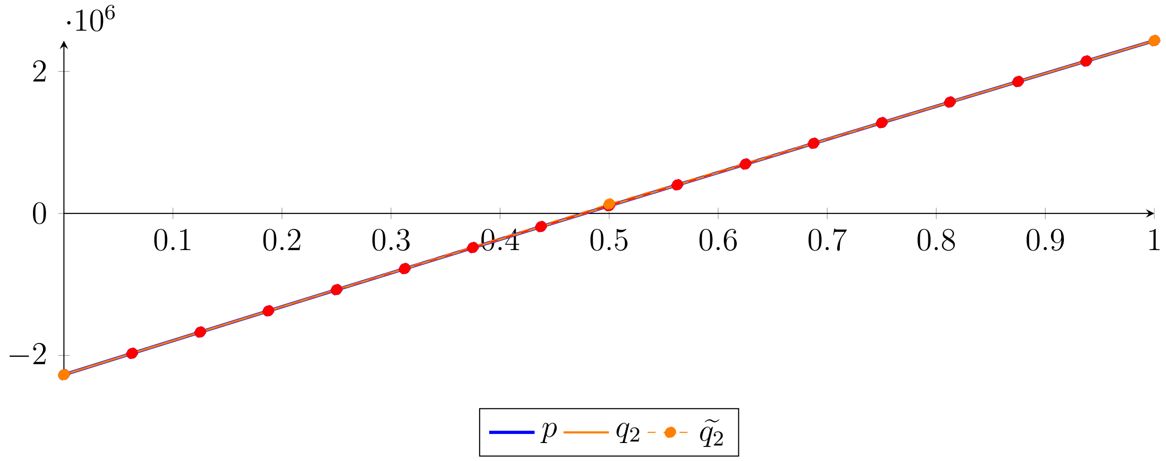
$$p = -2.82438\cdot 10^{-05}X^{16} + 2.90051\cdot 10^{-05}X^{15} + 0.000231285X^{14} + 0.000648014X^{13} + 0.00318916X^{12} + 0.00323204X^{11} + 0.00460533X^{10} + 0.00220012X^{9} - 0.000101882X^{8} + 5.32717\cdot 10^{-06}X^{7} - 0.00186824X^{6} + 0.132741X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874\cdot 10^{6}X - 2.2715\cdot 10^{6}$$
$$= -2.2715\cdot 10^{6}B_{0,16}(X) - 1.97096\cdot 10^{6}B_{1,16}(X) - 1.67127\cdot 10^{6}B_{2,16}(X) - 1.37244\cdot 10^{6}B_{3,16}(X) - 1.07448\cdot 10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806\cdot 10^{6}B_{12,16}(X) + 1.5682\cdot 10^{6}B_{13,16}(X) + 1.85747\cdot 10^{6}B_{14,16}(X) + 2.14586\cdot 10^{6}B_{15,16}(X) + 2.43338\cdot 10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265X^{2} + 4.80923\cdot 10^{6}X - 2.27154\cdot 10^{6}$$
$$= -2.27154\cdot 10^{6}B_{0,2} + 133069B_{1,2} + 2.43342\cdot 10^{6}B_{2,2}$$

$$\widetilde{q}_2 = 43556.7X^{16} - 347904X^{15} + 1.2616\cdot 10^{6}X^{14} - 2.74835\cdot 10^{6}X^{13} + 4.01042\cdot 10^{6}X^{12} - 4.13709\cdot 10^{6}X^{11} + 3.10251\cdot 10^{6}X^{10} - 1.71209\cdot 10^{6}X^{9} + 695531X^{8} - 205965X^{7} + 43591.2X^{6} - 6402.94X^{5} + 625.678X^{4} - 37.485X^{3} - 104264X^{2} + 4.80923\cdot 10^{6}X - 2.27154\cdot 10^{6}$$
$$= -2.27154\cdot 10^{6}B_{0,16} - 1.97097\cdot 10^{6}B_{1,16} - 1.67126\cdot 10^{6}B_{2,16} - 1.37242\cdot 10^{6}B_{3,16} - 1.07445\cdot 10^{6}B_{4,16} - 777350B_{5,16} - 481117B_{6,16} - 185755B_{7,16} + 108741B_{8,16} + 402365B_{9,16} + 695123B_{10,16} + 987010B_{11,16} + 1.27803\cdot 10^{6}B_{12,16} + 1.56818\cdot 10^{6}B_{13,16} + 1.85746\cdot 10^{6}B_{14,16} + 2.14587\cdot 10^{6}B_{15,16} + 2.43342\cdot 10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.
Bounding polynomials M and m:

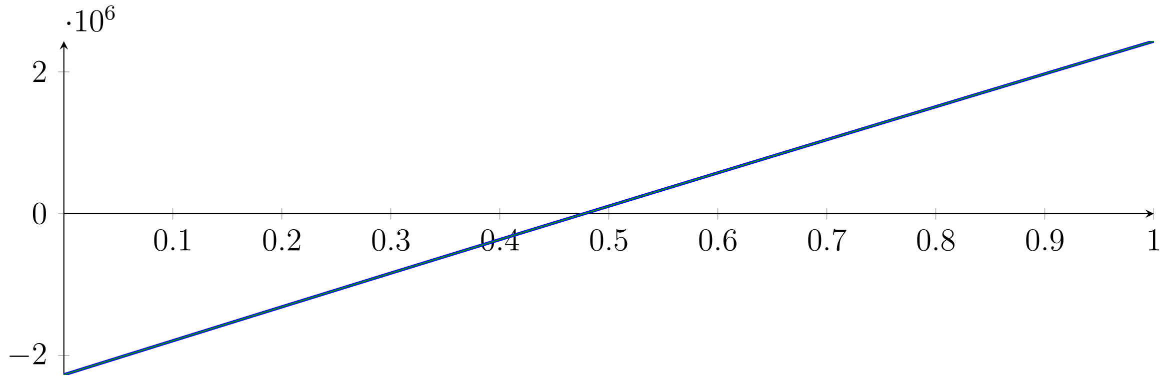
$$M = -104265X^{2} + 4.80923\cdot 10^{6}X - 2.2715\cdot 10^{6}$$

$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

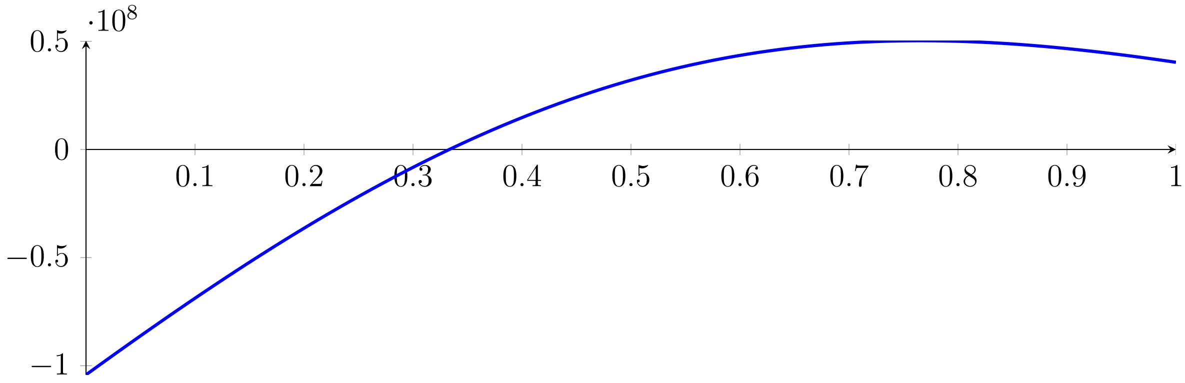
65.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

65.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

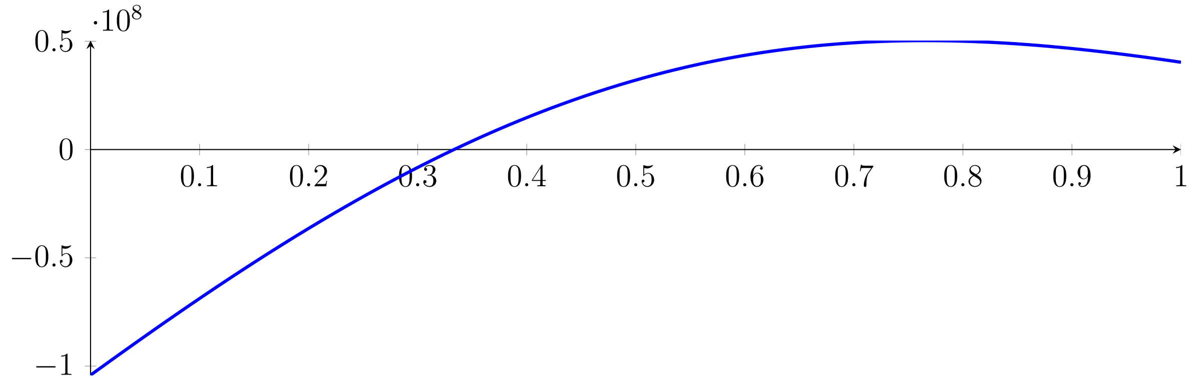
with precision $\varepsilon = 0.01$.

66 Running `CubeClip` on f_{16} with epsilon 2

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

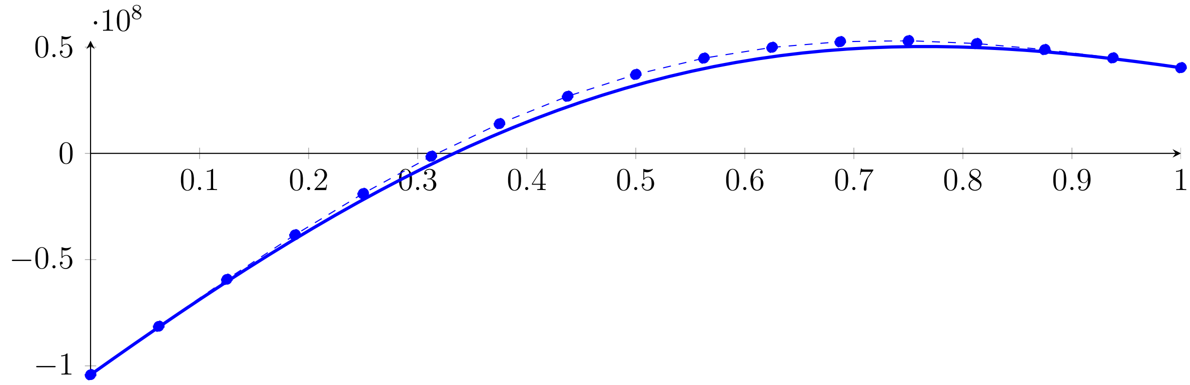
$$\begin{aligned} p = {} & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} \\ & + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \end{aligned}$$



66.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

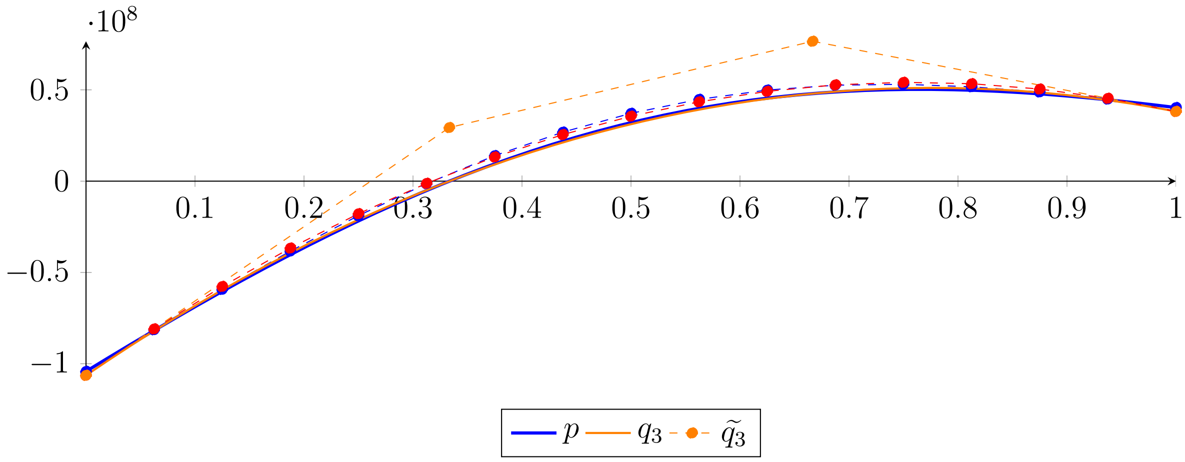
$$\begin{aligned} p = {} & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\ & + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823 \\ & \cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \\ = {} & -1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203 \\ & \cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X) \\ & + 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901 \\ & \cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X) \\ & + 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812\cdot10^6 X^3 - 2.65319\cdot10^8 X^2 + 4.07072\cdot10^8 X - 1.06335\cdot10^8$$
$$= -1.06335\cdot10^8 B_{0,3} + 2.93558\cdot10^7 B_{1,3} + 7.66068\cdot10^7 B_{2,3} + 3.81764\cdot10^7 B_{3,3}$$

$$\widetilde{q}_3 = 2.11088\cdot10^6 X^{16} - 1.70537\cdot10^7 X^{15} + 6.25913\cdot10^7 X^{14} - 1.38111\cdot10^8 X^{13} + 2.043\cdot10^8 X^{12} - 2.13799\cdot10^8 X^{11} + 1.62714\cdot10^8 X^{10} - 9.11092\cdot10^7 X^9 + 3.75254\cdot10^7 X^8 - 1.12529\cdot10^7 X^7 + 2.40899\cdot10^6 X^6 - 357156 X^5 + 34986.1 X^4 + 2.75602\cdot10^6 X^3 - 2.65318\cdot10^8 X^2 + 4.07072\cdot10^8 X - 1.06335\cdot10^8$$
$$= -1.06335\cdot10^8 B_{0,16} - 8.08928\cdot10^7 B_{1,16} - 5.76618\cdot10^7 B_{2,16} - 3.66369\cdot10^7 B_{3,16} - 1.78131\cdot10^7 B_{4,16} - 1.18553\cdot10^6 B_{5,16} + 1.32508\cdot10^7 B_{6,16} + 2.55007\cdot10^7 B_{7,16} + 3.55692\cdot10^7 B_{8,16} + 4.34611\cdot10^7 B_{9,16} + 4.91816\cdot10^7 B_{10,16} + 5.27352\cdot10^7 B_{11,16} + 5.41273\cdot10^7 B_{12,16} + 5.33624\cdot10^7 B_{13,16} + 5.04457\cdot10^7 B_{14,16} + 4.53821\cdot10^7 B_{15,16} + 3.81764\cdot10^7 B_{16,16}$$



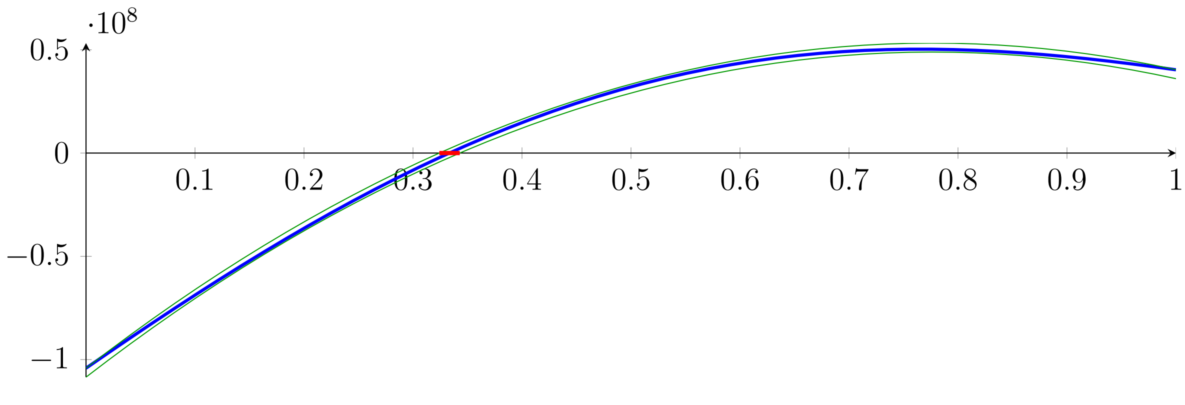
The maximum difference of the Bézier coefficients is $\delta = 2.16806\cdot10^6$.
Bounding polynomials M and m:

$$M = 2.75812\cdot10^6 X^3 - 2.65319\cdot10^8 X^2 + 4.07072\cdot10^8 X - 1.04167\cdot10^8$$
$$m = 2.75812\cdot10^6 X^3 - 2.65319\cdot10^8 X^2 + 4.07072\cdot10^8 X - 1.08503\cdot10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



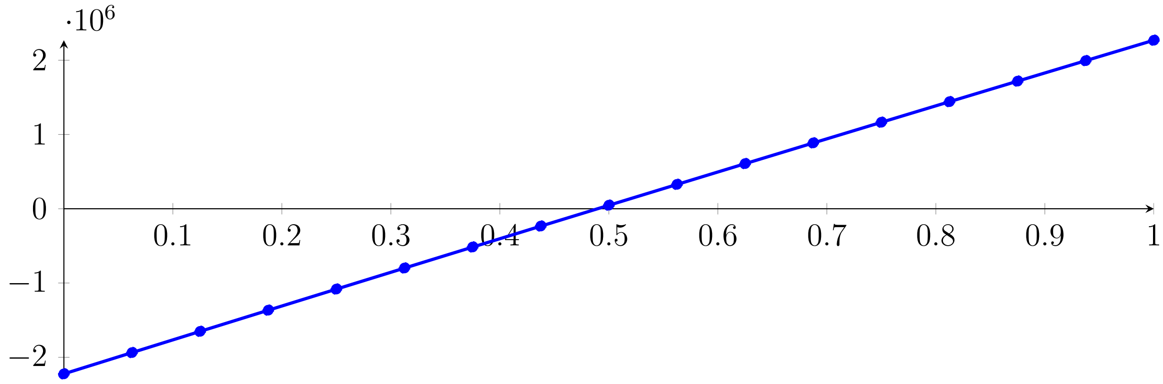
$$[0.324143, 0.342913]$$

Longest intersection interval: 0.0187703
$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

66.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

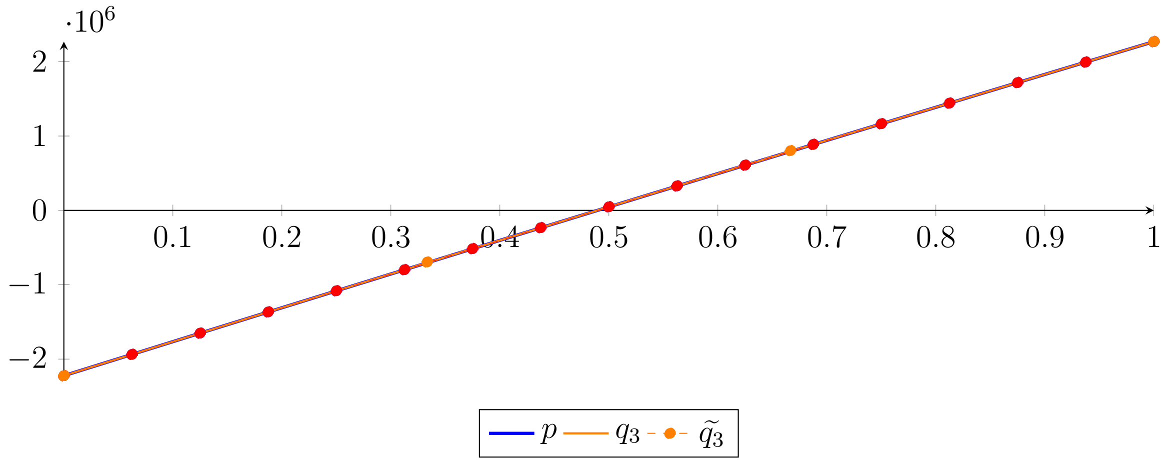
$$p = -1.93035\cdot 10^{-05}X^{16} + 1.08927\cdot 10^{-05}X^{15} + 0.000255816X^{14} + 0.000616983X^{13} + 0.00371715X^{12} + 0.00325035X^{11} + 0.00510875X^{10} + 0.00190713X^{9} + 0.000179792X^{8} + 1.06543\cdot 10^{-05}X^{7} - 0.00136482X^{6} + 0.104959X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6}$$
$$= -2.22335\cdot 10^{6}B_{0,16}(X) - 1.93666\cdot 10^{6}B_{1,16}(X) - 1.65074\cdot 10^{6}B_{2,16}(X) - 1.36561\cdot 10^{6}B_{3,16}(X) - 1.08127\cdot 10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X) + 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot 10^{6}B_{12,16}(X) + 1.44233\cdot 10^{6}B_{13,16}(X) + 1.71876\cdot 10^{6}B_{14,16}(X) + 1.99439\cdot 10^{6}B_{15,16}(X) + 2.26922\cdot 10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6}$$
$$= -2.22335\cdot 10^{6}B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot 10^{6}B_{3,3}$$

$$\widetilde{q}_3 = 15290.6X^{16} - 125412X^{15} + 468045X^{14} - 1.05198\cdot 10^{6}X^{13} + 1.588\cdot 10^{6}X^{12} - 1.69893\cdot 10^{6}X^{11} + 1.32382\cdot 10^{6}X^{10} - 759402X^{9} + 320067X^{8} - 97826.5X^{7} + 21197.8X^{6} - 3162.8X^{5} + 313.931X^{4} - 720.165X^{3} - 93879.2X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6}$$
$$= -2.22335\cdot 10^{6}B_{0,16} - 1.93666\cdot 10^{6}B_{1,16} - 1.65074\cdot 10^{6}B_{2,16} - 1.36561\cdot 10^{6}B_{3,16} - 1.08126\cdot 10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.7B_{8,16} + 328649B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot 10^{6}B_{12,16} + 1.44233\cdot 10^{6}B_{13,16} + 1.71876\cdot 10^{6}B_{14,16} + 1.99439\cdot 10^{6}B_{15,16} + 2.26922\cdot 10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 0.457751$.

Bounding polynomials M and m:

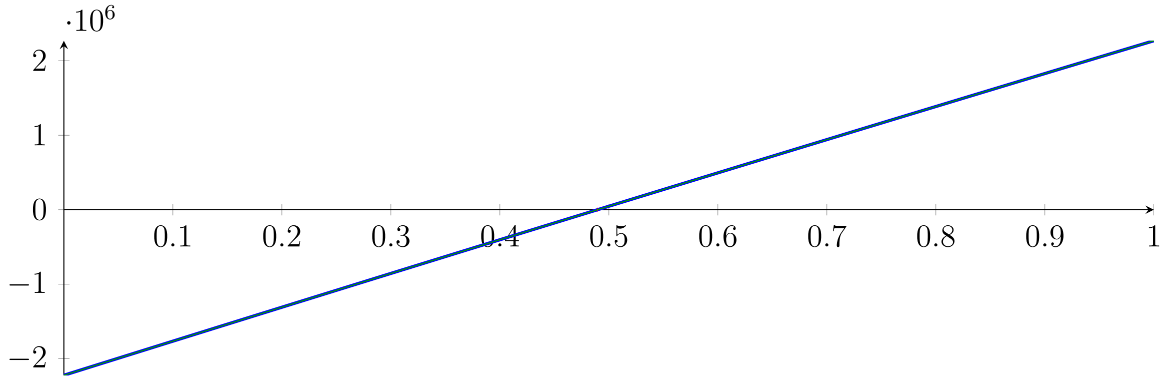
$$M = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6}$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $2.03684 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

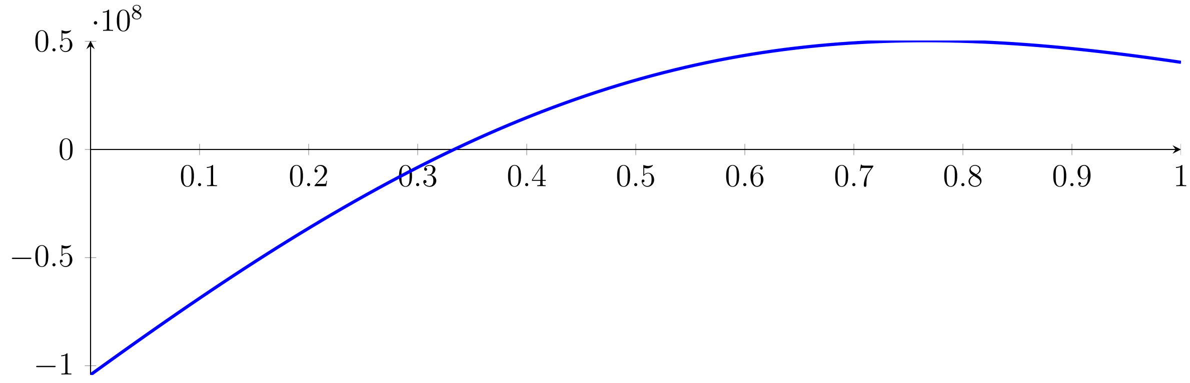
66.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

66.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

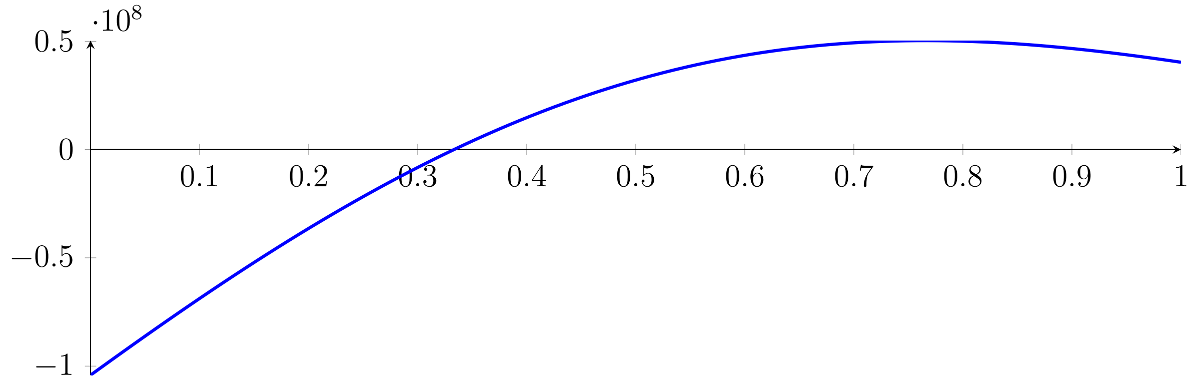
with precision $\varepsilon = 0.01$.

67 Running BezClip on f_{16} with epsilon 4

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^6 X^9 - 2.52262\cdot 10^6 X^8 - 1.43506\cdot 10^7 X^7 + 138542X^6 + 7.92823\cdot 10^7 X^5 + 3.97396\cdot 10^7 X^4 - 2.40625\cdot 10^8 X^3 - 8.33333\cdot 10^7 X^2 + 3.64583\cdot 10^8 X - 1.04167\cdot 10^8$$

Called BezClip with input polynomial on interval $[0, 1]$:

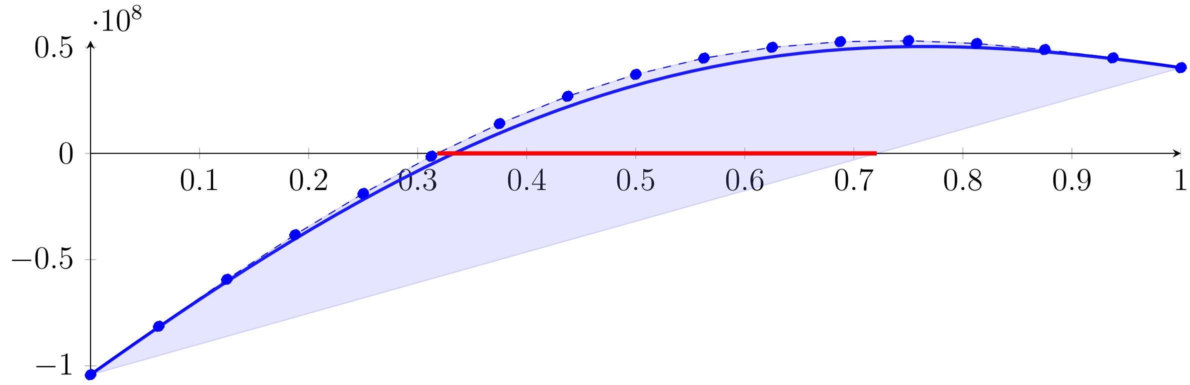
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^6 X^9 - 2.52262\cdot 10^6 X^8 - 1.43506\cdot 10^7 X^7 + 138542X^6 + 7.92823\cdot 10^7 X^5 + 3.97396\cdot 10^7 X^4 - 2.40625\cdot 10^8 X^3 - 8.33333\cdot 10^7 X^2 + 3.64583\cdot 10^8 X - 1.04167\cdot 10^8$$



67.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot 10^6 X^9 - 2.52262\cdot 10^6 X^8 - 1.43506\cdot 10^7 X^7 + 138542X^6 + 7.92823 \\ &\quad \cdot 10^7 X^5 + 3.97396\cdot 10^7 X^4 - 2.40625\cdot 10^8 X^3 - 8.33333\cdot 10^7 X^2 + 3.64583\cdot 10^8 X - 1.04167\cdot 10^8 \\ &= -1.04167\cdot 10^8 B_{0,16}(X) - 8.13802\cdot 10^7 B_{1,16}(X) - 5.92882\cdot 10^7 B_{2,16}(X) - 3.83203 \\ &\quad \cdot 10^7 B_{3,16}(X) - 1.88844\cdot 10^7 B_{4,16}(X) - 1.34837\cdot 10^6 B_{5,16}(X) + 1.39781\cdot 10^7 B_{6,16}(X) \\ &\quad + 2.68604\cdot 10^7 B_{7,16}(X) + 3.71532\cdot 10^7 B_{8,16}(X) + 4.48105\cdot 10^7 B_{9,16}(X) + 4.98901 \\ &\quad \cdot 10^7 B_{10,16}(X) + 5.25504\cdot 10^7 B_{11,16}(X) + 5.30407\cdot 10^7 B_{12,16}(X) + 5.16832\cdot 10^7 B_{13,16}(X) \\ &\quad + 4.88488\cdot 10^7 B_{14,16}(X) + 4.49297\cdot 10^7 B_{15,16}(X) + 4.03108\cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

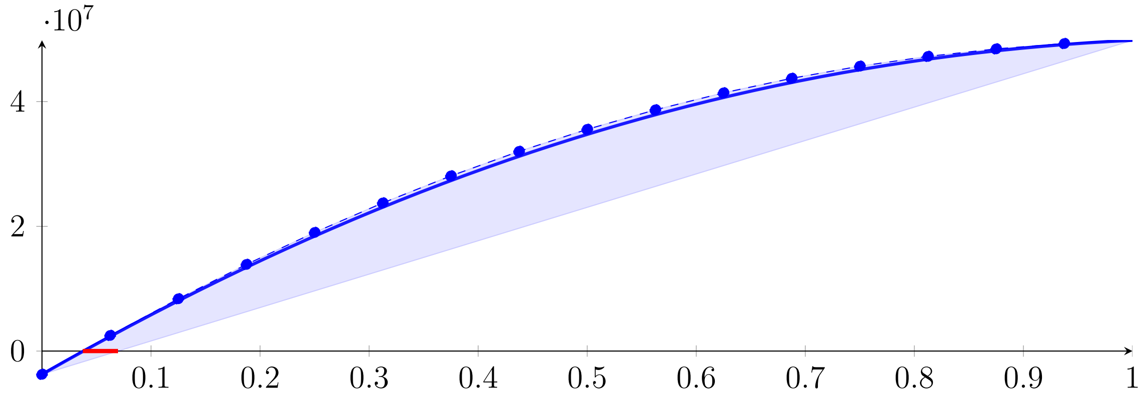
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

67.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -0.00483322X^{16} + 0.0186809X^{15} - 0.0194312X^{14} - 0.0738695X^{13} - 1.11673X^{12} - 5.0471X^{11} + 36.3082X^{10} + 692.914X^{9} + 1886.96X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5} + 3.91945\cdot 10^{6}X^{4} - 7.54561\cdot 10^{6}X^{3} - 4.29225\cdot 10^{7}X^{2} + 9.97982\cdot 10^{7}X - 3.73501\cdot 10^{6}$$
$$= -3.73501\cdot 10^{6}B_{0,16}(X) + 2.50237\cdot 10^{6}B_{1,16}(X) + 8.38207\cdot 10^{6}B_{2,16}(X) + 1.38906\cdot 10^{7}B_{3,16}(X) + 1.90167\cdot 10^{7}B_{4,16}(X) + 2.37512\cdot 10^{7}B_{5,16}(X) + 2.80875\cdot 10^{7}B_{6,16}(X) + 3.20213\cdot 10^{7}B_{7,16}(X) + 3.55508\cdot 10^{7}B_{8,16}(X) + 3.86768\cdot 10^{7}B_{9,16}(X) + 4.14025\cdot 10^{7}B_{10,16}(X) + 4.37336\cdot 10^{7}B_{11,16}(X) + 4.56783\cdot 10^{7}B_{12,16}(X) + 4.7247\cdot 10^{7}B_{13,16}(X) + 4.84524\cdot 10^{7}B_{14,16}(X) + 4.93093\cdot 10^{7}B_{15,16}(X) + 4.98343\cdot 10^{7}B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

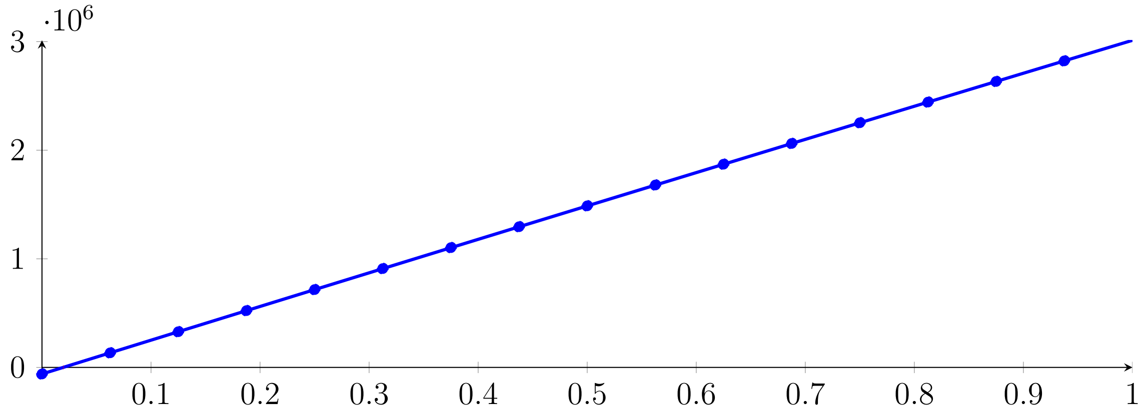
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

67.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -0.000205056X^{16} + 0.000776167X^{15} - 0.000863333X^{14} - 0.000217499X^{13} - 0.00809276X^{12} - 5.09773\cdot 10^{-05}X^{11} - 0.00564923X^{10} - 0.000162811X^{9} - 0.000215376X^{8} - 3.32948\cdot 10^{-07}X^{7} - 0.000259866X^{6} + 0.0161077X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot 10^{6}X - 60508.5$$
$$= -60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X) + 910202B_{5,16}(X) + 1.1032\cdot 10^{6}B_{6,16}(X) + 1.29582\cdot 10^{6}B_{7,16}(X) + 1.48805\cdot 10^{6}B_{8,16}(X) + 1.6799\cdot 10^{6}B_{9,16}(X) + 1.87137\cdot 10^{6}B_{10,16}(X) + 2.06245\cdot 10^{6}B_{11,16}(X) + 2.25315\cdot 10^{6}B_{12,16}(X) + 2.44346\cdot 10^{6}B_{13,16}(X) + 2.63339\cdot 10^{6}B_{14,16}(X) + 2.82293\cdot 10^{6}B_{15,16}(X) + 3.01209\cdot 10^{6}B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

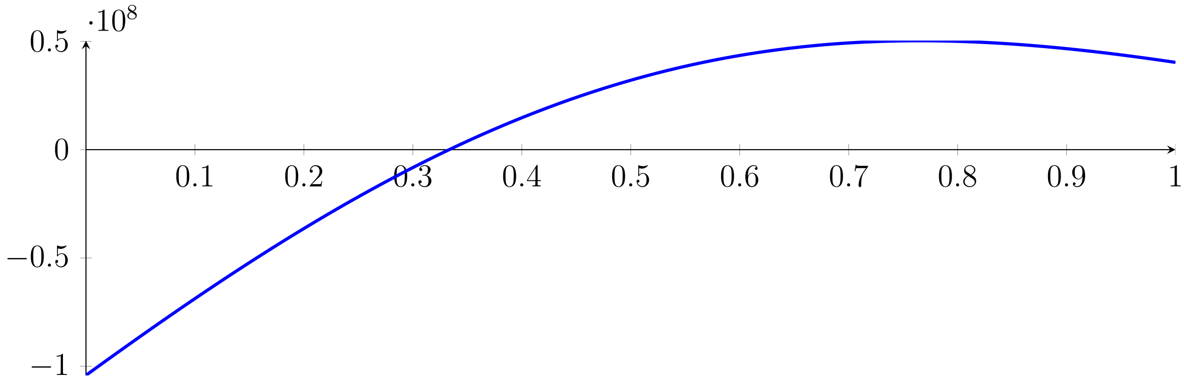
67.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Found root in interval $[0.333333, 0.333337]$ at recursion depth 4!

67.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^{6}X^{9} - 2.52262 \cdot 10^{6}X^{8} - 1.43506 \cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \cdot 10^{7}X^{5} + 3.97396 \cdot 10^{7}X^{4} - 2.40625 \cdot 10^{8}X^{3} - 8.33333 \cdot 10^{7}X^{2} + 3.64583 \cdot 10^{8}X - 1.04167 \cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333337]$$

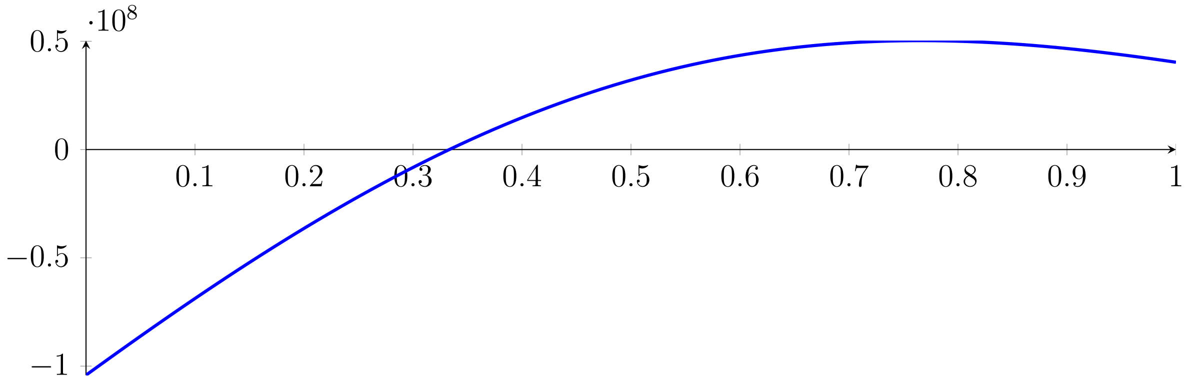
with precision $\varepsilon = 0.0001$.

68 Running QuadClip on f_{16} with epsilon 4

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

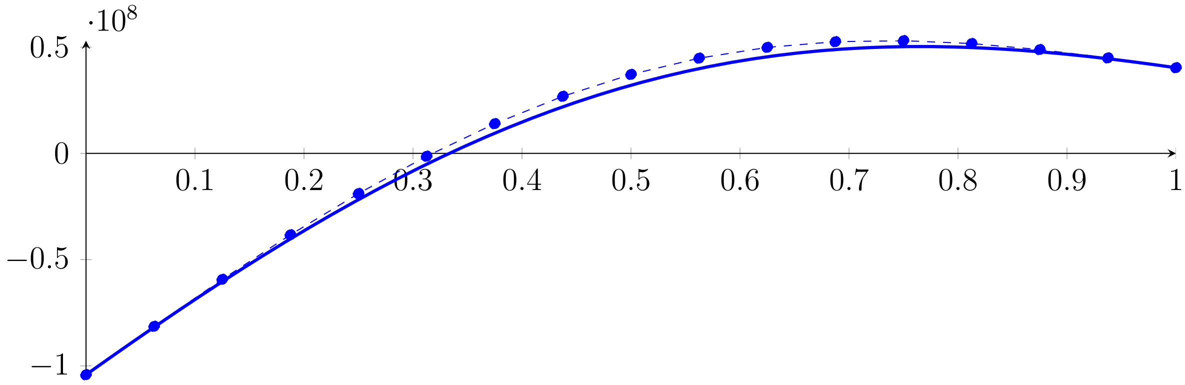
$$\begin{aligned}p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10}\\ &+ 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5}\\ &+ 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}\end{aligned}$$



68.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

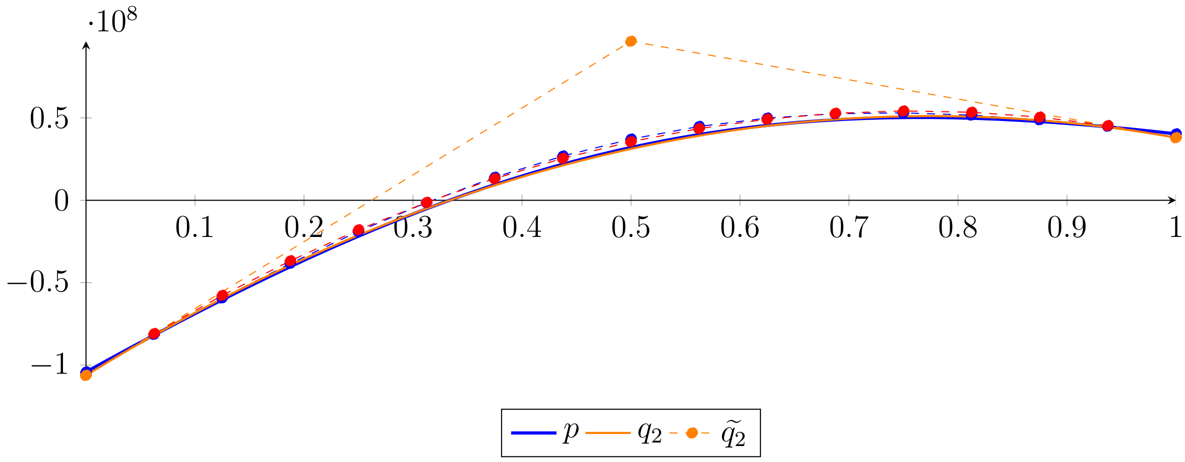
$$\begin{aligned}p = &-0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11}\\ &+ 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\\ &\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}\\ = &-1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203\\ &\cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X)\\ &+ 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901\\ &\cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X)\\ &+ 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X)\end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181\cdot10^8X^2 + 4.05417\cdot10^8X - 1.06197\cdot10^8$$
$$= -1.06197\cdot10^8B_{0,2} + 9.65115\cdot10^7B_{1,2} + 3.80384\cdot10^7B_{2,2}$$

$$\widetilde{q}_2 = -1.41787\cdot10^6X^{16} + 1.11761\cdot10^7X^{15} - 3.98898\cdot10^7X^{14} + 8.52437\cdot10^7X^{13} - 1.21528\cdot10^8X^{12} + 1.21946\cdot10^8X^{11} - 8.86062\cdot10^7X^{10} + 4.72904\cdot10^7X^9 - 1.86355\cdot10^7X^8 + 5.41059\cdot10^6X^7 - 1.14487\cdot10^6X^6 + 172240X^5 - 17636.6X^4 + 1159.74X^3 - 2.61181\cdot10^8X^2 + 4.05417\cdot10^8X - 1.06197\cdot10^8$$
$$= -1.06197\cdot10^8B_{0,16} - 8.08583\cdot10^7B_{1,16} - 5.76963\cdot10^7B_{2,16} - 3.67107\cdot10^7B_{3,16} - 1.79017\cdot10^7B_{4,16} - 1.26923\cdot10^6B_{5,16} + 1.31867\cdot10^7B_{6,16} + 2.54662\cdot10^7B_{7,16} + 3.55691\cdot10^7B_{8,16} + 4.34957\cdot10^7B_{9,16} + 4.92455\cdot10^7B_{10,16} + 5.2819\cdot10^7B_{11,16} + 5.42159\cdot10^7B_{12,16} + 5.34363\cdot10^7B_{13,16} + 5.04802\cdot10^7B_{14,16} + 4.53476\cdot10^7B_{15,16} + 3.80384\cdot10^7B_{16,16}$$



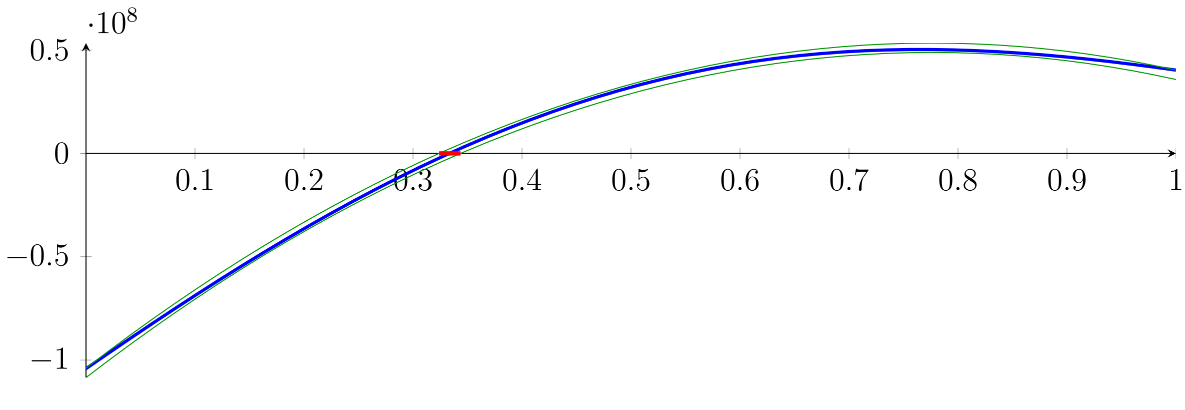
The maximum difference of the Bézier coefficients is $\delta = 2.27233\cdot10^6$.
Bounding polynomials M and m:

$$M = -2.61181\cdot10^8X^2 + 4.05417\cdot10^8X - 1.03924\cdot10^8$$
$$m = -2.61181\cdot10^8X^2 + 4.05417\cdot10^8X - 1.08469\cdot10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



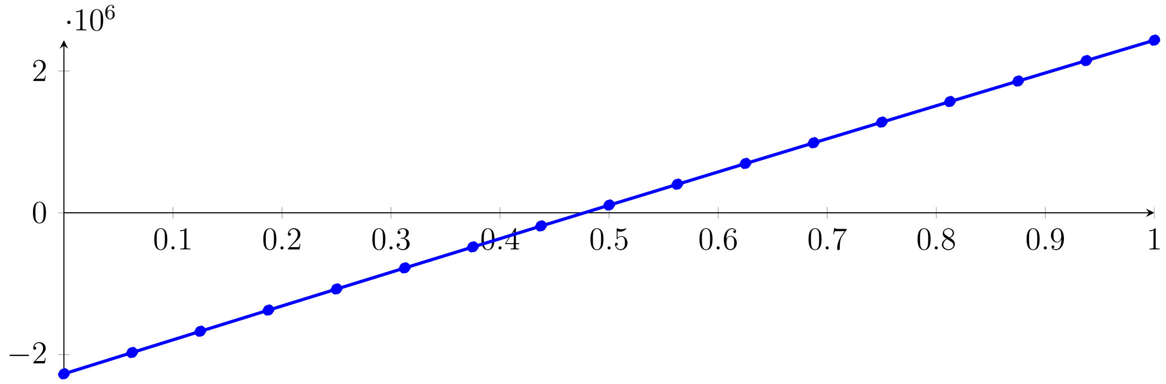
$$[0.323946, 0.343615]$$

Longest intersection interval: 0.0196686
$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

68.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

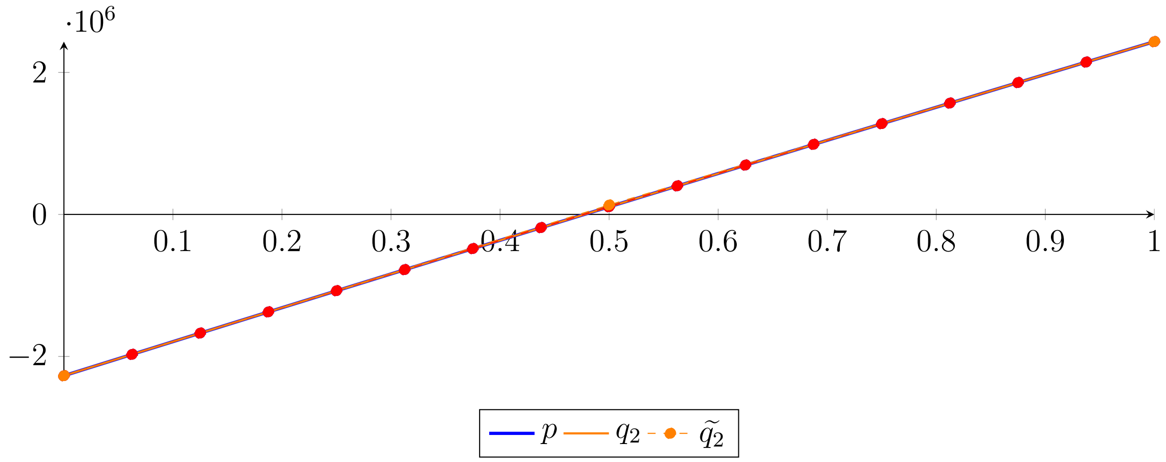
$$p = -2.82438 \cdot 10^{-05} X^{16} + 2.90051 \cdot 10^{-05} X^{15} + 0.000231285 X^{14} + 0.000648014 X^{13} + 0.00318916 X^{12} + 0.00323204 X^{11} + 0.00460533 X^{10} + 0.00220012 X^{9} - 0.000101882 X^{8} + 5.32717 \cdot 10^{-06} X^{7} - 0.00186824 X^{6} + 0.132741 X^{5} + 22.4437 X^{4} - 850.239 X^{3} - 103028 X^{2} + 4.80874 \cdot 10^{6} X - 2.2715 \cdot 10^{6}$$
$$= -2.2715 \cdot 10^{6} B_{0,16}(X) - 1.97096 \cdot 10^{6} B_{1,16}(X) - 1.67127 \cdot 10^{6} B_{2,16}(X) - 1.37244 \cdot 10^{6} B_{3,16}(X) - 1.07448 \cdot 10^{6} B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^{6} B_{12,16}(X) + 1.5682 \cdot 10^{6} B_{13,16}(X) + 1.85747 \cdot 10^{6} B_{14,16}(X) + 2.14586 \cdot 10^{6} B_{15,16}(X) + 2.43338 \cdot 10^{6} B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265 X^{2} + 4.80923 \cdot 10^{6} X - 2.27154 \cdot 10^{6}$$
$$= -2.27154 \cdot 10^{6} B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^{6} B_{2,2}$$

$$\widetilde{q}_2 = 43556.7 X^{16} - 347904 X^{15} + 1.2616 \cdot 10^{6} X^{14} - 2.74835 \cdot 10^{6} X^{13} + 4.01042 \cdot 10^{6} X^{12} - 4.13709 \cdot 10^{6} X^{11} + 3.10251 \cdot 10^{6} X^{10} - 1.71209 \cdot 10^{6} X^{9} + 695531 X^{8} - 205965 X^{7} + 43591.2 X^{6} - 6402.94 X^{5} + 625.678 X^{4} - 37.485 X^{3} - 104264 X^{2} + 4.80923 \cdot 10^{6} X - 2.27154 \cdot 10^{6}$$
$$= -2.27154 \cdot 10^{6} B_{0,16} - 1.97097 \cdot 10^{6} B_{1,16} - 1.67126 \cdot 10^{6} B_{2,16} - 1.37242 \cdot 10^{6} B_{3,16} - 1.07445 \cdot 10^{6} B_{4,16} - 777350 B_{5,16} - 481117 B_{6,16} - 185755 B_{7,16} + 108741 B_{8,16} + 402365 B_{9,16} + 695123 B_{10,16} + 987010 B_{11,16} + 1.27803 \cdot 10^{6} B_{12,16} + 1.56818 \cdot 10^{6} B_{13,16} + 1.85746 \cdot 10^{6} B_{14,16} + 2.14587 \cdot 10^{6} B_{15,16} + 2.43342 \cdot 10^{6} B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m:

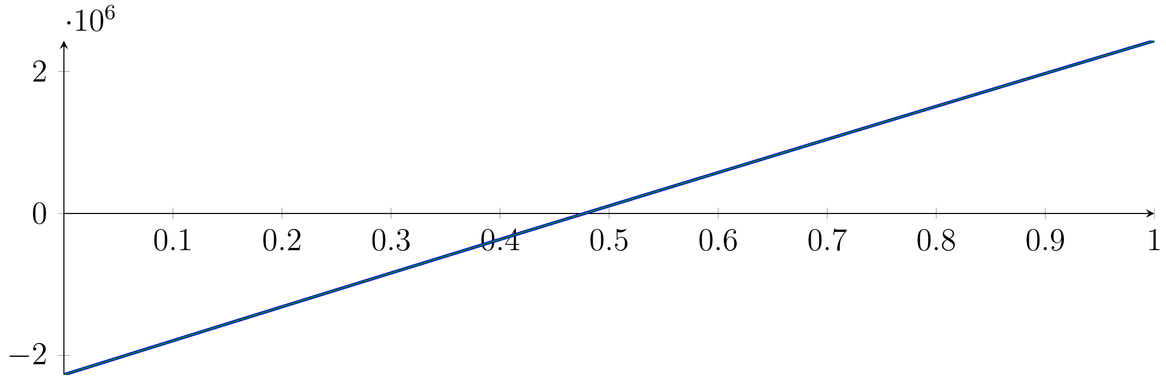
$$M = -104265 X^{2} + 4.80923 \cdot 10^{6} X - 2.2715 \cdot 10^{6}$$

$$m = -104265X^2 + 4.80923{\cdot}10^6X - 2.27159{\cdot}10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301{\cdot}10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

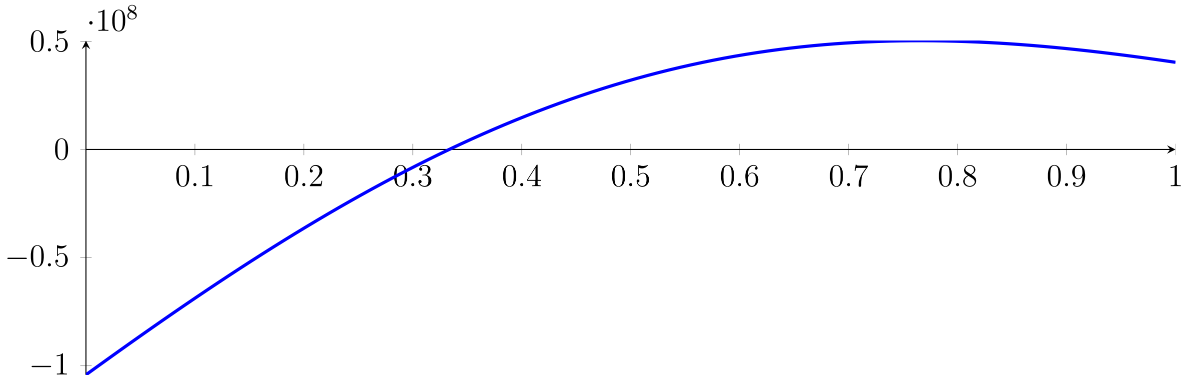
68.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

68.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

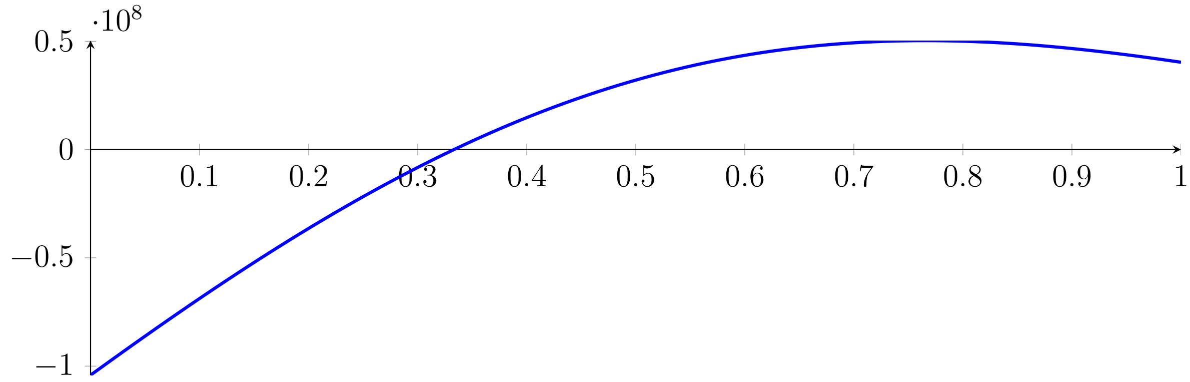
with precision $\varepsilon = 0.0001$.

69 Running CubeClip on f_{16} with epsilon 4

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

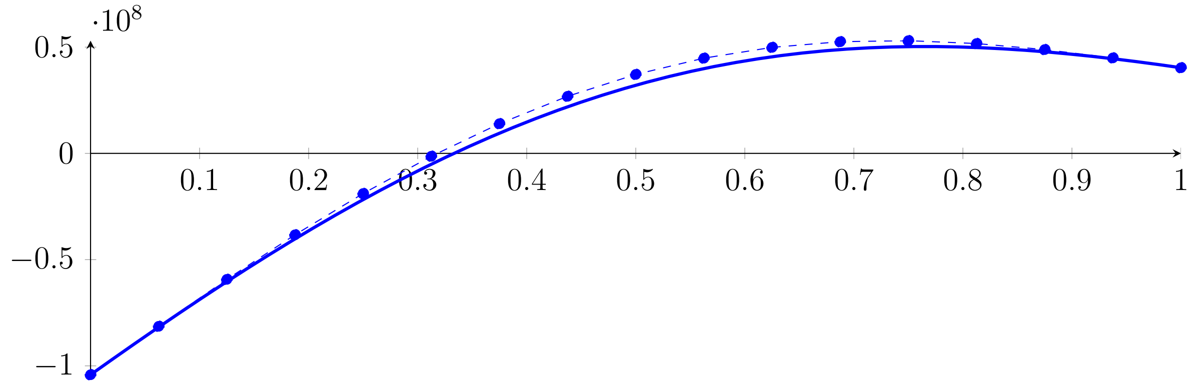
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} \\ &+ 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \end{aligned}$$



69.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

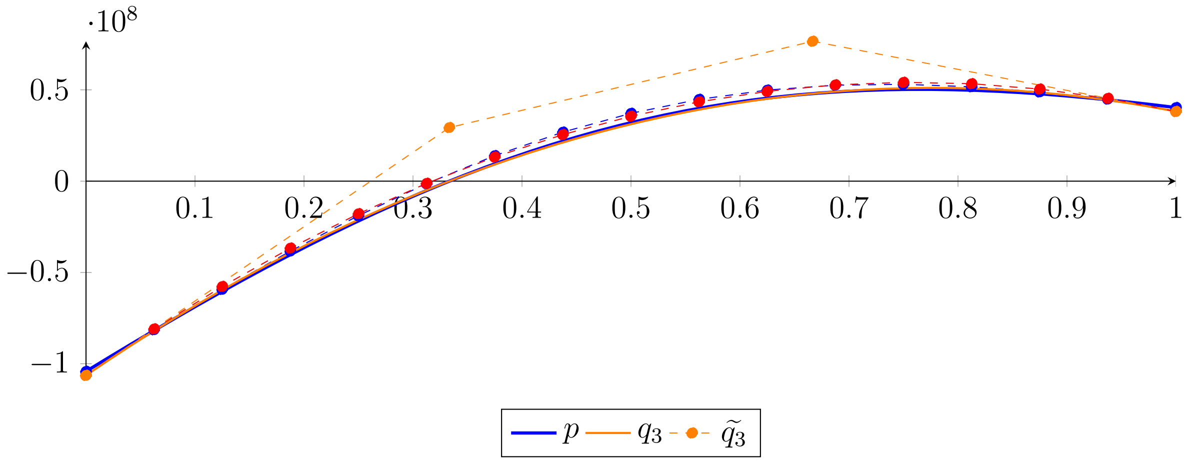
$$\begin{aligned} p = &-0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\ &+ 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \\ = &-1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203 \\ &\cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X) \\ &+ 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901 \\ &\cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X) \\ &+ 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.06335\cdot 10^8$$
$$= -1.06335\cdot 10^8 B_{0,3} + 2.93558\cdot 10^7 B_{1,3} + 7.66068\cdot 10^7 B_{2,3} + 3.81764\cdot 10^7 B_{3,3}$$

$$\widetilde{q}_3 = 2.11088\cdot 10^6 X^{16} - 1.70537\cdot 10^7 X^{15} + 6.25913\cdot 10^7 X^{14} - 1.38111\cdot 10^8 X^{13} + 2.043\cdot 10^8 X^{12} - 2.13799\cdot 10^8 X^{11} + 1.62714\cdot 10^8 X^{10} - 9.11092\cdot 10^7 X^9 + 3.75254\cdot 10^7 X^8 - 1.12529\cdot 10^7 X^7 + 2.40899\cdot 10^6 X^6 - 357156 X^5 + 34986.1 X^4 + 2.75602\cdot 10^6 X^3 - 2.65318\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.06335\cdot 10^8$$
$$= -1.06335\cdot 10^8 B_{0,16} - 8.08928\cdot 10^7 B_{1,16} - 5.76618\cdot 10^7 B_{2,16} - 3.66369\cdot 10^7 B_{3,16} - 1.78131\cdot 10^7 B_{4,16} - 1.18553\cdot 10^6 B_{5,16} + 1.32508\cdot 10^7 B_{6,16} + 2.55007\cdot 10^7 B_{7,16} + 3.55692\cdot 10^7 B_{8,16} + 4.34611\cdot 10^7 B_{9,16} + 4.91816\cdot 10^7 B_{10,16} + 5.27352\cdot 10^7 B_{11,16} + 5.41273\cdot 10^7 B_{12,16} + 5.33624\cdot 10^7 B_{13,16} + 5.04457\cdot 10^7 B_{14,16} + 4.53821\cdot 10^7 B_{15,16} + 3.81764\cdot 10^7 B_{16,16}$$



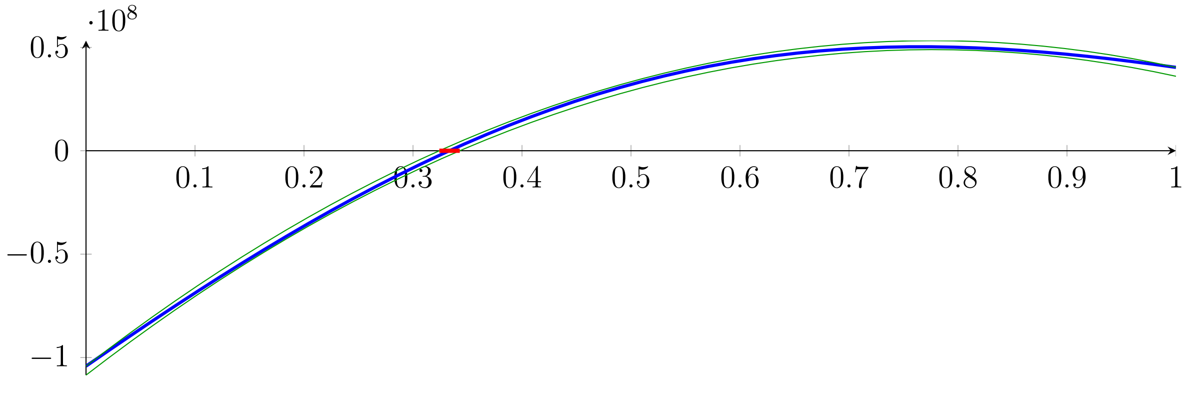
The maximum difference of the Bézier coefficients is $\delta = 2.16806\cdot 10^6$.
Bounding polynomials M and m:

$$M = 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.04167\cdot 10^8$$
$$m = 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.08503\cdot 10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



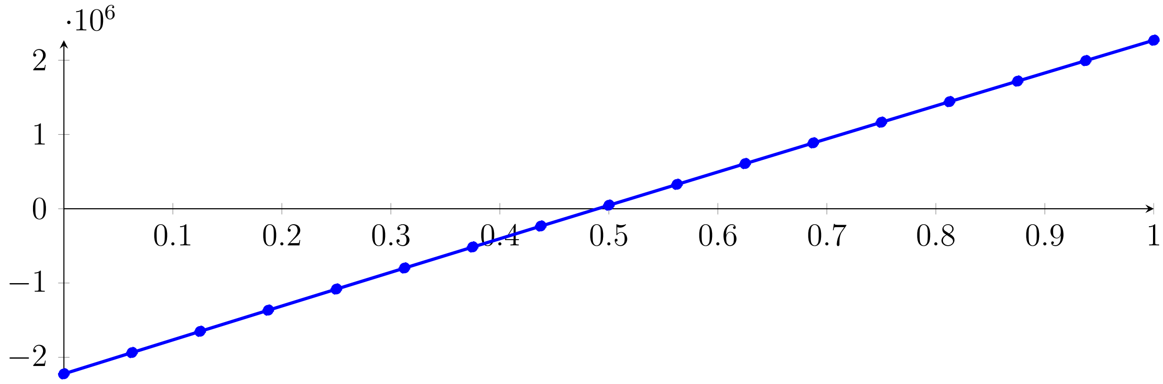
$$[0.324143, 0.342913]$$

Longest intersection interval: 0.0187703
$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

69.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

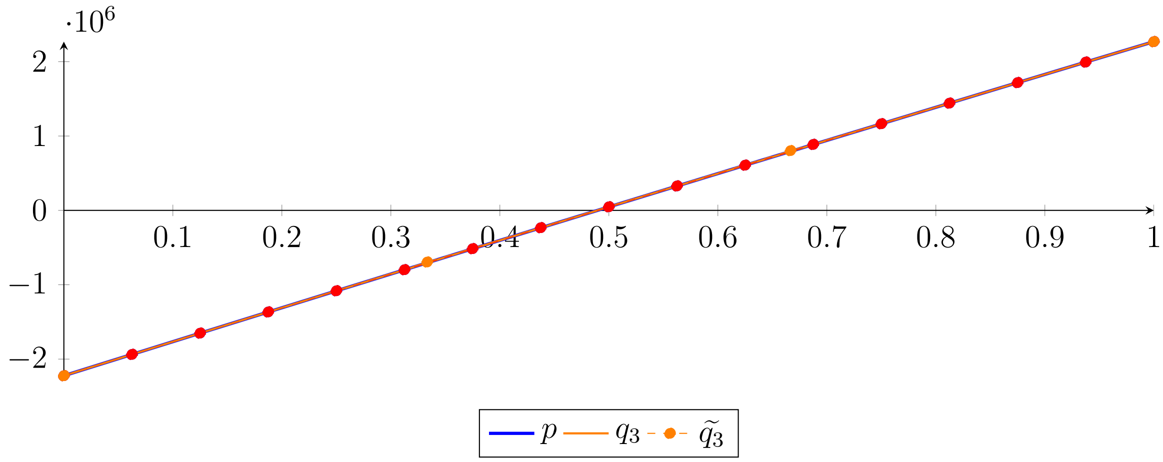
$$p = -1.93035\cdot10^{-05}X^{16} + 1.08927\cdot10^{-05}X^{15} + 0.000255816X^{14} + 0.000616983X^{13} + 0.00371715X^{12} + 0.00325035X^{11} + 0.00510875X^{10} + 0.00190713X^{9} + 0.000179792X^{8} + 1.06543\cdot10^{-05}X^{7} - 0.00136482X^{6} + 0.104959X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,16}(X) - 1.93666\cdot10^{6}B_{1,16}(X) - 1.65074\cdot10^{6}B_{2,16}(X) - 1.36561\cdot10^{6}B_{3,16}(X) - 1.08127\cdot10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X) + 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot10^{6}B_{12,16}(X) + 1.44233\cdot10^{6}B_{13,16}(X) + 1.71876\cdot10^{6}B_{14,16}(X) + 1.99439\cdot10^{6}B_{15,16}(X) + 2.26922\cdot10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot10^{6}B_{3,3}$$

$$\widetilde{q}_3 = 15290.6X^{16} - 125412X^{15} + 468045X^{14} - 1.05198\cdot10^{6}X^{13} + 1.588\cdot10^{6}X^{12} - 1.69893\cdot10^{6}X^{11} + 1.32382\cdot10^{6}X^{10} - 759402X^{9} + 320067X^{8} - 97826.5X^{7} + 21197.8X^{6} - 3162.8X^{5} + 313.931X^{4} - 720.165X^{3} - 93879.2X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,16} - 1.93666\cdot10^{6}B_{1,16} - 1.65074\cdot10^{6}B_{2,16} - 1.36561\cdot10^{6}B_{3,16} - 1.08126\cdot10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.7B_{8,16} + 328649B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot10^{6}B_{12,16} + 1.44233\cdot10^{6}B_{13,16} + 1.71876\cdot10^{6}B_{14,16} + 1.99439\cdot10^{6}B_{15,16} + 2.26922\cdot10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 0.457751$.

Bounding polynomials M and m:

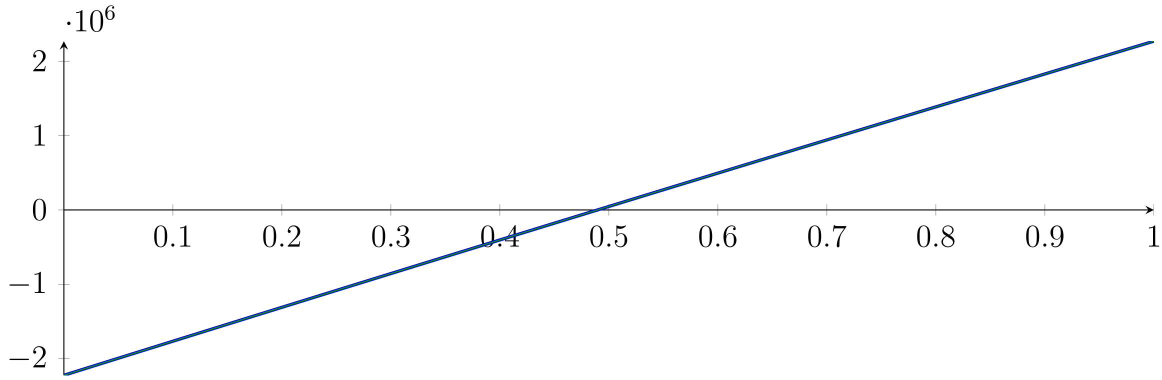
$$M = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715{\cdot}10^6X - 2.22335{\cdot}10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $2.03684{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

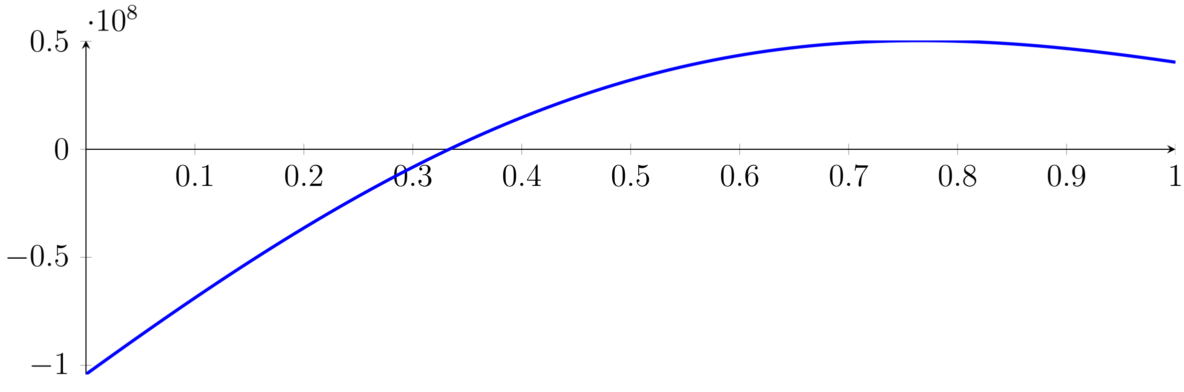
69.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

69.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0,1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

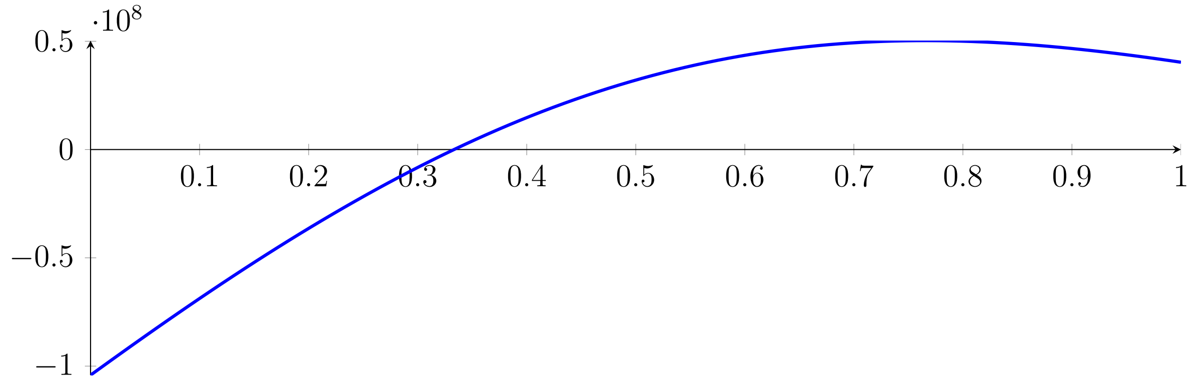
with precision $\varepsilon = 0.0001$.

70 Running `BezClip` on f_{16} with epsilon 8

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

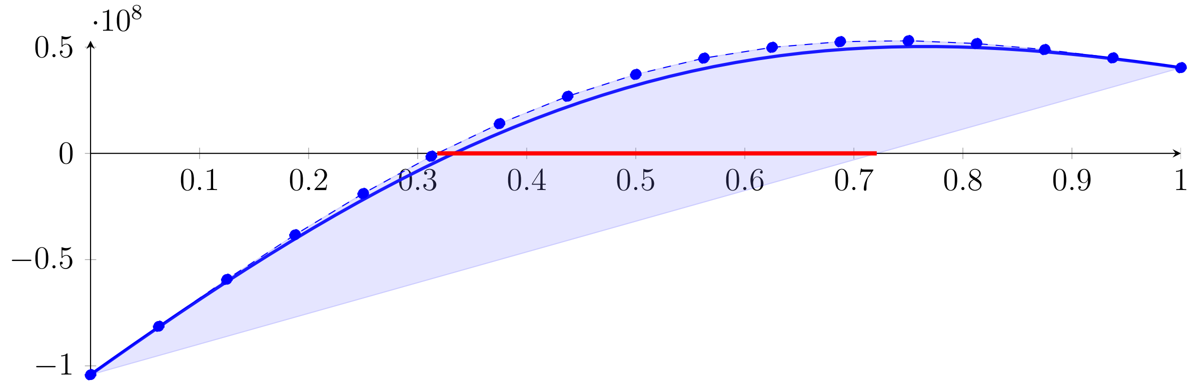
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



70.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\quad \cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \\ &= -1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

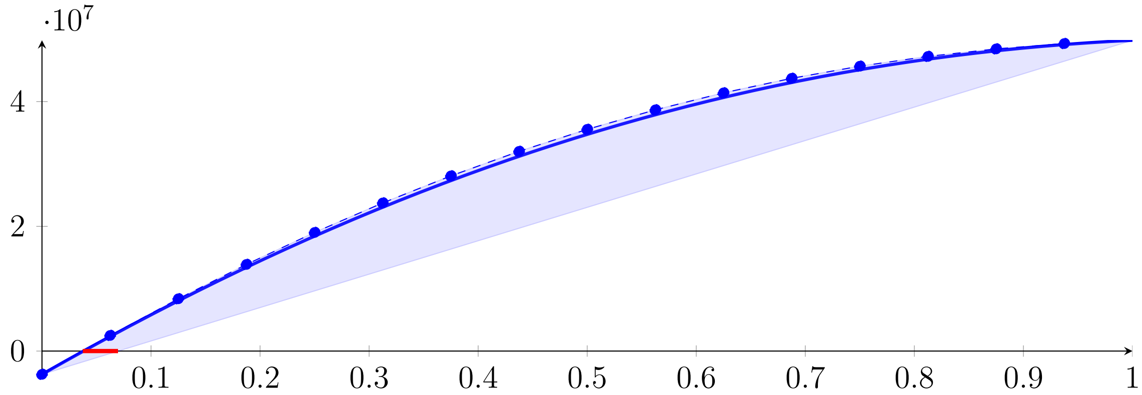
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

70.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-0.00483322X^{16} + 0.0186809X^{15} - 0.0194312X^{14} - 0.0738695X^{13} - 1.11673X^{12} \\ &- 5.0471X^{11} + 36.3082X^{10} + 692.914X^{9} + 1886.96X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5} \\ &+ 3.91945\cdot10^{6}X^{4} - 7.54561\cdot10^{6}X^{3} - 4.29225\cdot10^{7}X^{2} + 9.97982\cdot10^{7}X - 3.73501\cdot10^{6} \\ = &-3.73501\cdot10^{6}B_{0,16}(X) + 2.50237\cdot10^{6}B_{1,16}(X) + 8.38207\cdot10^{6}B_{2,16}(X) + 1.38906 \\ &\cdot10^{7}B_{3,16}(X) + 1.90167\cdot10^{7}B_{4,16}(X) + 2.37512\cdot10^{7}B_{5,16}(X) + 2.80875\cdot10^{7}B_{6,16}(X) \\ &+ 3.20213\cdot10^{7}B_{7,16}(X) + 3.55508\cdot10^{7}B_{8,16}(X) + 3.86768\cdot10^{7}B_{9,16}(X) + 4.14025 \\ &\cdot10^{7}B_{10,16}(X) + 4.37336\cdot10^{7}B_{11,16}(X) + 4.56783\cdot10^{7}B_{12,16}(X) + 4.7247\cdot10^{7}B_{13,16}(X) \\ &+ 4.84524\cdot10^{7}B_{14,16}(X) + 4.93093\cdot10^{7}B_{15,16}(X) + 4.98343\cdot10^{7}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

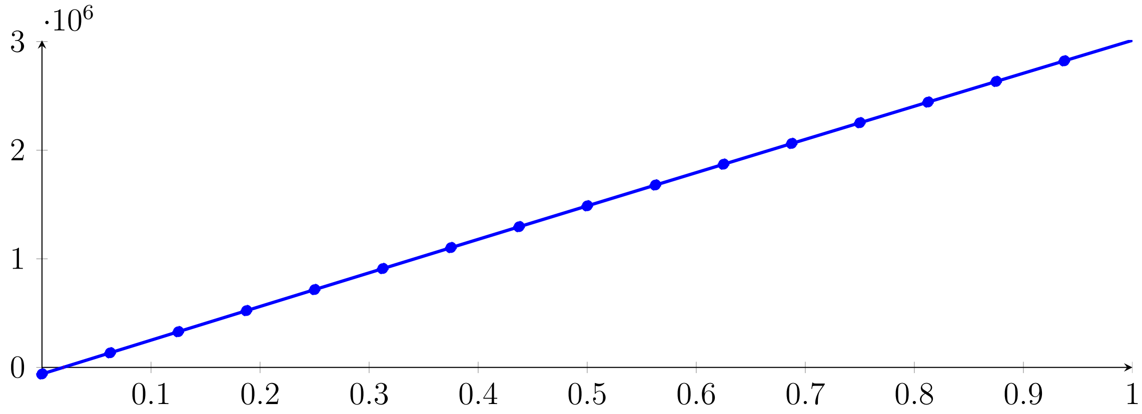
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

70.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-0.000205056X^{16} + 0.000776167X^{15} - 0.000863333X^{14} - 0.000217499X^{13} - 0.00809276X^{12} \\ &- 5.09773\cdot10^{-05}X^{11} - 0.00564923X^{10} - 0.000162811X^{9} - 0.000215376X^{8} - 3.32948\cdot10^{-07}X^{7} \\ &- 0.000259866X^{6} + 0.0161077X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot10^{6}X - 60508.5 \\ = &-60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X) \\ &+ 910202B_{5,16}(X) + 1.1032\cdot10^{6}B_{6,16}(X) + 1.29582\cdot10^{6}B_{7,16}(X) + 1.48805\cdot10^{6}B_{8,16}(X) \\ &+ 1.6799\cdot10^{6}B_{9,16}(X) + 1.87137\cdot10^{6}B_{10,16}(X) + 2.06245\cdot10^{6}B_{11,16}(X) + 2.25315\cdot10^{6}B_{12,16}(X) \\ &+ 2.44346\cdot10^{6}B_{13,16}(X) + 2.63339\cdot10^{6}B_{14,16}(X) + 2.82293\cdot10^{6}B_{15,16}(X) + 3.01209\cdot10^{6}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

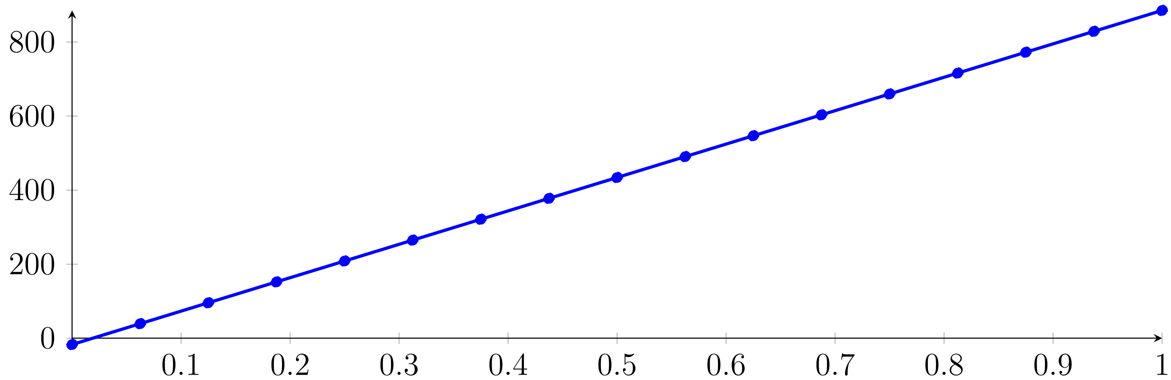
$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

70.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-5.9692\cdot 10^{-08}X^{16} + 2.16103\cdot 10^{-07}X^{15} - 2.28456\cdot 10^{-07}X^{14} - 1.17238\cdot 10^{-07}X^{13} \\ &- 2.29525\cdot 10^{-06}X^{12} - 8.31778\cdot 10^{-08}X^{11} - 1.74251\cdot 10^{-06}X^{10} - 9.42919\cdot 10^{-08}X^{9} \\ &- 7.38891\cdot 10^{-08}X^{8} + 3.25144\cdot 10^{-09}X^{7} - 2.61741\cdot 10^{-08}X^{6} + 7.44876\cdot 10^{-10}X^{5} \\ &- 2.58638\cdot 10^{-10}X^{4} - 5.65024\cdot 10^{-09}X^{3} - 0.00382618X^{2} + 902.448X - 17.178 \\ = &-17.178B_{0,16}(X) + 39.225B_{1,16}(X) + 95.6279B_{2,16}(X) + 152.031B_{3,16}(X) + 208.434B_{4,16}(X) \\ &+ 264.837B_{5,16}(X) + 321.24B_{6,16}(X) + 377.642B_{7,16}(X) + 434.045B_{8,16}(X) \\ &+ 490.448B_{9,16}(X) + 546.851B_{10,16}(X) + 603.253B_{11,16}(X) + 659.656B_{12,16}(X) \\ &+ 716.059B_{13,16}(X) + 772.461B_{14,16}(X) + 828.864B_{15,16}(X) + 885.266B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045\cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

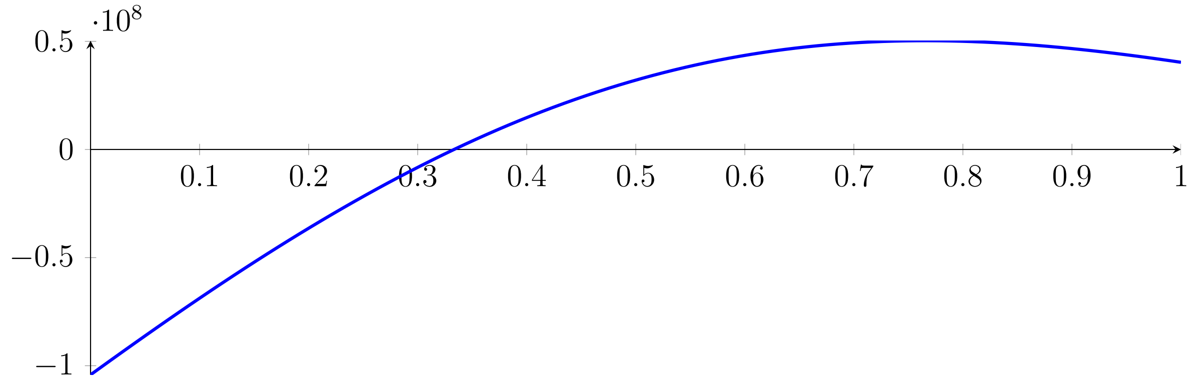
70.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

70.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^{6}X^{9} - 2.52262 \cdot 10^{6}X^{8} - 1.43506 \cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \cdot 10^{7}X^{5} + 3.97396 \cdot 10^{7}X^{4} - 2.40625 \cdot 10^{8}X^{3} - 8.33333 \cdot 10^{7}X^{2} + 3.64583 \cdot 10^{8}X - 1.04167 \cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

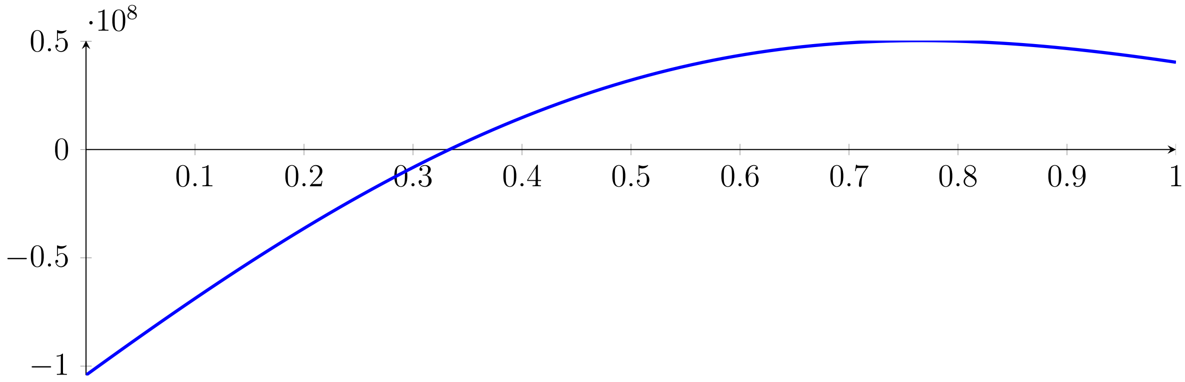
with precision $\varepsilon = 1 \cdot 10^{-08}$.

71 Running QuadClip on f_{16} with epsilon 8

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

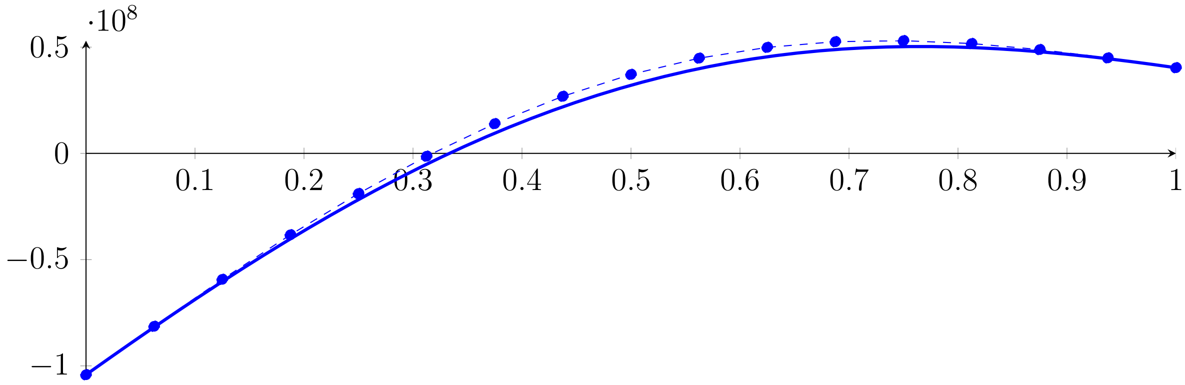
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} \\ &+ 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \end{aligned}$$



71.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

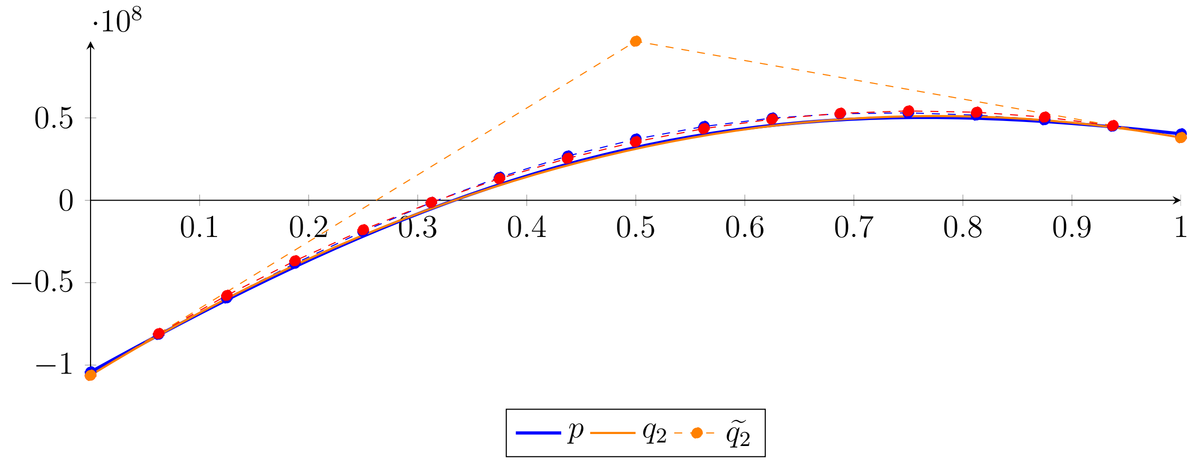
$$\begin{aligned} p = &-0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\ &+ 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ = &-1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ &\cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ &+ 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ &\cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ &+ 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181\cdot10^8X^2 + 4.05417\cdot10^8X - 1.06197\cdot10^8$$
$$= -1.06197\cdot10^8B_{0,2} + 9.65115\cdot10^7B_{1,2} + 3.80384\cdot10^7B_{2,2}$$

$$\widetilde{q}_2 = -1.41787\cdot10^6X^{16} + 1.11761\cdot10^7X^{15} - 3.98898\cdot10^7X^{14} + 8.52437\cdot10^7X^{13} - 1.21528\cdot10^8X^{12} + 1.21946\cdot10^8X^{11} - 8.86062\cdot10^7X^{10} + 4.72904\cdot10^7X^9 - 1.86355\cdot10^7X^8 + 5.41059\cdot10^6X^7 - 1.14487\cdot10^6X^6 + 172240X^5 - 17636.6X^4 + 1159.74X^3 - 2.61181\cdot10^8X^2 + 4.05417\cdot10^8X - 1.06197\cdot10^8$$
$$= -1.06197\cdot10^8B_{0,16} - 8.08583\cdot10^7B_{1,16} - 5.76963\cdot10^7B_{2,16} - 3.67107\cdot10^7B_{3,16} - 1.79017\cdot10^7B_{4,16} - 1.26923\cdot10^6B_{5,16} + 1.31867\cdot10^7B_{6,16} + 2.54662\cdot10^7B_{7,16} + 3.55691\cdot10^7B_{8,16} + 4.34957\cdot10^7B_{9,16} + 4.92455\cdot10^7B_{10,16} + 5.2819\cdot10^7B_{11,16} + 5.42159\cdot10^7B_{12,16} + 5.34363\cdot10^7B_{13,16} + 5.04802\cdot10^7B_{14,16} + 4.53476\cdot10^7B_{15,16} + 3.80384\cdot10^7B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233\cdot10^6$.

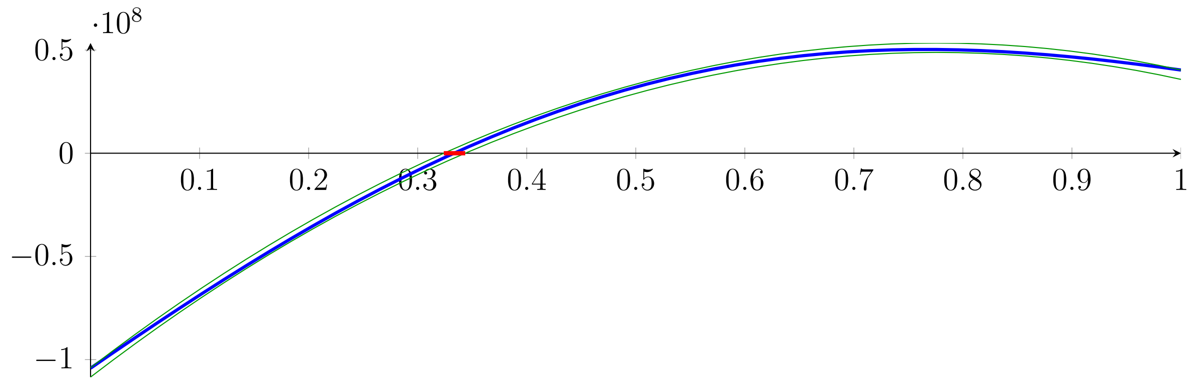
Bounding polynomials M and m:

$$M = -2.61181\cdot10^8X^2 + 4.05417\cdot10^8X - 1.03924\cdot10^8$$
$$m = -2.61181\cdot10^8X^2 + 4.05417\cdot10^8X - 1.08469\cdot10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

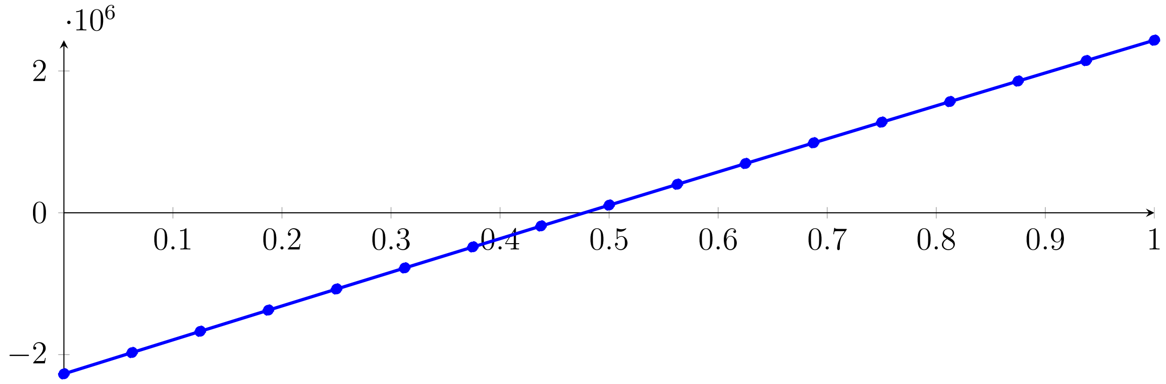
Longest intersection interval: 0.0196686

$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

71.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

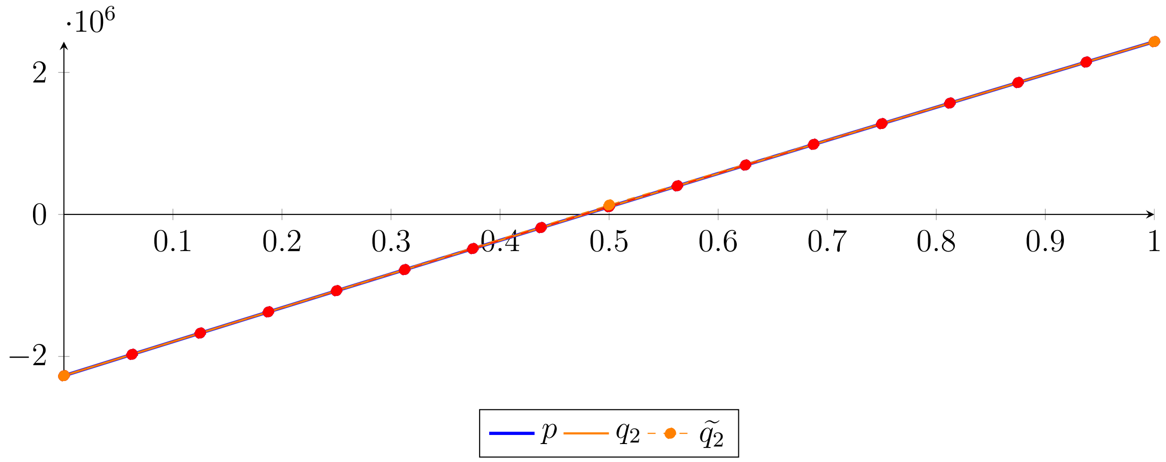
$$p = -2.82438\cdot 10^{-05}X^{16} + 2.90051\cdot 10^{-05}X^{15} + 0.000231285X^{14} + 0.000648014X^{13} + 0.00318916X^{12} + 0.00323204X^{11} + 0.00460533X^{10} + 0.00220012X^{9} - 0.000101882X^{8} + 5.32717\cdot 10^{-06}X^{7} - 0.00186824X^{6} + 0.132741X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874\cdot 10^{6}X - 2.2715\cdot 10^{6}$$
$$= -2.2715\cdot 10^{6}B_{0,16}(X) - 1.97096\cdot 10^{6}B_{1,16}(X) - 1.67127\cdot 10^{6}B_{2,16}(X) - 1.37244\cdot 10^{6}B_{3,16}(X) - 1.07448\cdot 10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806\cdot 10^{6}B_{12,16}(X) + 1.5682\cdot 10^{6}B_{13,16}(X) + 1.85747\cdot 10^{6}B_{14,16}(X) + 2.14586\cdot 10^{6}B_{15,16}(X) + 2.43338\cdot 10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265X^{2} + 4.80923\cdot 10^{6}X - 2.27154\cdot 10^{6}$$
$$= -2.27154\cdot 10^{6}B_{0,2} + 133069B_{1,2} + 2.43342\cdot 10^{6}B_{2,2}$$

$$\widetilde{q}_2 = 43556.7X^{16} - 347904X^{15} + 1.2616\cdot 10^{6}X^{14} - 2.74835\cdot 10^{6}X^{13} + 4.01042\cdot 10^{6}X^{12} - 4.13709\cdot 10^{6}X^{11} + 3.10251\cdot 10^{6}X^{10} - 1.71209\cdot 10^{6}X^{9} + 695531X^{8} - 205965X^{7} + 43591.2X^{6} - 6402.94X^{5} + 625.678X^{4} - 37.485X^{3} - 104264X^{2} + 4.80923\cdot 10^{6}X - 2.27154\cdot 10^{6}$$
$$= -2.27154\cdot 10^{6}B_{0,16} - 1.97097\cdot 10^{6}B_{1,16} - 1.67126\cdot 10^{6}B_{2,16} - 1.37242\cdot 10^{6}B_{3,16} - 1.07445\cdot 10^{6}B_{4,16} - 777350B_{5,16} - 481117B_{6,16} - 185755B_{7,16} + 108741B_{8,16} + 402365B_{9,16} + 695123B_{10,16} + 987010B_{11,16} + 1.27803\cdot 10^{6}B_{12,16} + 1.56818\cdot 10^{6}B_{13,16} + 1.85746\cdot 10^{6}B_{14,16} + 2.14587\cdot 10^{6}B_{15,16} + 2.43342\cdot 10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m:

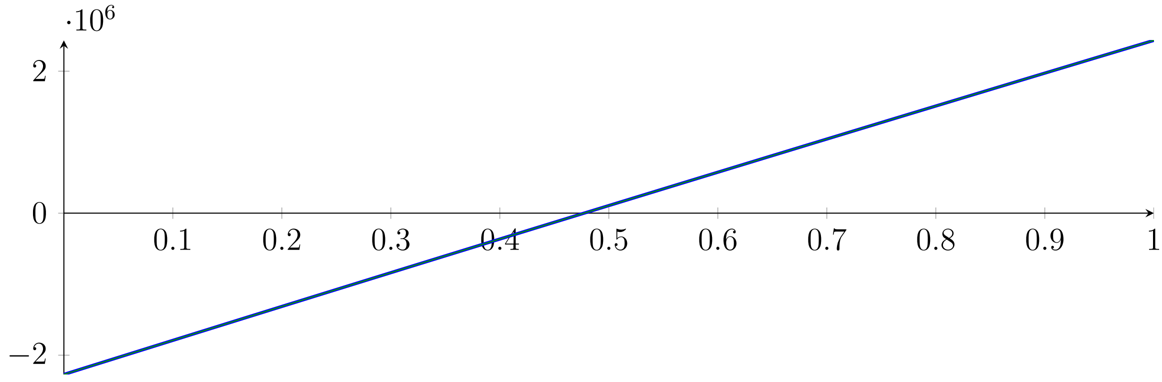
$$M = -104265X^{2} + 4.80923\cdot 10^{6}X - 2.2715\cdot 10^{6}$$

$$m = -104265X^2 + 4.80923{\cdot}10^6 X - 2.27159{\cdot}10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



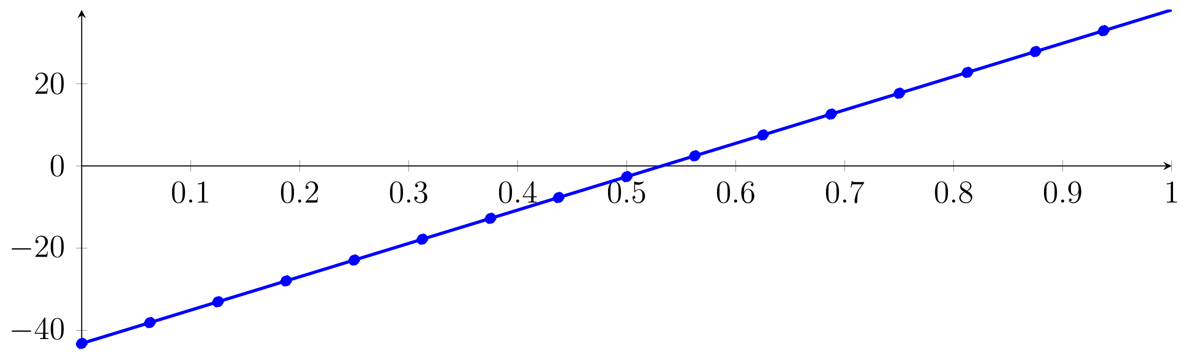
$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

71.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

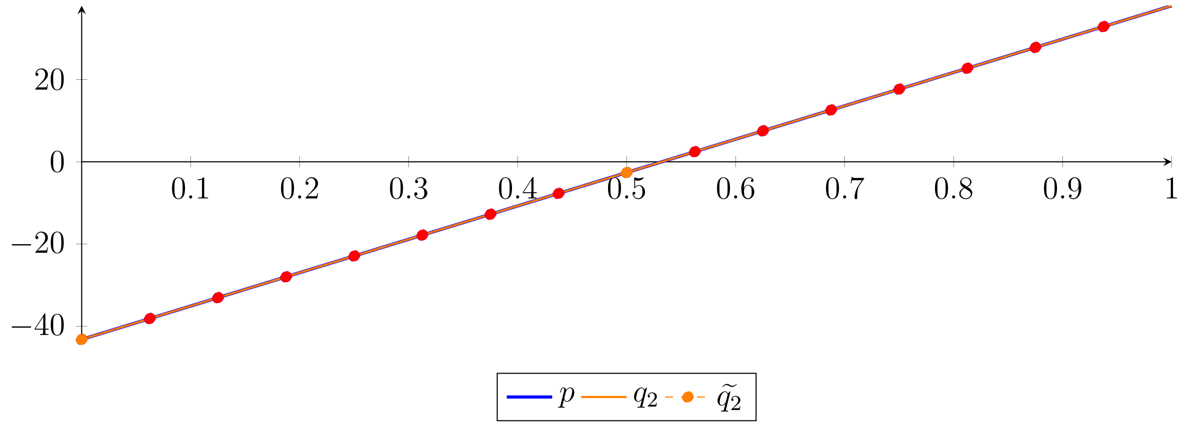
$$\begin{aligned} p = {} & 8.74252{\cdot}10^{-11}X^{16} - 1.56979{\cdot}10^{-09}X^{15} + 6.68479{\cdot}10^{-09}X^{14} + 1.20008{\cdot}10^{-08}X^{13} + 9.07301{\cdot}10^{-08}X^{12} \\ & + 5.58657{\cdot}10^{-08}X^{11} + 1.13801{\cdot}10^{-07}X^{10} + 3.70665{\cdot}10^{-08}X^{9} + 7.31575{\cdot}10^{-10}X^{8} + 1.30058{\cdot}10^{-09}X^{7} \\ & + 5.00722{\cdot}10^{-09}X^{6} + 1.24146{\cdot}10^{-10}X^{5} + 1.03455{\cdot}10^{-10}X^{4} - 3.09388{\cdot}10^{-05}X^{2} + 81.1505X - 43.1911 \\ = {} & -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\ & - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68777B_{7,16}(X) - 2.61587B_{8,16}(X) \\ & + 2.45604B_{9,16}(X) + 7.52794B_{10,16}(X) + 12.5998B_{11,16}(X) + 17.6718B_{12,16}(X) \\ & + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09388{\cdot}10^{-05}X^2 + 81.1505X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 = {} & 0.721495X^{16} - 5.74915X^{15} + 20.7933X^{14} - 45.1627X^{13} + 65.6806X^{12} - 67.5044X^{11} \\ & + 50.4286X^{10} - 27.728X^{9} + 11.2318X^{8} - 3.32011X^{7} + 0.702408X^{6} - 0.103415X^{5} \\ & + 0.0102099X^{4} - 0.000624725X^{3} - 1.10834{\cdot}10^{-05}X^{2} + 81.1505X - 43.1911 \\ = {} & -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} \\ & - 12.7597B_{6,16} - 7.68779B_{7,16} - 2.61585B_{8,16} + 2.45602B_{9,16} + 7.52795B_{10,16} + 12.5998B_{11,16} \\ & + 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.57956 \cdot 10^{-05}$.

Bounding polynomials M and m:

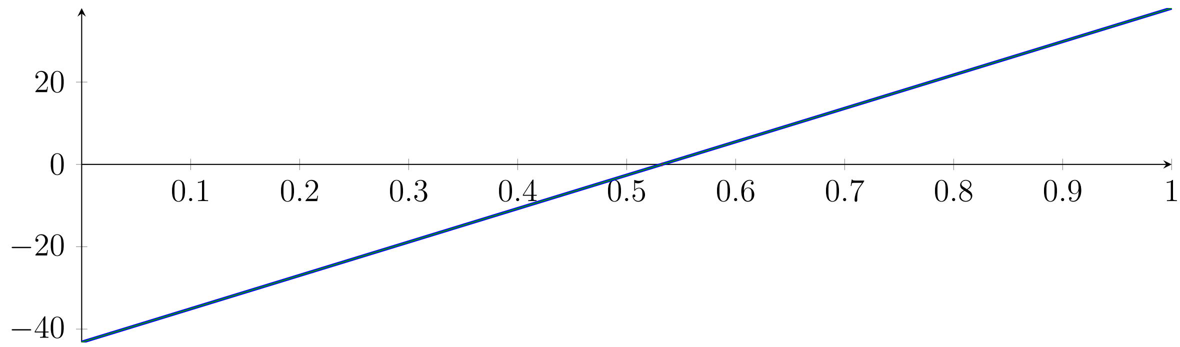
$$M = -3.09388 \cdot 10^{-05} X^2 + 81.1505X - 43.1911$$

$$m = -3.09388 \cdot 10^{-05} X^2 + 81.1505X - 43.1911$$

Root of M and m:

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\} \qquad N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

Longest intersection interval: $3.8903 \cdot 10^{-07}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

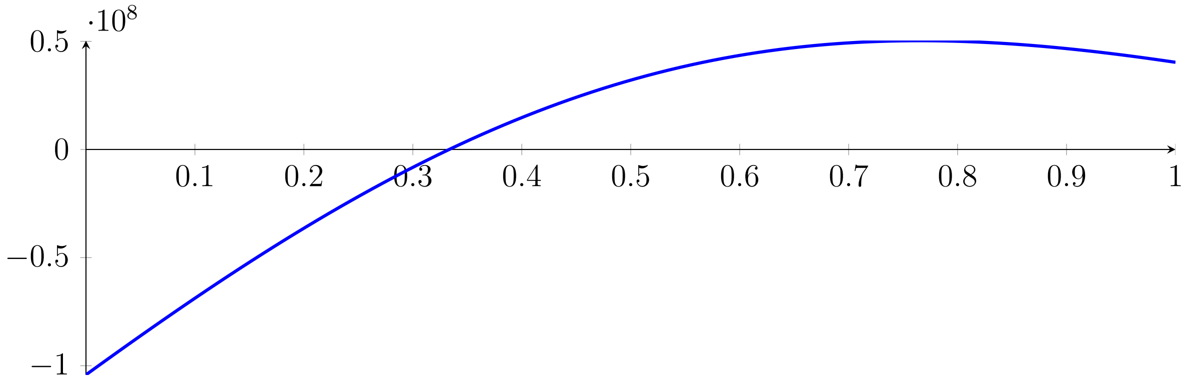
71.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

71.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1\cdot 10^{-08}$.

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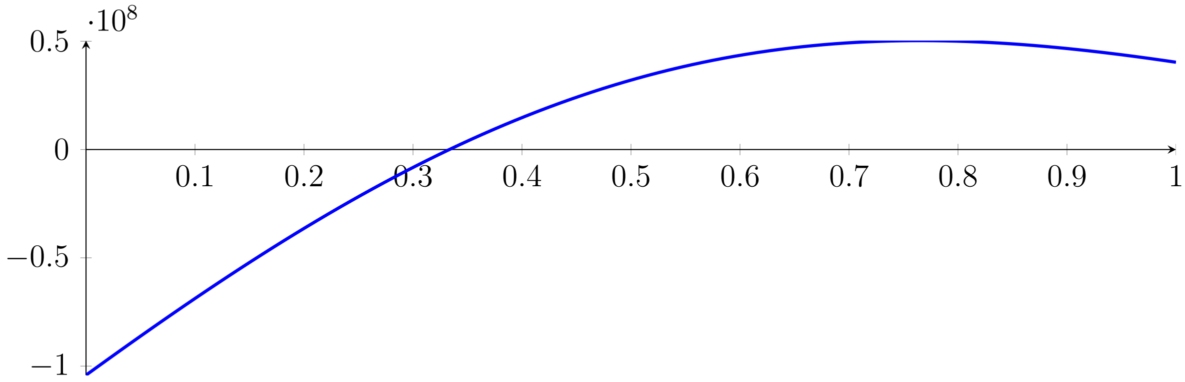
$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^6X^9 - 2.52262{\cdot}10^6X^8 - 1.43506{\cdot}10^7X^7 + 138542X^6 + 7.92823{\cdot}10^7X^5 + 3.97396{\cdot}10^7X^4 - 2.40625{\cdot}10^8X^3 - 8.33333{\cdot}10^7X^2 + 3.64583{\cdot}10^8X - 1.04167{\cdot}10^8$$

Called CubeClip with input polynomial on interval $[0, 1]$:

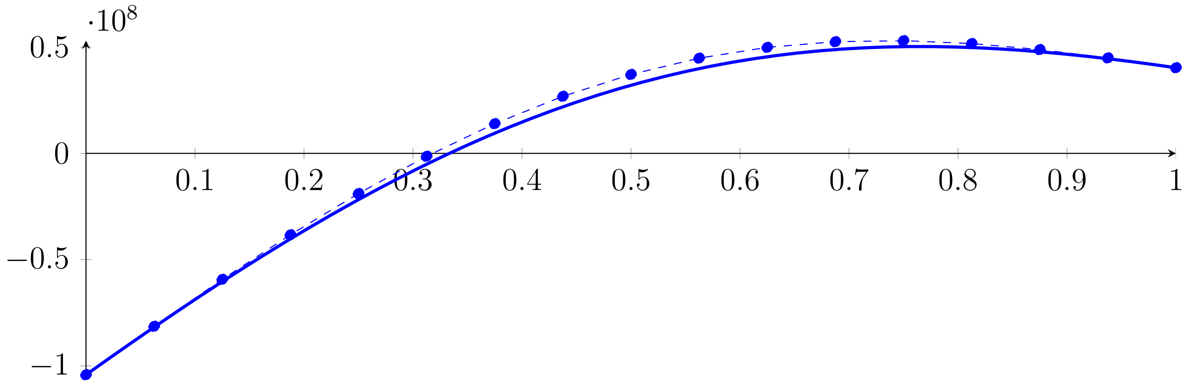
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^6X^9 - 2.52262{\cdot}10^6X^8 - 1.43506{\cdot}10^7X^7 + 138542X^6 + 7.92823{\cdot}10^7X^5 + 3.97396{\cdot}10^7X^4 - 2.40625{\cdot}10^8X^3 - 8.33333{\cdot}10^7X^2 + 3.64583{\cdot}10^8X - 1.04167{\cdot}10^8$$



72.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

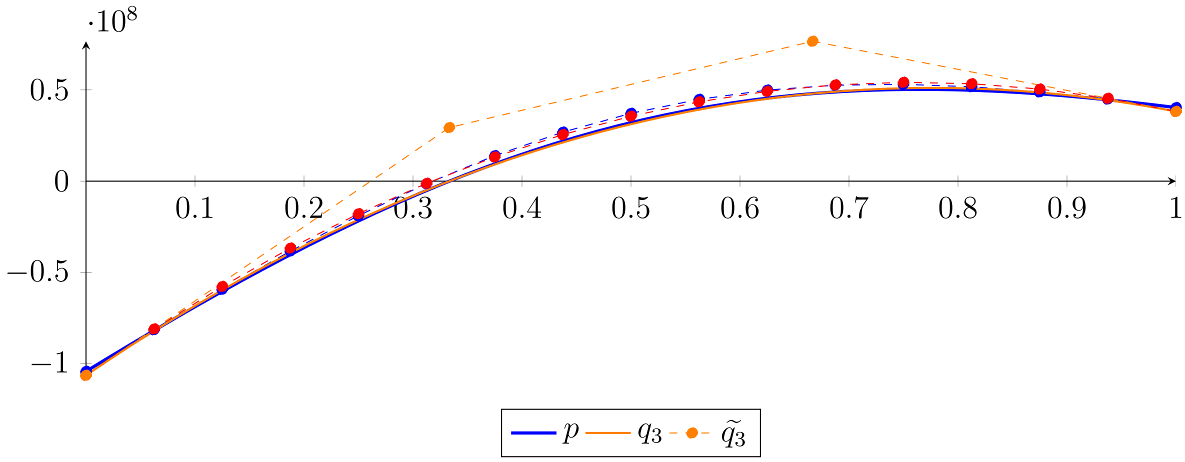
$$\begin{aligned} p &= -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} + 451673X^{10} + 1.12172{\cdot}10^6X^9 - 2.52262{\cdot}10^6X^8 - 1.43506{\cdot}10^7X^7 + 138542X^6 + 7.92823{\cdot}10^7X^5 + 3.97396{\cdot}10^7X^4 - 2.40625{\cdot}10^8X^3 - 8.33333{\cdot}10^7X^2 + 3.64583{\cdot}10^8X - 1.04167{\cdot}10^8 \\ &= -1.04167{\cdot}10^8B_{0,16}(X) - 8.13802{\cdot}10^7B_{1,16}(X) - 5.92882{\cdot}10^7B_{2,16}(X) - 3.83203{\cdot}10^7B_{3,16}(X) - 1.88844{\cdot}10^7B_{4,16}(X) - 1.34837{\cdot}10^6B_{5,16}(X) + 1.39781{\cdot}10^7B_{6,16}(X) + 2.68604{\cdot}10^7B_{7,16}(X) + 3.71532{\cdot}10^7B_{8,16}(X) + 4.48105{\cdot}10^7B_{9,16}(X) + 4.98901{\cdot}10^7B_{10,16}(X) + 5.25504{\cdot}10^7B_{11,16}(X) + 5.30407{\cdot}10^7B_{12,16}(X) + 5.16832{\cdot}10^7B_{13,16}(X) + 4.88488{\cdot}10^7B_{14,16}(X) + 4.49297{\cdot}10^7B_{15,16}(X) + 4.03108{\cdot}10^7B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.06335{\cdot}10^8$$
$$= -1.06335{\cdot}10^8 B_{0,3} + 2.93558{\cdot}10^7 B_{1,3} + 7.66068{\cdot}10^7 B_{2,3} + 3.81764{\cdot}10^7 B_{3,3}$$

$$\widetilde{q}_3 = 2.11088{\cdot}10^6 X^{16} - 1.70537{\cdot}10^7 X^{15} + 6.25913{\cdot}10^7 X^{14} - 1.38111{\cdot}10^8 X^{13} + 2.043{\cdot}10^8 X^{12} - 2.13799{\cdot}10^8 X^{11} + 1.62714{\cdot}10^8 X^{10} - 9.11092{\cdot}10^7 X^9 + 3.75254{\cdot}10^7 X^8 - 1.12529{\cdot}10^7 X^7 + 2.40899{\cdot}10^6 X^6 - 357156 X^5 + 34986.1 X^4 + 2.75602{\cdot}10^6 X^3 - 2.65318{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.06335{\cdot}10^8$$
$$= -1.06335{\cdot}10^8 B_{0,16} - 8.08928{\cdot}10^7 B_{1,16} - 5.76618{\cdot}10^7 B_{2,16} - 3.66369{\cdot}10^7 B_{3,16} - 1.78131{\cdot}10^7 B_{4,16} - 1.18553{\cdot}10^6 B_{5,16} + 1.32508{\cdot}10^7 B_{6,16} + 2.55007{\cdot}10^7 B_{7,16} + 3.55692{\cdot}10^7 B_{8,16} + 4.34611{\cdot}10^7 B_{9,16} + 4.91816{\cdot}10^7 B_{10,16} + 5.27352{\cdot}10^7 B_{11,16} + 5.41273{\cdot}10^7 B_{12,16} + 5.33624{\cdot}10^7 B_{13,16} + 5.04457{\cdot}10^7 B_{14,16} + 4.53821{\cdot}10^7 B_{15,16} + 3.81764{\cdot}10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806{\cdot}10^6$.

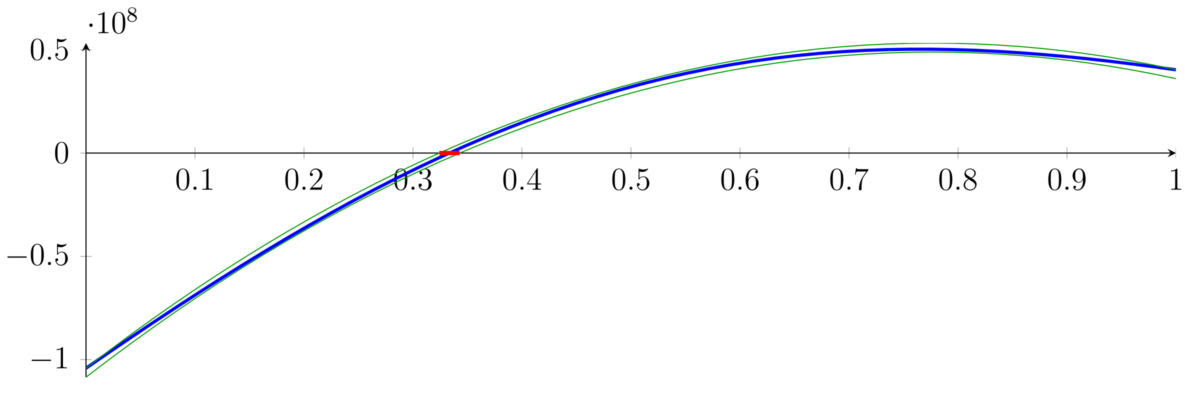
Bounding polynomials M and m:

$$M = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.04167{\cdot}10^8$$
$$m = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.08503{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

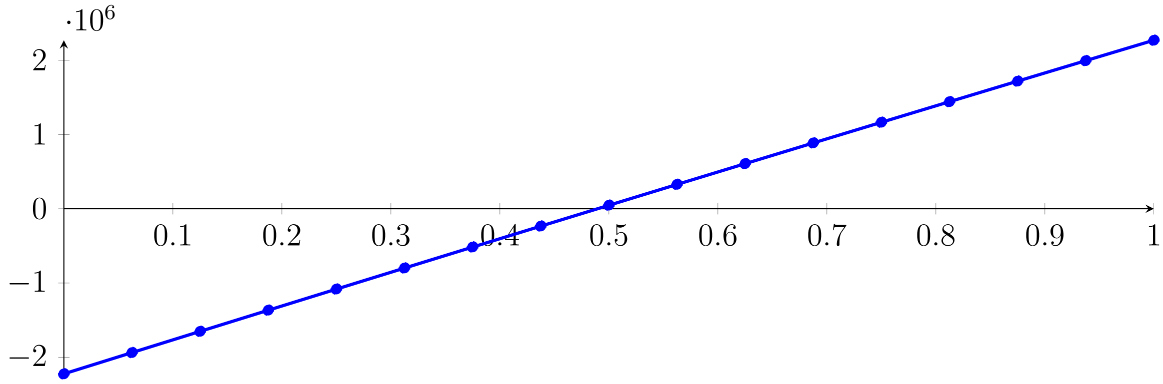
Longest intersection interval: 0.0187703

$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

72.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

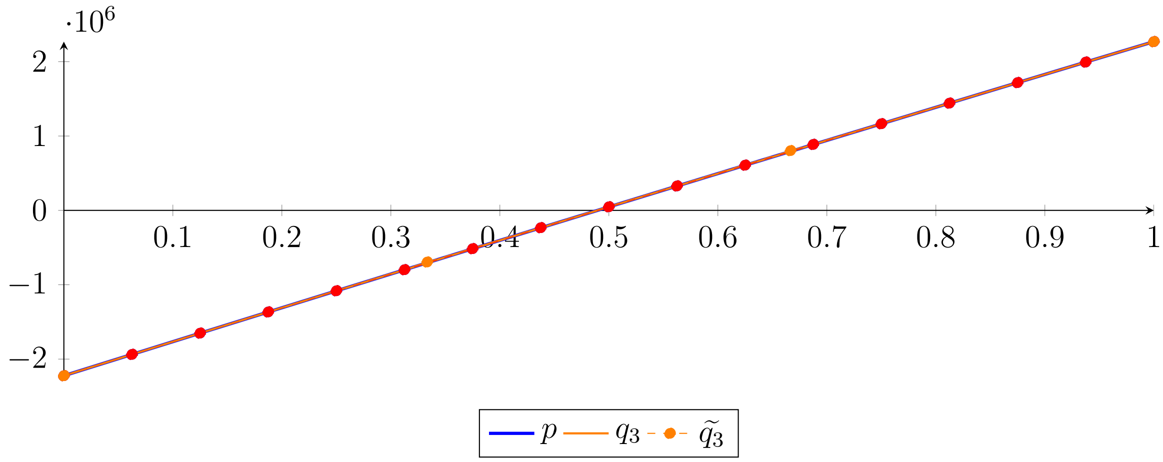
$$p = -1.93035 \cdot 10^{-05} X^{16} + 1.08927 \cdot 10^{-05} X^{15} + 0.000255816 X^{14} + 0.000616983 X^{13} + 0.00371715 X^{12} + 0.00325035 X^{11} + 0.00510875 X^{10} + 0.00190713 X^{9} + 0.000179792 X^{8} + 1.06543 \cdot 10^{-05} X^{7} - 0.00136482 X^{6} + 0.104959 X^{5} + 18.6215 X^{4} - 738.202 X^{3} - 93855.7 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$
$$= -2.22335 \cdot 10^{6} B_{0,16}(X) - 1.93666 \cdot 10^{6} B_{1,16}(X) - 1.65074 \cdot 10^{6} B_{2,16}(X) - 1.36561 \cdot 10^{6} B_{3,16}(X) - 1.08127 \cdot 10^{6} B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^{6} B_{12,16}(X) + 1.44233 \cdot 10^{6} B_{13,16}(X) + 1.71876 \cdot 10^{6} B_{14,16}(X) + 1.99439 \cdot 10^{6} B_{15,16}(X) + 2.26922 \cdot 10^{6} B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -700.673 X^{3} - 93879.9 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$
$$= -2.22335 \cdot 10^{6} B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^{6} B_{3,3}$$

$$\widetilde{q}_3 = 15290.6 X^{16} - 125412 X^{15} + 468045 X^{14} - 1.05198 \cdot 10^{6} X^{13} + 1.588 \cdot 10^{6} X^{12} - 1.69893 \cdot 10^{6} X^{11} + 1.32382 \cdot 10^{6} X^{10} - 759402 X^{9} + 320067 X^{8} - 97826.5 X^{7} + 21197.8 X^{6} - 3162.8 X^{5} + 313.931 X^{4} - 720.165 X^{3} - 93879.2 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$
$$= -2.22335 \cdot 10^{6} B_{0,16} - 1.93666 \cdot 10^{6} B_{1,16} - 1.65074 \cdot 10^{6} B_{2,16} - 1.36561 \cdot 10^{6} B_{3,16} - 1.08126 \cdot 10^{6} B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.7 B_{8,16} + 328649 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^{6} B_{12,16} + 1.44233 \cdot 10^{6} B_{13,16} + 1.71876 \cdot 10^{6} B_{14,16} + 1.99439 \cdot 10^{6} B_{15,16} + 2.26922 \cdot 10^{6} B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 0.457751$.

Bounding polynomials M and m:

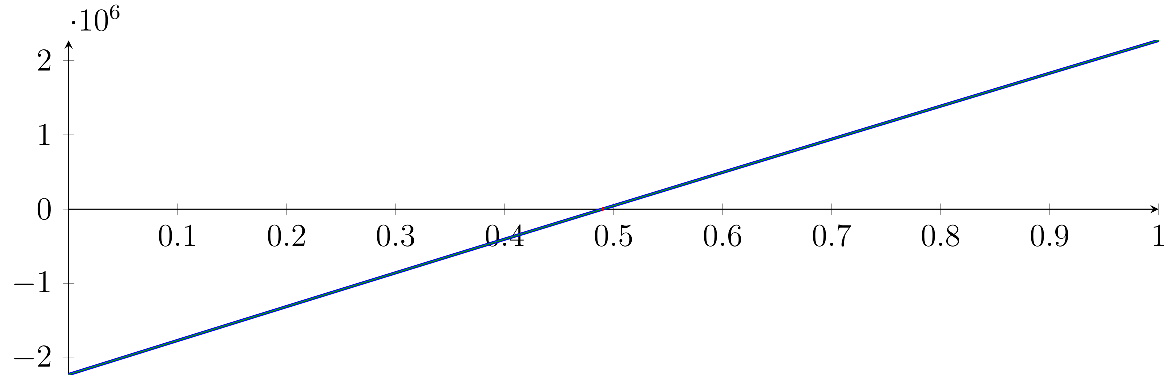
$$M = -700.673 X^{3} - 93879.9 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715{\cdot}10^6X - 2.22335{\cdot}10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $2.03684{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

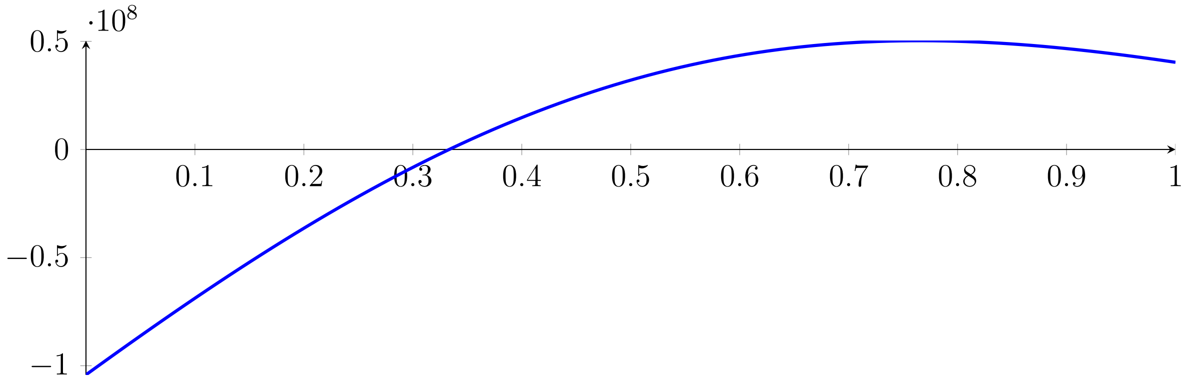
72.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

72.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} + 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1{\cdot}10^{-08}$.

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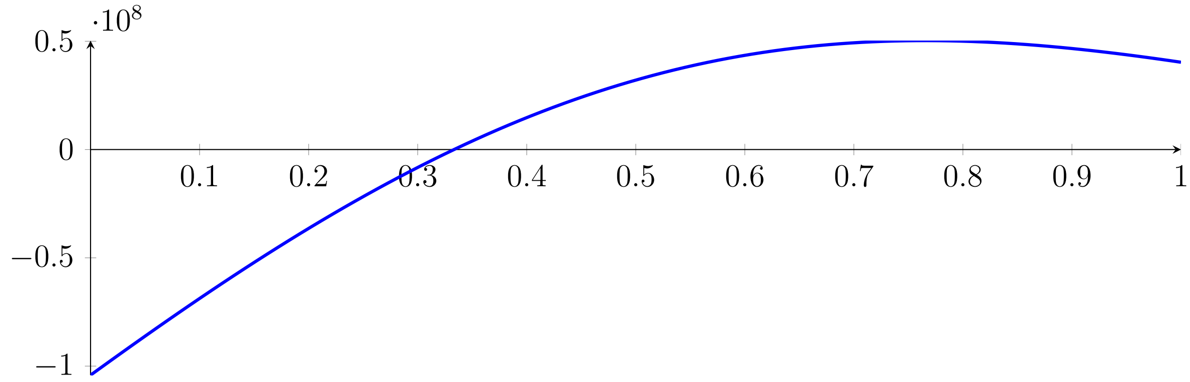
$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^6X^9 - 2.52262\cdot 10^6X^8 - 1.43506\cdot 10^7X^7 + 138542X^6 + 7.92823\cdot 10^7X^5 + 3.97396\cdot 10^7X^4 - 2.40625\cdot 10^8X^3 - 8.33333\cdot 10^7X^2 + 3.64583\cdot 10^8X - 1.04167\cdot 10^8$$

Called BezClip with input polynomial on interval $[0, 1]$:

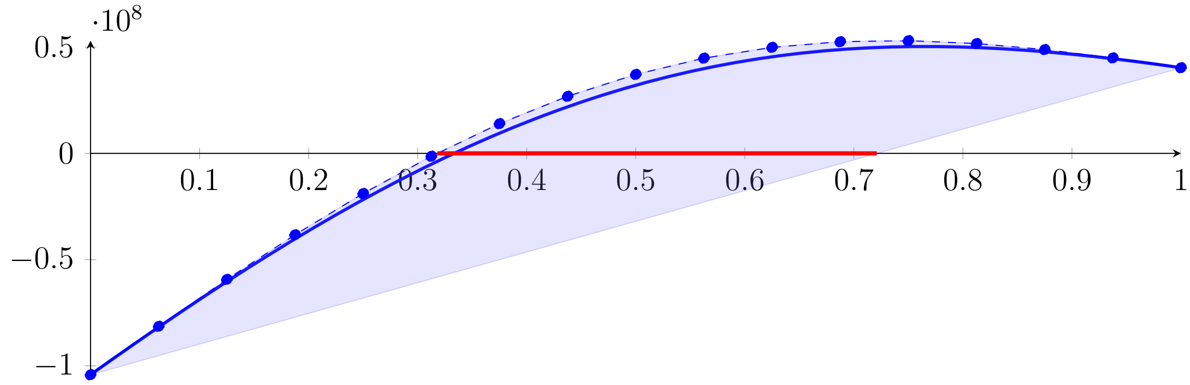
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^6X^9 - 2.52262\cdot 10^6X^8 - 1.43506\cdot 10^7X^7 + 138542X^6 + 7.92823\cdot 10^7X^5 + 3.97396\cdot 10^7X^4 - 2.40625\cdot 10^8X^3 - 8.33333\cdot 10^7X^2 + 3.64583\cdot 10^8X - 1.04167\cdot 10^8$$



73.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot 10^6X^9 - 2.52262\cdot 10^6X^8 - 1.43506\cdot 10^7X^7 + 138542X^6 + 7.92823 \\ &\quad \cdot 10^7X^5 + 3.97396\cdot 10^7X^4 - 2.40625\cdot 10^8X^3 - 8.33333\cdot 10^7X^2 + 3.64583\cdot 10^8X - 1.04167\cdot 10^8 \\ &= -1.04167\cdot 10^8B_{0,16}(X) - 8.13802\cdot 10^7B_{1,16}(X) - 5.92882\cdot 10^7B_{2,16}(X) - 3.83203 \\ &\quad \cdot 10^7B_{3,16}(X) - 1.88844\cdot 10^7B_{4,16}(X) - 1.34837\cdot 10^6B_{5,16}(X) + 1.39781\cdot 10^7B_{6,16}(X) \\ &\quad + 2.68604\cdot 10^7B_{7,16}(X) + 3.71532\cdot 10^7B_{8,16}(X) + 4.48105\cdot 10^7B_{9,16}(X) + 4.98901 \\ &\quad \cdot 10^7B_{10,16}(X) + 5.25504\cdot 10^7B_{11,16}(X) + 5.30407\cdot 10^7B_{12,16}(X) + 5.16832\cdot 10^7B_{13,16}(X) \\ &\quad + 4.88488\cdot 10^7B_{14,16}(X) + 4.49297\cdot 10^7B_{15,16}(X) + 4.03108\cdot 10^7B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

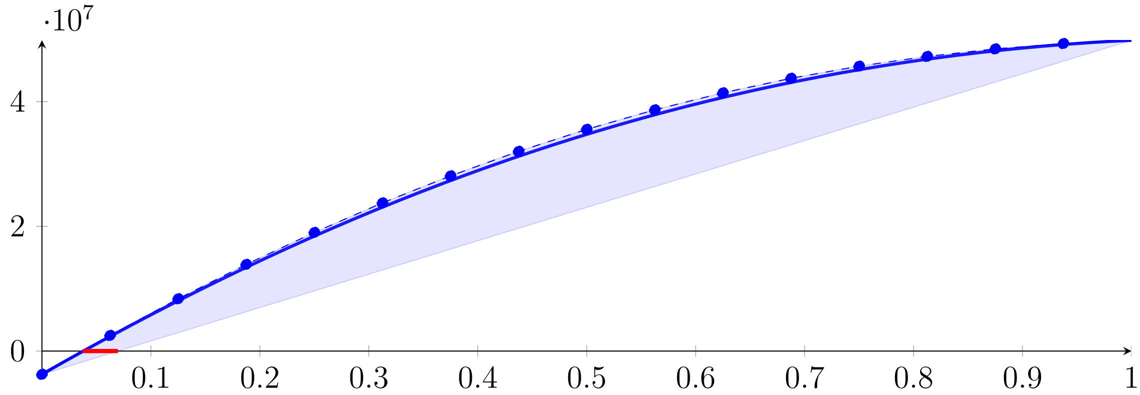
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

73.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -0.00483322X^{16} + 0.0186809X^{15} - 0.0194312X^{14} - 0.0738695X^{13} - 1.11673X^{12} \\ &\quad - 5.0471X^{11} + 36.3082X^{10} + 692.914X^{9} + 1886.96X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5} \\ &\quad + 3.91945\cdot 10^{6}X^{4} - 7.54561\cdot 10^{6}X^{3} - 4.29225\cdot 10^{7}X^{2} + 9.97982\cdot 10^{7}X - 3.73501\cdot 10^{6} \\ &= -3.73501\cdot 10^{6}B_{0,16}(X) + 2.50237\cdot 10^{6}B_{1,16}(X) + 8.38207\cdot 10^{6}B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^{7}B_{3,16}(X) + 1.90167\cdot 10^{7}B_{4,16}(X) + 2.37512\cdot 10^{7}B_{5,16}(X) + 2.80875\cdot 10^{7}B_{6,16}(X) \\ &\quad + 3.20213\cdot 10^{7}B_{7,16}(X) + 3.55508\cdot 10^{7}B_{8,16}(X) + 3.86768\cdot 10^{7}B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^{7}B_{10,16}(X) + 4.37336\cdot 10^{7}B_{11,16}(X) + 4.56783\cdot 10^{7}B_{12,16}(X) + 4.7247\cdot 10^{7}B_{13,16}(X) \\ &\quad + 4.84524\cdot 10^{7}B_{14,16}(X) + 4.93093\cdot 10^{7}B_{15,16}(X) + 4.98343\cdot 10^{7}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

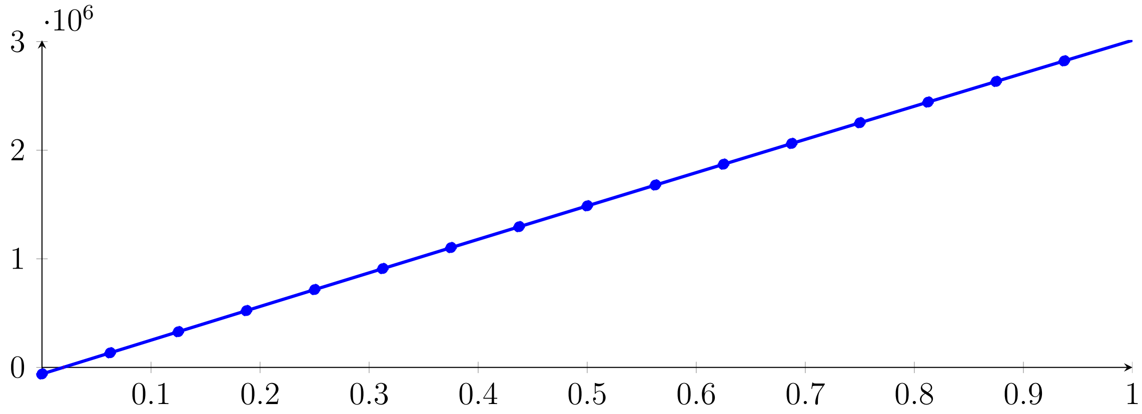
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

73.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -0.000205056X^{16} + 0.000776167X^{15} - 0.000863333X^{14} - 0.000217499X^{13} - 0.00809276X^{12} \\ &\quad - 5.09773\cdot 10^{-05}X^{11} - 0.00564923X^{10} - 0.000162811X^{9} - 0.000215376X^{8} - 3.32948\cdot 10^{-07}X^{7} \\ &\quad - 0.000259866X^{6} + 0.0161077X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot 10^{6}X - 60508.5 \\ &= -60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X) \\ &\quad + 910202B_{5,16}(X) + 1.1032\cdot 10^{6}B_{6,16}(X) + 1.29582\cdot 10^{6}B_{7,16}(X) + 1.48805\cdot 10^{6}B_{8,16}(X) \\ &\quad + 1.6799\cdot 10^{6}B_{9,16}(X) + 1.87137\cdot 10^{6}B_{10,16}(X) + 2.06245\cdot 10^{6}B_{11,16}(X) + 2.25315\cdot 10^{6}B_{12,16}(X) \\ &\quad + 2.44346\cdot 10^{6}B_{13,16}(X) + 2.63339\cdot 10^{6}B_{14,16}(X) + 2.82293\cdot 10^{6}B_{15,16}(X) + 3.01209\cdot 10^{6}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

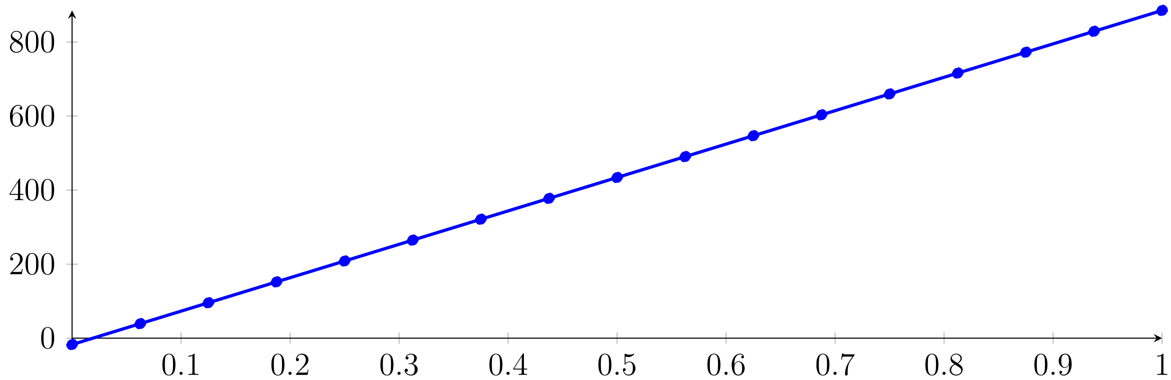
$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

73.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -5.9692\cdot 10^{-08}X^{16} + 2.16103\cdot 10^{-07}X^{15} - 2.28456\cdot 10^{-07}X^{14} - 1.17238\cdot 10^{-07}X^{13} \\ & - 2.29525\cdot 10^{-06}X^{12} - 8.31778\cdot 10^{-08}X^{11} - 1.74251\cdot 10^{-06}X^{10} - 9.42919\cdot 10^{-08}X^{9} \\ & - 7.38891\cdot 10^{-08}X^{8} + 3.25144\cdot 10^{-09}X^{7} - 2.61741\cdot 10^{-08}X^{6} + 7.44876\cdot 10^{-10}X^{5} \\ & - 2.58638\cdot 10^{-10}X^{4} - 5.65024\cdot 10^{-09}X^{3} - 0.00382618X^{2} + 902.448X - 17.178 \\ = & -17.178B_{0,16}(X) + 39.225B_{1,16}(X) + 95.6279B_{2,16}(X) + 152.031B_{3,16}(X) + 208.434B_{4,16}(X) \\ & + 264.837B_{5,16}(X) + 321.24B_{6,16}(X) + 377.642B_{7,16}(X) + 434.045B_{8,16}(X) \\ & + 490.448B_{9,16}(X) + 546.851B_{10,16}(X) + 603.253B_{11,16}(X) + 659.656B_{12,16}(X) \\ & + 716.059B_{13,16}(X) + 772.461B_{14,16}(X) + 828.864B_{15,16}(X) + 885.266B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

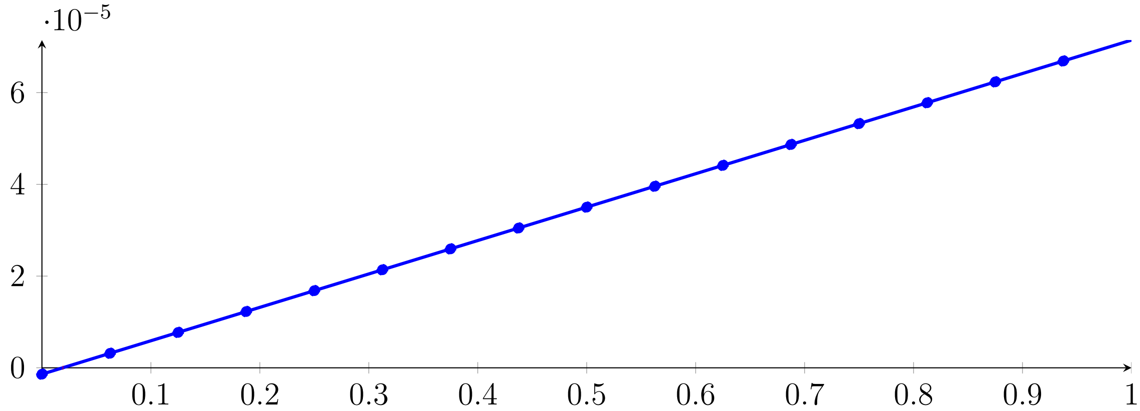
$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045\cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

73.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -4.80379\cdot10^{-15}X^{16} + 1.74137\cdot10^{-14}X^{15} - 1.72656\cdot10^{-14}X^{14} - 6.84186\cdot10^{-15}X^{13} - 1.91627\cdot10^{-13}X^{12} - 4.7358\cdot10^{-15}X^{11} - 1.33436\cdot10^{-13}X^{10} - 1.97677\cdot10^{-15}X^{9} - 7.4565\cdot10^{-15}X^{8} - 1.16281\cdot10^{-16}X^{7} - 1.98065\cdot10^{-15}X^{6} + 1.47994\cdot10^{-17}X^{5} - 1.84992\cdot10^{-17}X^{4} - 2.48011\cdot10^{-17}X^{2} + 7.28316\cdot10^{-05}X - 1.38634\cdot10^{-06}$$
$$= -1.38634\cdot10^{-06}B_{0,16}(X) + 3.16564\cdot10^{-06}B_{1,16}(X) + 7.71761\cdot10^{-06}B_{2,16}(X) + 1.22696\cdot10^{-05}B_{3,16}(X) + 1.68216\cdot10^{-05}B_{4,16}(X) + 2.13735\cdot10^{-05}B_{5,16}(X) + 2.59255\cdot10^{-05}B_{6,16}(X) + 3.04775\cdot10^{-05}B_{7,16}(X) + 3.50295\cdot10^{-05}B_{8,16}(X) + 3.95814\cdot10^{-05}B_{9,16}(X) + 4.41334\cdot10^{-05}B_{10,16}(X) + 4.86854\cdot10^{-05}B_{11,16}(X) + 5.32374\cdot10^{-05}B_{12,16}(X) + 5.77893\cdot10^{-05}B_{13,16}(X) + 6.23413\cdot10^{-05}B_{14,16}(X) + 6.68933\cdot10^{-05}B_{15,16}(X) + 7.14453\cdot10^{-05}B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.50521\cdot10^{-15}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

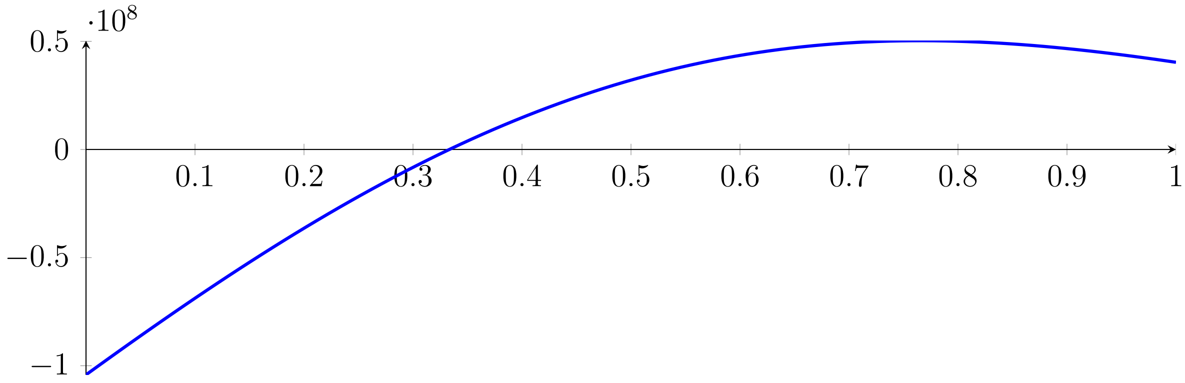
73.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

73.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0,1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

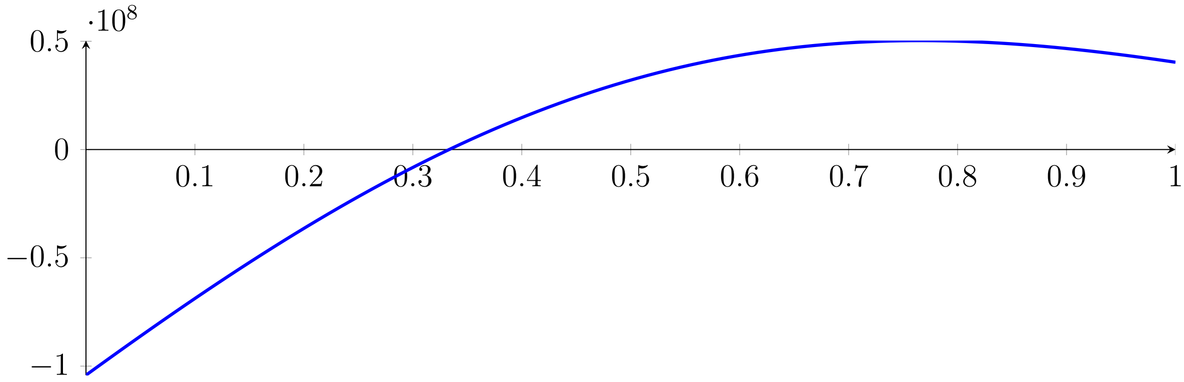
with precision $\varepsilon = 1\cdot 10^{-16}$.

74 Running QuadClip on f_{16} with epsilon 16

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

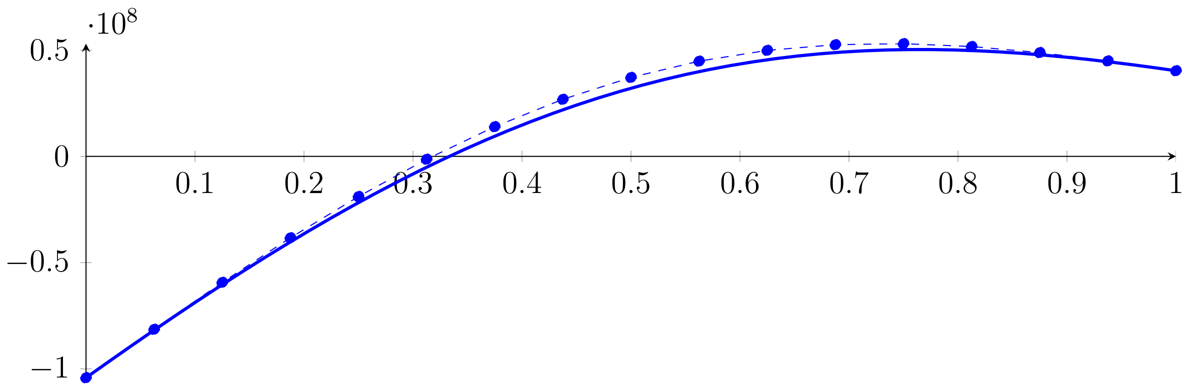
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



74.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

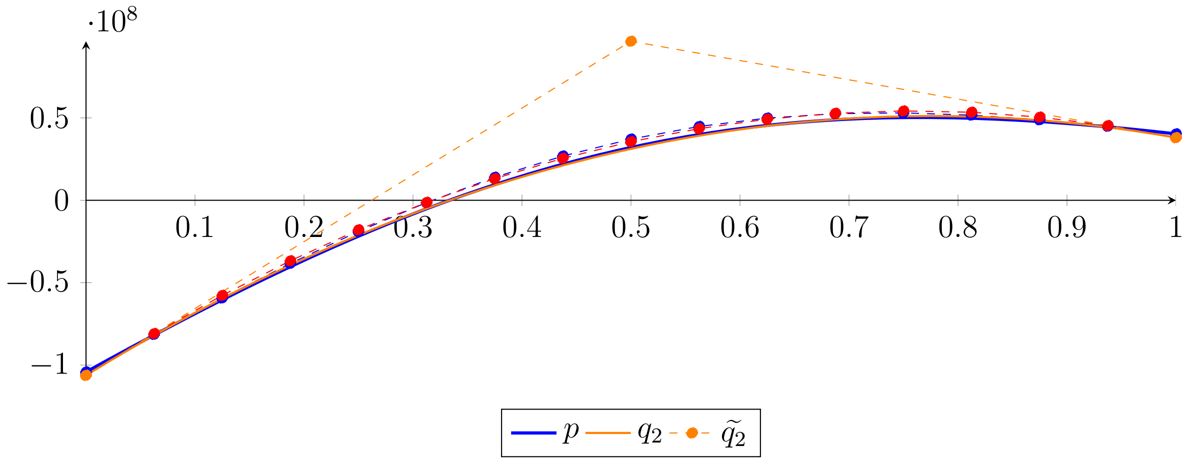
$$p = -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$
$$= -1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203\cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) + 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901\cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) + 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.06197{\cdot}10^8$$
$$= -1.06197{\cdot}10^8 B_{0,2} + 9.65115{\cdot}10^7 B_{1,2} + 3.80384{\cdot}10^7 B_{2,2}$$

$$\widetilde{q}_2 = -1.41787{\cdot}10^6 X^{16} + 1.11761{\cdot}10^7 X^{15} - 3.98898{\cdot}10^7 X^{14} + 8.52437{\cdot}10^7 X^{13} - 1.21528{\cdot}10^8 X^{12} + 1.21946{\cdot}10^8 X^{11} - 8.86062{\cdot}10^7 X^{10} + 4.72904{\cdot}10^7 X^9 - 1.86355{\cdot}10^7 X^8 + 5.41059{\cdot}10^6 X^7 - 1.14487{\cdot}10^6 X^6 + 172240 X^5 - 17636.6 X^4 + 1159.74 X^3 - 2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.06197{\cdot}10^8$$
$$= -1.06197{\cdot}10^8 B_{0,16} - 8.08583{\cdot}10^7 B_{1,16} - 5.76963{\cdot}10^7 B_{2,16} - 3.67107{\cdot}10^7 B_{3,16} - 1.79017{\cdot}10^7 B_{4,16} - 1.26923{\cdot}10^6 B_{5,16} + 1.31867{\cdot}10^7 B_{6,16} + 2.54662{\cdot}10^7 B_{7,16} + 3.55691{\cdot}10^7 B_{8,16} + 4.34957{\cdot}10^7 B_{9,16} + 4.92455{\cdot}10^7 B_{10,16} + 5.2819{\cdot}10^7 B_{11,16} + 5.42159{\cdot}10^7 B_{12,16} + 5.34363{\cdot}10^7 B_{13,16} + 5.04802{\cdot}10^7 B_{14,16} + 4.53476{\cdot}10^7 B_{15,16} + 3.80384{\cdot}10^7 B_{16,16}$$



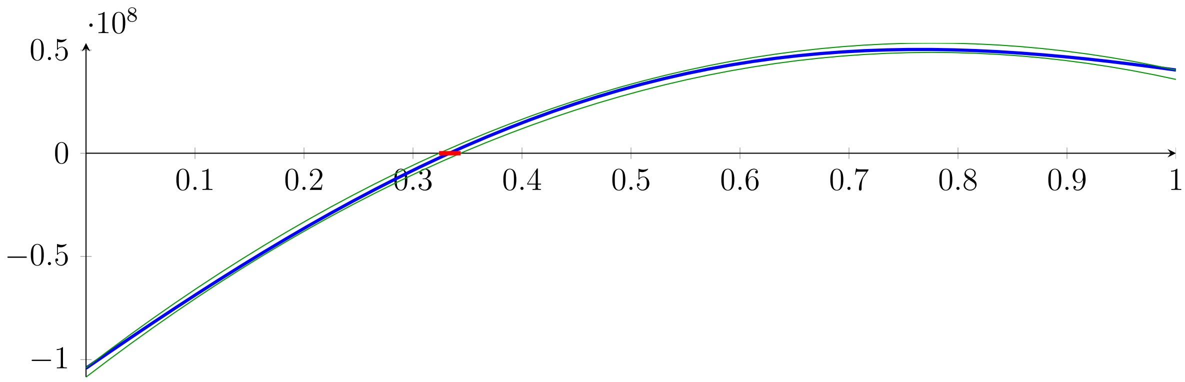
The maximum difference of the Bézier coefficients is $\delta = 2.27233{\cdot}10^6$.
Bounding polynomials M and m:

$$M = -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.03924{\cdot}10^8$$
$$m = -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.08469{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



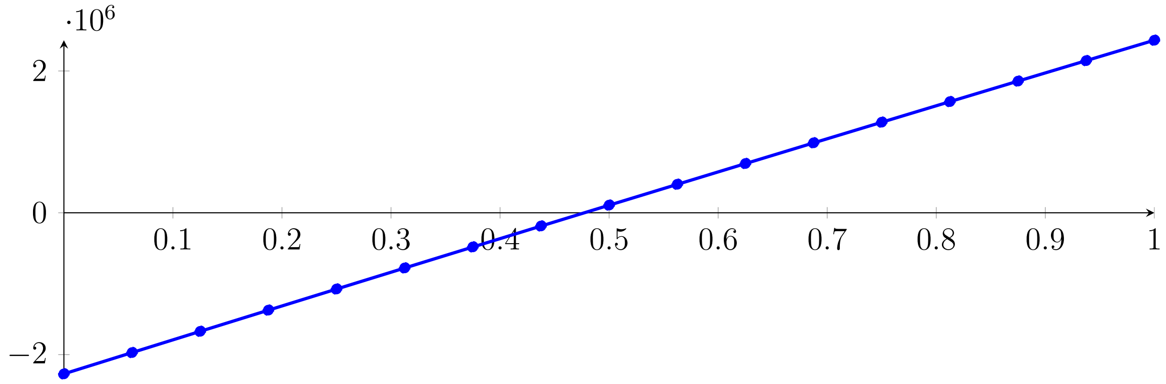
$$[0.323946, 0.343615]$$

Longest intersection interval: 0.0196686
$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

74.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

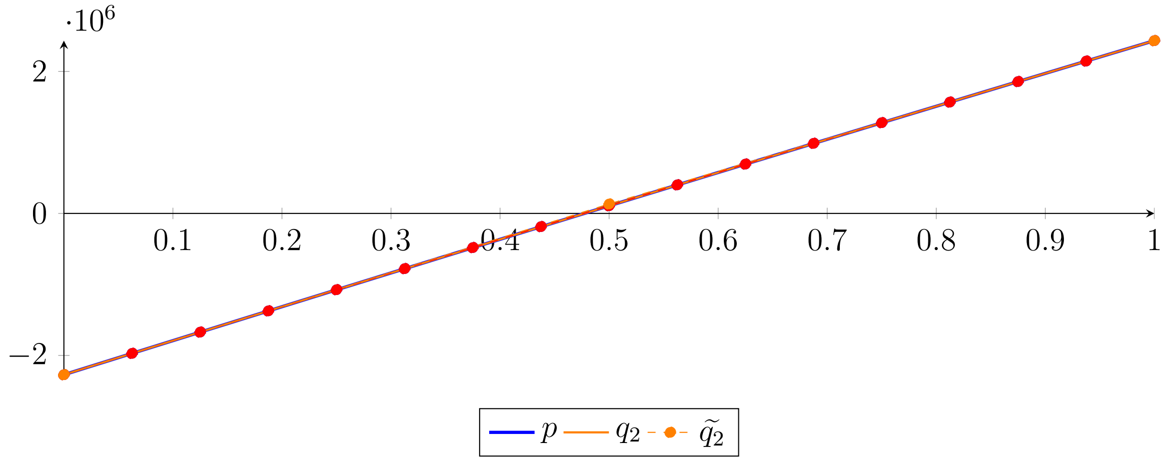
$$\begin{aligned} p = & -2.82438\cdot 10^{-05}X^{16} + 2.90051\cdot 10^{-05}X^{15} + 0.000231285X^{14} + 0.000648014X^{13} + 0.00318916X^{12} \\ & + 0.00323204X^{11} + 0.00460533X^{10} + 0.00220012X^{9} - 0.000101882X^{8} + 5.32717\cdot 10^{-06}X^{7} \\ & - 0.00186824X^{6} + 0.132741X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874\cdot 10^{6}X - 2.2715\cdot 10^{6} \\ = & -2.2715\cdot 10^{6}B_{0,16}(X) - 1.97096\cdot 10^{6}B_{1,16}(X) - 1.67127\cdot 10^{6}B_{2,16}(X) - 1.37244\cdot 10^{6}B_{3,16}(X) \\ & - 1.07448\cdot 10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) \\ & + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806\cdot 10^{6}B_{12,16}(X) + 1.5682 \\ & \cdot 10^{6}B_{13,16}(X) + 1.85747\cdot 10^{6}B_{14,16}(X) + 2.14586\cdot 10^{6}B_{15,16}(X) + 2.43338\cdot 10^{6}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -104265X^{2} + 4.80923\cdot 10^{6}X - 2.27154\cdot 10^{6} \\ &= -2.27154\cdot 10^{6}B_{0,2} + 133069B_{1,2} + 2.43342\cdot 10^{6}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 = & 43556.7X^{16} - 347904X^{15} + 1.2616\cdot 10^{6}X^{14} - 2.74835\cdot 10^{6}X^{13} + 4.01042\cdot 10^{6}X^{12} - 4.13709 \\ & \cdot 10^{6}X^{11} + 3.10251\cdot 10^{6}X^{10} - 1.71209\cdot 10^{6}X^{9} + 695531X^{8} - 205965X^{7} + 43591.2X^{6} \\ & - 6402.94X^{5} + 625.678X^{4} - 37.485X^{3} - 104264X^{2} + 4.80923\cdot 10^{6}X - 2.27154\cdot 10^{6} \\ = & -2.27154\cdot 10^{6}B_{0,16} - 1.97097\cdot 10^{6}B_{1,16} - 1.67126\cdot 10^{6}B_{2,16} - 1.37242\cdot 10^{6}B_{3,16} \\ & - 1.07445\cdot 10^{6}B_{4,16} - 777350B_{5,16} - 481117B_{6,16} - 185755B_{7,16} + 108741B_{8,16} \\ & + 402365B_{9,16} + 695123B_{10,16} + 987010B_{11,16} + 1.27803\cdot 10^{6}B_{12,16} + 1.56818 \\ & \cdot 10^{6}B_{13,16} + 1.85746\cdot 10^{6}B_{14,16} + 2.14587\cdot 10^{6}B_{15,16} + 2.43342\cdot 10^{6}B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m:

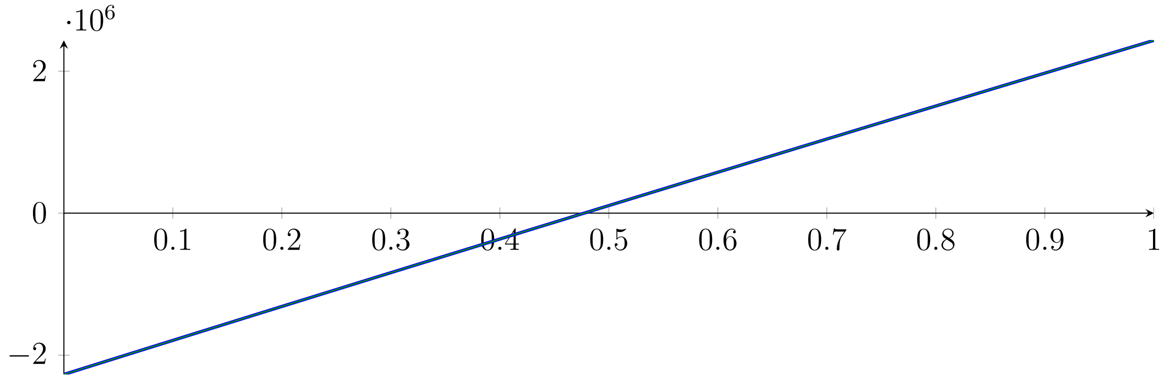
$$M = -104265X^{2} + 4.80923\cdot 10^{6}X - 2.2715\cdot 10^{6}$$

$$m = -104265X^2 + 4.80923{\cdot}10^6 X - 2.27159{\cdot}10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

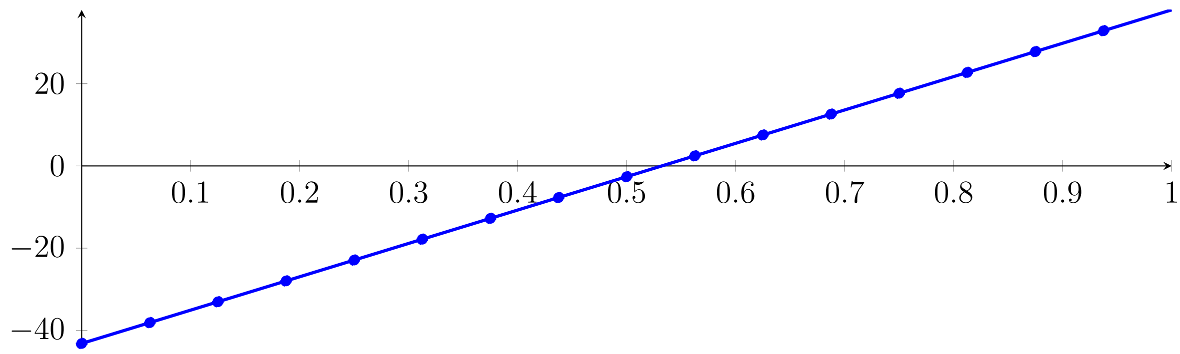
Longest intersection interval: $1.72301 \cdot 10^{-05}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

74.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

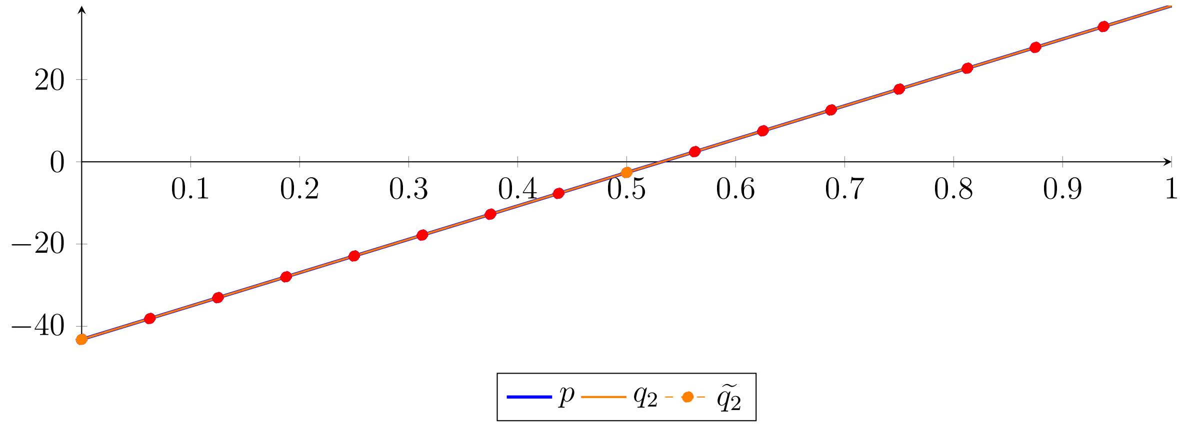
$$\begin{aligned}
p = {} & 8.74252{\cdot}10^{-11}X^{16} - 1.56979{\cdot}10^{-09}X^{15} + 6.68479{\cdot}10^{-09}X^{14} + 1.20008{\cdot}10^{-08}X^{13} + 9.07301{\cdot}10^{-08}X^{12} \\
& + 5.58657{\cdot}10^{-08}X^{11} + 1.13801{\cdot}10^{-07}X^{10} + 3.70665{\cdot}10^{-08}X^{9} + 7.31575{\cdot}10^{-10}X^{8} + 1.30058{\cdot}10^{-09}X^{7} \\
& + 5.00722{\cdot}10^{-09}X^{6} + 1.24146{\cdot}10^{-10}X^{5} + 1.03455{\cdot}10^{-10}X^{4} - 3.09388{\cdot}10^{-05}X^{2} + 81.1505X - 43.1911 \\
= {} & -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\
& - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68777B_{7,16}(X) - 2.61587B_{8,16}(X) \\
& + 2.45604B_{9,16}(X) + 7.52794B_{10,16}(X) + 12.5998B_{11,16}(X) + 17.6718B_{12,16}(X) \\
& + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X)
\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
q_2 &= -3.09388{\cdot}10^{-05}X^2 + 81.1505X - 43.1911 \\
&= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2}
\end{aligned}$$

$$\begin{aligned}
\tilde{q}_2 = {} & 0.721495X^{16} - 5.74915X^{15} + 20.7933X^{14} - 45.1627X^{13} + 65.6806X^{12} - 67.5044X^{11} \\
& + 50.4286X^{10} - 27.728X^{9} + 11.2318X^{8} - 3.32011X^{7} + 0.702408X^{6} - 0.103415X^{5} \\
& + 0.0102099X^{4} - 0.000624725X^{3} - 1.10834{\cdot}10^{-05}X^{2} + 81.1505X - 43.1911 \\
= {} & -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} \\
& - 12.7597B_{6,16} - 7.68779B_{7,16} - 2.61585B_{8,16} + 2.45602B_{9,16} + 7.52795B_{10,16} + 12.5998B_{11,16} \\
& + 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.57956{\cdot}10^{-05}$.

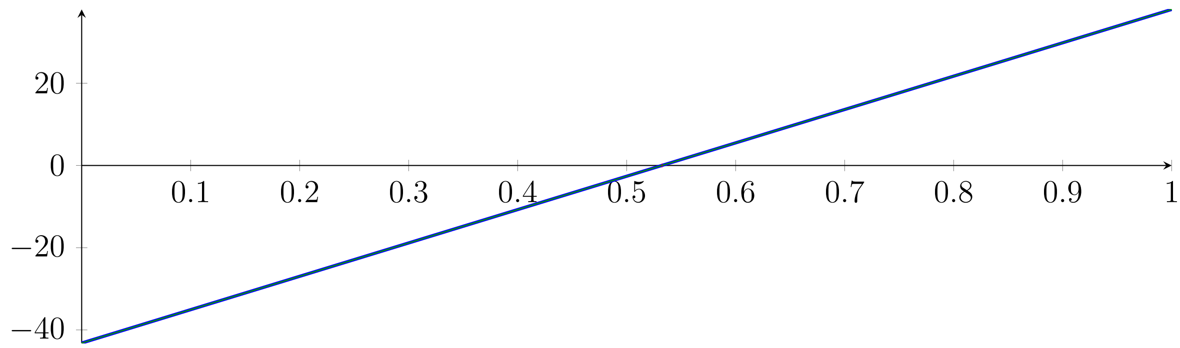
Bounding polynomials M and m:

$$M = -3.09388{\cdot}10^{-05}X^2 + 81.1505X - 43.1911$$
$$m = -3.09388{\cdot}10^{-05}X^2 + 81.1505X - 43.1911$$

Root of M and m:

$$N(M) = \{0.532235, 2.62293{\cdot}10^{6}\} \qquad N(m) = \{0.532235, 2.62293{\cdot}10^{6}\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

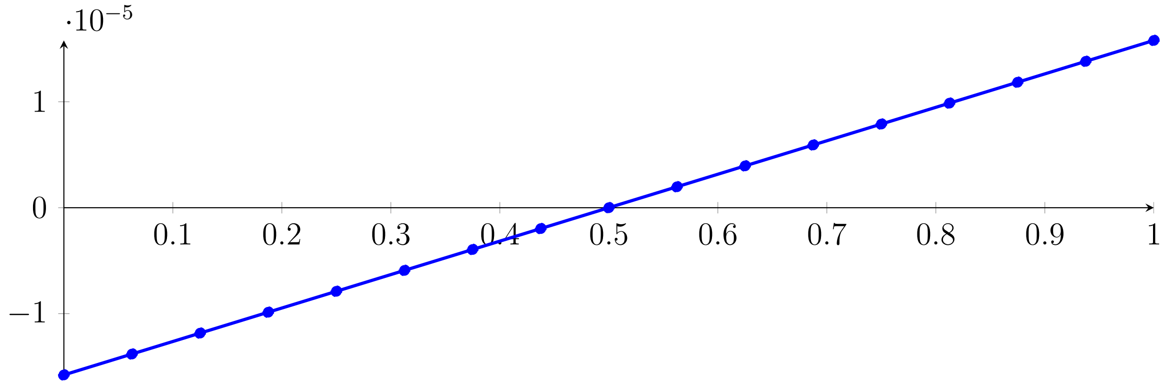
Longest intersection interval: $3.8903{\cdot}10^{-07}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

74.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

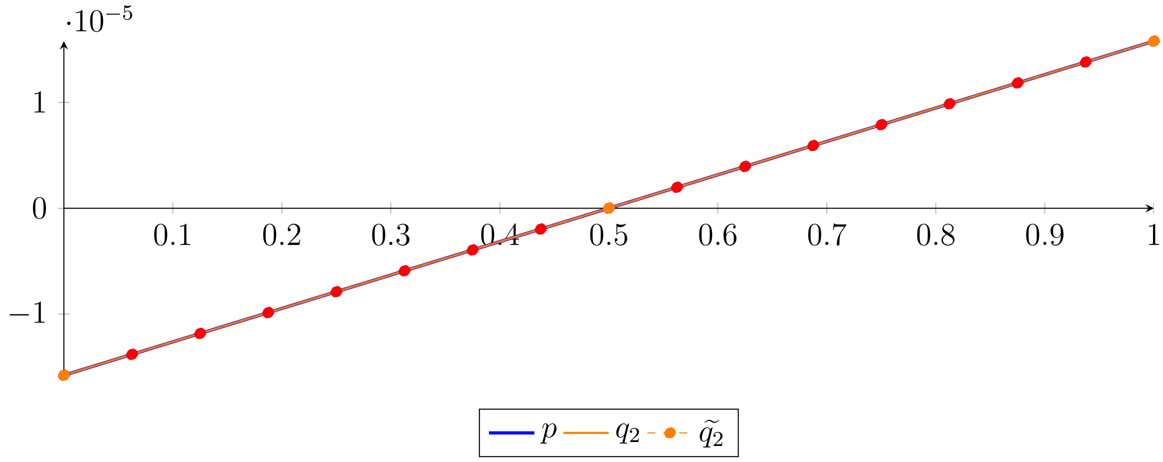
$$\begin{aligned}
p = &-1.04409{\cdot}10^{-16}X^{16} - 1.53089{\cdot}10^{-16}X^{15} + 1.74665{\cdot}10^{-15}X^{14} + 5.46438{\cdot}10^{-15}X^{13} \\
&+ 2.56522{\cdot}10^{-14}X^{12} + 2.15479{\cdot}10^{-14}X^{11} + 3.51633{\cdot}10^{-14}X^{10} + 1.67444 \\
&{\cdot}10^{-14}X^{9} - 8.72105{\cdot}10^{-16}X^{8} + 1.41087{\cdot}10^{-15}X^{6} + 5.91974{\cdot}10^{-17}X^{5} + 4.93312 \\
&{\cdot}10^{-17}X^{4} + 3.79471{\cdot}10^{-18}X^{3} - 4.87891{\cdot}10^{-18}X^{2} + 3.157{\cdot}10^{-05}X - 1.57804{\cdot}10^{-05} \\
= &-1.57804{\cdot}10^{-05}B_{0,16}(X) - 1.38073{\cdot}10^{-05}B_{1,16}(X) - 1.18341{\cdot}10^{-05}B_{2,16}(X) - 9.86101 \\
&{\cdot}10^{-06}B_{3,16}(X) - 7.88788{\cdot}10^{-06}B_{4,16}(X) - 5.91476{\cdot}10^{-06}B_{5,16}(X) - 3.94163{\cdot}10^{-06}B_{6,16}(X) \\
&- 1.96851{\cdot}10^{-06}B_{7,16}(X) + 4.61501{\cdot}10^{-09}B_{8,16}(X) + 1.97774{\cdot}10^{-06}B_{9,16}(X) + 3.95086 \\
&{\cdot}10^{-06}B_{10,16}(X) + 5.92399{\cdot}10^{-06}B_{11,16}(X) + 7.89711{\cdot}10^{-06}B_{12,16}(X) + 9.87024{\cdot}10^{-06}B_{13,16}(X) \\
&+ 1.18434{\cdot}10^{-05}B_{14,16}(X) + 1.38165{\cdot}10^{-05}B_{15,16}(X) + 1.57896{\cdot}10^{-05}B_{16,16}(X)
\end{aligned}$$



Degree reduction and raising:

$$q_2 = -4.65868{\cdot}10^{-18}X^2 + 3.157{\cdot}10^{-05}X - 1.57804{\cdot}10^{-05}$$
$$= -1.57804{\cdot}10^{-05}B_{0,2} + 4.61501{\cdot}10^{-09}B_{1,2} + 1.57896{\cdot}10^{-05}B_{2,2}$$

$$\widetilde{q_2} = 2.92413{\cdot}10^{-07}X^{16} - 2.33332{\cdot}10^{-06}X^{15} + 8.45203{\cdot}10^{-06}X^{14} - 1.83895{\cdot}10^{-05}X^{13} + 2.67963{\cdot}10^{-05}X^{12} - 2.75995{\cdot}10^{-05}X^{11} + 2.06638{\cdot}10^{-05}X^{10} - 1.13854{\cdot}10^{-05}X^{9} + 4.61944{\cdot}10^{-06}X^{8} - 1.36687{\cdot}10^{-06}X^{7} + 2.89249{\cdot}10^{-07}X^{6} - 4.25295{\cdot}10^{-08}X^{5} + 4.17283{\cdot}10^{-09}X^{4} - 2.52119{\cdot}10^{-10}X^{3} + 7.82992{\cdot}10^{-12}X^{2} + 3.157{\cdot}10^{-05}X - 1.57804{\cdot}10^{-05}$$
$$= -1.57804{\cdot}10^{-05}B_{0,16} - 1.38073{\cdot}10^{-05}B_{1,16} - 1.18341{\cdot}10^{-05}B_{2,16} - 9.86101{\cdot}10^{-06}B_{3,16} - 7.88788{\cdot}10^{-06}B_{4,16} - 5.91476{\cdot}10^{-06}B_{5,16} - 3.94163{\cdot}10^{-06}B_{6,16} - 1.96851{\cdot}10^{-06}B_{7,16} + 4.62125{\cdot}10^{-09}B_{8,16} + 1.97773{\cdot}10^{-06}B_{9,16} + 3.95087{\cdot}10^{-06}B_{10,16} + 5.92399{\cdot}10^{-06}B_{11,16} + 7.89711{\cdot}10^{-06}B_{12,16} + 9.87024{\cdot}10^{-06}B_{13,16} + 1.18434{\cdot}10^{-05}B_{14,16} + 1.38165{\cdot}10^{-05}B_{15,16} + 1.57896{\cdot}10^{-05}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 6.24192{\cdot}10^{-12}$.

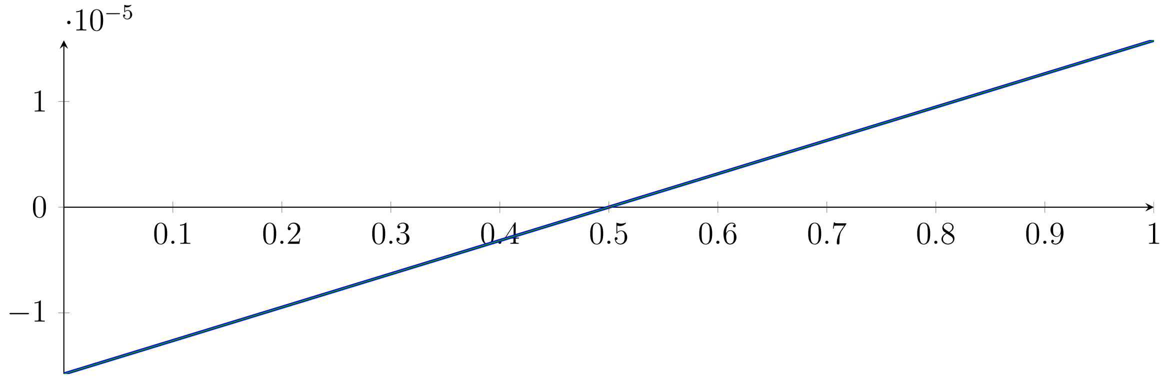
Bounding polynomials M and m:

$$M = -4.65868{\cdot}10^{-18}X^2 + 3.157{\cdot}10^{-05}X - 1.57804{\cdot}10^{-05}$$
$$m = -4.65868{\cdot}10^{-18}X^2 + 3.157{\cdot}10^{-05}X - 1.57804{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{0.499636, 6.77659{\cdot}10^{12}\} \qquad N(m) = \{0.500364, 6.77659{\cdot}10^{12}\}$$

Intersection intervals:



$[0.499636, 0.500364]$

Longest intersection interval: 0.000727273
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

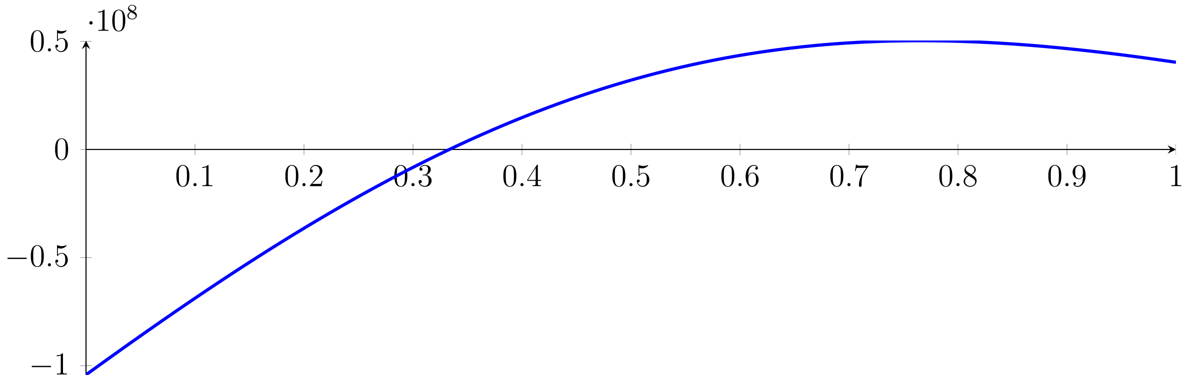
74.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

74.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

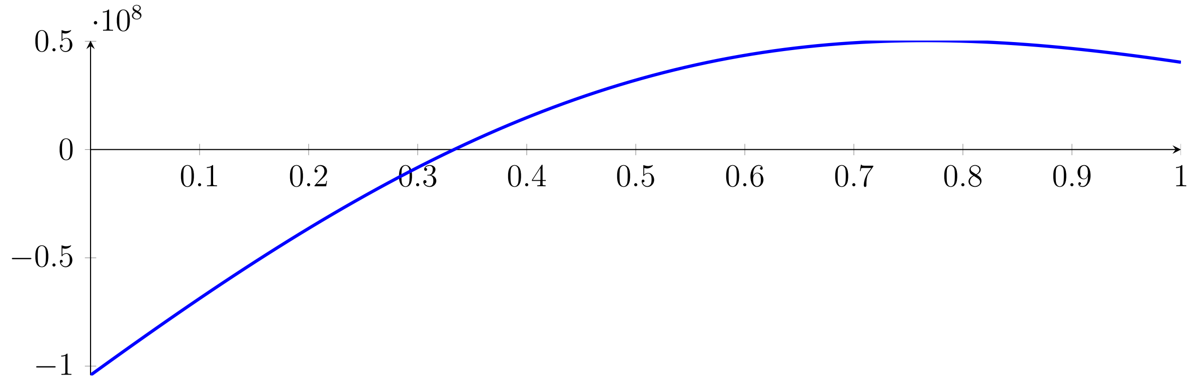
with precision $\varepsilon = 1\cdot10^{-16}$.

75 Running CubeClip on f_{16} with epsilon 16

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

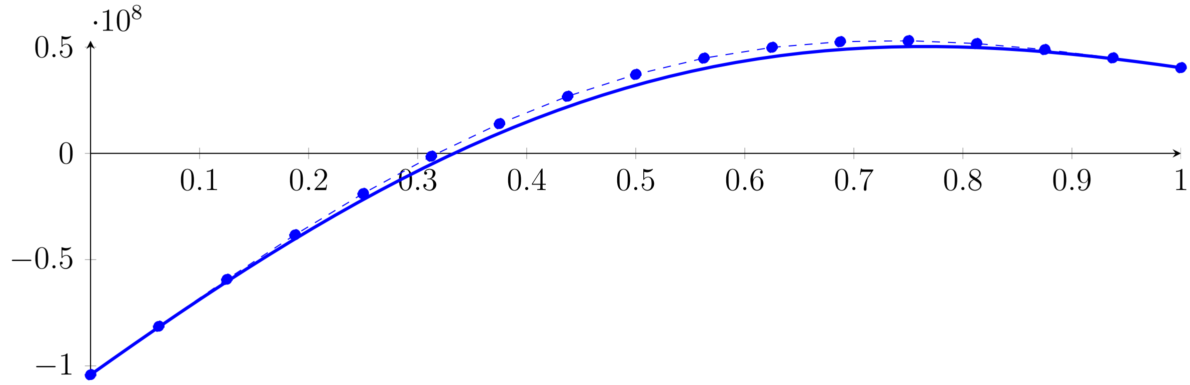
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} \\ &+ 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \end{aligned}$$



75.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

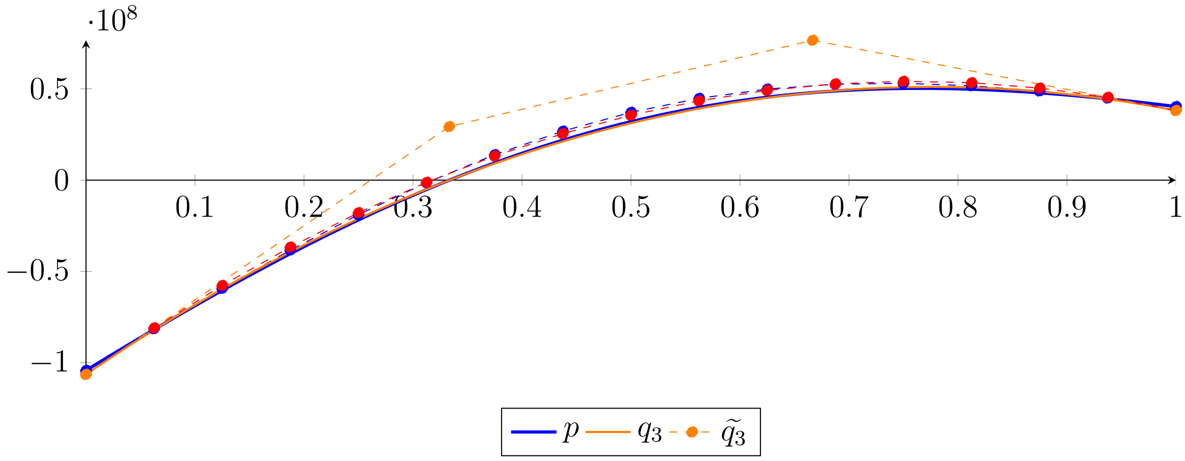
$$\begin{aligned} p = &-0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\ &+ 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \\ = &-1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203 \\ &\cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X) \\ &+ 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901 \\ &\cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X) \\ &+ 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.06335\cdot 10^8$$
$$= -1.06335\cdot 10^8 B_{0,3} + 2.93558\cdot 10^7 B_{1,3} + 7.66068\cdot 10^7 B_{2,3} + 3.81764\cdot 10^7 B_{3,3}$$

$$\widetilde{q}_3 = 2.11088\cdot 10^6 X^{16} - 1.70537\cdot 10^7 X^{15} + 6.25913\cdot 10^7 X^{14} - 1.38111\cdot 10^8 X^{13} + 2.043\cdot 10^8 X^{12} - 2.13799 \cdot 10^8 X^{11} + 1.62714\cdot 10^8 X^{10} - 9.11092\cdot 10^7 X^9 + 3.75254\cdot 10^7 X^8 - 1.12529\cdot 10^7 X^7 + 2.40899\cdot 10^6 X^6 - 357156 X^5 + 34986.1 X^4 + 2.75602\cdot 10^6 X^3 - 2.65318\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.06335\cdot 10^8$$
$$= -1.06335\cdot 10^8 B_{0,16} - 8.08928\cdot 10^7 B_{1,16} - 5.76618\cdot 10^7 B_{2,16} - 3.66369\cdot 10^7 B_{3,16} - 1.78131 \cdot 10^7 B_{4,16} - 1.18553\cdot 10^6 B_{5,16} + 1.32508\cdot 10^7 B_{6,16} + 2.55007\cdot 10^7 B_{7,16} + 3.55692\cdot 10^7 B_{8,16} + 4.34611\cdot 10^7 B_{9,16} + 4.91816\cdot 10^7 B_{10,16} + 5.27352\cdot 10^7 B_{11,16} + 5.41273\cdot 10^7 B_{12,16} + 5.33624\cdot 10^7 B_{13,16} + 5.04457\cdot 10^7 B_{14,16} + 4.53821\cdot 10^7 B_{15,16} + 3.81764\cdot 10^7 B_{16,16}$$



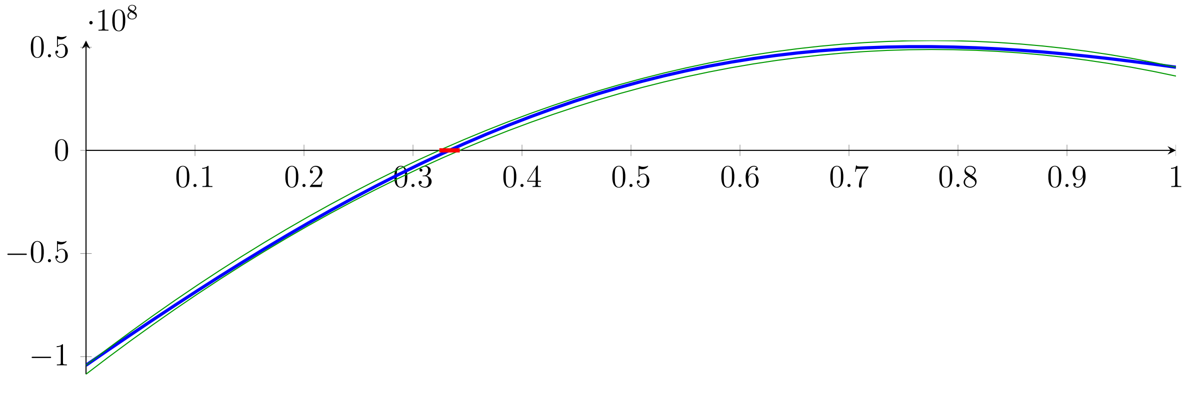
The maximum difference of the Bézier coefficients is $\delta = 2.16806\cdot 10^6$.
Bounding polynomials M and m:

$$M = 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.04167\cdot 10^8$$
$$m = 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.08503\cdot 10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



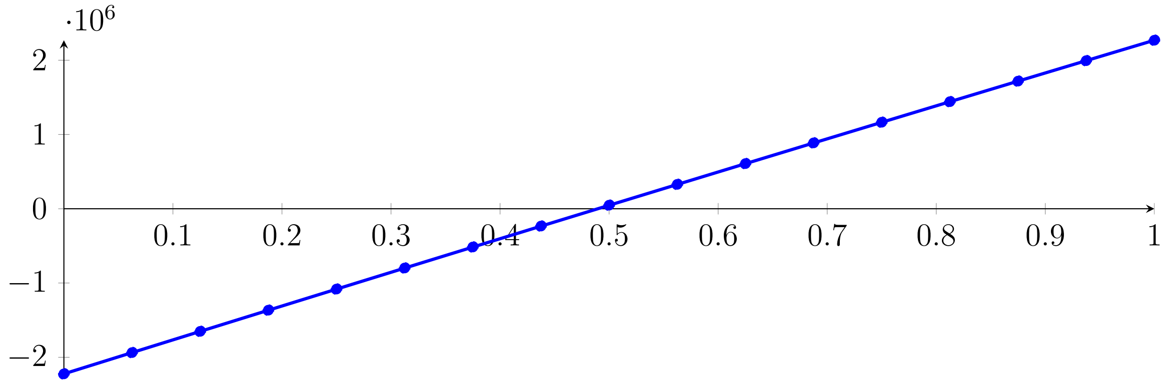
$$[0.324143, 0.342913]$$

Longest intersection interval: 0.0187703
$\Longrightarrow$ Selective recursion: interval 1: [0.324143, 0.342913],

75.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

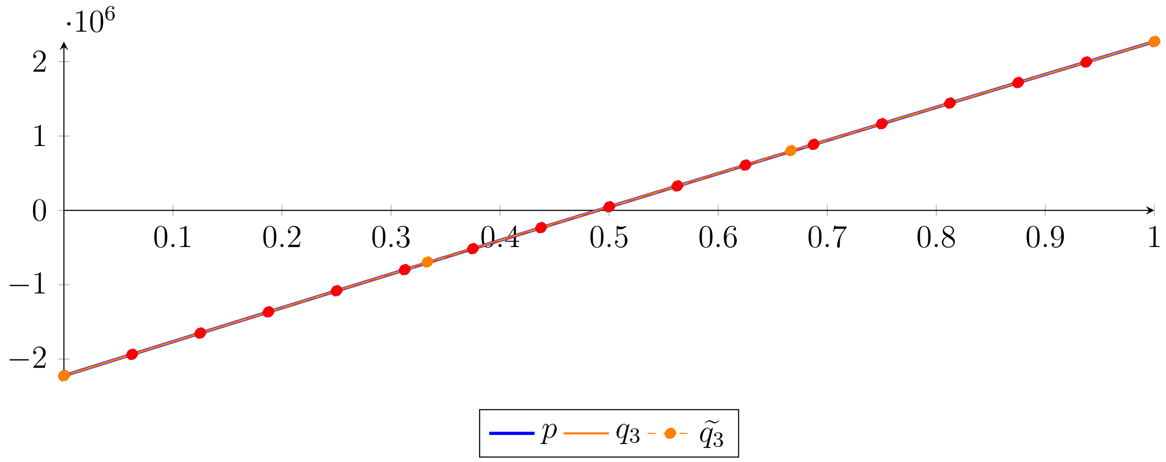
$$p = -1.93035\cdot 10^{-05}X^{16} + 1.08927\cdot 10^{-05}X^{15} + 0.000255816X^{14} + 0.000616983X^{13} + 0.00371715X^{12} + 0.00325035X^{11} + 0.00510875X^{10} + 0.00190713X^{9} + 0.000179792X^{8} + 1.06543\cdot 10^{-05}X^{7} - 0.00136482X^{6} + 0.104959X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6}$$
$$= -2.22335\cdot 10^{6}B_{0,16}(X) - 1.93666\cdot 10^{6}B_{1,16}(X) - 1.65074\cdot 10^{6}B_{2,16}(X) - 1.36561\cdot 10^{6}B_{3,16}(X) - 1.08127\cdot 10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X) + 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot 10^{6}B_{12,16}(X) + 1.44233\cdot 10^{6}B_{13,16}(X) + 1.71876\cdot 10^{6}B_{14,16}(X) + 1.99439\cdot 10^{6}B_{15,16}(X) + 2.26922\cdot 10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6}$$
$$= -2.22335\cdot 10^{6}B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot 10^{6}B_{3,3}$$

$$\widetilde{q}_3 = 15290.6X^{16} - 125412X^{15} + 468045X^{14} - 1.05198\cdot 10^{6}X^{13} + 1.588\cdot 10^{6}X^{12} - 1.69893\cdot 10^{6}X^{11} + 1.32382\cdot 10^{6}X^{10} - 759402X^{9} + 320067X^{8} - 97826.5X^{7} + 21197.8X^{6} - 3162.8X^{5} + 313.931X^{4} - 720.165X^{3} - 93879.2X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6}$$
$$= -2.22335\cdot 10^{6}B_{0,16} - 1.93666\cdot 10^{6}B_{1,16} - 1.65074\cdot 10^{6}B_{2,16} - 1.36561\cdot 10^{6}B_{3,16} - 1.08126\cdot 10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.7B_{8,16} + 328649B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot 10^{6}B_{12,16} + 1.44233\cdot 10^{6}B_{13,16} + 1.71876\cdot 10^{6}B_{14,16} + 1.99439\cdot 10^{6}B_{15,16} + 2.26922\cdot 10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 0.457751$.

Bounding polynomials M and m:

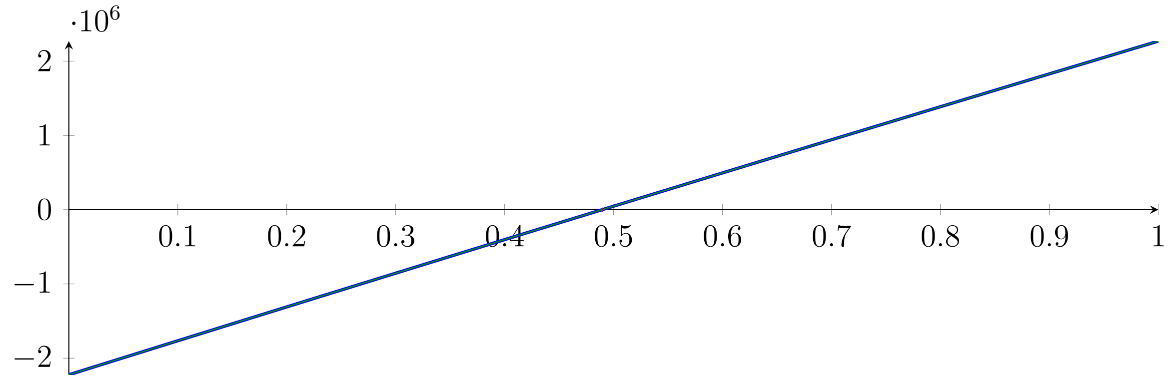
$$M = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6}$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715\cdot 10^6 X - 2.22335\cdot 10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



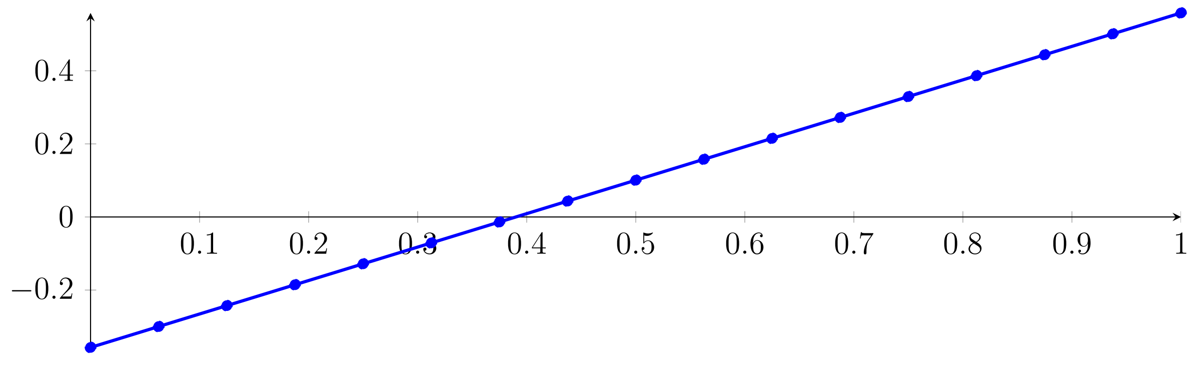
$$[0.489616, 0.489616]$$

Longest intersection interval: $2.03684\cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

75.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

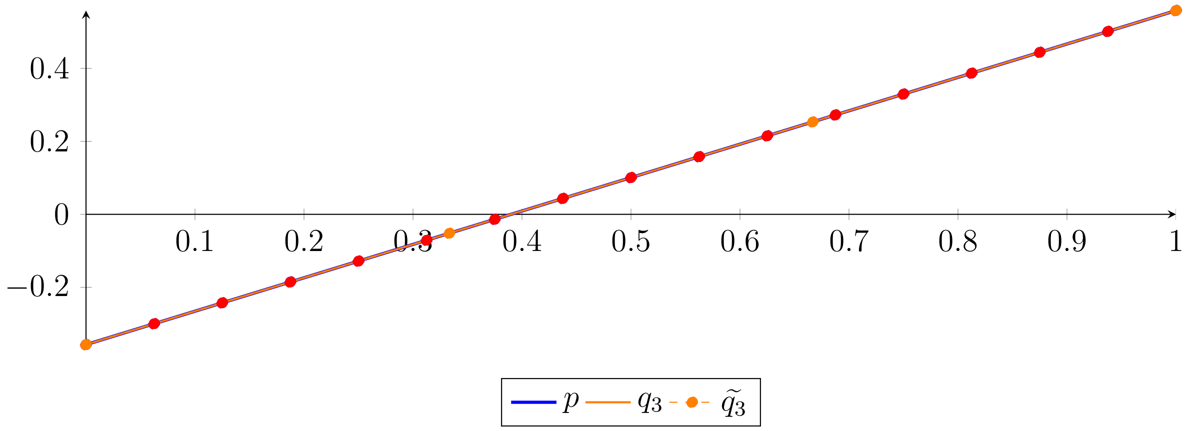
$$\begin{aligned} p = &-1.56399\cdot 10^{-11}X^{16} + 4.58016\cdot 10^{-11}X^{15} - 3.19744\cdot 10^{-12}X^{14} + 6.54055\cdot 10^{-11}X^{13} \\ &+ 1.05072\cdot 10^{-10}X^{12} + 4.59728\cdot 10^{-10}X^{11} + 5.01434\cdot 10^{-10}X^{10} + 2.99742\cdot 10^{-10}X^{9} \\ &+ 1.14309\cdot 10^{-11}X^{8} - 5.08038\cdot 10^{-12}X^{7} + 3.37845\cdot 10^{-11}X^{6} - 9.69891\cdot 10^{-13}X^{5} \\ &+ 4.04121\cdot 10^{-13}X^{4} + 6.21725\cdot 10^{-14}X^{3} - 3.93767\cdot 10^{-09}X^{2} + 0.915501X - 0.357072 \\ = &-0.357072B_{0,16}(X) - 0.299853B_{1,16}(X) - 0.242635B_{2,16}(X) - 0.185416B_{3,16}(X) \\ &- 0.128197B_{4,16}(X) - 0.0709781B_{5,16}(X) - 0.0137592B_{6,16}(X) \\ &+ 0.0434596B_{7,16}(X) + 0.100678B_{8,16}(X) + 0.157897B_{9,16}(X) + 0.215116B_{10,16}(X) \\ &+ 0.272335B_{11,16}(X) + 0.329554B_{12,16}(X) + 0.386773B_{13,16}(X) \\ &+ 0.443991B_{14,16}(X) + 0.50121B_{15,16}(X) + 0.558429B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 1.05471\cdot 10^{-15}X^3 - 3.93767\cdot 10^{-09}X^2 + 0.915501X - 0.357072 \\ &= -0.357072B_{0,3} - 0.0519051B_{1,3} + 0.253262B_{2,3} + 0.558429B_{3,3} \end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= 0.00291222X^{16} - 0.0241801X^{15} + 0.0914452X^{14} - 0.208537X^{13} + 0.319778X^{12} - 0.347745X^{11} \\ &\quad + 0.275244X^{10} - 0.159971X^{9} + 0.0679818X^{8} - 0.0208072X^{7} + 0.00447629X^{6} - 0.000654783X^{5} \\ &\quad + 6.22034\cdot 10^{-05}X^{4} - 3.60145\cdot 10^{-06}X^{3} + 9.78811\cdot 10^{-08}X^{2} + 0.915501X - 0.357072 \\ &= -0.357072B_{0,16} - 0.299853B_{1,16} - 0.242635B_{2,16} - 0.185416B_{3,16} - 0.128197B_{4,16} \\ &\quad - 0.0709781B_{5,16} - 0.0137592B_{6,16} + 0.0434595B_{7,16} + 0.100678B_{8,16} \\ &\quad + 0.157897B_{9,16} + 0.215116B_{10,16} + 0.272335B_{11,16} + 0.329554B_{12,16} \\ &\quad + 0.386773B_{13,16} + 0.443991B_{14,16} + 0.50121B_{15,16} + 0.558429B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.5212\cdot 10^{-08}$.

Bounding polynomials M and m:

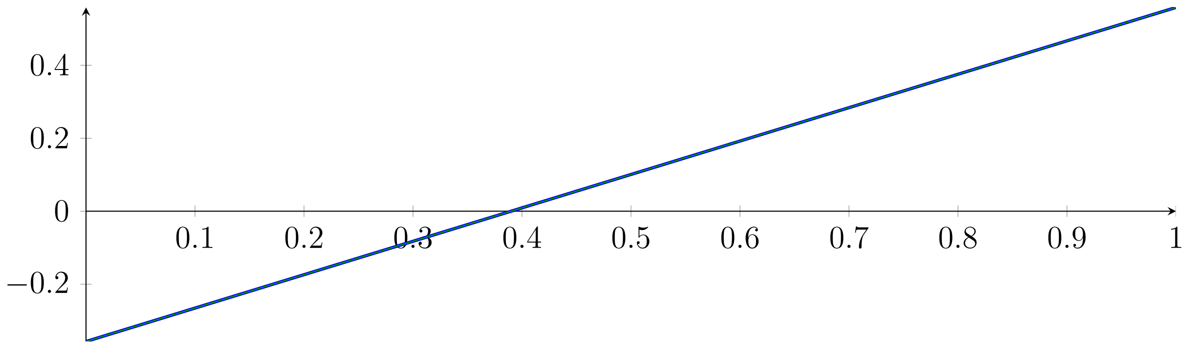
$$M = 9.99201\cdot 10^{-16}X^{3} - 3.93767\cdot 10^{-09}X^{2} + 0.915501X - 0.357072$$

$$m = 1.22125\cdot 10^{-15}X^{3} - 3.93767\cdot 10^{-09}X^{2} + 0.915501X - 0.357072$$

Root of M and m:

$$N(M) = \{0.390029\} \qquad N(m) = \{0.390029\}$$

Intersection intervals:



$$[0.390029, 0.390029]$$

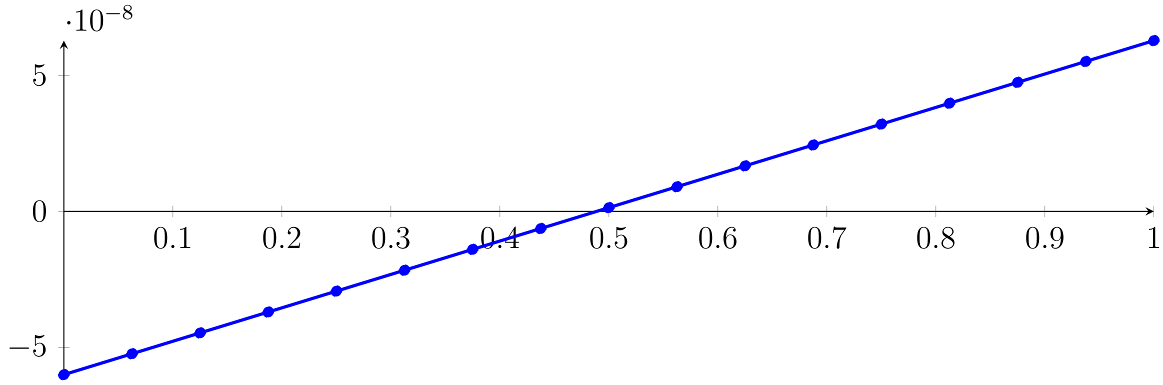
Longest intersection interval: $1.3411\cdot 10^{-07}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

75.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

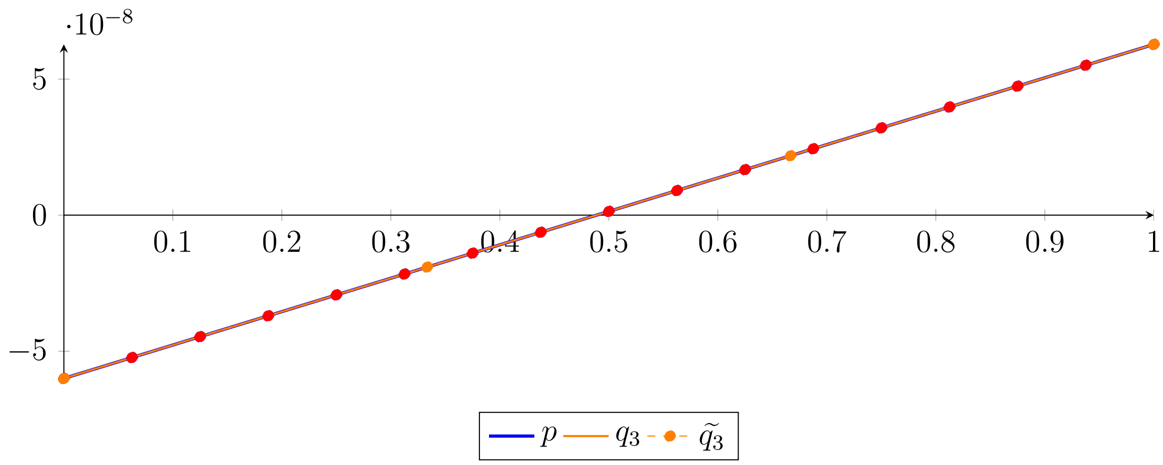
$$p = -5.65183\cdot10^{-19}X^{16} + 5.13302\cdot10^{-19}X^{15} + 6.37816\cdot10^{-18}X^{14} + 1.69576\cdot10^{-17}X^{13} + 9.4423\cdot10^{-17}X^{12} + 8.0934\cdot10^{-17}X^{11} + 1.37357\cdot10^{-16}X^{10} + 6.23797\cdot10^{-17}X^{9} + 1.36266\cdot10^{-18}X^{8} + 6.05629\cdot10^{-19}X^{7} + 5.08728\cdot10^{-18}X^{6} + 2.3124\cdot10^{-19}X^{5} + 1.44525\cdot10^{-19}X^{4} - 7.41154\cdot10^{-21}X^{3} + 1.22778\cdot10^{-07}X - 6.00146\cdot10^{-08}$$
$$= -6.00146\cdot10^{-08}B_{0,16}(X) - 5.2341\cdot10^{-08}B_{1,16}(X) - 4.46674\cdot10^{-08}B_{2,16}(X) - 3.69937\cdot10^{-08}B_{3,16}(X) - 2.93201\cdot10^{-08}B_{4,16}(X) - 2.16464\cdot10^{-08}B_{5,16}(X) - 1.39728\cdot10^{-08}B_{6,16}(X) - 6.29913\cdot10^{-09}B_{7,16}(X) + 1.37451\cdot10^{-09}B_{8,16}(X) + 9.04815\cdot10^{-09}B_{9,16}(X) + 1.67218\cdot10^{-08}B_{10,16}(X) + 2.43954\cdot10^{-08}B_{11,16}(X) + 3.20691\cdot10^{-08}B_{12,16}(X) + 3.97427\cdot10^{-08}B_{13,16}(X) + 4.74164\cdot10^{-08}B_{14,16}(X) + 5.509\cdot10^{-08}B_{15,16}(X) + 6.27637\cdot10^{-08}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = 2.51463\cdot10^{-22}X^{3} - 3.17637\cdot10^{-22}X^{2} + 1.22778\cdot10^{-07}X - 6.00146\cdot10^{-08}$$
$$= -6.00146\cdot10^{-08}B_{0,3} - 1.90885\cdot10^{-08}B_{1,3} + 2.18376\cdot10^{-08}B_{2,3} + 6.27637\cdot10^{-08}B_{3,3}$$

$$\widetilde{q}_3 = 4.01426\cdot10^{-10}X^{16} - 3.29623\cdot10^{-09}X^{15} + 1.2317\cdot10^{-08}X^{14} - 2.77213\cdot10^{-08}X^{13} + 4.19074\cdot10^{-08}X^{12} - 4.49047\cdot10^{-08}X^{11} + 3.50457\cdot10^{-08}X^{10} - 2.01341\cdot10^{-08}X^{9} + 8.49676\cdot10^{-09}X^{8} - 2.5992\cdot10^{-09}X^{7} + 5.63364\cdot10^{-10}X^{6} - 8.40231\cdot10^{-11}X^{5} + 8.33259\cdot10^{-12}X^{4} - 5.16965\cdot10^{-13}X^{3} + 1.7395\cdot10^{-14}X^{2} + 1.22778\cdot10^{-07}X - 6.00146\cdot10^{-08}$$
$$= -6.00146\cdot10^{-08}B_{0,16} - 5.2341\cdot10^{-08}B_{1,16} - 4.46674\cdot10^{-08}B_{2,16} - 3.69937\cdot10^{-08}B_{3,16} - 2.93201\cdot10^{-08}B_{4,16} - 2.16464\cdot10^{-08}B_{5,16} - 1.39728\cdot10^{-08}B_{6,16} - 6.29914\cdot10^{-09}B_{7,16} + 1.37452\cdot10^{-09}B_{8,16} + 9.04815\cdot10^{-09}B_{9,16} + 1.67218\cdot10^{-08}B_{10,16} + 2.43954\cdot10^{-08}B_{11,16} + 3.20691\cdot10^{-08}B_{12,16} + 3.97427\cdot10^{-08}B_{13,16} + 4.74164\cdot10^{-08}B_{14,16} + 5.509\cdot10^{-08}B_{15,16} + 6.27637\cdot10^{-08}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 7.90061 \cdot 10^{-15}$.
Bounding polynomials M and m:

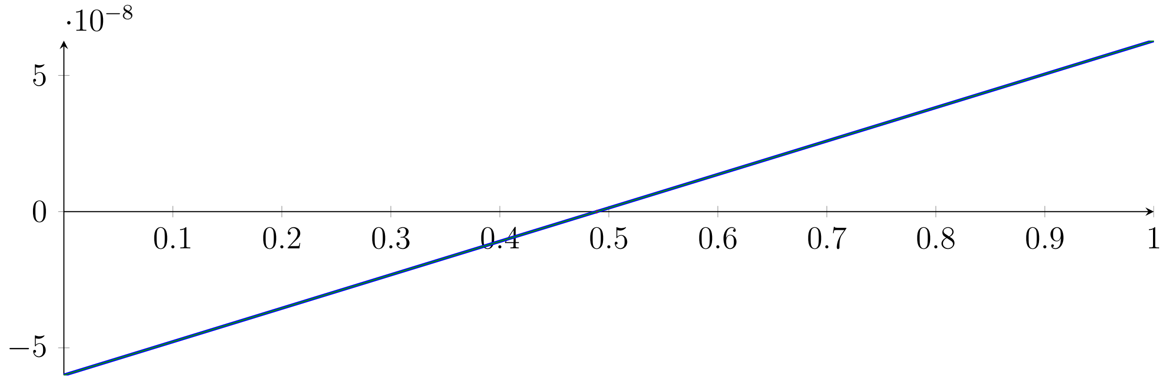
$$M = 2.38228 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08}$$

$$m = 2.51463 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08}$$

Root of M and m:

$$N(M) = \{0.488805\} \qquad N(m) = \{0.488805\}$$

Intersection intervals:



$$[0.488805, 0.488805]$$

Longest intersection interval: $1.3086 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

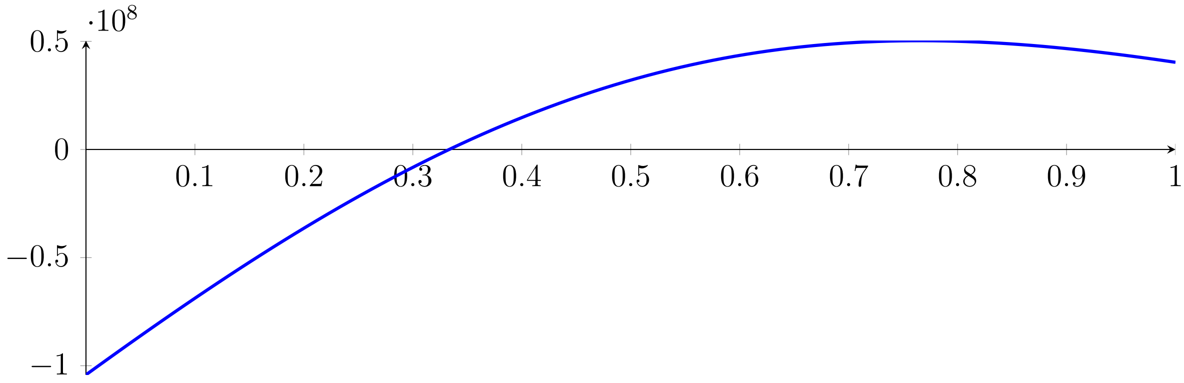
75.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

75.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1\cdot10^{-16}$.

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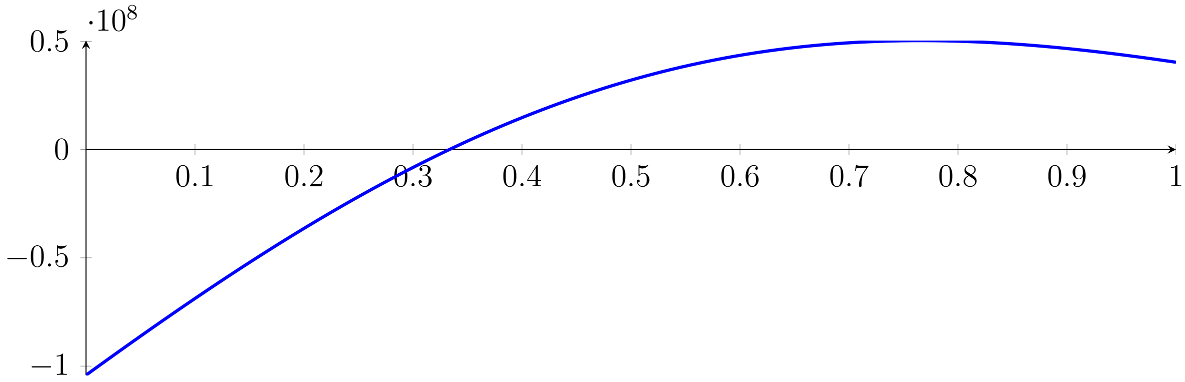
$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^6X^9 - 2.52262{\cdot}10^6X^8 - 1.43506{\cdot}10^7X^7 + 138542X^6 + 7.92823{\cdot}10^7X^5 + 3.97396{\cdot}10^7X^4 - 2.40625{\cdot}10^8X^3 - 8.33333{\cdot}10^7X^2 + 3.64583{\cdot}10^8X - 1.04167{\cdot}10^8$$

Called BezClip with input polynomial on interval $[0,1]$:

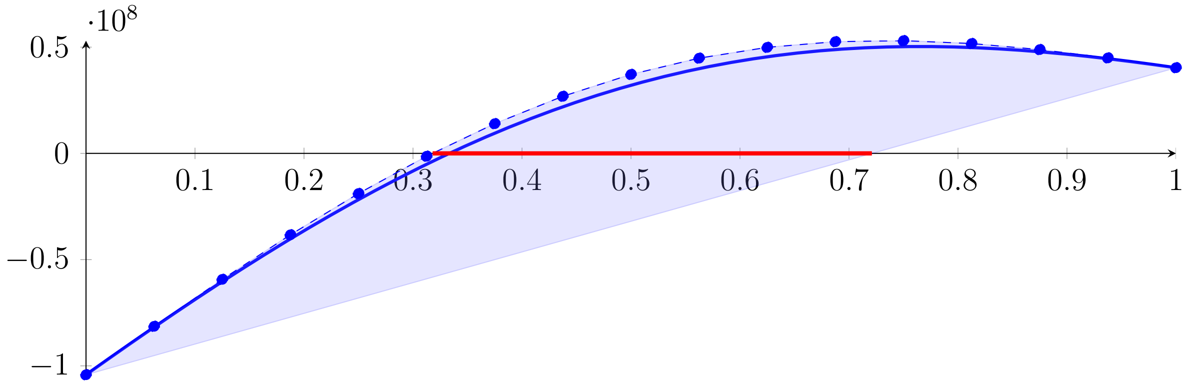
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^6X^9 - 2.52262{\cdot}10^6X^8 - 1.43506{\cdot}10^7X^7 + 138542X^6 + 7.92823{\cdot}10^7X^5 + 3.97396{\cdot}10^7X^4 - 2.40625{\cdot}10^8X^3 - 8.33333{\cdot}10^7X^2 + 3.64583{\cdot}10^8X - 1.04167{\cdot}10^8$$



76.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\ &\quad + 451673X^{10} + 1.12172{\cdot}10^6X^9 - 2.52262{\cdot}10^6X^8 - 1.43506{\cdot}10^7X^7 + 138542X^6 + 7.92823 \\ &\quad {\cdot}10^7X^5 + 3.97396{\cdot}10^7X^4 - 2.40625{\cdot}10^8X^3 - 8.33333{\cdot}10^7X^2 + 3.64583{\cdot}10^8X - 1.04167{\cdot}10^8 \\ &= -1.04167{\cdot}10^8B_{0,16}(X) - 8.13802{\cdot}10^7B_{1,16}(X) - 5.92882{\cdot}10^7B_{2,16}(X) - 3.83203 \\ &\quad {\cdot}10^7B_{3,16}(X) - 1.88844{\cdot}10^7B_{4,16}(X) - 1.34837{\cdot}10^6B_{5,16}(X) + 1.39781{\cdot}10^7B_{6,16}(X) \\ &\quad + 2.68604{\cdot}10^7B_{7,16}(X) + 3.71532{\cdot}10^7B_{8,16}(X) + 4.48105{\cdot}10^7B_{9,16}(X) + 4.98901 \\ &\quad {\cdot}10^7B_{10,16}(X) + 5.25504{\cdot}10^7B_{11,16}(X) + 5.30407{\cdot}10^7B_{12,16}(X) + 5.16832{\cdot}10^7B_{13,16}(X) \\ &\quad + 4.88488{\cdot}10^7B_{14,16}(X) + 4.49297{\cdot}10^7B_{15,16}(X) + 4.03108{\cdot}10^7B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

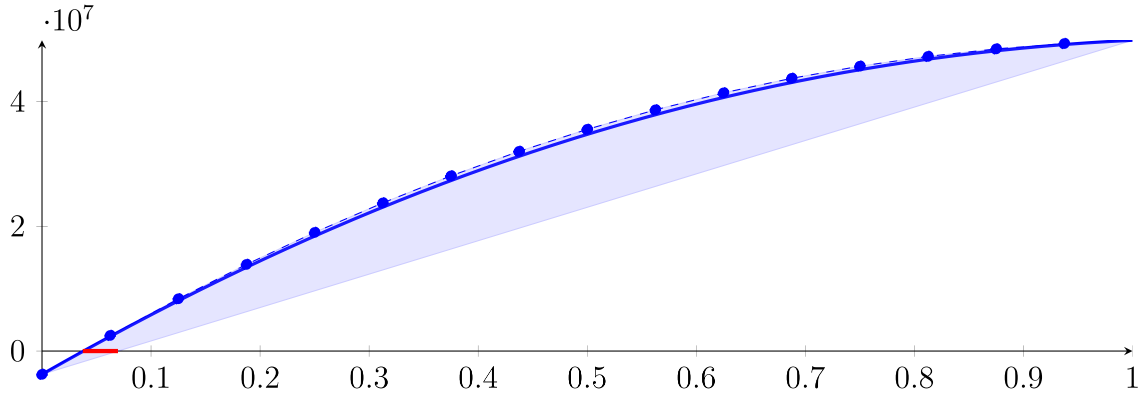
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

76.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -0.00483322X^{16} + 0.0186809X^{15} - 0.0194312X^{14} - 0.0738695X^{13} - 1.11673X^{12} \\ & - 5.0471X^{11} + 36.3082X^{10} + 692.914X^{9} + 1886.96X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5} \\ & + 3.91945 \cdot 10^{6}X^{4} - 7.54561 \cdot 10^{6}X^{3} - 4.29225 \cdot 10^{7}X^{2} + 9.97982 \cdot 10^{7}X - 3.73501 \cdot 10^{6} \\ = & -3.73501 \cdot 10^{6}B_{0,16}(X) + 2.50237 \cdot 10^{6}B_{1,16}(X) + 8.38207 \cdot 10^{6}B_{2,16}(X) + 1.38906 \\ & \cdot 10^{7}B_{3,16}(X) + 1.90167 \cdot 10^{7}B_{4,16}(X) + 2.37512 \cdot 10^{7}B_{5,16}(X) + 2.80875 \cdot 10^{7}B_{6,16}(X) \\ & + 3.20213 \cdot 10^{7}B_{7,16}(X) + 3.55508 \cdot 10^{7}B_{8,16}(X) + 3.86768 \cdot 10^{7}B_{9,16}(X) + 4.14025 \\ & \cdot 10^{7}B_{10,16}(X) + 4.37336 \cdot 10^{7}B_{11,16}(X) + 4.56783 \cdot 10^{7}B_{12,16}(X) + 4.7247 \cdot 10^{7}B_{13,16}(X) \\ & + 4.84524 \cdot 10^{7}B_{14,16}(X) + 4.93093 \cdot 10^{7}B_{15,16}(X) + 4.98343 \cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

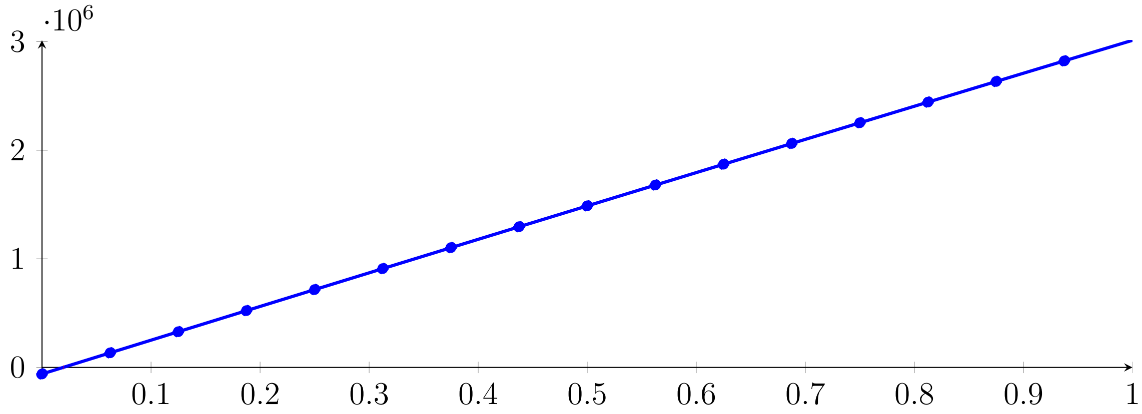
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

76.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -0.000205056X^{16} + 0.000776167X^{15} - 0.000863333X^{14} - 0.000217499X^{13} - 0.00809276X^{12} \\ & - 5.09773 \cdot 10^{-05}X^{11} - 0.00564923X^{10} - 0.000162811X^{9} - 0.000215376X^{8} - 3.32948 \cdot 10^{-07}X^{7} \\ & - 0.000259866X^{6} + 0.0161077X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845 \cdot 10^{6}X - 60508.5 \\ = & -60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X) \\ & + 910202B_{5,16}(X) + 1.1032 \cdot 10^{6}B_{6,16}(X) + 1.29582 \cdot 10^{6}B_{7,16}(X) + 1.48805 \cdot 10^{6}B_{8,16}(X) \\ & + 1.6799 \cdot 10^{6}B_{9,16}(X) + 1.87137 \cdot 10^{6}B_{10,16}(X) + 2.06245 \cdot 10^{6}B_{11,16}(X) + 2.25315 \cdot 10^{6}B_{12,16}(X) \\ & + 2.44346 \cdot 10^{6}B_{13,16}(X) + 2.63339 \cdot 10^{6}B_{14,16}(X) + 2.82293 \cdot 10^{6}B_{15,16}(X) + 3.01209 \cdot 10^{6}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

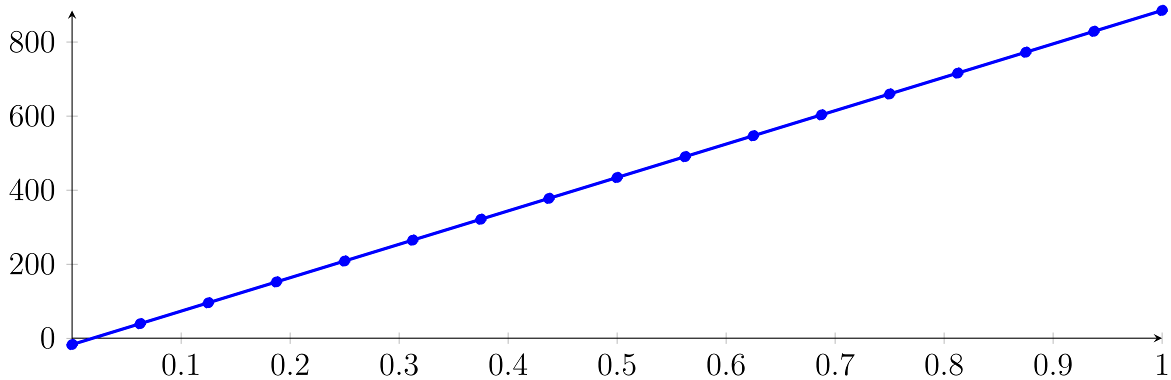
$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

76.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -5.9692 \cdot 10^{-08} X^{16} + 2.16103 \cdot 10^{-07} X^{15} - 2.28456 \cdot 10^{-07} X^{14} - 1.17238 \cdot 10^{-07} X^{13} \\ & - 2.29525 \cdot 10^{-06} X^{12} - 8.31778 \cdot 10^{-08} X^{11} - 1.74251 \cdot 10^{-06} X^{10} - 9.42919 \cdot 10^{-08} X^{9} \\ & - 7.38891 \cdot 10^{-08} X^{8} + 3.25144 \cdot 10^{-09} X^{7} - 2.61741 \cdot 10^{-08} X^{6} + 7.44876 \cdot 10^{-10} X^{5} \\ & - 2.58638 \cdot 10^{-10} X^{4} - 5.65024 \cdot 10^{-09} X^{3} - 0.00382618 X^{2} + 902.448 X - 17.178 \\ = & -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\ & + 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\ & + 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\ & + 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

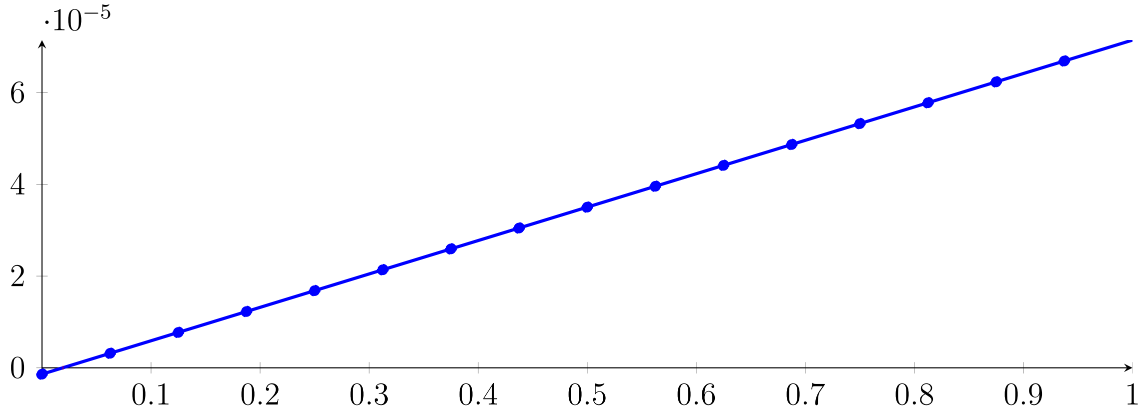
$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045 \cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

76.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -4.80379 \cdot 10^{-15} X^{16} + 1.74137 \cdot 10^{-14} X^{15} - 1.72656 \cdot 10^{-14} X^{14} - 6.84186 \cdot 10^{-15} X^{13} - 1.91627 \cdot 10^{-13} X^{12} - 4.7358 \cdot 10^{-15} X^{11} - 1.33436 \cdot 10^{-13} X^{10} - 1.97677 \cdot 10^{-15} X^{9} - 7.4565 \cdot 10^{-15} X^{8} - 1.16281 \cdot 10^{-16} X^{7} - 1.98065 \cdot 10^{-15} X^{6} + 1.47994 \cdot 10^{-17} X^{5} - 1.84992 \cdot 10^{-17} X^{4} - 2.48011 \cdot 10^{-17} X^{2} + 7.28316 \cdot 10^{-05} X - 1.38634 \cdot 10^{-06}$$
$$= -1.38634 \cdot 10^{-06} B_{0,16}(X) + 3.16564 \cdot 10^{-06} B_{1,16}(X) + 7.71761 \cdot 10^{-06} B_{2,16}(X) + 1.22696 \cdot 10^{-05} B_{3,16}(X) + 1.68216 \cdot 10^{-05} B_{4,16}(X) + 2.13735 \cdot 10^{-05} B_{5,16}(X) + 2.59255 \cdot 10^{-05} B_{6,16}(X) + 3.04775 \cdot 10^{-05} B_{7,16}(X) + 3.50295 \cdot 10^{-05} B_{8,16}(X) + 3.95814 \cdot 10^{-05} B_{9,16}(X) + 4.41334 \cdot 10^{-05} B_{10,16}(X) + 4.86854 \cdot 10^{-05} B_{11,16}(X) + 5.32374 \cdot 10^{-05} B_{12,16}(X) + 5.77893 \cdot 10^{-05} B_{13,16}(X) + 6.23413 \cdot 10^{-05} B_{14,16}(X) + 6.68933 \cdot 10^{-05} B_{15,16}(X) + 7.14453 \cdot 10^{-05} B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.50521 \cdot 10^{-15}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

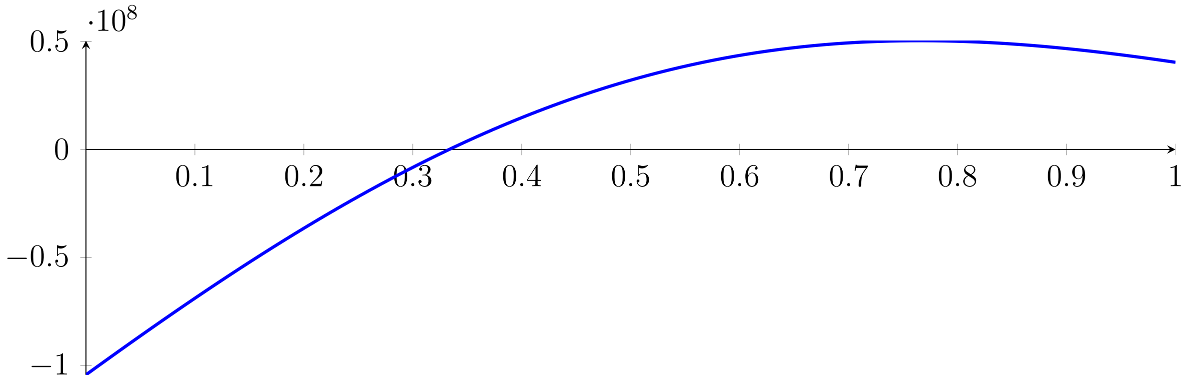
76.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

76.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} + 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1{\cdot}10^{-32}$.

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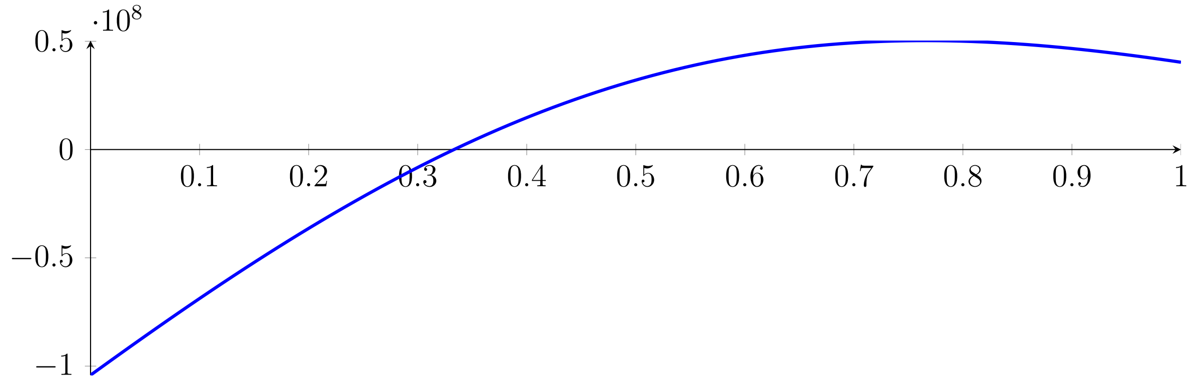
$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

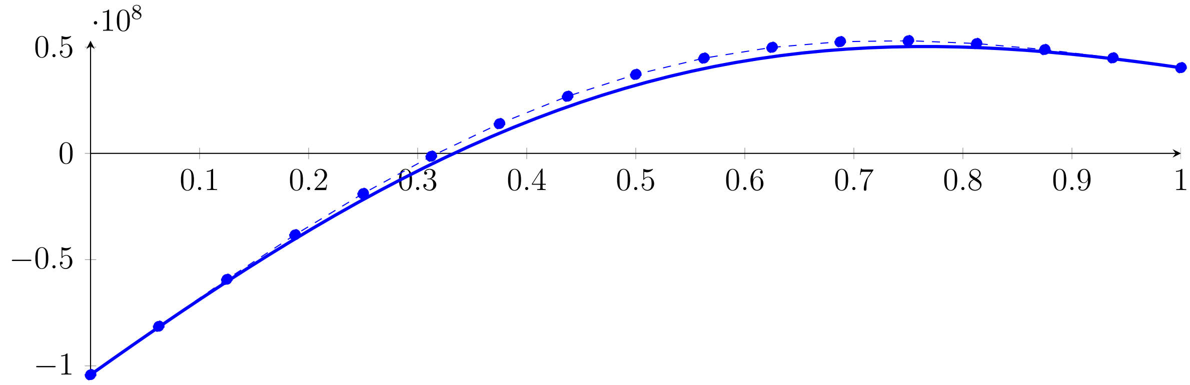
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



77.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

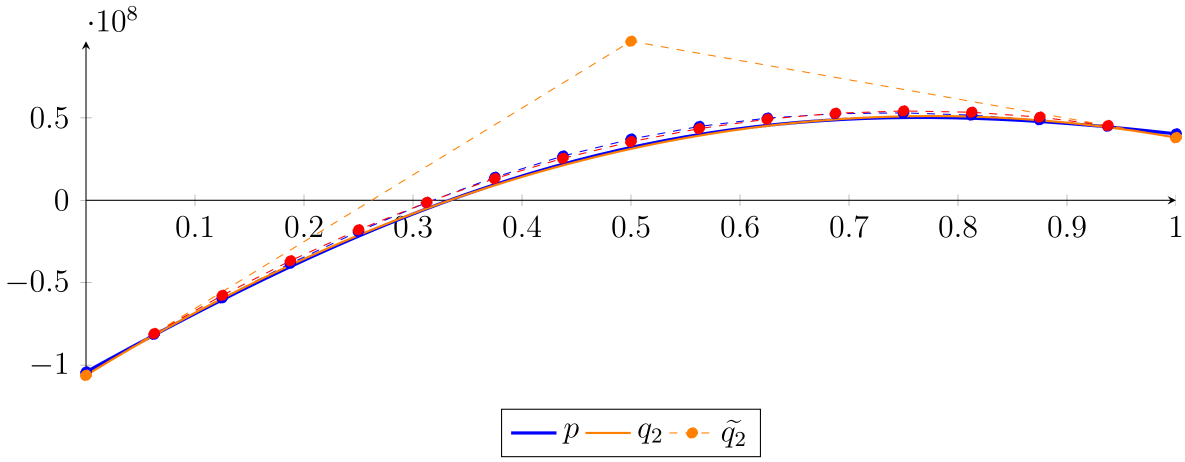
$$\begin{aligned} p &= -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ &= -1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203\cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) + 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901\cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) + 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$
$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\widetilde{q}_2 = -1.41787 \cdot 10^6 X^{16} + 1.11761 \cdot 10^7 X^{15} - 3.98898 \cdot 10^7 X^{14} + 8.52437 \cdot 10^7 X^{13} - 1.21528 \cdot 10^8 X^{12} + 1.21946 \cdot 10^8 X^{11} - 8.86062 \cdot 10^7 X^{10} + 4.72904 \cdot 10^7 X^9 - 1.86355 \cdot 10^7 X^8 + 5.41059 \cdot 10^6 X^7 - 1.14487 \cdot 10^6 X^6 + 172240 X^5 - 17636.6 X^4 + 1159.74 X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$
$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017 \cdot 10^7 B_{4,16} - 1.26923 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55691 \cdot 10^7 B_{8,16} + 4.34957 \cdot 10^7 B_{9,16} + 4.92455 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16} + 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



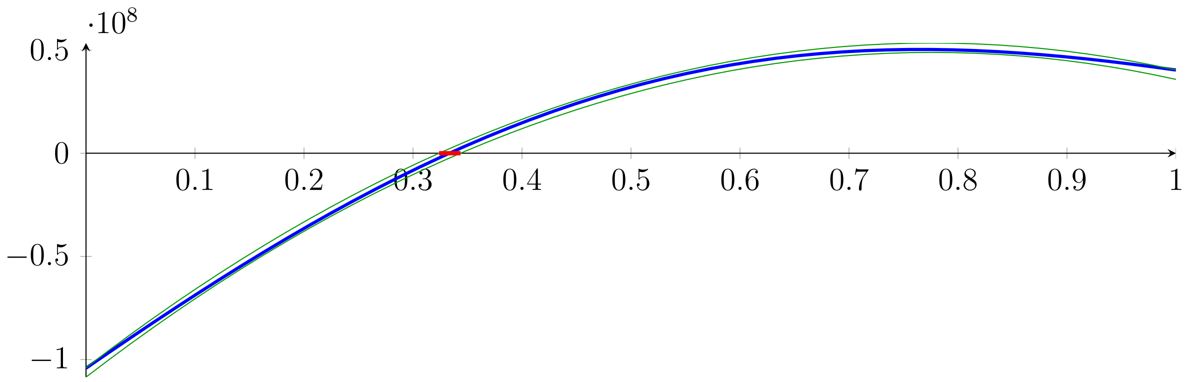
The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.
Bounding polynomials M and m:

$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$
$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



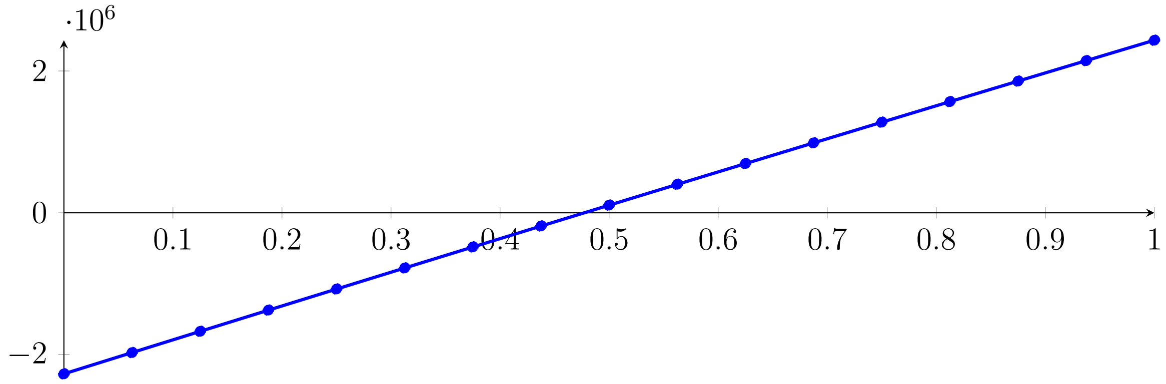
$$[0.323946, 0.343615]$$

Longest intersection interval: 0.0196686
$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

77.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

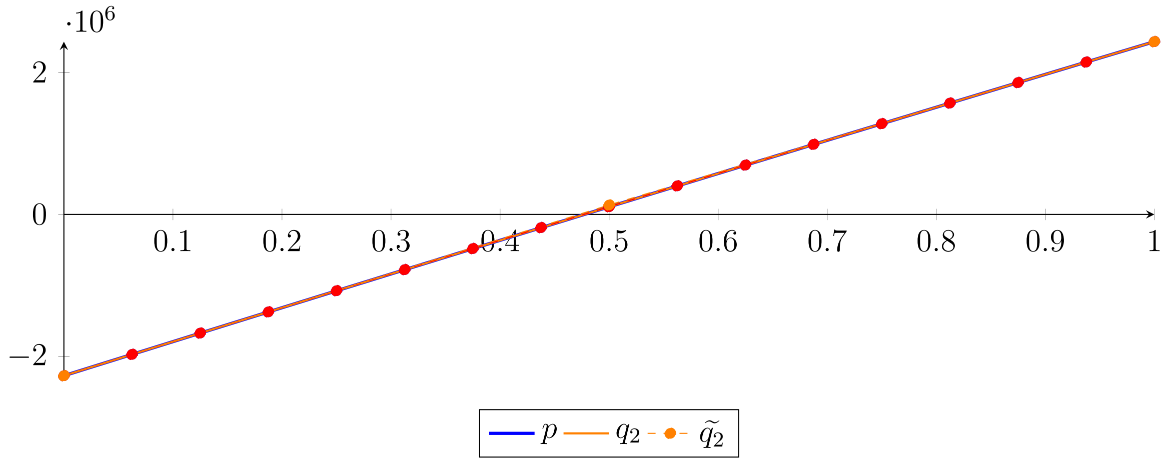
$$\begin{aligned}p &= -2.82438\cdot10^{-05}X^{16} + 2.90051\cdot10^{-05}X^{15} + 0.000231285X^{14} + 0.000648014X^{13} + 0.00318916X^{12}\\ &\quad + 0.00323204X^{11} + 0.00460533X^{10} + 0.00220012X^{9} - 0.000101882X^{8} + 5.32717\cdot10^{-06}X^{7}\\ &\quad - 0.00186824X^{6} + 0.132741X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874\cdot10^{6}X - 2.2715\cdot10^{6}\\ &= -2.2715\cdot10^{6}B_{0,16}(X) - 1.97096\cdot10^{6}B_{1,16}(X) - 1.67127\cdot10^{6}B_{2,16}(X) - 1.37244\cdot10^{6}B_{3,16}(X)\\ &\quad - 1.07448\cdot10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X)\\ &\quad + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806\cdot10^{6}B_{12,16}(X) + 1.5682\\ &\quad \cdot10^{6}B_{13,16}(X) + 1.85747\cdot10^{6}B_{14,16}(X) + 2.14586\cdot10^{6}B_{15,16}(X) + 2.43338\cdot10^{6}B_{16,16}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -104265X^{2} + 4.80923\cdot10^{6}X - 2.27154\cdot10^{6}\\ &= -2.27154\cdot10^{6}B_{0,2} + 133069B_{1,2} + 2.43342\cdot10^{6}B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_2 &= 43556.7X^{16} - 347904X^{15} + 1.2616\cdot10^{6}X^{14} - 2.74835\cdot10^{6}X^{13} + 4.01042\cdot10^{6}X^{12} - 4.13709\\ &\quad \cdot10^{6}X^{11} + 3.10251\cdot10^{6}X^{10} - 1.71209\cdot10^{6}X^{9} + 695531X^{8} - 205965X^{7} + 43591.2X^{6}\\ &\quad - 6402.94X^{5} + 625.678X^{4} - 37.485X^{3} - 104264X^{2} + 4.80923\cdot10^{6}X - 2.27154\cdot10^{6}\\ &= -2.27154\cdot10^{6}B_{0,16} - 1.97097\cdot10^{6}B_{1,16} - 1.67126\cdot10^{6}B_{2,16} - 1.37242\cdot10^{6}B_{3,16}\\ &\quad - 1.07445\cdot10^{6}B_{4,16} - 777350B_{5,16} - 481117B_{6,16} - 185755B_{7,16} + 108741B_{8,16}\\ &\quad + 402365B_{9,16} + 695123B_{10,16} + 987010B_{11,16} + 1.27803\cdot10^{6}B_{12,16} + 1.56818\\ &\quad \cdot10^{6}B_{13,16} + 1.85746\cdot10^{6}B_{14,16} + 2.14587\cdot10^{6}B_{15,16} + 2.43342\cdot10^{6}B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m:

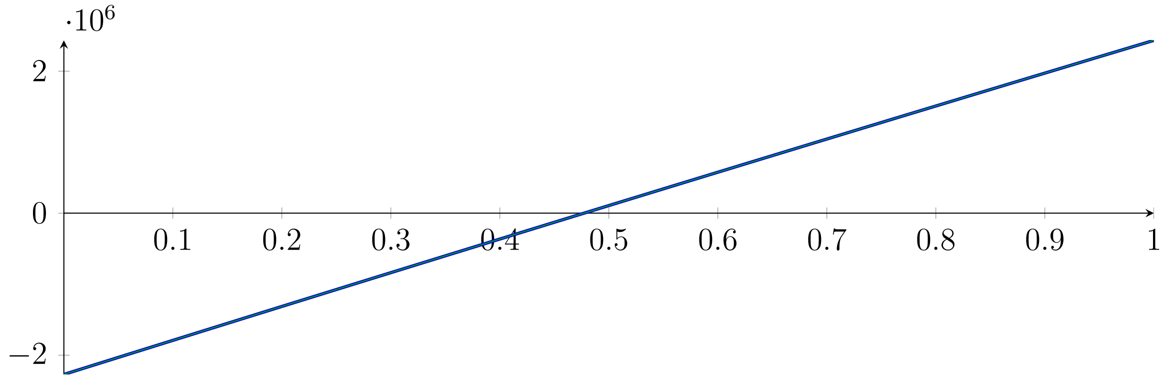
$$M = -104265X^{2} + 4.80923\cdot10^{6}X - 2.2715\cdot10^{6}$$

$$m = -104265X^2 + 4.80923{\cdot}10^6 X - 2.27159{\cdot}10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

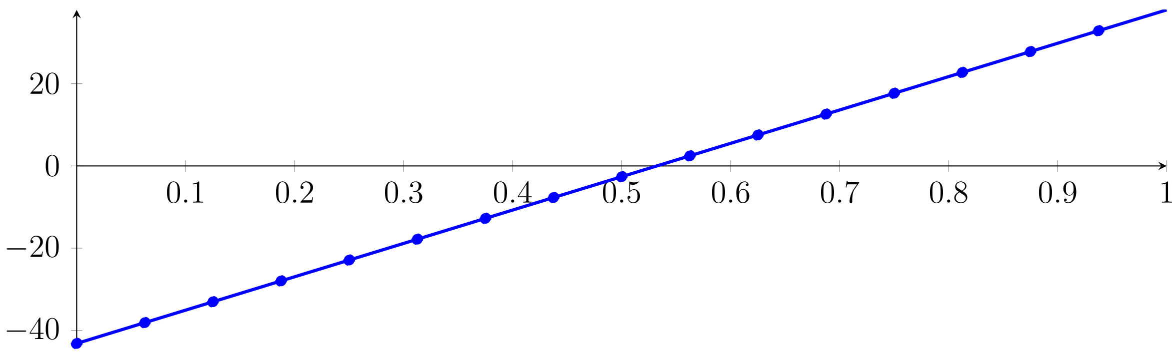
Longest intersection interval: $1.72301 \cdot 10^{-05}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

77.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

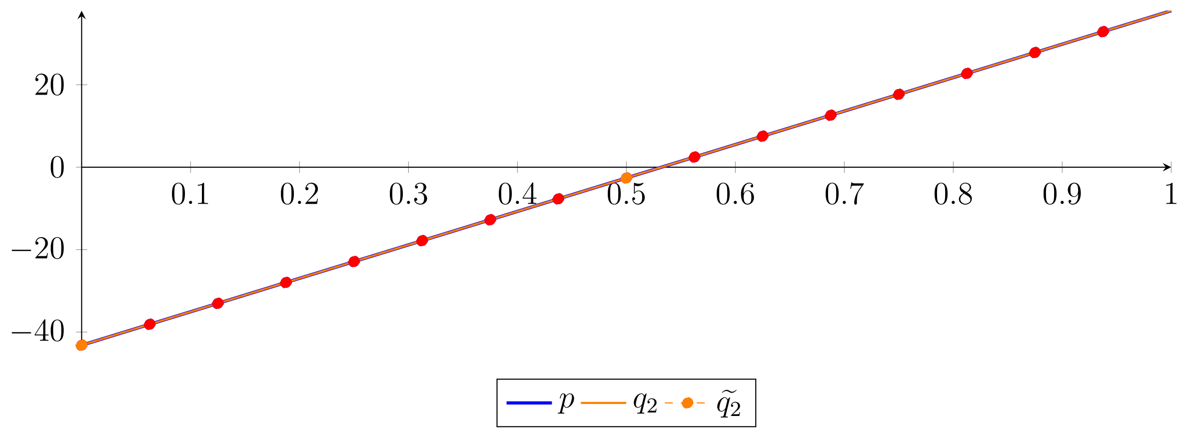
$$\begin{aligned} p = {} & 8.74252{\cdot}10^{-11}X^{16} - 1.56979{\cdot}10^{-09}X^{15} + 6.68479{\cdot}10^{-09}X^{14} + 1.20008{\cdot}10^{-08}X^{13} + 9.07301{\cdot}10^{-08}X^{12} \\ & + 5.58657{\cdot}10^{-08}X^{11} + 1.13801{\cdot}10^{-07}X^{10} + 3.70665{\cdot}10^{-08}X^{9} + 7.31575{\cdot}10^{-10}X^{8} + 1.30058{\cdot}10^{-09}X^{7} \\ & + 5.00722{\cdot}10^{-09}X^{6} + 1.24146{\cdot}10^{-10}X^{5} + 1.03455{\cdot}10^{-10}X^{4} - 3.09388{\cdot}10^{-05}X^{2} + 81.1505X - 43.1911 \\ = {} & -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\ & - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68777B_{7,16}(X) - 2.61587B_{8,16}(X) \\ & + 2.45604B_{9,16}(X) + 7.52794B_{10,16}(X) + 12.5998B_{11,16}(X) + 17.6718B_{12,16}(X) \\ & + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09388{\cdot}10^{-05}X^2 + 81.1505X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 = {} & 0.721495X^{16} - 5.74915X^{15} + 20.7933X^{14} - 45.1627X^{13} + 65.6806X^{12} - 67.5044X^{11} \\ & + 50.4286X^{10} - 27.728X^{9} + 11.2318X^{8} - 3.32011X^{7} + 0.702408X^{6} - 0.103415X^{5} \\ & + 0.0102099X^{4} - 0.000624725X^{3} - 1.10834{\cdot}10^{-05}X^{2} + 81.1505X - 43.1911 \\ = {} & -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} \\ & - 12.7597B_{6,16} - 7.68779B_{7,16} - 2.61585B_{8,16} + 2.45602B_{9,16} + 7.52795B_{10,16} + 12.5998B_{11,16} \\ & + 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16} \end{aligned}$$



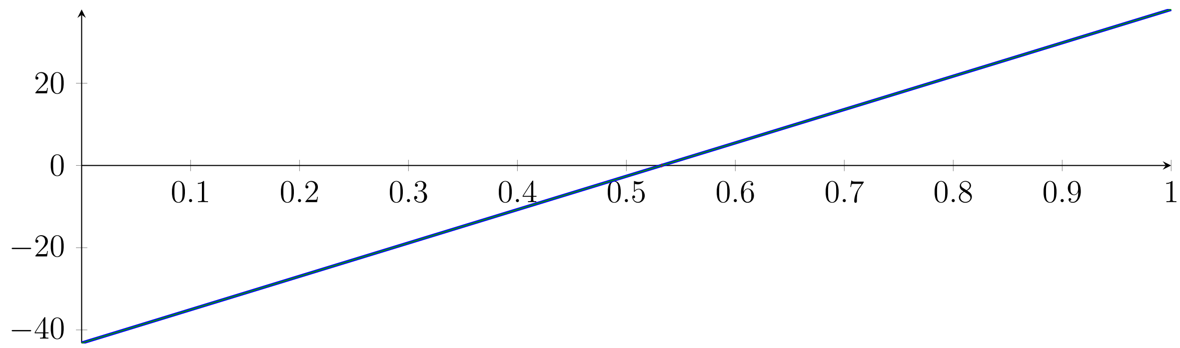
The maximum difference of the Bézier coefficients is $\delta = 1.57956 \cdot 10^{-05}$.
Bounding polynomials M and m:

$$M = -3.09388 \cdot 10^{-05} X^2 + 81.1505X - 43.1911$$
$$m = -3.09388 \cdot 10^{-05} X^2 + 81.1505X - 43.1911$$

Root of M and m:

$$N(M) = \{0.532235, 2.62293 \cdot 10^{6}\} \qquad N(m) = \{0.532235, 2.62293 \cdot 10^{6}\}$$

Intersection intervals:



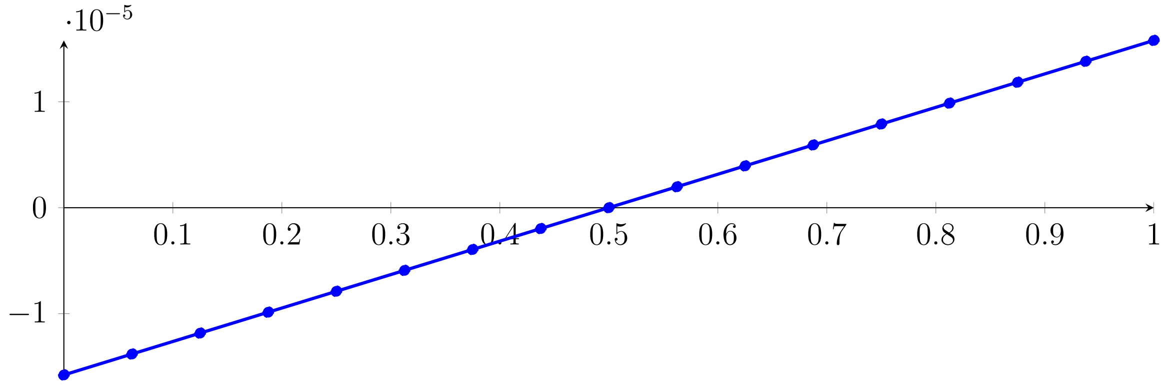
$[0.532235, 0.532235]$

Longest intersection interval: $3.8903 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

77.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

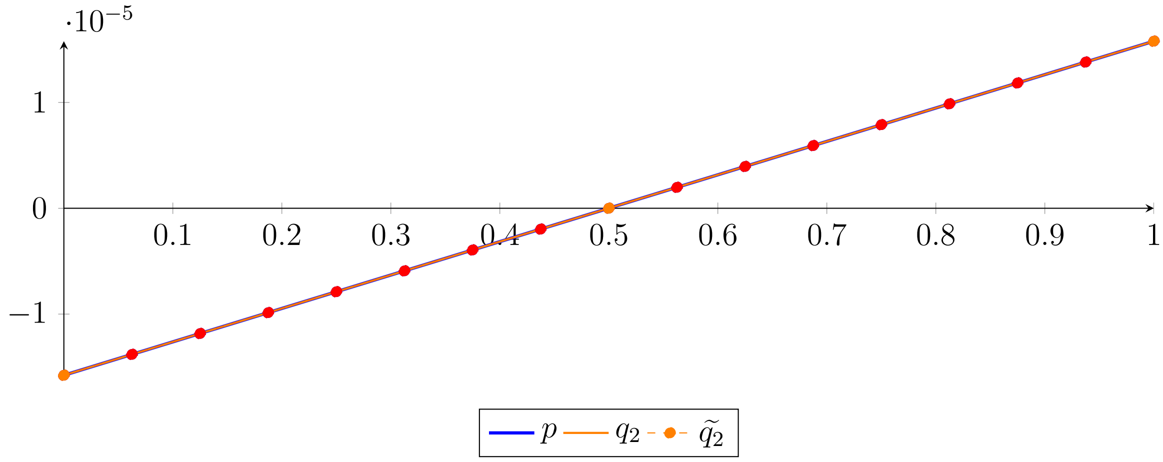
$$p = -1.04409 \cdot 10^{-16} X^{16} - 1.53089 \cdot 10^{-16} X^{15} + 1.74665 \cdot 10^{-15} X^{14} + 5.46438 \cdot 10^{-15} X^{13} + 2.56522 \cdot 10^{-14} X^{12} + 2.15479 \cdot 10^{-14} X^{11} + 3.51633 \cdot 10^{-14} X^{10} + 1.67444 \cdot 10^{-14} X^{9} - 8.72105 \cdot 10^{-16} X^{8} + 1.41087 \cdot 10^{-15} X^{6} + 5.91974 \cdot 10^{-17} X^{5} + 4.93312 \cdot 10^{-17} X^{4} + 3.79471 \cdot 10^{-18} X^{3} - 4.87891 \cdot 10^{-18} X^{2} + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$
$$= -1.57804 \cdot 10^{-05} B_{0,16}(X) - 1.38073 \cdot 10^{-05} B_{1,16}(X) - 1.18341 \cdot 10^{-05} B_{2,16}(X) - 9.86101 \cdot 10^{-06} B_{3,16}(X) - 7.88788 \cdot 10^{-06} B_{4,16}(X) - 5.91476 \cdot 10^{-06} B_{5,16}(X) - 3.94163 \cdot 10^{-06} B_{6,16}(X) - 1.96851 \cdot 10^{-06} B_{7,16}(X) + 4.61501 \cdot 10^{-09} B_{8,16}(X) + 1.97774 \cdot 10^{-06} B_{9,16}(X) + 3.95086 \cdot 10^{-06} B_{10,16}(X) + 5.92399 \cdot 10^{-06} B_{11,16}(X) + 7.89711 \cdot 10^{-06} B_{12,16}(X) + 9.87024 \cdot 10^{-06} B_{13,16}(X) + 1.18434 \cdot 10^{-05} B_{14,16}(X) + 1.38165 \cdot 10^{-05} B_{15,16}(X) + 1.57896 \cdot 10^{-05} B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -4.65868 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$
$$= -1.57804 \cdot 10^{-05} B_{0,2} + 4.61501 \cdot 10^{-09} B_{1,2} + 1.57896 \cdot 10^{-05} B_{2,2}$$

$$\widetilde{q}_2 = 2.92413 \cdot 10^{-07} X^{16} - 2.33332 \cdot 10^{-06} X^{15} + 8.45203 \cdot 10^{-06} X^{14} - 1.83895 \cdot 10^{-05} X^{13} + 2.67963 \cdot 10^{-05} X^{12} - 2.75995 \cdot 10^{-05} X^{11} + 2.06638 \cdot 10^{-05} X^{10} - 1.13854 \cdot 10^{-05} X^{9} + 4.61944 \cdot 10^{-06} X^{8} - 1.36687 \cdot 10^{-06} X^{7} + 2.89249 \cdot 10^{-07} X^{6} - 4.25295 \cdot 10^{-08} X^{5} + 4.17283 \cdot 10^{-09} X^{4} - 2.52119 \cdot 10^{-10} X^{3} + 7.82992 \cdot 10^{-12} X^{2} + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$
$$= -1.57804 \cdot 10^{-05} B_{0,16} - 1.38073 \cdot 10^{-05} B_{1,16} - 1.18341 \cdot 10^{-05} B_{2,16} - 9.86101 \cdot 10^{-06} B_{3,16} - 7.88788 \cdot 10^{-06} B_{4,16} - 5.91476 \cdot 10^{-06} B_{5,16} - 3.94163 \cdot 10^{-06} B_{6,16} - 1.96851 \cdot 10^{-06} B_{7,16} + 4.62125 \cdot 10^{-09} B_{8,16} + 1.97773 \cdot 10^{-06} B_{9,16} + 3.95087 \cdot 10^{-06} B_{10,16} + 5.92399 \cdot 10^{-06} B_{11,16} + 7.89711 \cdot 10^{-06} B_{12,16} + 9.87024 \cdot 10^{-06} B_{13,16} + 1.18434 \cdot 10^{-05} B_{14,16} + 1.38165 \cdot 10^{-05} B_{15,16} + 1.57896 \cdot 10^{-05} B_{16,16}$$



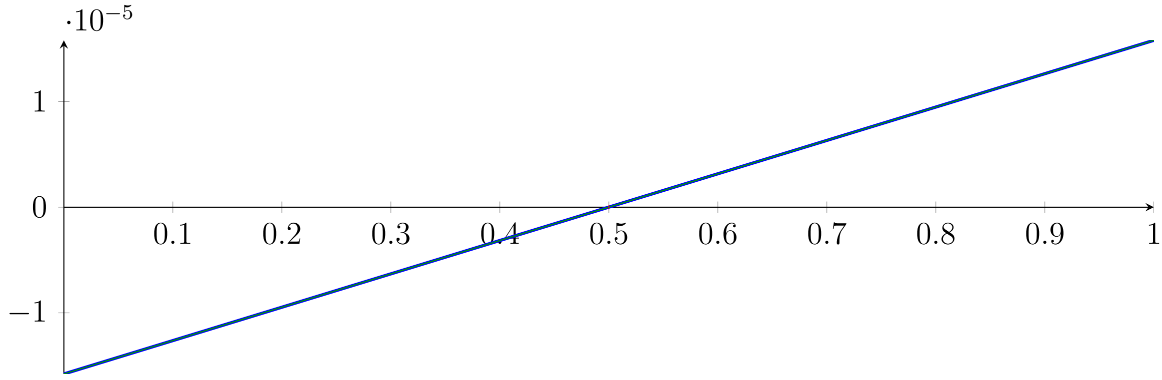
The maximum difference of the Bézier coefficients is $\delta = 6.24192 \cdot 10^{-12}$.
Bounding polynomials M and m:

$$M = -4.65868 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$
$$m = -4.65868 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.499636, 6.77659 \cdot 10^{12}\} \qquad N(m) = \{0.500364, 6.77659 \cdot 10^{12}\}$$

Intersection intervals:



$[0.499636, 0.500364]$

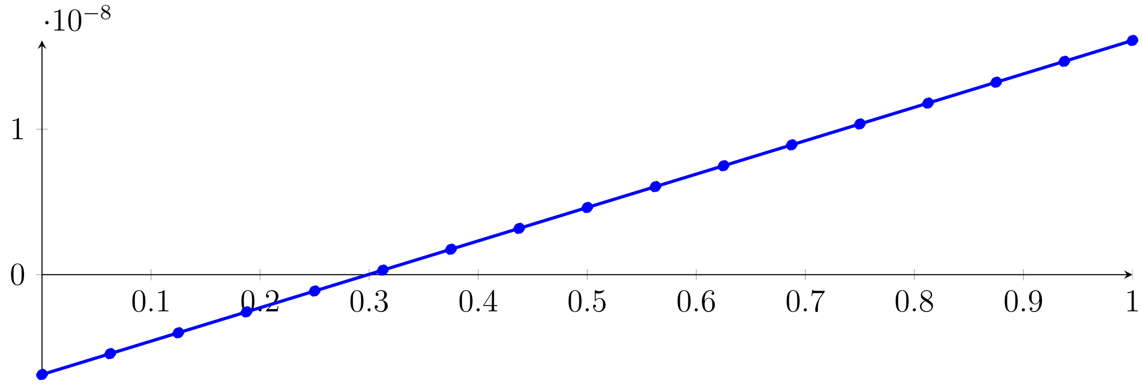
Longest intersection interval: 0.000727273

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

77.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

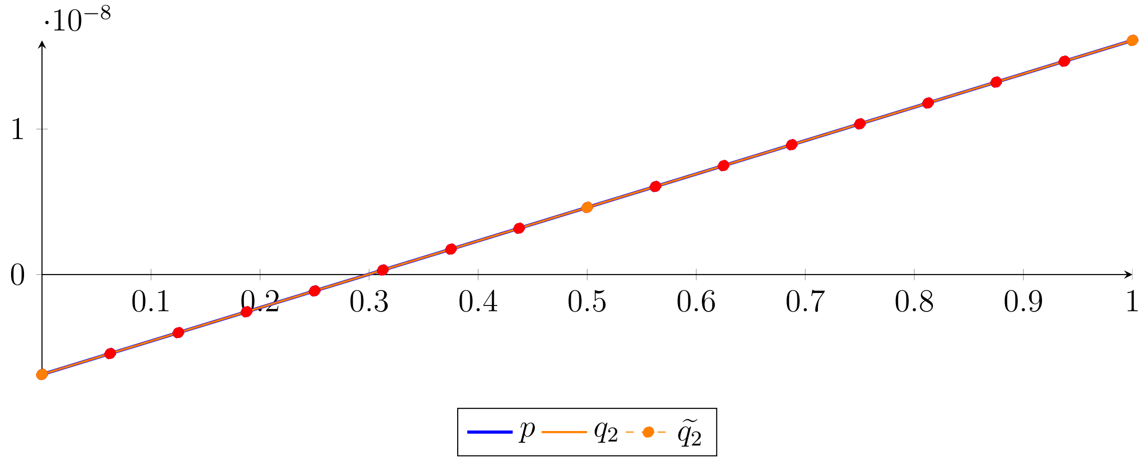
$$\begin{aligned} p = &-6.76104\cdot 10^{-19}X^{16} + 2.21478\cdot 10^{-18}X^{15} - 1.59295\cdot 10^{-18}X^{14} + 1.03762\cdot 10^{-18}X^{13} \\ &- 1.34649\cdot 10^{-17}X^{12} + 9.65427\cdot 10^{-18}X^{11} - 2.75561\cdot 10^{-18}X^{10} + 5.90488\cdot 10^{-18}X^{9} \\ &- 8.51665\cdot 10^{-19}X^{8} + 3.02814\cdot 10^{-19}X^{7} + 2.64962\cdot 10^{-19}X^{6} + 2.8905\cdot 10^{-20}X^{5} \\ &+ 6.02187\cdot 10^{-21}X^{4} + 9.26442\cdot 10^{-22}X^{3} + 2.296\cdot 10^{-08}X - 6.86499\cdot 10^{-09} \\ = &-6.86499\cdot 10^{-09}B_{0,16}(X) - 5.42999\cdot 10^{-09}B_{1,16}(X) - 3.99499\cdot 10^{-09}B_{2,16}(X) \\ &- 2.55999\cdot 10^{-09}B_{3,16}(X) - 1.12499\cdot 10^{-09}B_{4,16}(X) + 3.10008\cdot 10^{-10}B_{5,16}(X) + 1.74501 \\ &\cdot 10^{-09}B_{6,16}(X) + 3.18001\cdot 10^{-09}B_{7,16}(X) + 4.61501\cdot 10^{-09}B_{8,16}(X) + 6.05001\cdot 10^{-09}B_{9,16}(X) \\ &+ 7.48501\cdot 10^{-09}B_{10,16}(X) + 8.92001\cdot 10^{-09}B_{11,16}(X) + 1.0355\cdot 10^{-08}B_{12,16}(X) + 1.179 \\ &\cdot 10^{-08}B_{13,16}(X) + 1.3225\cdot 10^{-08}B_{14,16}(X) + 1.466\cdot 10^{-08}B_{15,16}(X) + 1.6095\cdot 10^{-08}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 = &\ 1.40621\cdot 10^{-23}X^2 + 2.296\cdot 10^{-08}X - 6.86499\cdot 10^{-09} \\ = &-6.86499\cdot 10^{-09}B_{0,2} + 4.61501\cdot 10^{-09}B_{1,2} + 1.6095\cdot 10^{-08}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 = &\ 2.65578\cdot 10^{-10}X^{16} - 2.13329\cdot 10^{-09}X^{15} + 7.78369\cdot 10^{-09}X^{14} - 1.70728\cdot 10^{-08}X^{13} \\ &+ 2.51029\cdot 10^{-08}X^{12} - 2.61095\cdot 10^{-08}X^{11} + 1.97446\cdot 10^{-08}X^{10} - 1.09796\cdot 10^{-08}X^{9} \\ &+ 4.48709\cdot 10^{-09}X^{8} - 1.33355\cdot 10^{-09}X^{7} + 2.82501\cdot 10^{-10}X^{6} - 4.12963\cdot 10^{-11}X^{5} + 3.94088 \\ &\cdot 10^{-12}X^{4} - 2.24328\cdot 10^{-13}X^{3} + 6.17064\cdot 10^{-15}X^{2} + 2.296\cdot 10^{-08}X - 6.86499\cdot 10^{-09} \\ = &-6.86499\cdot 10^{-09}B_{0,16} - 5.42999\cdot 10^{-09}B_{1,16} - 3.99499\cdot 10^{-09}B_{2,16} - 2.55999\cdot 10^{-09}B_{3,16} \\ &- 1.12499\cdot 10^{-09}B_{4,16} + 3.10006\cdot 10^{-10}B_{5,16} + 1.74501\cdot 10^{-09}B_{6,16} + 3.18\cdot 10^{-09}B_{7,16} + 4.61501 \\ &\cdot 10^{-09}B_{8,16} + 6.05\cdot 10^{-09}B_{9,16} + 7.48501\cdot 10^{-09}B_{10,16} + 8.92\cdot 10^{-09}B_{11,16} + 1.0355\cdot 10^{-08}B_{12,16} \\ &+ 1.179\cdot 10^{-08}B_{13,16} + 1.3225\cdot 10^{-08}B_{14,16} + 1.466\cdot 10^{-08}B_{15,16} + 1.6095\cdot 10^{-08}B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.35405 \cdot 10^{-15}$.
Bounding polynomials M and m:

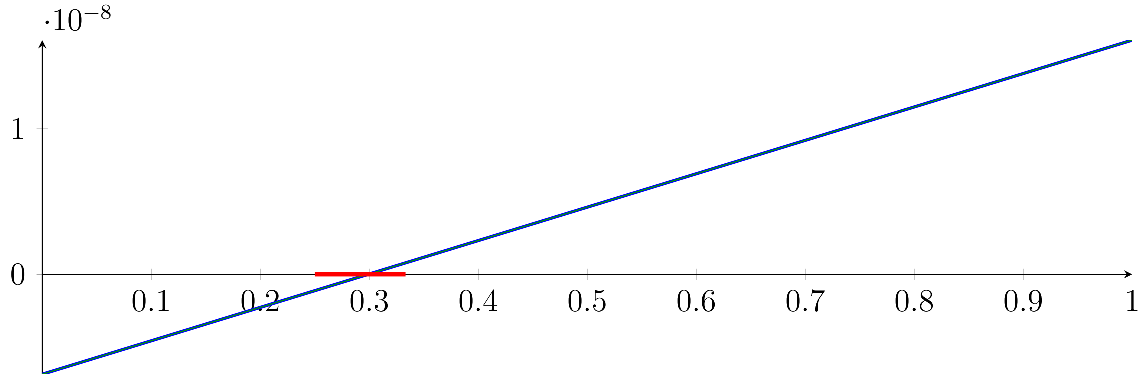
$$M = 1.32349 \cdot 10^{-23} X^2 + 2.296 \cdot 10^{-08} X - 6.86498 \cdot 10^{-09}$$

$$m = 1.48893 \cdot 10^{-23} X^2 + 2.296 \cdot 10^{-08} X - 6.865 \cdot 10^{-09}$$

Root of M and m:

$$N(M) = \{-1.73481 \cdot 10^{15}, 0.25\} \qquad N(m) = \{-1.54205 \cdot 10^{15}, 0.333333\}$$

Intersection intervals:



$$[0.333333, 0.25]$$

Longest intersection interval: -0.0833333
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

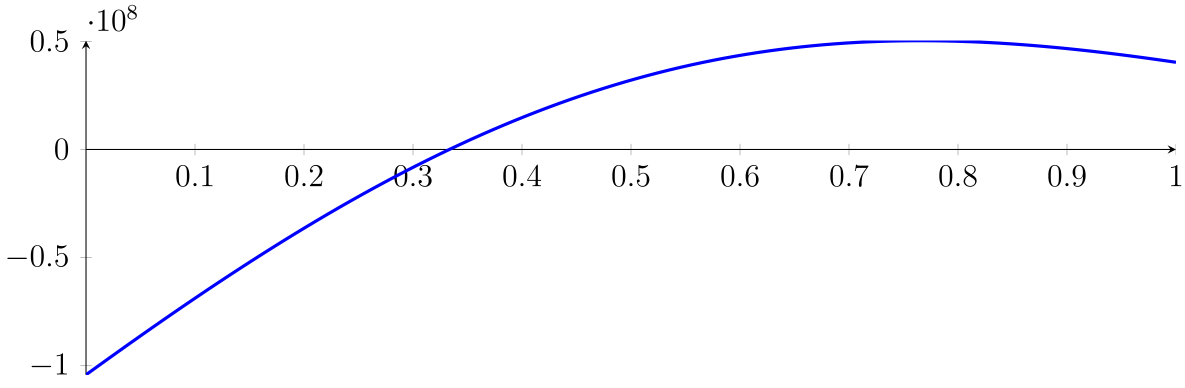
77.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

77.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

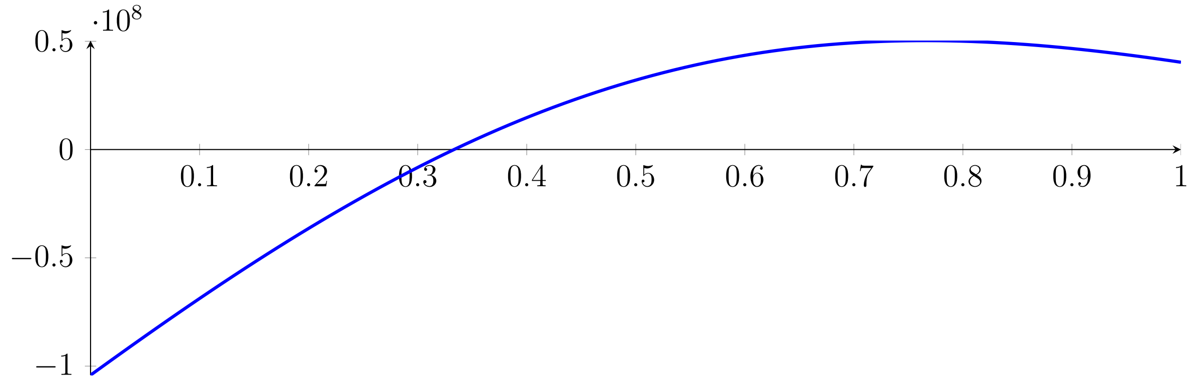
with precision $\varepsilon = 1\cdot10^{-32}$.

78 Running `CubeClip` on f_{16} with epsilon 32

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called `CubeClip` with input polynomial on interval $[0,1]$:

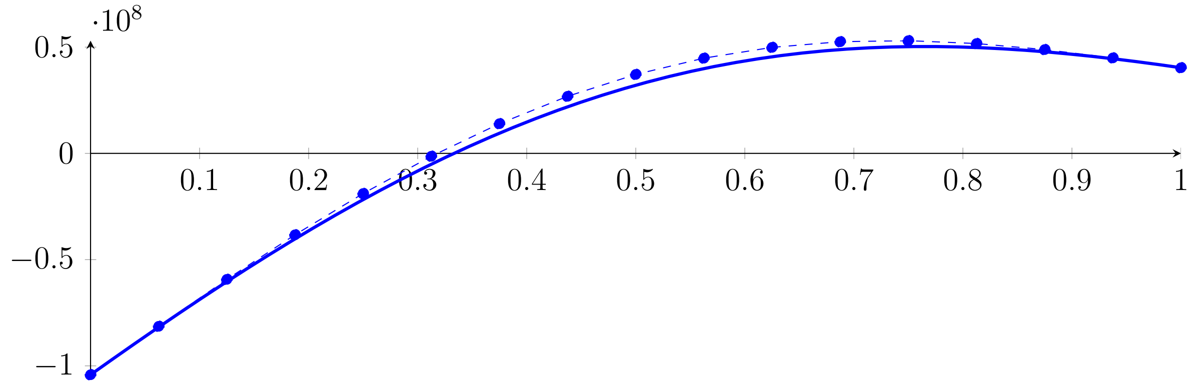
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



78.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

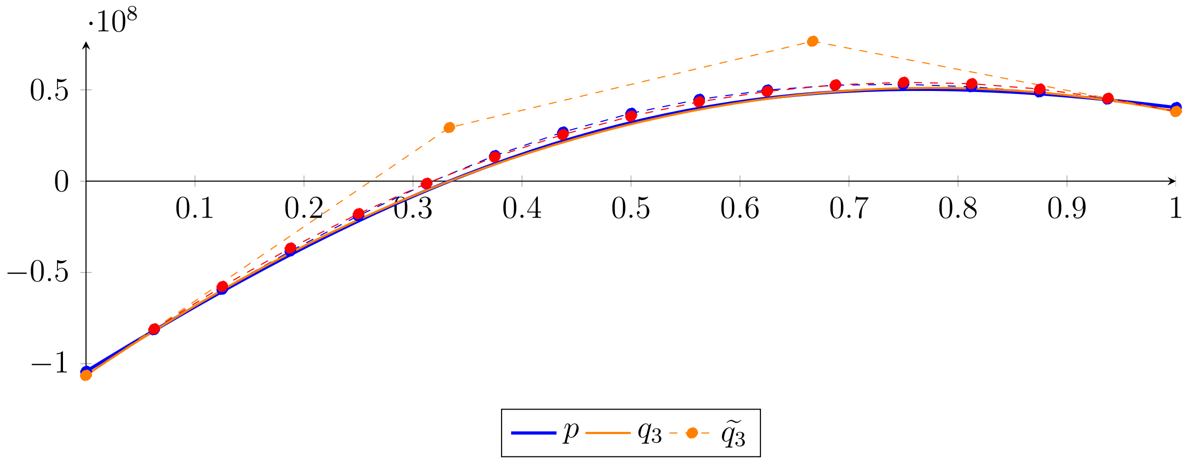
$$\begin{aligned} p &= -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\quad \cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ &= -1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.06335{\cdot}10^8$$
$$= -1.06335{\cdot}10^8B_{0,3} + 2.93558{\cdot}10^7B_{1,3} + 7.66068{\cdot}10^7B_{2,3} + 3.81764{\cdot}10^7B_{3,3}$$

$$\widetilde{q}_3 = 2.11088{\cdot}10^6X^{16} - 1.70537{\cdot}10^7X^{15} + 6.25913{\cdot}10^7X^{14} - 1.38111{\cdot}10^8X^{13} + 2.043{\cdot}10^8X^{12} - 2.13799{\cdot}10^8X^{11} + 1.62714{\cdot}10^8X^{10} - 9.11092{\cdot}10^7X^9 + 3.75254{\cdot}10^7X^8 - 1.12529{\cdot}10^7X^7 + 2.40899{\cdot}10^6X^6 - 357156X^5 + 34986.1X^4 + 2.75602{\cdot}10^6X^3 - 2.65318{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.06335{\cdot}10^8$$
$$= -1.06335{\cdot}10^8B_{0,16} - 8.08928{\cdot}10^7B_{1,16} - 5.76618{\cdot}10^7B_{2,16} - 3.66369{\cdot}10^7B_{3,16} - 1.78131{\cdot}10^7B_{4,16} - 1.18553{\cdot}10^6B_{5,16} + 1.32508{\cdot}10^7B_{6,16} + 2.55007{\cdot}10^7B_{7,16} + 3.55692{\cdot}10^7B_{8,16} + 4.34611{\cdot}10^7B_{9,16} + 4.91816{\cdot}10^7B_{10,16} + 5.27352{\cdot}10^7B_{11,16} + 5.41273{\cdot}10^7B_{12,16} + 5.33624{\cdot}10^7B_{13,16} + 5.04457{\cdot}10^7B_{14,16} + 4.53821{\cdot}10^7B_{15,16} + 3.81764{\cdot}10^7B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806{\cdot}10^6$.

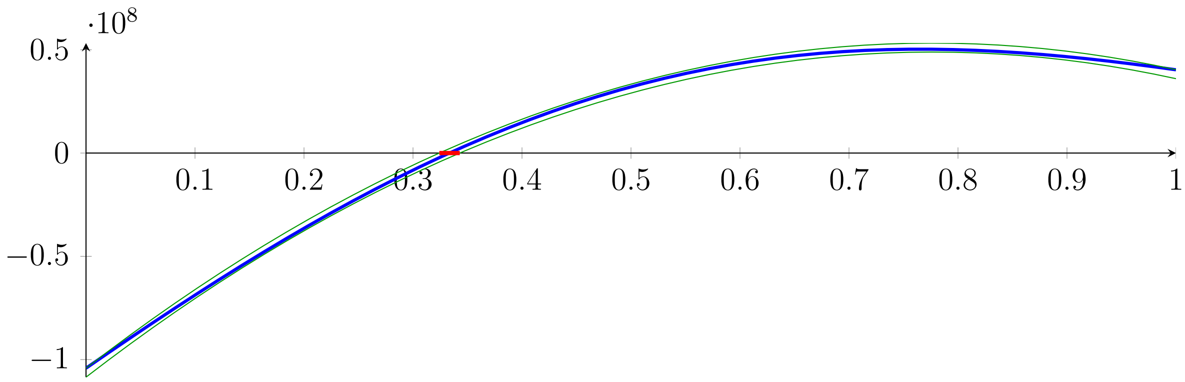
Bounding polynomials M and m:

$$M = 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.04167{\cdot}10^8$$
$$m = 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.08503{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

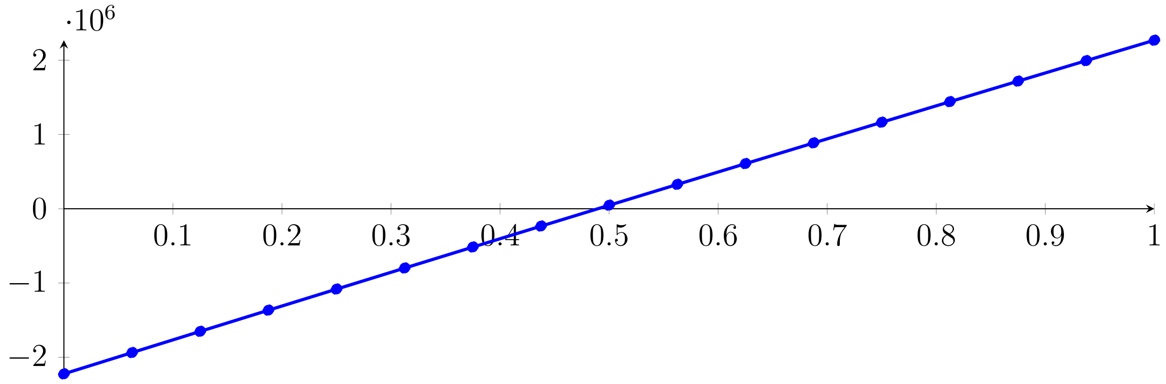
Longest intersection interval: 0.0187703

$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

78.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

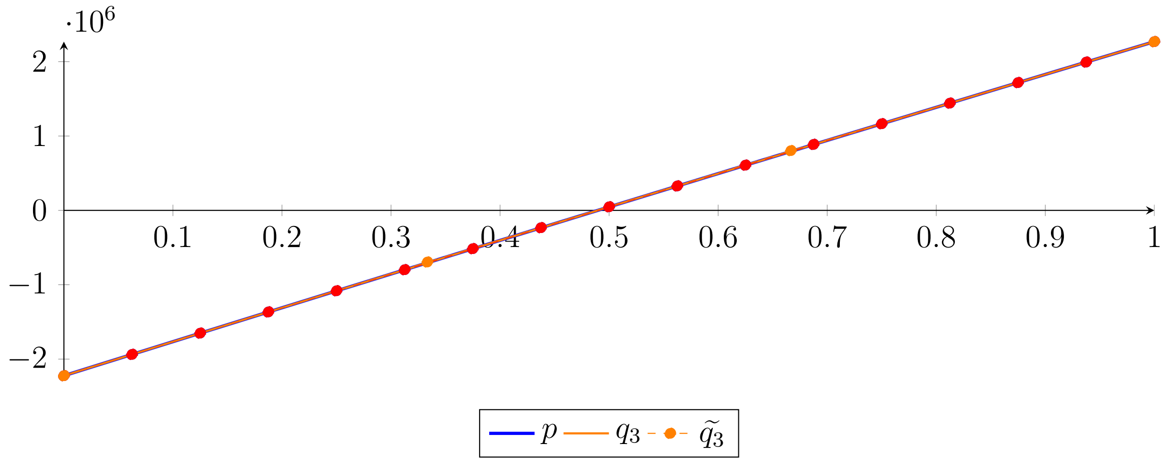
$$p = -1.93035\cdot10^{-05}X^{16} + 1.08927\cdot10^{-05}X^{15} + 0.000255816X^{14} + 0.000616983X^{13} + 0.00371715X^{12} + 0.00325035X^{11} + 0.00510875X^{10} + 0.00190713X^{9} + 0.000179792X^{8} + 1.06543\cdot10^{-05}X^{7} - 0.00136482X^{6} + 0.104959X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,16}(X) - 1.93666\cdot10^{6}B_{1,16}(X) - 1.65074\cdot10^{6}B_{2,16}(X) - 1.36561\cdot10^{6}B_{3,16}(X) - 1.08127\cdot10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X) + 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot10^{6}B_{12,16}(X) + 1.44233\cdot10^{6}B_{13,16}(X) + 1.71876\cdot10^{6}B_{14,16}(X) + 1.99439\cdot10^{6}B_{15,16}(X) + 2.26922\cdot10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot10^{6}B_{3,3}$$

$$\widetilde{q}_3 = 15290.6X^{16} - 125412X^{15} + 468045X^{14} - 1.05198\cdot10^{6}X^{13} + 1.588\cdot10^{6}X^{12} - 1.69893\cdot10^{6}X^{11} + 1.32382\cdot10^{6}X^{10} - 759402X^{9} + 320067X^{8} - 97826.5X^{7} + 21197.8X^{6} - 3162.8X^{5} + 313.931X^{4} - 720.165X^{3} - 93879.2X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,16} - 1.93666\cdot10^{6}B_{1,16} - 1.65074\cdot10^{6}B_{2,16} - 1.36561\cdot10^{6}B_{3,16} - 1.08126\cdot10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.7B_{8,16} + 328649B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot10^{6}B_{12,16} + 1.44233\cdot10^{6}B_{13,16} + 1.71876\cdot10^{6}B_{14,16} + 1.99439\cdot10^{6}B_{15,16} + 2.26922\cdot10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 0.457751$.

Bounding polynomials M and m:

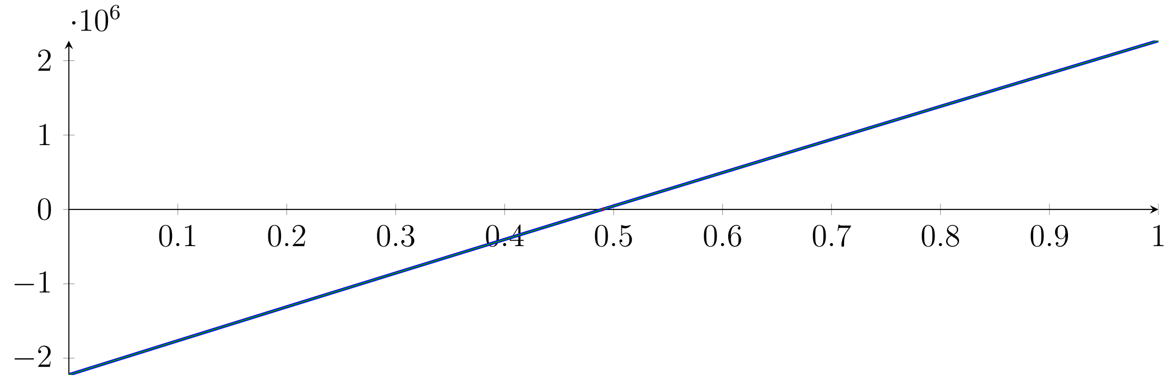
$$M = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715{\cdot}10^6X - 2.22335{\cdot}10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



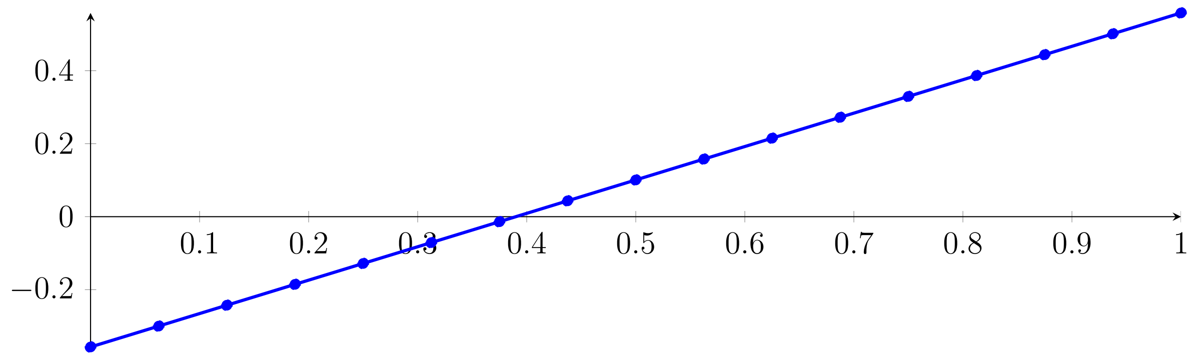
$$[0.489616, 0.489616]$$

Longest intersection interval: $2.03684{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

78.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

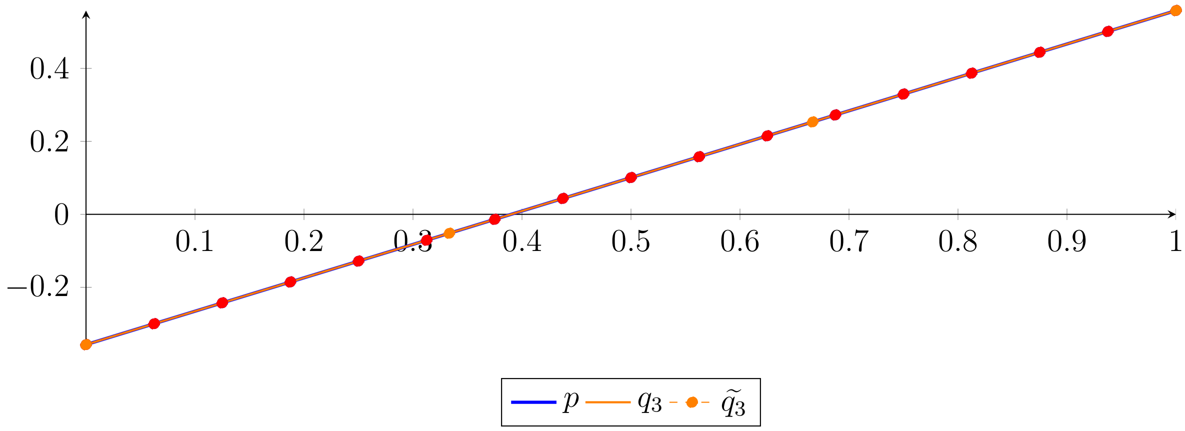
$$\begin{aligned}
p = &-1.56399{\cdot}10^{-11}X^{16} + 4.58016{\cdot}10^{-11}X^{15} - 3.19744{\cdot}10^{-12}X^{14} + 6.54055{\cdot}10^{-11}X^{13} \\
&+ 1.05072{\cdot}10^{-10}X^{12} + 4.59728{\cdot}10^{-10}X^{11} + 5.01434{\cdot}10^{-10}X^{10} + 2.99742{\cdot}10^{-10}X^{9} \\
&+ 1.14309{\cdot}10^{-11}X^{8} - 5.08038{\cdot}10^{-12}X^{7} + 3.37845{\cdot}10^{-11}X^{6} - 9.69891{\cdot}10^{-13}X^{5} \\
&+ 4.04121{\cdot}10^{-13}X^{4} + 6.21725{\cdot}10^{-14}X^{3} - 3.93767{\cdot}10^{-09}X^{2} + 0.915501X - 0.357072 \\
= &-0.357072B_{0,16}(X) - 0.299853B_{1,16}(X) - 0.242635B_{2,16}(X) - 0.185416B_{3,16}(X) \\
&- 0.128197B_{4,16}(X) - 0.0709781B_{5,16}(X) - 0.0137592B_{6,16}(X) \\
&+ 0.0434596B_{7,16}(X) + 0.100678B_{8,16}(X) + 0.157897B_{9,16}(X) + 0.215116B_{10,16}(X) \\
&+ 0.272335B_{11,16}(X) + 0.329554B_{12,16}(X) + 0.386773B_{13,16}(X) \\
&+ 0.443991B_{14,16}(X) + 0.50121B_{15,16}(X) + 0.558429B_{16,16}(X)
\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
q_3 &= 1.05471{\cdot}10^{-15}X^3 - 3.93767{\cdot}10^{-09}X^2 + 0.915501X - 0.357072 \\
&= -0.357072B_{0,3} - 0.0519051B_{1,3} + 0.253262B_{2,3} + 0.558429B_{3,3}
\end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= 0.00291222X^{16} - 0.0241801X^{15} + 0.0914452X^{14} - 0.208537X^{13} + 0.319778X^{12} - 0.347745X^{11} \\ &\quad + 0.275244X^{10} - 0.159971X^{9} + 0.0679818X^{8} - 0.0208072X^{7} + 0.00447629X^{6} - 0.000654783X^{5} \\ &\quad + 6.22034\cdot 10^{-05}X^{4} - 3.60145\cdot 10^{-06}X^{3} + 9.78811\cdot 10^{-08}X^{2} + 0.915501X - 0.357072 \\ &= -0.357072B_{0,16} - 0.299853B_{1,16} - 0.242635B_{2,16} - 0.185416B_{3,16} - 0.128197B_{4,16} \\ &\quad - 0.0709781B_{5,16} - 0.0137592B_{6,16} + 0.0434595B_{7,16} + 0.100678B_{8,16} \\ &\quad + 0.157897B_{9,16} + 0.215116B_{10,16} + 0.272335B_{11,16} + 0.329554B_{12,16} \\ &\quad + 0.386773B_{13,16} + 0.443991B_{14,16} + 0.50121B_{15,16} + 0.558429B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.5212\cdot 10^{-08}$.

Bounding polynomials M and m:

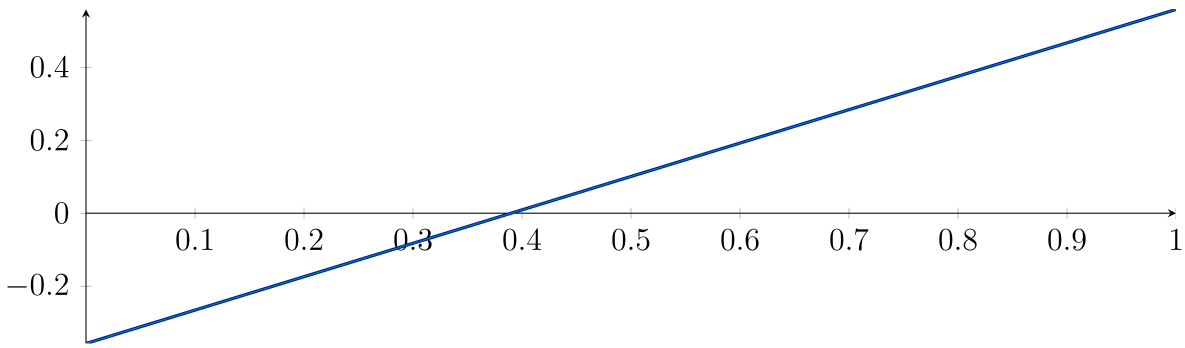
$$M = 9.99201\cdot 10^{-16}X^{3} - 3.93767\cdot 10^{-09}X^{2} + 0.915501X - 0.357072$$

$$m = 1.22125\cdot 10^{-15}X^{3} - 3.93767\cdot 10^{-09}X^{2} + 0.915501X - 0.357072$$

Root of M and m:

$$N(M) = \{0.390029\} \qquad N(m) = \{0.390029\}$$

Intersection intervals:



$$[0.390029, 0.390029]$$

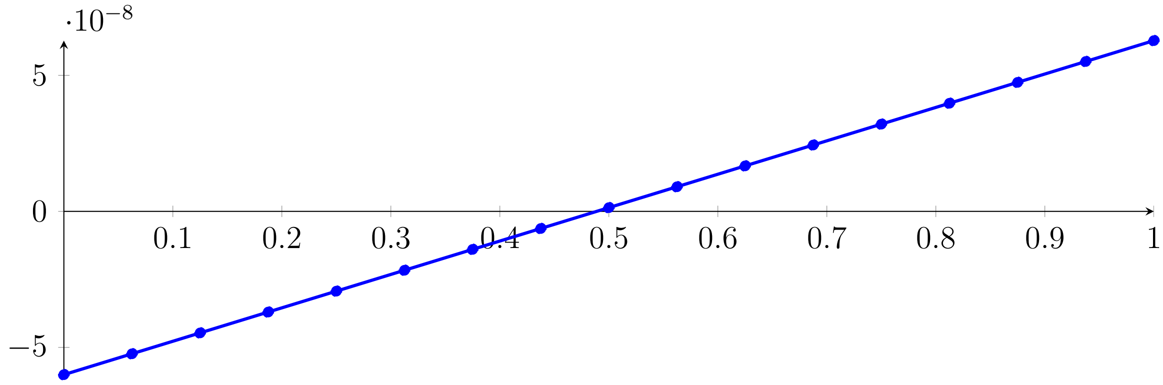
Longest intersection interval: $1.3411\cdot 10^{-07}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

78.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

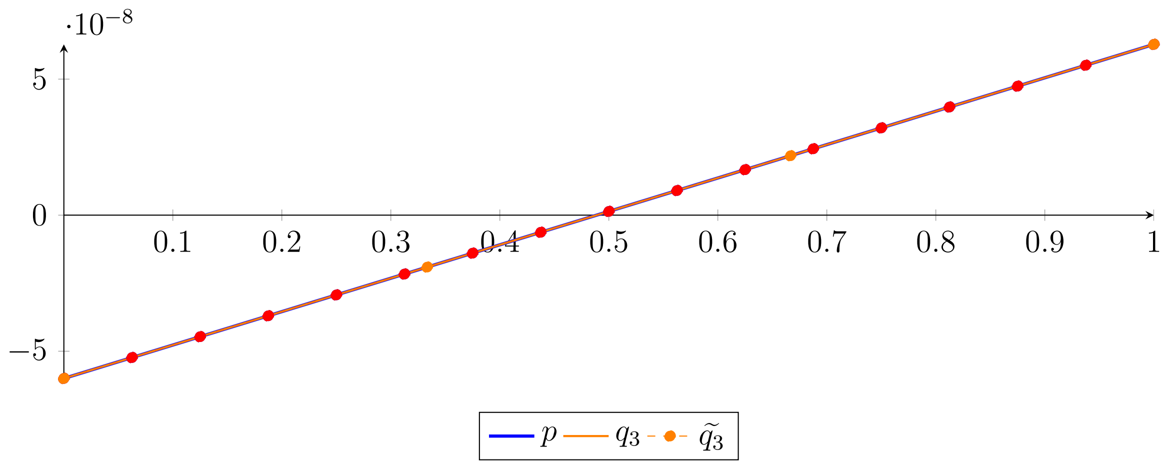
$$p = -5.65183\cdot10^{-19}X^{16} + 5.13302\cdot10^{-19}X^{15} + 6.37816\cdot10^{-18}X^{14} + 1.69576\cdot10^{-17}X^{13} + 9.4423\cdot10^{-17}X^{12} + 8.0934\cdot10^{-17}X^{11} + 1.37357\cdot10^{-16}X^{10} + 6.23797\cdot10^{-17}X^{9} + 1.36266\cdot10^{-18}X^{8} + 6.05629\cdot10^{-19}X^{7} + 5.08728\cdot10^{-18}X^{6} + 2.3124\cdot10^{-19}X^{5} + 1.44525\cdot10^{-19}X^{4} - 7.41154\cdot10^{-21}X^{3} + 1.22778\cdot10^{-07}X - 6.00146\cdot10^{-08}$$
$$= -6.00146\cdot10^{-08}B_{0,16}(X) - 5.2341\cdot10^{-08}B_{1,16}(X) - 4.46674\cdot10^{-08}B_{2,16}(X) - 3.69937\cdot10^{-08}B_{3,16}(X) - 2.93201\cdot10^{-08}B_{4,16}(X) - 2.16464\cdot10^{-08}B_{5,16}(X) - 1.39728\cdot10^{-08}B_{6,16}(X) - 6.29913\cdot10^{-09}B_{7,16}(X) + 1.37451\cdot10^{-09}B_{8,16}(X) + 9.04815\cdot10^{-09}B_{9,16}(X) + 1.67218\cdot10^{-08}B_{10,16}(X) + 2.43954\cdot10^{-08}B_{11,16}(X) + 3.20691\cdot10^{-08}B_{12,16}(X) + 3.97427\cdot10^{-08}B_{13,16}(X) + 4.74164\cdot10^{-08}B_{14,16}(X) + 5.509\cdot10^{-08}B_{15,16}(X) + 6.27637\cdot10^{-08}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = 2.51463\cdot10^{-22}X^{3} - 3.17637\cdot10^{-22}X^{2} + 1.22778\cdot10^{-07}X - 6.00146\cdot10^{-08}$$
$$= -6.00146\cdot10^{-08}B_{0,3} - 1.90885\cdot10^{-08}B_{1,3} + 2.18376\cdot10^{-08}B_{2,3} + 6.27637\cdot10^{-08}B_{3,3}$$

$$\widetilde{q}_3 = 4.01426\cdot10^{-10}X^{16} - 3.29623\cdot10^{-09}X^{15} + 1.2317\cdot10^{-08}X^{14} - 2.77213\cdot10^{-08}X^{13} + 4.19074\cdot10^{-08}X^{12} - 4.49047\cdot10^{-08}X^{11} + 3.50457\cdot10^{-08}X^{10} - 2.01341\cdot10^{-08}X^{9} + 8.49676\cdot10^{-09}X^{8} - 2.5992\cdot10^{-09}X^{7} + 5.63364\cdot10^{-10}X^{6} - 8.40231\cdot10^{-11}X^{5} + 8.33259\cdot10^{-12}X^{4} - 5.16965\cdot10^{-13}X^{3} + 1.7395\cdot10^{-14}X^{2} + 1.22778\cdot10^{-07}X - 6.00146\cdot10^{-08}$$
$$= -6.00146\cdot10^{-08}B_{0,16} - 5.2341\cdot10^{-08}B_{1,16} - 4.46674\cdot10^{-08}B_{2,16} - 3.69937\cdot10^{-08}B_{3,16} - 2.93201\cdot10^{-08}B_{4,16} - 2.16464\cdot10^{-08}B_{5,16} - 1.39728\cdot10^{-08}B_{6,16} - 6.29914\cdot10^{-09}B_{7,16} + 1.37452\cdot10^{-09}B_{8,16} + 9.04815\cdot10^{-09}B_{9,16} + 1.67218\cdot10^{-08}B_{10,16} + 2.43954\cdot10^{-08}B_{11,16} + 3.20691\cdot10^{-08}B_{12,16} + 3.97427\cdot10^{-08}B_{13,16} + 4.74164\cdot10^{-08}B_{14,16} + 5.509\cdot10^{-08}B_{15,16} + 6.27637\cdot10^{-08}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 7.90061 \cdot 10^{-15}$.
Bounding polynomials M and m:

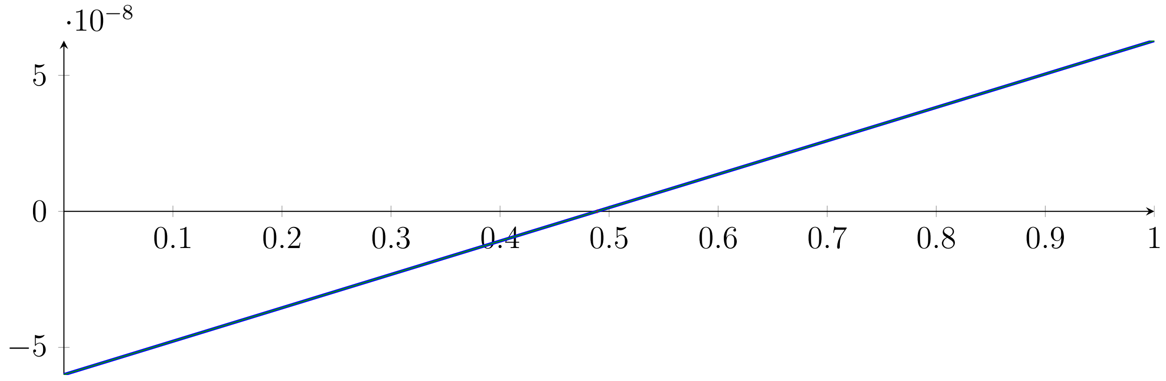
$$M = 2.38228 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08}$$

$$m = 2.51463 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08}$$

Root of M and m:

$$N(M) = \{0.488805\} \qquad N(m) = \{0.488805\}$$

Intersection intervals:



$$[0.488805, 0.488805]$$

Longest intersection interval: $1.3086 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

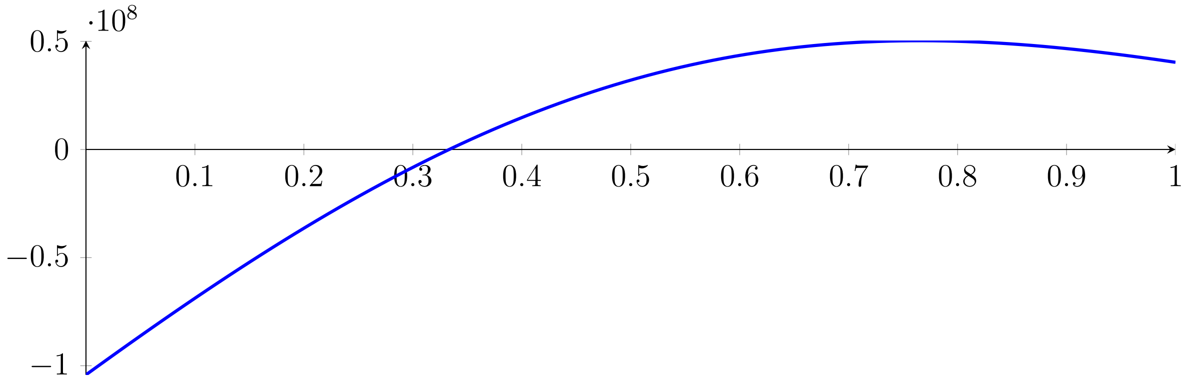
78.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

78.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

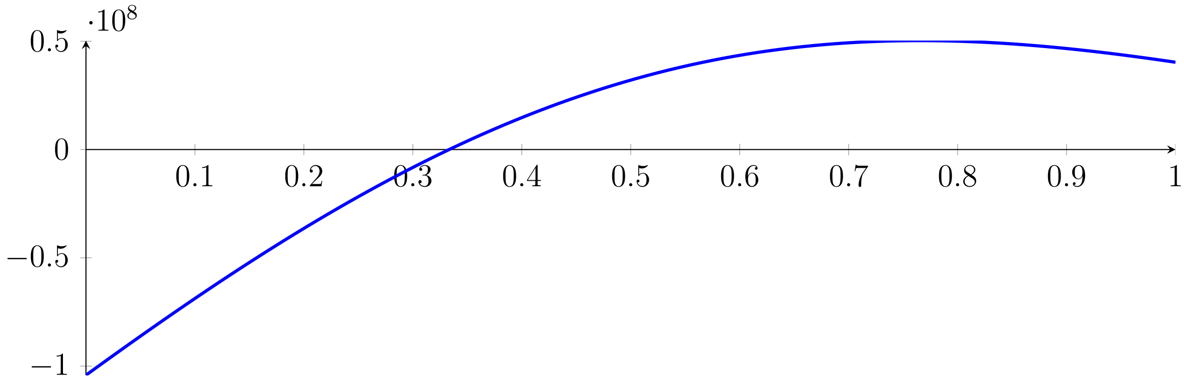
with precision $\varepsilon = 1\cdot 10^{-32}$.

79 Running `BezClip` on f_{16} with epsilon 64

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

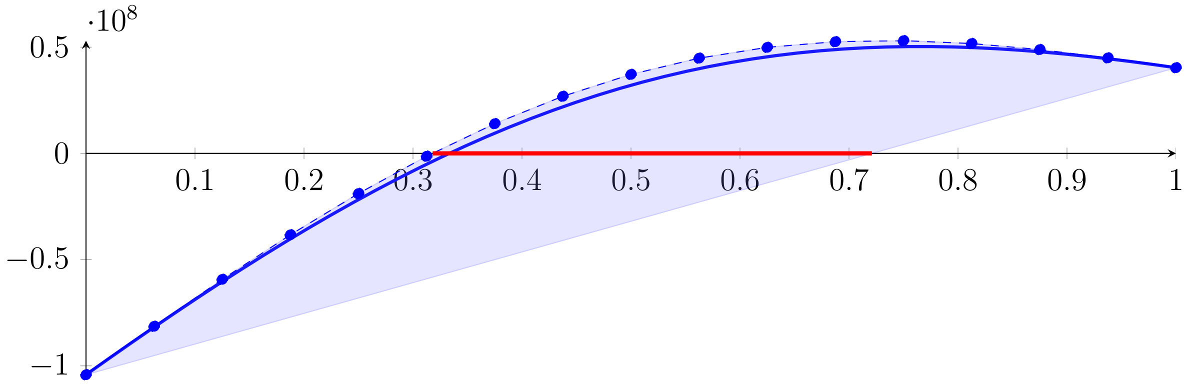
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



79.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$
$$= -1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203\cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X) + 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901\cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X) + 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

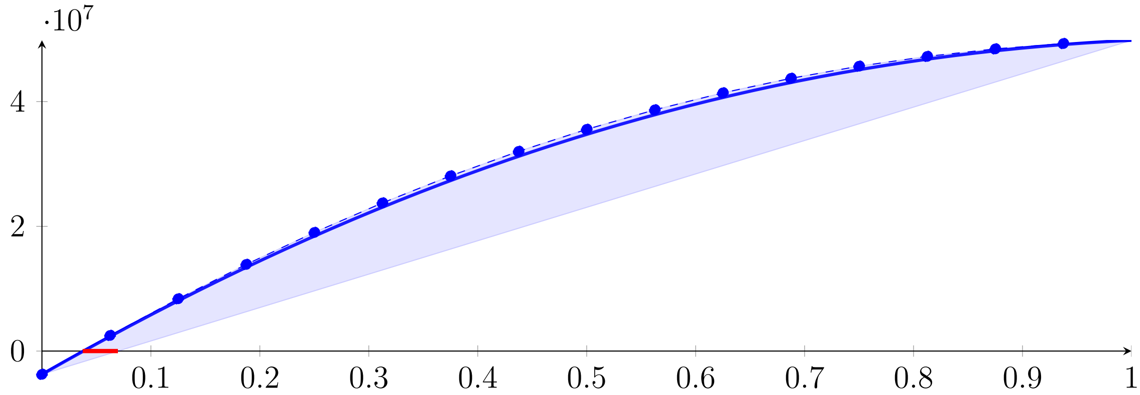
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

79.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p = &-0.00483322X^{16} + 0.0186809X^{15} - 0.0194312X^{14} - 0.0738695X^{13} - 1.11673X^{12} \\ &- 5.0471X^{11} + 36.3082X^{10} + 692.914X^{9} + 1886.96X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5} \\ &+ 3.91945\cdot 10^{6}X^{4} - 7.54561\cdot 10^{6}X^{3} - 4.29225\cdot 10^{7}X^{2} + 9.97982\cdot 10^{7}X - 3.73501\cdot 10^{6} \\ = &-3.73501\cdot 10^{6}B_{0,16}(X) + 2.50237\cdot 10^{6}B_{1,16}(X) + 8.38207\cdot 10^{6}B_{2,16}(X) + 1.38906 \\ &\cdot 10^{7}B_{3,16}(X) + 1.90167\cdot 10^{7}B_{4,16}(X) + 2.37512\cdot 10^{7}B_{5,16}(X) + 2.80875\cdot 10^{7}B_{6,16}(X) \\ &+ 3.20213\cdot 10^{7}B_{7,16}(X) + 3.55508\cdot 10^{7}B_{8,16}(X) + 3.86768\cdot 10^{7}B_{9,16}(X) + 4.14025 \\ &\cdot 10^{7}B_{10,16}(X) + 4.37336\cdot 10^{7}B_{11,16}(X) + 4.56783\cdot 10^{7}B_{12,16}(X) + 4.7247\cdot 10^{7}B_{13,16}(X) \\ &+ 4.84524\cdot 10^{7}B_{14,16}(X) + 4.93093\cdot 10^{7}B_{15,16}(X) + 4.98343\cdot 10^{7}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

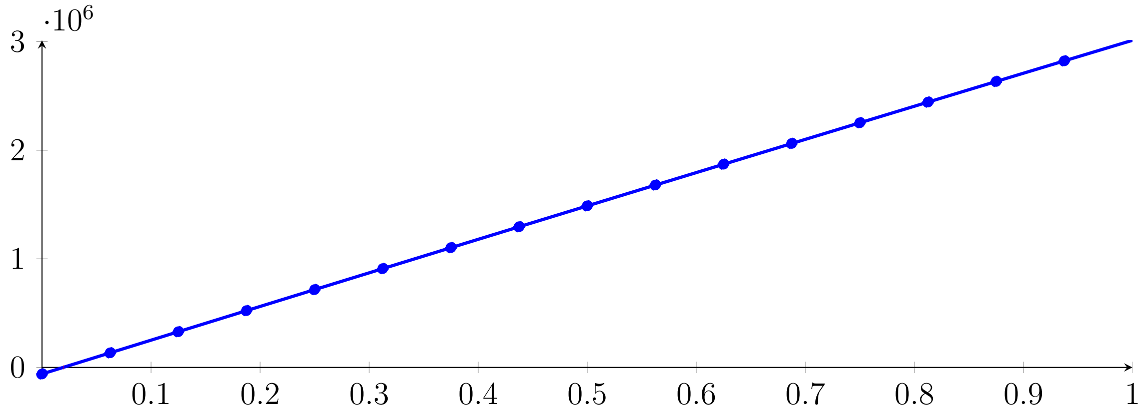
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

79.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p = &-0.000205056X^{16} + 0.000776167X^{15} - 0.000863333X^{14} - 0.000217499X^{13} - 0.00809276X^{12} \\ &- 5.09773\cdot 10^{-05}X^{11} - 0.00564923X^{10} - 0.000162811X^{9} - 0.000215376X^{8} - 3.32948\cdot 10^{-07}X^{7} \\ &- 0.000259866X^{6} + 0.0161077X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot 10^{6}X - 60508.5 \\ = &-60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X) \\ &+ 910202B_{5,16}(X) + 1.1032\cdot 10^{6}B_{6,16}(X) + 1.29582\cdot 10^{6}B_{7,16}(X) + 1.48805\cdot 10^{6}B_{8,16}(X) \\ &+ 1.6799\cdot 10^{6}B_{9,16}(X) + 1.87137\cdot 10^{6}B_{10,16}(X) + 2.06245\cdot 10^{6}B_{11,16}(X) + 2.25315\cdot 10^{6}B_{12,16}(X) \\ &+ 2.44346\cdot 10^{6}B_{13,16}(X) + 2.63339\cdot 10^{6}B_{14,16}(X) + 2.82293\cdot 10^{6}B_{15,16}(X) + 3.01209\cdot 10^{6}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

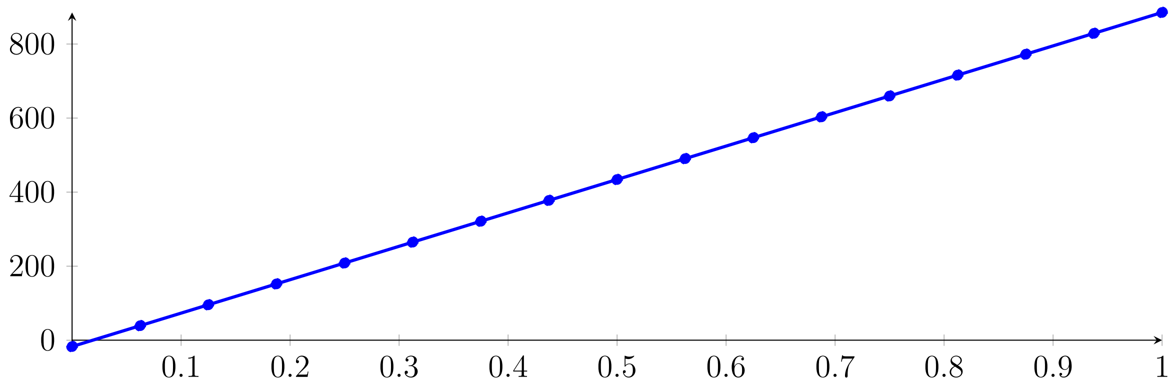
Longest intersection interval: 0.000289554

$\Longrightarrow$ Selective recursion: interval 1: [0.333333, 0.333337],

79.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333337]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p = &-5.9692\cdot 10^{-08}X^{16} + 2.16103\cdot 10^{-07}X^{15} - 2.28456\cdot 10^{-07}X^{14} - 1.17238\cdot 10^{-07}X^{13}\\ &- 2.29525\cdot 10^{-06}X^{12} - 8.31778\cdot 10^{-08}X^{11} - 1.74251\cdot 10^{-06}X^{10} - 9.42919\cdot 10^{-08}X^{9}\\ &- 7.38891\cdot 10^{-08}X^{8} + 3.25144\cdot 10^{-09}X^{7} - 2.61741\cdot 10^{-08}X^{6} + 7.44876\cdot 10^{-10}X^{5}\\ &- 2.58638\cdot 10^{-10}X^{4} - 5.65024\cdot 10^{-09}X^{3} - 0.00382618X^{2} + 902.448X - 17.178\\ = &-17.178B_{0,16}(X) + 39.225B_{1,16}(X) + 95.6279B_{2,16}(X) + 152.031B_{3,16}(X) + 208.434B_{4,16}(X)\\ &+ 264.837B_{5,16}(X) + 321.24B_{6,16}(X) + 377.642B_{7,16}(X) + 434.045B_{8,16}(X)\\ &+ 490.448B_{9,16}(X) + 546.851B_{10,16}(X) + 603.253B_{11,16}(X) + 659.656B_{12,16}(X)\\ &+ 716.059B_{13,16}(X) + 772.461B_{14,16}(X) + 828.864B_{15,16}(X) + 885.266B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

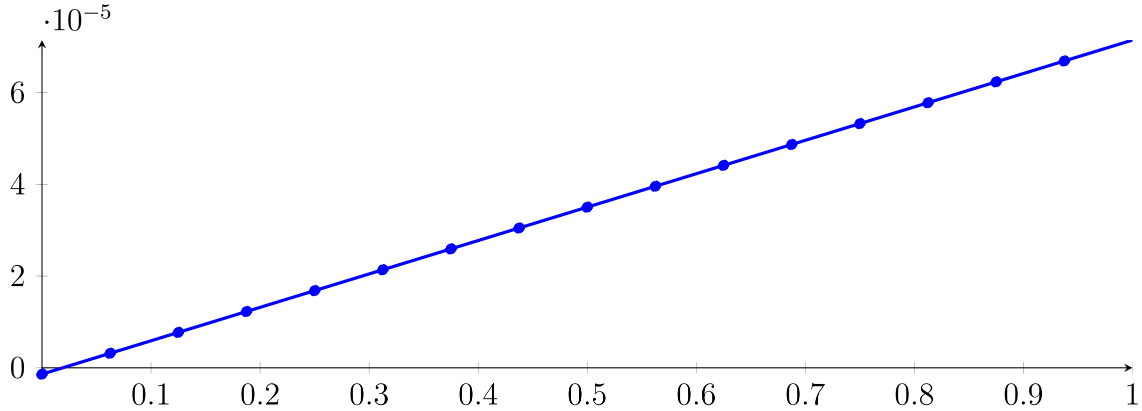
Longest intersection interval: $8.07045\cdot 10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: [0.333333, 0.333333],

79.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -4.80379\cdot10^{-15}X^{16} + 1.74137\cdot10^{-14}X^{15} - 1.72656\cdot10^{-14}X^{14} - 6.84186\cdot10^{-15}X^{13} - 1.91627\cdot10^{-13}X^{12} - 4.7358\cdot10^{-15}X^{11} - 1.33436\cdot10^{-13}X^{10} - 1.97677\cdot10^{-15}X^{9} - 7.4565\cdot10^{-15}X^{8} - 1.16281\cdot10^{-16}X^{7} - 1.98065\cdot10^{-15}X^{6} + 1.47994\cdot10^{-17}X^{5} - 1.84992\cdot10^{-17}X^{4} - 2.48011\cdot10^{-17}X^{2} + 7.28316\cdot10^{-05}X - 1.38634\cdot10^{-06}$$
$$= -1.38634\cdot10^{-06}B_{0,16}(X) + 3.16564\cdot10^{-06}B_{1,16}(X) + 7.71761\cdot10^{-06}B_{2,16}(X) + 1.22696\cdot10^{-05}B_{3,16}(X) + 1.68216\cdot10^{-05}B_{4,16}(X) + 2.13735\cdot10^{-05}B_{5,16}(X) + 2.59255\cdot10^{-05}B_{6,16}(X) + 3.04775\cdot10^{-05}B_{7,16}(X) + 3.50295\cdot10^{-05}B_{8,16}(X) + 3.95814\cdot10^{-05}B_{9,16}(X) + 4.41334\cdot10^{-05}B_{10,16}(X) + 4.86854\cdot10^{-05}B_{11,16}(X) + 5.32374\cdot10^{-05}B_{12,16}(X) + 5.77893\cdot10^{-05}B_{13,16}(X) + 6.23413\cdot10^{-05}B_{14,16}(X) + 6.68933\cdot10^{-05}B_{15,16}(X) + 7.14453\cdot10^{-05}B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.50521\cdot10^{-15}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

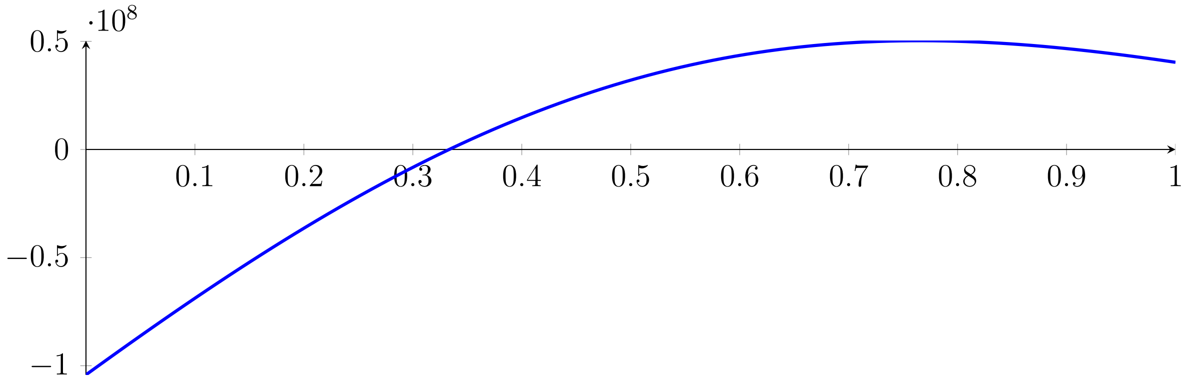
79.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

79.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

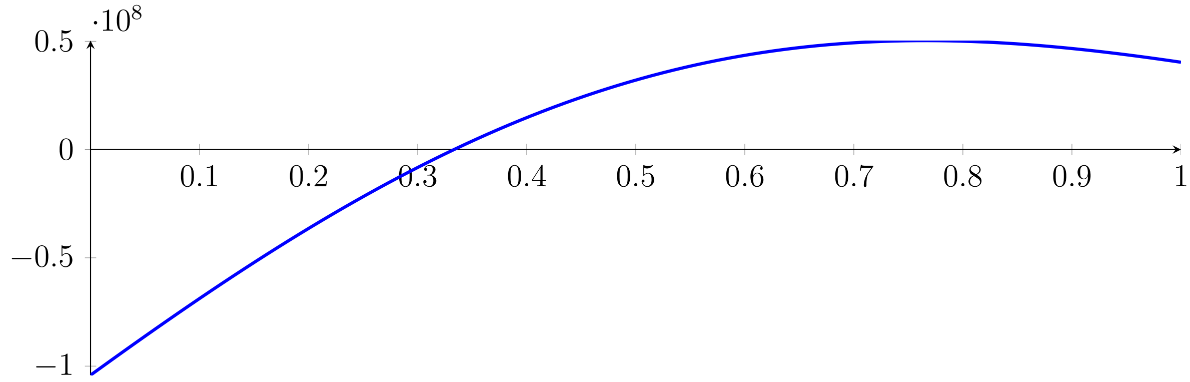
with precision $\varepsilon = 1\cdot 10^{-64}$.

80 Running QuadClip on f_{16} with epsilon 64

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

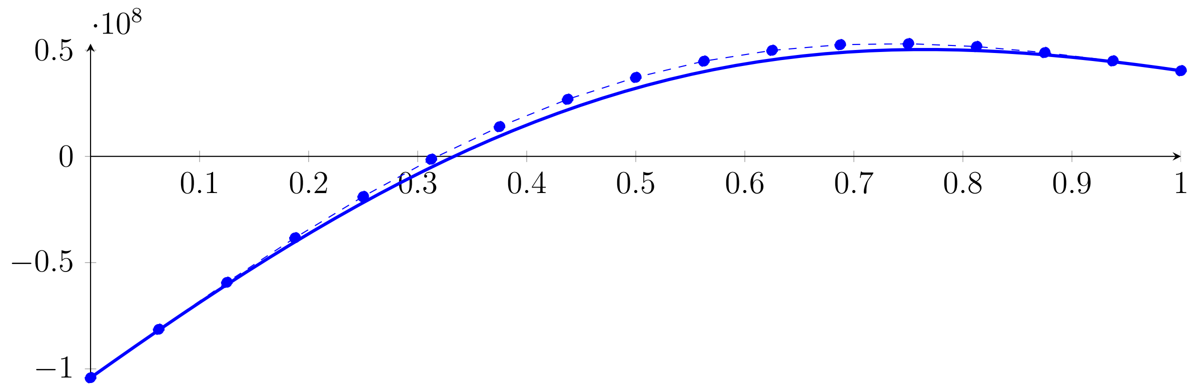
$$\begin{aligned}p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10}\\ &+ 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5}\\ &+ 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}\end{aligned}$$



80.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

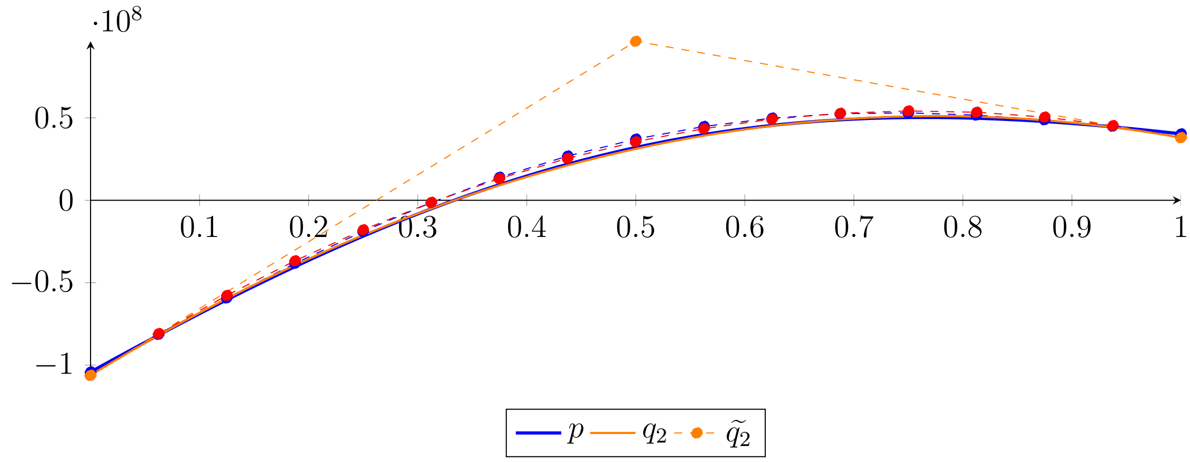
$$\begin{aligned}p = &-0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11}\\ &+ 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\\ &\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}\\ = &-1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203\\ &\cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X)\\ &+ 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901\\ &\cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X)\\ &+ 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -2.61181{\cdot}10^8X^2 + 4.05417{\cdot}10^8X - 1.06197{\cdot}10^8 \\ &= -1.06197{\cdot}10^8B_{0,2} + 9.65115{\cdot}10^7B_{1,2} + 3.80384{\cdot}10^7B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.41787{\cdot}10^6X^{16} + 1.11761{\cdot}10^7X^{15} - 3.98898{\cdot}10^7X^{14} + 8.52437{\cdot}10^7X^{13} - 1.21528{\cdot}10^8X^{12} \\ &\quad + 1.21946{\cdot}10^8X^{11} - 8.86062{\cdot}10^7X^{10} + 4.72904{\cdot}10^7X^9 - 1.86355{\cdot}10^7X^8 + 5.41059{\cdot}10^6X^7 - 1.14487 \\ &\quad {\cdot}10^6X^6 + 172240X^5 - 17636.6X^4 + 1159.74X^3 - 2.61181{\cdot}10^8X^2 + 4.05417{\cdot}10^8X - 1.06197{\cdot}10^8 \\ &= -1.06197{\cdot}10^8B_{0,16} - 8.08583{\cdot}10^7B_{1,16} - 5.76963{\cdot}10^7B_{2,16} - 3.67107{\cdot}10^7B_{3,16} - 1.79017 \\ &\quad {\cdot}10^7B_{4,16} - 1.26923{\cdot}10^6B_{5,16} + 1.31867{\cdot}10^7B_{6,16} + 2.54662{\cdot}10^7B_{7,16} + 3.55691{\cdot}10^7B_{8,16} \\ &\quad + 4.34957{\cdot}10^7B_{9,16} + 4.92455{\cdot}10^7B_{10,16} + 5.2819{\cdot}10^7B_{11,16} + 5.42159{\cdot}10^7B_{12,16} \\ &\quad + 5.34363{\cdot}10^7B_{13,16} + 5.04802{\cdot}10^7B_{14,16} + 4.53476{\cdot}10^7B_{15,16} + 3.80384{\cdot}10^7B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233{\cdot}10^6$.

Bounding polynomials M and m:

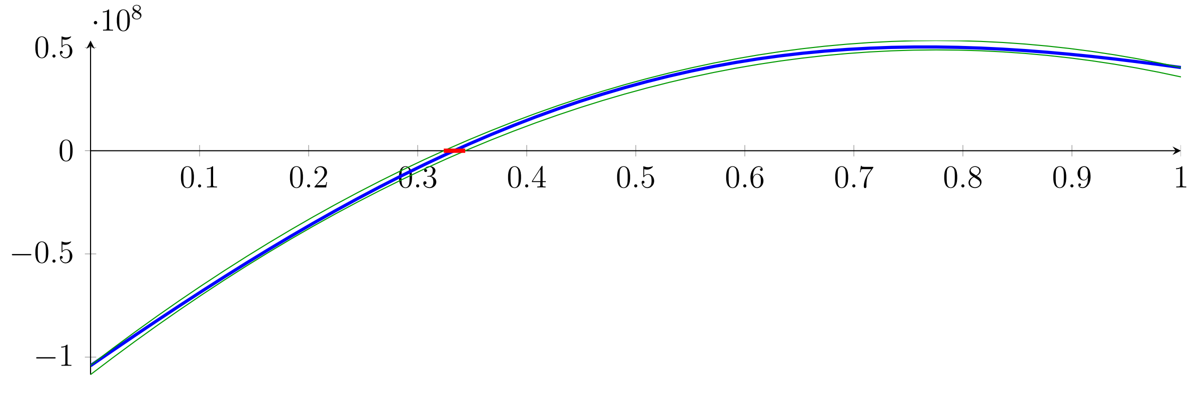
$$M = -2.61181{\cdot}10^8X^2 + 4.05417{\cdot}10^8X - 1.03924{\cdot}10^8$$

$$m = -2.61181{\cdot}10^8X^2 + 4.05417{\cdot}10^8X - 1.08469{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

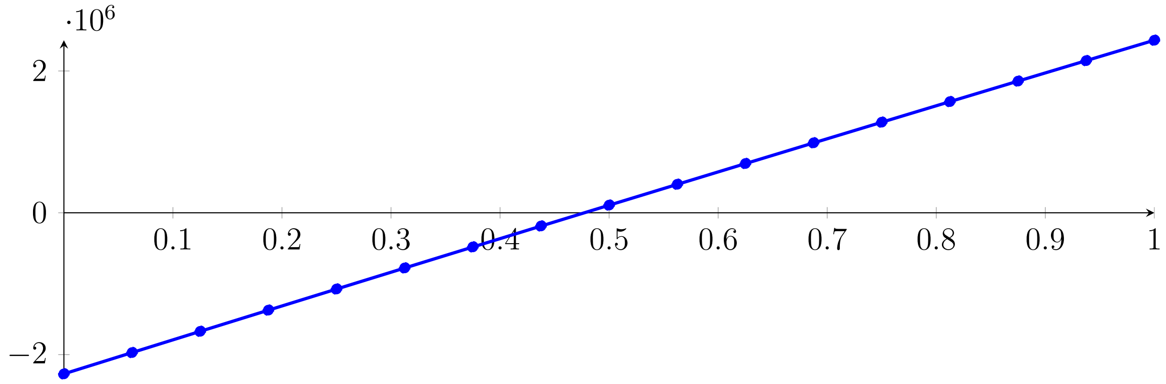
Longest intersection interval: 0.0196686

$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

80.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

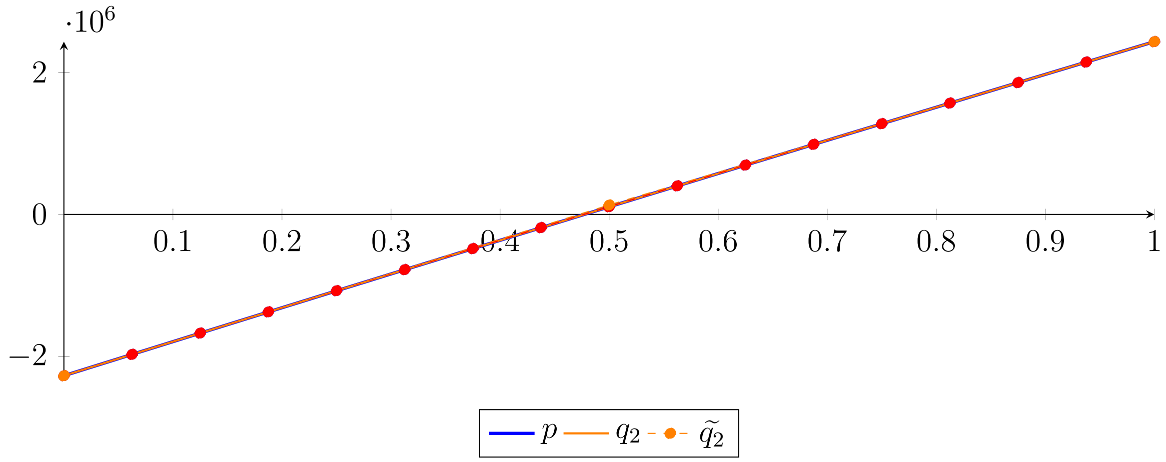
$$\begin{aligned} p = & -2.82438\cdot 10^{-05}X^{16} + 2.90051\cdot 10^{-05}X^{15} + 0.000231285X^{14} + 0.000648014X^{13} + 0.00318916X^{12} \\ & + 0.00323204X^{11} + 0.00460533X^{10} + 0.00220012X^{9} - 0.000101882X^{8} + 5.32717\cdot 10^{-06}X^{7} \\ & - 0.00186824X^{6} + 0.132741X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874\cdot 10^{6}X - 2.2715\cdot 10^{6} \\ = & -2.2715\cdot 10^{6}B_{0,16}(X) - 1.97096\cdot 10^{6}B_{1,16}(X) - 1.67127\cdot 10^{6}B_{2,16}(X) - 1.37244\cdot 10^{6}B_{3,16}(X) \\ & - 1.07448\cdot 10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) \\ & + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806\cdot 10^{6}B_{12,16}(X) + 1.5682 \\ & \cdot 10^{6}B_{13,16}(X) + 1.85747\cdot 10^{6}B_{14,16}(X) + 2.14586\cdot 10^{6}B_{15,16}(X) + 2.43338\cdot 10^{6}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -104265X^{2} + 4.80923\cdot 10^{6}X - 2.27154\cdot 10^{6} \\ &= -2.27154\cdot 10^{6}B_{0,2} + 133069B_{1,2} + 2.43342\cdot 10^{6}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 = & \, 43556.7X^{16} - 347904X^{15} + 1.2616\cdot 10^{6}X^{14} - 2.74835\cdot 10^{6}X^{13} + 4.01042\cdot 10^{6}X^{12} - 4.13709 \\ & \cdot 10^{6}X^{11} + 3.10251\cdot 10^{6}X^{10} - 1.71209\cdot 10^{6}X^{9} + 695531X^{8} - 205965X^{7} + 43591.2X^{6} \\ & - 6402.94X^{5} + 625.678X^{4} - 37.485X^{3} - 104264X^{2} + 4.80923\cdot 10^{6}X - 2.27154\cdot 10^{6} \\ = & -2.27154\cdot 10^{6}B_{0,16} - 1.97097\cdot 10^{6}B_{1,16} - 1.67126\cdot 10^{6}B_{2,16} - 1.37242\cdot 10^{6}B_{3,16} \\ & - 1.07445\cdot 10^{6}B_{4,16} - 777350B_{5,16} - 481117B_{6,16} - 185755B_{7,16} + 108741B_{8,16} \\ & + 402365B_{9,16} + 695123B_{10,16} + 987010B_{11,16} + 1.27803\cdot 10^{6}B_{12,16} + 1.56818 \\ & \cdot 10^{6}B_{13,16} + 1.85746\cdot 10^{6}B_{14,16} + 2.14587\cdot 10^{6}B_{15,16} + 2.43342\cdot 10^{6}B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.
Bounding polynomials M and m:

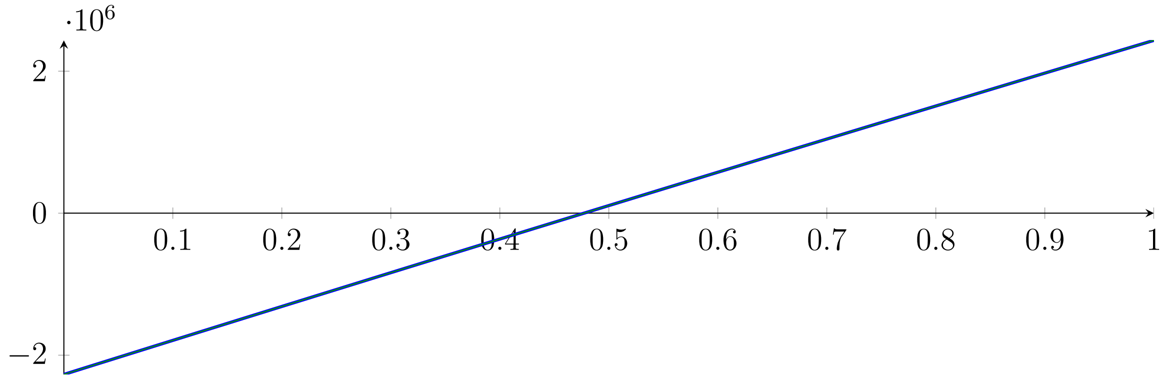
$$M = -104265X^{2} + 4.80923\cdot 10^{6}X - 2.2715\cdot 10^{6}$$

$$m = -104265X^2 + 4.80923{\cdot}10^6 X - 2.27159{\cdot}10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$[0.47726, 0.477278]$

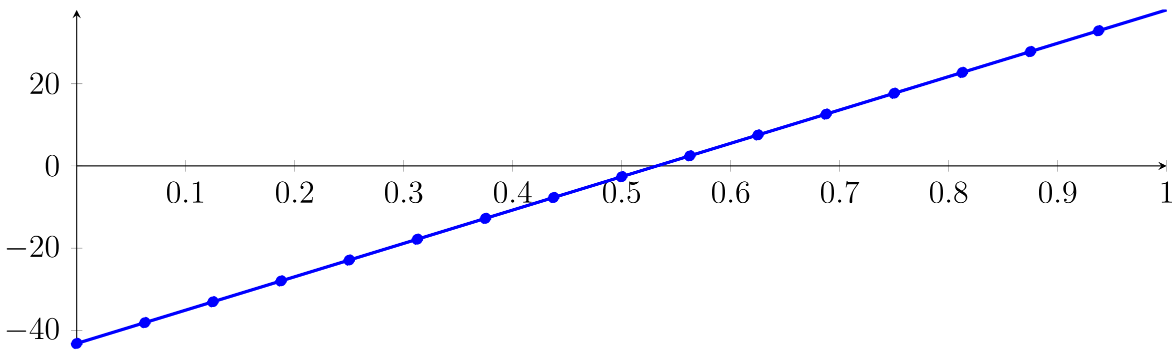
Longest intersection interval: $1.72301 \cdot 10^{-05}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

80.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

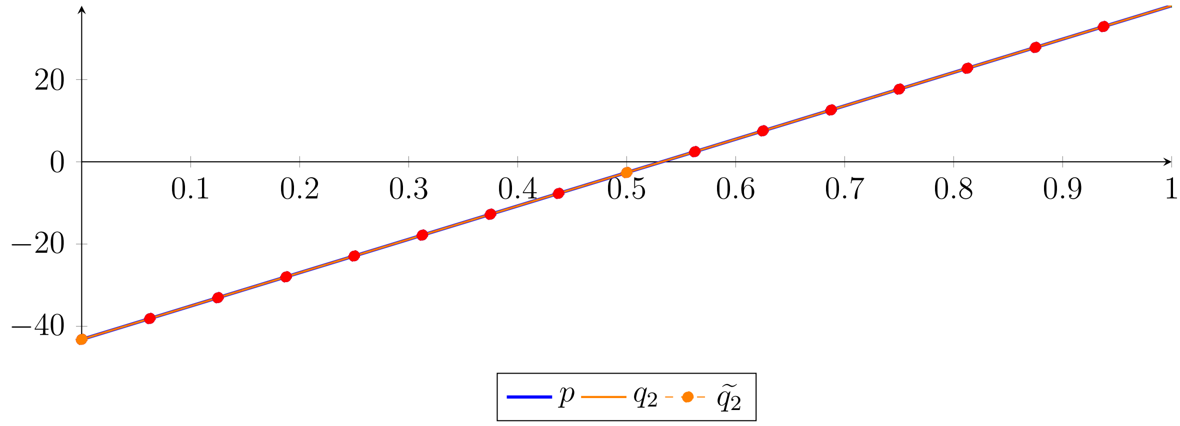
$$\begin{aligned}
p = {} & 8.74252{\cdot}10^{-11}X^{16} - 1.56979{\cdot}10^{-09}X^{15} + 6.68479{\cdot}10^{-09}X^{14} + 1.20008{\cdot}10^{-08}X^{13} + 9.07301{\cdot}10^{-08}X^{12} \\
& + 5.58657{\cdot}10^{-08}X^{11} + 1.13801{\cdot}10^{-07}X^{10} + 3.70665{\cdot}10^{-08}X^{9} + 7.31575{\cdot}10^{-10}X^{8} + 1.30058{\cdot}10^{-09}X^{7} \\
& + 5.00722{\cdot}10^{-09}X^{6} + 1.24146{\cdot}10^{-10}X^{5} + 1.03455{\cdot}10^{-10}X^{4} - 3.09388{\cdot}10^{-05}X^{2} + 81.1505X - 43.1911 \\
= {} & -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\
& - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68777B_{7,16}(X) - 2.61587B_{8,16}(X) \\
& + 2.45604B_{9,16}(X) + 7.52794B_{10,16}(X) + 12.5998B_{11,16}(X) + 17.6718B_{12,16}(X) \\
& + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X)
\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
q_2 &= -3.09388{\cdot}10^{-05}X^2 + 81.1505X - 43.1911 \\
&= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2}
\end{aligned}$$

$$\begin{aligned}
\widetilde{q}_2 = {} & 0.721495X^{16} - 5.74915X^{15} + 20.7933X^{14} - 45.1627X^{13} + 65.6806X^{12} - 67.5044X^{11} \\
& + 50.4286X^{10} - 27.728X^{9} + 11.2318X^{8} - 3.32011X^{7} + 0.702408X^{6} - 0.103415X^{5} \\
& + 0.0102099X^{4} - 0.000624725X^{3} - 1.10834{\cdot}10^{-05}X^{2} + 81.1505X - 43.1911 \\
= {} & -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} \\
& - 12.7597B_{6,16} - 7.68779B_{7,16} - 2.61585B_{8,16} + 2.45602B_{9,16} + 7.52795B_{10,16} + 12.5998B_{11,16} \\
& + 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.57956 \cdot 10^{-05}$.
Bounding polynomials M and m:

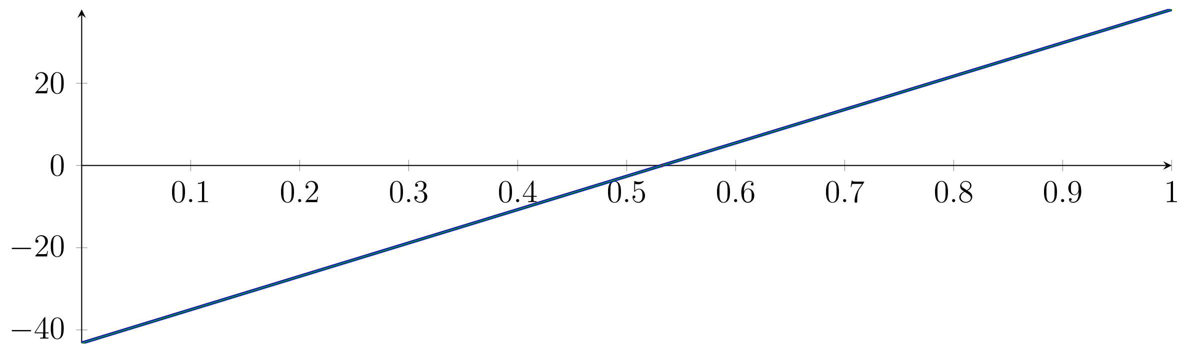
$$M = -3.09388 \cdot 10^{-05} X^2 + 81.1505X - 43.1911$$

$$m = -3.09388 \cdot 10^{-05} X^2 + 81.1505X - 43.1911$$

Root of M and m:

$$N(M) = \{0.532235, 2.62293 \cdot 10^{6}\} \qquad N(m) = \{0.532235, 2.62293 \cdot 10^{6}\}$$

Intersection intervals:



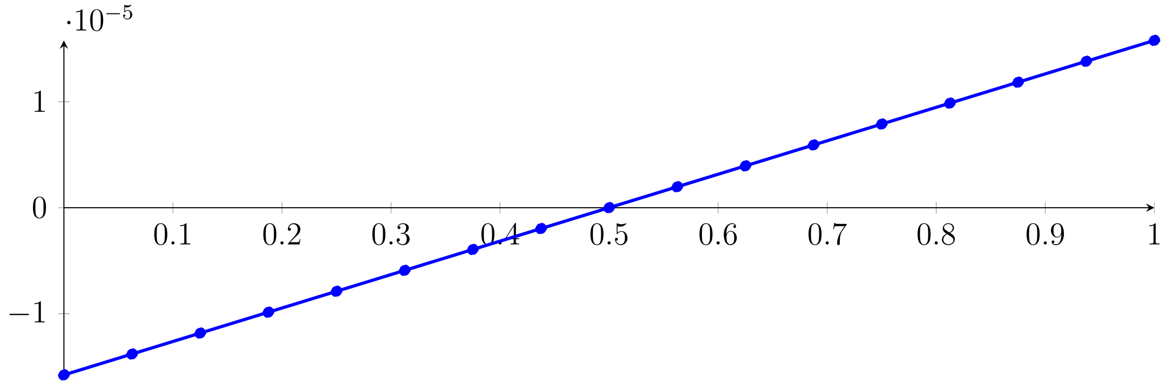
$[0.532235, 0.532235]$

Longest intersection interval: $3.8903 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

80.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

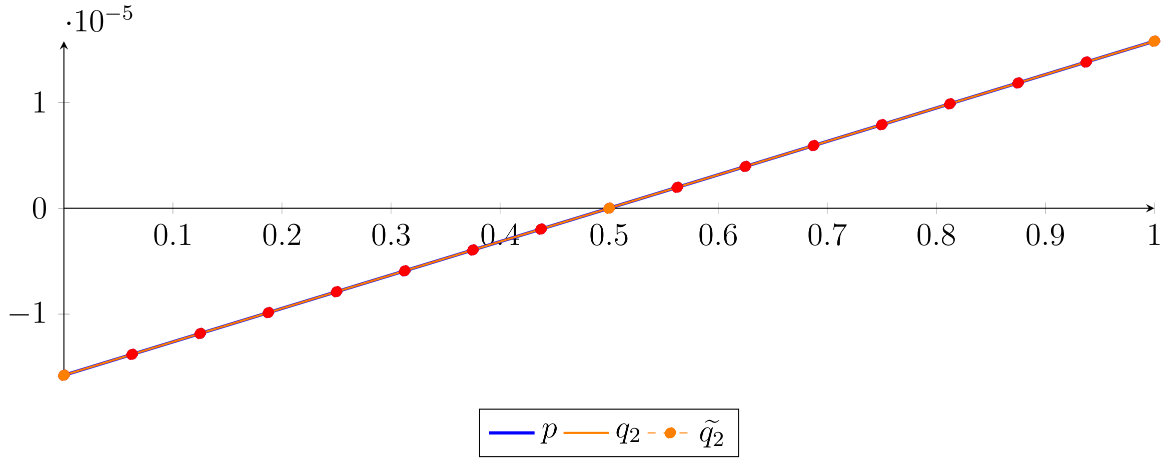
$$\begin{aligned} p = &-1.04409 \cdot 10^{-16} X^{16} - 1.53089 \cdot 10^{-16} X^{15} + 1.74665 \cdot 10^{-15} X^{14} + 5.46438 \cdot 10^{-15} X^{13} \\ &+ 2.56522 \cdot 10^{-14} X^{12} + 2.15479 \cdot 10^{-14} X^{11} + 3.51633 \cdot 10^{-14} X^{10} + 1.67444 \\ &\cdot 10^{-14} X^{9} - 8.72105 \cdot 10^{-16} X^{8} + 1.41087 \cdot 10^{-15} X^{6} + 5.91974 \cdot 10^{-17} X^{5} + 4.93312 \\ &\cdot 10^{-17} X^{4} + 3.79471 \cdot 10^{-18} X^{3} - 4.87891 \cdot 10^{-18} X^{2} + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05} \\ = &-1.57804 \cdot 10^{-05} B_{0,16}(X) - 1.38073 \cdot 10^{-05} B_{1,16}(X) - 1.18341 \cdot 10^{-05} B_{2,16}(X) - 9.86101 \\ &\cdot 10^{-06} B_{3,16}(X) - 7.88788 \cdot 10^{-06} B_{4,16}(X) - 5.91476 \cdot 10^{-06} B_{5,16}(X) - 3.94163 \cdot 10^{-06} B_{6,16}(X) \\ &- 1.96851 \cdot 10^{-06} B_{7,16}(X) + 4.61501 \cdot 10^{-09} B_{8,16}(X) + 1.97774 \cdot 10^{-06} B_{9,16}(X) + 3.95086 \\ &\cdot 10^{-06} B_{10,16}(X) + 5.92399 \cdot 10^{-06} B_{11,16}(X) + 7.89711 \cdot 10^{-06} B_{12,16}(X) + 9.87024 \cdot 10^{-06} B_{13,16}(X) \\ &+ 1.18434 \cdot 10^{-05} B_{14,16}(X) + 1.38165 \cdot 10^{-05} B_{15,16}(X) + 1.57896 \cdot 10^{-05} B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -4.65868{\cdot}10^{-18}X^2 + 3.157{\cdot}10^{-05}X - 1.57804{\cdot}10^{-05} \\ &= -1.57804{\cdot}10^{-05}B_{0,2} + 4.61501{\cdot}10^{-09}B_{1,2} + 1.57896{\cdot}10^{-05}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= 2.92413{\cdot}10^{-07}X^{16} - 2.33332{\cdot}10^{-06}X^{15} + 8.45203{\cdot}10^{-06}X^{14} - 1.83895{\cdot}10^{-05}X^{13} \\ &\quad + 2.67963{\cdot}10^{-05}X^{12} - 2.75995{\cdot}10^{-05}X^{11} + 2.06638{\cdot}10^{-05}X^{10} - 1.13854{\cdot}10^{-05}X^{9} \\ &\quad + 4.61944{\cdot}10^{-06}X^{8} - 1.36687{\cdot}10^{-06}X^{7} + 2.89249{\cdot}10^{-07}X^{6} - 4.25295{\cdot}10^{-08}X^{5} + 4.17283 \\ &\quad {\cdot}10^{-09}X^{4} - 2.52119{\cdot}10^{-10}X^{3} + 7.82992{\cdot}10^{-12}X^{2} + 3.157{\cdot}10^{-05}X - 1.57804{\cdot}10^{-05} \\ &= -1.57804{\cdot}10^{-05}B_{0,16} - 1.38073{\cdot}10^{-05}B_{1,16} - 1.18341{\cdot}10^{-05}B_{2,16} - 9.86101{\cdot}10^{-06}B_{3,16} - 7.88788 \\ &\quad {\cdot}10^{-06}B_{4,16} - 5.91476{\cdot}10^{-06}B_{5,16} - 3.94163{\cdot}10^{-06}B_{6,16} - 1.96851{\cdot}10^{-06}B_{7,16} + 4.62125{\cdot}10^{-09}B_{8,16} \\ &\quad + 1.97773{\cdot}10^{-06}B_{9,16} + 3.95087{\cdot}10^{-06}B_{10,16} + 5.92399{\cdot}10^{-06}B_{11,16} + 7.89711{\cdot}10^{-06}B_{12,16} \\ &\quad + 9.87024{\cdot}10^{-06}B_{13,16} + 1.18434{\cdot}10^{-05}B_{14,16} + 1.38165{\cdot}10^{-05}B_{15,16} + 1.57896{\cdot}10^{-05}B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.24192{\cdot}10^{-12}$.

Bounding polynomials M and m:

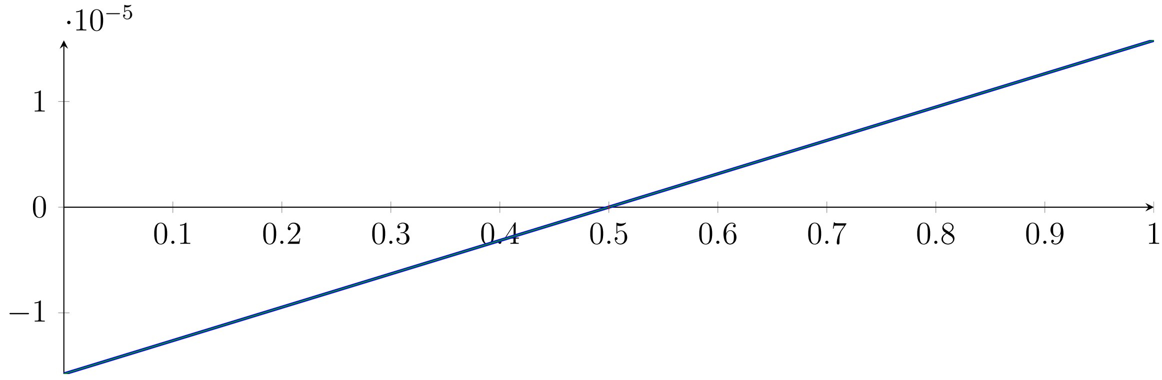
$$M = -4.65868{\cdot}10^{-18}X^2 + 3.157{\cdot}10^{-05}X - 1.57804{\cdot}10^{-05}$$

$$m = -4.65868{\cdot}10^{-18}X^2 + 3.157{\cdot}10^{-05}X - 1.57804{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{0.499636, 6.77659{\cdot}10^{12}\} \qquad N(m) = \{0.500364, 6.77659{\cdot}10^{12}\}$$

Intersection intervals:



$[0.499636, 0.500364]$

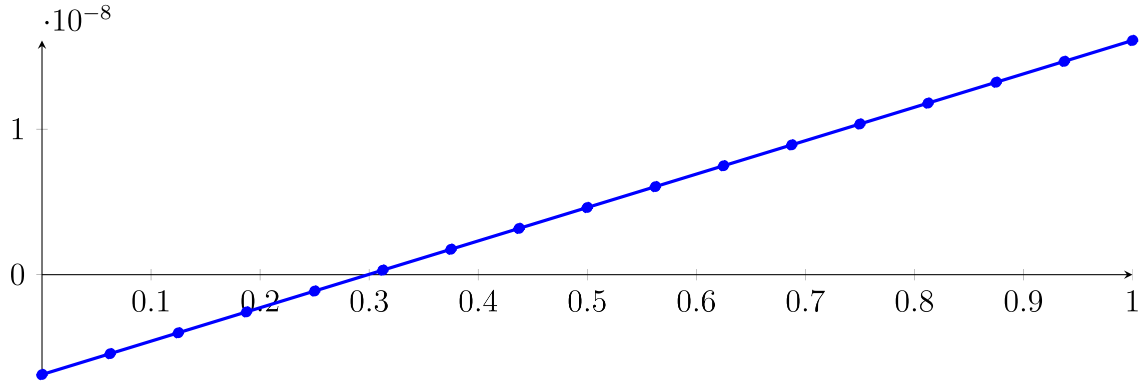
Longest intersection interval: 0.000727273

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

80.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

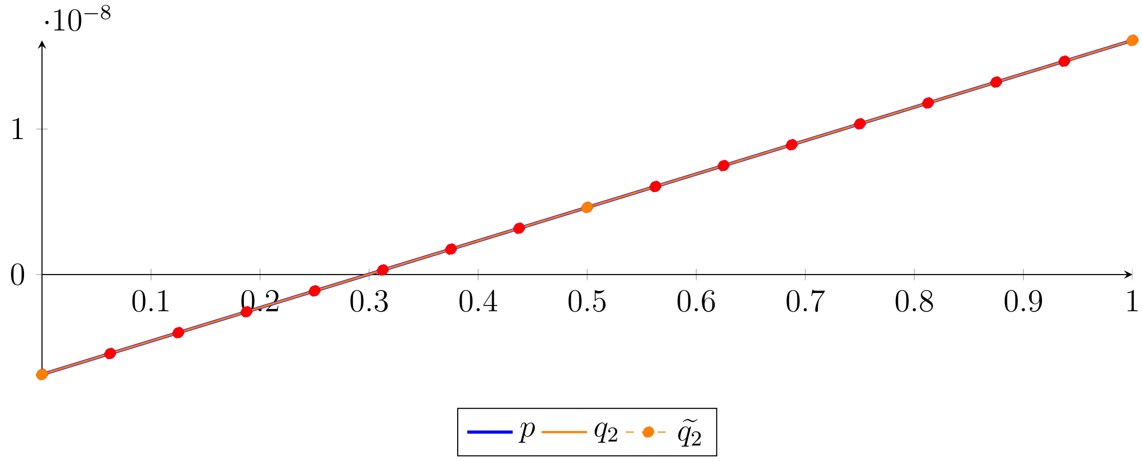
$$\begin{aligned} p = & -6.76104\cdot 10^{-19}X^{16} + 2.21478\cdot 10^{-18}X^{15} - 1.59295\cdot 10^{-18}X^{14} + 1.03762\cdot 10^{-18}X^{13} \\ & - 1.34649\cdot 10^{-17}X^{12} + 9.65427\cdot 10^{-18}X^{11} - 2.75561\cdot 10^{-18}X^{10} + 5.90488\cdot 10^{-18}X^{9} \\ & - 8.51665\cdot 10^{-19}X^{8} + 3.02814\cdot 10^{-19}X^{7} + 2.64962\cdot 10^{-19}X^{6} + 2.8905\cdot 10^{-20}X^{5} \\ & + 6.02187\cdot 10^{-21}X^{4} + 9.26442\cdot 10^{-22}X^{3} + 2.296\cdot 10^{-08}X - 6.86499\cdot 10^{-09} \\ = & -6.86499\cdot 10^{-09}B_{0,16}(X) - 5.42999\cdot 10^{-09}B_{1,16}(X) - 3.99499\cdot 10^{-09}B_{2,16}(X) \\ & - 2.55999\cdot 10^{-09}B_{3,16}(X) - 1.12499\cdot 10^{-09}B_{4,16}(X) + 3.10008\cdot 10^{-10}B_{5,16}(X) + 1.74501 \\ & \cdot 10^{-09}B_{6,16}(X) + 3.18001\cdot 10^{-09}B_{7,16}(X) + 4.61501\cdot 10^{-09}B_{8,16}(X) + 6.05001\cdot 10^{-09}B_{9,16}(X) \\ & + 7.48501\cdot 10^{-09}B_{10,16}(X) + 8.92001\cdot 10^{-09}B_{11,16}(X) + 1.0355\cdot 10^{-08}B_{12,16}(X) + 1.179 \\ & \cdot 10^{-08}B_{13,16}(X) + 1.3225\cdot 10^{-08}B_{14,16}(X) + 1.466\cdot 10^{-08}B_{15,16}(X) + 1.6095\cdot 10^{-08}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 = & \ 1.40621\cdot 10^{-23}X^2 + 2.296\cdot 10^{-08}X - 6.86499\cdot 10^{-09} \\ = & -6.86499\cdot 10^{-09}B_{0,2} + 4.61501\cdot 10^{-09}B_{1,2} + 1.6095\cdot 10^{-08}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 = & \ 2.65578\cdot 10^{-10}X^{16} - 2.13329\cdot 10^{-09}X^{15} + 7.78369\cdot 10^{-09}X^{14} - 1.70728\cdot 10^{-08}X^{13} \\ & + 2.51029\cdot 10^{-08}X^{12} - 2.61095\cdot 10^{-08}X^{11} + 1.97446\cdot 10^{-08}X^{10} - 1.09796\cdot 10^{-08}X^{9} \\ & + 4.48709\cdot 10^{-09}X^{8} - 1.33355\cdot 10^{-09}X^{7} + 2.82501\cdot 10^{-10}X^{6} - 4.12963\cdot 10^{-11}X^{5} + 3.94088 \\ & \cdot 10^{-12}X^{4} - 2.24328\cdot 10^{-13}X^{3} + 6.17064\cdot 10^{-15}X^{2} + 2.296\cdot 10^{-08}X - 6.86499\cdot 10^{-09} \\ = & -6.86499\cdot 10^{-09}B_{0,16} - 5.42999\cdot 10^{-09}B_{1,16} - 3.99499\cdot 10^{-09}B_{2,16} - 2.55999\cdot 10^{-09}B_{3,16} \\ & - 1.12499\cdot 10^{-09}B_{4,16} + 3.10006\cdot 10^{-10}B_{5,16} + 1.74501\cdot 10^{-09}B_{6,16} + 3.18\cdot 10^{-09}B_{7,16} + 4.61501 \\ & \cdot 10^{-09}B_{8,16} + 6.05\cdot 10^{-09}B_{9,16} + 7.48501\cdot 10^{-09}B_{10,16} + 8.92\cdot 10^{-09}B_{11,16} + 1.0355\cdot 10^{-08}B_{12,16} \\ & + 1.179\cdot 10^{-08}B_{13,16} + 1.3225\cdot 10^{-08}B_{14,16} + 1.466\cdot 10^{-08}B_{15,16} + 1.6095\cdot 10^{-08}B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.35405 \cdot 10^{-15}$.
Bounding polynomials M and m:

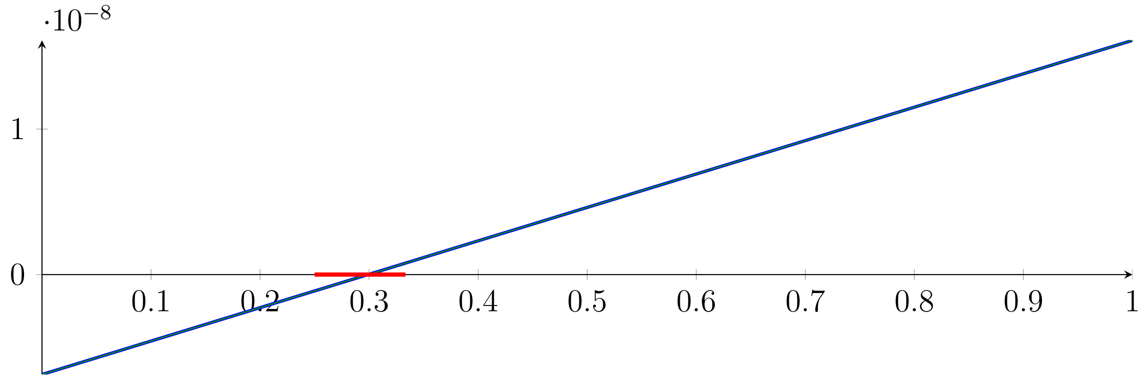
$$M = 1.32349 \cdot 10^{-23} X^2 + 2.296 \cdot 10^{-08} X - 6.86498 \cdot 10^{-09}$$

$$m = 1.48893 \cdot 10^{-23} X^2 + 2.296 \cdot 10^{-08} X - 6.865 \cdot 10^{-09}$$

Root of M and m:

$$N(M) = \{-1.73481 \cdot 10^{15}, 0.25\} \qquad N(m) = \{-1.54205 \cdot 10^{15}, 0.333333\}$$

Intersection intervals:



$$[0.333333, 0.25]$$

Longest intersection interval: -0.0833333
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

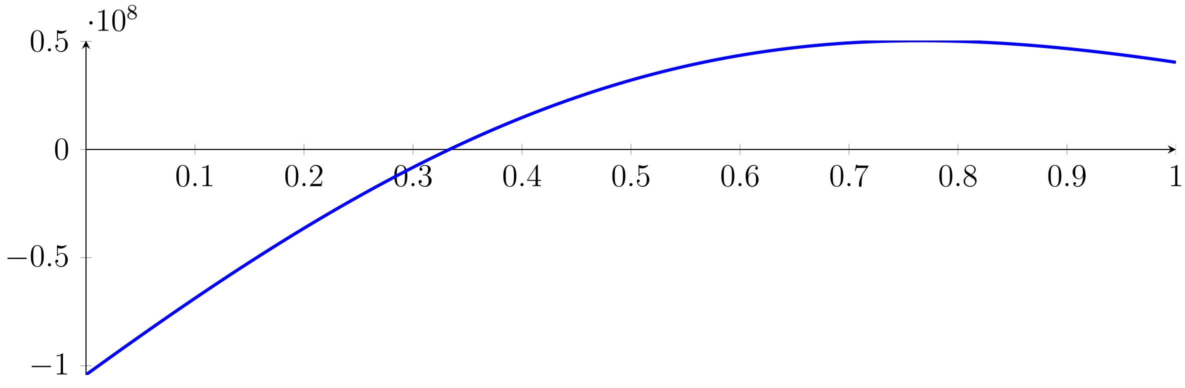
80.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

80.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

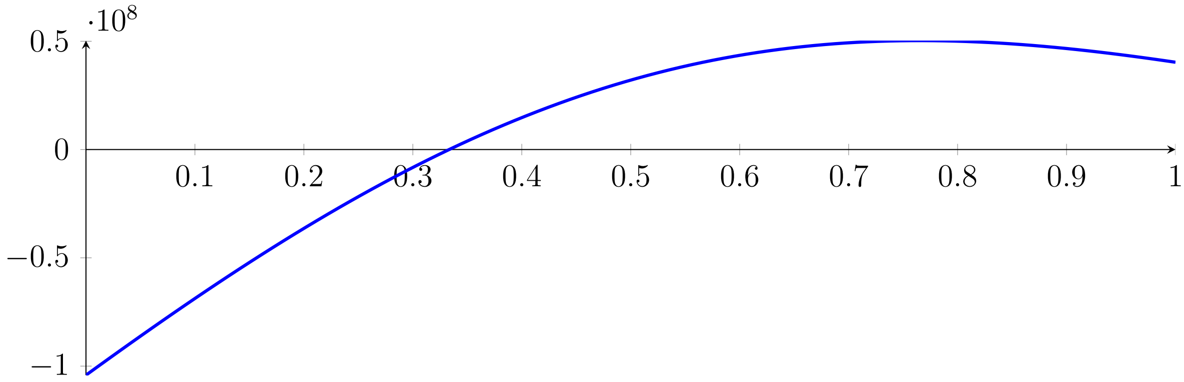
with precision $\varepsilon = 1\cdot10^{-64}$.

81 Running `CubeClip` on f_{16} with epsilon 64

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

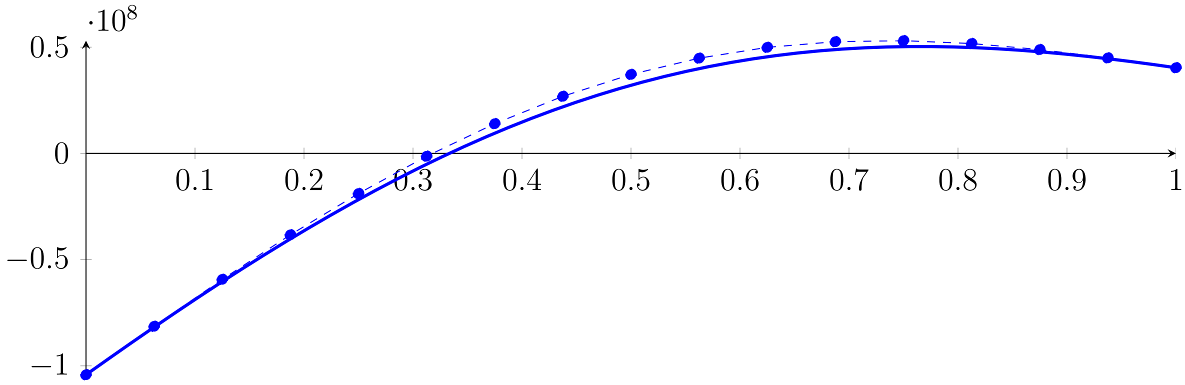
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



81.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

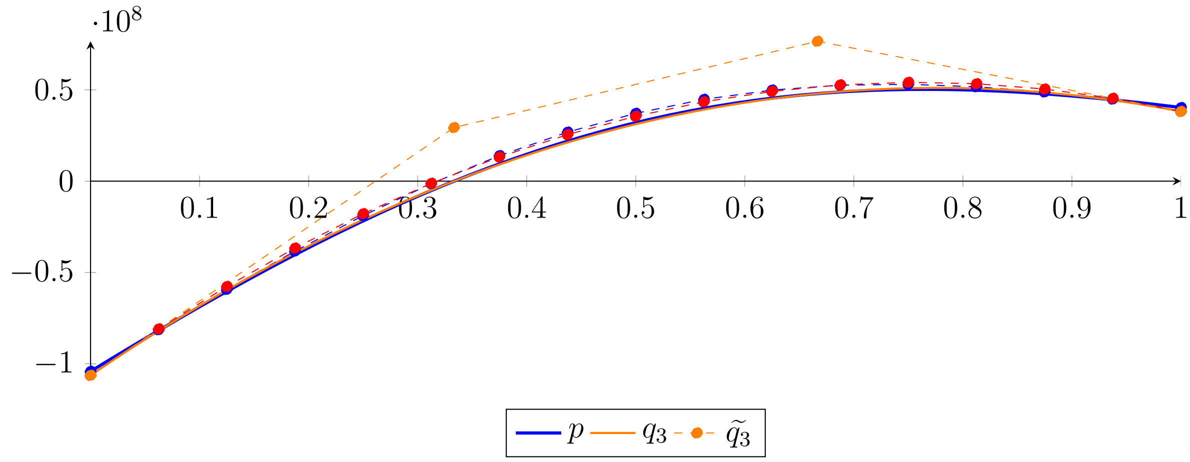
$$p = -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$
$$= -1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203\cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) + 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901\cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) + 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.06335{\cdot}10^8$$
$$= -1.06335{\cdot}10^8B_{0,3} + 2.93558{\cdot}10^7B_{1,3} + 7.66068{\cdot}10^7B_{2,3} + 3.81764{\cdot}10^7B_{3,3}$$

$$\widetilde{q}_3 = 2.11088{\cdot}10^6X^{16} - 1.70537{\cdot}10^7X^{15} + 6.25913{\cdot}10^7X^{14} - 1.38111{\cdot}10^8X^{13} + 2.043{\cdot}10^8X^{12} - 2.13799{\cdot}10^8X^{11} + 1.62714{\cdot}10^8X^{10} - 9.11092{\cdot}10^7X^9 + 3.75254{\cdot}10^7X^8 - 1.12529{\cdot}10^7X^7 + 2.40899{\cdot}10^6X^6 - 357156X^5 + 34986.1X^4 + 2.75602{\cdot}10^6X^3 - 2.65318{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.06335{\cdot}10^8$$
$$= -1.06335{\cdot}10^8B_{0,16} - 8.08928{\cdot}10^7B_{1,16} - 5.76618{\cdot}10^7B_{2,16} - 3.66369{\cdot}10^7B_{3,16} - 1.78131{\cdot}10^7B_{4,16} - 1.18553{\cdot}10^6B_{5,16} + 1.32508{\cdot}10^7B_{6,16} + 2.55007{\cdot}10^7B_{7,16} + 3.55692{\cdot}10^7B_{8,16} + 4.34611{\cdot}10^7B_{9,16} + 4.91816{\cdot}10^7B_{10,16} + 5.27352{\cdot}10^7B_{11,16} + 5.41273{\cdot}10^7B_{12,16} + 5.33624{\cdot}10^7B_{13,16} + 5.04457{\cdot}10^7B_{14,16} + 4.53821{\cdot}10^7B_{15,16} + 3.81764{\cdot}10^7B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806{\cdot}10^6$.

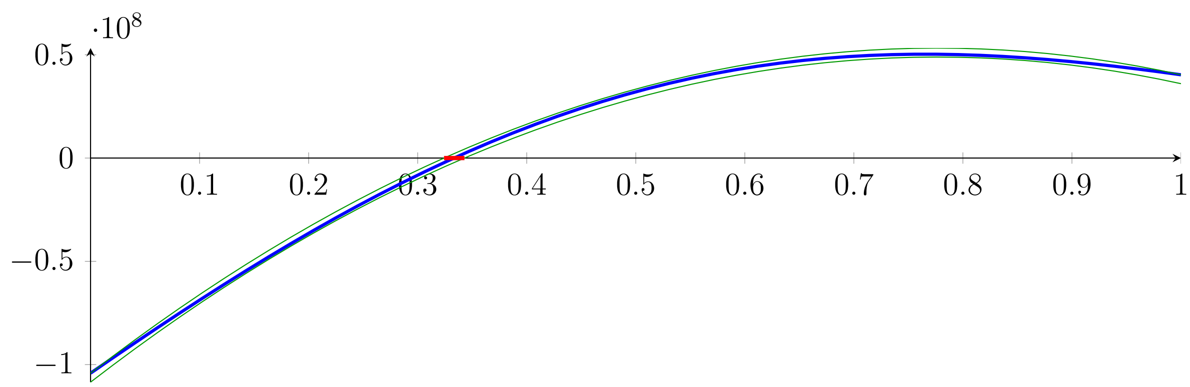
Bounding polynomials M and m:

$$M = 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.04167{\cdot}10^8$$
$$m = 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.08503{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

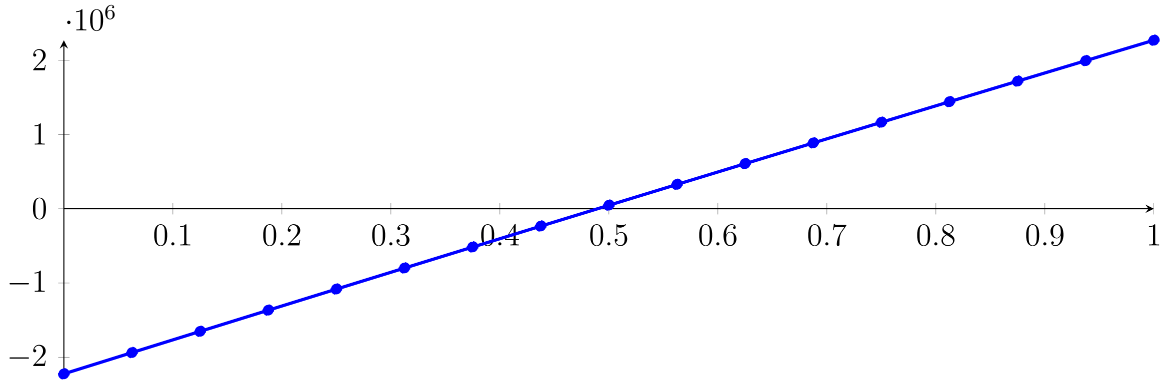
Longest intersection interval: 0.0187703

$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

81.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

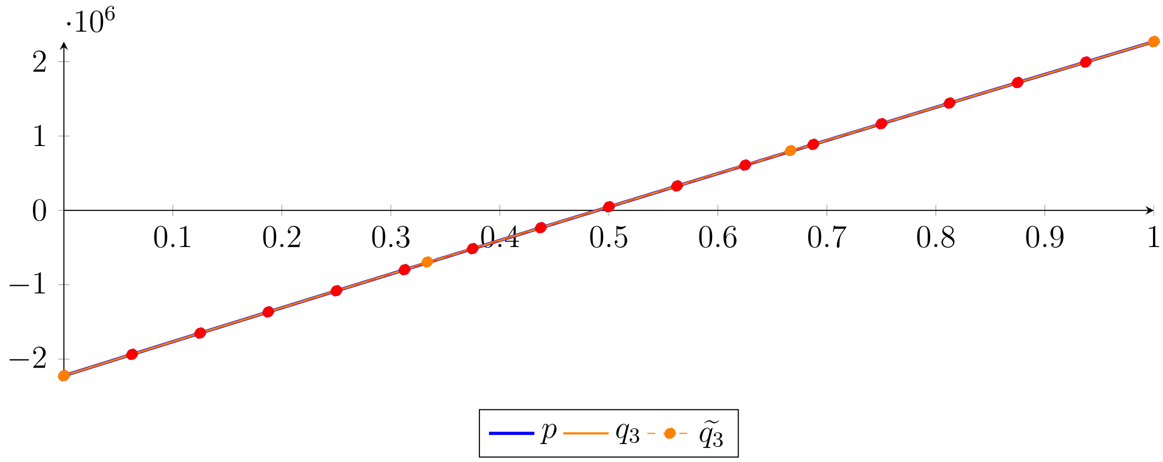
$$\begin{aligned}p = &-1.93035\cdot10^{-05}X^{16} + 1.08927\cdot10^{-05}X^{15} + 0.000255816X^{14} + 0.000616983X^{13} + 0.00371715X^{12}\\&+ 0.00325035X^{11} + 0.00510875X^{10} + 0.00190713X^{9} + 0.000179792X^{8} + 1.06543\cdot10^{-05}X^{7}\\&-0.00136482X^{6} + 0.104959X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}\\= &-2.22335\cdot10^{6}B_{0,16}(X) - 1.93666\cdot10^{6}B_{1,16}(X) - 1.65074\cdot10^{6}B_{2,16}(X) - 1.36561\cdot10^{6}B_{3,16}(X)\\&- 1.08127\cdot10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X)\\&+ 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot10^{6}B_{12,16}(X) + 1.44233\\&\cdot10^{6}B_{13,16}(X) + 1.71876\cdot10^{6}B_{14,16}(X) + 1.99439\cdot10^{6}B_{15,16}(X) + 2.26922\cdot10^{6}B_{16,16}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}\\&= -2.22335\cdot10^{6}B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot10^{6}B_{3,3}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= 15290.6X^{16} - 125412X^{15} + 468045X^{14} - 1.05198\cdot10^{6}X^{13} + 1.588\cdot10^{6}X^{12} - 1.69893\\&\cdot10^{6}X^{11} + 1.32382\cdot10^{6}X^{10} - 759402X^{9} + 320067X^{8} - 97826.5X^{7} + 21197.8X^{6}\\&- 3162.8X^{5} + 313.931X^{4} - 720.165X^{3} - 93879.2X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}\\&= -2.22335\cdot10^{6}B_{0,16} - 1.93666\cdot10^{6}B_{1,16} - 1.65074\cdot10^{6}B_{2,16} - 1.36561\cdot10^{6}B_{3,16}\\&- 1.08126\cdot10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.7B_{8,16}\\&+ 328649B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot10^{6}B_{12,16} + 1.44233\\&\cdot10^{6}B_{13,16} + 1.71876\cdot10^{6}B_{14,16} + 1.99439\cdot10^{6}B_{15,16} + 2.26922\cdot10^{6}B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.457751$.
Bounding polynomials M and m:

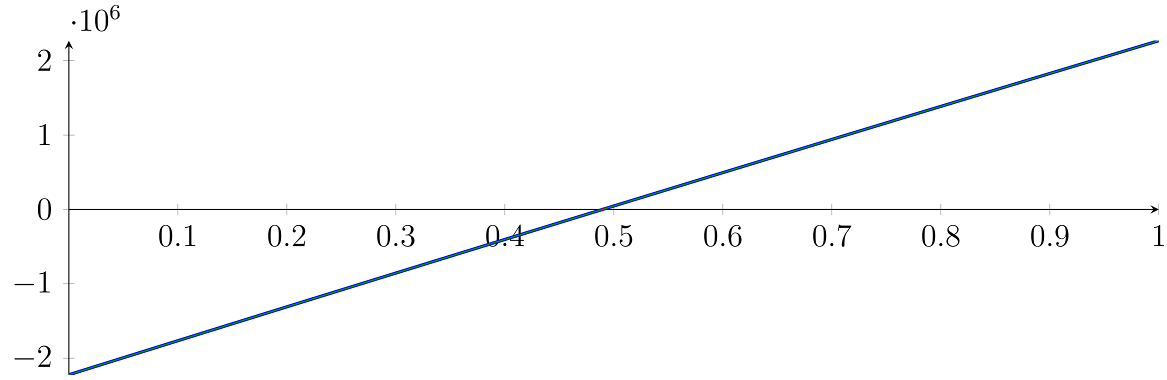
$$M = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715{\cdot}10^6X - 2.22335{\cdot}10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



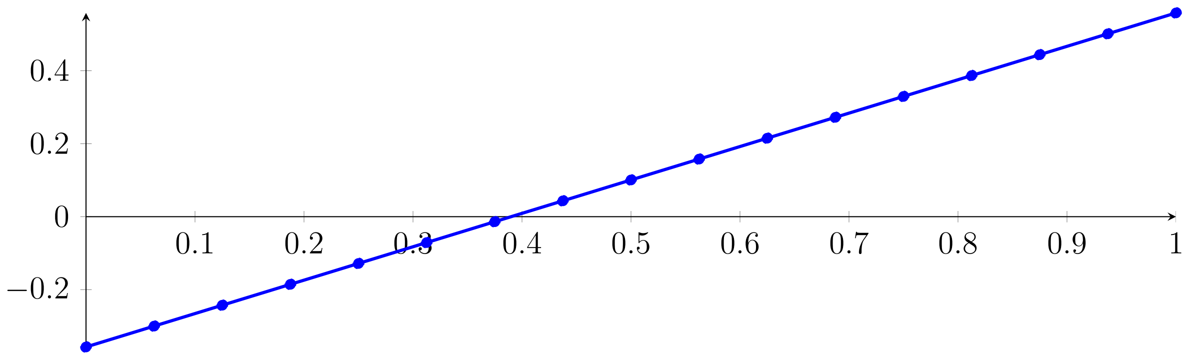
$$[0.489616, 0.489616]$$

Longest intersection interval: $2.03684{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

81.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

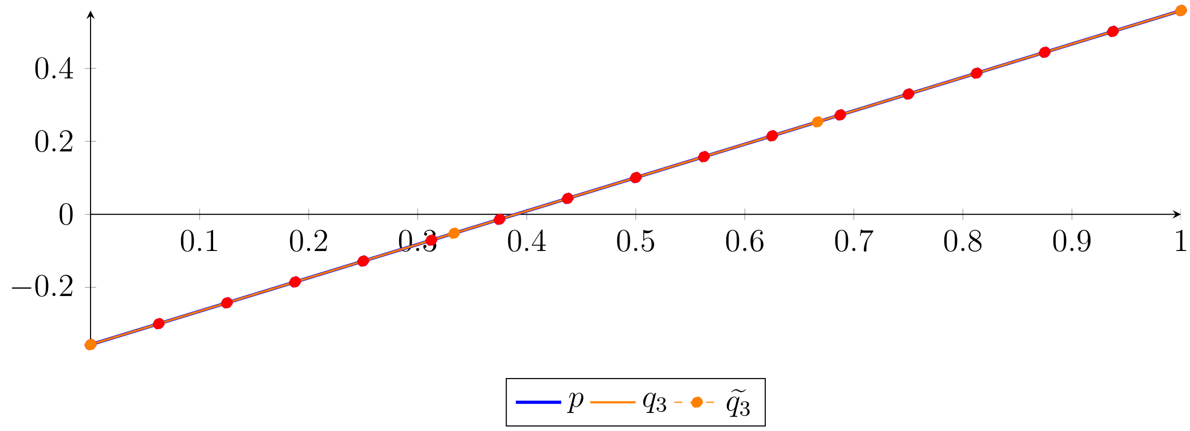
$$\begin{aligned} p = &-1.56399{\cdot}10^{-11}X^{16} + 4.58016{\cdot}10^{-11}X^{15} - 3.19744{\cdot}10^{-12}X^{14} + 6.54055{\cdot}10^{-11}X^{13} \\ &+ 1.05072{\cdot}10^{-10}X^{12} + 4.59728{\cdot}10^{-10}X^{11} + 5.01434{\cdot}10^{-10}X^{10} + 2.99742{\cdot}10^{-10}X^{9} \\ &+ 1.14309{\cdot}10^{-11}X^{8} - 5.08038{\cdot}10^{-12}X^{7} + 3.37845{\cdot}10^{-11}X^{6} - 9.69891{\cdot}10^{-13}X^{5} \\ &+ 4.04121{\cdot}10^{-13}X^{4} + 6.21725{\cdot}10^{-14}X^{3} - 3.93767{\cdot}10^{-09}X^{2} + 0.915501X - 0.357072 \\ = &-0.357072B_{0,16}(X) - 0.299853B_{1,16}(X) - 0.242635B_{2,16}(X) - 0.185416B_{3,16}(X) \\ &- 0.128197B_{4,16}(X) - 0.0709781B_{5,16}(X) - 0.0137592B_{6,16}(X) \\ &+ 0.0434596B_{7,16}(X) + 0.100678B_{8,16}(X) + 0.157897B_{9,16}(X) + 0.215116B_{10,16}(X) \\ &+ 0.272335B_{11,16}(X) + 0.329554B_{12,16}(X) + 0.386773B_{13,16}(X) \\ &+ 0.443991B_{14,16}(X) + 0.50121B_{15,16}(X) + 0.558429B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 1.05471{\cdot}10^{-15}X^3 - 3.93767{\cdot}10^{-09}X^2 + 0.915501X - 0.357072 \\ &= -0.357072B_{0,3} - 0.0519051B_{1,3} + 0.253262B_{2,3} + 0.558429B_{3,3} \end{aligned}$$

$$\widetilde{q}_3 = 0.00291222X^{16} - 0.0241801X^{15} + 0.0914452X^{14} - 0.208537X^{13} + 0.319778X^{12} - 0.347745X^{11} + 0.275244X^{10} - 0.159971X^{9} + 0.0679818X^{8} - 0.0208072X^{7} + 0.00447629X^{6} - 0.000654783X^{5} + 6.22034{\cdot}10^{-05}X^{4} - 3.60145{\cdot}10^{-06}X^{3} + 9.78811{\cdot}10^{-08}X^{2} + 0.915501X - 0.357072$$
$$= -0.357072B_{0,16} - 0.299853B_{1,16} - 0.242635B_{2,16} - 0.185416B_{3,16} - 0.128197B_{4,16} - 0.0709781B_{5,16} - 0.0137592B_{6,16} + 0.0434595B_{7,16} + 0.100678B_{8,16} + 0.157897B_{9,16} + 0.215116B_{10,16} + 0.272335B_{11,16} + 0.329554B_{12,16} + 0.386773B_{13,16} + 0.443991B_{14,16} + 0.50121B_{15,16} + 0.558429B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 6.5212{\cdot}10^{-08}$.

Bounding polynomials M and m:

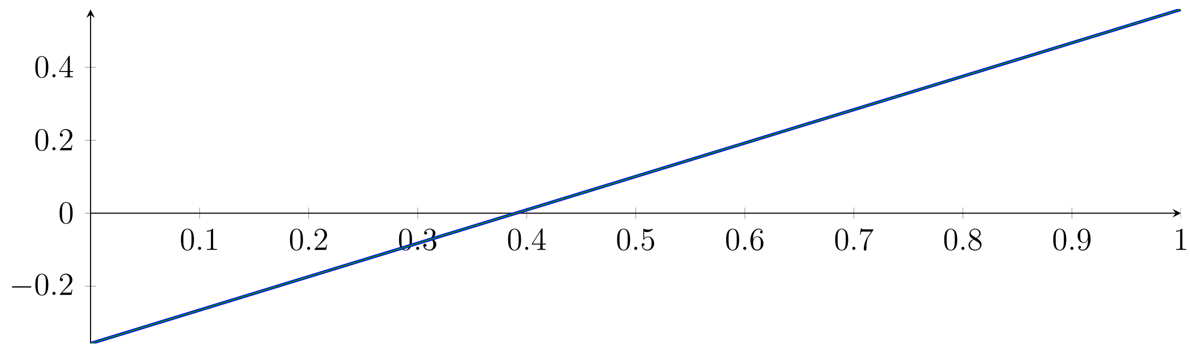
$$M = 9.99201{\cdot}10^{-16}X^3 - 3.93767{\cdot}10^{-09}X^2 + 0.915501X - 0.357072$$

$$m = 1.22125{\cdot}10^{-15}X^3 - 3.93767{\cdot}10^{-09}X^2 + 0.915501X - 0.357072$$

Root of M and m:

$$N(M) = \{0.390029\} \qquad N(m) = \{0.390029\}$$

Intersection intervals:



$$[0.390029, 0.390029]$$

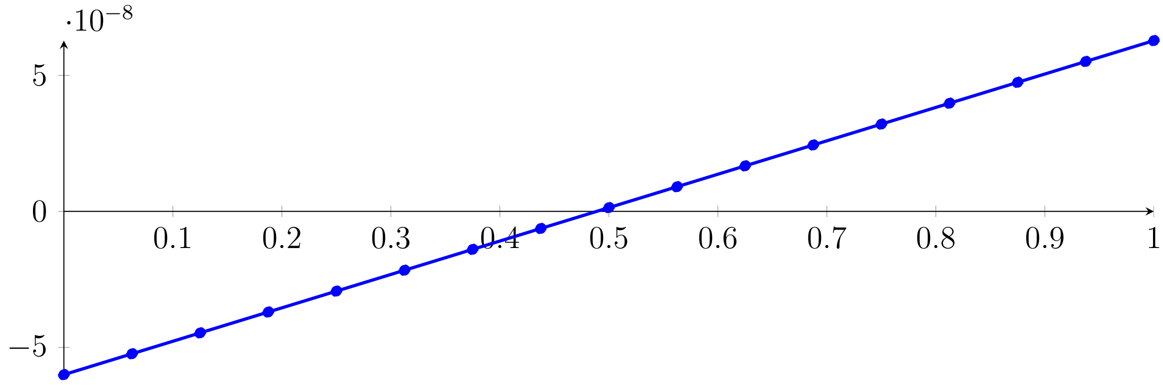
Longest intersection interval: $1.3411{\cdot}10^{-07}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

81.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

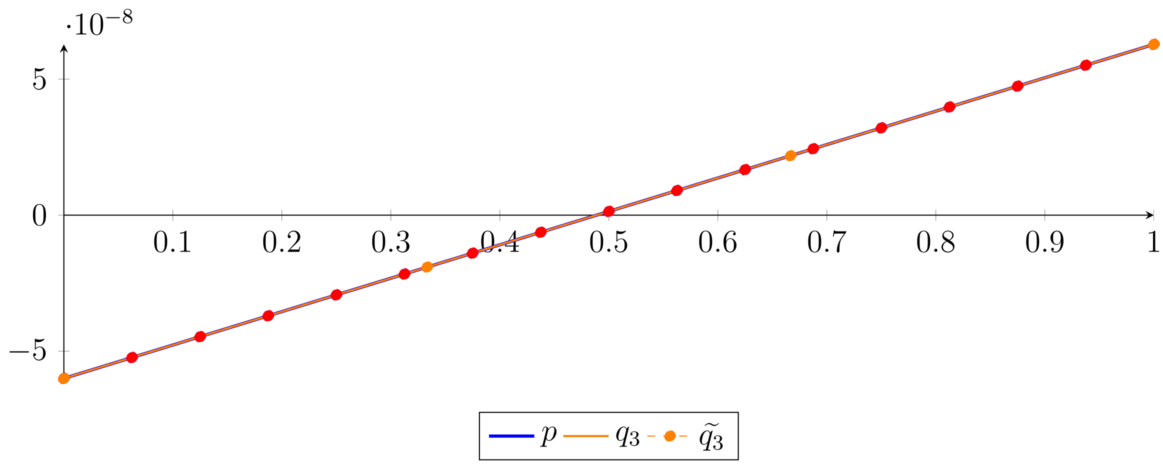
$$p = -5.65183\cdot10^{-19}X^{16} + 5.13302\cdot10^{-19}X^{15} + 6.37816\cdot10^{-18}X^{14} + 1.69576\cdot10^{-17}X^{13} + 9.4423\cdot10^{-17}X^{12} + 8.0934\cdot10^{-17}X^{11} + 1.37357\cdot10^{-16}X^{10} + 6.23797\cdot10^{-17}X^{9} + 1.36266\cdot10^{-18}X^{8} + 6.05629\cdot10^{-19}X^{7} + 5.08728\cdot10^{-18}X^{6} + 2.3124\cdot10^{-19}X^{5} + 1.44525\cdot10^{-19}X^{4} - 7.41154\cdot10^{-21}X^{3} + 1.22778\cdot10^{-07}X - 6.00146\cdot10^{-08}$$
$$= -6.00146\cdot10^{-08}B_{0,16}(X) - 5.2341\cdot10^{-08}B_{1,16}(X) - 4.46674\cdot10^{-08}B_{2,16}(X) - 3.69937\cdot10^{-08}B_{3,16}(X) - 2.93201\cdot10^{-08}B_{4,16}(X) - 2.16464\cdot10^{-08}B_{5,16}(X) - 1.39728\cdot10^{-08}B_{6,16}(X) - 6.29913\cdot10^{-09}B_{7,16}(X) + 1.37451\cdot10^{-09}B_{8,16}(X) + 9.04815\cdot10^{-09}B_{9,16}(X) + 1.67218\cdot10^{-08}B_{10,16}(X) + 2.43954\cdot10^{-08}B_{11,16}(X) + 3.20691\cdot10^{-08}B_{12,16}(X) + 3.97427\cdot10^{-08}B_{13,16}(X) + 4.74164\cdot10^{-08}B_{14,16}(X) + 5.509\cdot10^{-08}B_{15,16}(X) + 6.27637\cdot10^{-08}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = 2.51463\cdot10^{-22}X^{3} - 3.17637\cdot10^{-22}X^{2} + 1.22778\cdot10^{-07}X - 6.00146\cdot10^{-08}$$
$$= -6.00146\cdot10^{-08}B_{0,3} - 1.90885\cdot10^{-08}B_{1,3} + 2.18376\cdot10^{-08}B_{2,3} + 6.27637\cdot10^{-08}B_{3,3}$$

$$\widetilde{q}_3 = 4.01426\cdot10^{-10}X^{16} - 3.29623\cdot10^{-09}X^{15} + 1.2317\cdot10^{-08}X^{14} - 2.77213\cdot10^{-08}X^{13} + 4.19074\cdot10^{-08}X^{12} - 4.49047\cdot10^{-08}X^{11} + 3.50457\cdot10^{-08}X^{10} - 2.01341\cdot10^{-08}X^{9} + 8.49676\cdot10^{-09}X^{8} - 2.5992\cdot10^{-09}X^{7} + 5.63364\cdot10^{-10}X^{6} - 8.40231\cdot10^{-11}X^{5} + 8.33259\cdot10^{-12}X^{4} - 5.16965\cdot10^{-13}X^{3} + 1.7395\cdot10^{-14}X^{2} + 1.22778\cdot10^{-07}X - 6.00146\cdot10^{-08}$$
$$= -6.00146\cdot10^{-08}B_{0,16} - 5.2341\cdot10^{-08}B_{1,16} - 4.46674\cdot10^{-08}B_{2,16} - 3.69937\cdot10^{-08}B_{3,16} - 2.93201\cdot10^{-08}B_{4,16} - 2.16464\cdot10^{-08}B_{5,16} - 1.39728\cdot10^{-08}B_{6,16} - 6.29914\cdot10^{-09}B_{7,16} + 1.37452\cdot10^{-09}B_{8,16} + 9.04815\cdot10^{-09}B_{9,16} + 1.67218\cdot10^{-08}B_{10,16} + 2.43954\cdot10^{-08}B_{11,16} + 3.20691\cdot10^{-08}B_{12,16} + 3.97427\cdot10^{-08}B_{13,16} + 4.74164\cdot10^{-08}B_{14,16} + 5.509\cdot10^{-08}B_{15,16} + 6.27637\cdot10^{-08}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 7.90061 \cdot 10^{-15}$.
Bounding polynomials M and m:

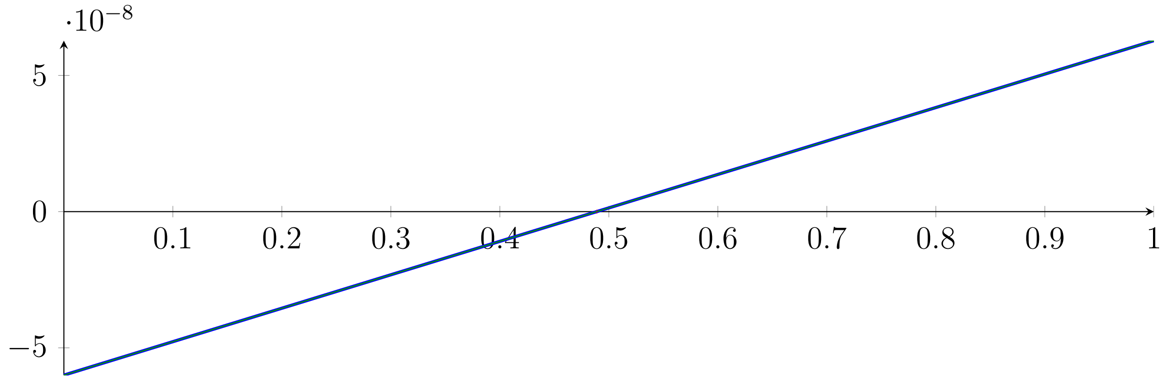
$$M = 2.38228 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08}$$

$$m = 2.51463 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08}$$

Root of M and m:

$$N(M) = \{0.488805\} \qquad N(m) = \{0.488805\}$$

Intersection intervals:



$$[0.488805, 0.488805]$$

Longest intersection interval: $1.3086 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

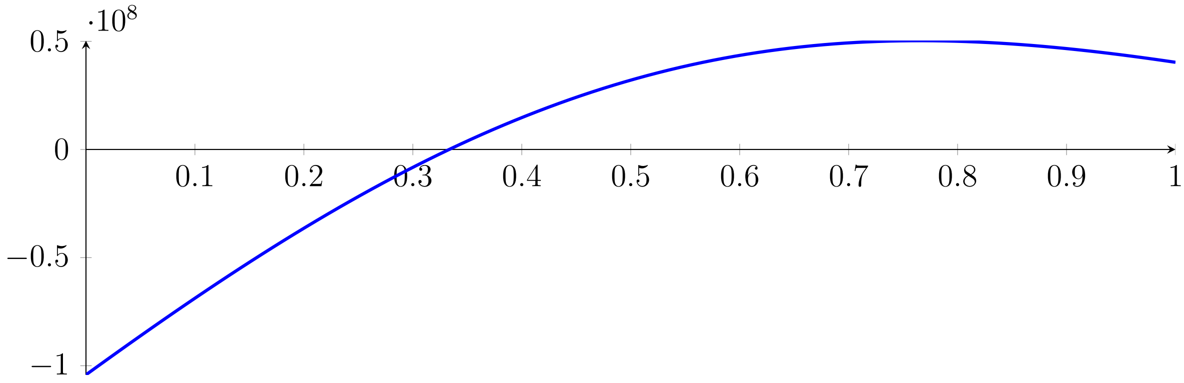
81.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

81.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

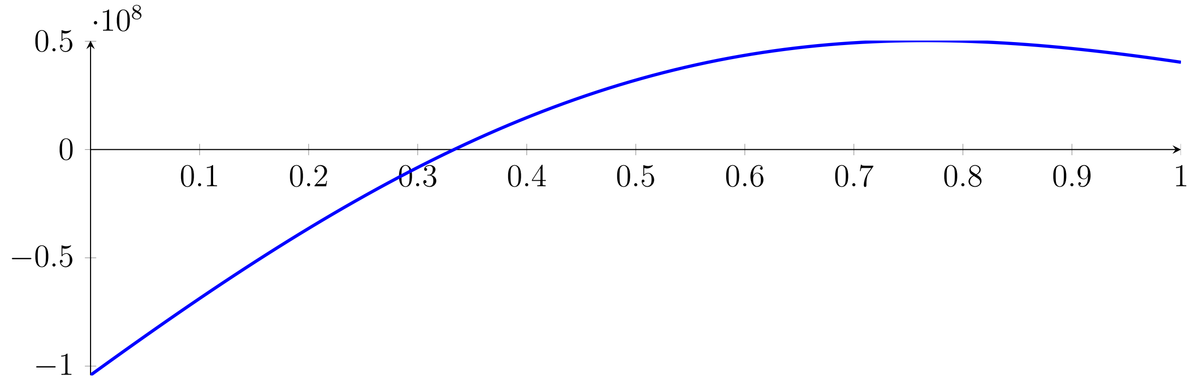
with precision $\varepsilon = 1\cdot 10^{-64}$.

82 Running BezClip on f_{16} with epsilon 128

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^6X^9 - 2.52262{\cdot}10^6X^8 - 1.43506{\cdot}10^7X^7 + 138542X^6 + 7.92823{\cdot}10^7X^5 + 3.97396{\cdot}10^7X^4 - 2.40625{\cdot}10^8X^3 - 8.33333{\cdot}10^7X^2 + 3.64583{\cdot}10^8X - 1.04167{\cdot}10^8$$

Called BezClip with input polynomial on interval $[0, 1]$:

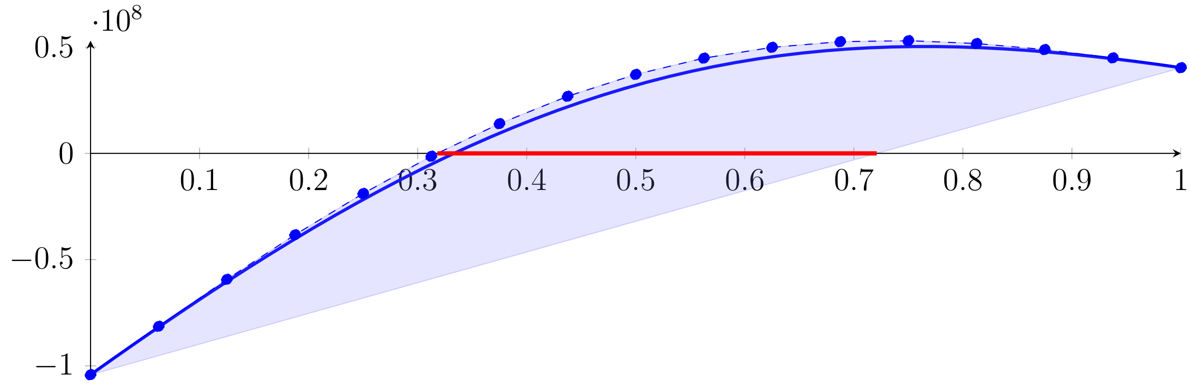
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^6X^9 - 2.52262{\cdot}10^6X^8 - 1.43506{\cdot}10^7X^7 + 138542X^6 + 7.92823{\cdot}10^7X^5 + 3.97396{\cdot}10^7X^4 - 2.40625{\cdot}10^8X^3 - 8.33333{\cdot}10^7X^2 + 3.64583{\cdot}10^8X - 1.04167{\cdot}10^8$$



82.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\ &\quad + 451673X^{10} + 1.12172{\cdot}10^6X^9 - 2.52262{\cdot}10^6X^8 - 1.43506{\cdot}10^7X^7 + 138542X^6 + 7.92823 \\ &\quad {\cdot}10^7X^5 + 3.97396{\cdot}10^7X^4 - 2.40625{\cdot}10^8X^3 - 8.33333{\cdot}10^7X^2 + 3.64583{\cdot}10^8X - 1.04167{\cdot}10^8 \\ &= -1.04167{\cdot}10^8B_{0,16}(X) - 8.13802{\cdot}10^7B_{1,16}(X) - 5.92882{\cdot}10^7B_{2,16}(X) - 3.83203 \\ &\quad {\cdot}10^7B_{3,16}(X) - 1.88844{\cdot}10^7B_{4,16}(X) - 1.34837{\cdot}10^6B_{5,16}(X) + 1.39781{\cdot}10^7B_{6,16}(X) \\ &\quad + 2.68604{\cdot}10^7B_{7,16}(X) + 3.71532{\cdot}10^7B_{8,16}(X) + 4.48105{\cdot}10^7B_{9,16}(X) + 4.98901 \\ &\quad {\cdot}10^7B_{10,16}(X) + 5.25504{\cdot}10^7B_{11,16}(X) + 5.30407{\cdot}10^7B_{12,16}(X) + 5.16832{\cdot}10^7B_{13,16}(X) \\ &\quad + 4.88488{\cdot}10^7B_{14,16}(X) + 4.49297{\cdot}10^7B_{15,16}(X) + 4.03108{\cdot}10^7B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

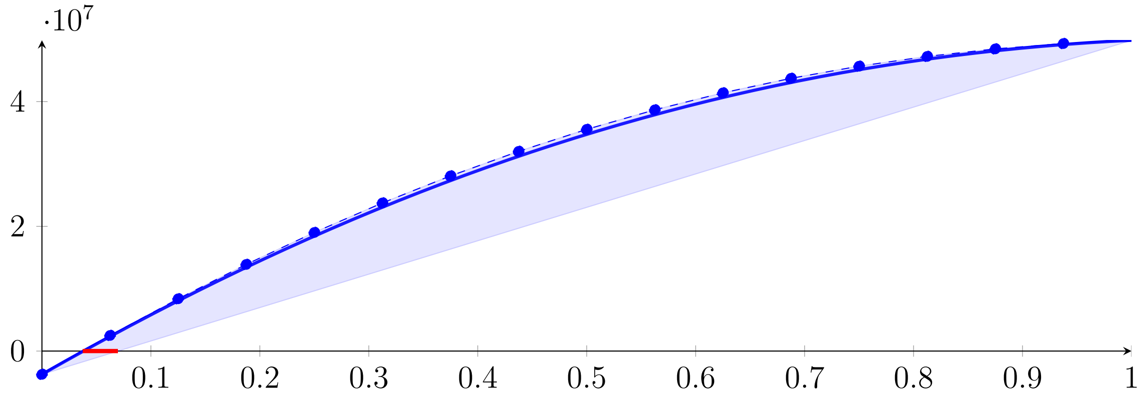
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

82.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -0.00483322X^{16} + 0.0186809X^{15} - 0.0194312X^{14} - 0.0738695X^{13} - 1.11673X^{12}\\ &\quad - 5.0471X^{11} + 36.3082X^{10} + 692.914X^{9} + 1886.96X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5}\\ &\quad + 3.91945\cdot 10^{6}X^{4} - 7.54561\cdot 10^{6}X^{3} - 4.29225\cdot 10^{7}X^{2} + 9.97982\cdot 10^{7}X - 3.73501\cdot 10^{6}\\ &= -3.73501\cdot 10^{6}B_{0,16}(X) + 2.50237\cdot 10^{6}B_{1,16}(X) + 8.38207\cdot 10^{6}B_{2,16}(X) + 1.38906\\ &\quad \cdot 10^{7}B_{3,16}(X) + 1.90167\cdot 10^{7}B_{4,16}(X) + 2.37512\cdot 10^{7}B_{5,16}(X) + 2.80875\cdot 10^{7}B_{6,16}(X)\\ &\quad + 3.20213\cdot 10^{7}B_{7,16}(X) + 3.55508\cdot 10^{7}B_{8,16}(X) + 3.86768\cdot 10^{7}B_{9,16}(X) + 4.14025\\ &\quad \cdot 10^{7}B_{10,16}(X) + 4.37336\cdot 10^{7}B_{11,16}(X) + 4.56783\cdot 10^{7}B_{12,16}(X) + 4.7247\cdot 10^{7}B_{13,16}(X)\\ &\quad + 4.84524\cdot 10^{7}B_{14,16}(X) + 4.93093\cdot 10^{7}B_{15,16}(X) + 4.98343\cdot 10^{7}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

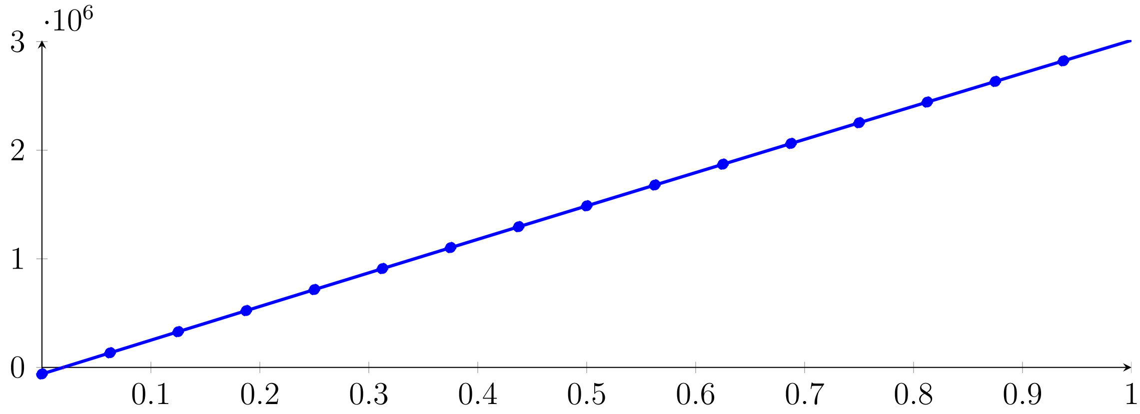
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

82.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -0.000205056X^{16} + 0.000776167X^{15} - 0.000863333X^{14} - 0.000217499X^{13} - 0.00809276X^{12}\\ &\quad - 5.09773\cdot 10^{-05}X^{11} - 0.00564923X^{10} - 0.000162811X^{9} - 0.000215376X^{8} - 3.32948\cdot 10^{-07}X^{7}\\ &\quad - 0.000259866X^{6} + 0.0161077X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot 10^{6}X - 60508.5\\ &= -60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X)\\ &\quad + 910202B_{5,16}(X) + 1.1032\cdot 10^{6}B_{6,16}(X) + 1.29582\cdot 10^{6}B_{7,16}(X) + 1.48805\cdot 10^{6}B_{8,16}(X)\\ &\quad + 1.6799\cdot 10^{6}B_{9,16}(X) + 1.87137\cdot 10^{6}B_{10,16}(X) + 2.06245\cdot 10^{6}B_{11,16}(X) + 2.25315\cdot 10^{6}B_{12,16}(X)\\ &\quad + 2.44346\cdot 10^{6}B_{13,16}(X) + 2.63339\cdot 10^{6}B_{14,16}(X) + 2.82293\cdot 10^{6}B_{15,16}(X) + 3.01209\cdot 10^{6}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

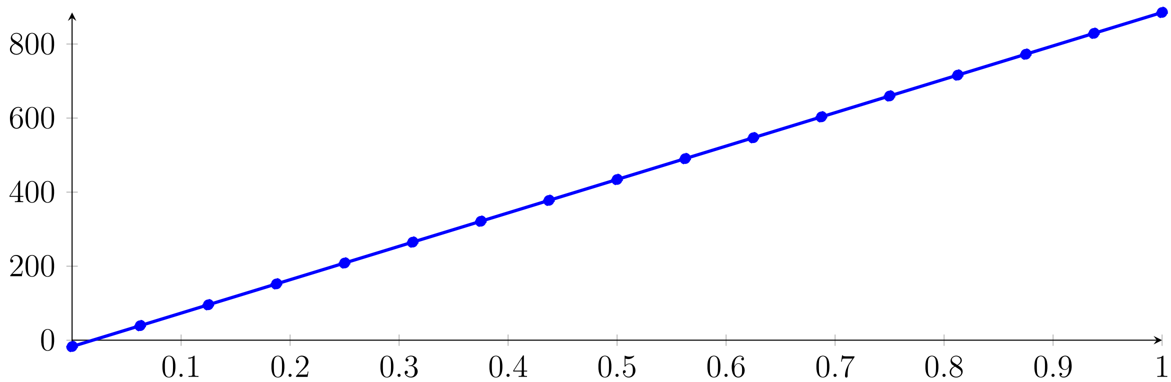
$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

82.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p = & -5.9692 \cdot 10^{-08} X^{16} + 2.16103 \cdot 10^{-07} X^{15} - 2.28456 \cdot 10^{-07} X^{14} - 1.17238 \cdot 10^{-07} X^{13} \\ & - 2.29525 \cdot 10^{-06} X^{12} - 8.31778 \cdot 10^{-08} X^{11} - 1.74251 \cdot 10^{-06} X^{10} - 9.42919 \cdot 10^{-08} X^{9} \\ & - 7.38891 \cdot 10^{-08} X^{8} + 3.25144 \cdot 10^{-09} X^{7} - 2.61741 \cdot 10^{-08} X^{6} + 7.44876 \cdot 10^{-10} X^{5} \\ & - 2.58638 \cdot 10^{-10} X^{4} - 5.65024 \cdot 10^{-09} X^{3} - 0.00382618 X^{2} + 902.448 X - 17.178 \\ = & -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\ & + 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\ & + 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\ & + 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

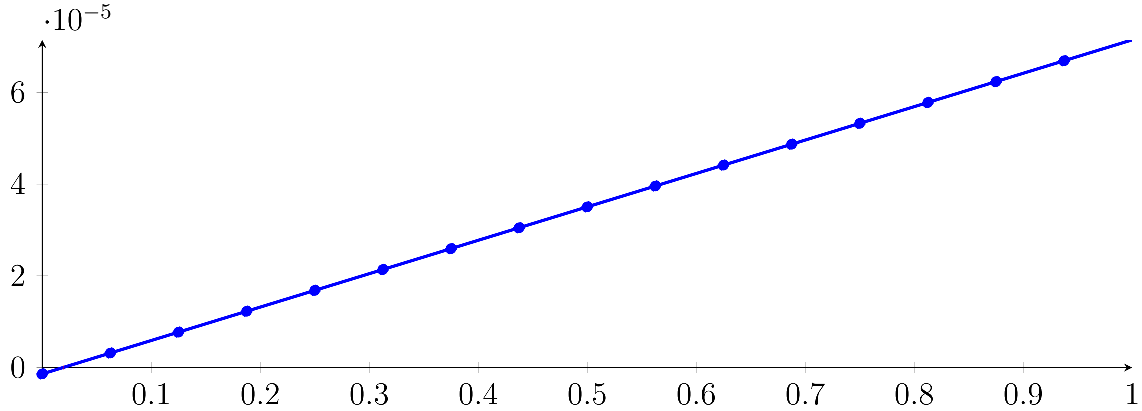
$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045 \cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

82.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p = &-4.80379\cdot10^{-15}X^{16} + 1.74137\cdot10^{-14}X^{15} - 1.72656\cdot10^{-14}X^{14} - 6.84186\cdot10^{-15}X^{13}\\ &- 1.91627\cdot10^{-13}X^{12} - 4.7358\cdot10^{-15}X^{11} - 1.33436\cdot10^{-13}X^{10} - 1.97677\cdot10^{-15}X^{9}\\ &- 7.4565\cdot10^{-15}X^{8} - 1.16281\cdot10^{-16}X^{7} - 1.98065\cdot10^{-15}X^{6} + 1.47994\cdot10^{-17}X^{5}\\ &- 1.84992\cdot10^{-17}X^{4} - 2.48011\cdot10^{-17}X^{2} + 7.28316\cdot10^{-05}X - 1.38634\cdot10^{-06}\\ = &-1.38634\cdot10^{-06}B_{0,16}(X) + 3.16564\cdot10^{-06}B_{1,16}(X) + 7.71761\cdot10^{-06}B_{2,16}(X) + 1.22696\\ &\cdot10^{-05}B_{3,16}(X) + 1.68216\cdot10^{-05}B_{4,16}(X) + 2.13735\cdot10^{-05}B_{5,16}(X) + 2.59255\cdot10^{-05}B_{6,16}(X)\\ &+ 3.04775\cdot10^{-05}B_{7,16}(X) + 3.50295\cdot10^{-05}B_{8,16}(X) + 3.95814\cdot10^{-05}B_{9,16}(X) + 4.41334\\ &\cdot10^{-05}B_{10,16}(X) + 4.86854\cdot10^{-05}B_{11,16}(X) + 5.32374\cdot10^{-05}B_{12,16}(X) + 5.77893\cdot10^{-05}B_{13,16}(X)\\ &+ 6.23413\cdot10^{-05}B_{14,16}(X) + 6.68933\cdot10^{-05}B_{15,16}(X) + 7.14453\cdot10^{-05}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.50521\cdot10^{-15}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

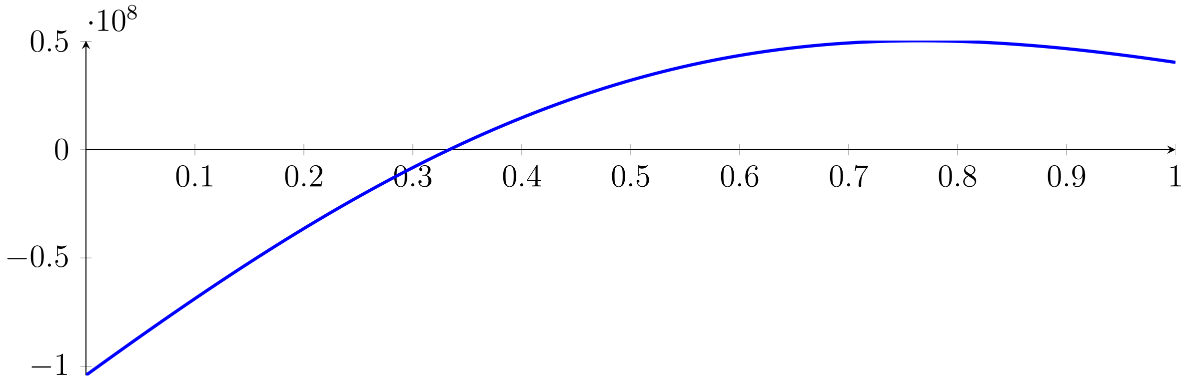
82.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

82.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

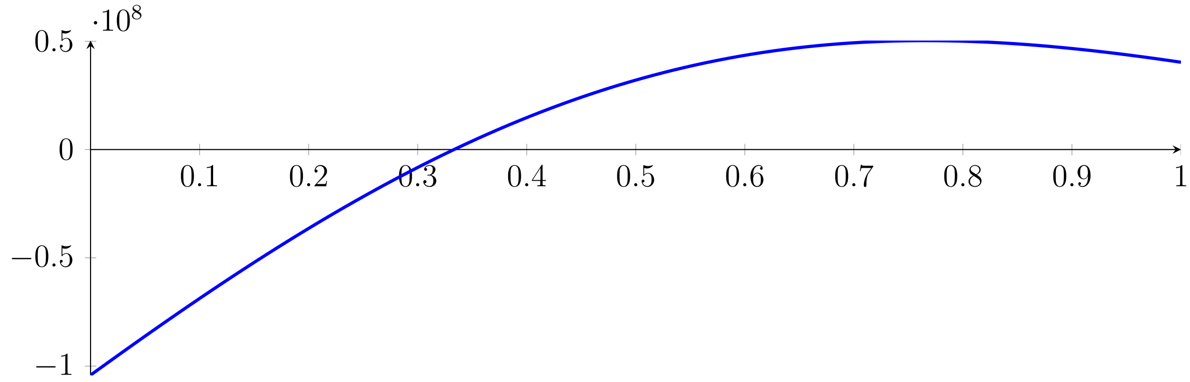
with precision $\varepsilon = 1\cdot10^{-128}$.

83 Running QuadClip on f_{16} with epsilon 128

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

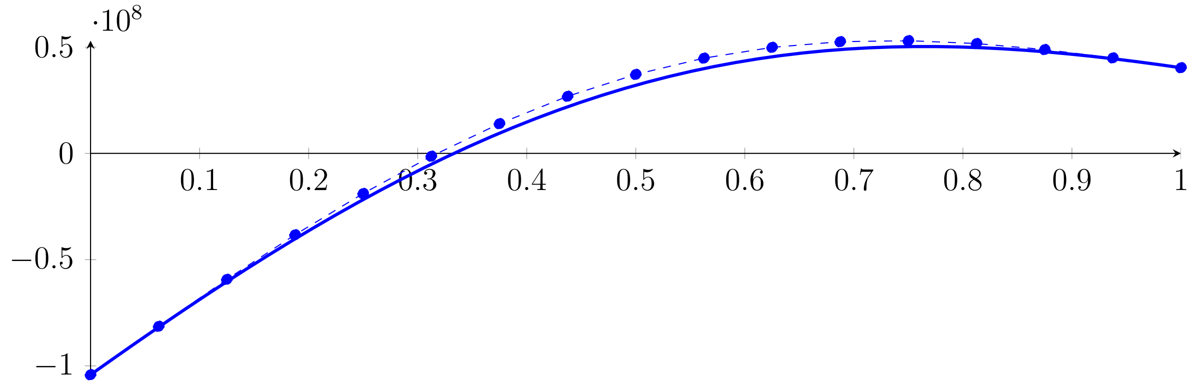
$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} \\ & + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \end{aligned}$$



83.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

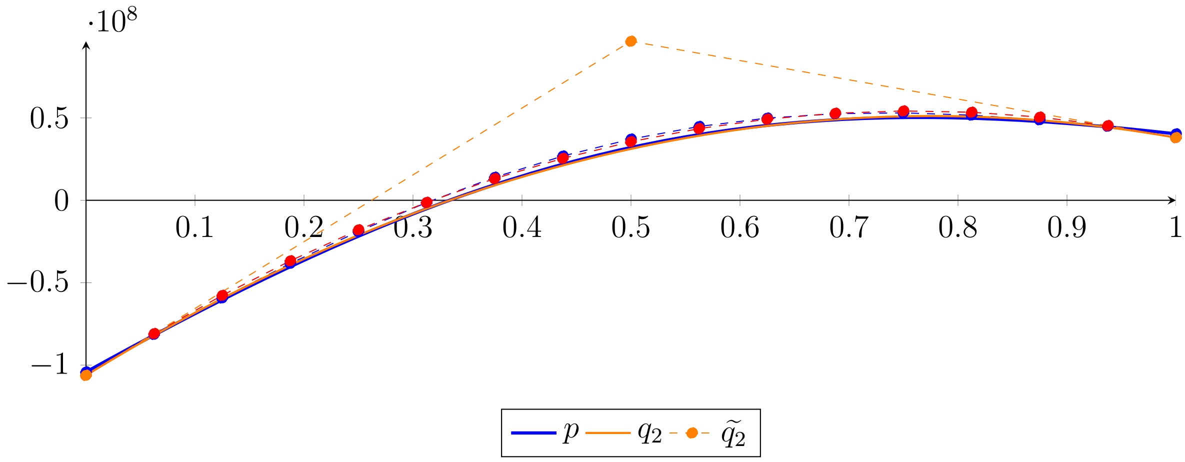
$$\begin{aligned} p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\ & + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \\ & \cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ = & -1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ & \cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ & + 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ & \cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ & + 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -2.61181\cdot 10^8 X^2 + 4.05417\cdot 10^8 X - 1.06197\cdot 10^8 \\ &= -1.06197\cdot 10^8 B_{0,2} + 9.65115\cdot 10^7 B_{1,2} + 3.80384\cdot 10^7 B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.41787\cdot 10^6 X^{16} + 1.11761\cdot 10^7 X^{15} - 3.98898\cdot 10^7 X^{14} + 8.52437\cdot 10^7 X^{13} - 1.21528\cdot 10^8 X^{12} \\ &\quad + 1.21946\cdot 10^8 X^{11} - 8.86062\cdot 10^7 X^{10} + 4.72904\cdot 10^7 X^9 - 1.86355\cdot 10^7 X^8 + 5.41059\cdot 10^6 X^7 - 1.14487 \\ &\quad \cdot 10^6 X^6 + 172240 X^5 - 17636.6 X^4 + 1159.74 X^3 - 2.61181\cdot 10^8 X^2 + 4.05417\cdot 10^8 X - 1.06197\cdot 10^8 \\ &= -1.06197\cdot 10^8 B_{0,16} - 8.08583\cdot 10^7 B_{1,16} - 5.76963\cdot 10^7 B_{2,16} - 3.67107\cdot 10^7 B_{3,16} - 1.79017 \\ &\quad \cdot 10^7 B_{4,16} - 1.26923\cdot 10^6 B_{5,16} + 1.31867\cdot 10^7 B_{6,16} + 2.54662\cdot 10^7 B_{7,16} + 3.55691\cdot 10^7 B_{8,16} \\ &\quad + 4.34957\cdot 10^7 B_{9,16} + 4.92455\cdot 10^7 B_{10,16} + 5.2819\cdot 10^7 B_{11,16} + 5.42159\cdot 10^7 B_{12,16} \\ &\quad + 5.34363\cdot 10^7 B_{13,16} + 5.04802\cdot 10^7 B_{14,16} + 4.53476\cdot 10^7 B_{15,16} + 3.80384\cdot 10^7 B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233\cdot 10^6$.

Bounding polynomials M and m:

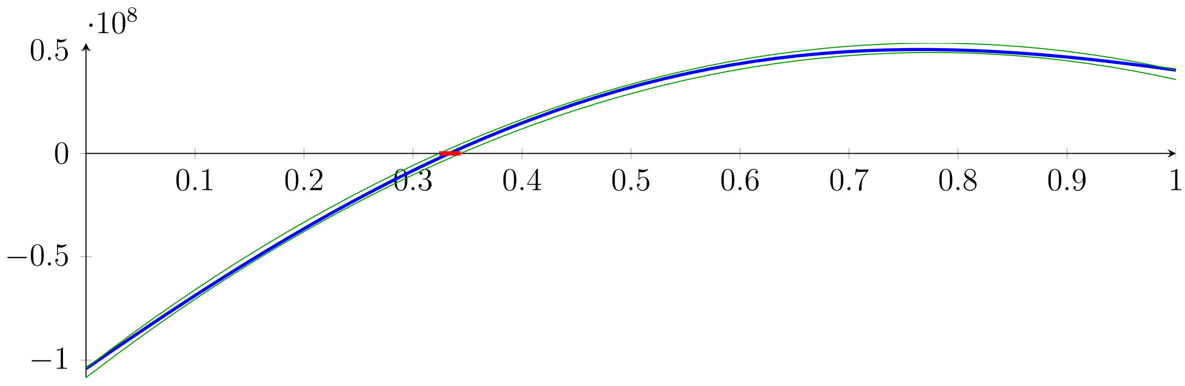
$$M = -2.61181\cdot 10^8 X^2 + 4.05417\cdot 10^8 X - 1.03924\cdot 10^8$$

$$m = -2.61181\cdot 10^8 X^2 + 4.05417\cdot 10^8 X - 1.08469\cdot 10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

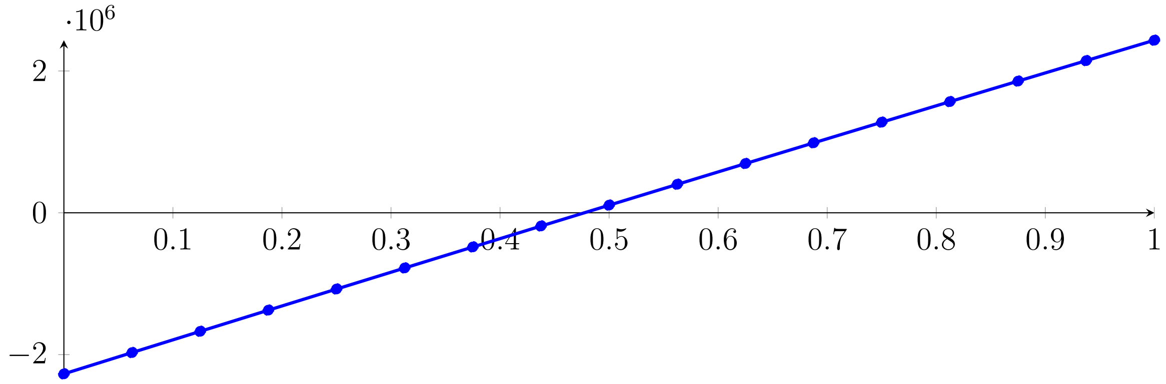
Longest intersection interval: 0.0196686

$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

83.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

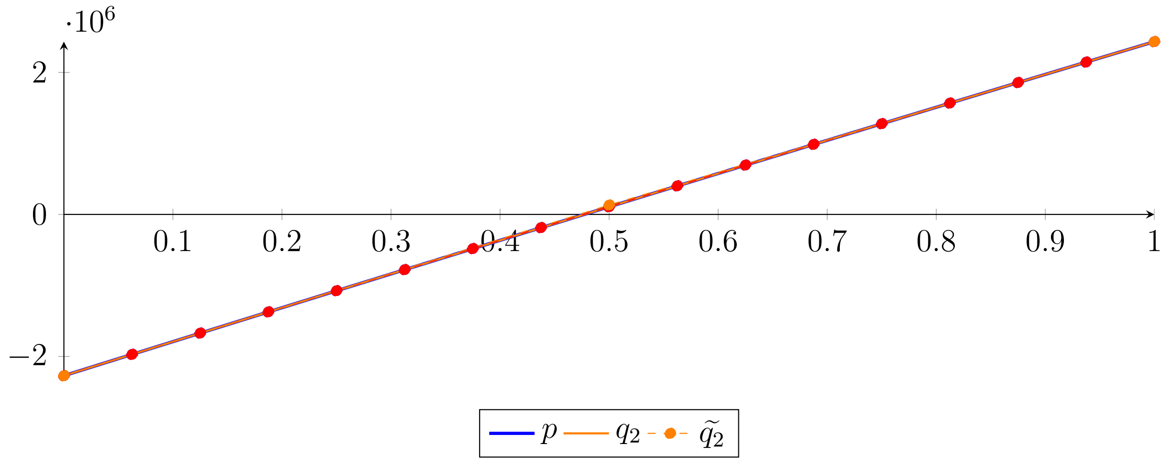
$$\begin{aligned} p = & -2.82438\cdot 10^{-05}X^{16} + 2.90051\cdot 10^{-05}X^{15} + 0.000231285X^{14} + 0.000648014X^{13} + 0.00318916X^{12} \\ & + 0.00323204X^{11} + 0.00460533X^{10} + 0.00220012X^{9} - 0.000101882X^{8} + 5.32717\cdot 10^{-06}X^{7} \\ & - 0.00186824X^{6} + 0.132741X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874\cdot 10^{6}X - 2.2715\cdot 10^{6} \\ = & -2.2715\cdot 10^{6}B_{0,16}(X) - 1.97096\cdot 10^{6}B_{1,16}(X) - 1.67127\cdot 10^{6}B_{2,16}(X) - 1.37244\cdot 10^{6}B_{3,16}(X) \\ & - 1.07448\cdot 10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) \\ & + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806\cdot 10^{6}B_{12,16}(X) + 1.5682 \\ & \cdot 10^{6}B_{13,16}(X) + 1.85747\cdot 10^{6}B_{14,16}(X) + 2.14586\cdot 10^{6}B_{15,16}(X) + 2.43338\cdot 10^{6}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -104265X^{2} + 4.80923\cdot 10^{6}X - 2.27154\cdot 10^{6} \\ &= -2.27154\cdot 10^{6}B_{0,2} + 133069B_{1,2} + 2.43342\cdot 10^{6}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 = & \, 43556.7X^{16} - 347904X^{15} + 1.2616\cdot 10^{6}X^{14} - 2.74835\cdot 10^{6}X^{13} + 4.01042\cdot 10^{6}X^{12} - 4.13709 \\ & \cdot 10^{6}X^{11} + 3.10251\cdot 10^{6}X^{10} - 1.71209\cdot 10^{6}X^{9} + 695531X^{8} - 205965X^{7} + 43591.2X^{6} \\ & - 6402.94X^{5} + 625.678X^{4} - 37.485X^{3} - 104264X^{2} + 4.80923\cdot 10^{6}X - 2.27154\cdot 10^{6} \\ = & -2.27154\cdot 10^{6}B_{0,16} - 1.97097\cdot 10^{6}B_{1,16} - 1.67126\cdot 10^{6}B_{2,16} - 1.37242\cdot 10^{6}B_{3,16} \\ & - 1.07445\cdot 10^{6}B_{4,16} - 777350B_{5,16} - 481117B_{6,16} - 185755B_{7,16} + 108741B_{8,16} \\ & + 402365B_{9,16} + 695123B_{10,16} + 987010B_{11,16} + 1.27803\cdot 10^{6}B_{12,16} + 1.56818 \\ & \cdot 10^{6}B_{13,16} + 1.85746\cdot 10^{6}B_{14,16} + 2.14587\cdot 10^{6}B_{15,16} + 2.43342\cdot 10^{6}B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m:

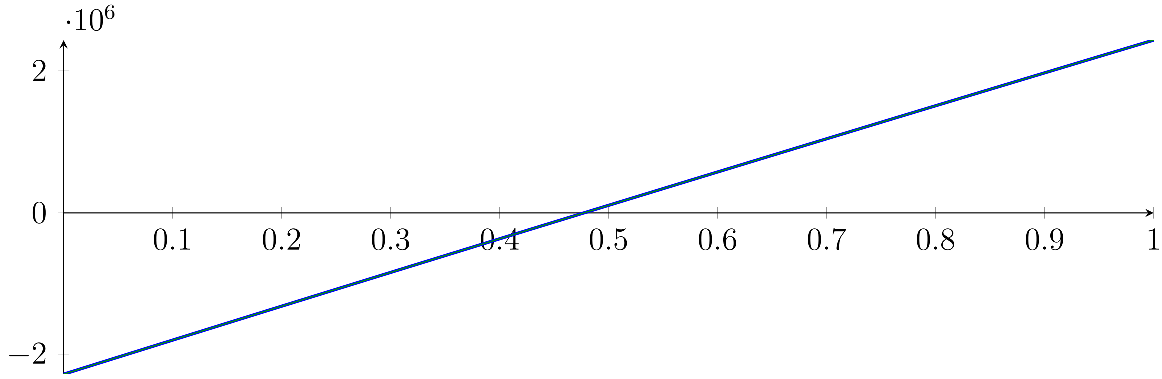
$$M = -104265X^{2} + 4.80923\cdot 10^{6}X - 2.2715\cdot 10^{6}$$

$$m = -104265X^2 + 4.80923{\cdot}10^6X - 2.27159{\cdot}10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



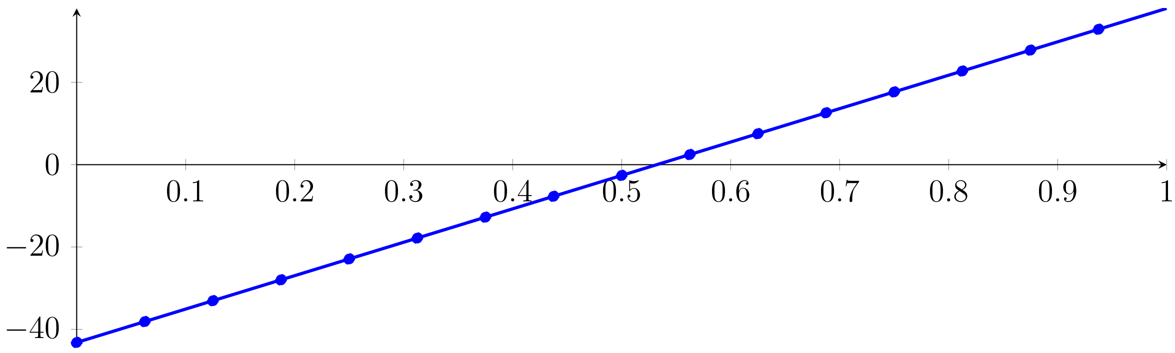
$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301\cdot10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

83.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

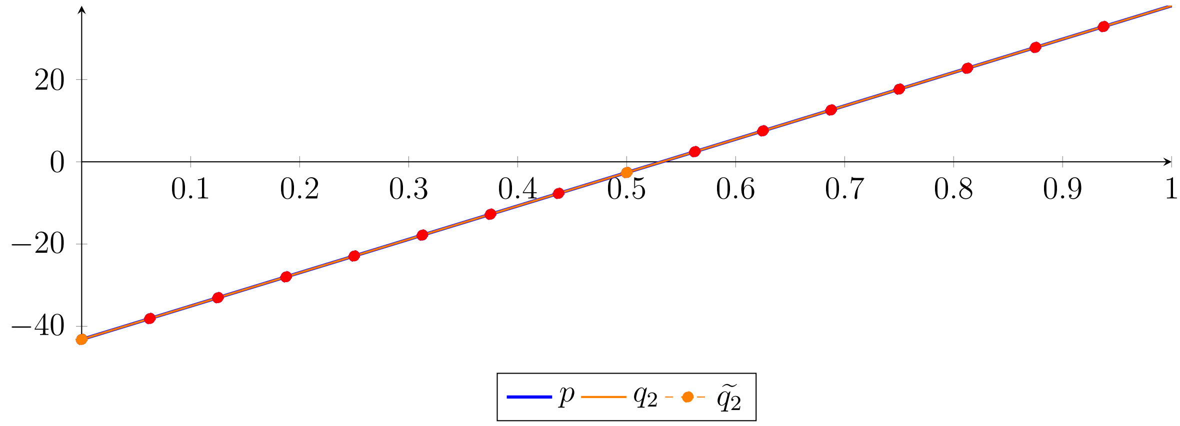
$$\begin{aligned}
p = {} & 8.74252{\cdot}10^{-11}X^{16} - 1.56979{\cdot}10^{-09}X^{15} + 6.68479{\cdot}10^{-09}X^{14} + 1.20008{\cdot}10^{-08}X^{13} + 9.07301{\cdot}10^{-08}X^{12} \\
& + 5.58657{\cdot}10^{-08}X^{11} + 1.13801{\cdot}10^{-07}X^{10} + 3.70665{\cdot}10^{-08}X^{9} + 7.31575{\cdot}10^{-10}X^{8} + 1.30058{\cdot}10^{-09}X^{7} \\
& + 5.00722{\cdot}10^{-09}X^{6} + 1.24146{\cdot}10^{-10}X^{5} + 1.03455{\cdot}10^{-10}X^{4} - 3.09388{\cdot}10^{-05}X^{2} + 81.1505X - 43.1911 \\
= {} & -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\
& - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68777B_{7,16}(X) - 2.61587B_{8,16}(X) \\
& + 2.45604B_{9,16}(X) + 7.52794B_{10,16}(X) + 12.5998B_{11,16}(X) + 17.6718B_{12,16}(X) \\
& + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X)
\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
q_2 &= -3.09388{\cdot}10^{-05}X^2 + 81.1505X - 43.1911 \\
&= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2}
\end{aligned}$$

$$\begin{aligned}
\widetilde{q}_2 = {} & 0.721495X^{16} - 5.74915X^{15} + 20.7933X^{14} - 45.1627X^{13} + 65.6806X^{12} - 67.5044X^{11} \\
& + 50.4286X^{10} - 27.728X^{9} + 11.2318X^{8} - 3.32011X^{7} + 0.702408X^{6} - 0.103415X^{5} \\
& + 0.0102099X^{4} - 0.000624725X^{3} - 1.10834{\cdot}10^{-05}X^{2} + 81.1505X - 43.1911 \\
= {} & -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} \\
& - 12.7597B_{6,16} - 7.68779B_{7,16} - 2.61585B_{8,16} + 2.45602B_{9,16} + 7.52795B_{10,16} + 12.5998B_{11,16} \\
& + 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16}
\end{aligned}$$



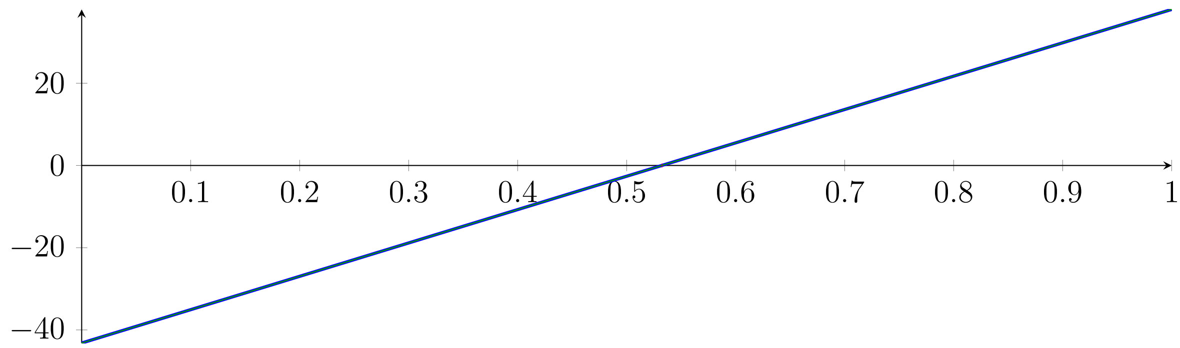
The maximum difference of the Bézier coefficients is $\delta = 1.57956{\cdot}10^{-05}$.
Bounding polynomials M and m:

$$M = -3.09388{\cdot}10^{-05}X^2 + 81.1505X - 43.1911$$
$$m = -3.09388{\cdot}10^{-05}X^2 + 81.1505X - 43.1911$$

Root of M and m:

$$N(M) = \{0.532235, 2.62293{\cdot}10^{6}\} \qquad N(m) = \{0.532235, 2.62293{\cdot}10^{6}\}$$

Intersection intervals:



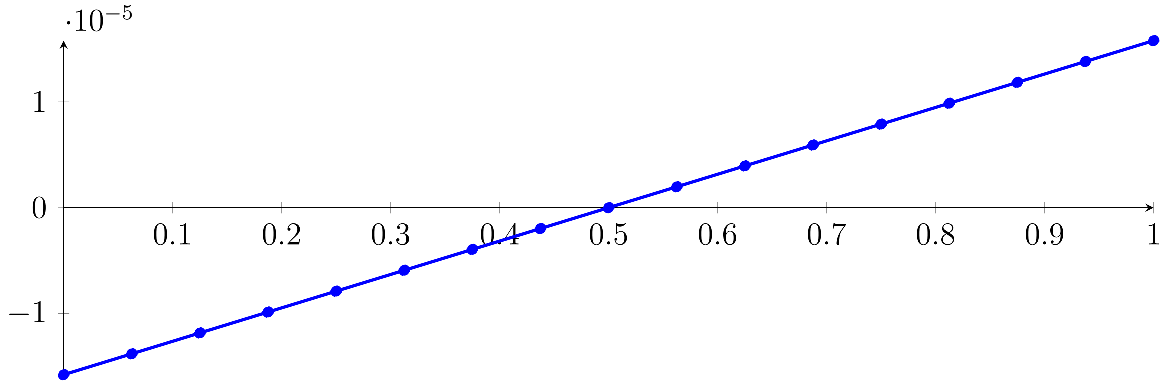
$$[0.532235, 0.532235]$$

Longest intersection interval: $3.8903{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

83.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

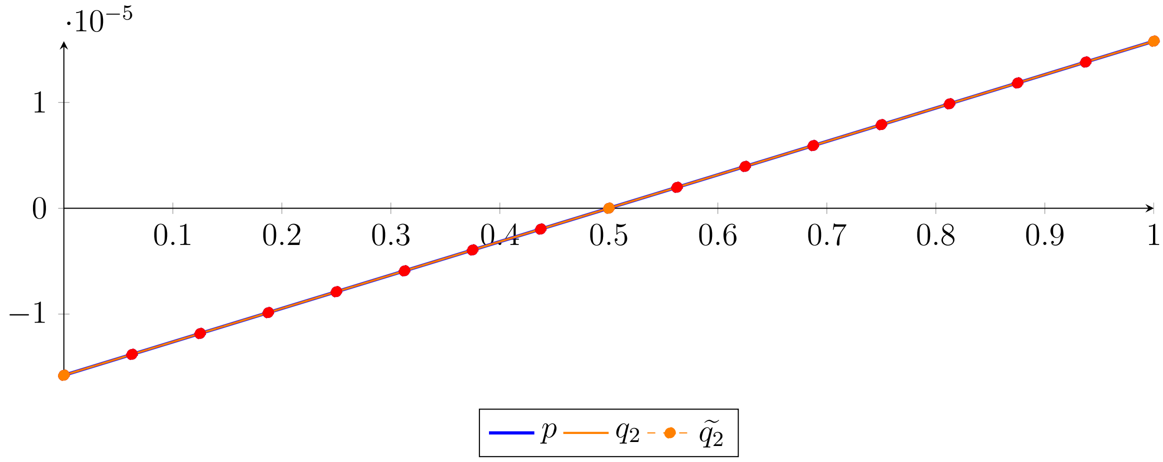
$$\begin{aligned} p = &-1.04409{\cdot}10^{-16}X^{16} - 1.53089{\cdot}10^{-16}X^{15} + 1.74665{\cdot}10^{-15}X^{14} + 5.46438{\cdot}10^{-15}X^{13} \\ &+ 2.56522{\cdot}10^{-14}X^{12} + 2.15479{\cdot}10^{-14}X^{11} + 3.51633{\cdot}10^{-14}X^{10} + 1.67444 \\ &{\cdot}10^{-14}X^{9} - 8.72105{\cdot}10^{-16}X^{8} + 1.41087{\cdot}10^{-15}X^{6} + 5.91974{\cdot}10^{-17}X^{5} + 4.93312 \\ &{\cdot}10^{-17}X^{4} + 3.79471{\cdot}10^{-18}X^{3} - 4.87891{\cdot}10^{-18}X^{2} + 3.157{\cdot}10^{-05}X - 1.57804{\cdot}10^{-05} \\ = &-1.57804{\cdot}10^{-05}B_{0,16}(X) - 1.38073{\cdot}10^{-05}B_{1,16}(X) - 1.18341{\cdot}10^{-05}B_{2,16}(X) - 9.86101 \\ &{\cdot}10^{-06}B_{3,16}(X) - 7.88788{\cdot}10^{-06}B_{4,16}(X) - 5.91476{\cdot}10^{-06}B_{5,16}(X) - 3.94163{\cdot}10^{-06}B_{6,16}(X) \\ &- 1.96851{\cdot}10^{-06}B_{7,16}(X) + 4.61501{\cdot}10^{-09}B_{8,16}(X) + 1.97774{\cdot}10^{-06}B_{9,16}(X) + 3.95086 \\ &{\cdot}10^{-06}B_{10,16}(X) + 5.92399{\cdot}10^{-06}B_{11,16}(X) + 7.89711{\cdot}10^{-06}B_{12,16}(X) + 9.87024{\cdot}10^{-06}B_{13,16}(X) \\ &+ 1.18434{\cdot}10^{-05}B_{14,16}(X) + 1.38165{\cdot}10^{-05}B_{15,16}(X) + 1.57896{\cdot}10^{-05}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -4.65868\cdot 10^{-18}X^2 + 3.157\cdot 10^{-05}X - 1.57804\cdot 10^{-05} \\ &= -1.57804\cdot 10^{-05}B_{0,2} + 4.61501\cdot 10^{-09}B_{1,2} + 1.57896\cdot 10^{-05}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 2.92413\cdot 10^{-07}X^{16} - 2.33332\cdot 10^{-06}X^{15} + 8.45203\cdot 10^{-06}X^{14} - 1.83895\cdot 10^{-05}X^{13} \\ &\quad + 2.67963\cdot 10^{-05}X^{12} - 2.75995\cdot 10^{-05}X^{11} + 2.06638\cdot 10^{-05}X^{10} - 1.13854\cdot 10^{-05}X^9 \\ &\quad + 4.61944\cdot 10^{-06}X^8 - 1.36687\cdot 10^{-06}X^7 + 2.89249\cdot 10^{-07}X^6 - 4.25295\cdot 10^{-08}X^5 + 4.17283 \\ &\quad \cdot 10^{-09}X^4 - 2.52119\cdot 10^{-10}X^3 + 7.82992\cdot 10^{-12}X^2 + 3.157\cdot 10^{-05}X - 1.57804\cdot 10^{-05} \\ &= -1.57804\cdot 10^{-05}B_{0,16} - 1.38073\cdot 10^{-05}B_{1,16} - 1.18341\cdot 10^{-05}B_{2,16} - 9.86101\cdot 10^{-06}B_{3,16} - 7.88788 \\ &\quad \cdot 10^{-06}B_{4,16} - 5.91476\cdot 10^{-06}B_{5,16} - 3.94163\cdot 10^{-06}B_{6,16} - 1.96851\cdot 10^{-06}B_{7,16} + 4.62125\cdot 10^{-09}B_{8,16} \\ &\quad + 1.97773\cdot 10^{-06}B_{9,16} + 3.95087\cdot 10^{-06}B_{10,16} + 5.92399\cdot 10^{-06}B_{11,16} + 7.89711\cdot 10^{-06}B_{12,16} \\ &\quad + 9.87024\cdot 10^{-06}B_{13,16} + 1.18434\cdot 10^{-05}B_{14,16} + 1.38165\cdot 10^{-05}B_{15,16} + 1.57896\cdot 10^{-05}B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.24192\cdot 10^{-12}$.

Bounding polynomials M and m:

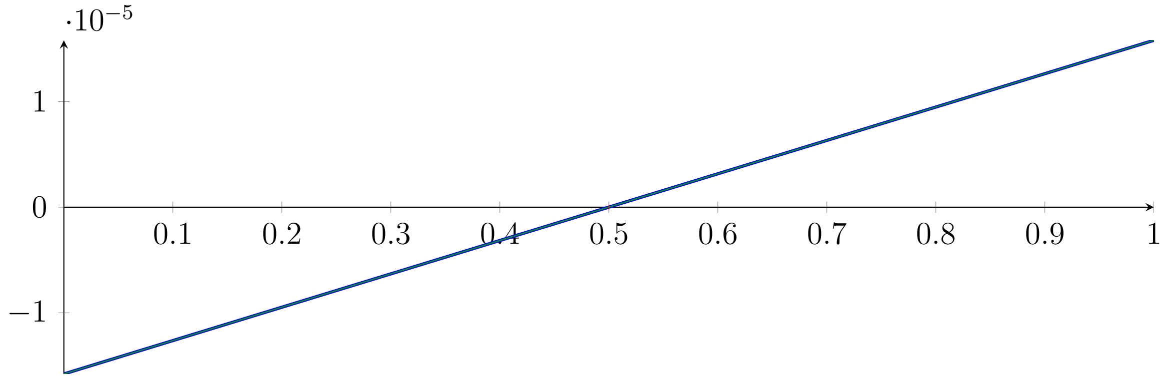
$$M = -4.65868\cdot 10^{-18}X^2 + 3.157\cdot 10^{-05}X - 1.57804\cdot 10^{-05}$$

$$m = -4.65868\cdot 10^{-18}X^2 + 3.157\cdot 10^{-05}X - 1.57804\cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.499636, 6.77659\cdot 10^{12}\} \qquad N(m) = \{0.500364, 6.77659\cdot 10^{12}\}$$

Intersection intervals:



$[0.499636, 0.500364]$

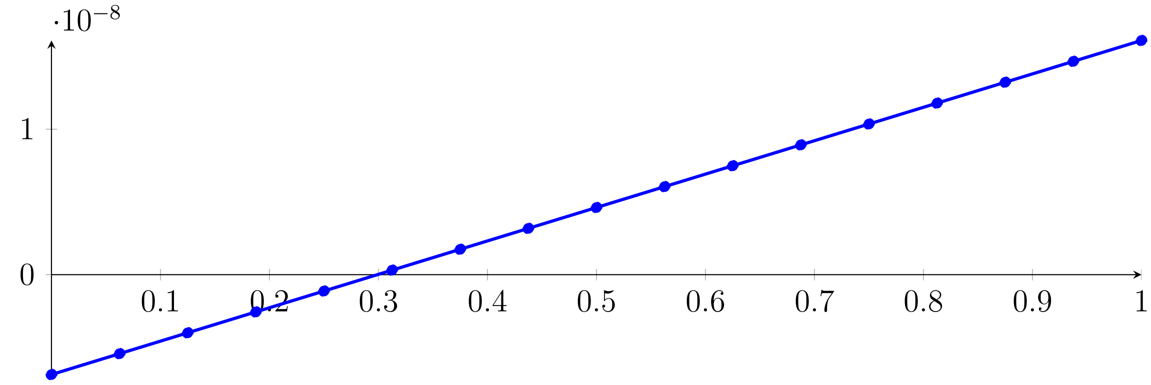
Longest intersection interval: 0.000727273

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

83.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

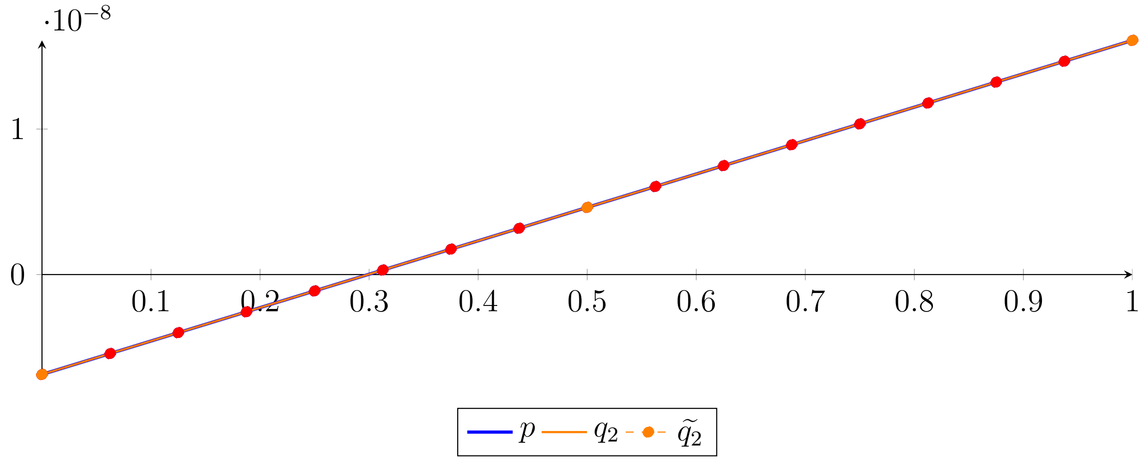
$$p = -6.76104 \cdot 10^{-19} X^{16} + 2.21478 \cdot 10^{-18} X^{15} - 1.59295 \cdot 10^{-18} X^{14} + 1.03762 \cdot 10^{-18} X^{13} - 1.34649 \cdot 10^{-17} X^{12} + 9.65427 \cdot 10^{-18} X^{11} - 2.75561 \cdot 10^{-18} X^{10} + 5.90488 \cdot 10^{-18} X^{9} - 8.51665 \cdot 10^{-19} X^{8} + 3.02814 \cdot 10^{-19} X^{7} + 2.64962 \cdot 10^{-19} X^{6} + 2.8905 \cdot 10^{-20} X^{5} + 6.02187 \cdot 10^{-21} X^{4} + 9.26442 \cdot 10^{-22} X^{3} + 2.296 \cdot 10^{-08} X - 6.86499 \cdot 10^{-09}$$
$$= -6.86499 \cdot 10^{-09} B_{0,16}(X) - 5.42999 \cdot 10^{-09} B_{1,16}(X) - 3.99499 \cdot 10^{-09} B_{2,16}(X) - 2.55999 \cdot 10^{-09} B_{3,16}(X) - 1.12499 \cdot 10^{-09} B_{4,16}(X) + 3.10008 \cdot 10^{-10} B_{5,16}(X) + 1.74501 \cdot 10^{-09} B_{6,16}(X) + 3.18001 \cdot 10^{-09} B_{7,16}(X) + 4.61501 \cdot 10^{-09} B_{8,16}(X) + 6.05001 \cdot 10^{-09} B_{9,16}(X) + 7.48501 \cdot 10^{-09} B_{10,16}(X) + 8.92001 \cdot 10^{-09} B_{11,16}(X) + 1.0355 \cdot 10^{-08} B_{12,16}(X) + 1.179 \cdot 10^{-08} B_{13,16}(X) + 1.3225 \cdot 10^{-08} B_{14,16}(X) + 1.466 \cdot 10^{-08} B_{15,16}(X) + 1.6095 \cdot 10^{-08} B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = 1.40621 \cdot 10^{-23} X^2 + 2.296 \cdot 10^{-08} X - 6.86499 \cdot 10^{-09}$$
$$= -6.86499 \cdot 10^{-09} B_{0,2} + 4.61501 \cdot 10^{-09} B_{1,2} + 1.6095 \cdot 10^{-08} B_{2,2}$$

$$\widetilde{q}_2 = 2.65578 \cdot 10^{-10} X^{16} - 2.13329 \cdot 10^{-09} X^{15} + 7.78369 \cdot 10^{-09} X^{14} - 1.70728 \cdot 10^{-08} X^{13} + 2.51029 \cdot 10^{-08} X^{12} - 2.61095 \cdot 10^{-08} X^{11} + 1.97446 \cdot 10^{-08} X^{10} - 1.09796 \cdot 10^{-08} X^{9} + 4.48709 \cdot 10^{-09} X^{8} - 1.33355 \cdot 10^{-09} X^{7} + 2.82501 \cdot 10^{-10} X^{6} - 4.12963 \cdot 10^{-11} X^{5} + 3.94088 \cdot 10^{-12} X^{4} - 2.24328 \cdot 10^{-13} X^{3} + 6.17064 \cdot 10^{-15} X^{2} + 2.296 \cdot 10^{-08} X - 6.86499 \cdot 10^{-09}$$
$$= -6.86499 \cdot 10^{-09} B_{0,16} - 5.42999 \cdot 10^{-09} B_{1,16} - 3.99499 \cdot 10^{-09} B_{2,16} - 2.55999 \cdot 10^{-09} B_{3,16} - 1.12499 \cdot 10^{-09} B_{4,16} + 3.10006 \cdot 10^{-10} B_{5,16} + 1.74501 \cdot 10^{-09} B_{6,16} + 3.18 \cdot 10^{-09} B_{7,16} + 4.61501 \cdot 10^{-09} B_{8,16} + 6.05 \cdot 10^{-09} B_{9,16} + 7.48501 \cdot 10^{-09} B_{10,16} + 8.92 \cdot 10^{-09} B_{11,16} + 1.0355 \cdot 10^{-08} B_{12,16} + 1.179 \cdot 10^{-08} B_{13,16} + 1.3225 \cdot 10^{-08} B_{14,16} + 1.466 \cdot 10^{-08} B_{15,16} + 1.6095 \cdot 10^{-08} B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 5.35405 \cdot 10^{-15}$.
Bounding polynomials M and m:

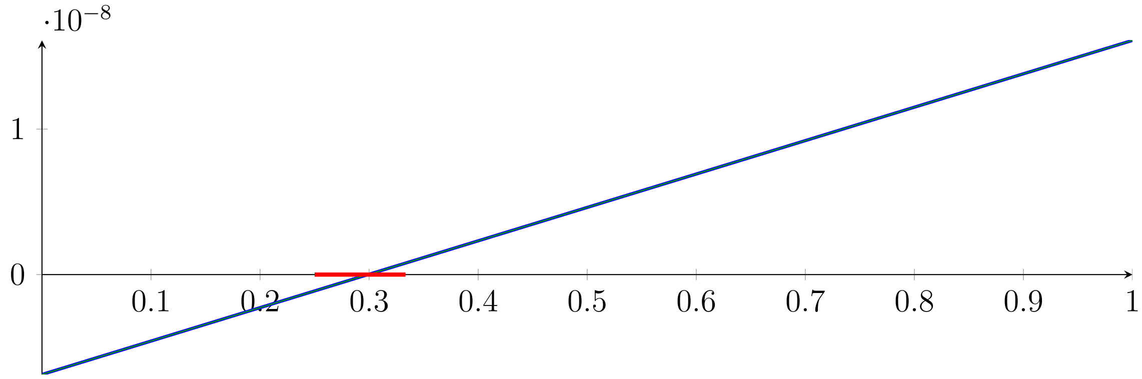
$$M = 1.32349 \cdot 10^{-23} X^2 + 2.296 \cdot 10^{-08} X - 6.86498 \cdot 10^{-09}$$

$$m = 1.48893 \cdot 10^{-23} X^2 + 2.296 \cdot 10^{-08} X - 6.865 \cdot 10^{-09}$$

Root of M and m:

$$N(M) = \{-1.73481 \cdot 10^{15}, 0.25\} \qquad N(m) = \{-1.54205 \cdot 10^{15}, 0.333333\}$$

Intersection intervals:



$$[0.333333, 0.25]$$

Longest intersection interval: -0.0833333
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

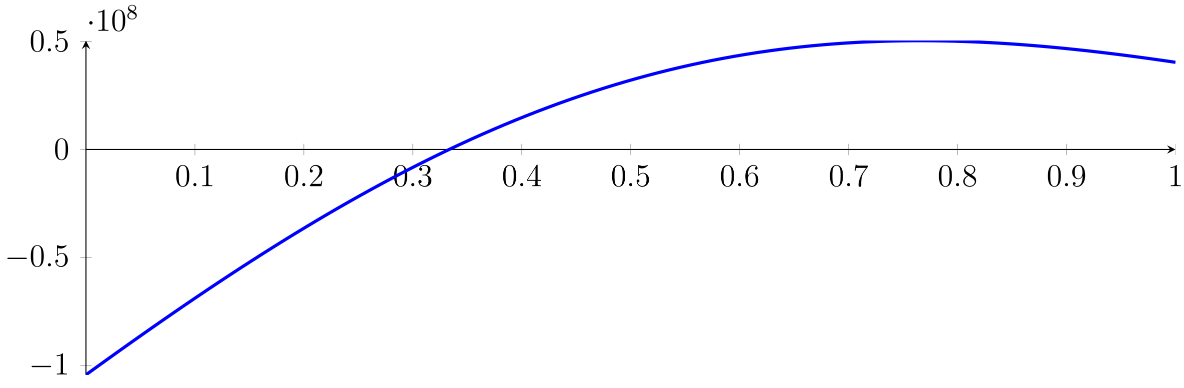
83.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

83.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

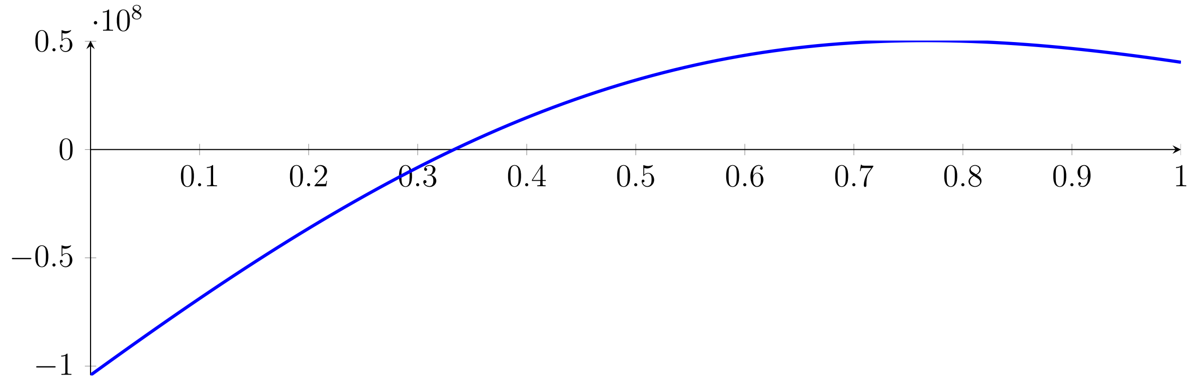
with precision $\varepsilon = 1\cdot10^{-128}$.

84 Running CubeClip on f_{16} with epsilon 128

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

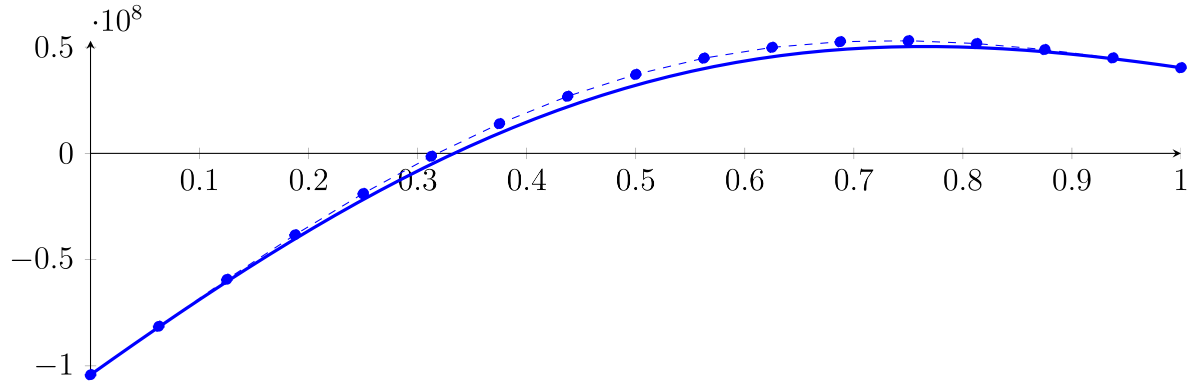
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} \\ &+ 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \end{aligned}$$



84.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

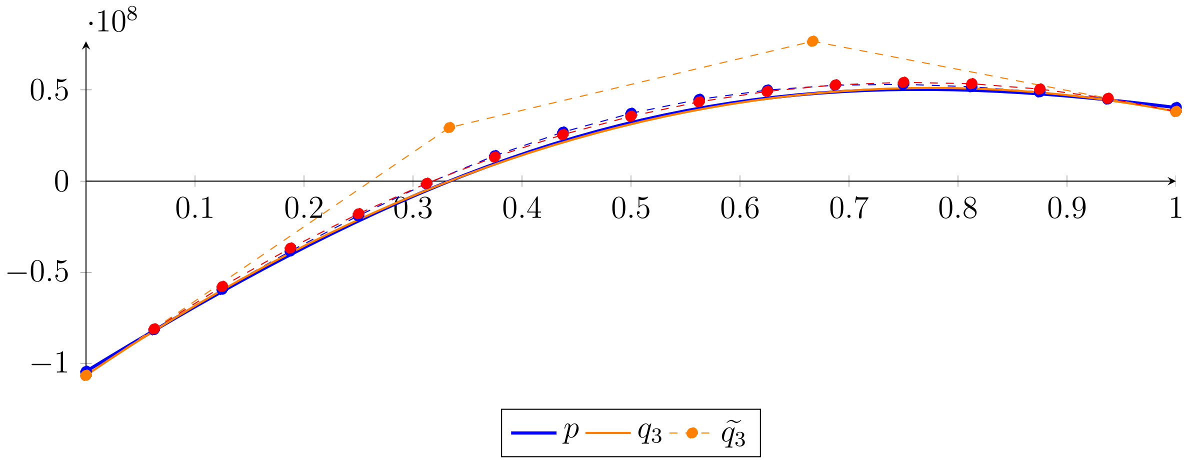
$$\begin{aligned} p = &-0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\ &+ 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ = &-1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ &\cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ &+ 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ &\cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ &+ 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.06335\cdot 10^8$$
$$= -1.06335\cdot 10^8 B_{0,3} + 2.93558\cdot 10^7 B_{1,3} + 7.66068\cdot 10^7 B_{2,3} + 3.81764\cdot 10^7 B_{3,3}$$

$$\widetilde{q}_3 = 2.11088\cdot 10^6 X^{16} - 1.70537\cdot 10^7 X^{15} + 6.25913\cdot 10^7 X^{14} - 1.38111\cdot 10^8 X^{13} + 2.043\cdot 10^8 X^{12} - 2.13799\cdot 10^8 X^{11} + 1.62714\cdot 10^8 X^{10} - 9.11092\cdot 10^7 X^9 + 3.75254\cdot 10^7 X^8 - 1.12529\cdot 10^7 X^7 + 2.40899\cdot 10^6 X^6 - 357156 X^5 + 34986.1 X^4 + 2.75602\cdot 10^6 X^3 - 2.65318\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.06335\cdot 10^8$$
$$= -1.06335\cdot 10^8 B_{0,16} - 8.08928\cdot 10^7 B_{1,16} - 5.76618\cdot 10^7 B_{2,16} - 3.66369\cdot 10^7 B_{3,16} - 1.78131\cdot 10^7 B_{4,16} - 1.18553\cdot 10^6 B_{5,16} + 1.32508\cdot 10^7 B_{6,16} + 2.55007\cdot 10^7 B_{7,16} + 3.55692\cdot 10^7 B_{8,16} + 4.34611\cdot 10^7 B_{9,16} + 4.91816\cdot 10^7 B_{10,16} + 5.27352\cdot 10^7 B_{11,16} + 5.41273\cdot 10^7 B_{12,16} + 5.33624\cdot 10^7 B_{13,16} + 5.04457\cdot 10^7 B_{14,16} + 4.53821\cdot 10^7 B_{15,16} + 3.81764\cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806\cdot 10^6$.

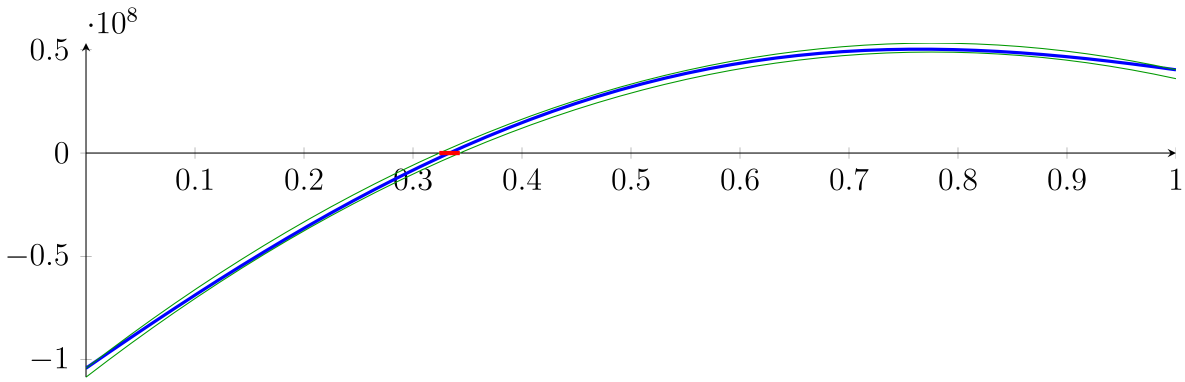
Bounding polynomials M and m:

$$M = 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.04167\cdot 10^8$$
$$m = 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.08503\cdot 10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

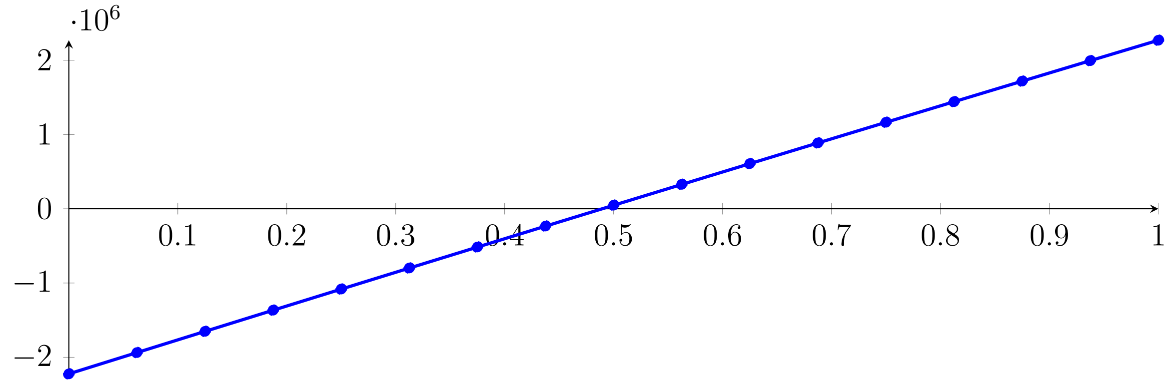
Longest intersection interval: 0.0187703

$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

84.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

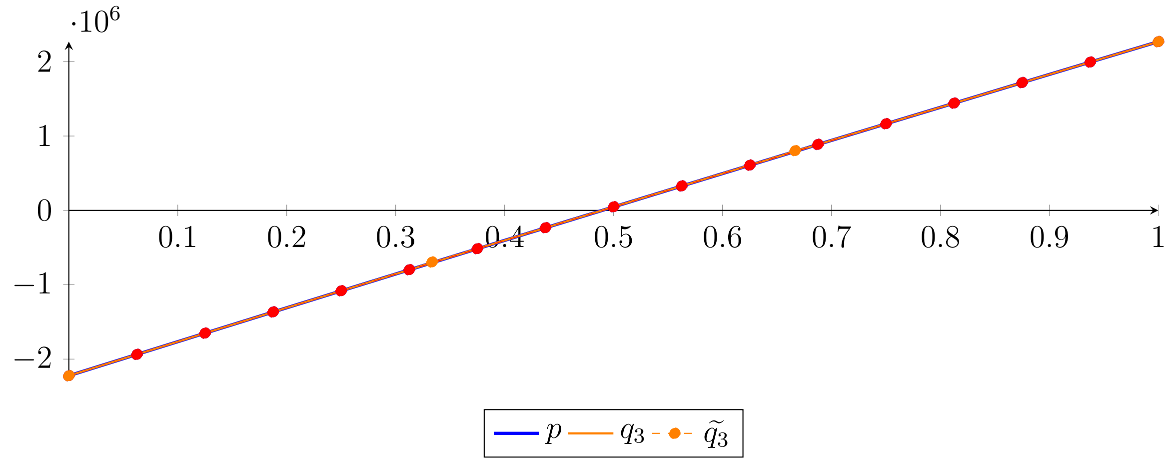
$$\begin{aligned} p = & -1.93035\cdot 10^{-05}X^{16} + 1.08927\cdot 10^{-05}X^{15} + 0.000255816X^{14} + 0.000616983X^{13} + 0.00371715X^{12} \\ & + 0.00325035X^{11} + 0.00510875X^{10} + 0.00190713X^{9} + 0.000179792X^{8} + 1.06543\cdot 10^{-05}X^{7} \\ & - 0.00136482X^{6} + 0.104959X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6} \\ = & -2.22335\cdot 10^{6}B_{0,16}(X) - 1.93666\cdot 10^{6}B_{1,16}(X) - 1.65074\cdot 10^{6}B_{2,16}(X) - 1.36561\cdot 10^{6}B_{3,16}(X) \\ & - 1.08127\cdot 10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X) \\ & + 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot 10^{6}B_{12,16}(X) + 1.44233 \\ & \cdot 10^{6}B_{13,16}(X) + 1.71876\cdot 10^{6}B_{14,16}(X) + 1.99439\cdot 10^{6}B_{15,16}(X) + 2.26922\cdot 10^{6}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 = & -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6} \\ = & -2.22335\cdot 10^{6}B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot 10^{6}B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 = & 15290.6X^{16} - 125412X^{15} + 468045X^{14} - 1.05198\cdot 10^{6}X^{13} + 1.588\cdot 10^{6}X^{12} - 1.69893 \\ & \cdot 10^{6}X^{11} + 1.32382\cdot 10^{6}X^{10} - 759402X^{9} + 320067X^{8} - 97826.5X^{7} + 21197.8X^{6} \\ & - 3162.8X^{5} + 313.931X^{4} - 720.165X^{3} - 93879.2X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6} \\ = & -2.22335\cdot 10^{6}B_{0,16} - 1.93666\cdot 10^{6}B_{1,16} - 1.65074\cdot 10^{6}B_{2,16} - 1.36561\cdot 10^{6}B_{3,16} \\ & - 1.08126\cdot 10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.7B_{8,16} \\ & + 328649B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot 10^{6}B_{12,16} + 1.44233 \\ & \cdot 10^{6}B_{13,16} + 1.71876\cdot 10^{6}B_{14,16} + 1.99439\cdot 10^{6}B_{15,16} + 2.26922\cdot 10^{6}B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.457751$.

Bounding polynomials M and m:

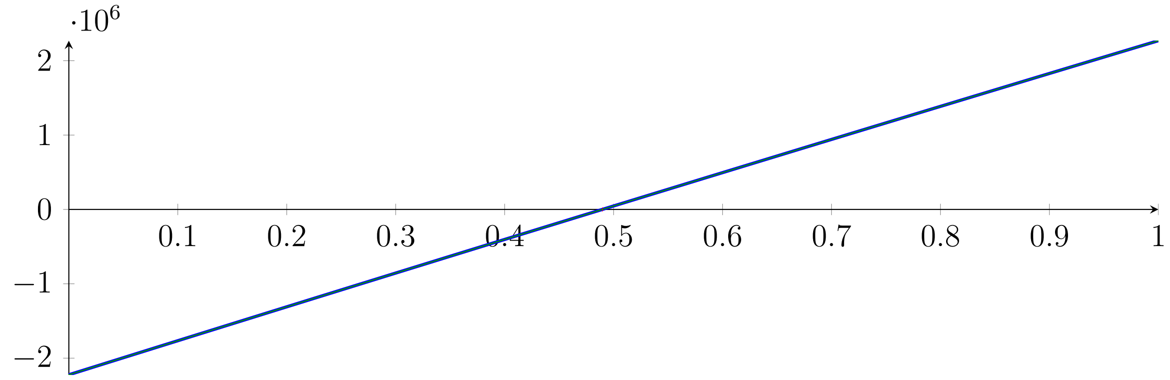
$$M = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6}$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715\cdot 10^6 X - 2.22335\cdot 10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



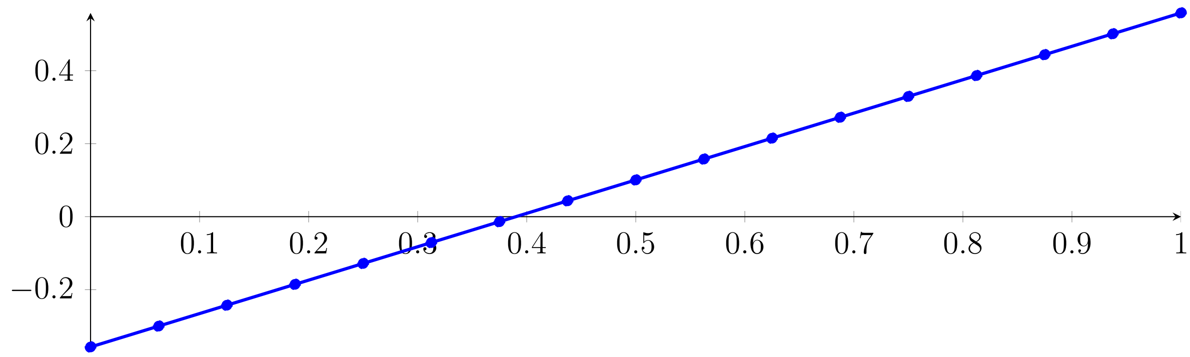
$$[0.489616, 0.489616]$$

Longest intersection interval: $2.03684\cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

84.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

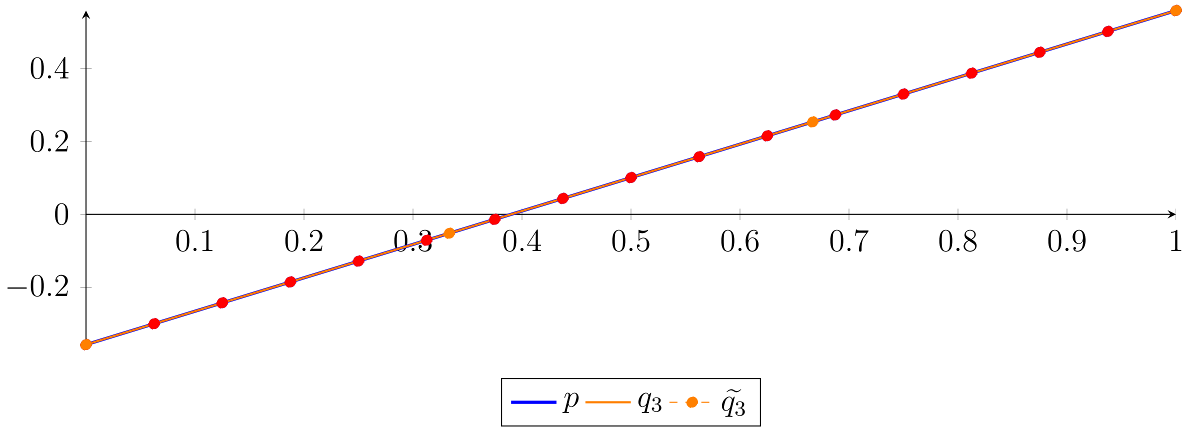
$$\begin{aligned}
p = &-1.56399\cdot 10^{-11}X^{16} + 4.58016\cdot 10^{-11}X^{15} - 3.19744\cdot 10^{-12}X^{14} + 6.54055\cdot 10^{-11}X^{13} \\
&+ 1.05072\cdot 10^{-10}X^{12} + 4.59728\cdot 10^{-10}X^{11} + 5.01434\cdot 10^{-10}X^{10} + 2.99742\cdot 10^{-10}X^{9} \\
&+ 1.14309\cdot 10^{-11}X^{8} - 5.08038\cdot 10^{-12}X^{7} + 3.37845\cdot 10^{-11}X^{6} - 9.69891\cdot 10^{-13}X^{5} \\
&+ 4.04121\cdot 10^{-13}X^{4} + 6.21725\cdot 10^{-14}X^{3} - 3.93767\cdot 10^{-09}X^{2} + 0.915501X - 0.357072 \\
= &-0.357072B_{0,16}(X) - 0.299853B_{1,16}(X) - 0.242635B_{2,16}(X) - 0.185416B_{3,16}(X) \\
&- 0.128197B_{4,16}(X) - 0.0709781B_{5,16}(X) - 0.0137592B_{6,16}(X) \\
&+ 0.0434596B_{7,16}(X) + 0.100678B_{8,16}(X) + 0.157897B_{9,16}(X) + 0.215116B_{10,16}(X) \\
&+ 0.272335B_{11,16}(X) + 0.329554B_{12,16}(X) + 0.386773B_{13,16}(X) \\
&+ 0.443991B_{14,16}(X) + 0.50121B_{15,16}(X) + 0.558429B_{16,16}(X)
\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
q_3 &= 1.05471\cdot 10^{-15}X^3 - 3.93767\cdot 10^{-09}X^2 + 0.915501X - 0.357072 \\
&= -0.357072B_{0,3} - 0.0519051B_{1,3} + 0.253262B_{2,3} + 0.558429B_{3,3}
\end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= 0.00291222X^{16} - 0.0241801X^{15} + 0.0914452X^{14} - 0.208537X^{13} + 0.319778X^{12} - 0.347745X^{11} \\ &\quad + 0.275244X^{10} - 0.159971X^{9} + 0.0679818X^{8} - 0.0208072X^{7} + 0.00447629X^{6} - 0.000654783X^{5} \\ &\quad + 6.22034\cdot 10^{-05}X^{4} - 3.60145\cdot 10^{-06}X^{3} + 9.78811\cdot 10^{-08}X^{2} + 0.915501X - 0.357072 \\ &= -0.357072B_{0,16} - 0.299853B_{1,16} - 0.242635B_{2,16} - 0.185416B_{3,16} - 0.128197B_{4,16} \\ &\quad - 0.0709781B_{5,16} - 0.0137592B_{6,16} + 0.0434595B_{7,16} + 0.100678B_{8,16} \\ &\quad + 0.157897B_{9,16} + 0.215116B_{10,16} + 0.272335B_{11,16} + 0.329554B_{12,16} \\ &\quad + 0.386773B_{13,16} + 0.443991B_{14,16} + 0.50121B_{15,16} + 0.558429B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.5212\cdot 10^{-08}$.

Bounding polynomials M and m:

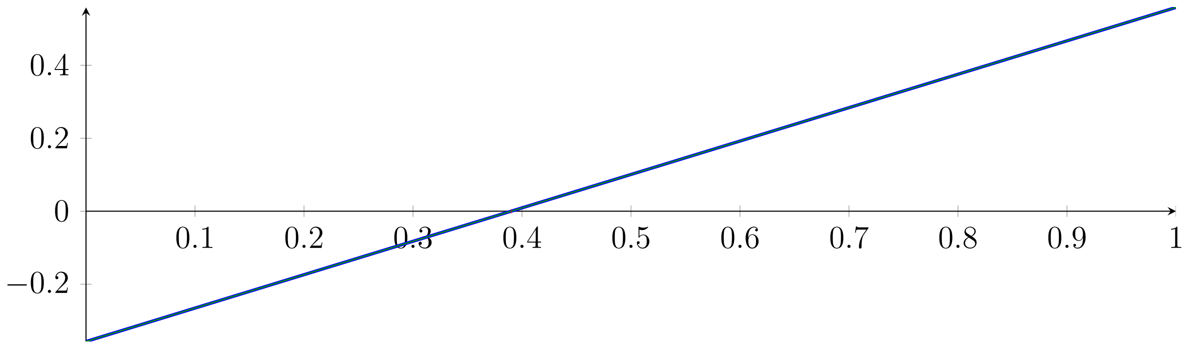
$$M = 9.99201\cdot 10^{-16}X^{3} - 3.93767\cdot 10^{-09}X^{2} + 0.915501X - 0.357072$$

$$m = 1.22125\cdot 10^{-15}X^{3} - 3.93767\cdot 10^{-09}X^{2} + 0.915501X - 0.357072$$

Root of M and m:

$$N(M) = \{0.390029\} \qquad N(m) = \{0.390029\}$$

Intersection intervals:



$$[0.390029, 0.390029]$$

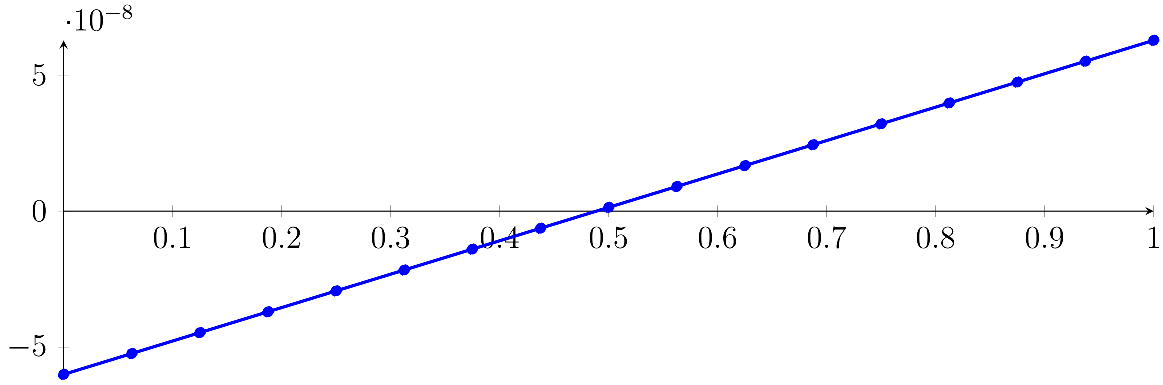
Longest intersection interval: $1.3411\cdot 10^{-07}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

84.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

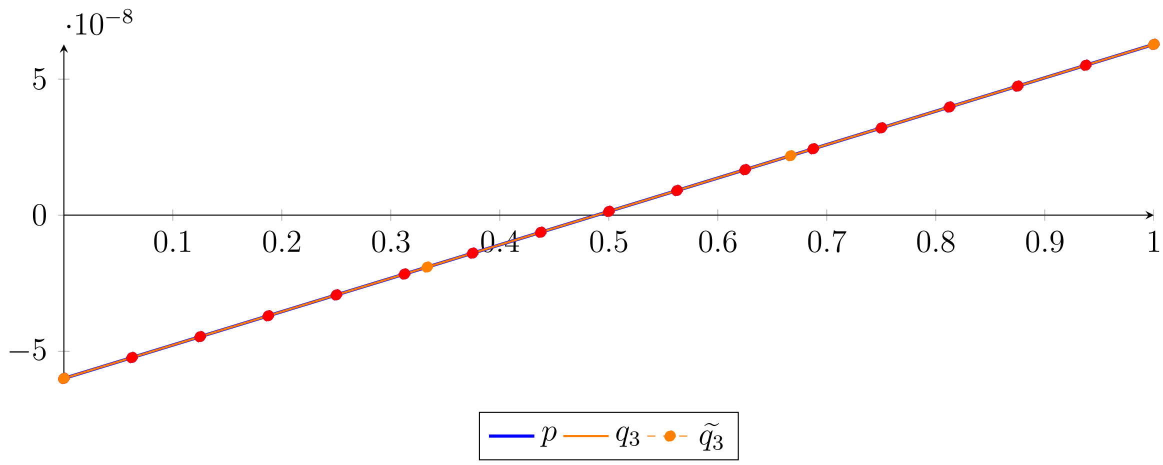
$$p = -5.65183\cdot10^{-19}X^{16} + 5.13302\cdot10^{-19}X^{15} + 6.37816\cdot10^{-18}X^{14} + 1.69576\cdot10^{-17}X^{13} + 9.4423\cdot10^{-17}X^{12} + 8.0934\cdot10^{-17}X^{11} + 1.37357\cdot10^{-16}X^{10} + 6.23797\cdot10^{-17}X^{9} + 1.36266\cdot10^{-18}X^{8} + 6.05629\cdot10^{-19}X^{7} + 5.08728\cdot10^{-18}X^{6} + 2.3124\cdot10^{-19}X^{5} + 1.44525\cdot10^{-19}X^{4} - 7.41154\cdot10^{-21}X^{3} + 1.22778\cdot10^{-07}X - 6.00146\cdot10^{-08}$$
$$= -6.00146\cdot10^{-08}B_{0,16}(X) - 5.2341\cdot10^{-08}B_{1,16}(X) - 4.46674\cdot10^{-08}B_{2,16}(X) - 3.69937\cdot10^{-08}B_{3,16}(X) - 2.93201\cdot10^{-08}B_{4,16}(X) - 2.16464\cdot10^{-08}B_{5,16}(X) - 1.39728\cdot10^{-08}B_{6,16}(X) - 6.29913\cdot10^{-09}B_{7,16}(X) + 1.37451\cdot10^{-09}B_{8,16}(X) + 9.04815\cdot10^{-09}B_{9,16}(X) + 1.67218\cdot10^{-08}B_{10,16}(X) + 2.43954\cdot10^{-08}B_{11,16}(X) + 3.20691\cdot10^{-08}B_{12,16}(X) + 3.97427\cdot10^{-08}B_{13,16}(X) + 4.74164\cdot10^{-08}B_{14,16}(X) + 5.509\cdot10^{-08}B_{15,16}(X) + 6.27637\cdot10^{-08}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = 2.51463\cdot10^{-22}X^3 - 3.17637\cdot10^{-22}X^2 + 1.22778\cdot10^{-07}X - 6.00146\cdot10^{-08}$$
$$= -6.00146\cdot10^{-08}B_{0,3} - 1.90885\cdot10^{-08}B_{1,3} + 2.18376\cdot10^{-08}B_{2,3} + 6.27637\cdot10^{-08}B_{3,3}$$

$$\widetilde{q}_3 = 4.01426\cdot10^{-10}X^{16} - 3.29623\cdot10^{-09}X^{15} + 1.2317\cdot10^{-08}X^{14} - 2.77213\cdot10^{-08}X^{13} + 4.19074\cdot10^{-08}X^{12} - 4.49047\cdot10^{-08}X^{11} + 3.50457\cdot10^{-08}X^{10} - 2.01341\cdot10^{-08}X^{9} + 8.49676\cdot10^{-09}X^{8} - 2.5992\cdot10^{-09}X^{7} + 5.63364\cdot10^{-10}X^{6} - 8.40231\cdot10^{-11}X^{5} + 8.33259\cdot10^{-12}X^{4} - 5.16965\cdot10^{-13}X^{3} + 1.7395\cdot10^{-14}X^{2} + 1.22778\cdot10^{-07}X - 6.00146\cdot10^{-08}$$
$$= -6.00146\cdot10^{-08}B_{0,16} - 5.2341\cdot10^{-08}B_{1,16} - 4.46674\cdot10^{-08}B_{2,16} - 3.69937\cdot10^{-08}B_{3,16} - 2.93201\cdot10^{-08}B_{4,16} - 2.16464\cdot10^{-08}B_{5,16} - 1.39728\cdot10^{-08}B_{6,16} - 6.29914\cdot10^{-09}B_{7,16} + 1.37452\cdot10^{-09}B_{8,16} + 9.04815\cdot10^{-09}B_{9,16} + 1.67218\cdot10^{-08}B_{10,16} + 2.43954\cdot10^{-08}B_{11,16} + 3.20691\cdot10^{-08}B_{12,16} + 3.97427\cdot10^{-08}B_{13,16} + 4.74164\cdot10^{-08}B_{14,16} + 5.509\cdot10^{-08}B_{15,16} + 6.27637\cdot10^{-08}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 7.90061 \cdot 10^{-15}$.
Bounding polynomials M and m:

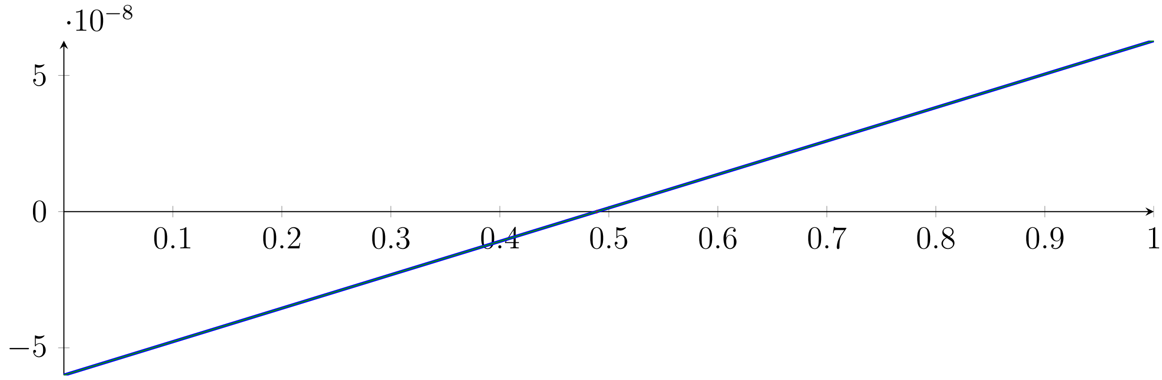
$$M = 2.38228 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08}$$

$$m = 2.51463 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08}$$

Root of M and m:

$$N(M) = \{0.488805\} \qquad N(m) = \{0.488805\}$$

Intersection intervals:



$$[0.488805, 0.488805]$$

Longest intersection interval: $1.3086 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

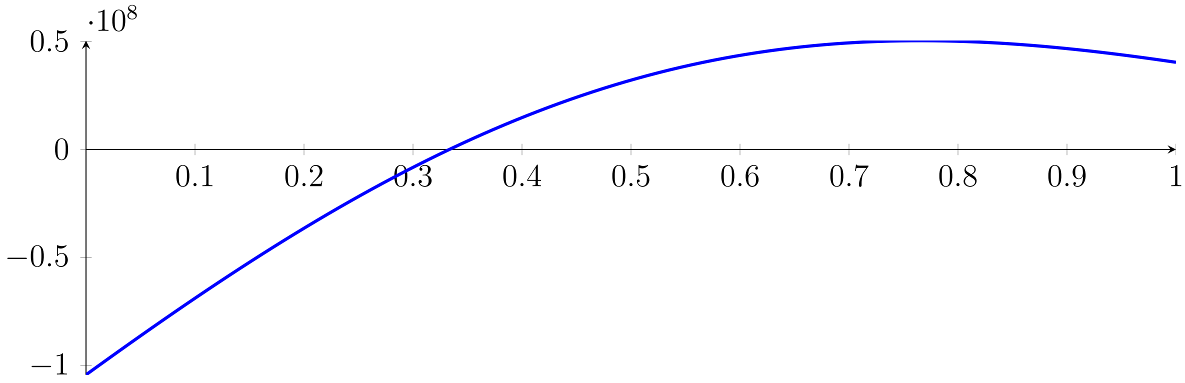
84.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

84.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-128}$.

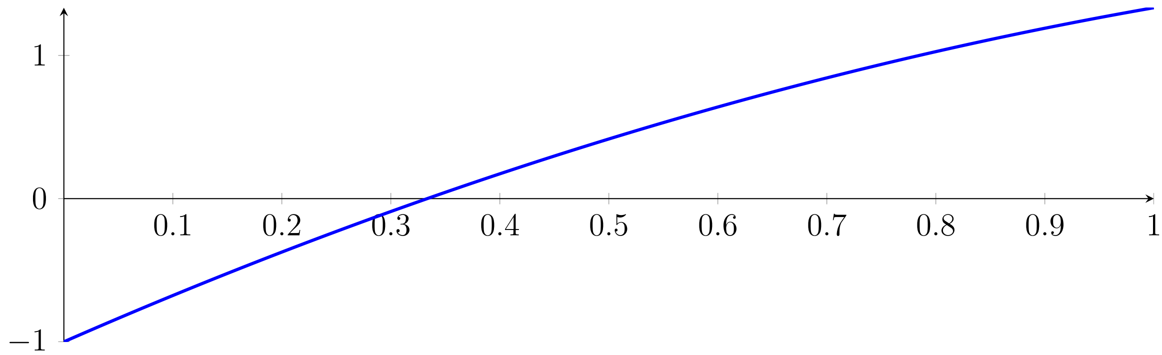
Part II
Numeric = long double

85 Running BezClip on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval [0, 1]:

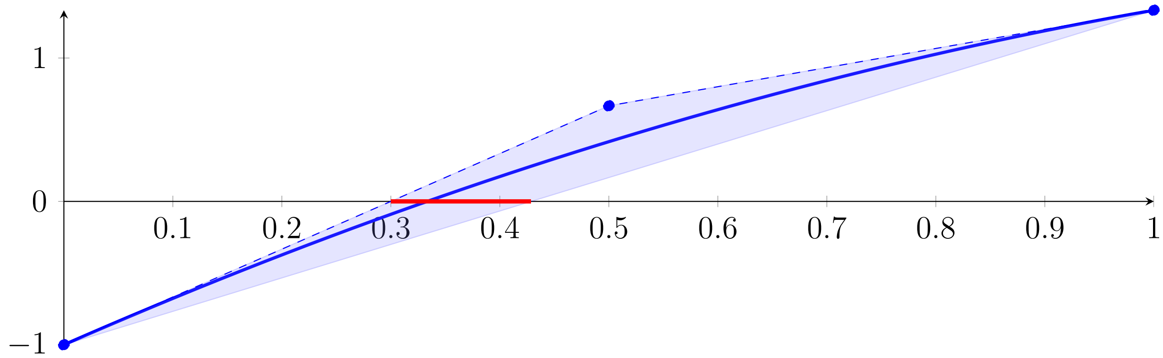
$$p = -1X^2 + 3.33333X - 1$$



85.1 Recursion Branch 1 for Input Interval [0, 1]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

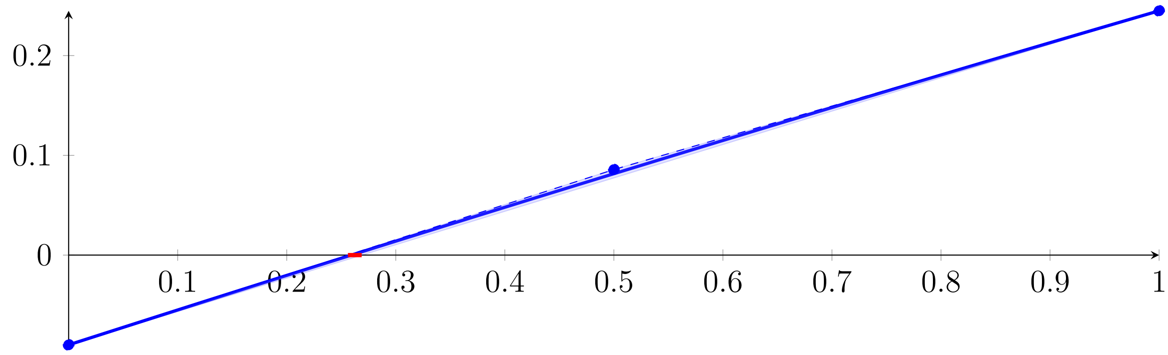
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
⟹ Selective recursion: interval 1: [0.3, 0.428571],

85.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

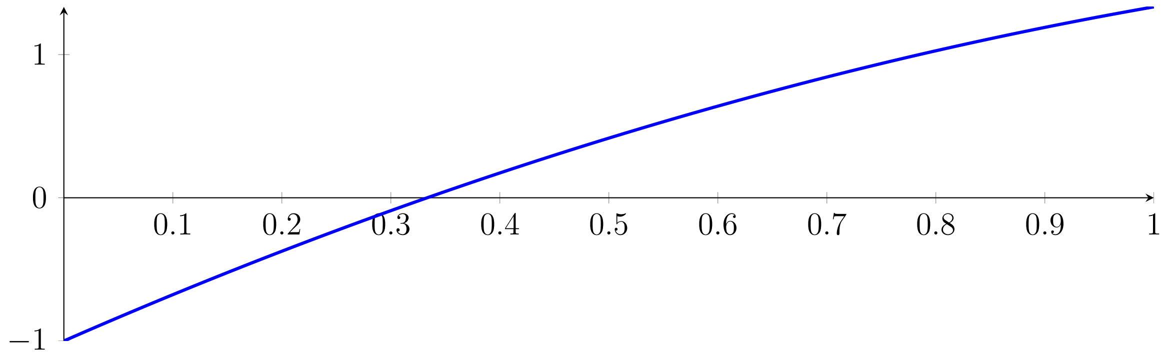
85.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Found root in interval $[0.332927, 0.334552]$ at recursion depth 3!

85.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.332927, 0.334552]$$

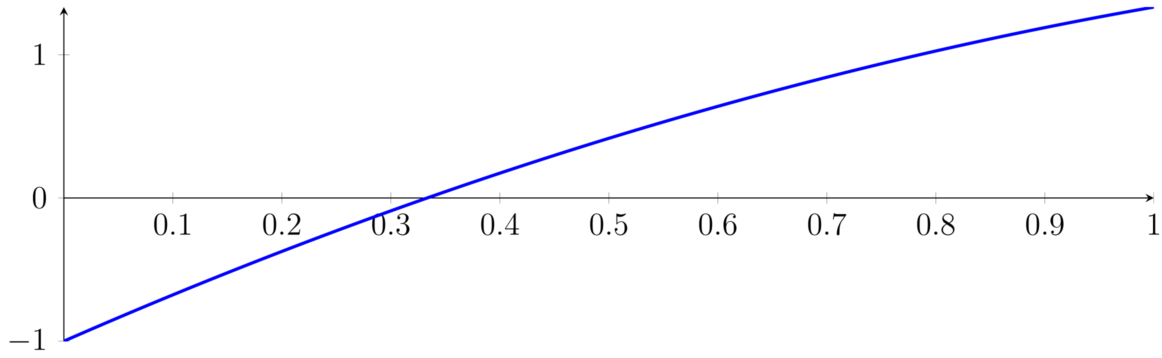
with precision $\varepsilon = 0.01$.

86 Running QuadClip on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

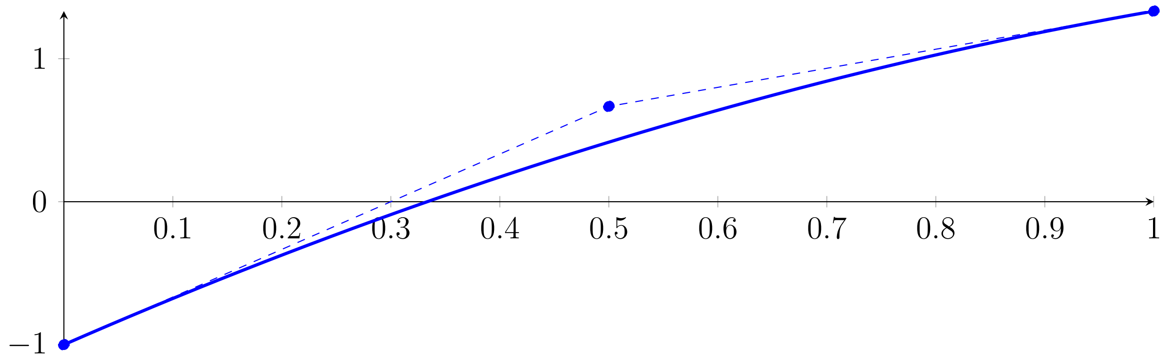
$$p = -1X^2 + 3.33333X - 1$$



86.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

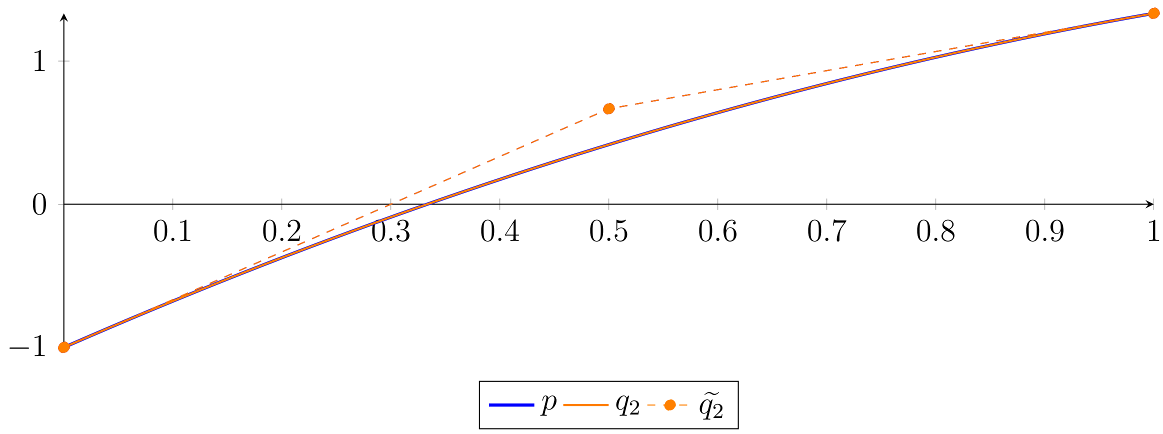
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.
Bounding polynomials M and m:

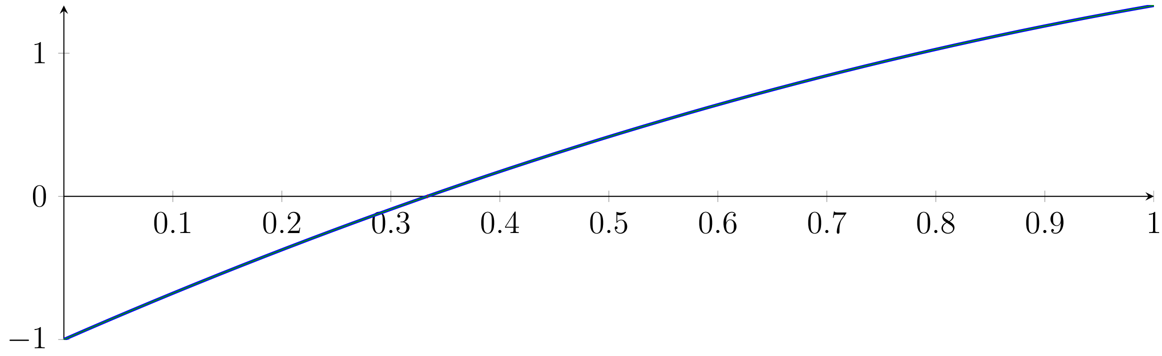
$$M = -1X^2 + 3.33333X - 1$$

$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $3.25261 \cdot 10^{-19}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

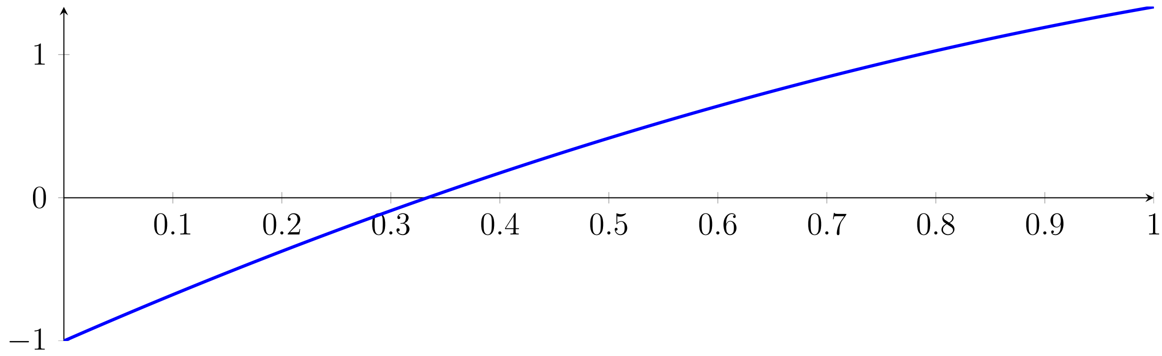
86.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

86.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

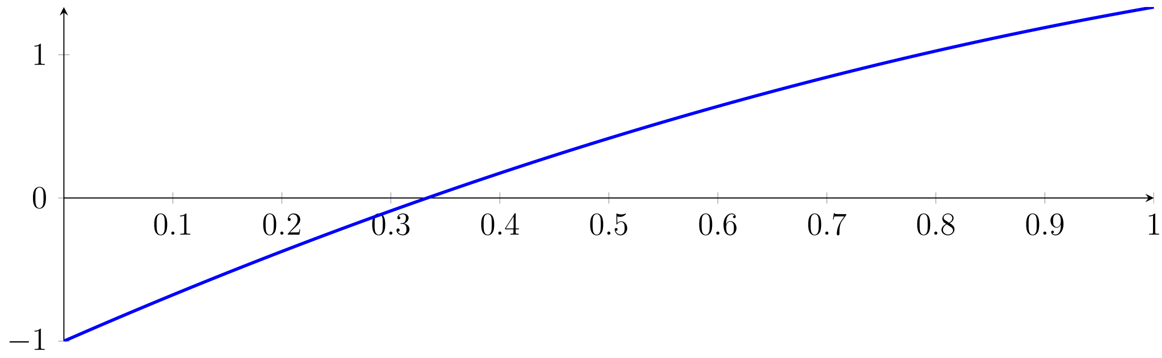
with precision $\varepsilon = 0.01$.

87 Running `CubeClip` on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

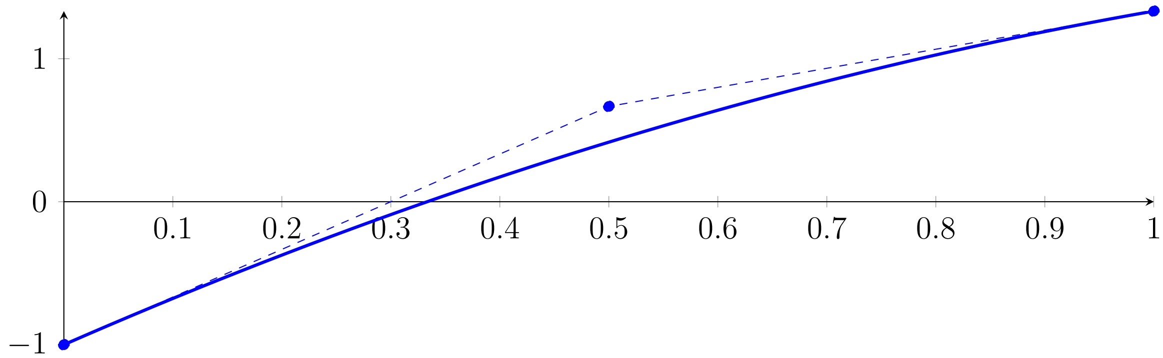
$$p = -1X^2 + 3.33333X - 1$$



87.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

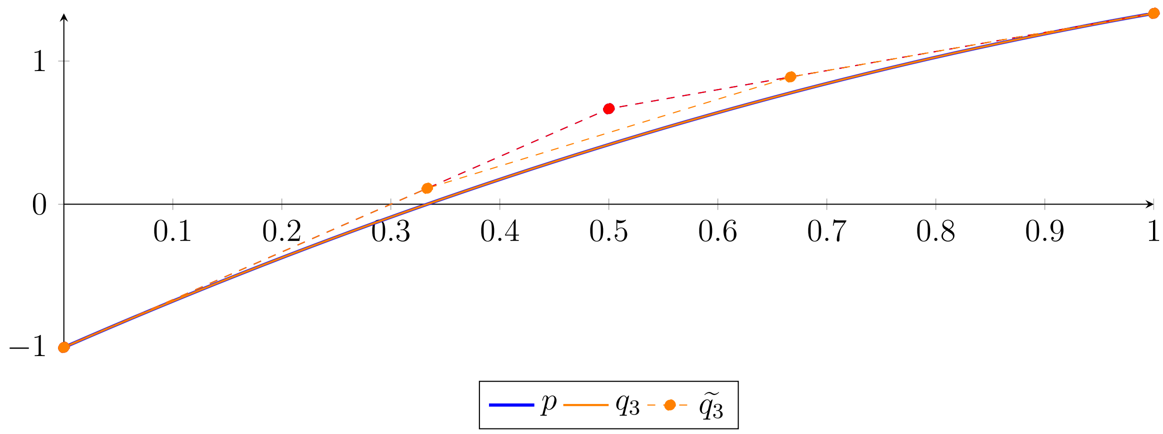
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.63715 \cdot 10^{-17}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.
Bounding polynomials M and m:

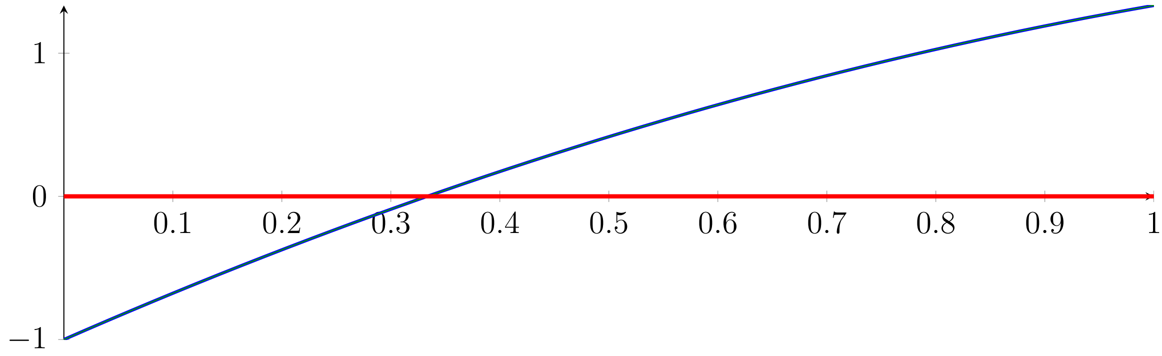
$$M = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

$$m = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{-6.10819 \cdot 10^{16}, 1.66602\} \qquad N(m) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

Intersection intervals:



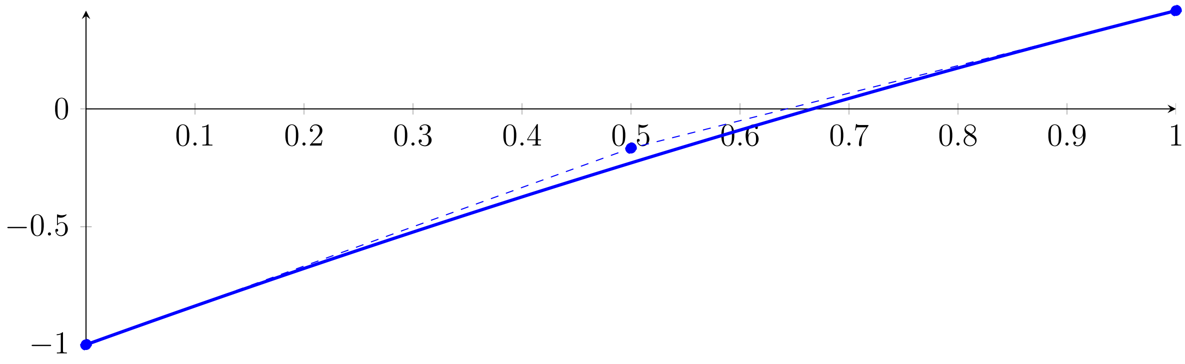
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

87.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

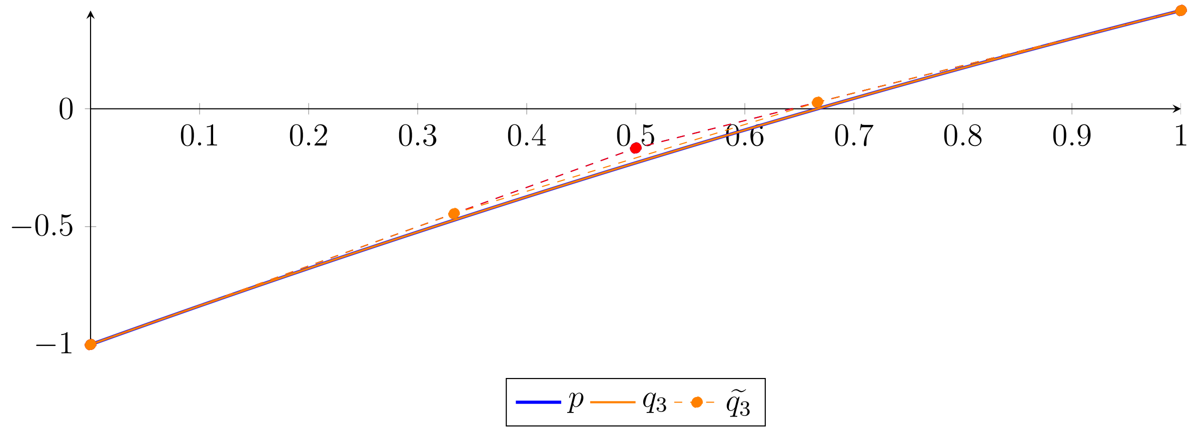
$$\begin{aligned} p &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2}(X) - 0.166667B_{1,2}(X) + 0.416667B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,3} - 0.444444B_{1,3} + 0.0277778B_{2,3} + 0.416667B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2} - 0.166667B_{1,2} + 0.416667B_{2,2} \end{aligned}$$



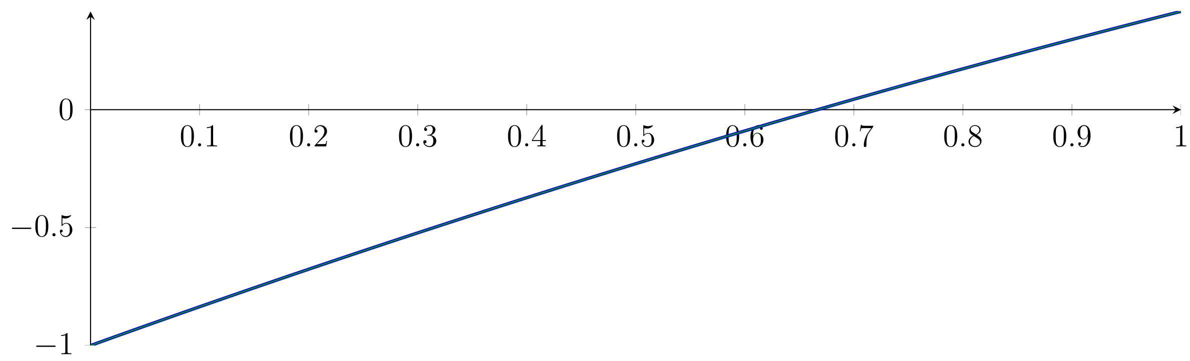
The maximum difference of the Bézier coefficients is $\delta = 7.58942 \cdot 10^{-19}$.
Bounding polynomials M and m:

$$M = -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1$$
$$m = -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1$$

Root of M and m:

$$N(M) = \{-2.32913 \cdot 10^{16}\} \qquad N(m) = \{-2.32913 \cdot 10^{16}\}$$

Intersection intervals:

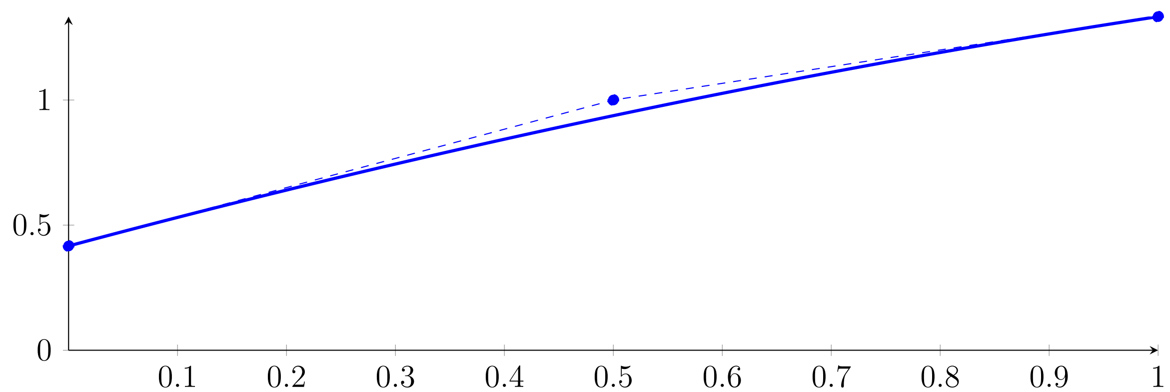


No intersection intervals with the x axis.

87.3 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

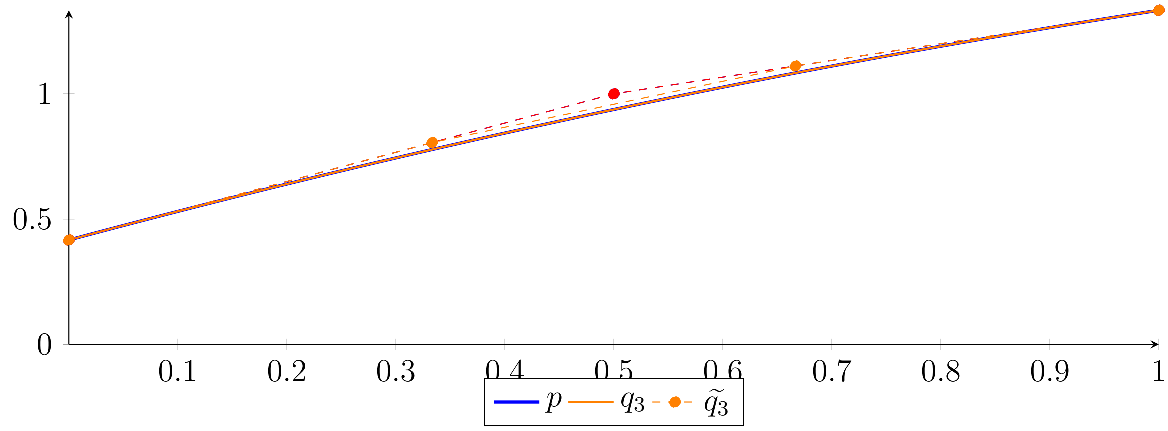
$$\begin{aligned} p &= -0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,2}(X) + 1B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -2.27682\cdot 10^{-18}X^3 - 0.25X^2 + 1.16667X + 0.416667$$
$$= 0.416667B_{0,3} + 0.805556B_{1,3} + 1.11111B_{2,3} + 1.33333B_{3,3}$$

$$\widetilde{q}_3 = -0.25X^2 + 1.16667X + 0.416667$$
$$= 0.416667B_{0,2} + 1B_{1,2} + 1.33333B_{2,2}$$



The maximum difference of the Bézier coefficients is $\delta = 1.30104\cdot 10^{-18}$.
Bounding polynomials M and m:

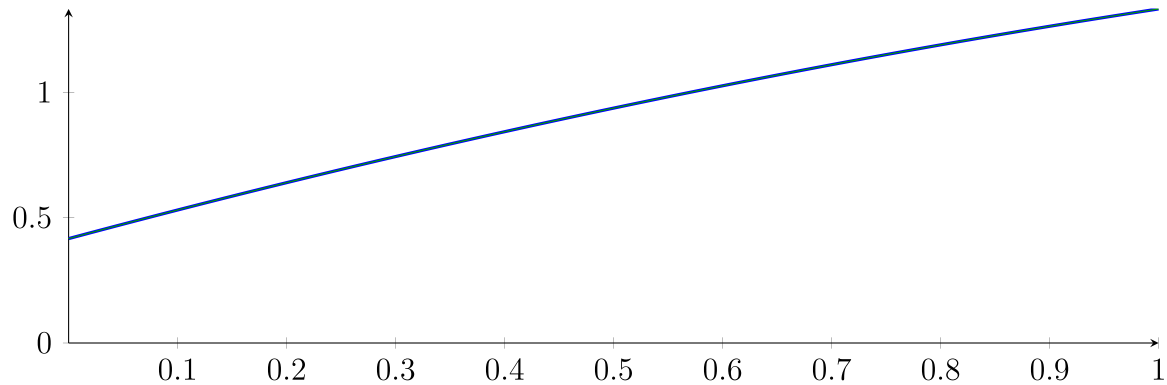
$$M = -2.27682\cdot 10^{-18}X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$m = -2.27682\cdot 10^{-18}X^3 - 0.25X^2 + 1.16667X + 0.416667$$

Root of M and m:

$$N(M) = \{-1.09802\cdot 10^{17}, -8.02734, 2.33594\} \qquad N(m) = \{-1.09802\cdot 10^{17}, -8.02734, 2.33594\}$$

Intersection intervals:

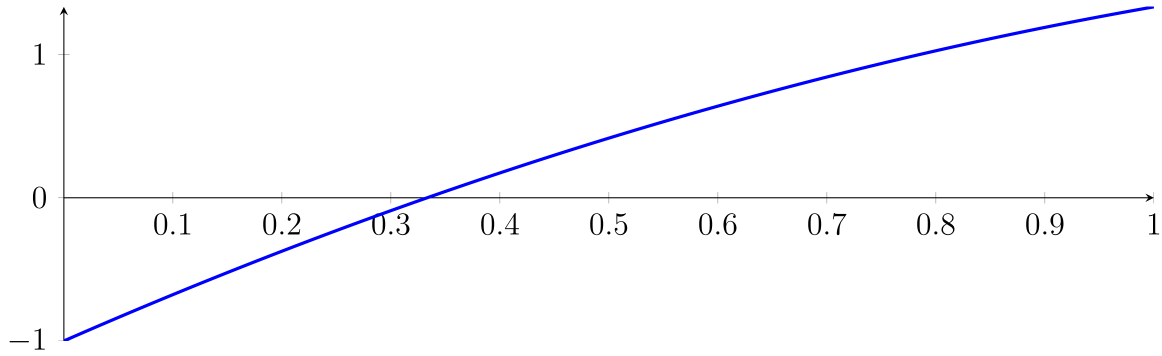


No intersection intervals with the x axis.

87.4 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

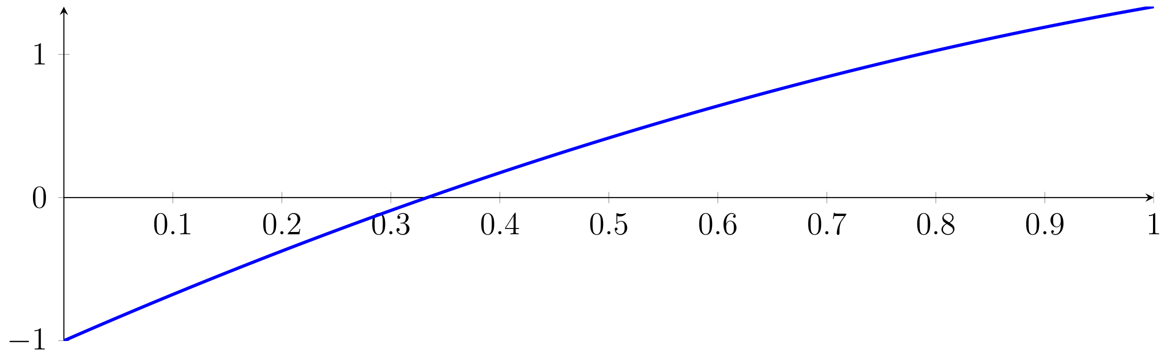
with precision $\varepsilon = 0.01$.

88 Running BezClip on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

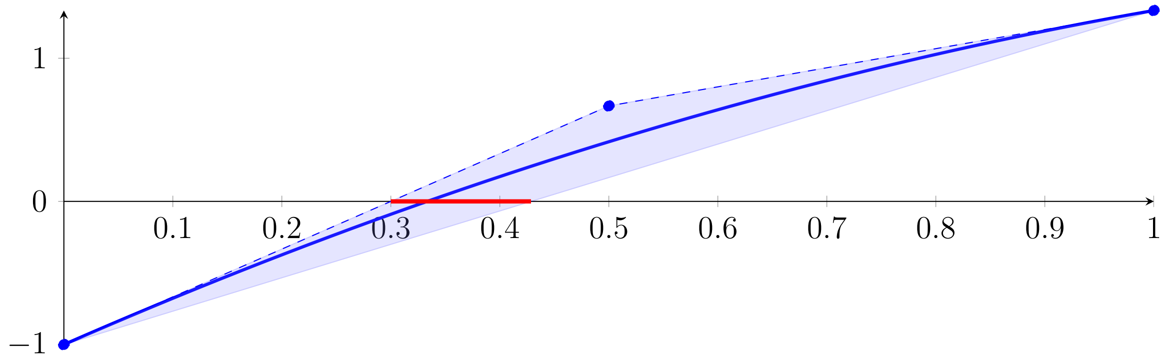
$$p = -1X^2 + 3.33333X - 1$$



88.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

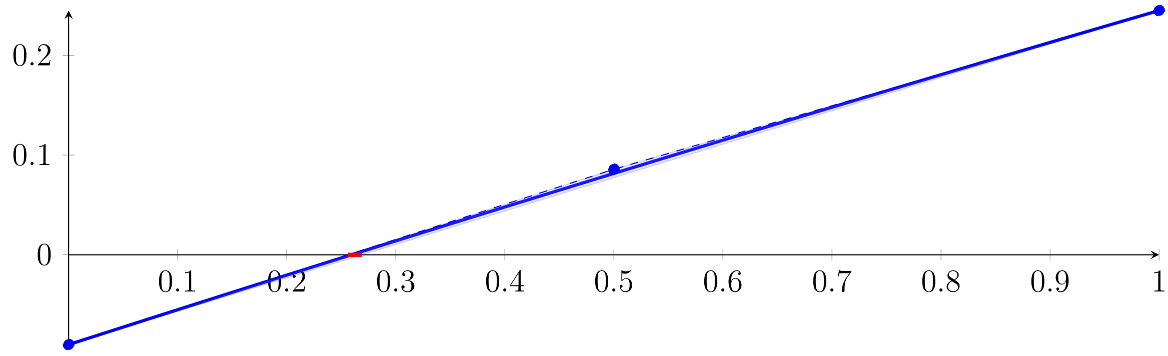
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

88.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

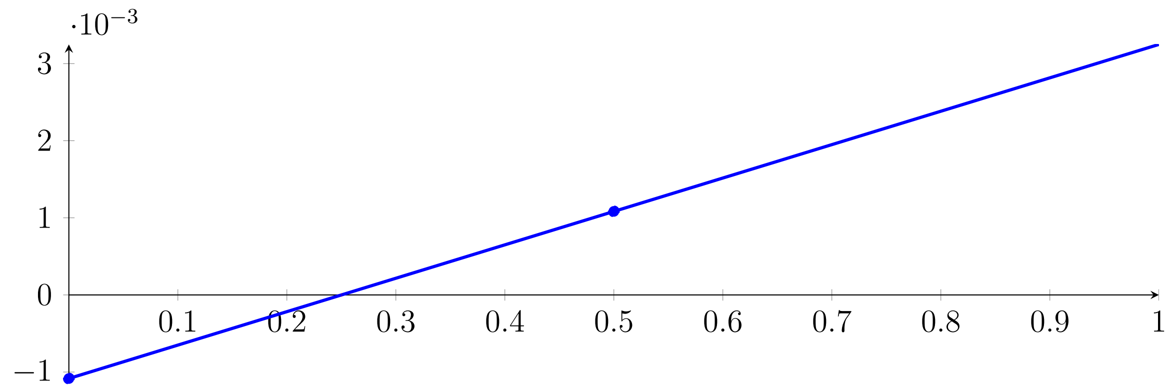
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

88.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151\cdot 10^{-06}X^2 + 0.00433538X - 0.00108418 \\ &= -0.00108418B_{0,2}(X) + 0.00108352B_{1,2}(X) + 0.00324857B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

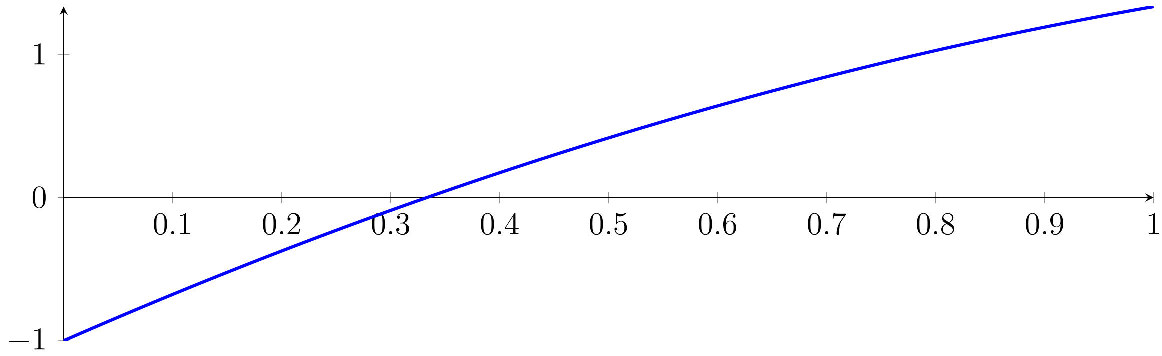
88.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Found root in interval $[0.333333, 0.333334]$ at recursion depth 4!

88.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333334]$$

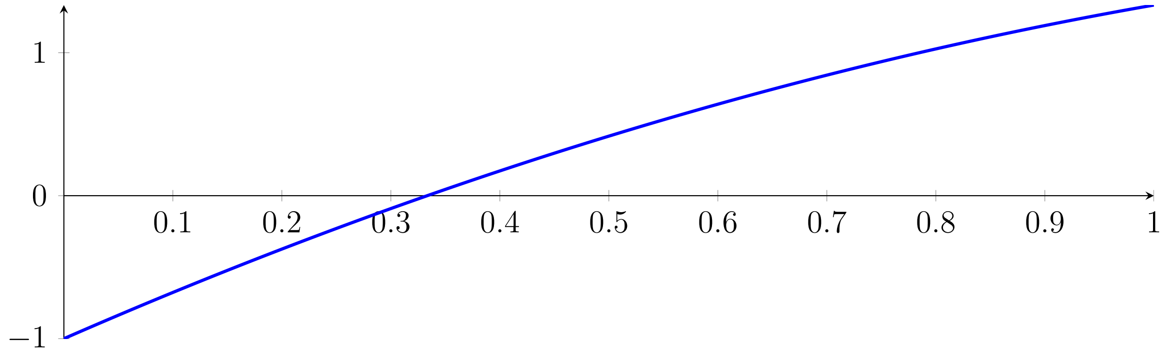
with precision $\varepsilon = 0.0001$.

89 Running QuadClip on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0,1]$:

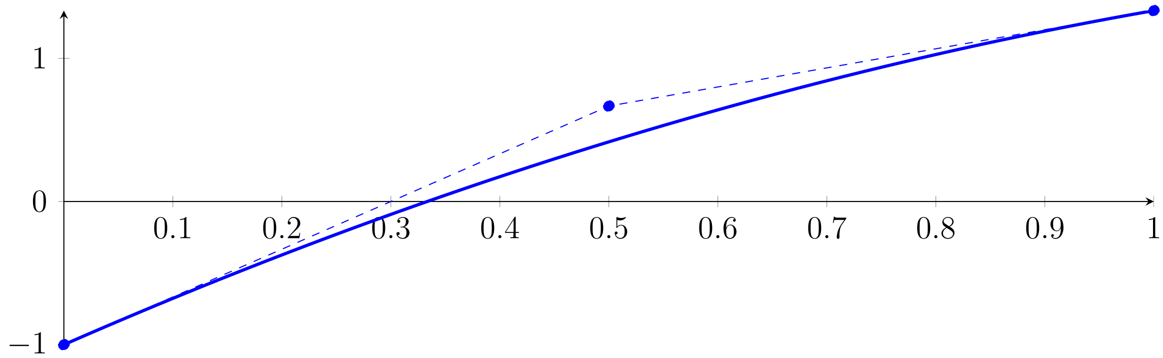
$$p = -1X^2 + 3.33333X - 1$$



89.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

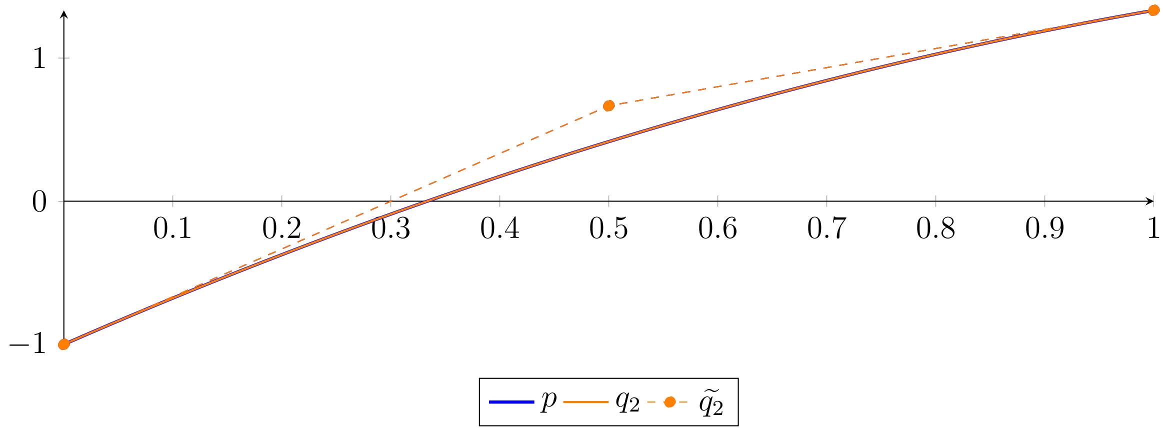
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.
Bounding polynomials M and m:

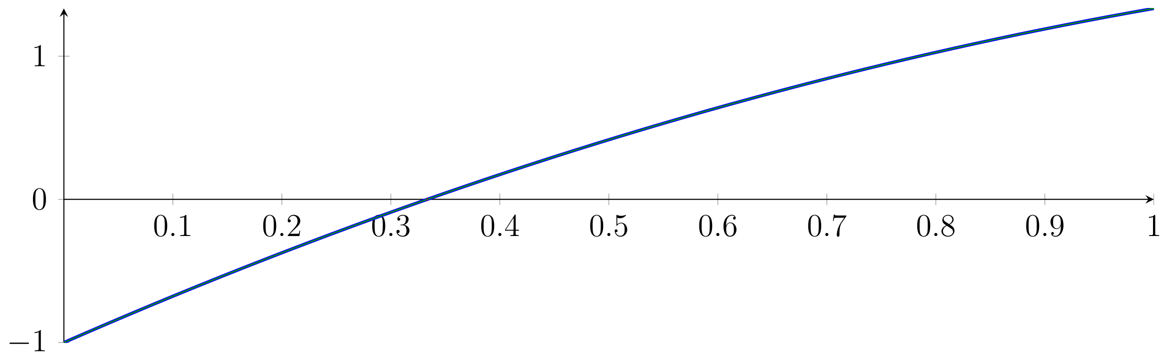
$$M = -1X^2 + 3.33333X - 1$$

$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $3.25261 \cdot 10^{-19}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

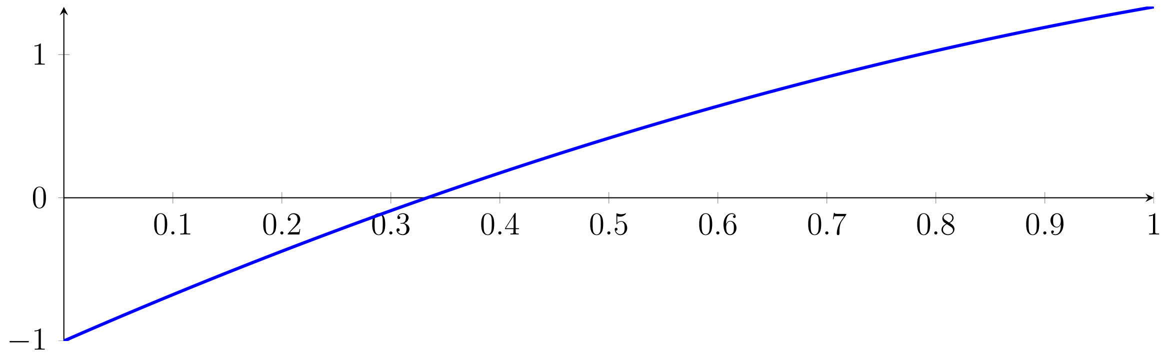
89.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

89.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

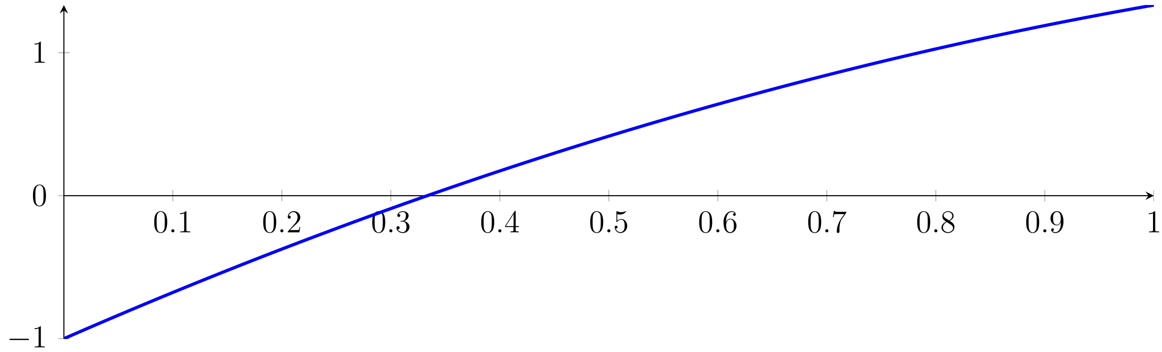
with precision $\varepsilon = 0.0001$.

90 Running CubeClip on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

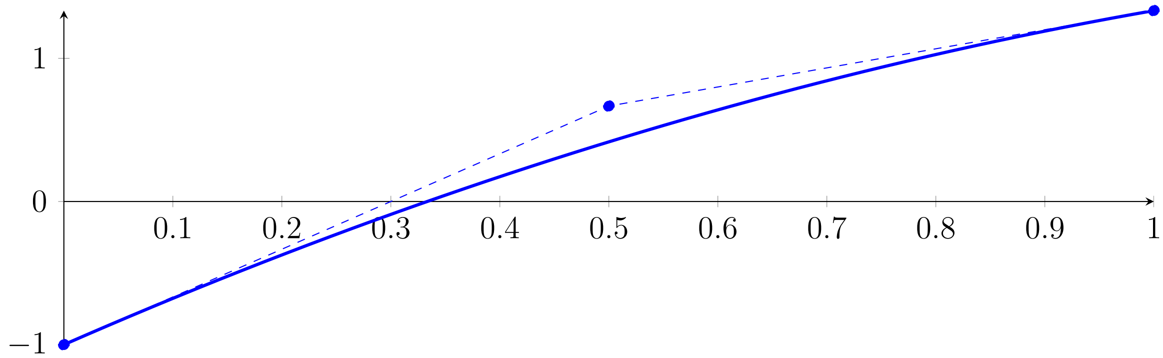
$$p = -1X^2 + 3.33333X - 1$$



90.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

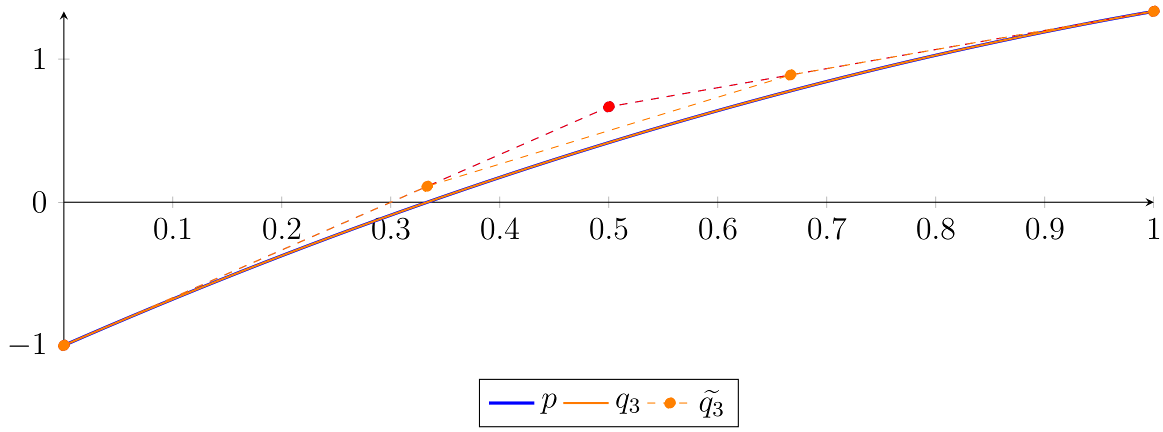
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.63715 \cdot 10^{-17}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



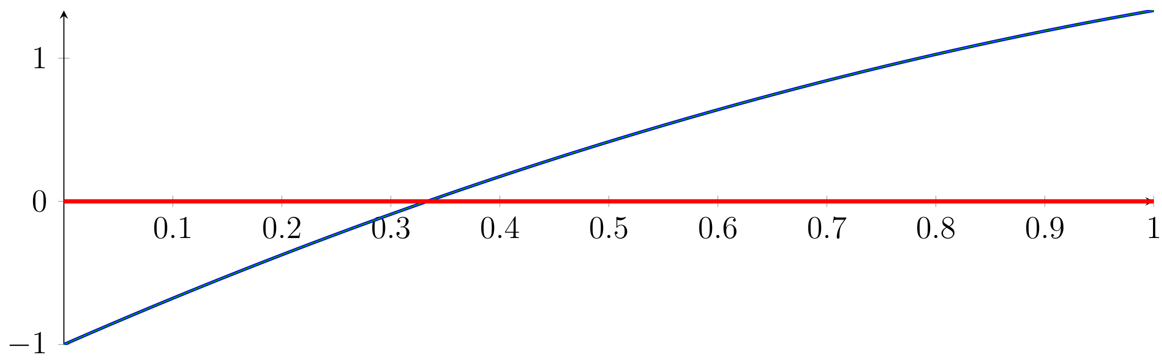
The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.
Bounding polynomials M and m:

$$M = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$
$$m = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{-6.10819 \cdot 10^{16}, 1.66602\} \qquad N(m) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

Intersection intervals:



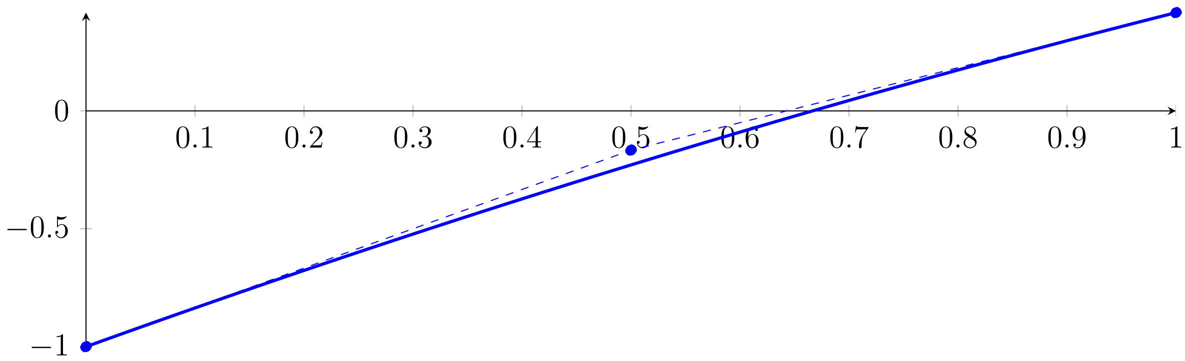
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

90.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

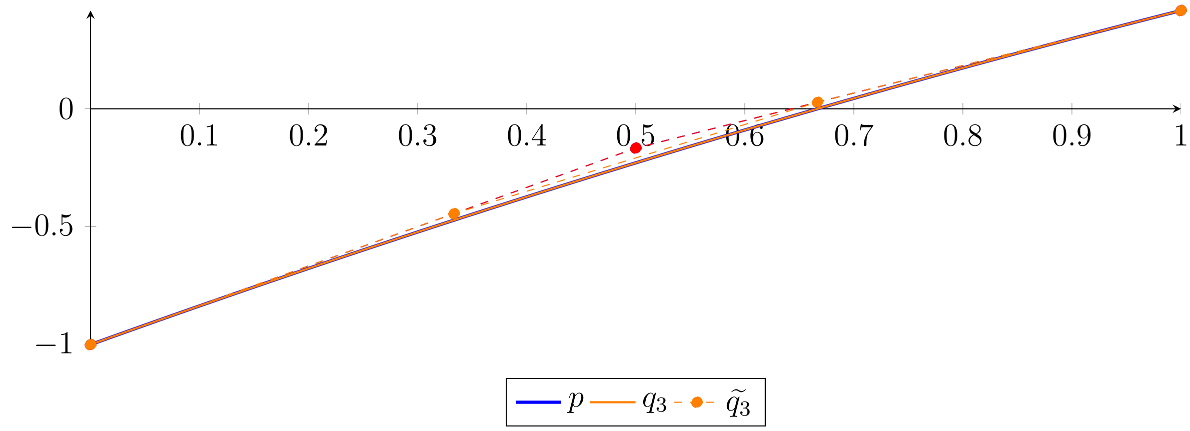
$$\begin{aligned} p &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2}(X) - 0.166667B_{1,2}(X) + 0.416667B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,3} - 0.444444B_{1,3} + 0.0277778B_{2,3} + 0.416667B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2} - 0.166667B_{1,2} + 0.416667B_{2,2} \end{aligned}$$



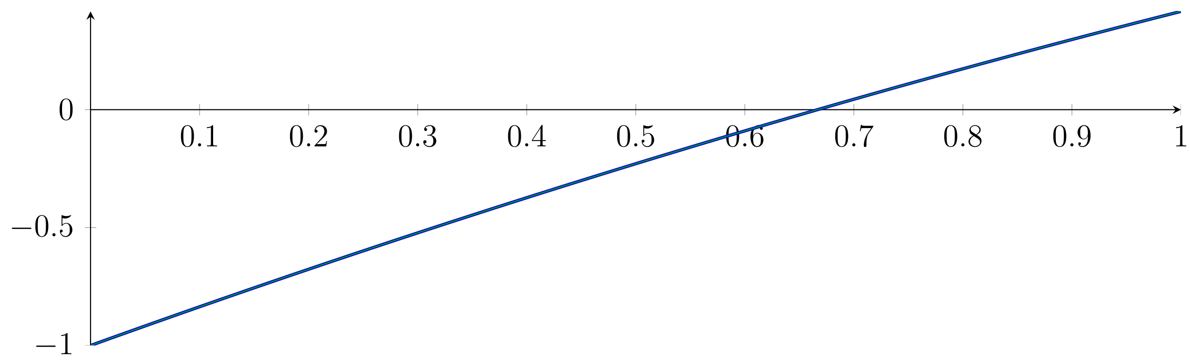
The maximum difference of the Bézier coefficients is $\delta = 7.58942 \cdot 10^{-19}$.
Bounding polynomials M and m:

$$M = -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1$$
$$m = -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1$$

Root of M and m:

$$N(M) = \{-2.32913 \cdot 10^{16}\} \qquad N(m) = \{-2.32913 \cdot 10^{16}\}$$

Intersection intervals:

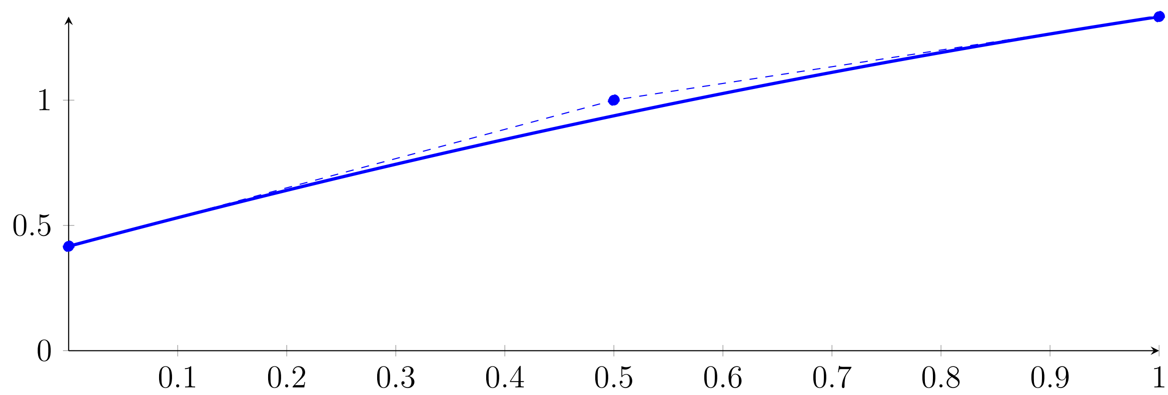


No intersection intervals with the x axis.

90.3 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

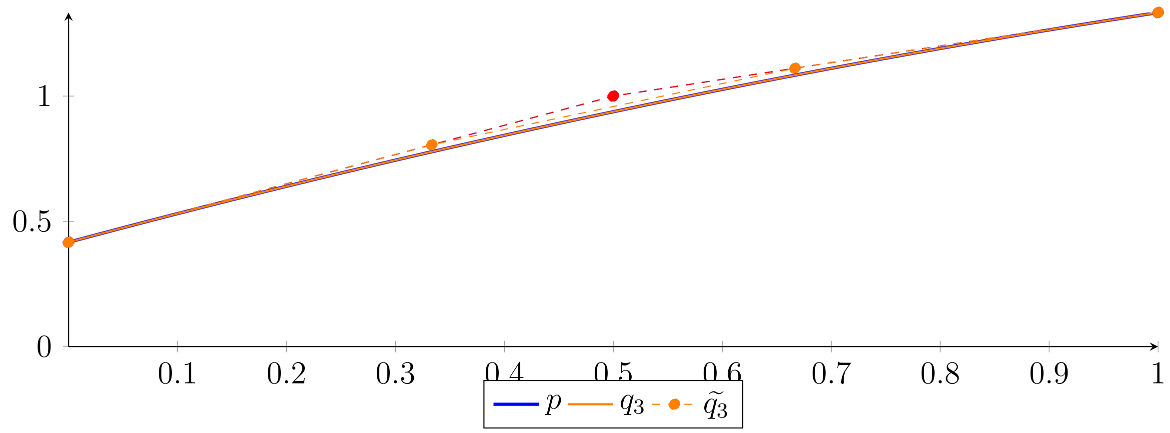
$$\begin{aligned} p &= -0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,2}(X) + 1B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -2.27682\cdot 10^{-18}X^3 - 0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,3} + 0.805556B_{1,3} + 1.11111B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,2} + 1B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.30104\cdot 10^{-18}$.
Bounding polynomials M and m:

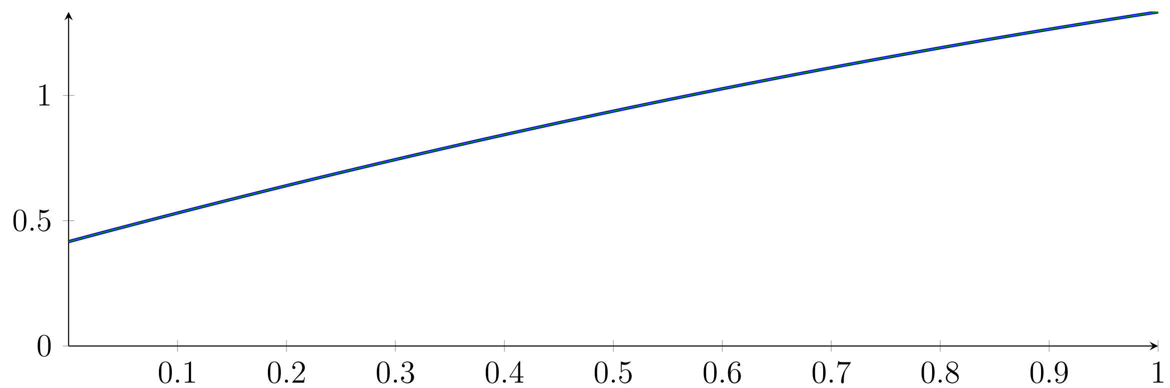
$$M = -2.27682\cdot 10^{-18}X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$m = -2.27682\cdot 10^{-18}X^3 - 0.25X^2 + 1.16667X + 0.416667$$

Root of M and m:

$$N(M) = \{-1.09802\cdot 10^{17}, -8.02734, 2.33594\} \qquad N(m) = \{-1.09802\cdot 10^{17}, -8.02734, 2.33594\}$$

Intersection intervals:

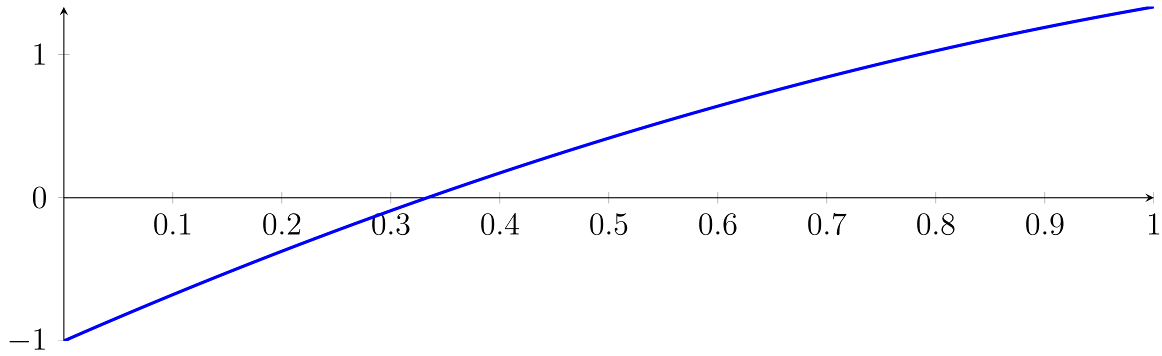


No intersection intervals with the x axis.

90.4 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

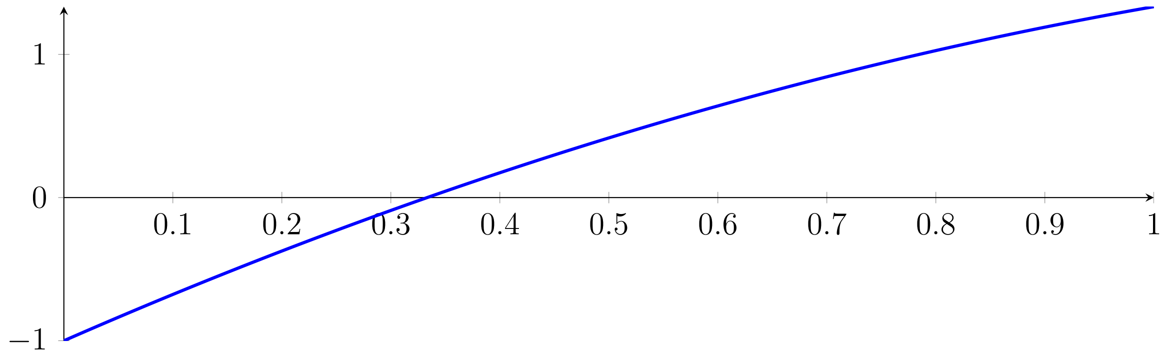
with precision $\varepsilon = 0.0001$.

91 Running BezClip on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

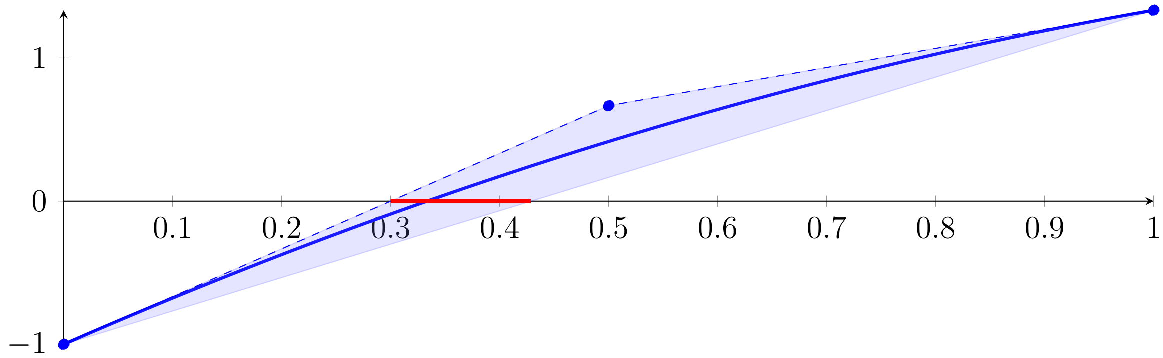
$$p = -1X^2 + 3.33333X - 1$$



91.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

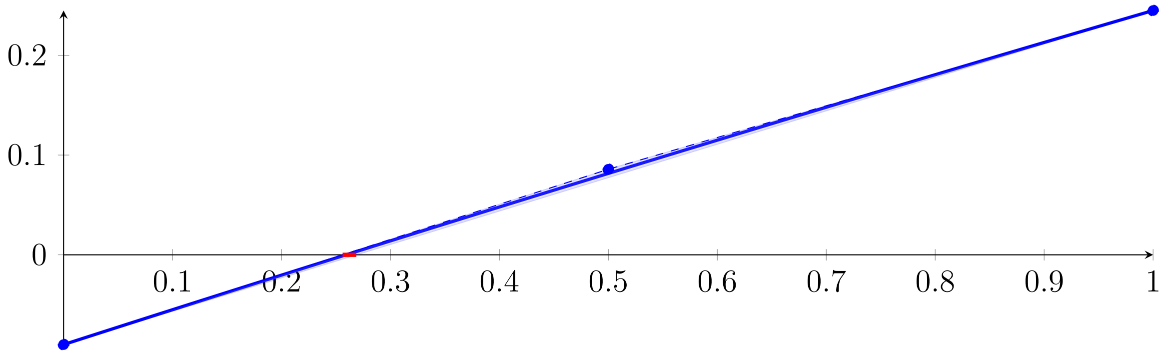
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

91.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

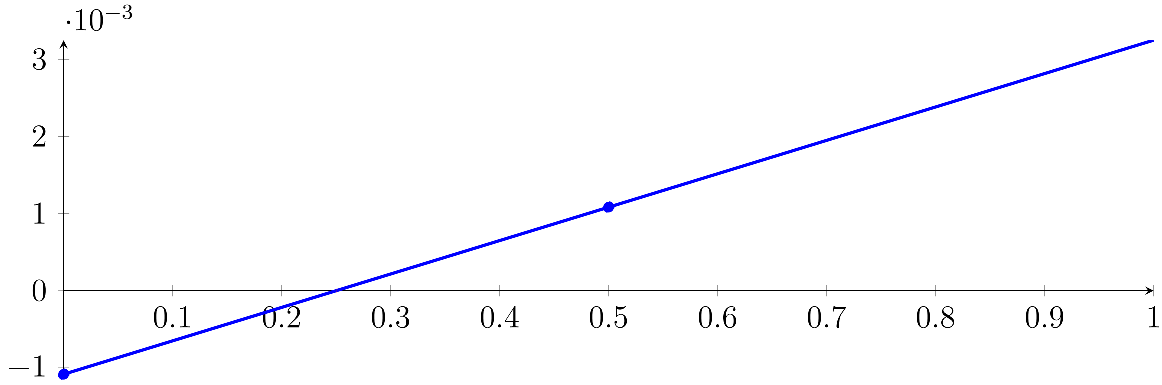
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

91.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

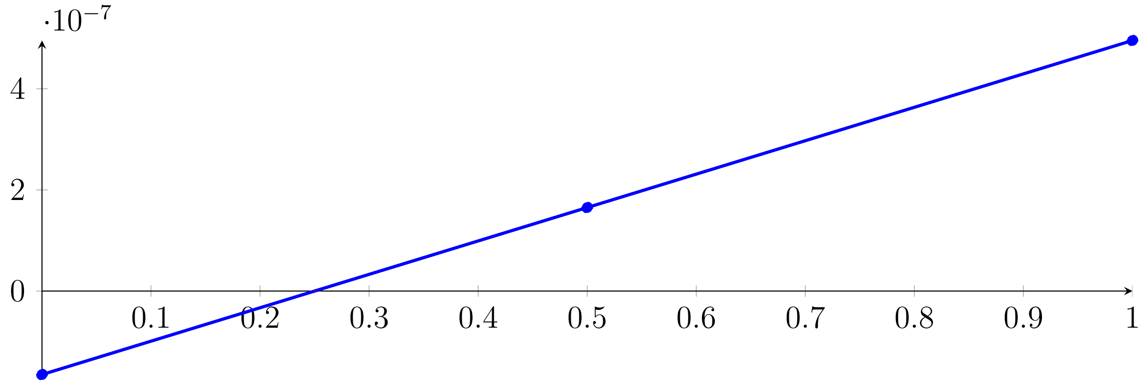
$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

91.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07}$$
$$= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $2.32306 \cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

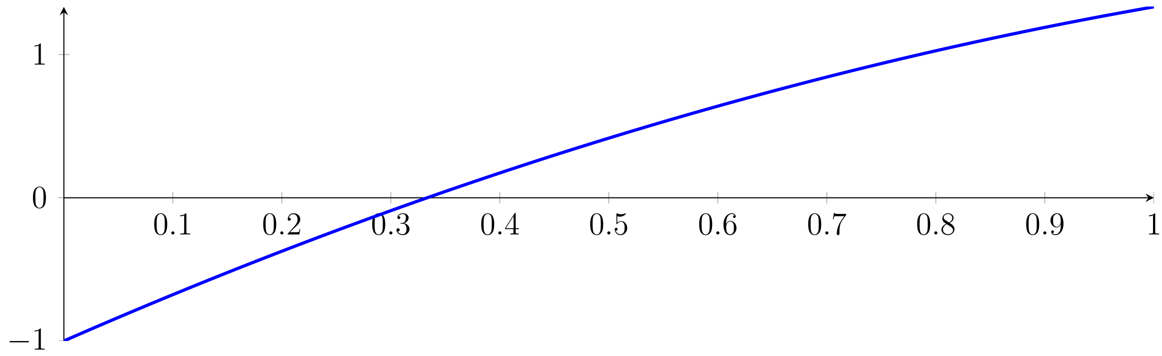
91.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

91.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

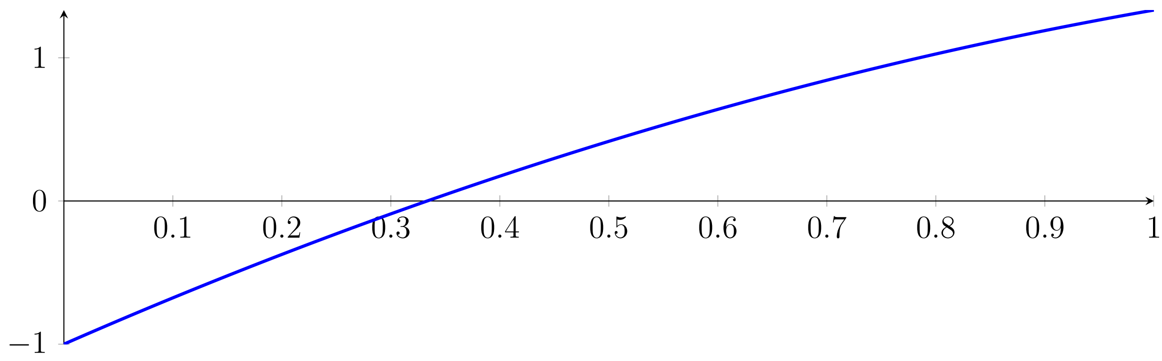
with precision $\varepsilon = 1 \cdot 10^{-08}$.

92 Running `QuadClip` on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called `QuadClip` with input polynomial on interval $[0, 1]$:

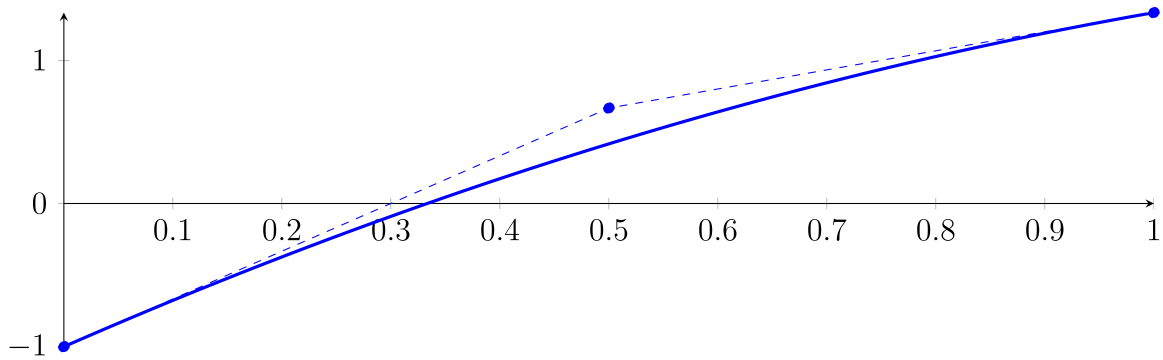
$$p = -1X^2 + 3.33333X - 1$$



92.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

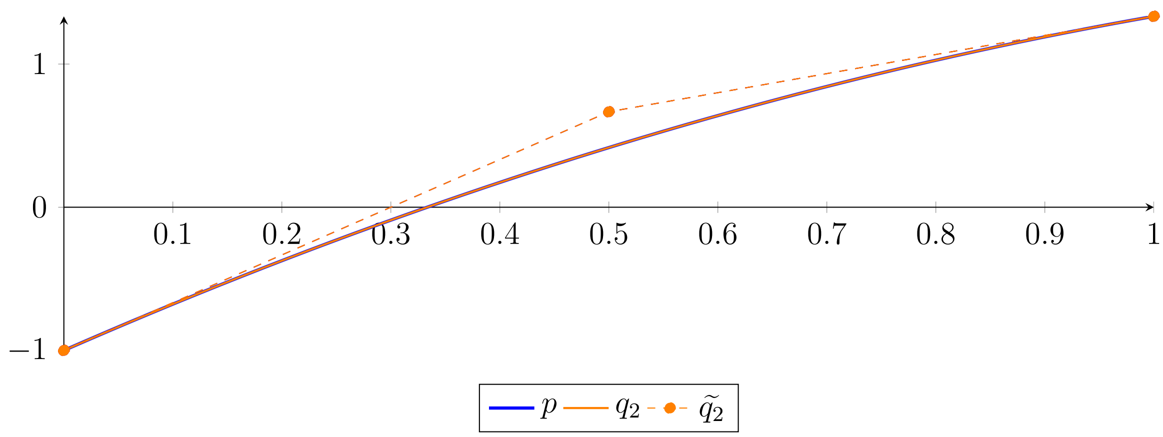
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.
Bounding polynomials M and m:

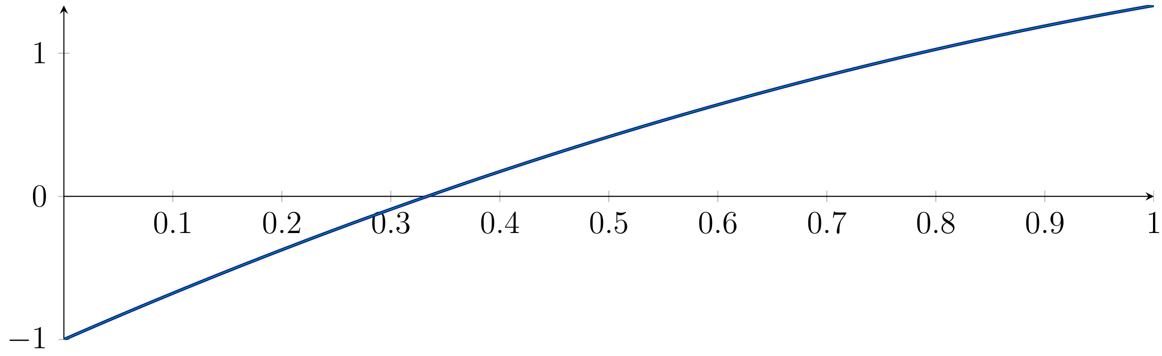
$$M = -1X^2 + 3.33333X - 1$$

$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $3.25261 \cdot 10^{-19}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

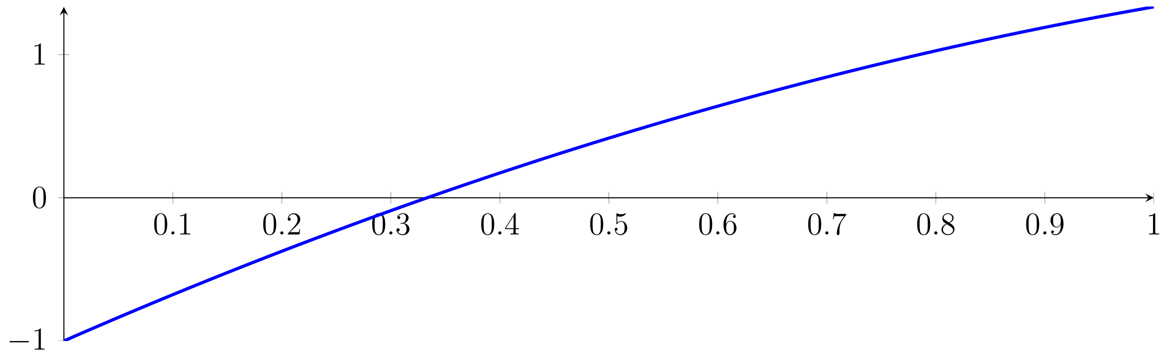
92.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

92.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

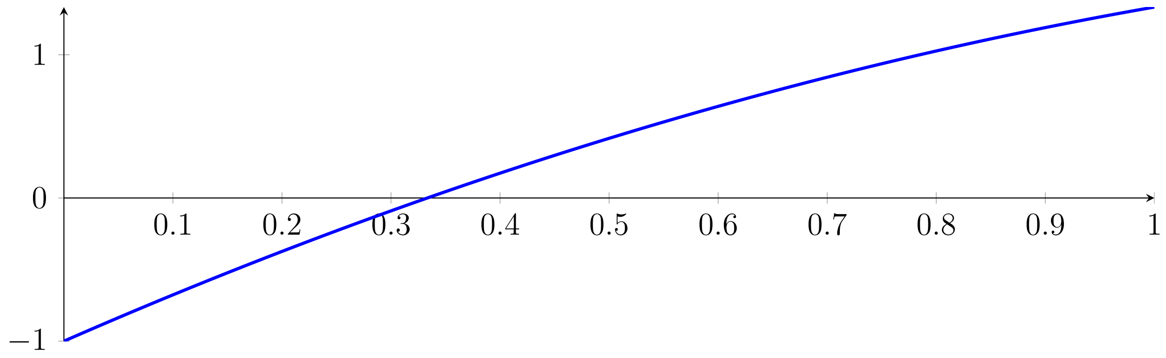
with precision $\varepsilon = 1 \cdot 10^{-08}$.

93 Running CubeClip on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

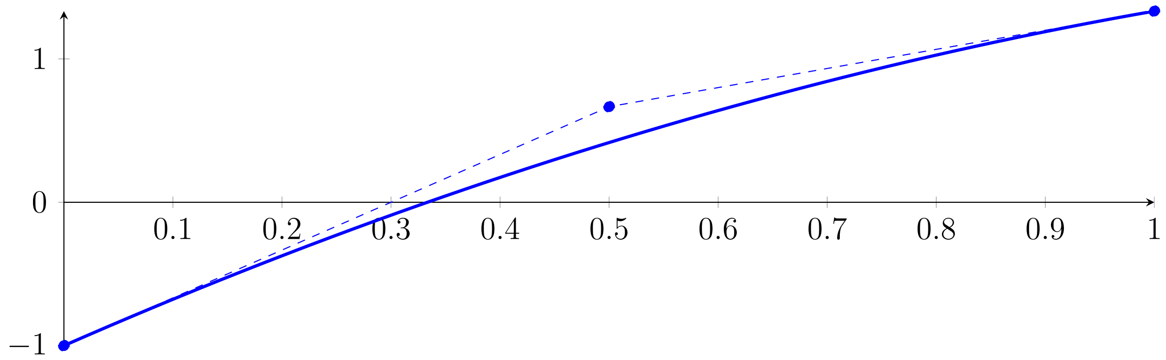
$$p = -1X^2 + 3.33333X - 1$$



93.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

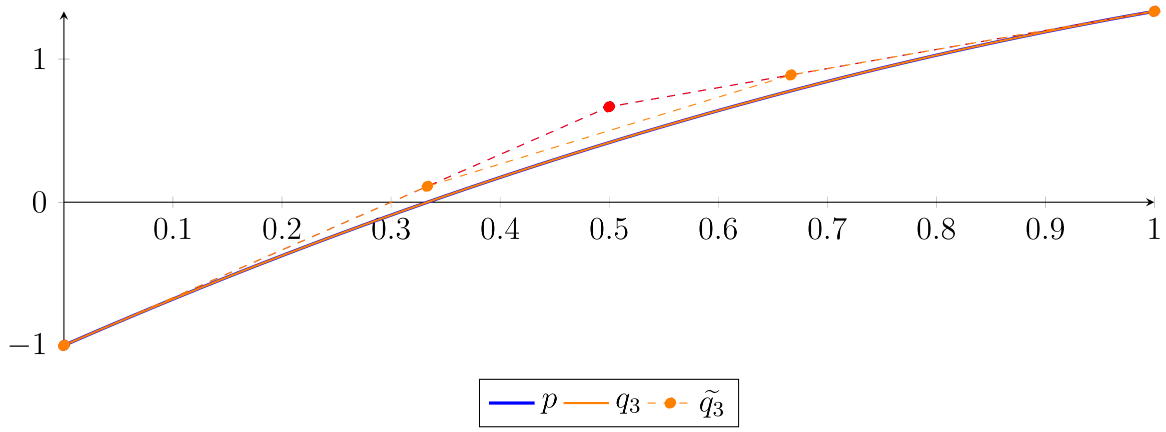
$$p = -1X^2 + 3.33333X - 1$$
$$= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X)$$



Degree reduction and raising:

$$q_3 = -1.63715 \cdot 10^{-17}X^3 - 1X^2 + 3.33333X - 1$$
$$= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3}$$

$$\tilde{q}_3 = -1X^2 + 3.33333X - 1$$
$$= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.
Bounding polynomials M and m:

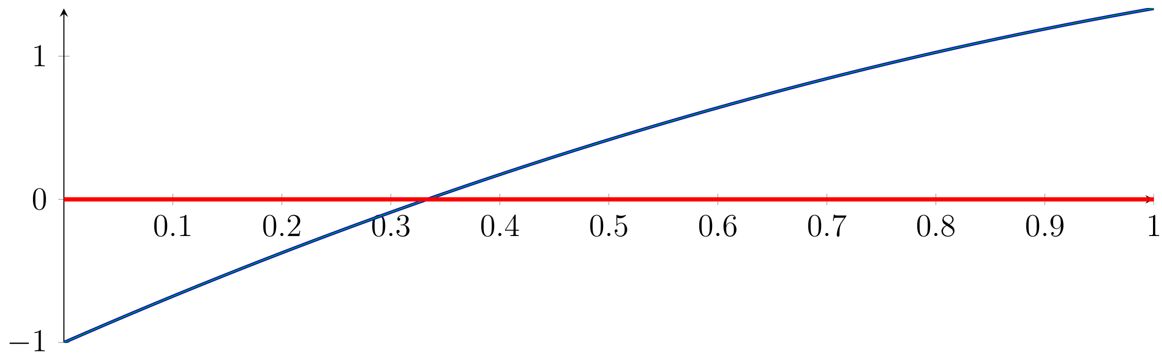
$$M = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

$$m = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{-6.10819 \cdot 10^{16}, 1.66602\} \qquad N(m) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

Intersection intervals:



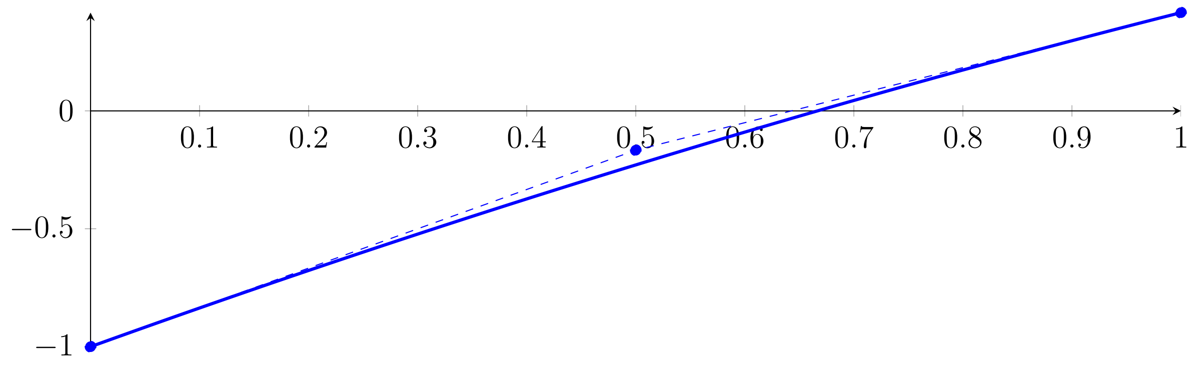
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

93.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

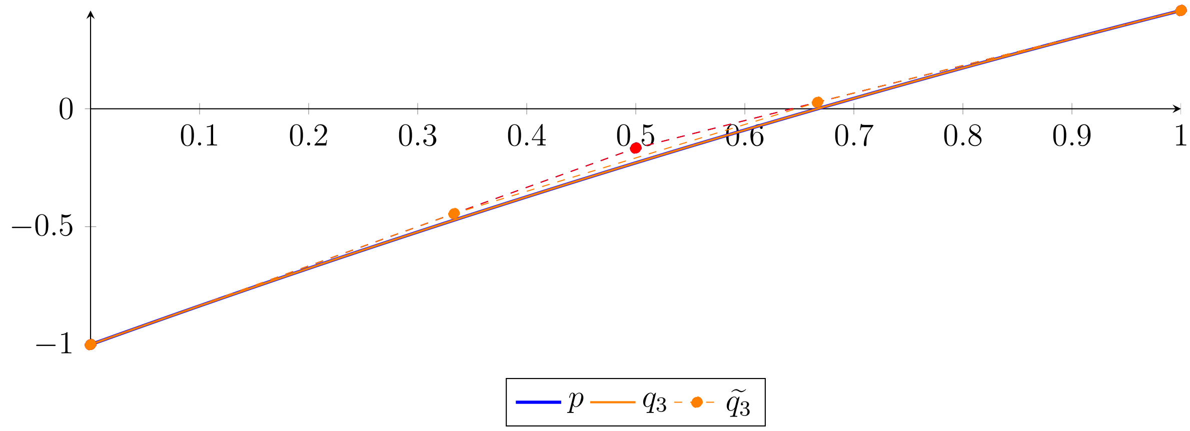
$$\begin{aligned} p &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2}(X) - 0.166667B_{1,2}(X) + 0.416667B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,3} - 0.444444B_{1,3} + 0.0277778B_{2,3} + 0.416667B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2} - 0.166667B_{1,2} + 0.416667B_{2,2} \end{aligned}$$



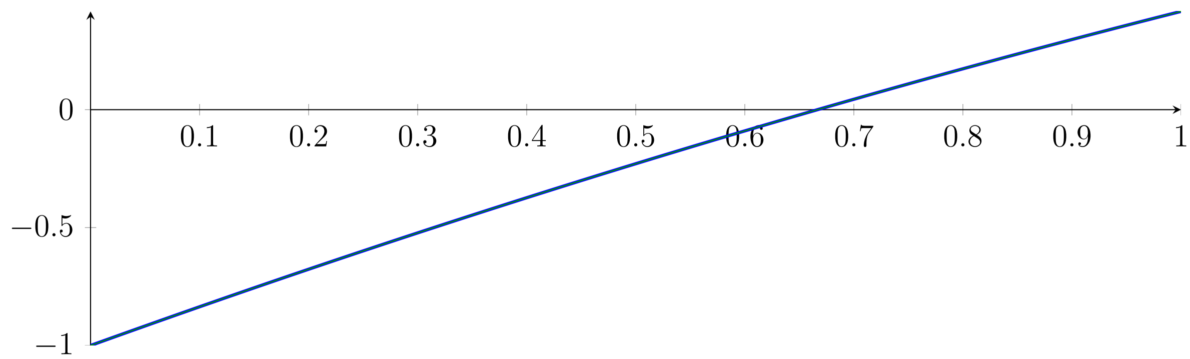
The maximum difference of the Bézier coefficients is $\delta = 7.58942 \cdot 10^{-19}$.
Bounding polynomials M and m:

$$M = -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1$$
$$m = -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1$$

Root of M and m:

$$N(M) = \{-2.32913 \cdot 10^{16}\} \qquad N(m) = \{-2.32913 \cdot 10^{16}\}$$

Intersection intervals:

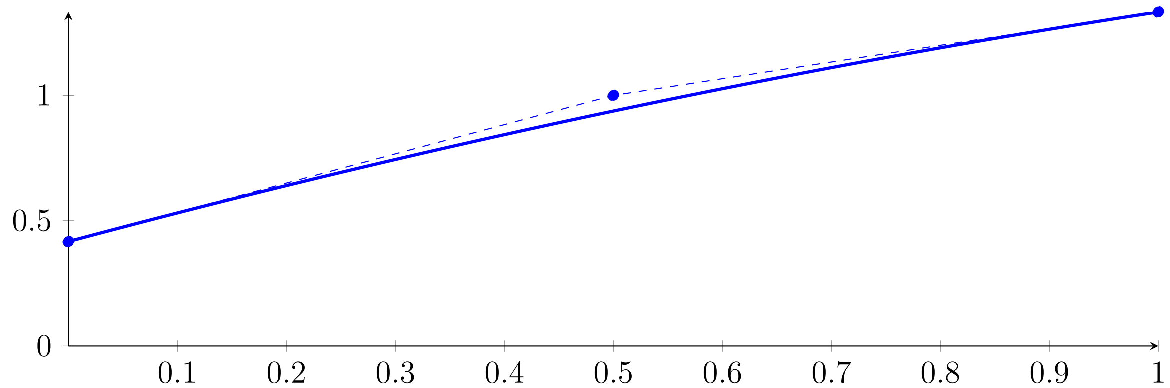


No intersection intervals with the x axis.

93.3 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

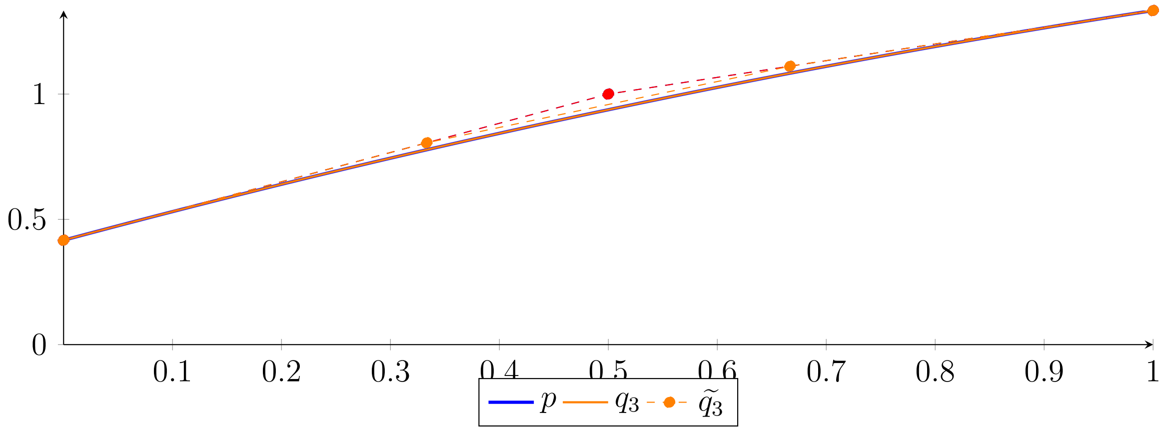
$$\begin{aligned} p &= -0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,2}(X) + 1B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,3} + 0.805556B_{1,3} + 1.11111B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,2} + 1B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.30104 \cdot 10^{-18}$.
Bounding polynomials M and m:

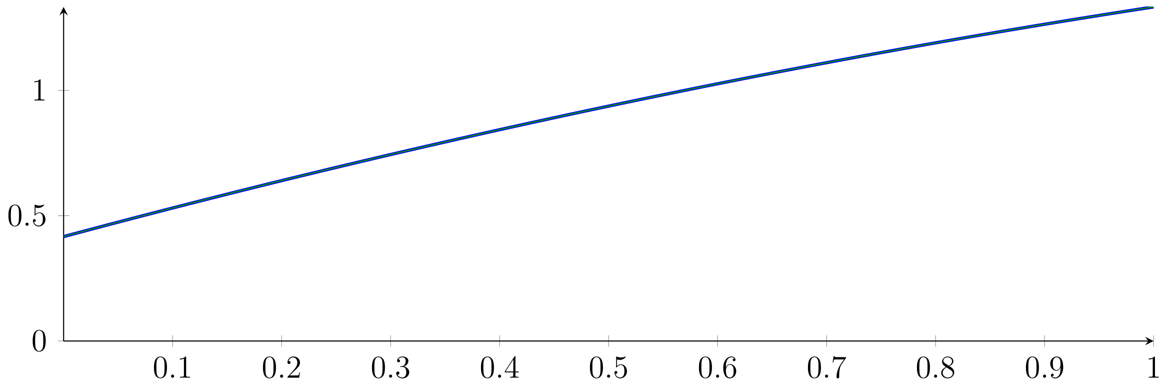
$$M = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$m = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

Root of M and m:

$$N(M) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\} \qquad N(m) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\}$$

Intersection intervals:

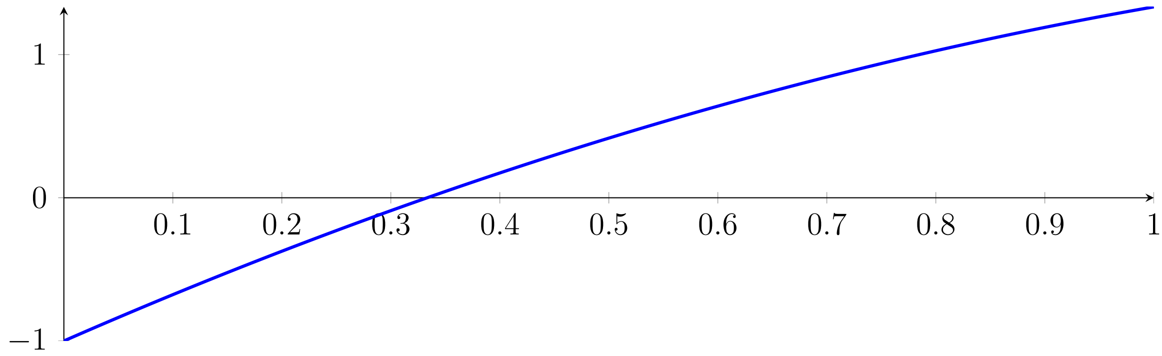


No intersection intervals with the x axis.

93.4 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

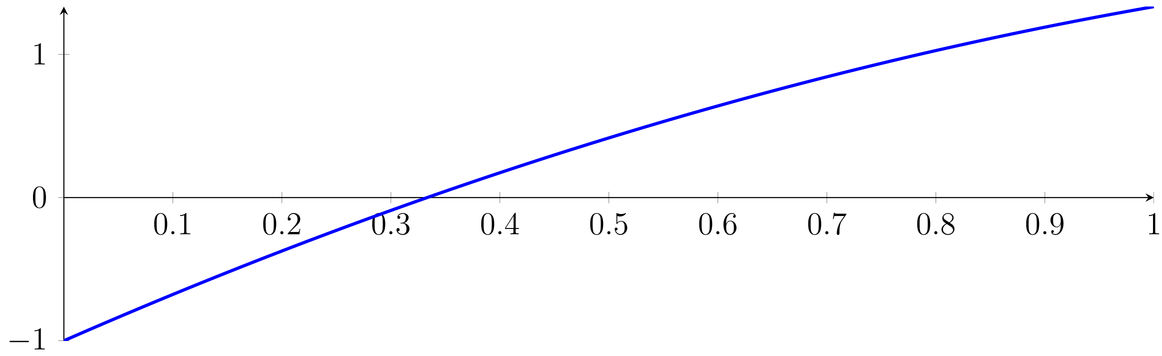
with precision $\varepsilon = 1 \cdot 10^{-08}$.

94 Running BezClip on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

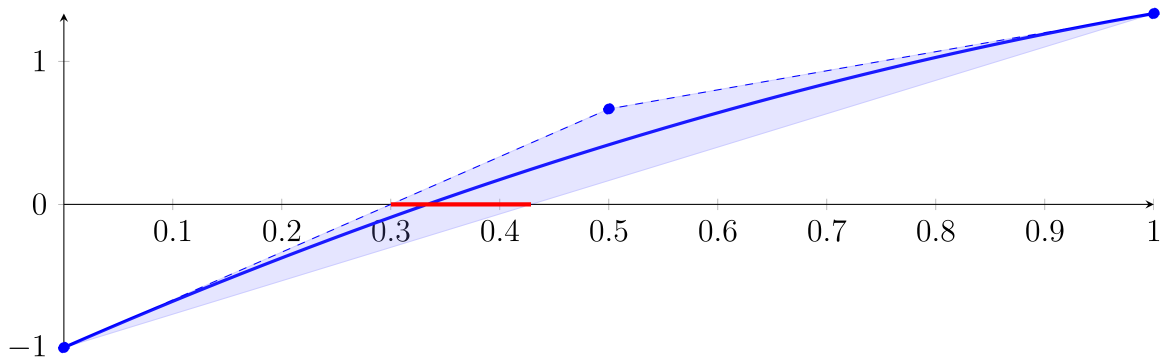
$$p = -1X^2 + 3.33333X - 1$$



94.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

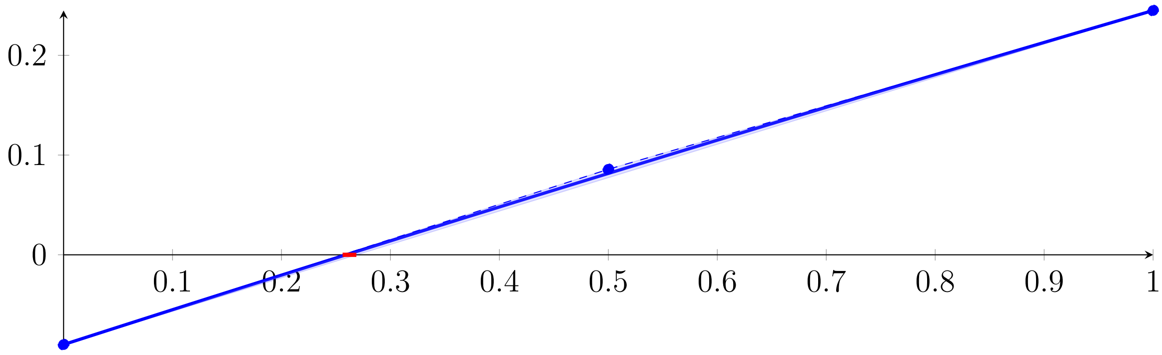
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

94.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

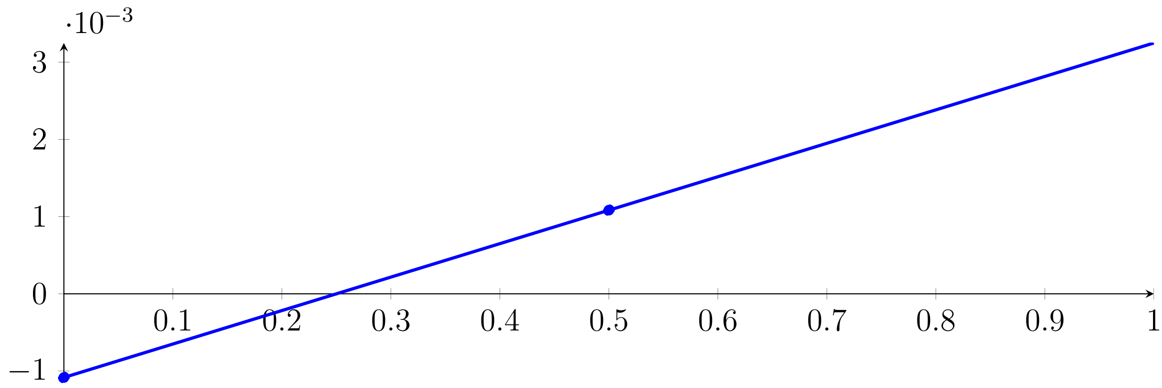
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

94.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538X - 0.00108418 \\ &= -0.00108418B_{0,2}(X) + 0.00108352B_{1,2}(X) + 0.00324857B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

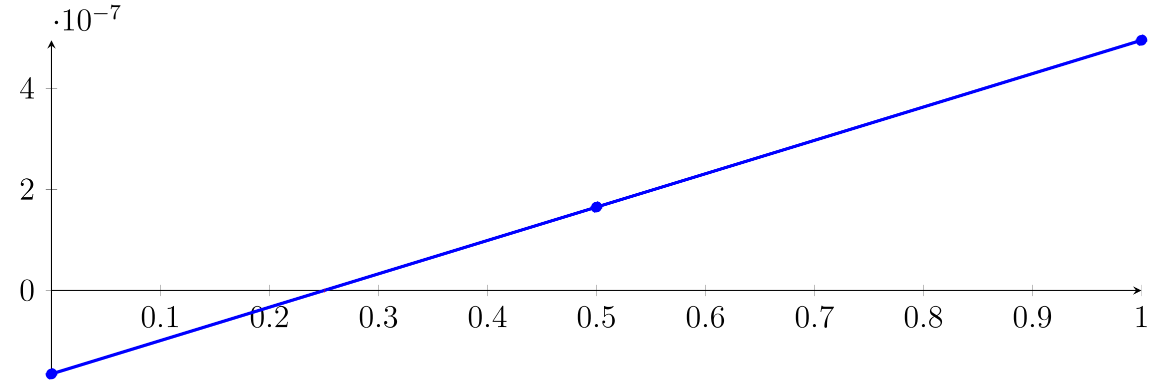
$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

94.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07}$$
$$= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

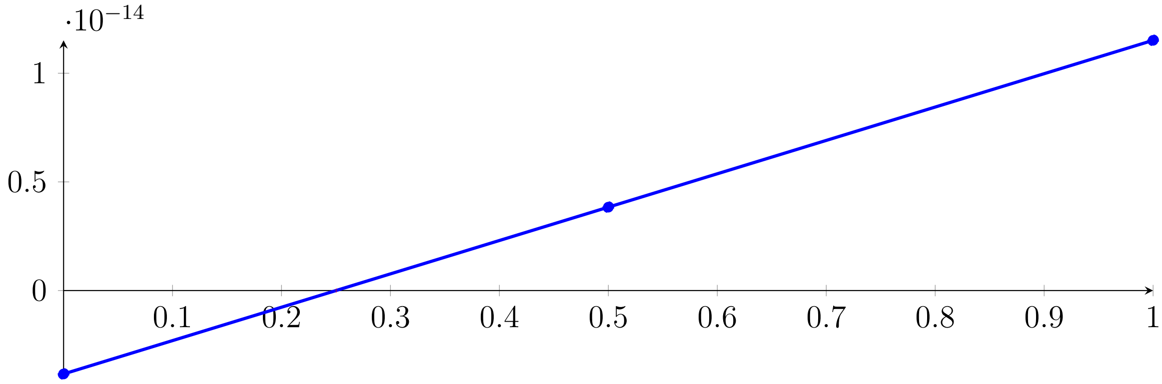
Longest intersection interval: $2.32306 \cdot 10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

94.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.31352 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15}$$
$$= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.39635 \cdot 10^{-16}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

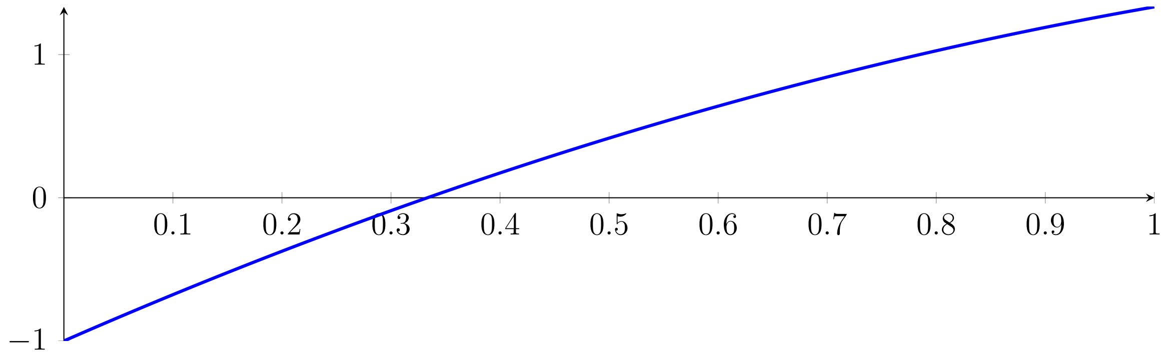
94.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

94.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

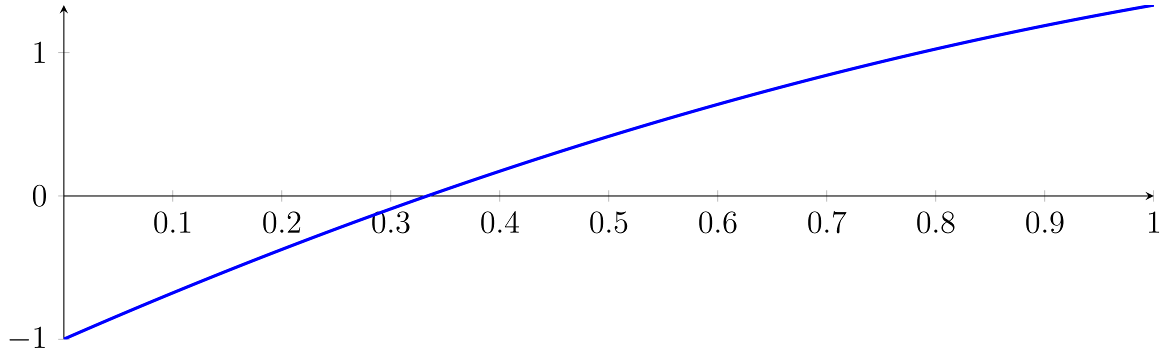
with precision $\varepsilon = 1 \cdot 10^{-16}$.

95 Running QuadClip on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

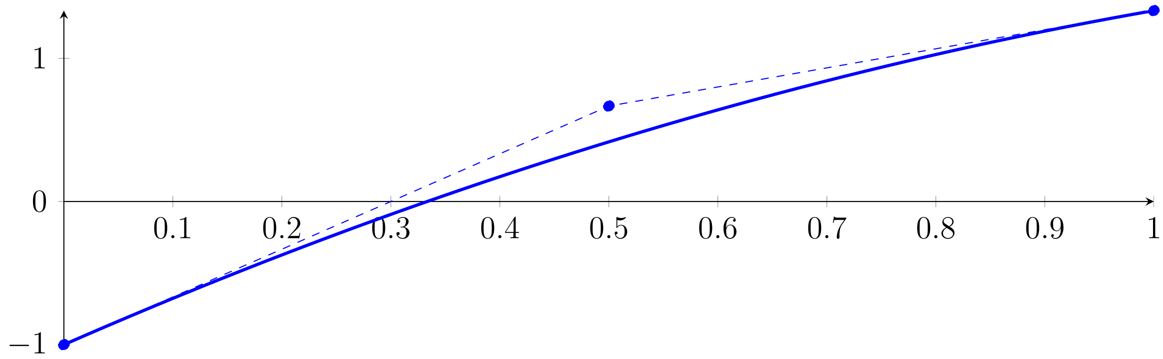
$$p = -1X^2 + 3.33333X - 1$$



95.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

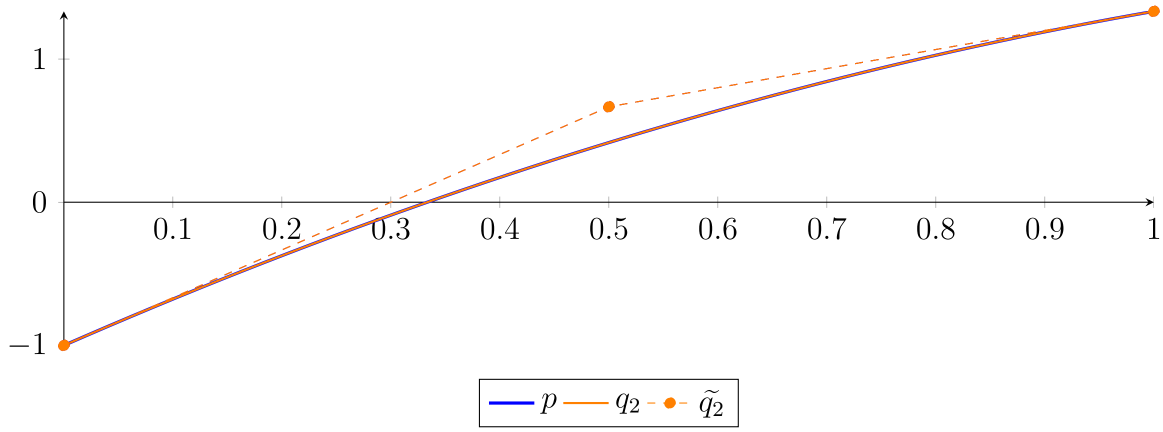
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.
Bounding polynomials M and m:

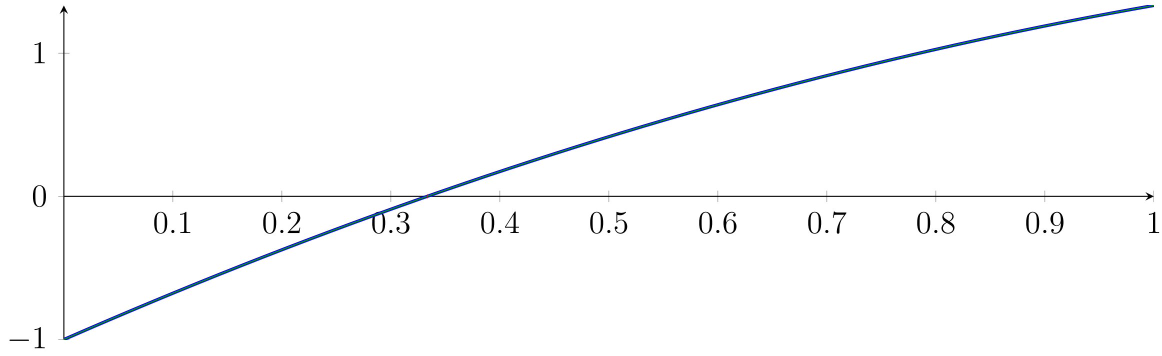
$$M = -1X^2 + 3.33333X - 1$$

$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $3.25261 \cdot 10^{-19}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

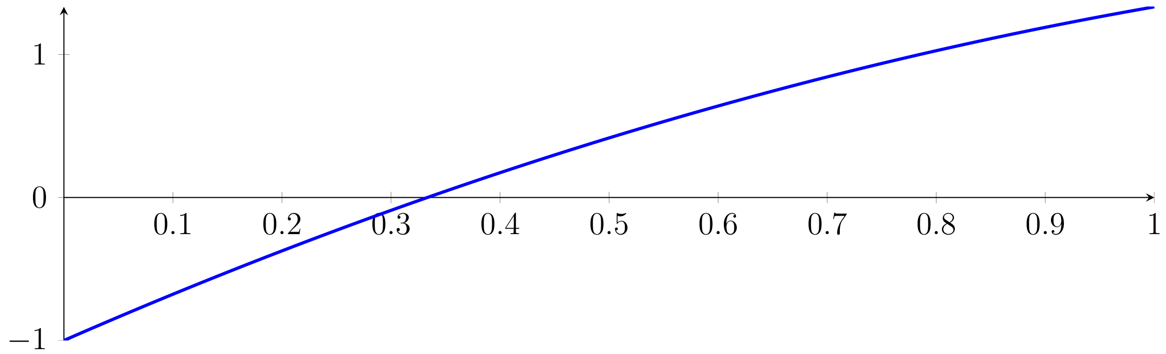
95.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

95.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

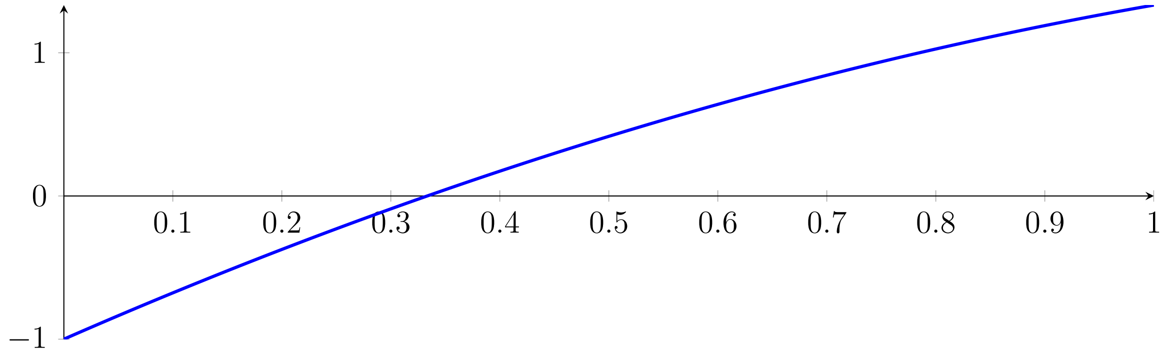
with precision $\varepsilon = 1 \cdot 10^{-16}$.

96 Running CubeClip on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

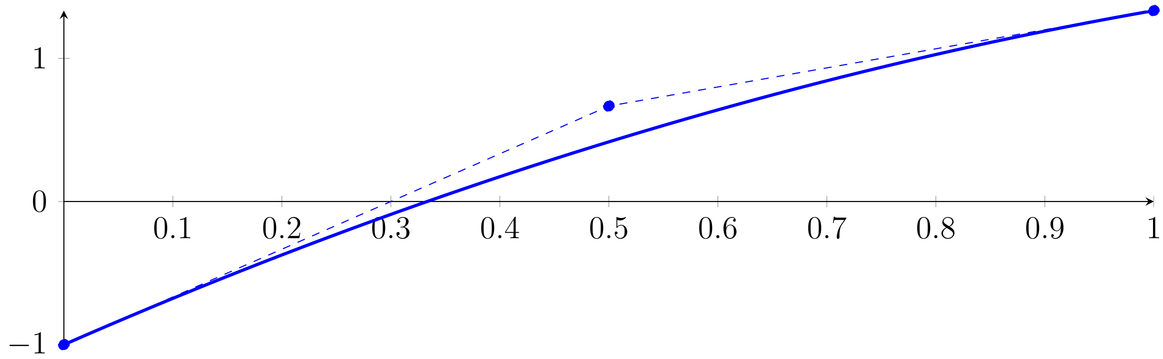
$$p = -1X^2 + 3.33333X - 1$$



96.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

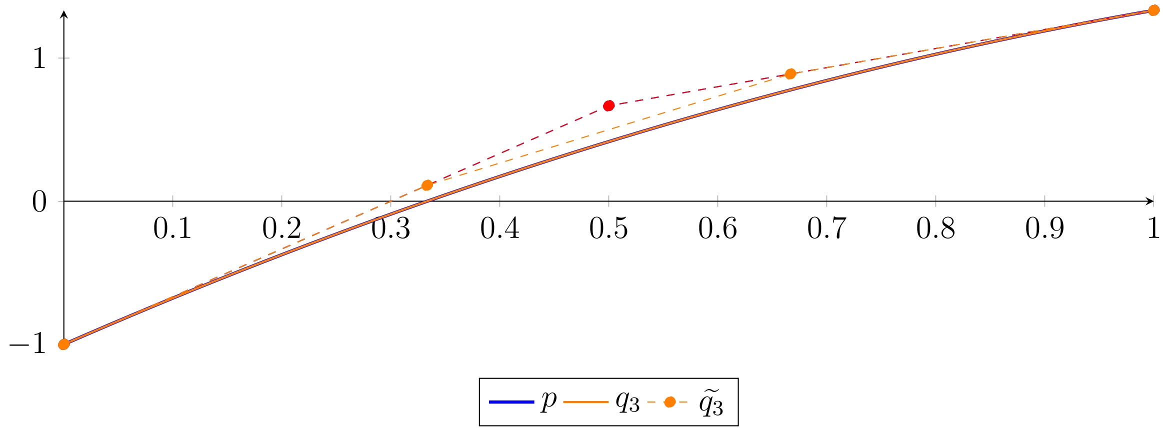
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.63715 \cdot 10^{-17}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.
Bounding polynomials M and m:

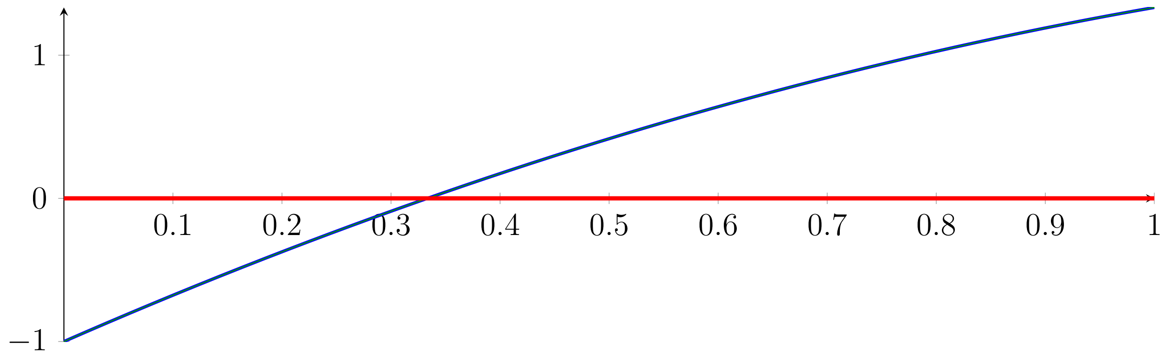
$$M = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

$$m = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{-6.10819 \cdot 10^{16}, 1.66602\} \qquad N(m) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

Intersection intervals:



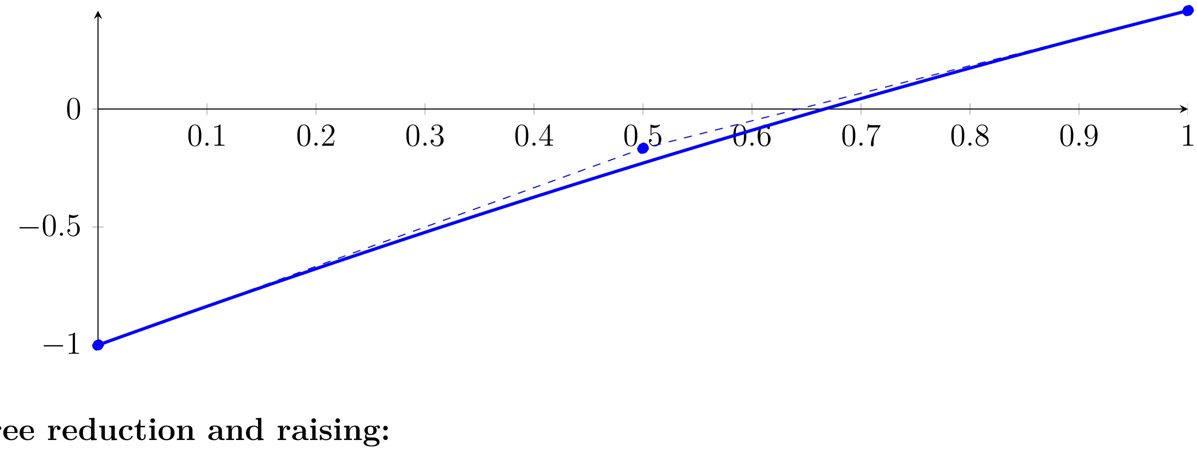
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

96.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

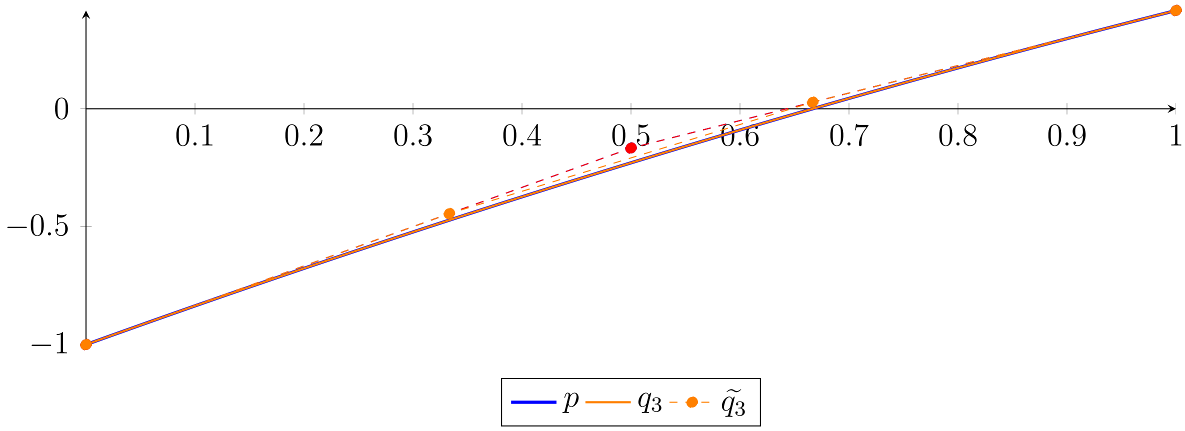
$$\begin{aligned} p &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2}(X) - 0.166667B_{1,2}(X) + 0.416667B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,3} - 0.444444B_{1,3} + 0.0277778B_{2,3} + 0.416667B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2} - 0.166667B_{1,2} + 0.416667B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.58942 \cdot 10^{-19}$.

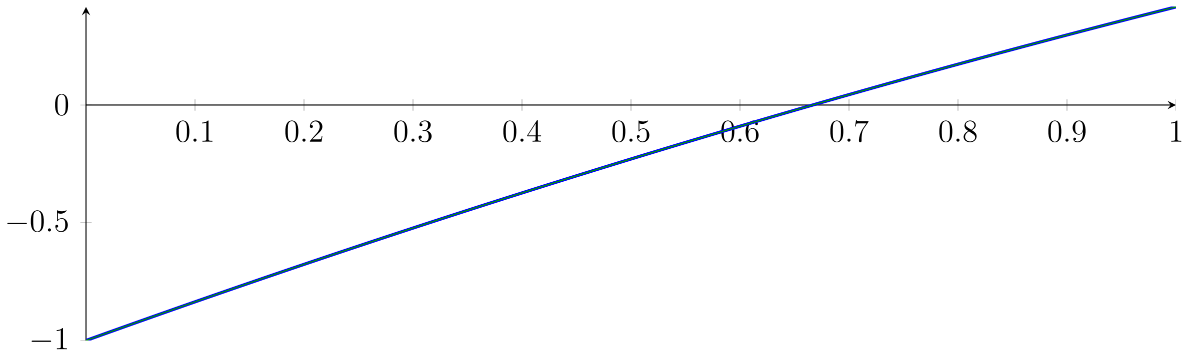
Bounding polynomials M and m:

$$M = -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1$$
$$m = -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1$$

Root of M and m:

$$N(M) = \{-2.32913 \cdot 10^{16}\} \qquad N(m) = \{-2.32913 \cdot 10^{16}\}$$

Intersection intervals:

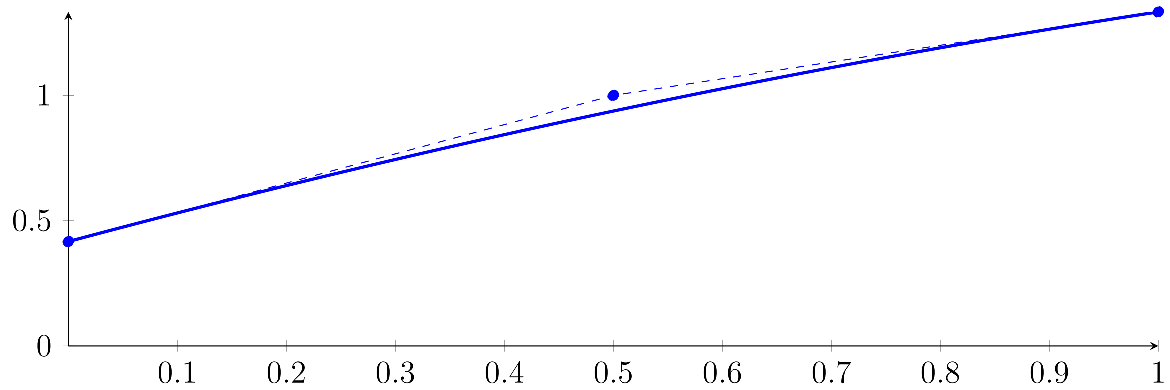


No intersection intervals with the x axis.

96.3 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

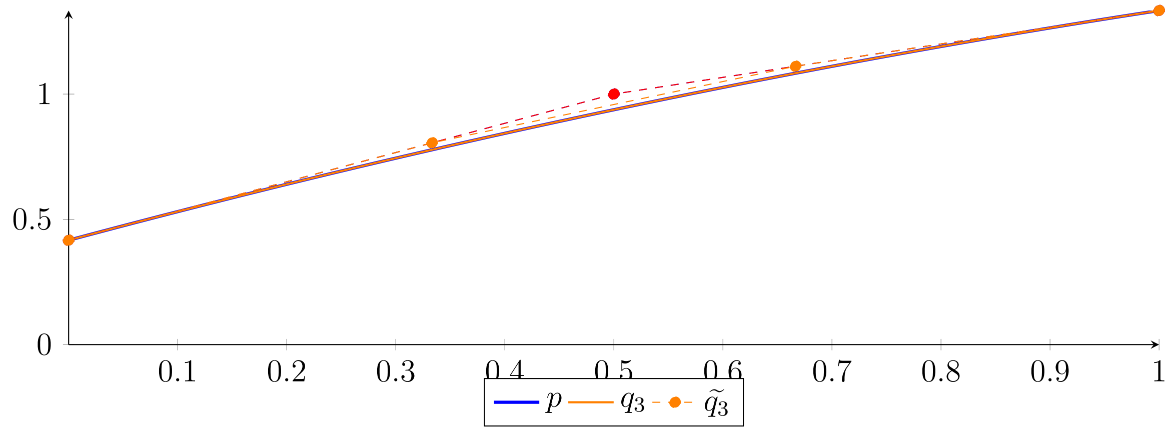
$$\begin{aligned} p &= -0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667 B_{0,2}(X) + 1 B_{1,2}(X) + 1.33333 B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -2.27682\cdot 10^{-18}X^3 - 0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,3} + 0.805556B_{1,3} + 1.11111B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,2} + 1B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.30104\cdot 10^{-18}$.
Bounding polynomials M and m:

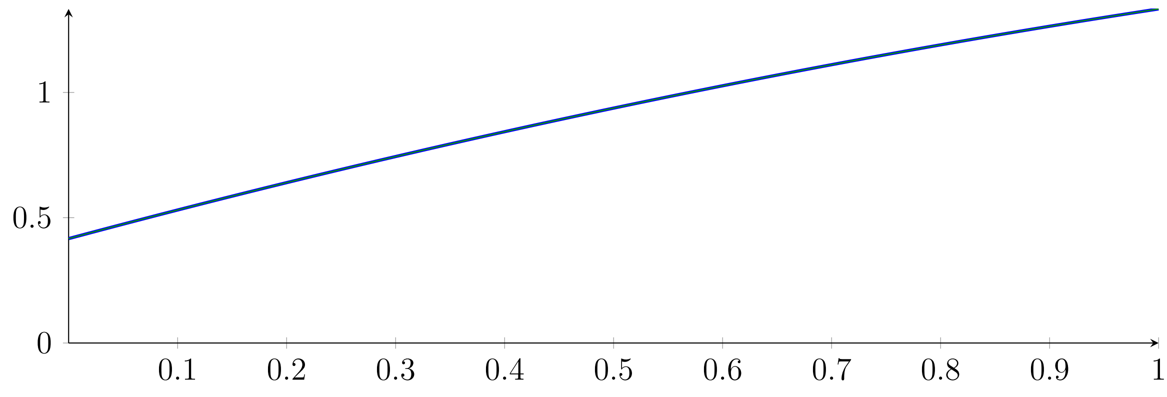
$$M = -2.27682\cdot 10^{-18}X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$m = -2.27682\cdot 10^{-18}X^3 - 0.25X^2 + 1.16667X + 0.416667$$

Root of M and m:

$$N(M) = \{-1.09802\cdot 10^{17}, -8.02734, 2.33594\} \qquad N(m) = \{-1.09802\cdot 10^{17}, -8.02734, 2.33594\}$$

Intersection intervals:

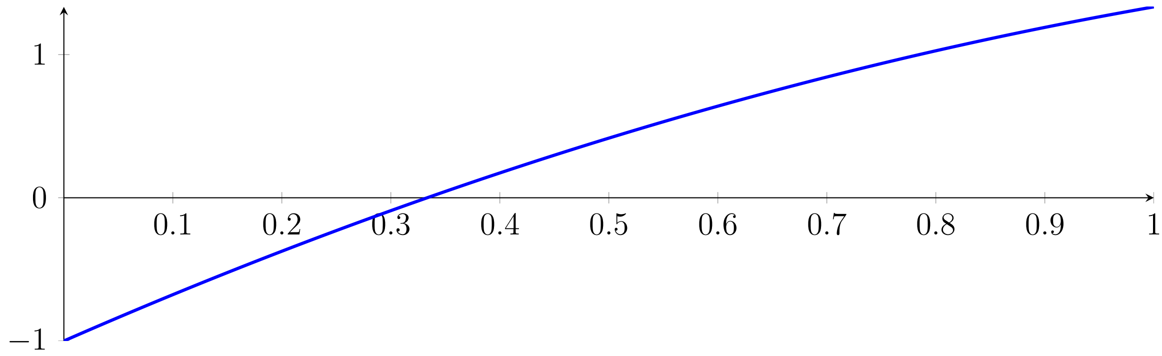


No intersection intervals with the x axis.

96.4 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

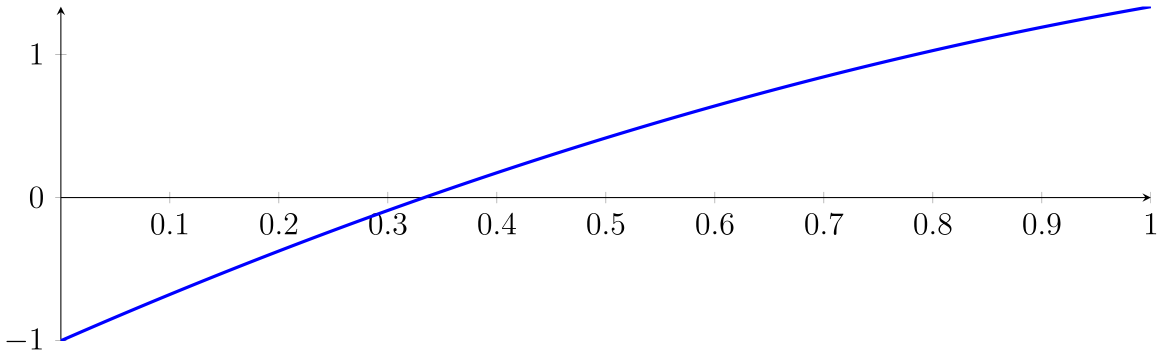
with precision $\varepsilon = 1 \cdot 10^{-16}$.

97 Running BezClip on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

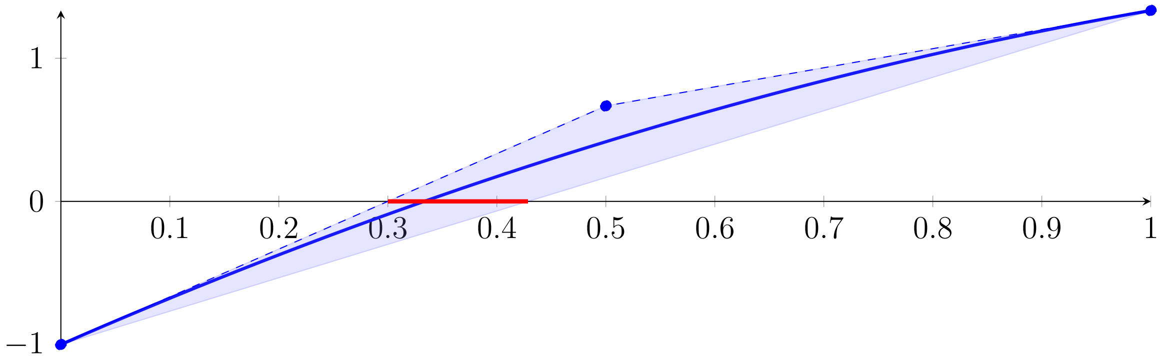
$$p = -1X^2 + 3.33333X - 1$$



97.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

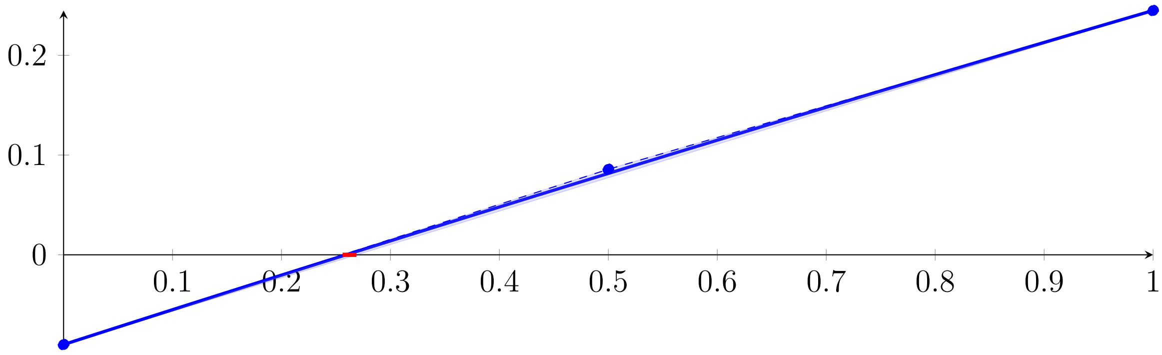
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

97.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

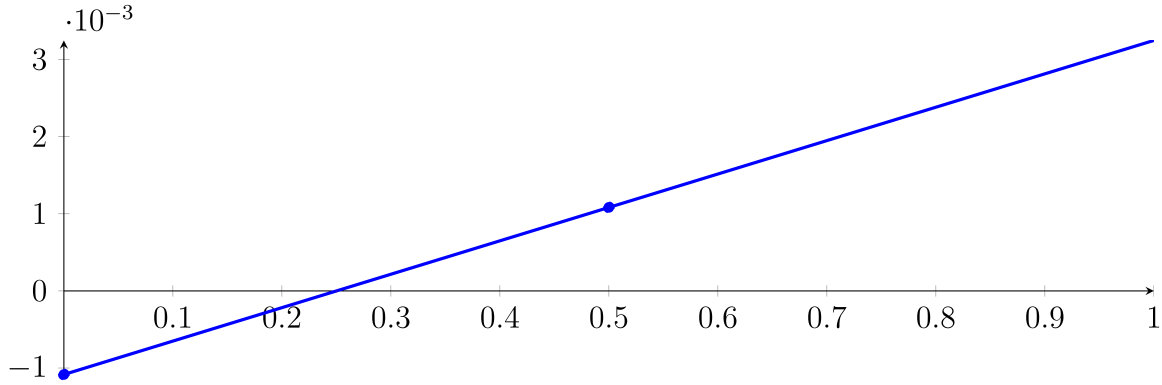
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

97.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -2.64151 \cdot 10^{-06} X^2 + 0.00433538X - 0.00108418$$
$$= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

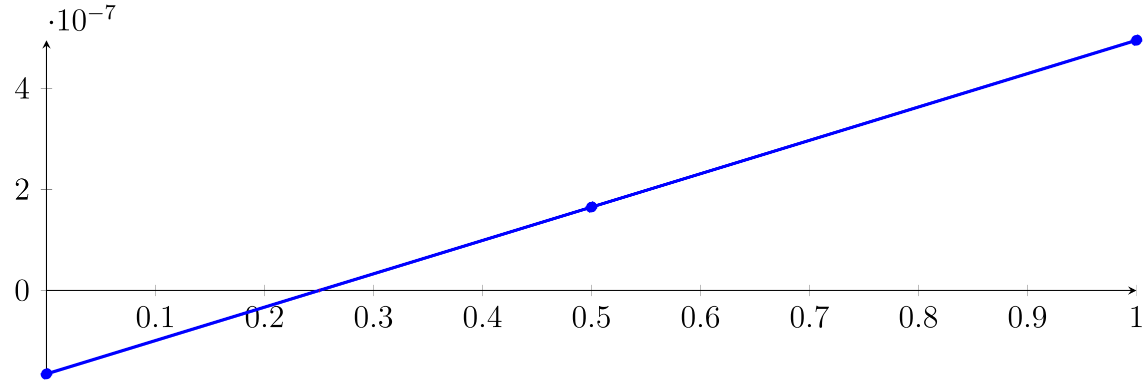
$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

97.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07}$$
$$= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

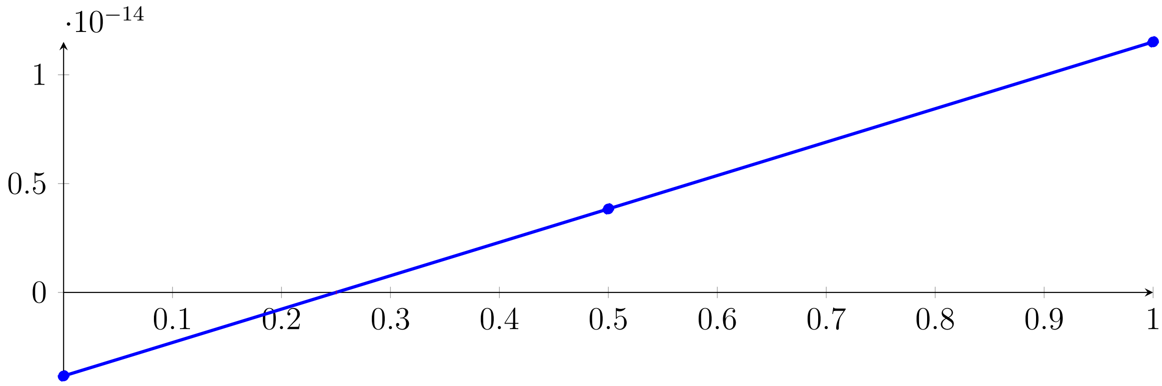
Longest intersection interval: $2.32306 \cdot 10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

97.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.31352 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15}$$
$$= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.39635 \cdot 10^{-16}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

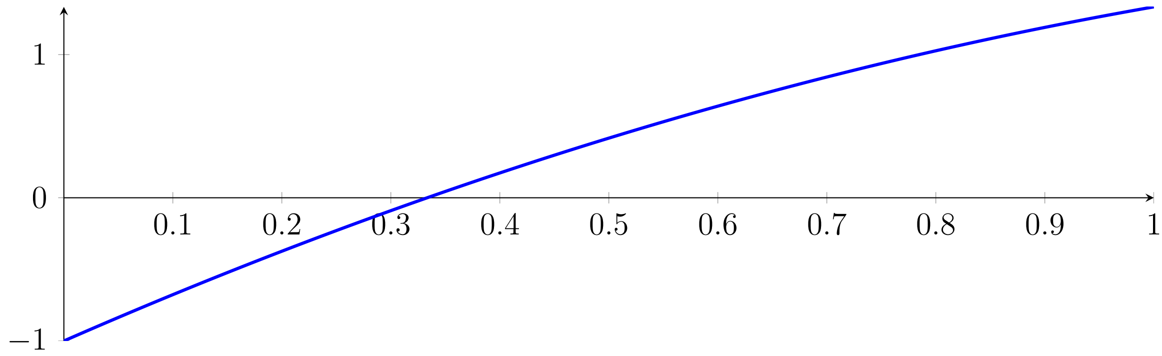
97.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

97.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

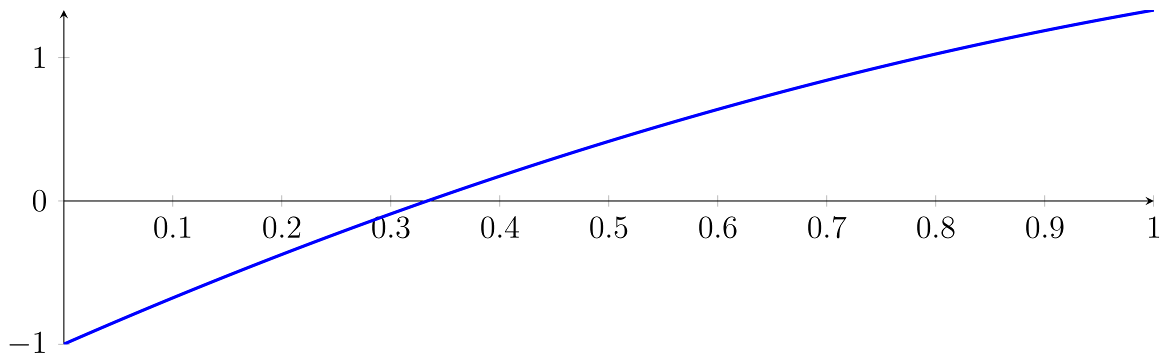
with precision $\varepsilon = 1 \cdot 10^{-32}$.

98 Running QuadClip on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

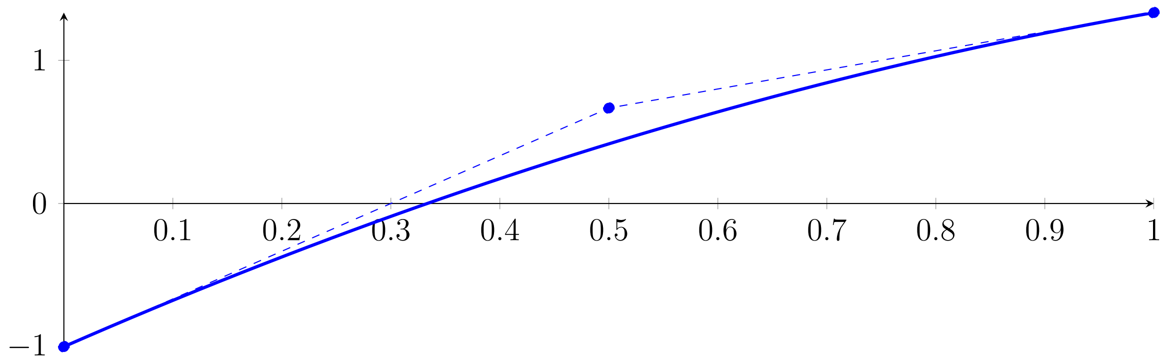
$$p = -1X^2 + 3.33333X - 1$$



98.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

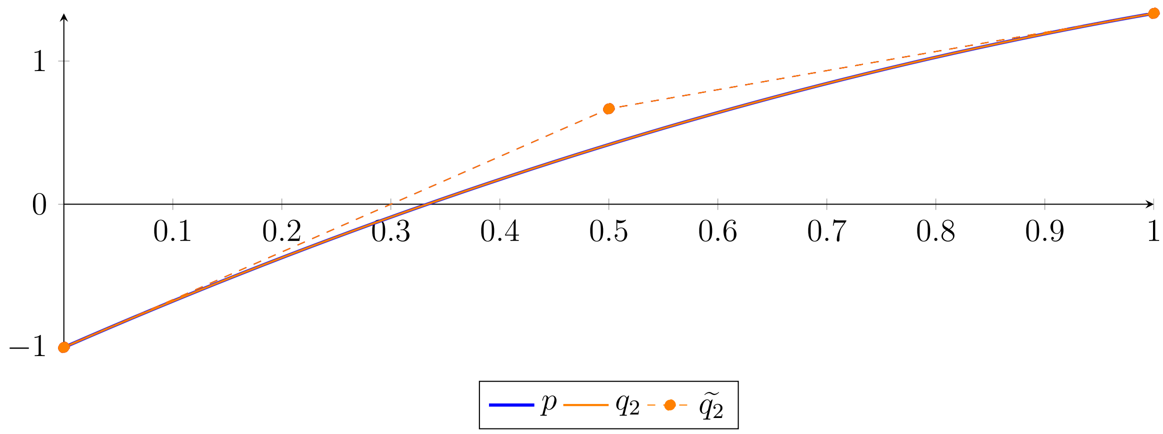
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.
Bounding polynomials M and m:

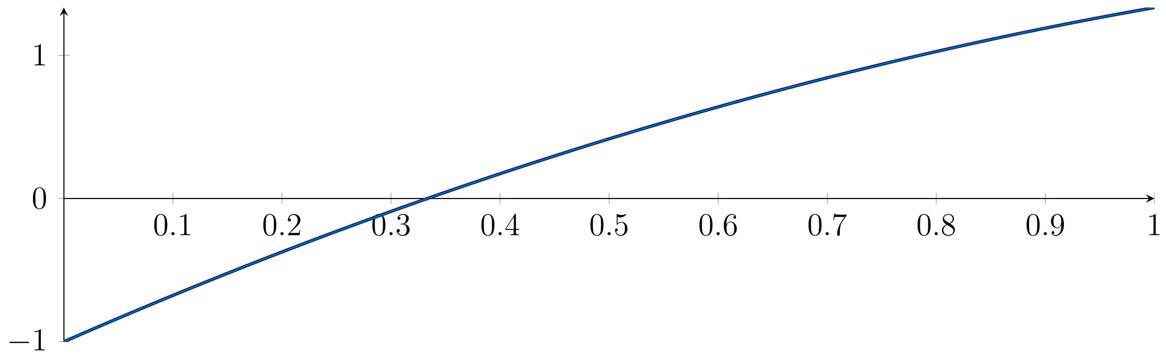
$$M = -1X^2 + 3.33333X - 1$$

$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



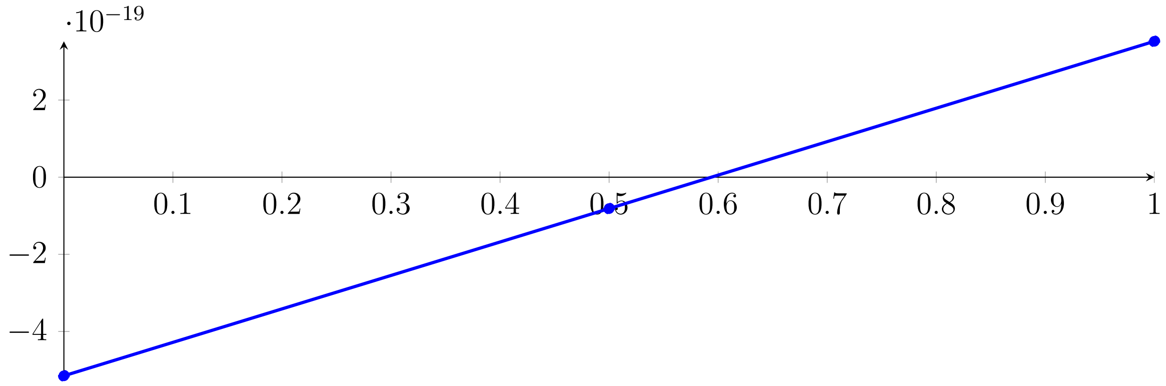
$$[0.333333, 0.333333]$$

Longest intersection interval: $3.25261 \cdot 10^{-19}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

98.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

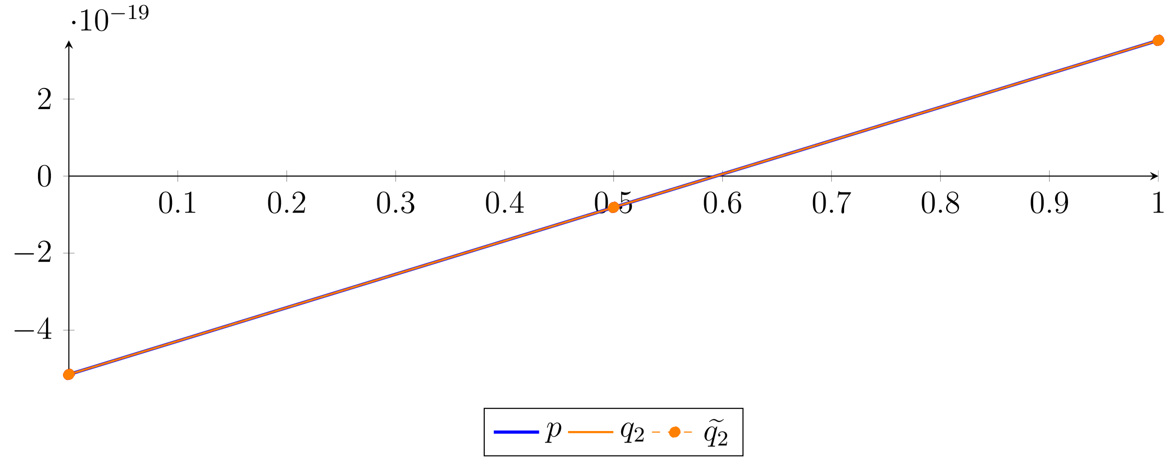
$$\begin{aligned} p &= -9.40395 \cdot 10^{-38} X^2 + 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2}(X) - 8.13152 \cdot 10^{-20} B_{1,2}(X) + 3.52366 \cdot 10^{-19} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2} - 8.13152 \cdot 10^{-20} B_{1,2} + 3.52366 \cdot 10^{-19} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 9.40395 \cdot 10^{-38} X^2 + 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2} - 8.13152 \cdot 10^{-20} B_{1,2} + 3.52366 \cdot 10^{-19} B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.29304{\cdot}10^{-37}$.

Bounding polynomials M and m:

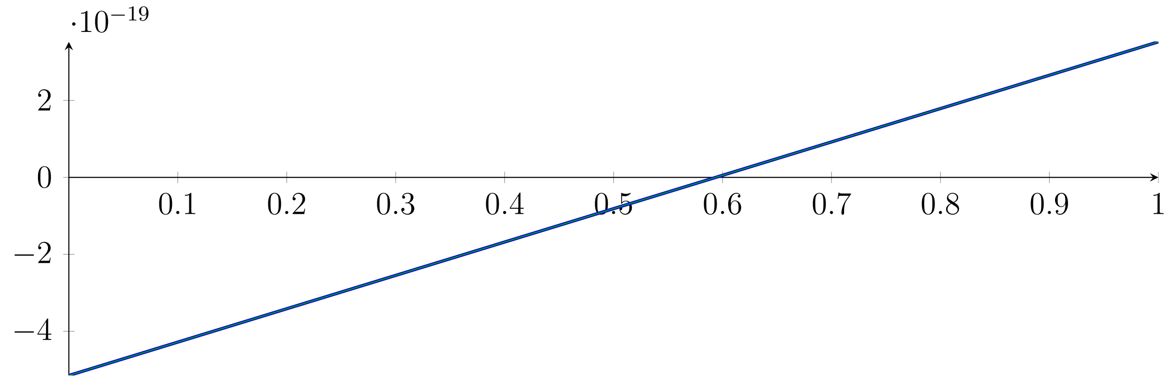
$$M = 8.67362{\cdot}10^{-19}X - 5.14996{\cdot}10^{-19}$$

$$m = 8.67362{\cdot}10^{-19}X - 5.14996{\cdot}10^{-19}$$

Root of M and m:

$$N(M) = \{\} \qquad N(m) = \{\}$$

Intersection intervals:

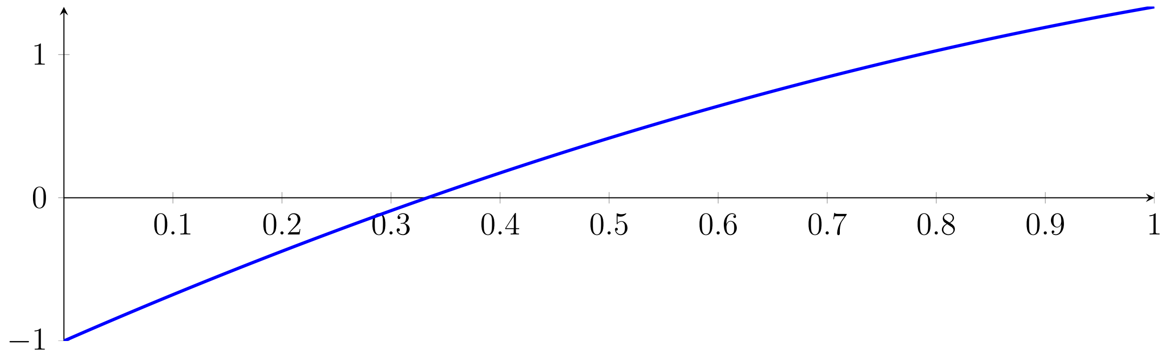


No intersection intervals with the x axis.

98.3 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

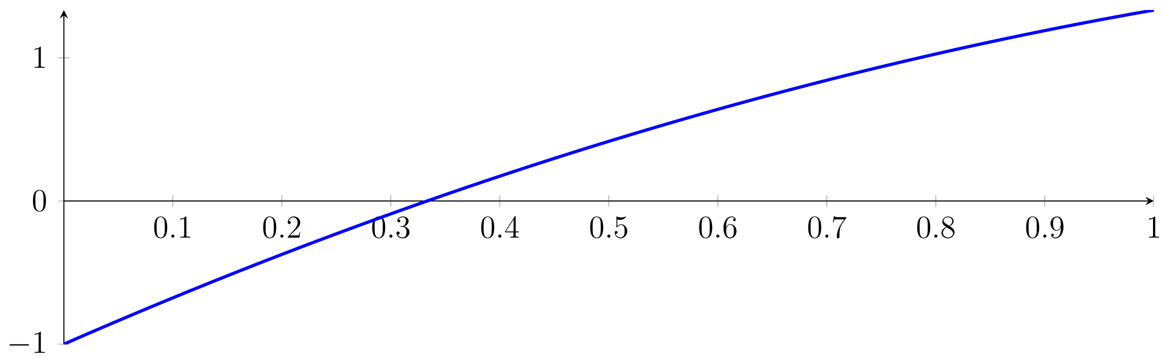
with precision $\varepsilon = 1{\cdot}10^{-32}$.

99 Running `CubeClip` on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

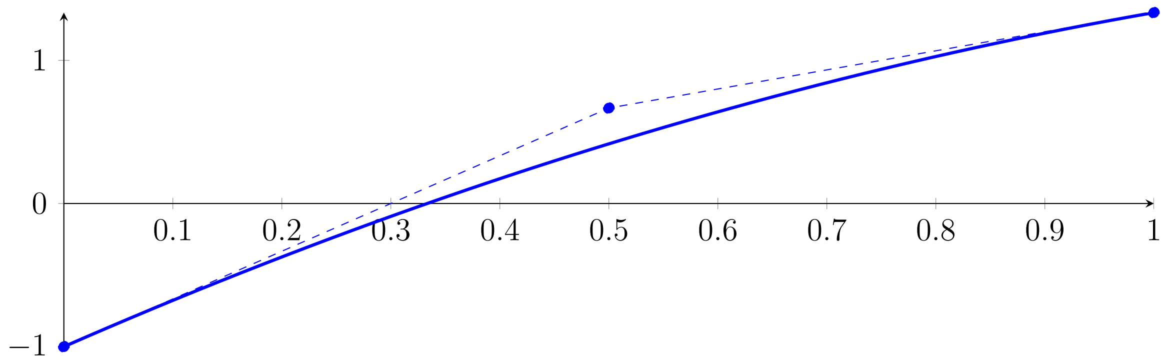
$$p = -1X^2 + 3.33333X - 1$$



99.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

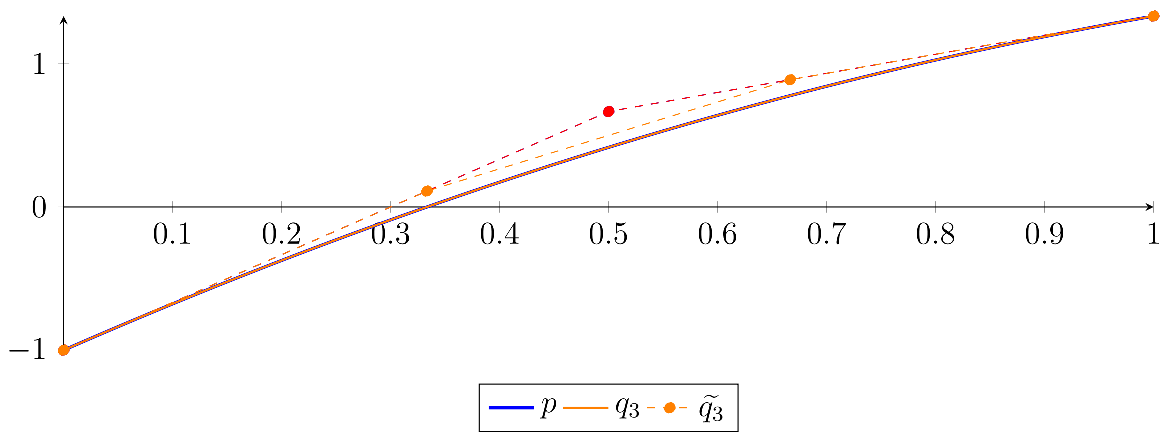
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.63715 \cdot 10^{-17}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681{\cdot}10^{-19}$.
Bounding polynomials M and m:

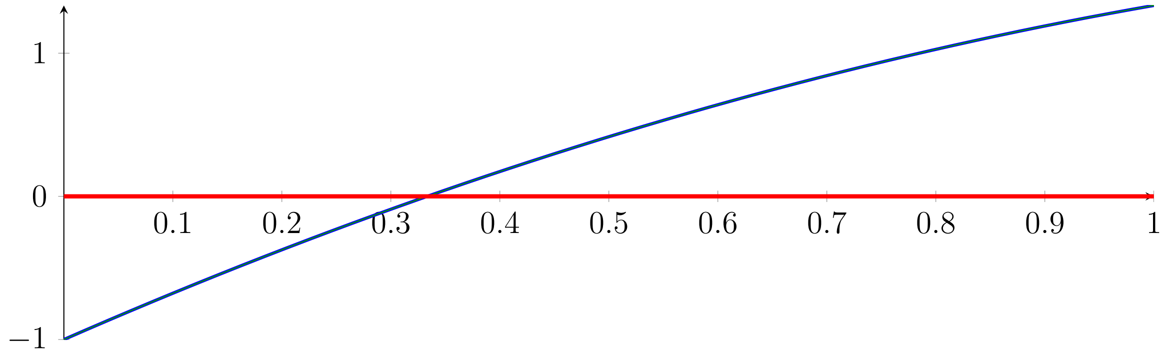
$$M = -1.63715{\cdot}10^{-17}X^3 - 1X^2 + 3.33333X - 1$$

$$m = -1.63715{\cdot}10^{-17}X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{-6.10819{\cdot}10^{16}, 1.66602\} \qquad N(m) = \{-6.10819{\cdot}10^{16}, 1.66602\}$$

Intersection intervals:



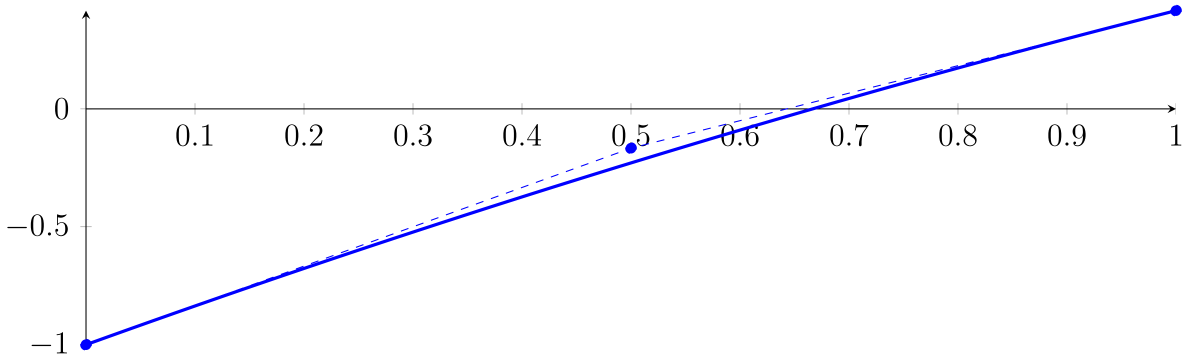
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

99.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

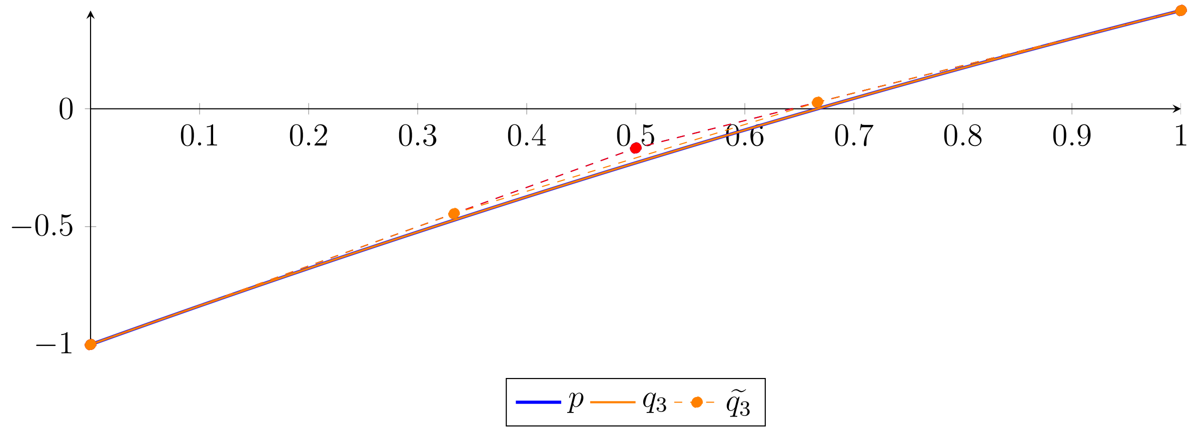
$$\begin{aligned} p &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2}(X) - 0.166667B_{1,2}(X) + 0.416667B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07336{\cdot}10^{-17}X^3 - 0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,3} - 0.444444B_{1,3} + 0.0277778B_{2,3} + 0.416667B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2} - 0.166667B_{1,2} + 0.416667B_{2,2} \end{aligned}$$



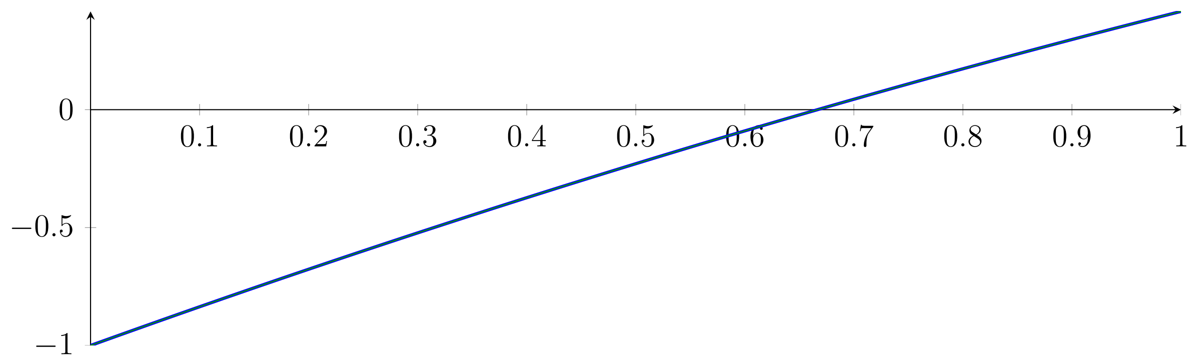
The maximum difference of the Bézier coefficients is $\delta = 7.58942{\cdot}10^{-19}$.
Bounding polynomials M and m:

$$M = -1.07336{\cdot}10^{-17}X^3 - 0.25X^2 + 1.66667X - 1$$
$$m = -1.07336{\cdot}10^{-17}X^3 - 0.25X^2 + 1.66667X - 1$$

Root of M and m:

$$N(M) = \{-2.32913{\cdot}10^{16}\} \qquad N(m) = \{-2.32913{\cdot}10^{16}\}$$

Intersection intervals:

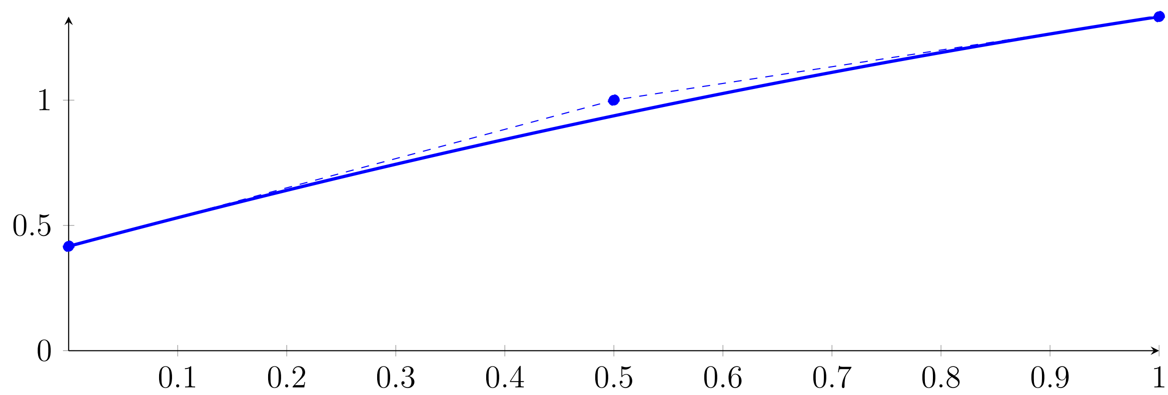


No intersection intervals with the x axis.

99.3 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

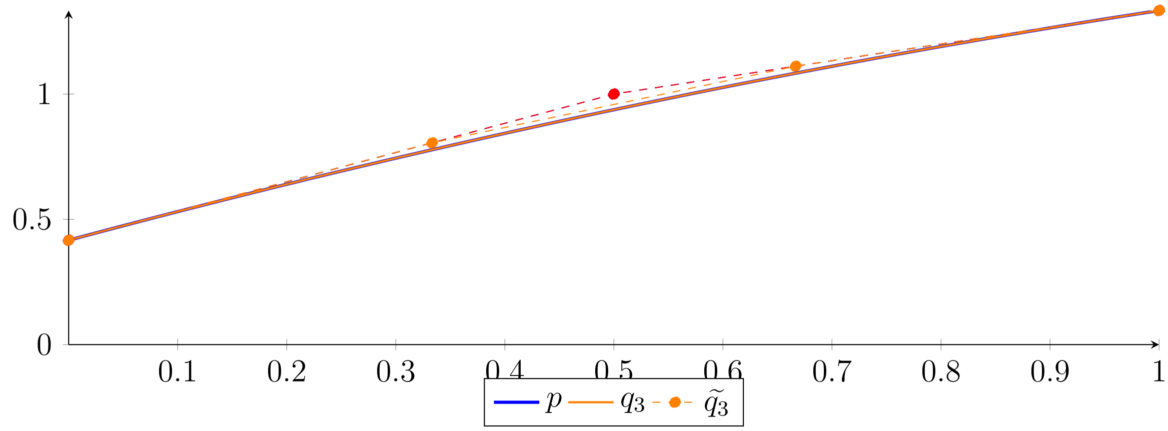
$$\begin{aligned} p &= -0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,2}(X) + 1B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -2.27682{\cdot}10^{-18}X^3 - 0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,3} + 0.805556B_{1,3} + 1.11111B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,2} + 1B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.30104{\cdot}10^{-18}$.
Bounding polynomials M and m:

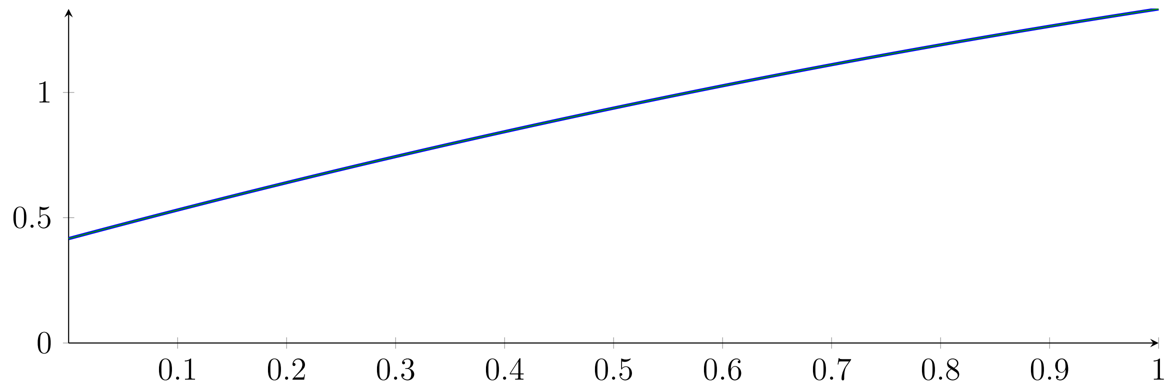
$$M = -2.27682{\cdot}10^{-18}X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$m = -2.27682{\cdot}10^{-18}X^3 - 0.25X^2 + 1.16667X + 0.416667$$

Root of M and m:

$$N(M) = \{-1.09802{\cdot}10^{17}, -8.02734, 2.33594\} \qquad N(m) = \{-1.09802{\cdot}10^{17}, -8.02734, 2.33594\}$$

Intersection intervals:

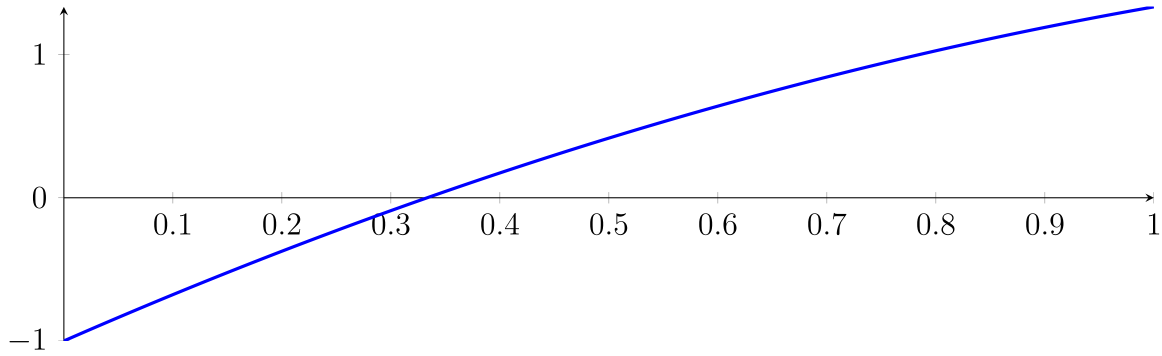


No intersection intervals with the x axis.

99.4 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

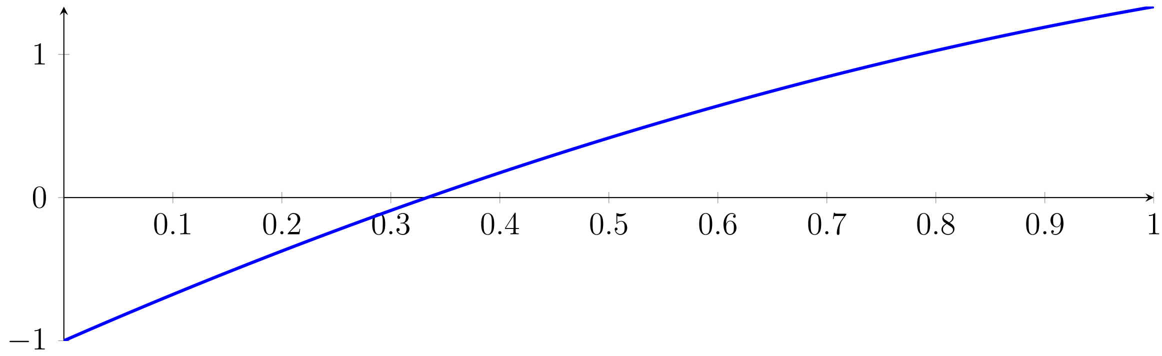
with precision $\varepsilon = 1 \cdot 10^{-32}$.

100 Running BezClip on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

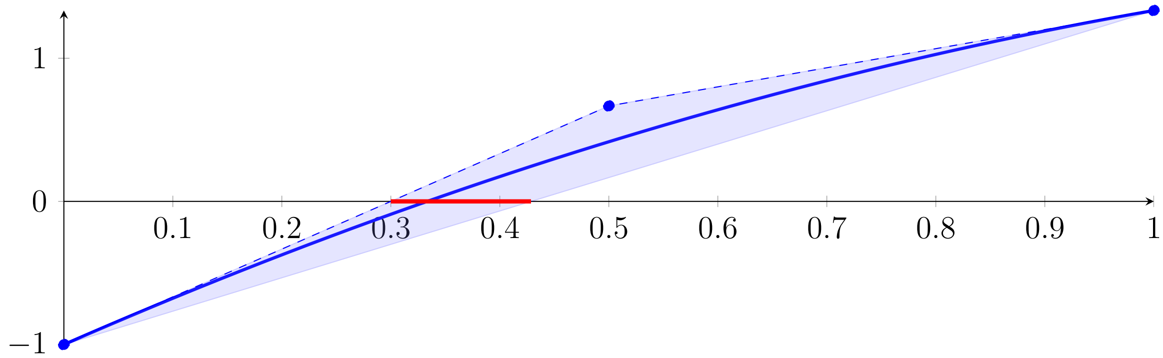
$$p = -1X^2 + 3.33333X - 1$$



100.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

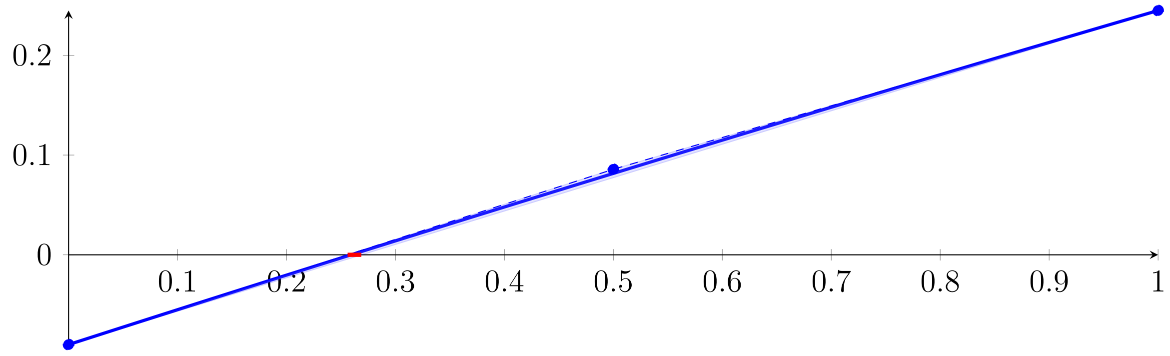
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

100.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

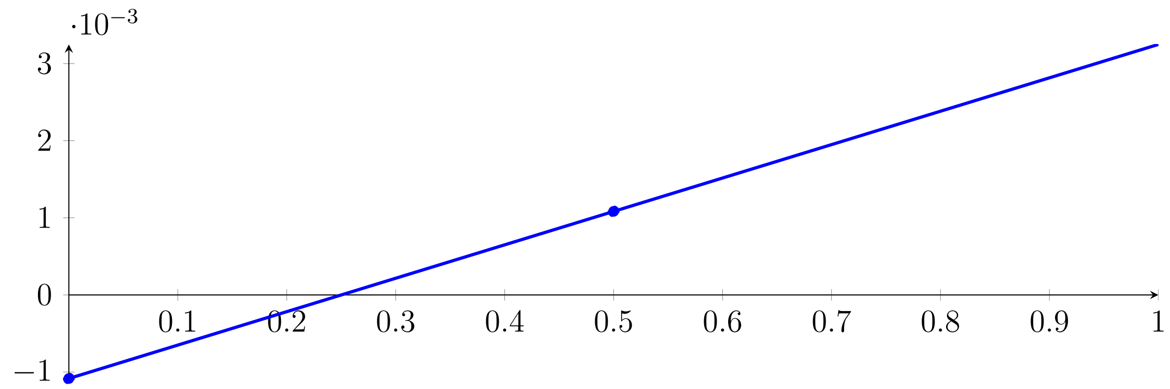
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

100.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -2.64151 \cdot 10^{-06} X^2 + 0.00433538X - 0.00108418$$
$$= -0.00108418B_{0,2}(X) + 0.00108352B_{1,2}(X) + 0.00324857B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

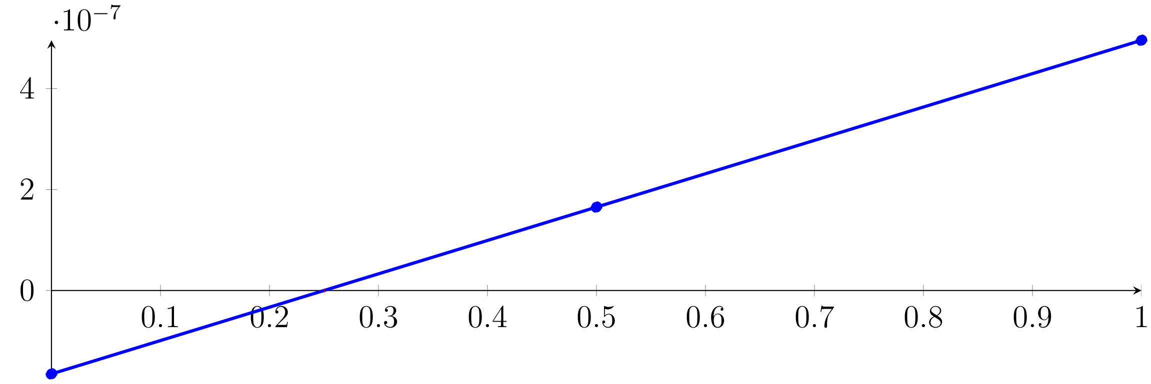
$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

100.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07}$$
$$= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

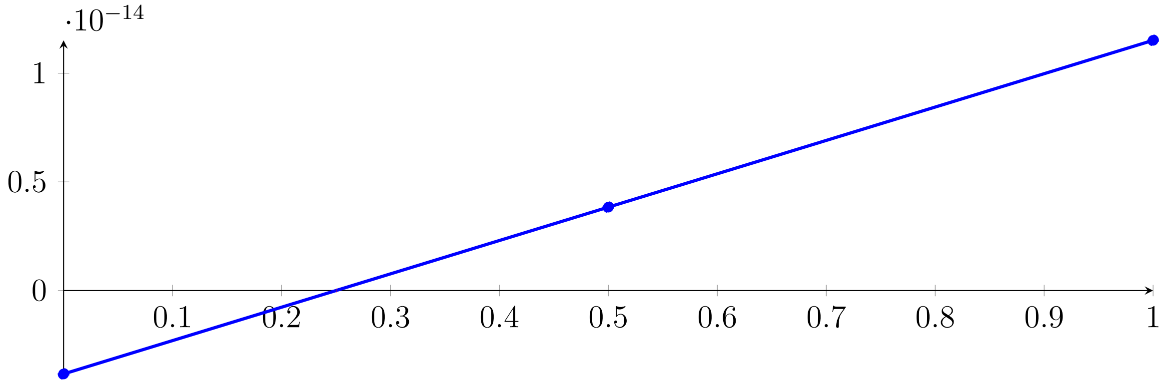
Longest intersection interval: $2.32306 \cdot 10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

100.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.31352 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15}$$
$$= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.39635 \cdot 10^{-16}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

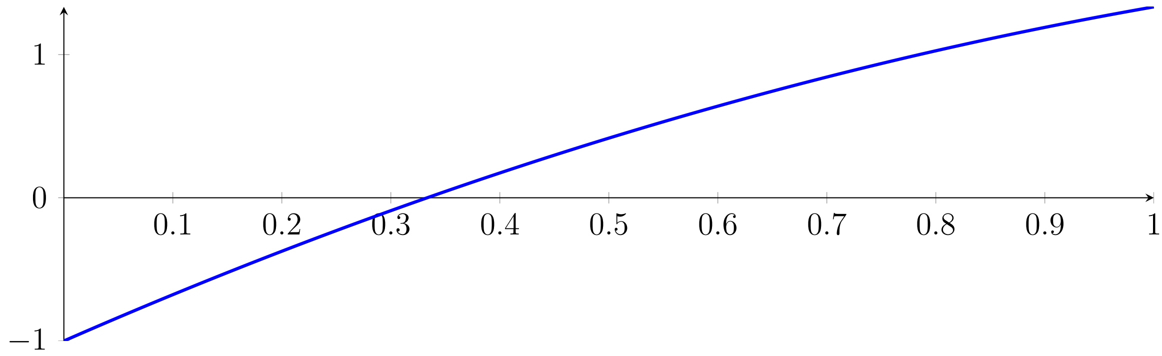
100.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

100.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

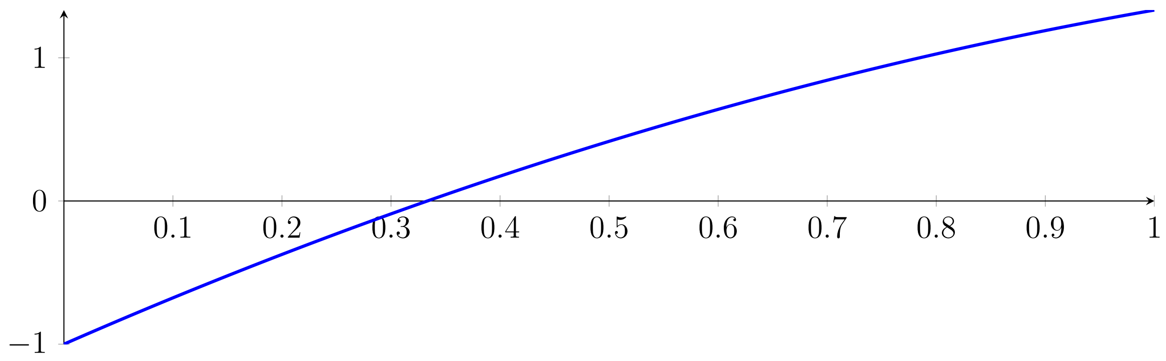
with precision $\varepsilon = 1 \cdot 10^{-64}$.

101 Running QuadClip on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

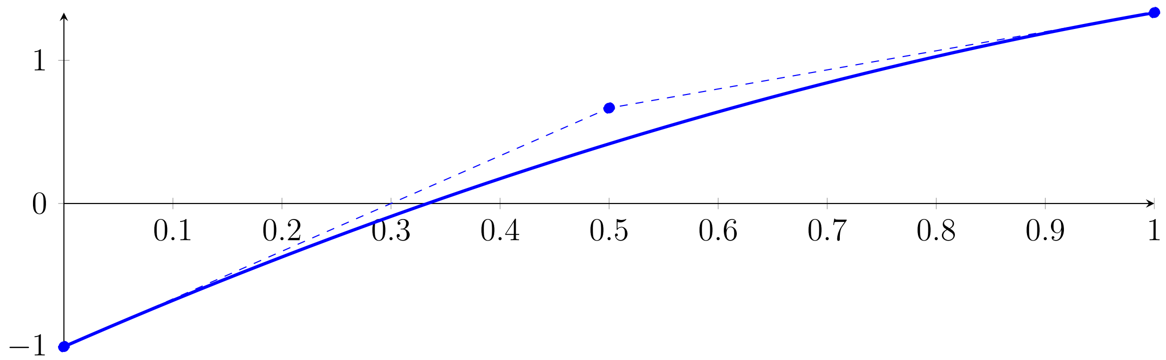
$$p = -1X^2 + 3.33333X - 1$$



101.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

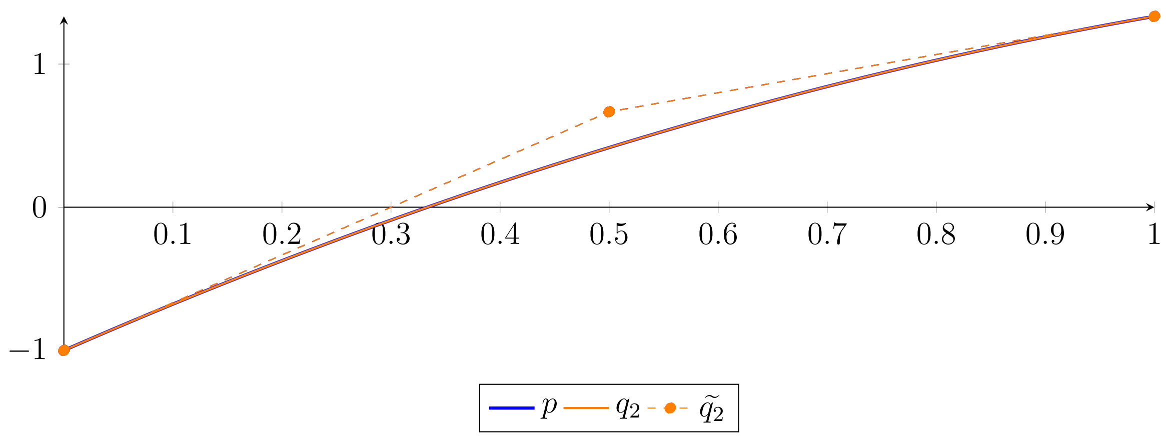
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



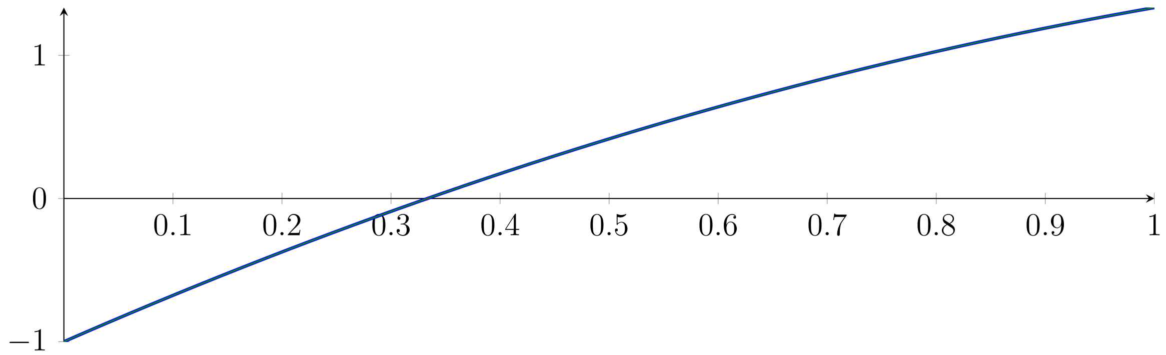
The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.
Bounding polynomials M and m:

$$M = -1X^2 + 3.33333X - 1$$
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



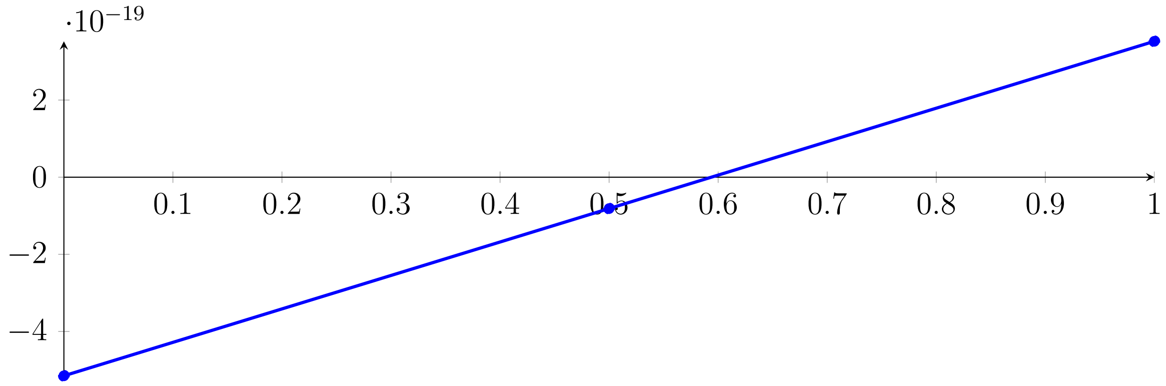
$$[0.333333, 0.333333]$$

Longest intersection interval: $3.25261 \cdot 10^{-19}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

101.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

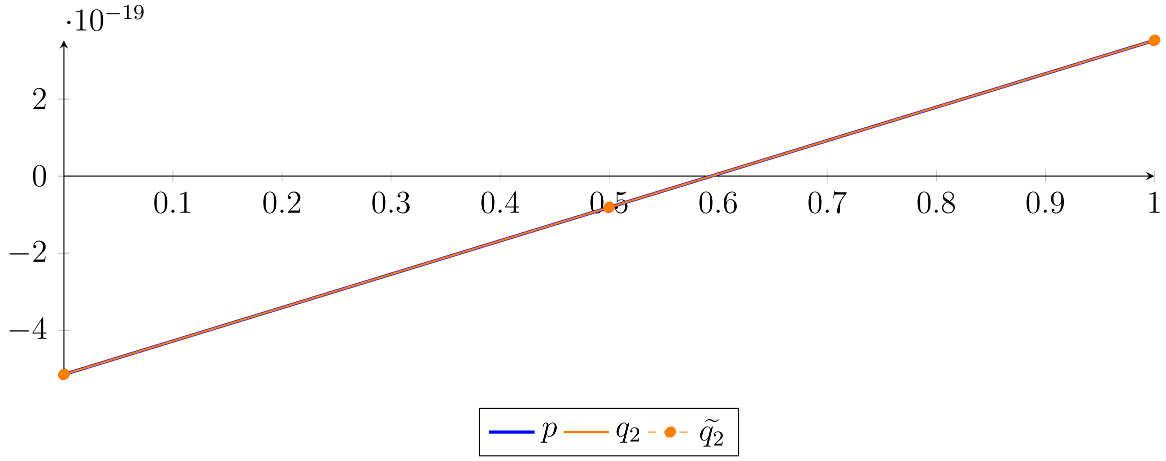
$$\begin{aligned} p &= -9.40395 \cdot 10^{-38} X^2 + 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2}(X) - 8.13152 \cdot 10^{-20} B_{1,2}(X) + 3.52366 \cdot 10^{-19} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2} - 8.13152 \cdot 10^{-20} B_{1,2} + 3.52366 \cdot 10^{-19} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 9.40395 \cdot 10^{-38} X^2 + 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2} - 8.13152 \cdot 10^{-20} B_{1,2} + 3.52366 \cdot 10^{-19} B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.29304{\cdot}10^{-37}$.
Bounding polynomials M and m:

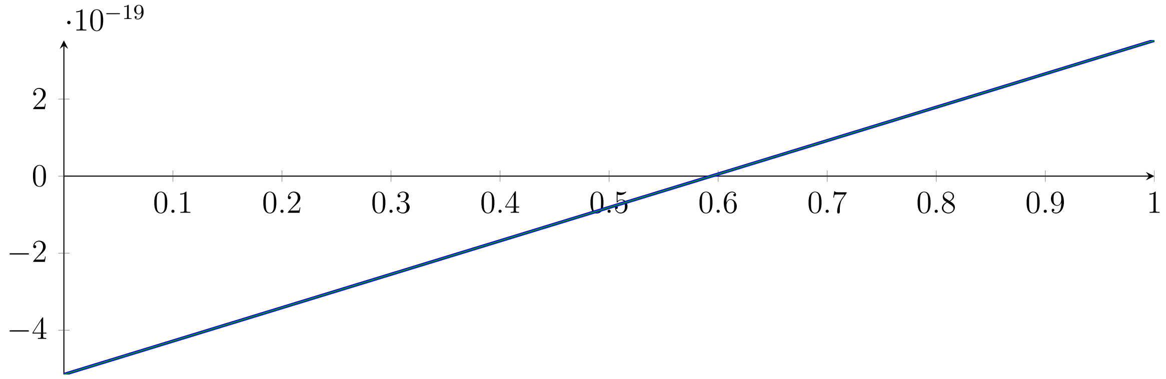
$$M = 8.67362{\cdot}10^{-19}X - 5.14996{\cdot}10^{-19}$$

$$m = 8.67362{\cdot}10^{-19}X - 5.14996{\cdot}10^{-19}$$

Root of M and m:

$$N(M) = \{\} \qquad N(m) = \{\}$$

Intersection intervals:

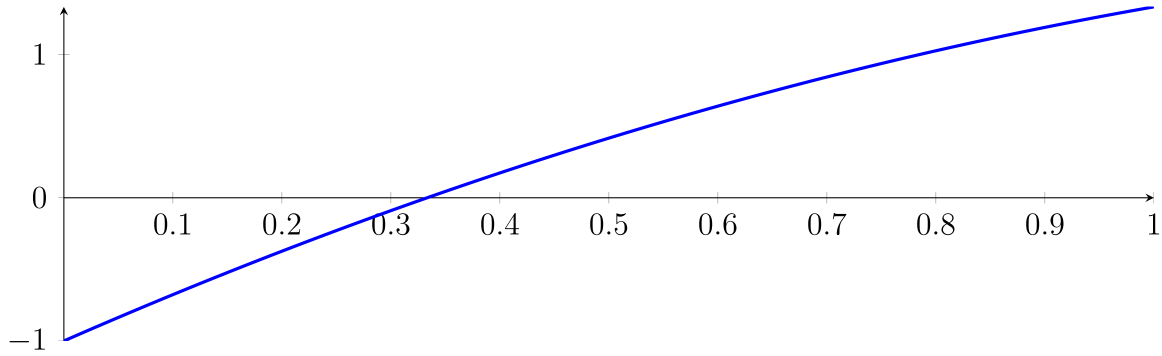


No intersection intervals with the x axis.

101.3 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

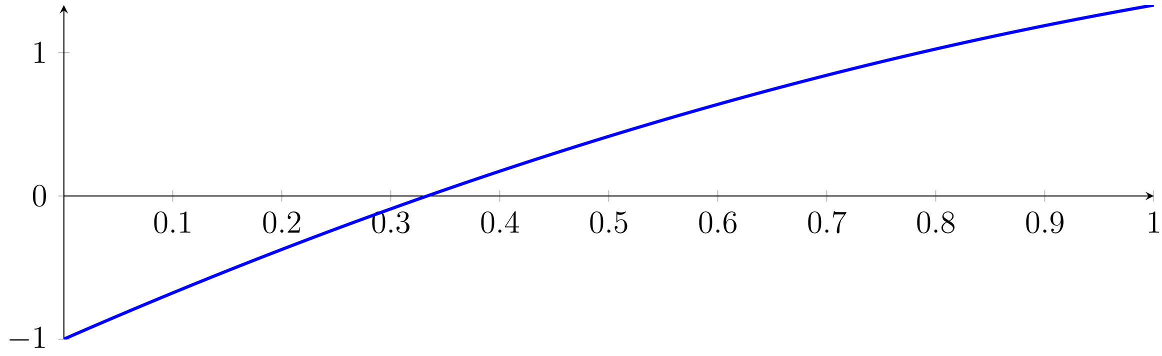
with precision $\varepsilon = 1 \cdot 10^{-64}$.

102 Running CubeClip on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

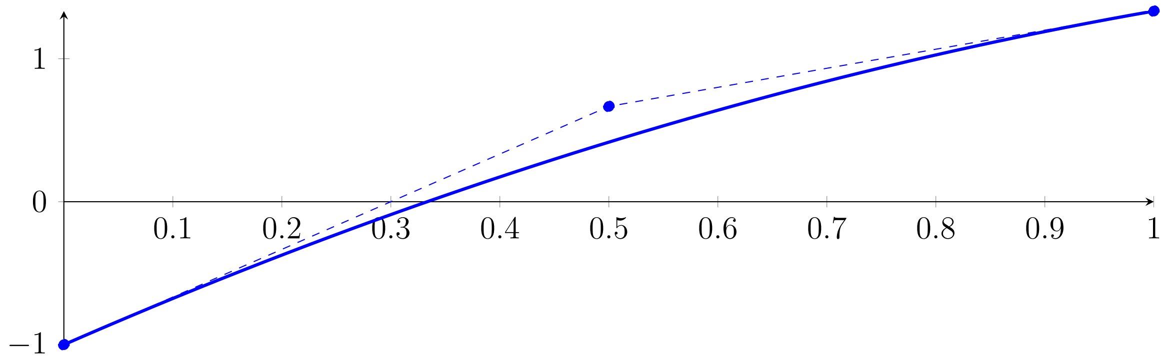
$$p = -1X^2 + 3.33333X - 1$$



102.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

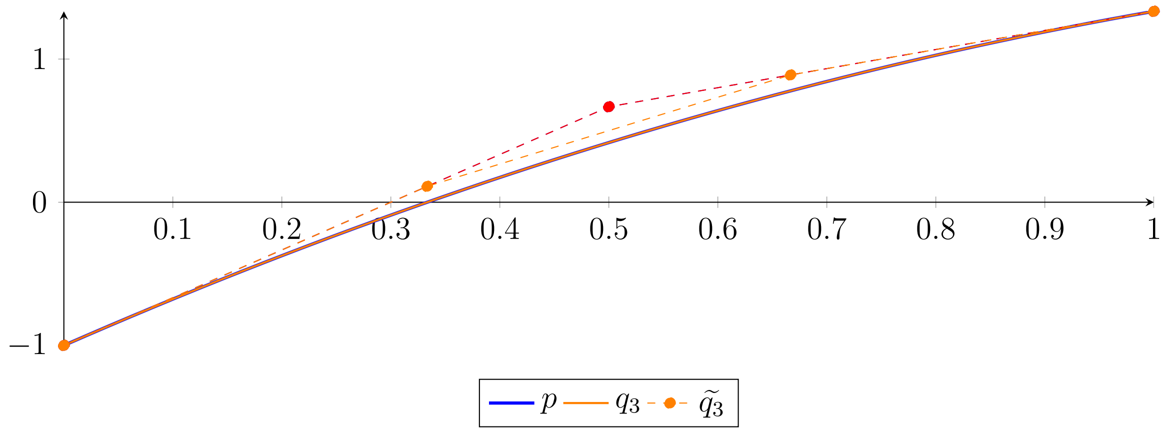
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.63715 \cdot 10^{-17}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681\cdot10^{-19}$.
Bounding polynomials M and m:

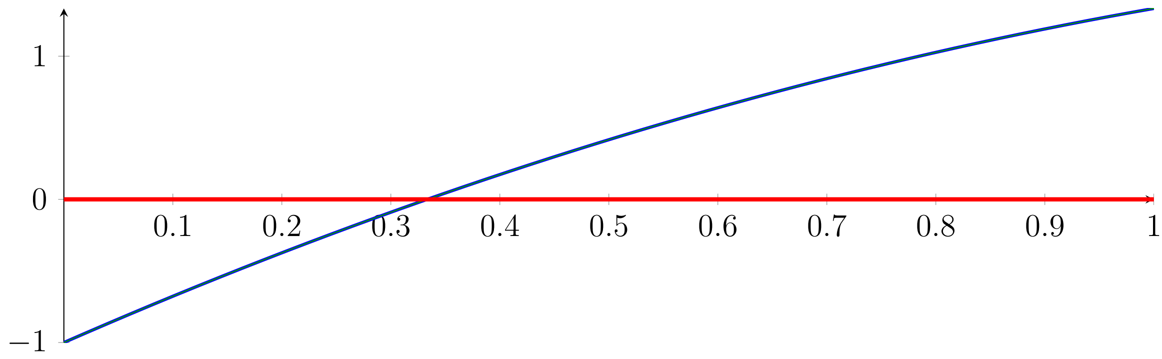
$$M = -1.63715\cdot10^{-17}X^3 - 1X^2 + 3.33333X - 1$$

$$m = -1.63715\cdot10^{-17}X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{-6.10819\cdot10^{16}, 1.66602\} \qquad N(m) = \{-6.10819\cdot10^{16}, 1.66602\}$$

Intersection intervals:



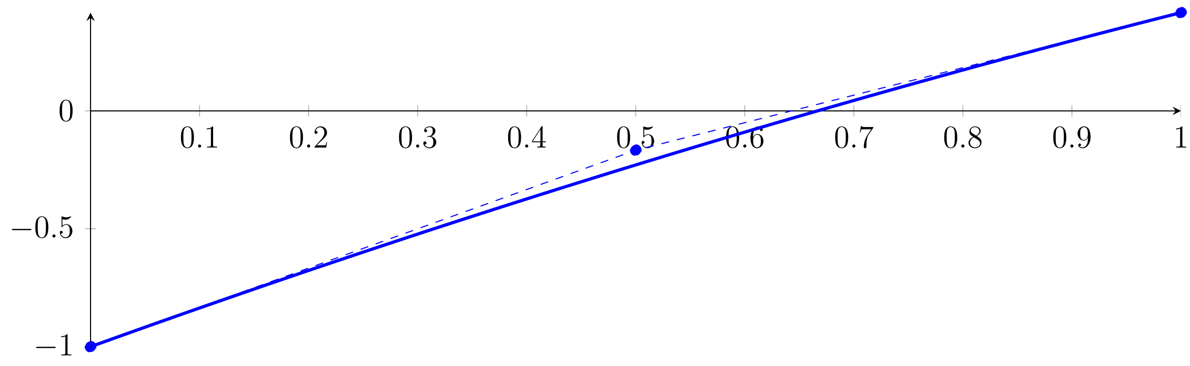
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

102.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

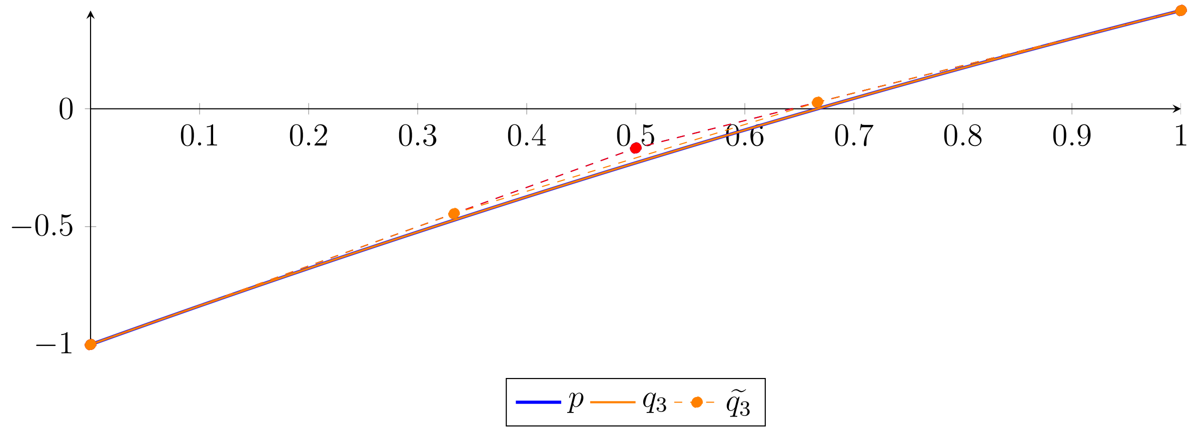
$$\begin{aligned} p &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2}(X) - 0.166667B_{1,2}(X) + 0.416667B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07336\cdot10^{-17}X^3 - 0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,3} - 0.444444B_{1,3} + 0.0277778B_{2,3} + 0.416667B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2} - 0.166667B_{1,2} + 0.416667B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.58942{\cdot}10^{-19}$.

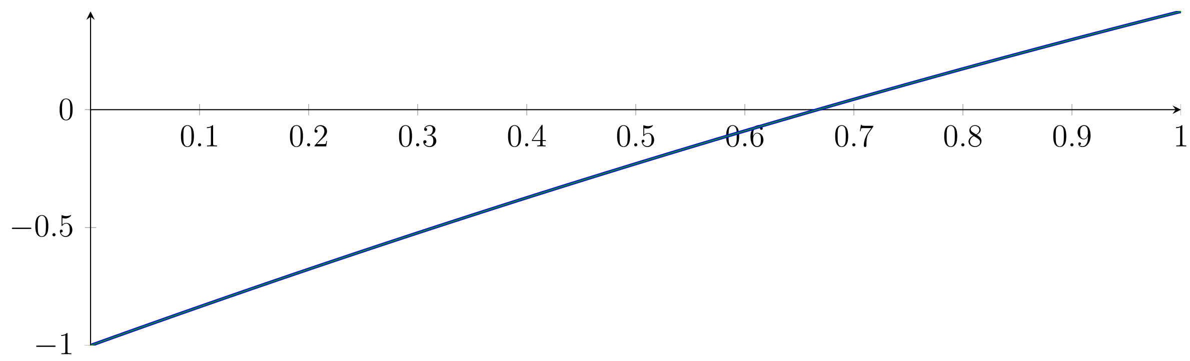
Bounding polynomials M and m:

$$M = -1.07336{\cdot}10^{-17}X^3 - 0.25X^2 + 1.66667X - 1$$
$$m = -1.07336{\cdot}10^{-17}X^3 - 0.25X^2 + 1.66667X - 1$$

Root of M and m:

$$N(M) = \{-2.32913{\cdot}10^{16}\} \qquad N(m) = \{-2.32913{\cdot}10^{16}\}$$

Intersection intervals:

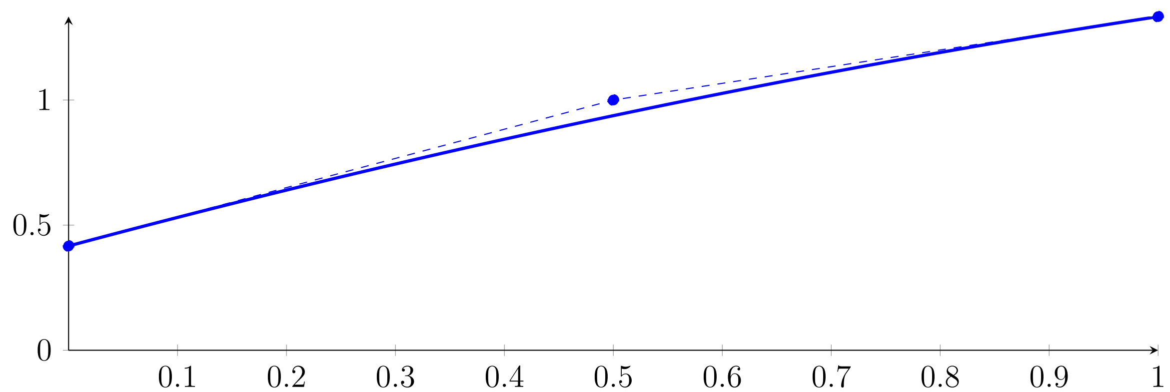


No intersection intervals with the x axis.

102.3 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

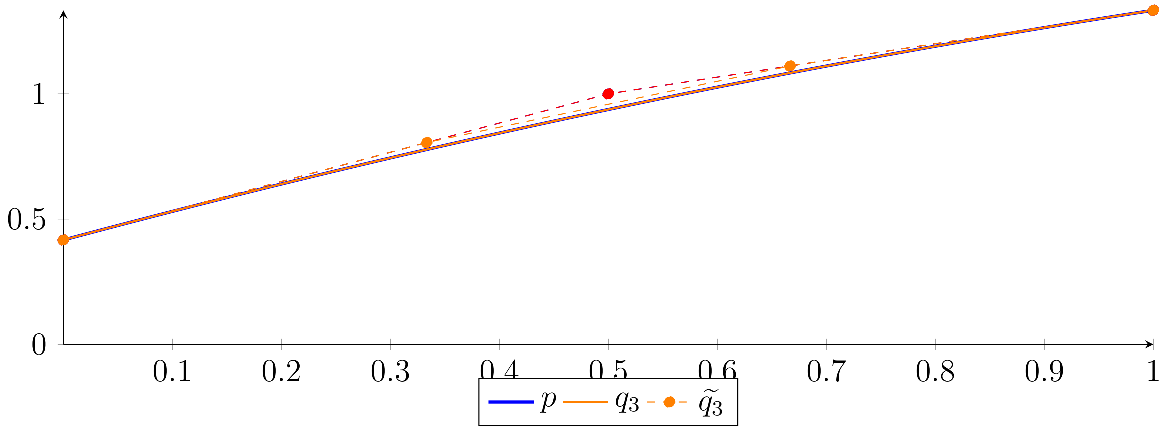
$$\begin{aligned} p &= -0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,2}(X) + 1B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667 B_{0,3} + 0.805556 B_{1,3} + 1.11111 B_{2,3} + 1.33333 B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667 B_{0,2} + 1 B_{1,2} + 1.33333 B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.30104 \cdot 10^{-18}$.
Bounding polynomials M and m:

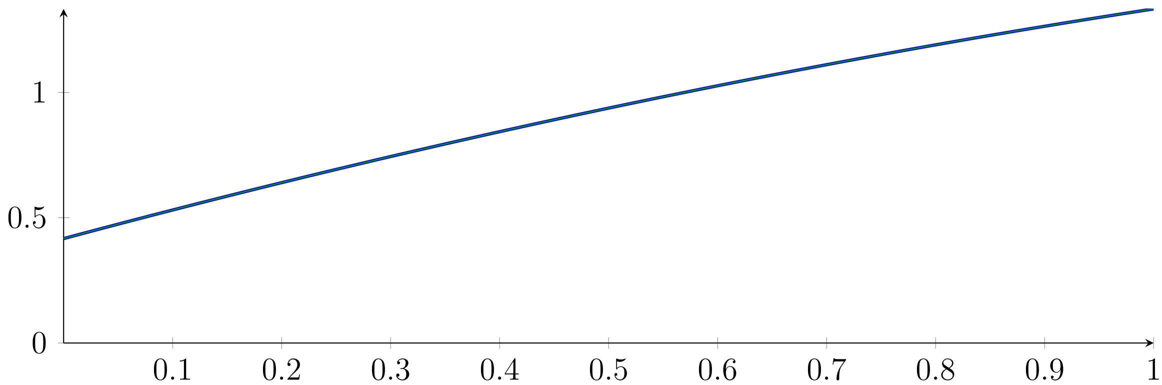
$$M = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$m = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

Root of M and m:

$$N(M) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\} \qquad N(m) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\}$$

Intersection intervals:

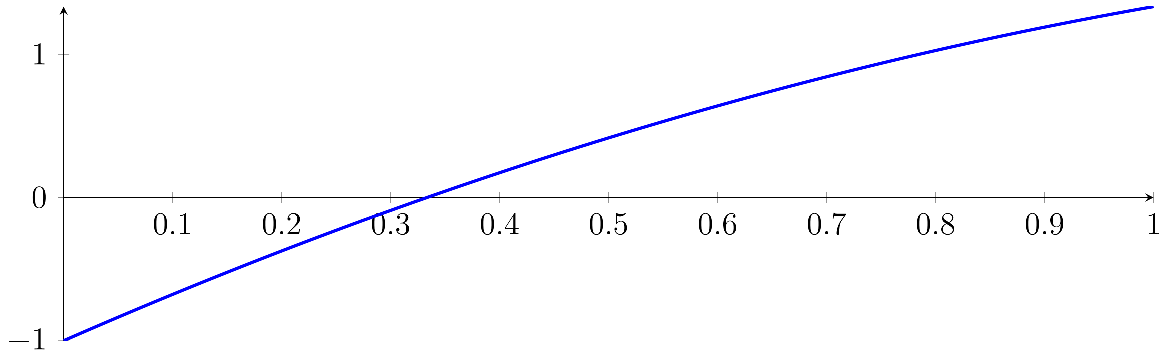


No intersection intervals with the x axis.

102.4 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

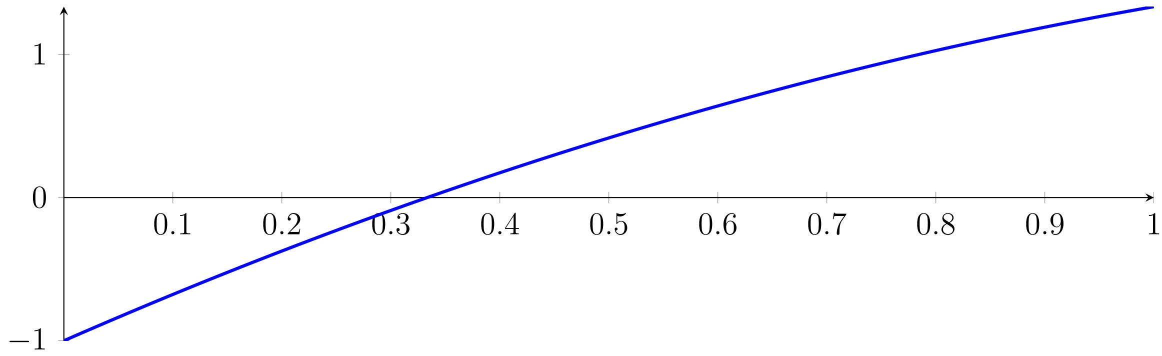
with precision $\varepsilon = 1 \cdot 10^{-64}$.

103 Running `BezClip` on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

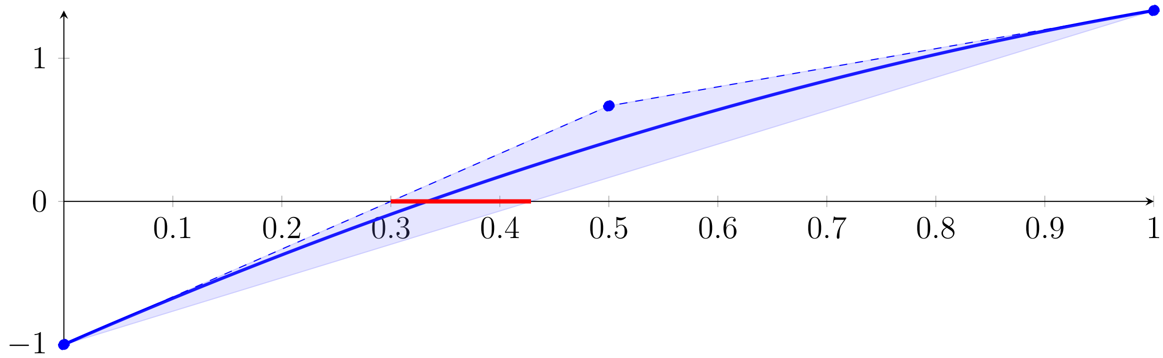
$$p = -1X^2 + 3.33333X - 1$$



103.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

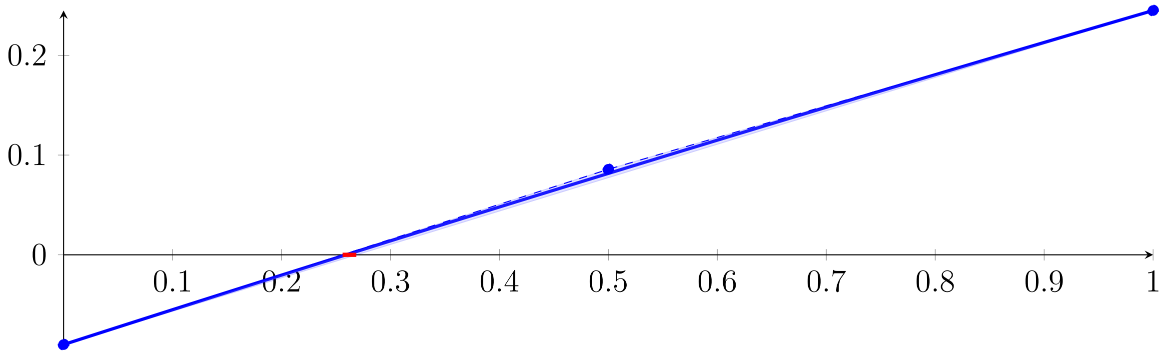
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

103.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

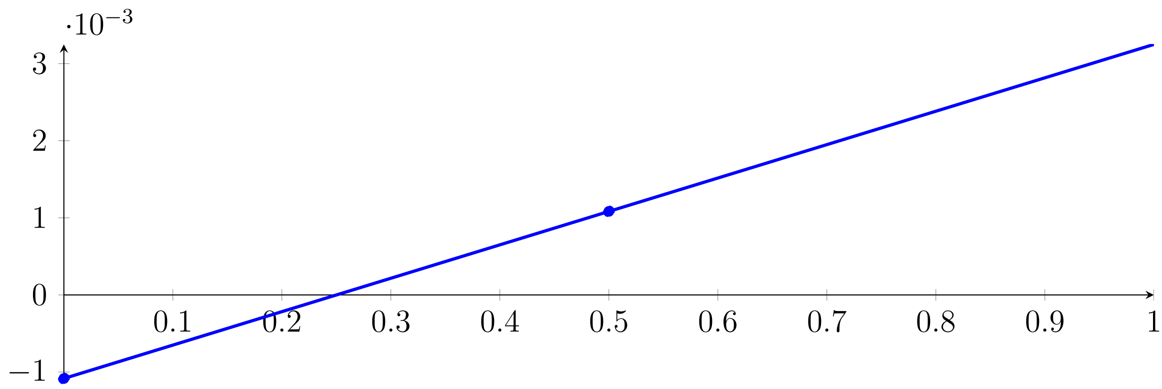
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

103.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

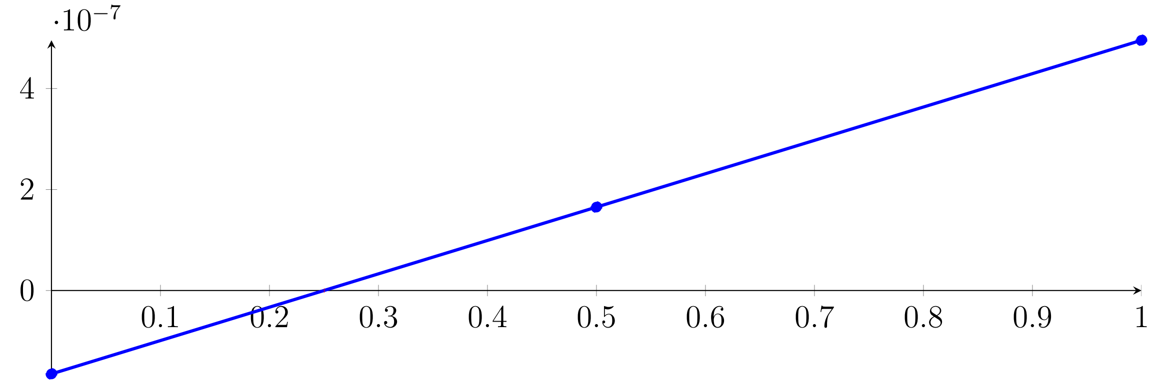
$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

103.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07}$$
$$= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

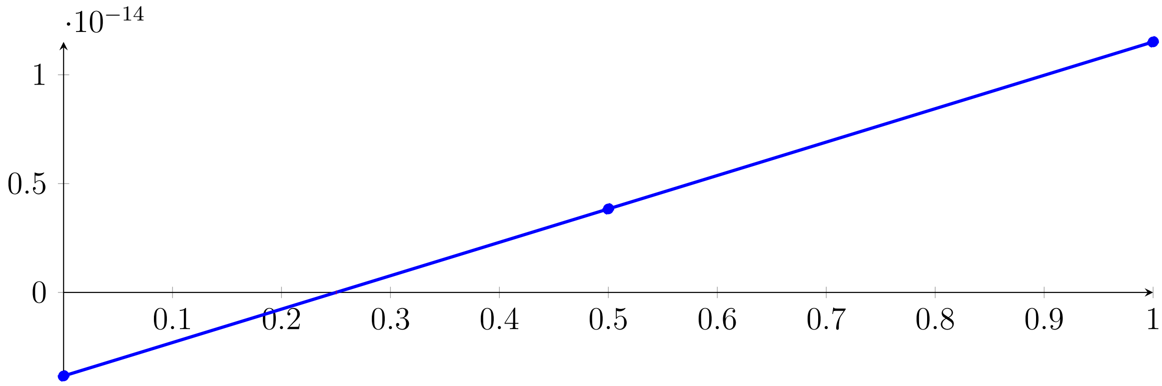
$$[0.25, 0.25]$$

Longest intersection interval: $2.32306 \cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

103.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.31352 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15}$$
$$= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.39635 \cdot 10^{-16}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

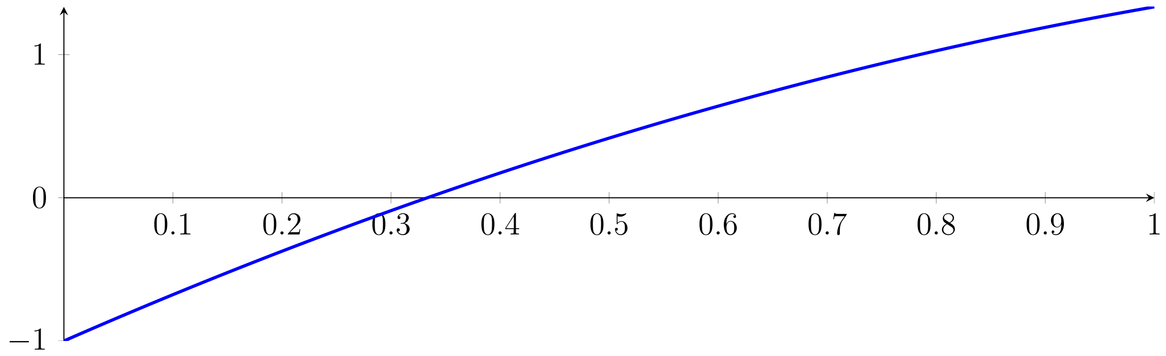
103.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

103.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

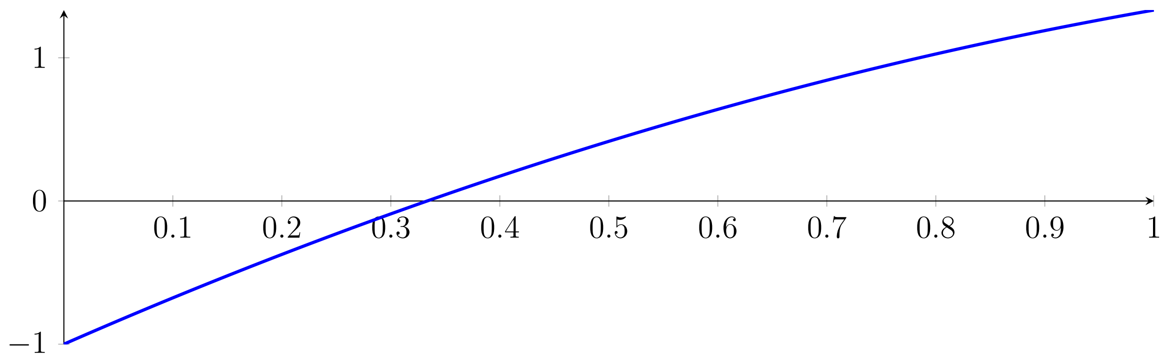
with precision $\varepsilon = 1 \cdot 10^{-128}$.

104 Running QuadClip on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0,1]$:

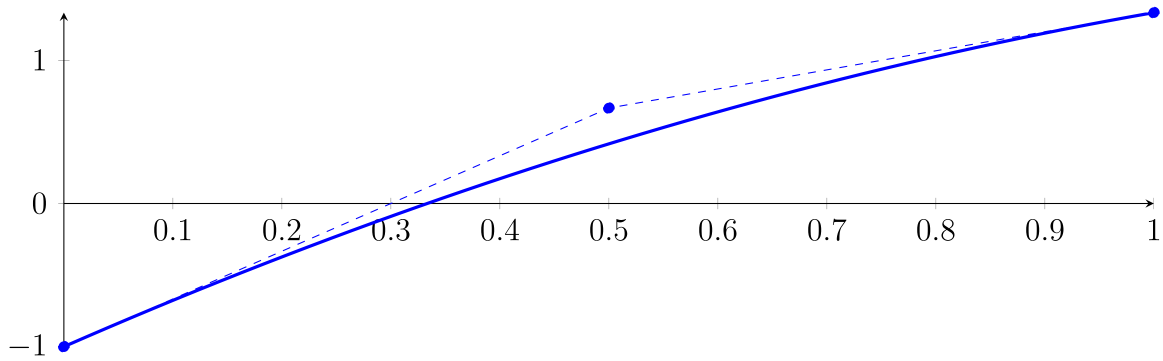
$$p = -1X^2 + 3.33333X - 1$$



104.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

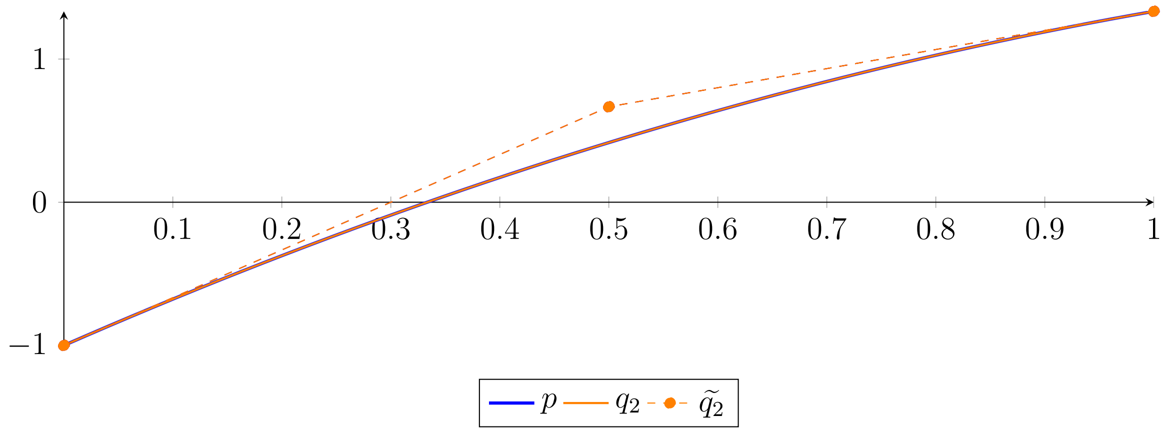
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.
Bounding polynomials M and m:

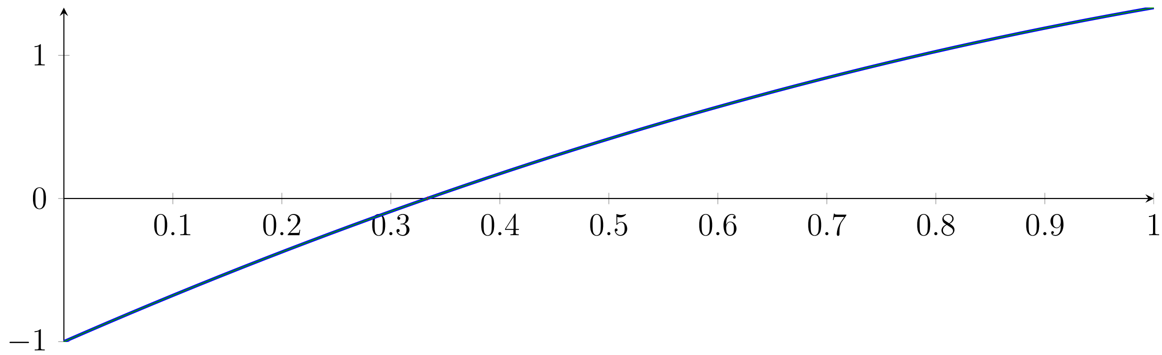
$$M = -1X^2 + 3.33333X - 1$$

$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



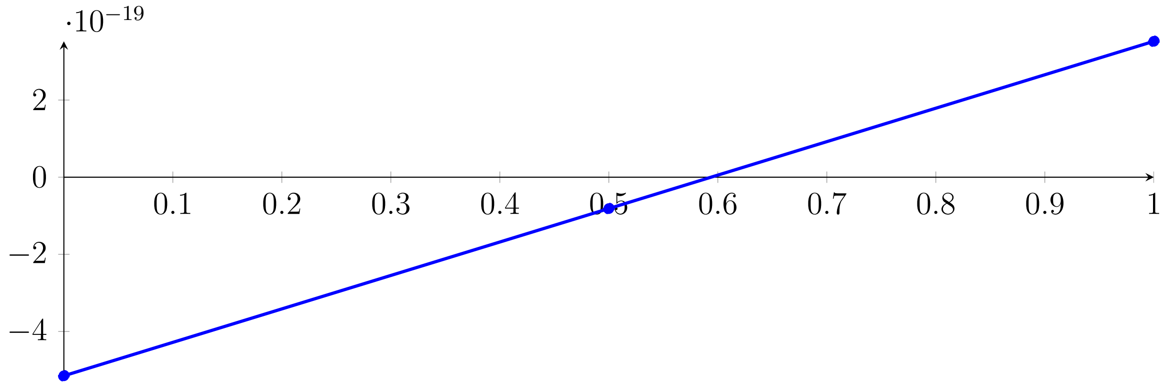
$$[0.333333, 0.333333]$$

Longest intersection interval: $3.25261 \cdot 10^{-19}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

104.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

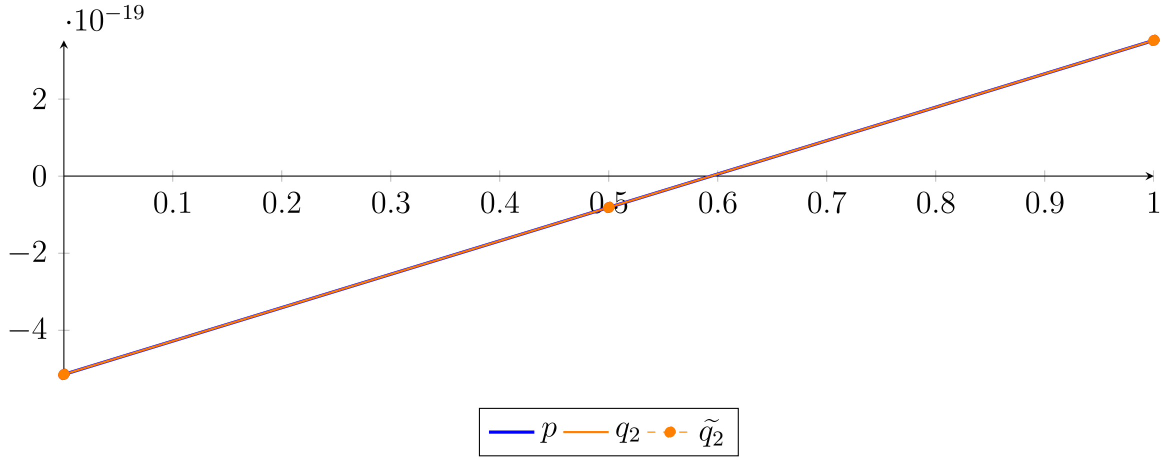
$$\begin{aligned} p &= -9.40395 \cdot 10^{-38} X^2 + 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2}(X) - 8.13152 \cdot 10^{-20} B_{1,2}(X) + 3.52366 \cdot 10^{-19} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2} - 8.13152 \cdot 10^{-20} B_{1,2} + 3.52366 \cdot 10^{-19} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 9.40395 \cdot 10^{-38} X^2 + 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2} - 8.13152 \cdot 10^{-20} B_{1,2} + 3.52366 \cdot 10^{-19} B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.29304{\cdot}10^{-37}$.

Bounding polynomials M and m:

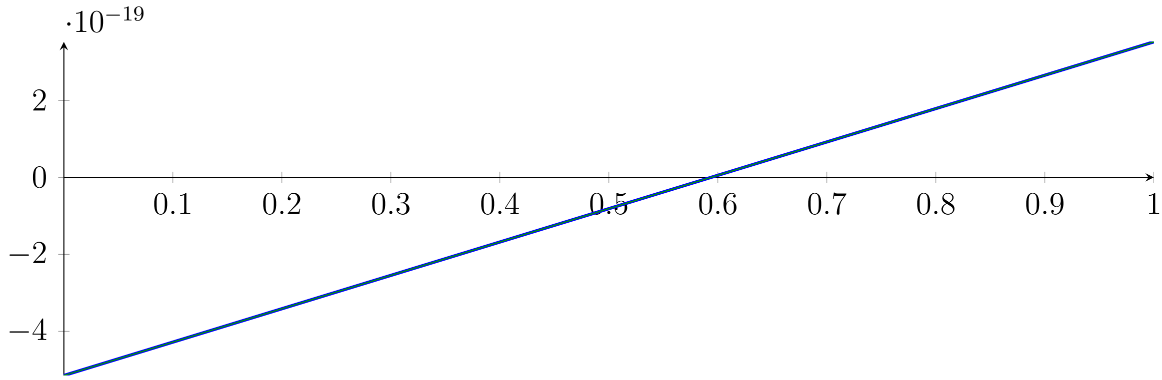
$$M = 8.67362{\cdot}10^{-19}X - 5.14996{\cdot}10^{-19}$$

$$m = 8.67362{\cdot}10^{-19}X - 5.14996{\cdot}10^{-19}$$

Root of M and m:

$$N(M) = \{\} \qquad N(m) = \{\}$$

Intersection intervals:

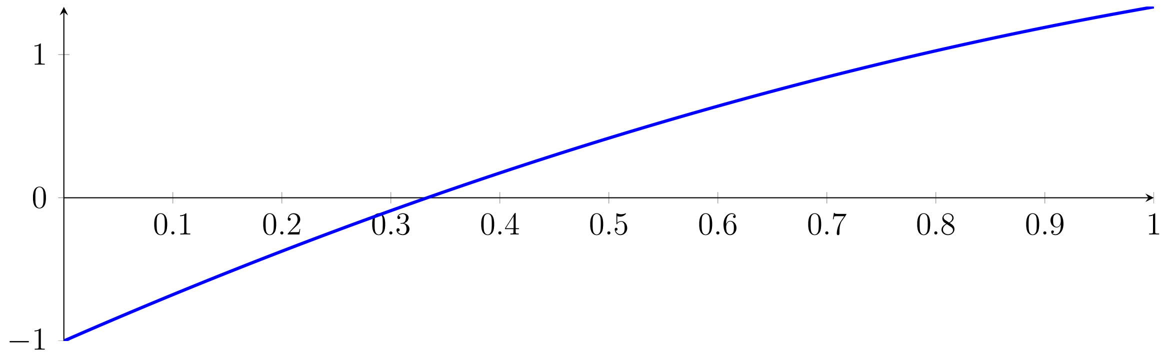


No intersection intervals with the x axis.

104.3 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

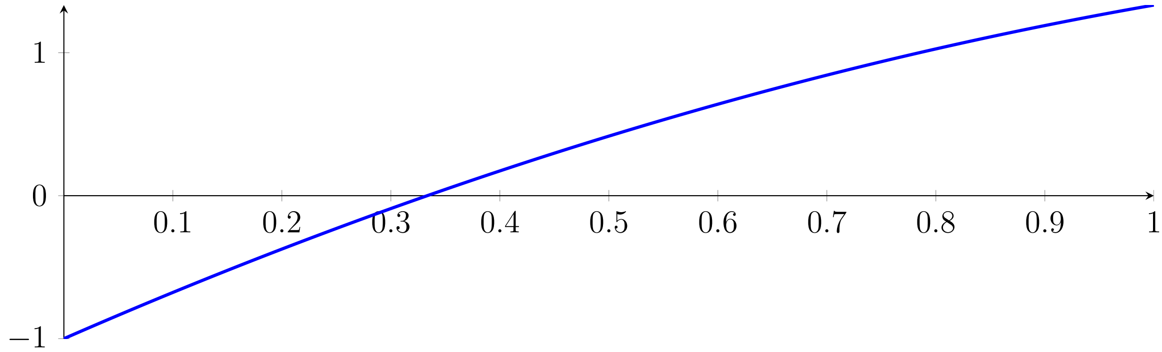
with precision $\varepsilon = 1{\cdot}10^{-128}$.

105 Running CubeClip on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

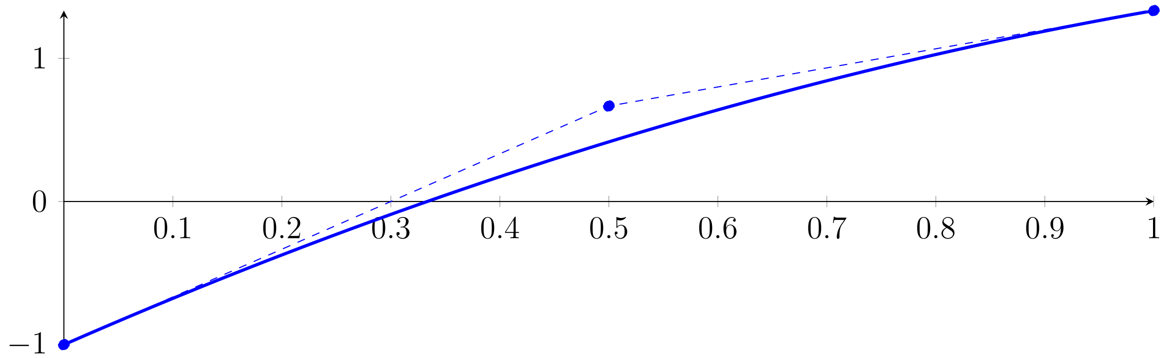
$$p = -1X^2 + 3.33333X - 1$$



105.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

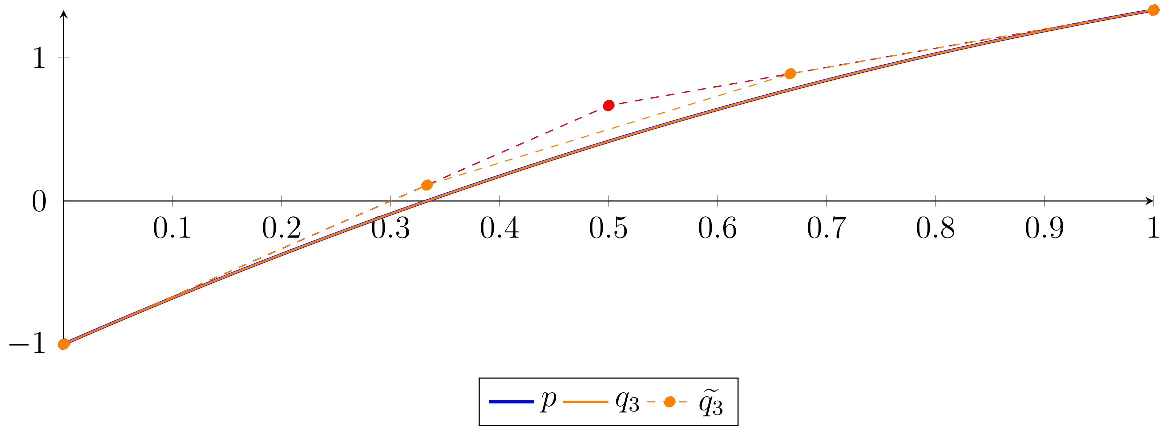
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.
Bounding polynomials M and m:

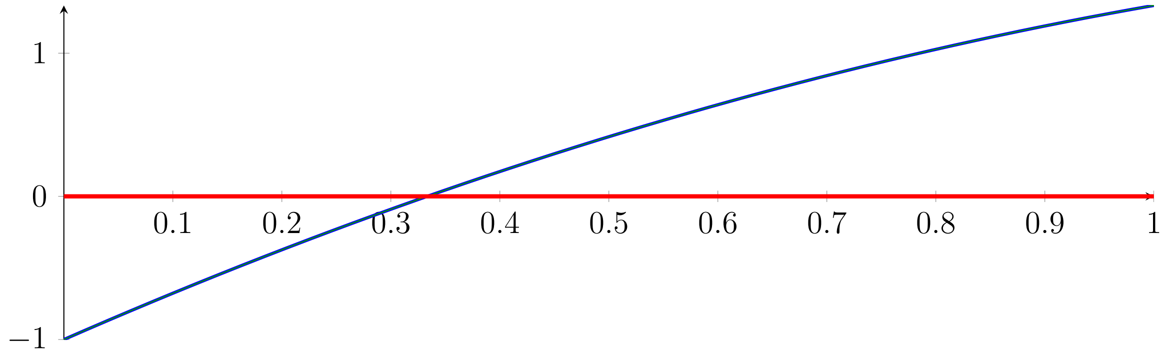
$$M = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

$$m = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{-6.10819 \cdot 10^{16}, 1.66602\} \qquad N(m) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

Intersection intervals:



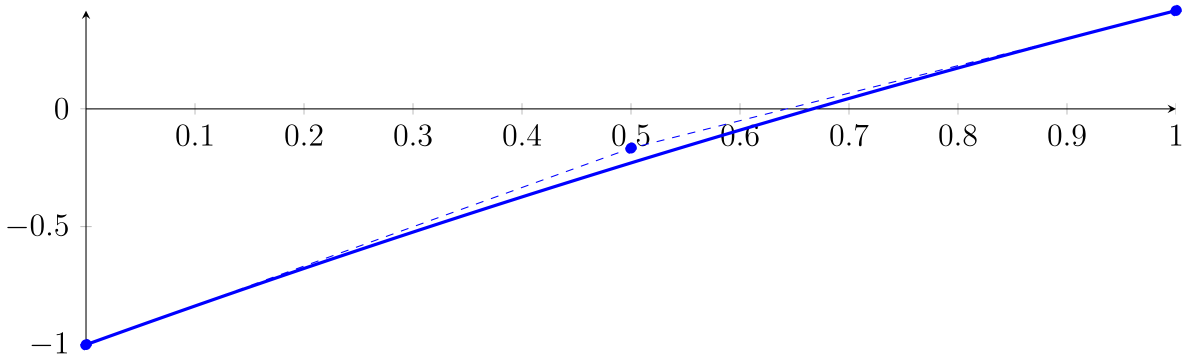
$[0, 1]$

Longest intersection interval: 1
$\Longrightarrow$ Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

105.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

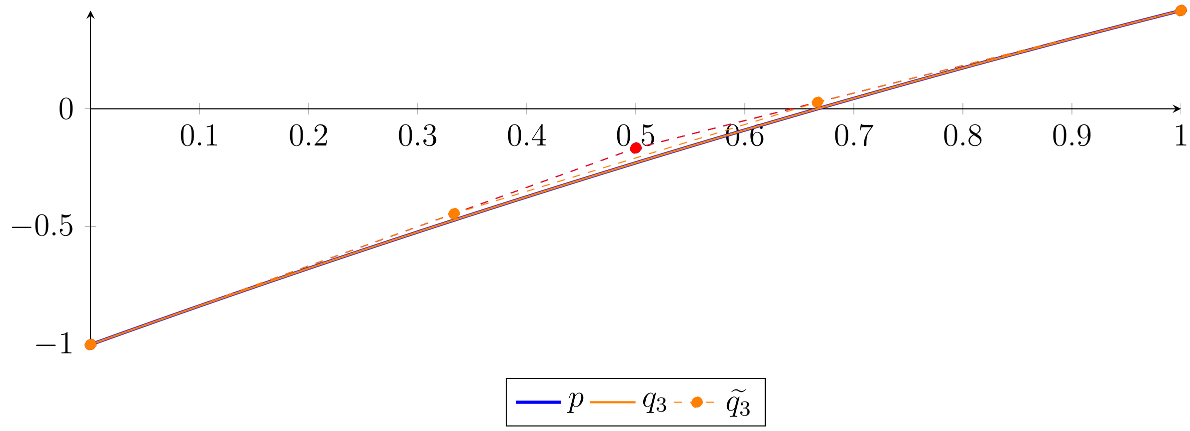
$$\begin{aligned} p &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2}(X) - 0.166667B_{1,2}(X) + 0.416667B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,3} - 0.444444B_{1,3} + 0.0277778B_{2,3} + 0.416667B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2} - 0.166667B_{1,2} + 0.416667B_{2,2} \end{aligned}$$



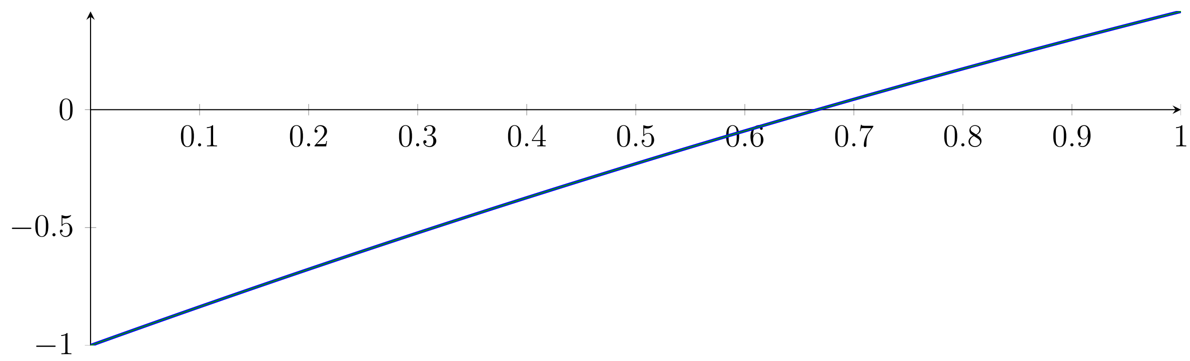
The maximum difference of the Bézier coefficients is $\delta = 7.58942 \cdot 10^{-19}$.
Bounding polynomials M and m:

$$M = -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1$$
$$m = -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1$$

Root of M and m:

$$N(M) = \{-2.32913 \cdot 10^{16}\} \qquad N(m) = \{-2.32913 \cdot 10^{16}\}$$

Intersection intervals:

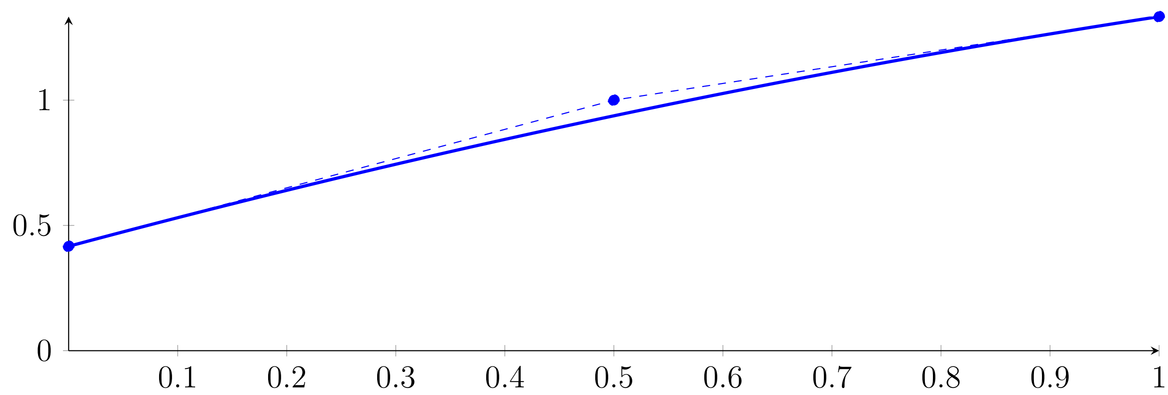


No intersection intervals with the x axis.

105.3 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

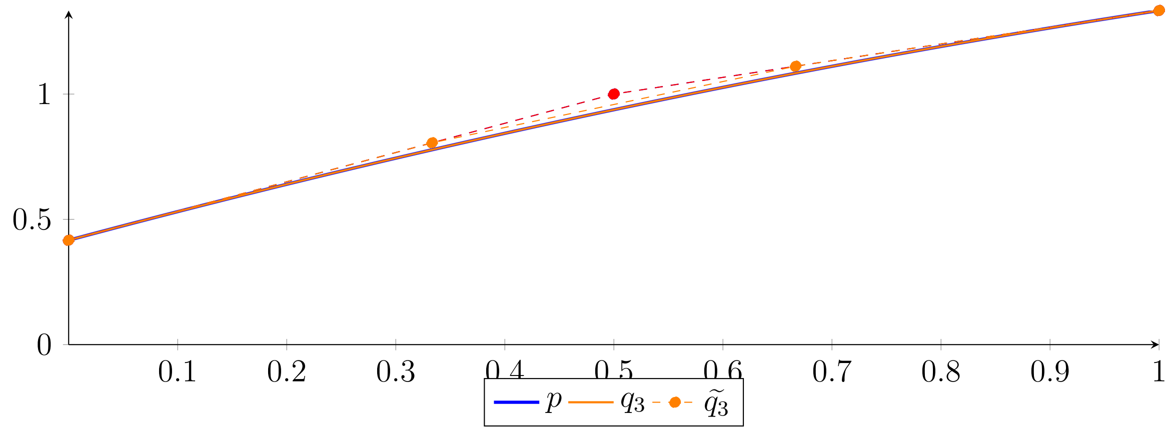
$$\begin{aligned} p &= -0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,2}(X) + 1B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -2.27682\cdot 10^{-18}X^3 - 0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,3} + 0.805556B_{1,3} + 1.11111B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,2} + 1B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.30104\cdot 10^{-18}$.

Bounding polynomials M and m:

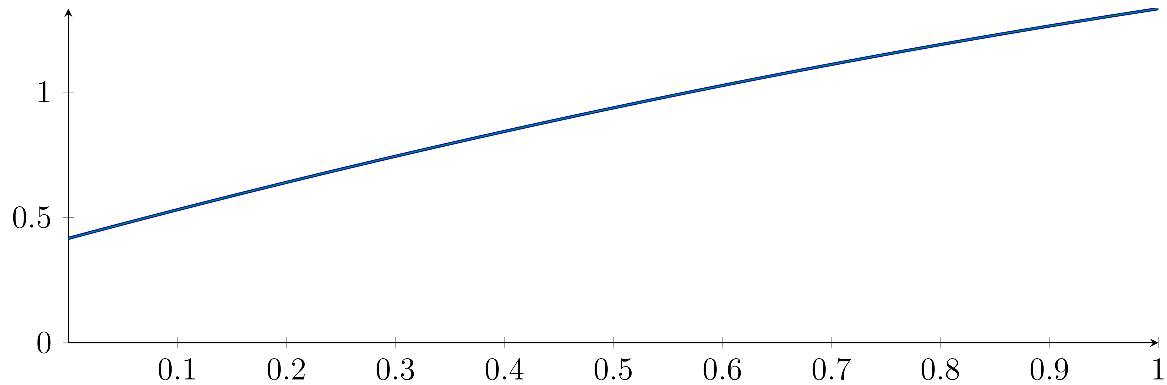
$$M = -2.27682\cdot 10^{-18}X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$m = -2.27682\cdot 10^{-18}X^3 - 0.25X^2 + 1.16667X + 0.416667$$

Root of M and m:

$$N(M) = \{-1.09802\cdot 10^{17}, -8.02734, 2.33594\} \qquad N(m) = \{-1.09802\cdot 10^{17}, -8.02734, 2.33594\}$$

Intersection intervals:

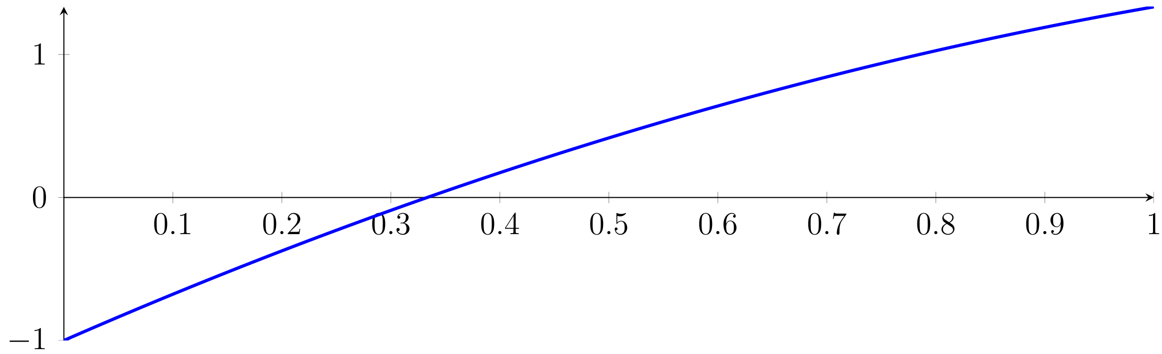


No intersection intervals with the x axis.

105.4 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

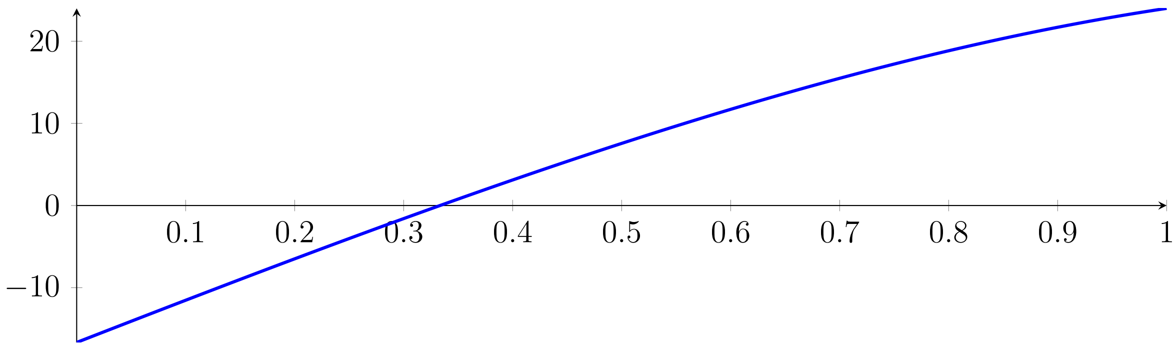
with precision $\varepsilon = 1 \cdot 10^{-128}$.

106 Running BezClip on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

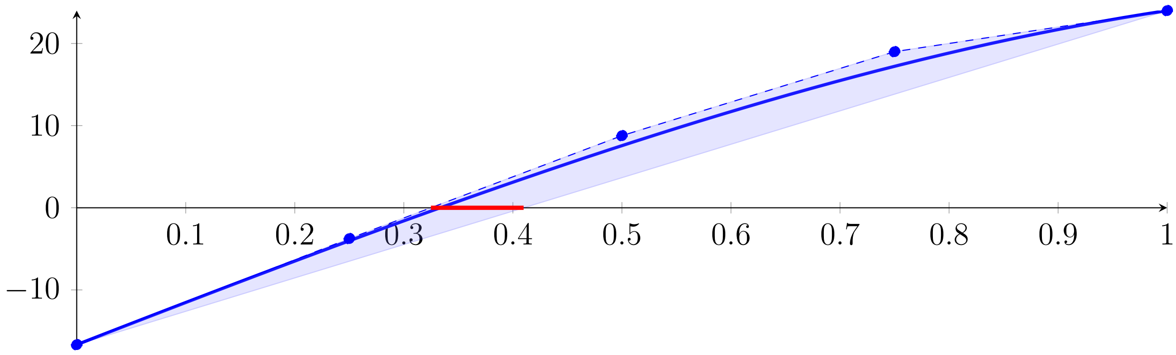
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



106.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

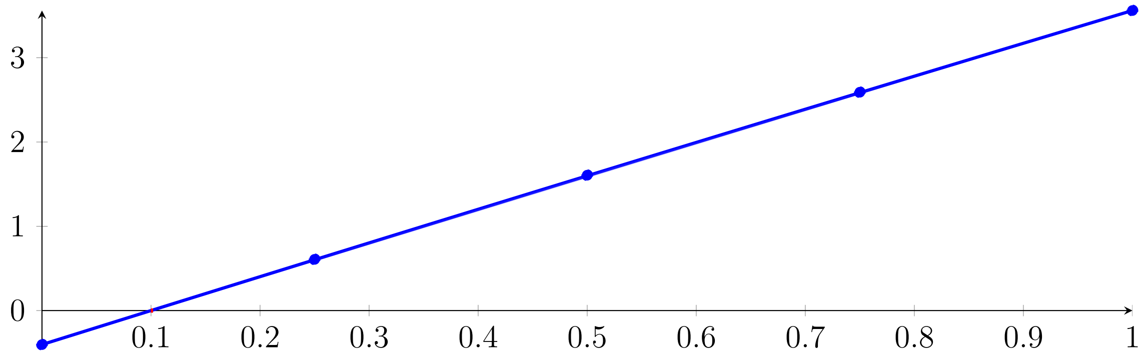
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

106.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

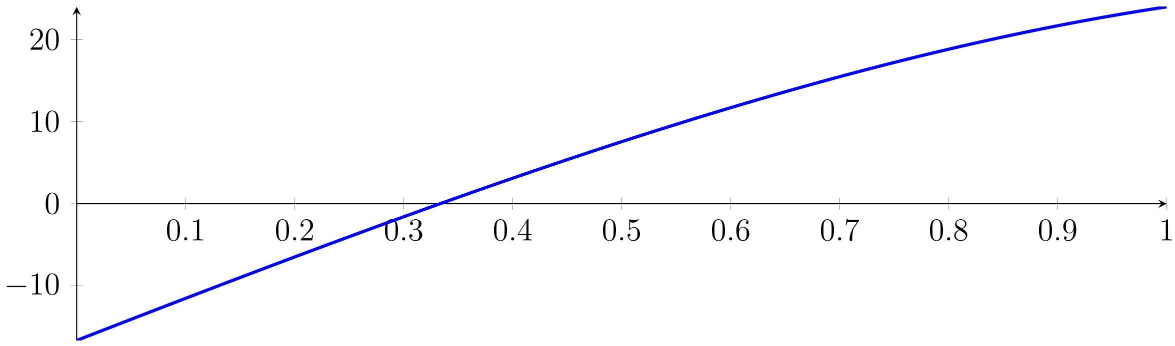
106.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Found root in interval $[0.333317, 0.333491]$ at recursion depth 3!

106.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333317, 0.333491]$$

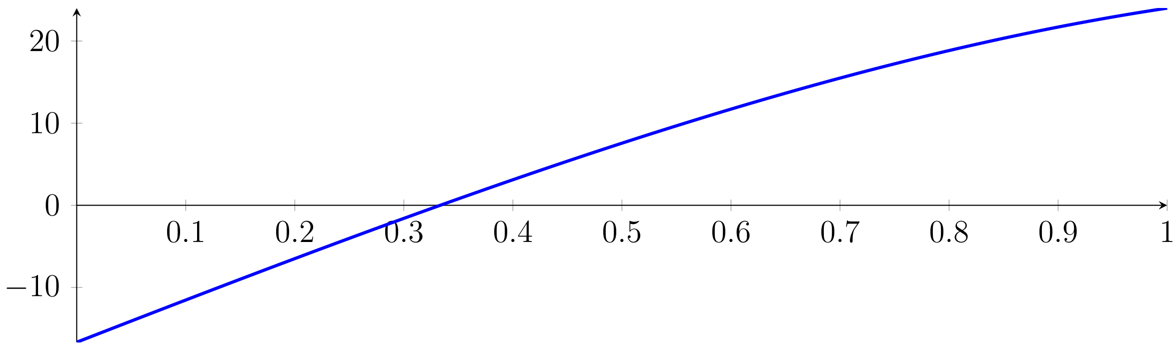
with precision $\varepsilon = 0.01$.

107 Running `QuadClip` on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called `QuadClip` with input polynomial on interval $[0,1]$:

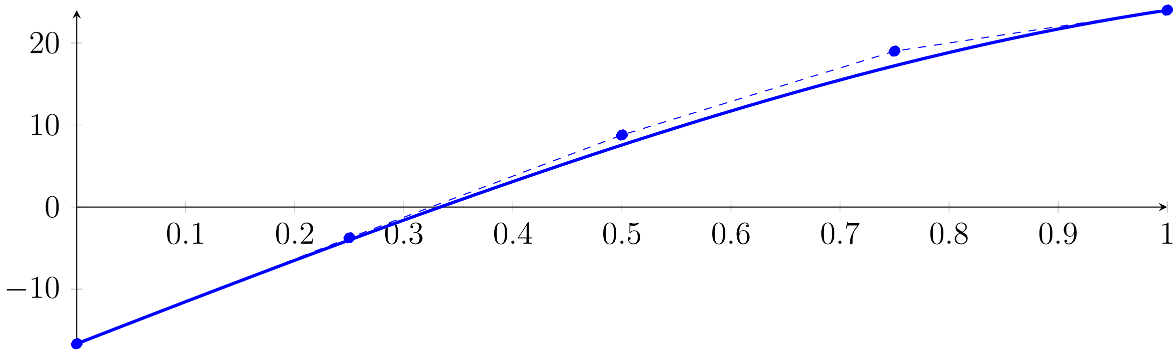
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



107.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

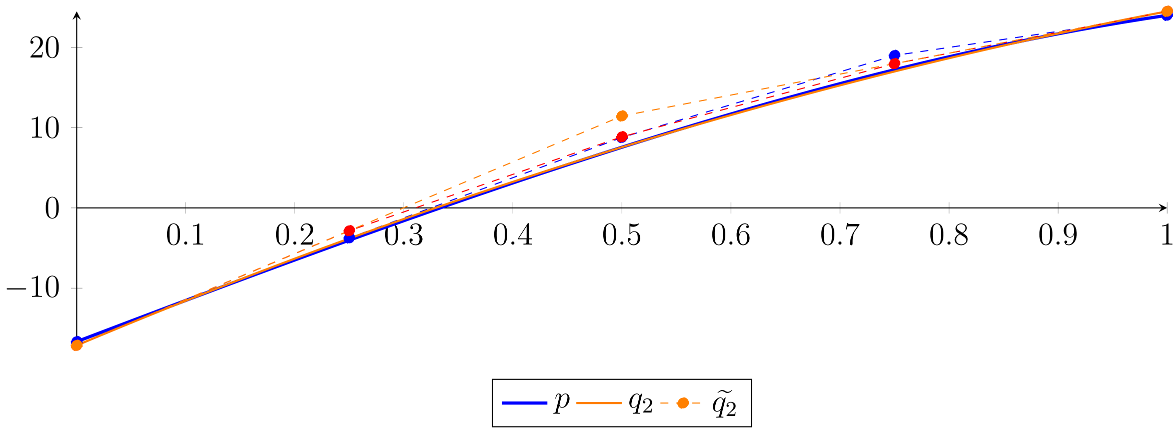
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -1.45023 \cdot 10^{-15}X^4 + 4.00374 \cdot 10^{-15}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



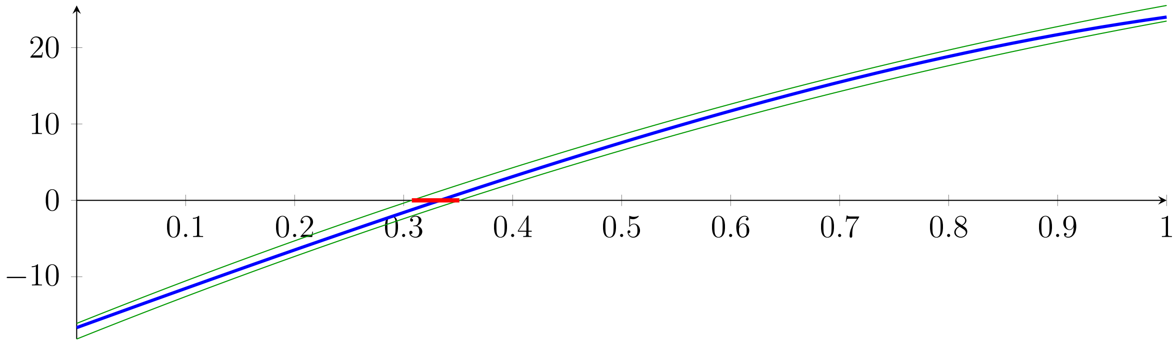
The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

$$M = -15.5476X^2 + 57.181X - 16.1119$$
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



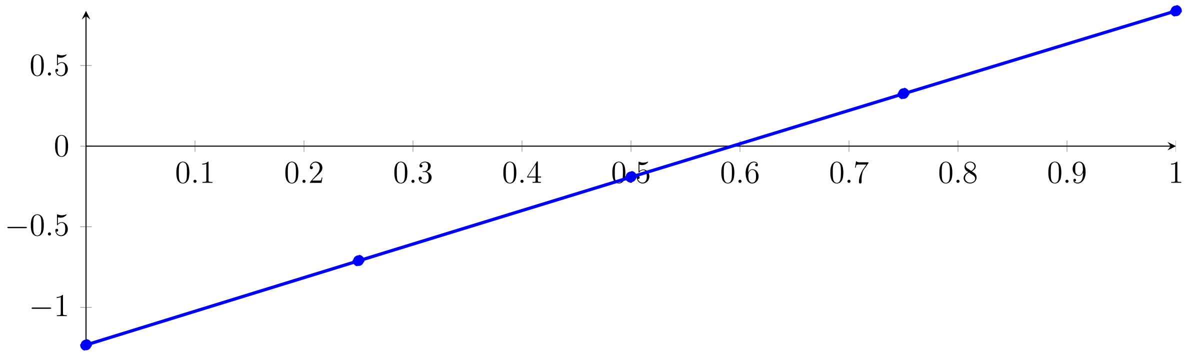
$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

107.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

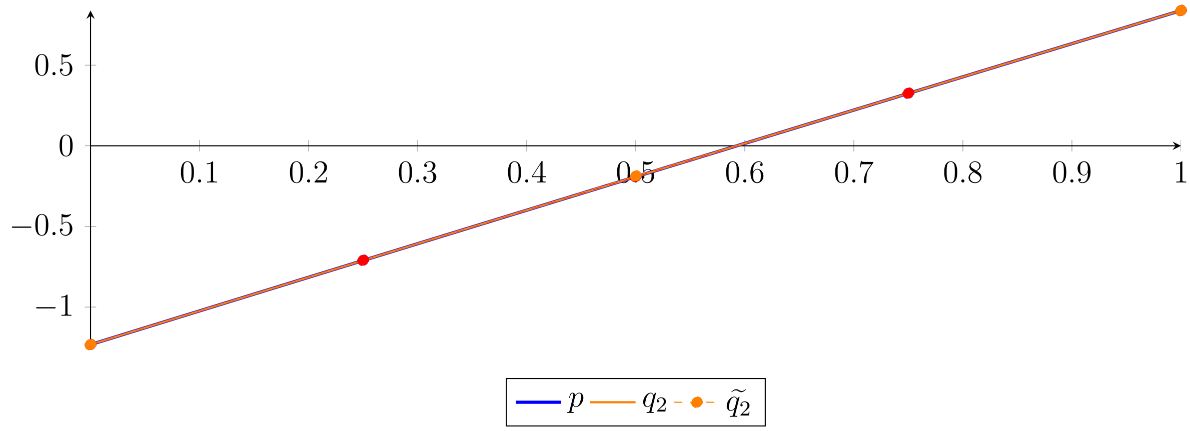
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.45283 \cdot 10^{-17} X^4 + 3.33934 \cdot 10^{-17} X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.
Bounding polynomials M and m:

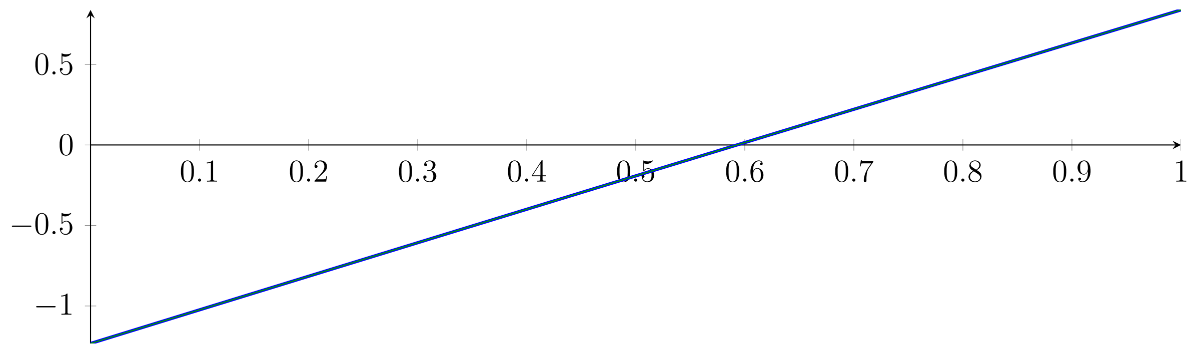
$$M = -0.020089X^2 + 2.09166X - 1.23274$$

$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

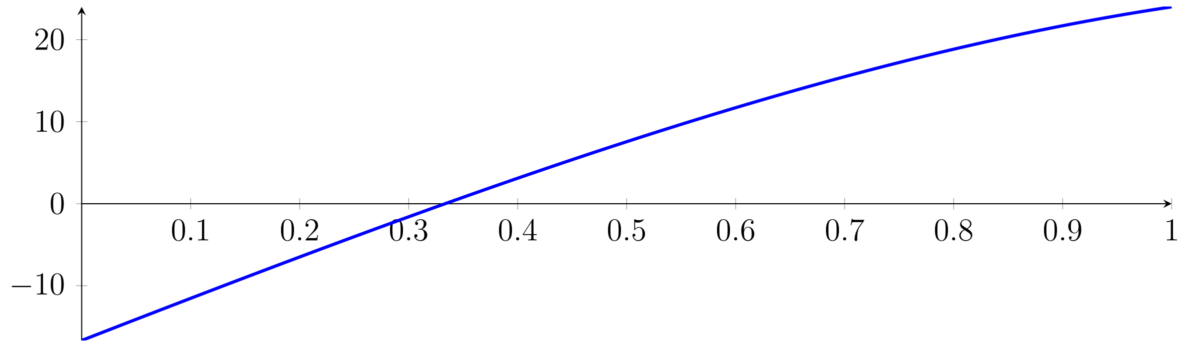
107.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Found root in interval $[0.333332, 0.333335]$ at recursion depth 3!

107.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333332, 0.333335]$$

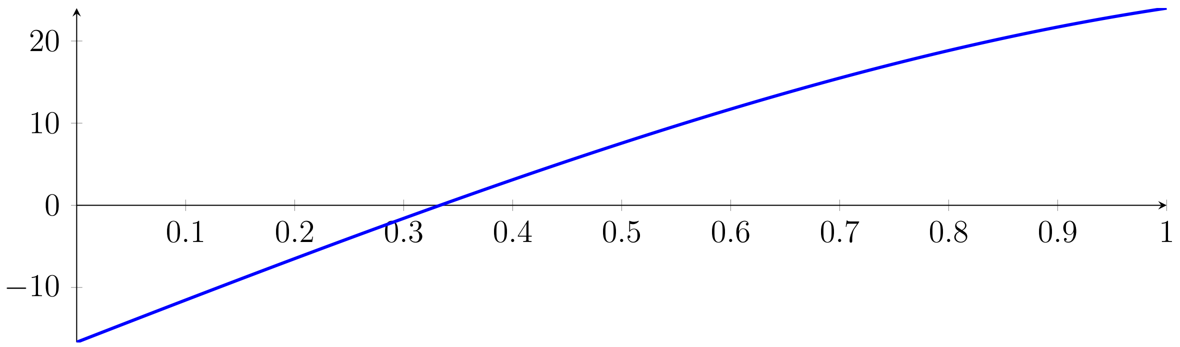
with precision $\varepsilon = 0.01$.

108 Running CubeClip on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

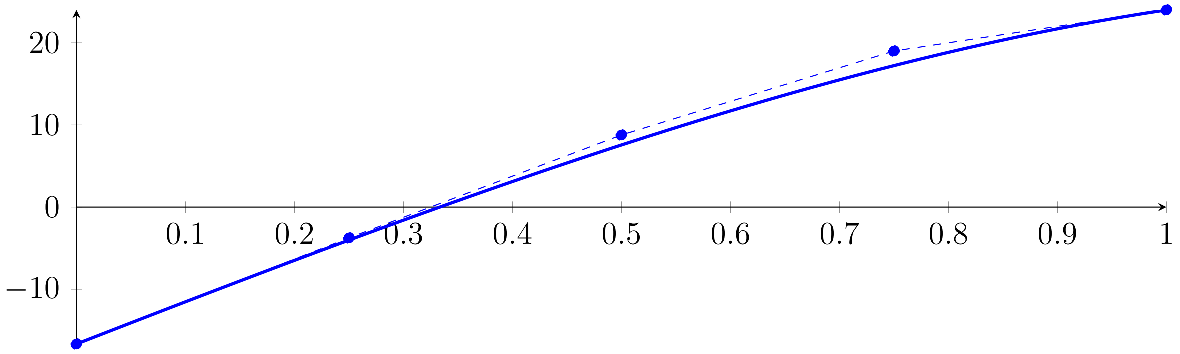
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



108.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

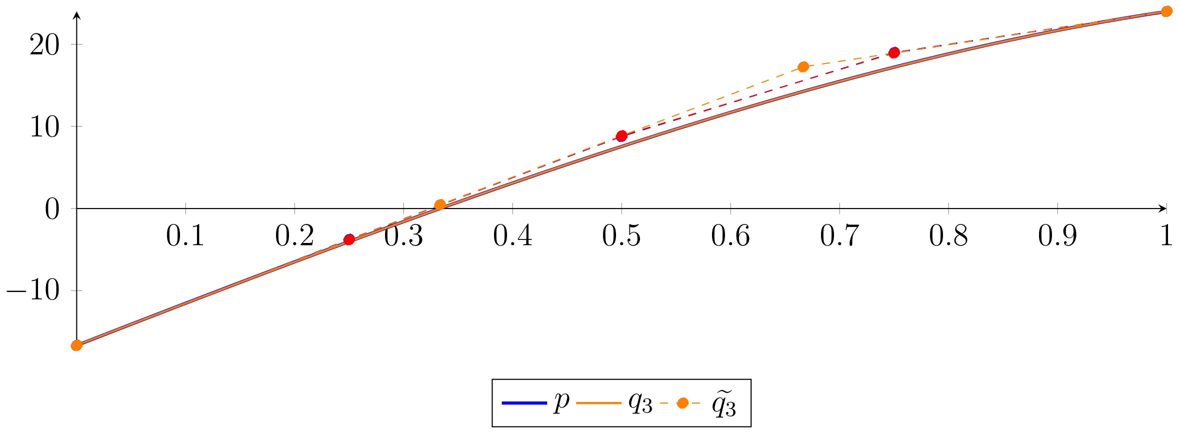
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 3.34802 \cdot 10^{-16}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



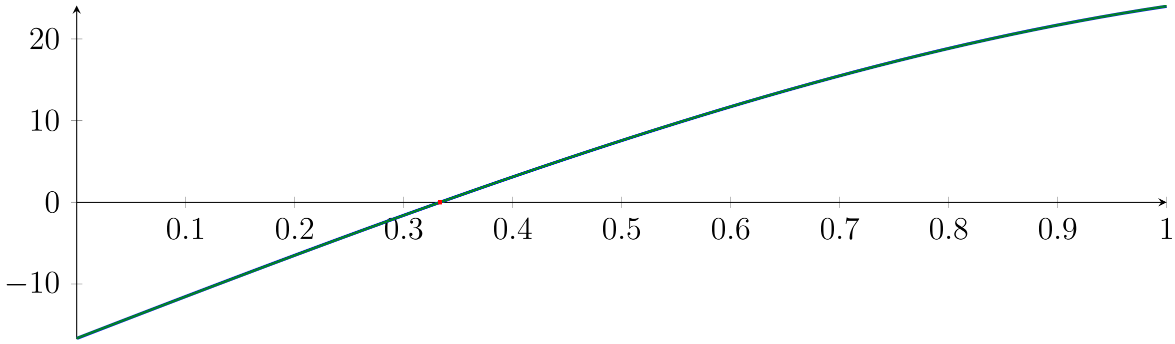
The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

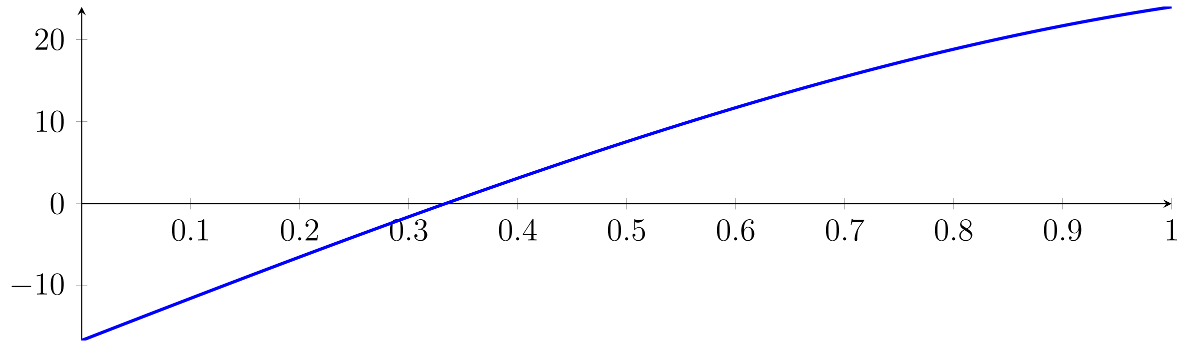
108.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Found root in interval $[0.331524, 0.335136]$ at recursion depth 2!

108.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.331524, 0.335136]$$

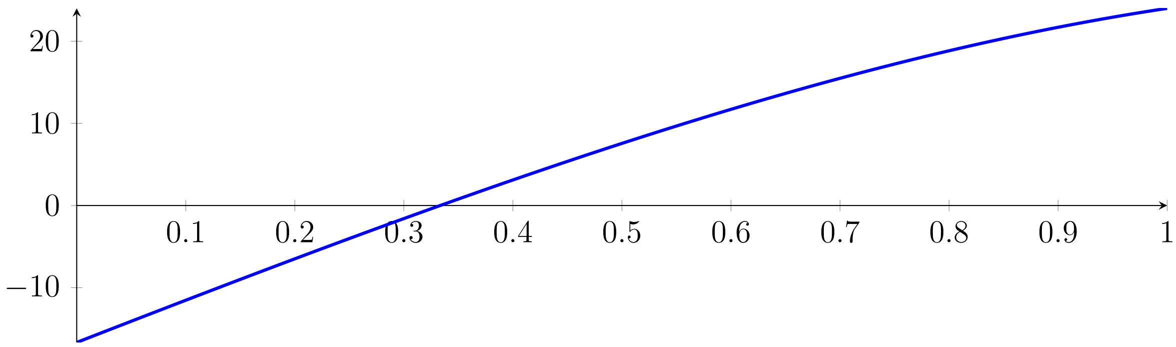
with precision $\varepsilon = 0.01$.

109 Running BezClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

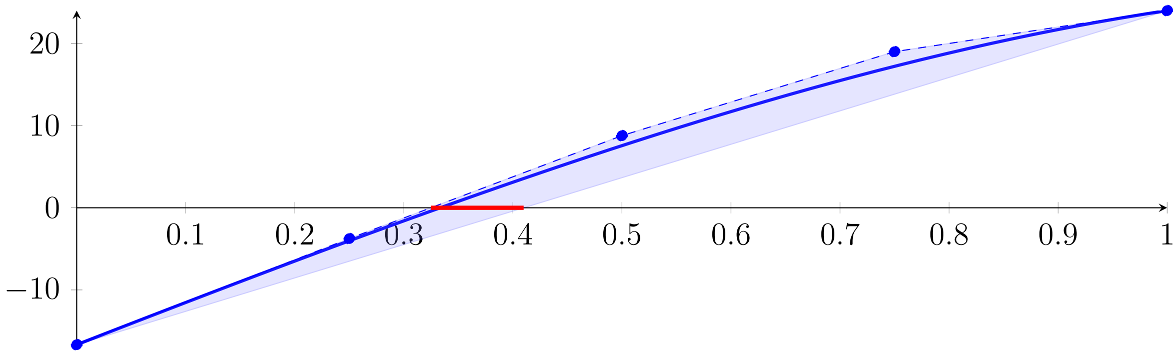
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



109.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

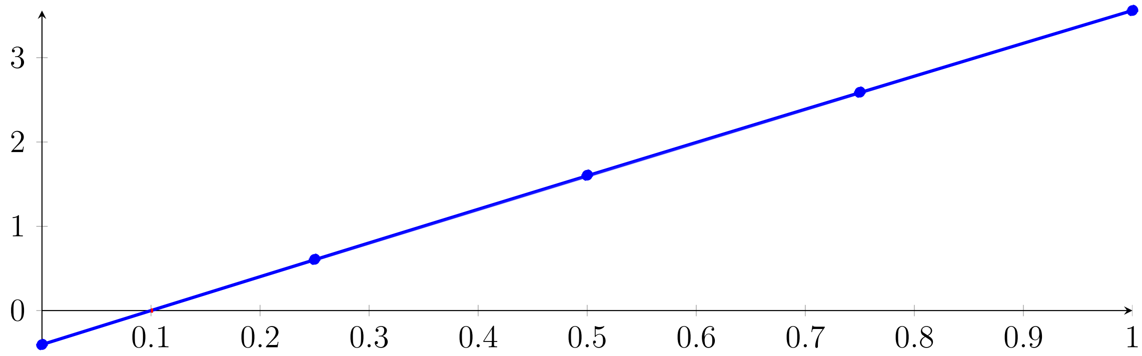
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

109.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

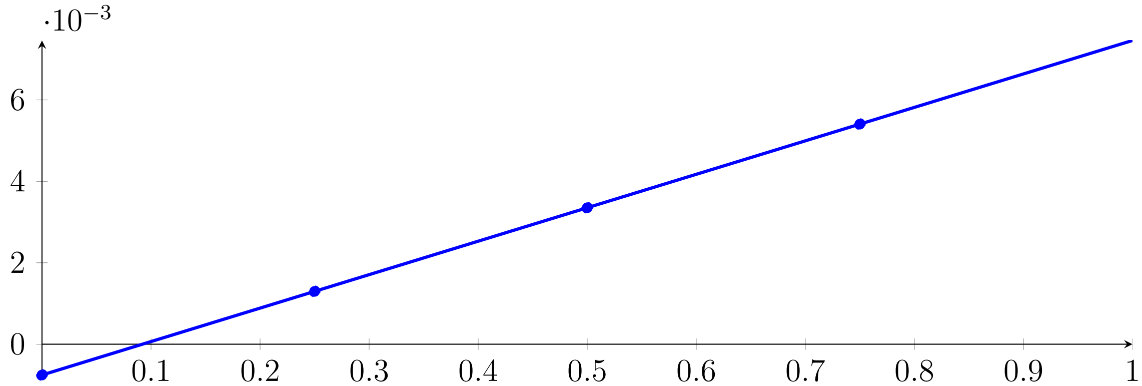
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\implies$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

109.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -9.01974 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ = & -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ & + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\implies$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

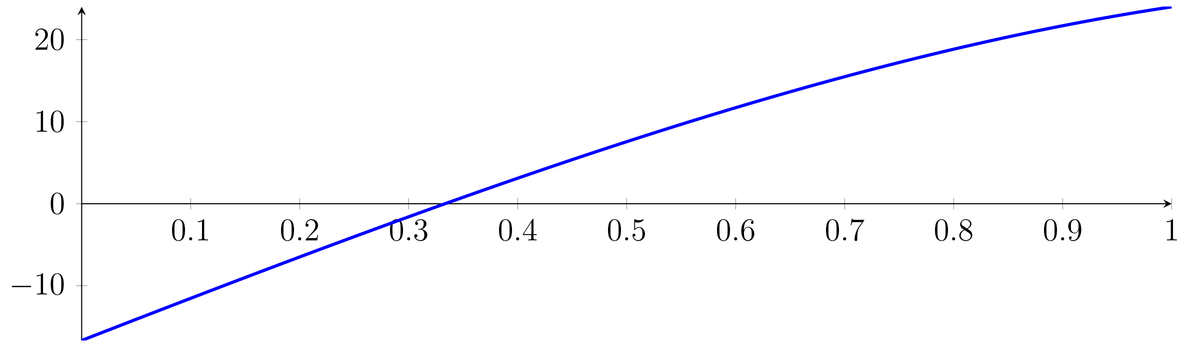
109.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

109.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

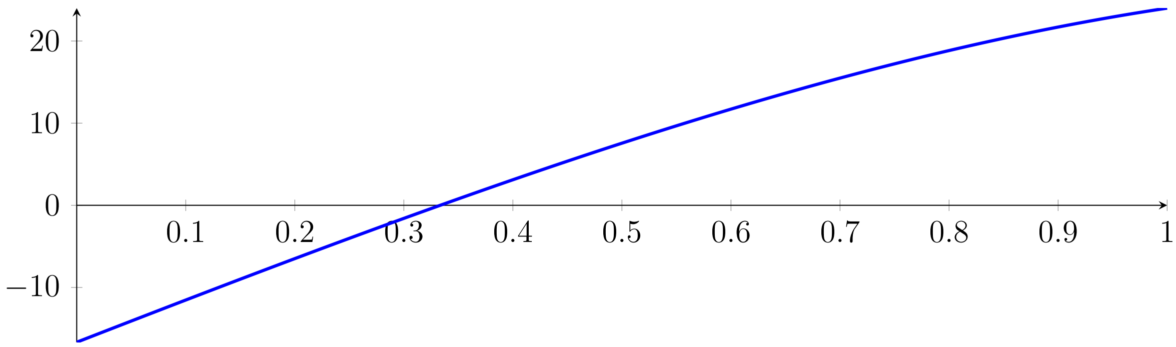
with precision $\varepsilon = 0.0001$.

110 Running QuadClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

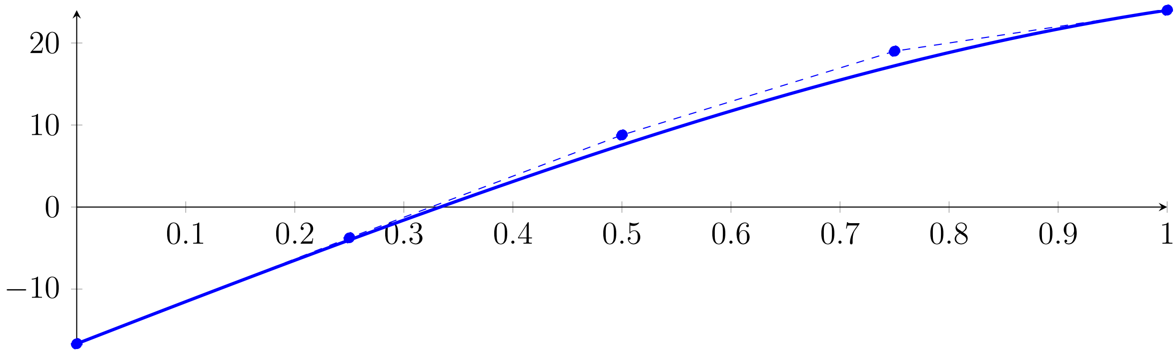
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



110.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

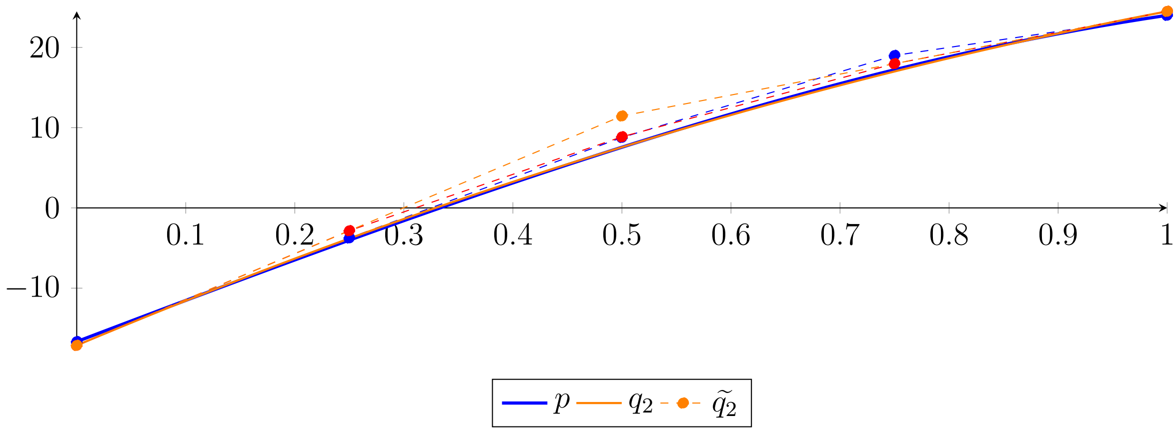
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.45023\cdot10^{-15}X^4 + 4.00374\cdot10^{-15}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



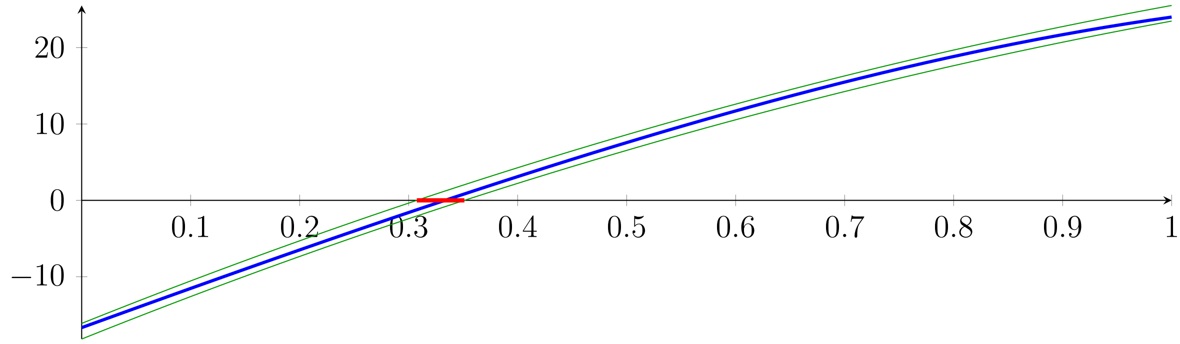
The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

$$M = -15.5476X^2 + 57.181X - 16.1119$$
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



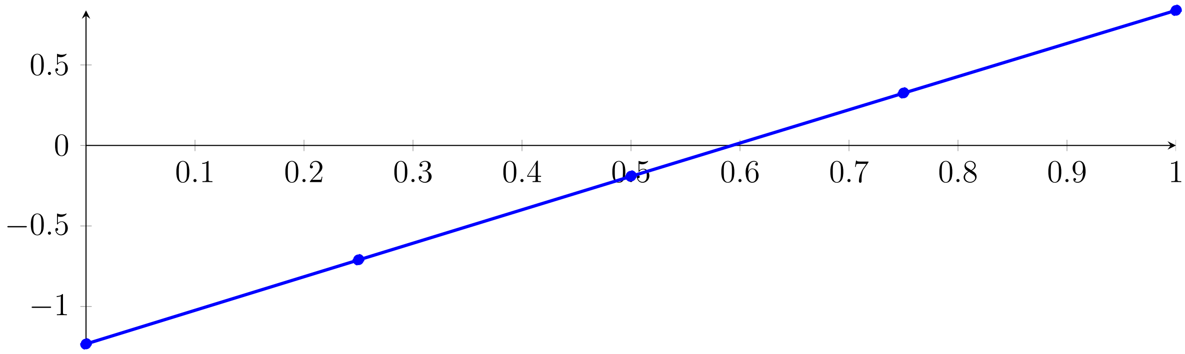
$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205
$\implies$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

110.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

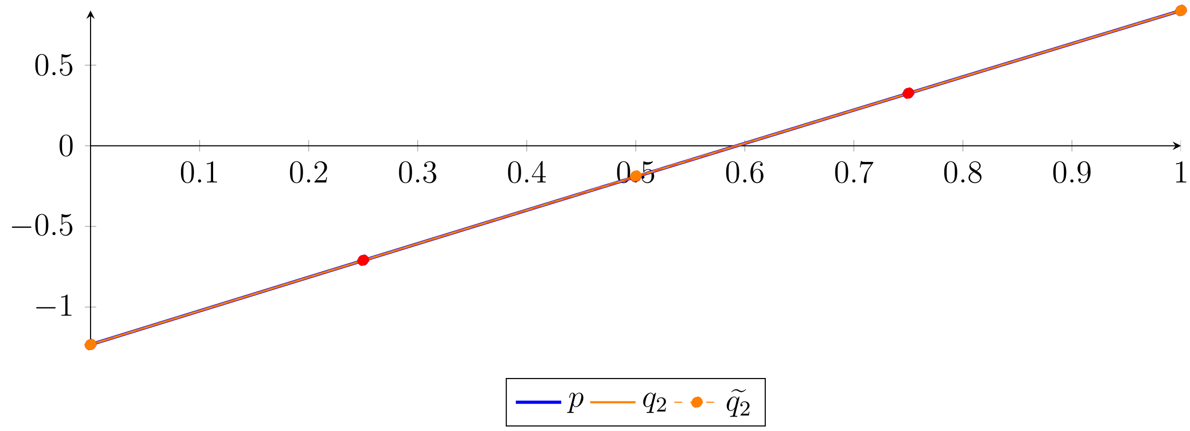
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.45283 \cdot 10^{-17} X^4 + 3.33934 \cdot 10^{-17} X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.
Bounding polynomials M and m:

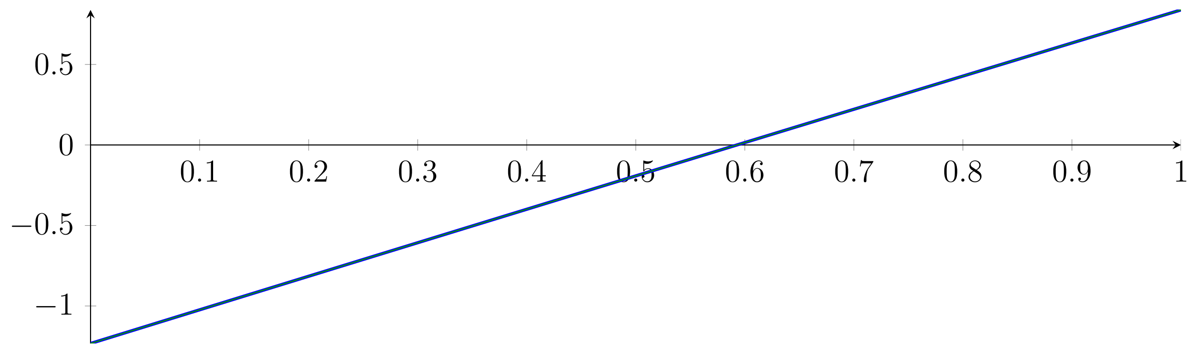
$$M = -0.020089X^2 + 2.09166X - 1.23274$$

$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

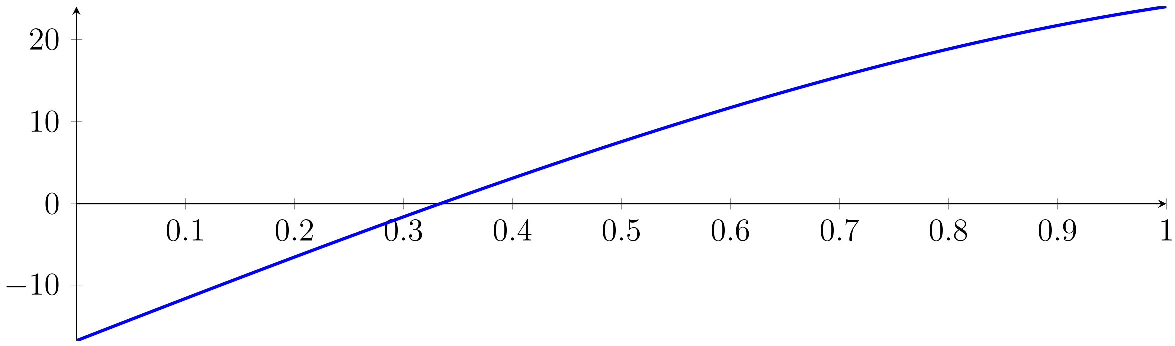
110.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Found root in interval $[0.333332, 0.333335]$ at recursion depth 3!

110.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333332, 0.333335]$$

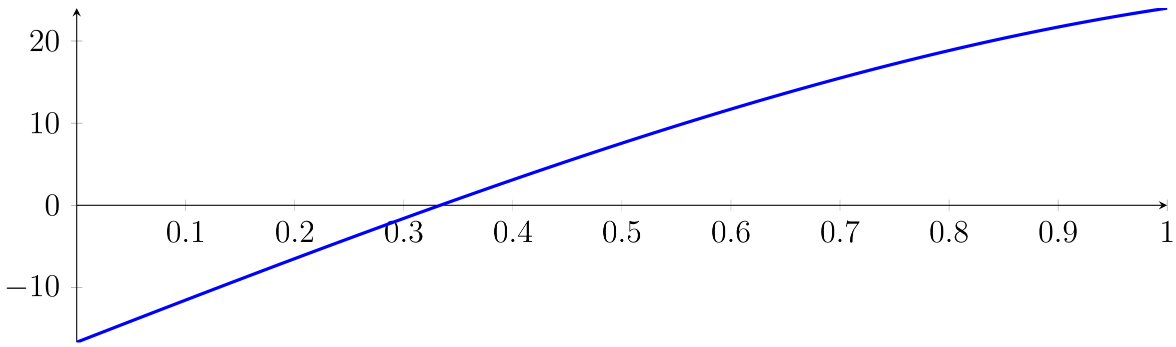
with precision $\varepsilon = 0.0001$.

111 Running CubeClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0,1]$:

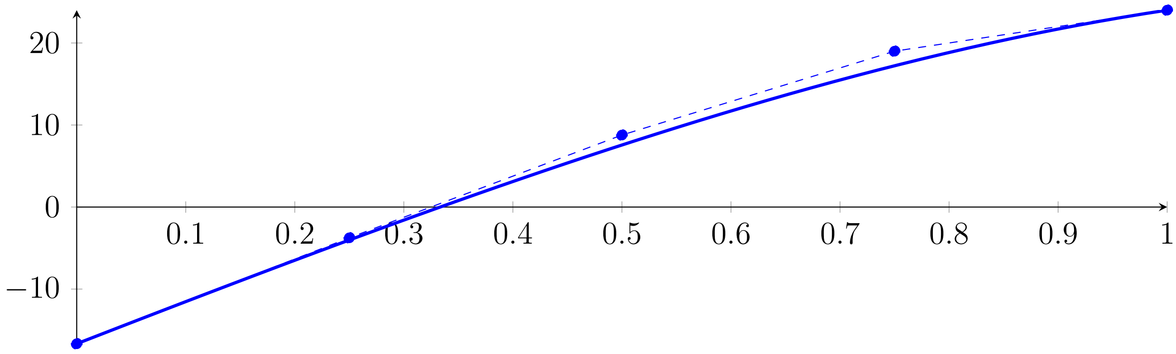
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



111.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

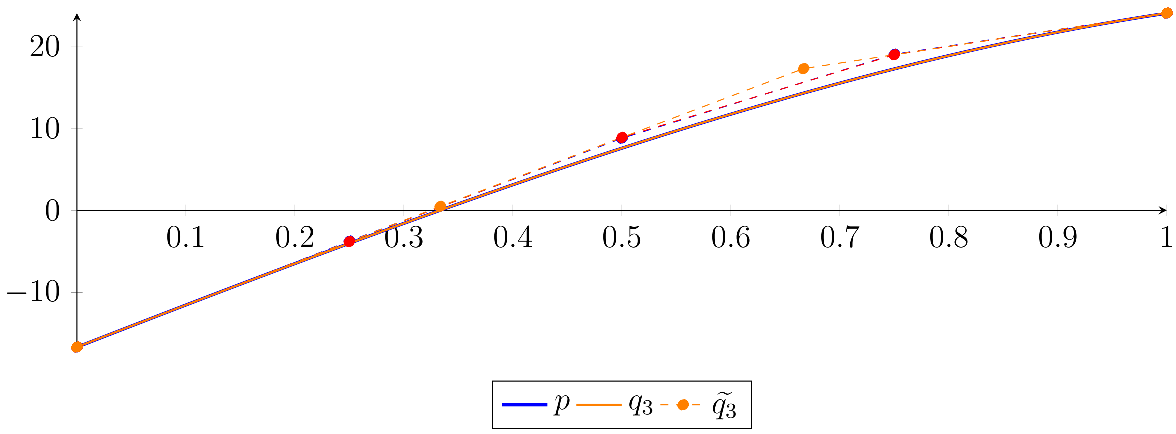
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= 3.34802 \cdot 10^{-16}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

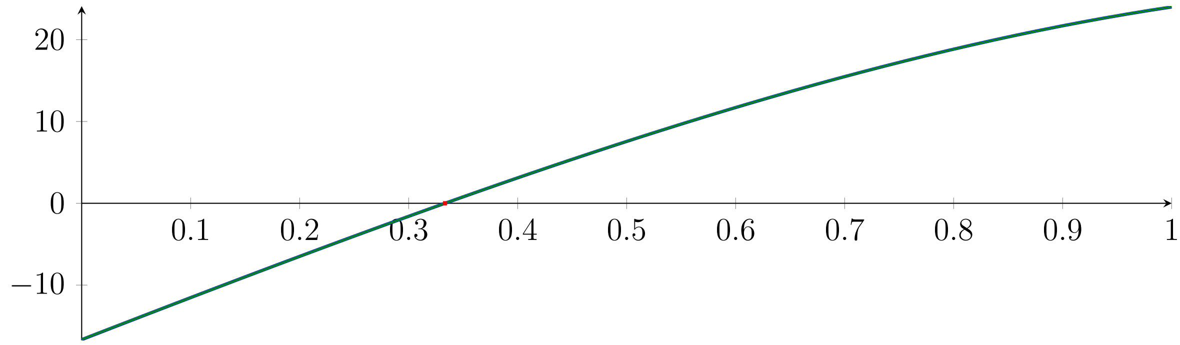
$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



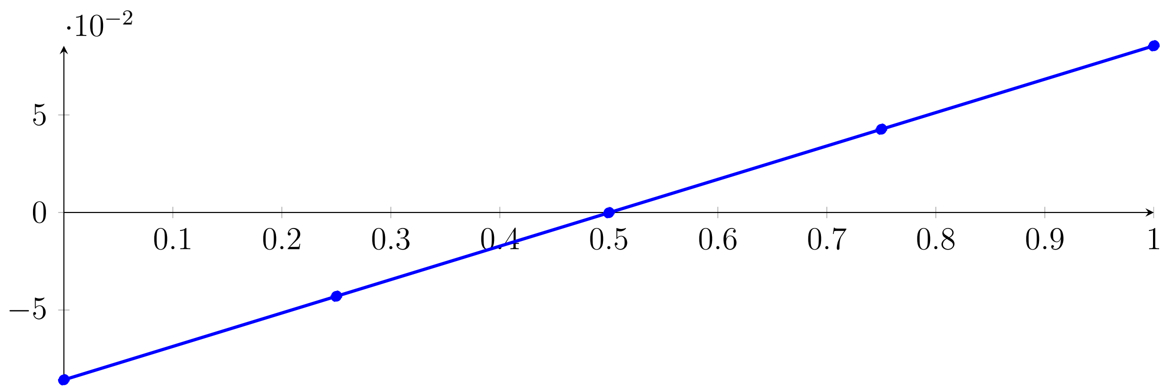
$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

111.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

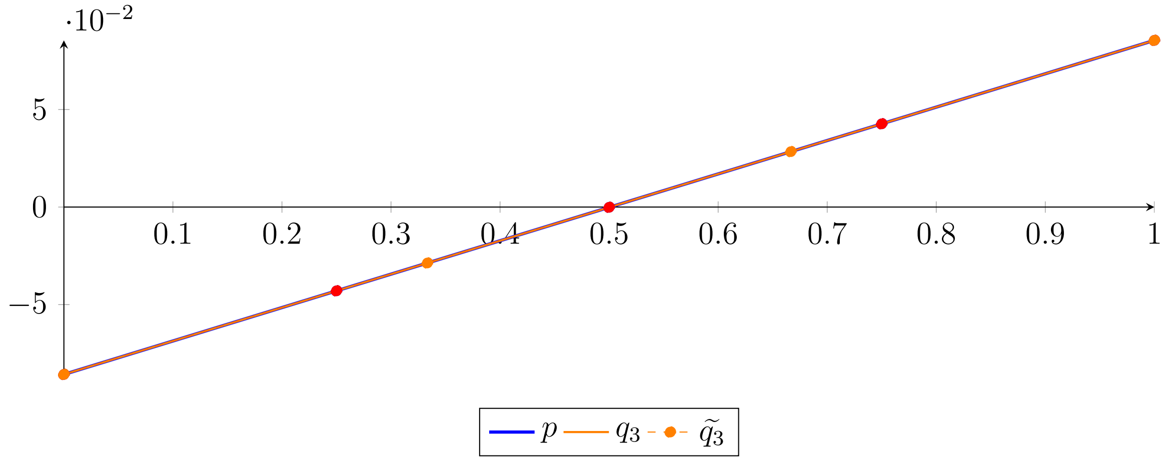
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4}(X) - 0.0429507B_{1,4}(X) - 0.000129666B_{2,4}(X) \\ &\quad + 0.0426682B_{3,4}(X) + 0.0854427B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,3} - 0.0286693B_{1,3} + 0.02841B_{2,3} + 0.0854427B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 1.99222 \cdot 10^{-18} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4} - 0.0429507B_{1,4} - 0.000129666B_{2,4} + 0.0426682B_{3,4} + 0.0854427B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.
Bounding polynomials M and m:

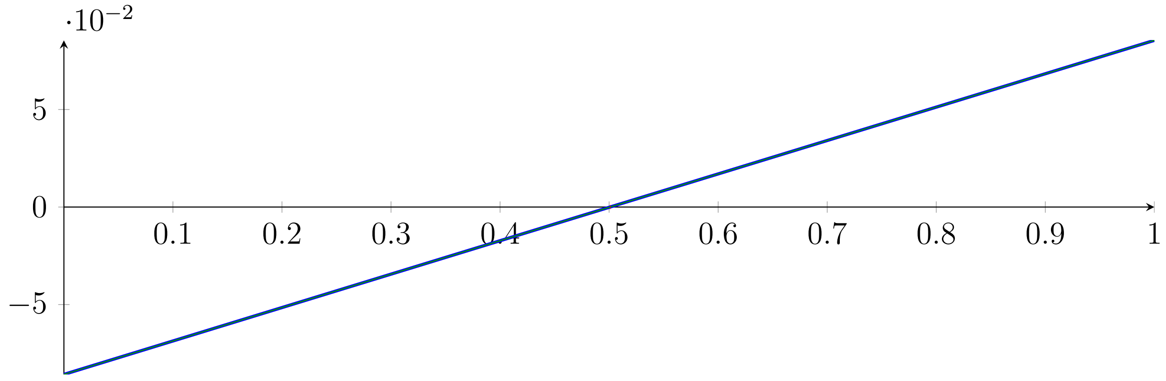
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

Longest intersection interval: $1.7041 \cdot 10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

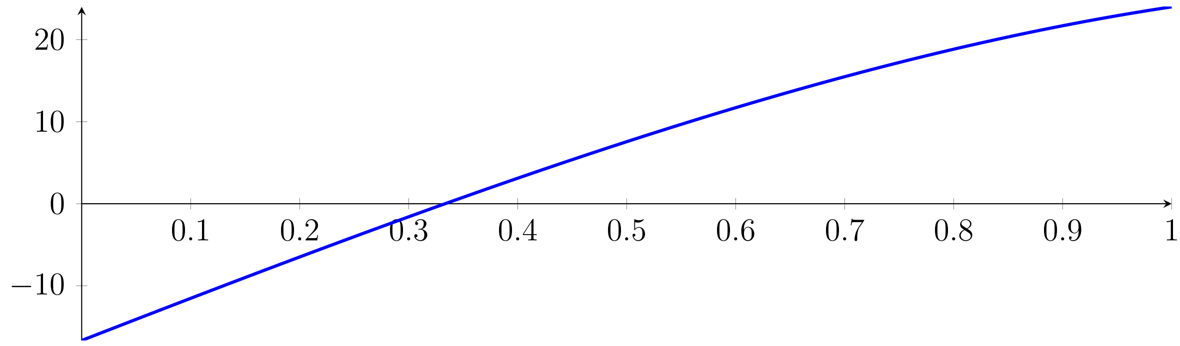
111.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

111.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

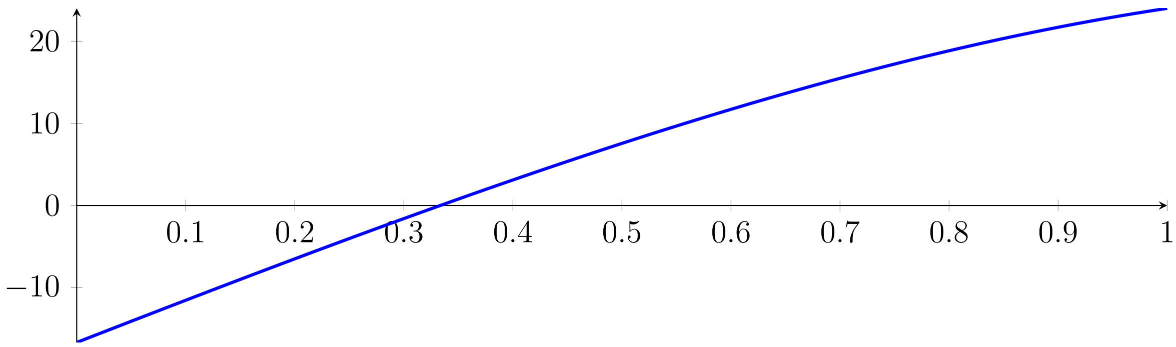
with precision $\varepsilon = 0.0001$.

112 Running `BezClip` on f_4 with epsilon 8

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

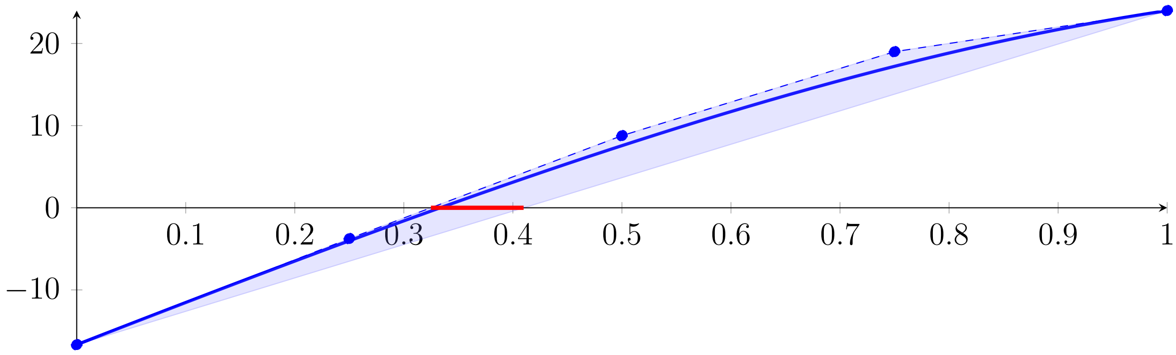
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



112.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

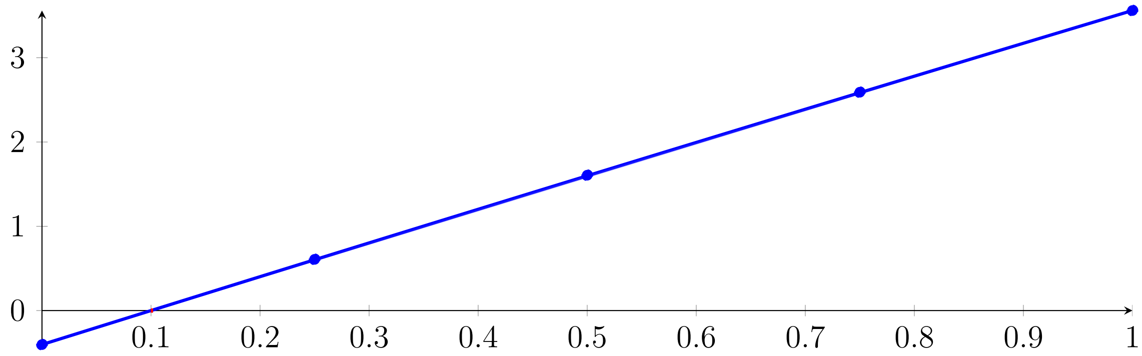
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

112.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

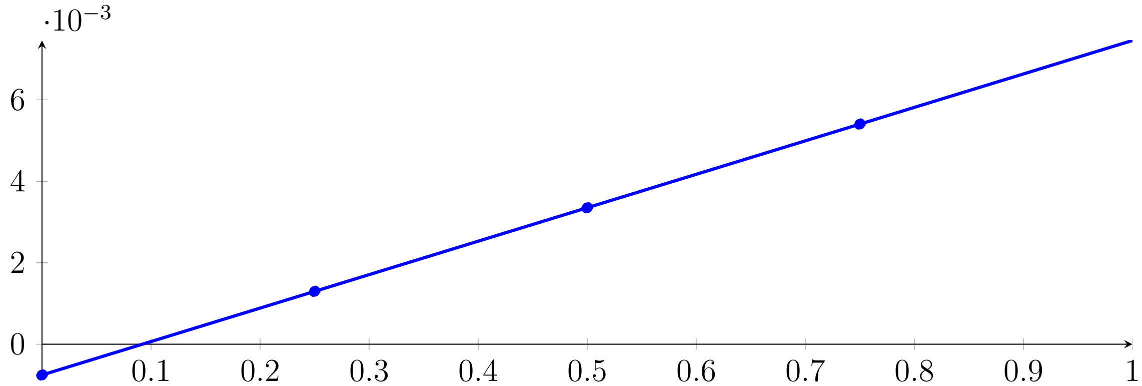
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

112.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01974 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

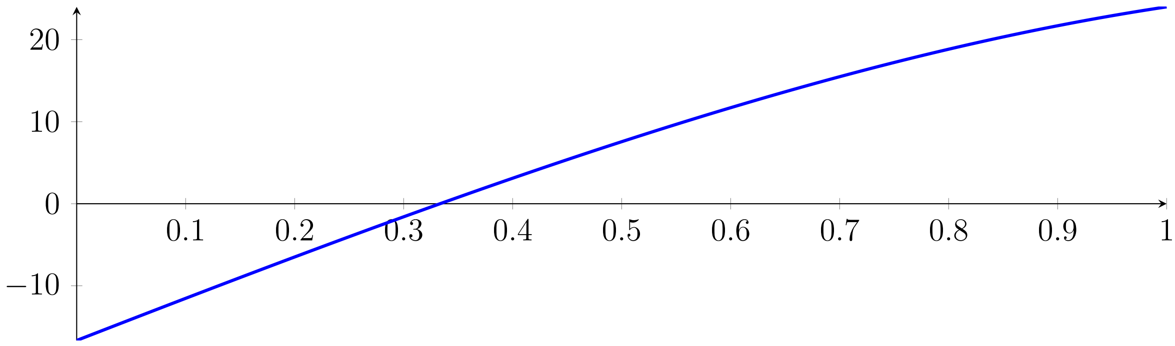
112.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

112.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

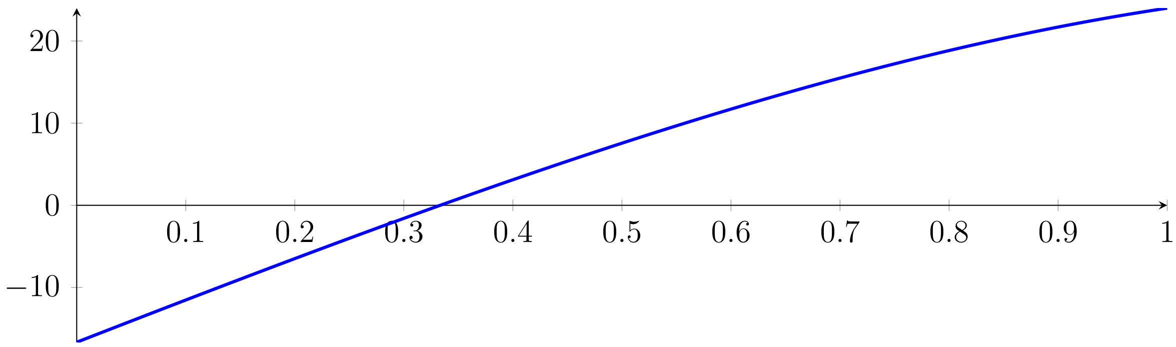
with precision $\varepsilon = 1 \cdot 10^{-08}$.

113 Running QuadClip on f_4 with epsilon 8

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0,1]$:

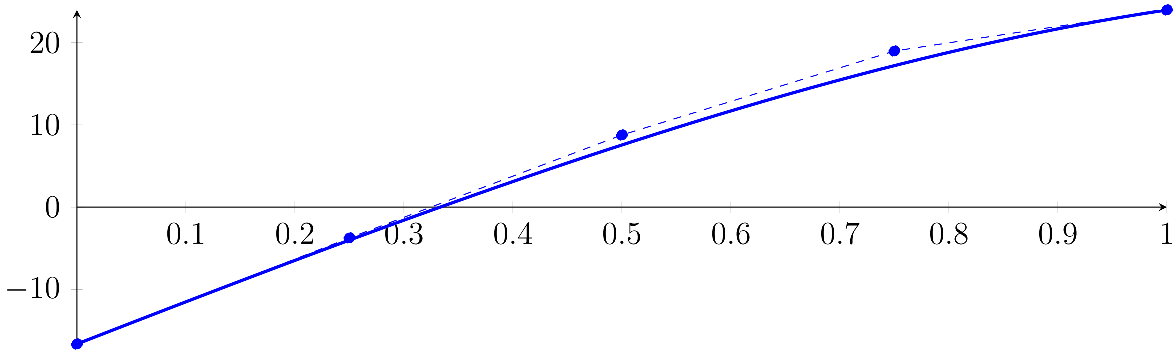
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



113.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

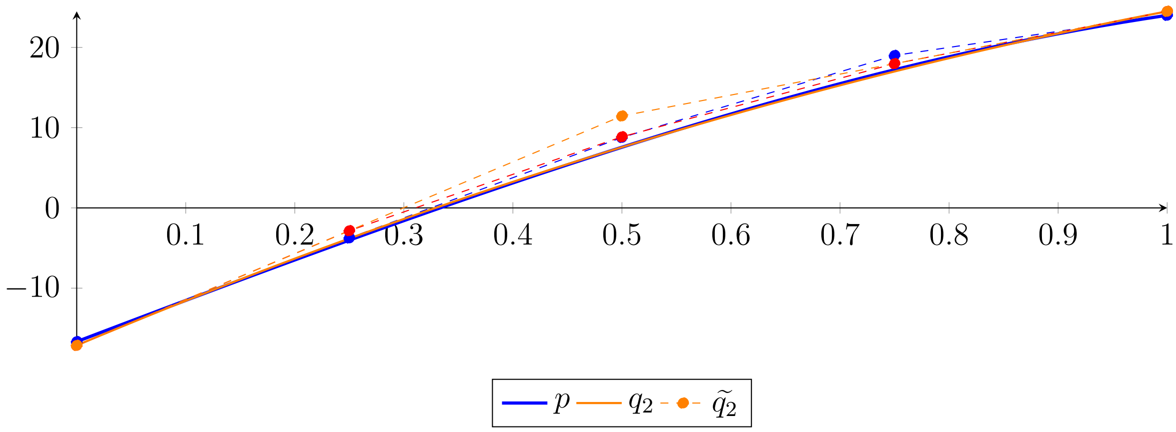
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.45023\cdot10^{-15}X^4 + 4.00374\cdot10^{-15}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



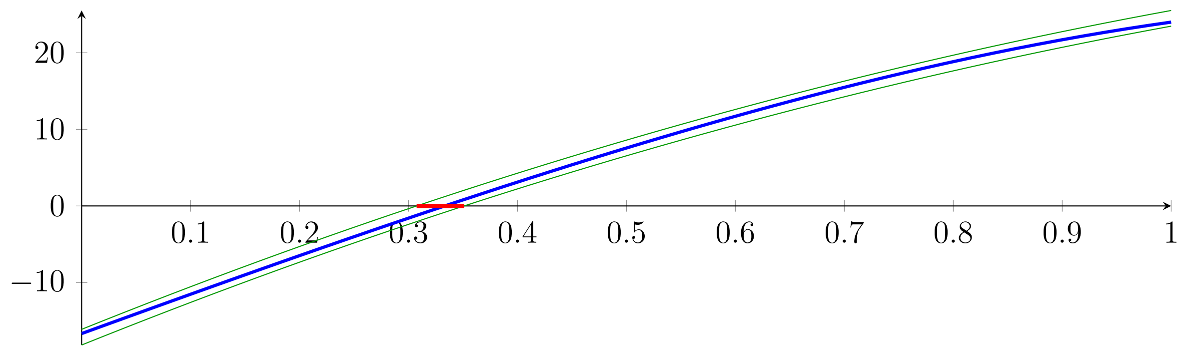
The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

$$M = -15.5476X^2 + 57.181X - 16.1119$$
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



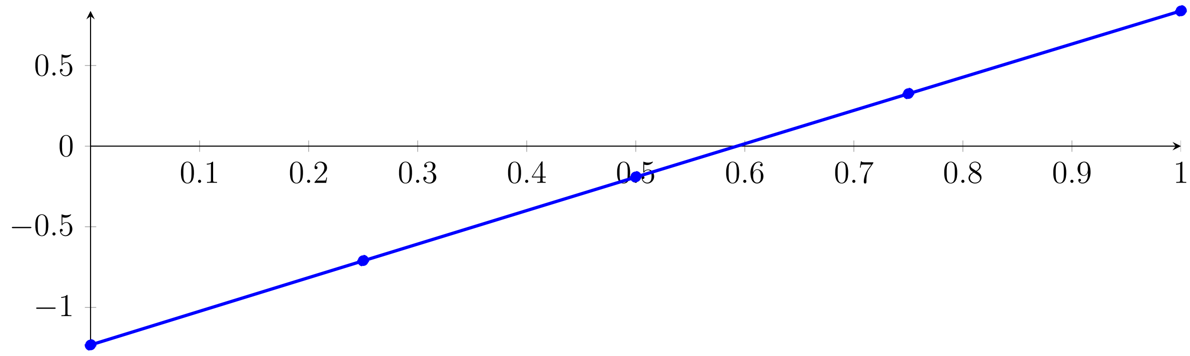
$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

113.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

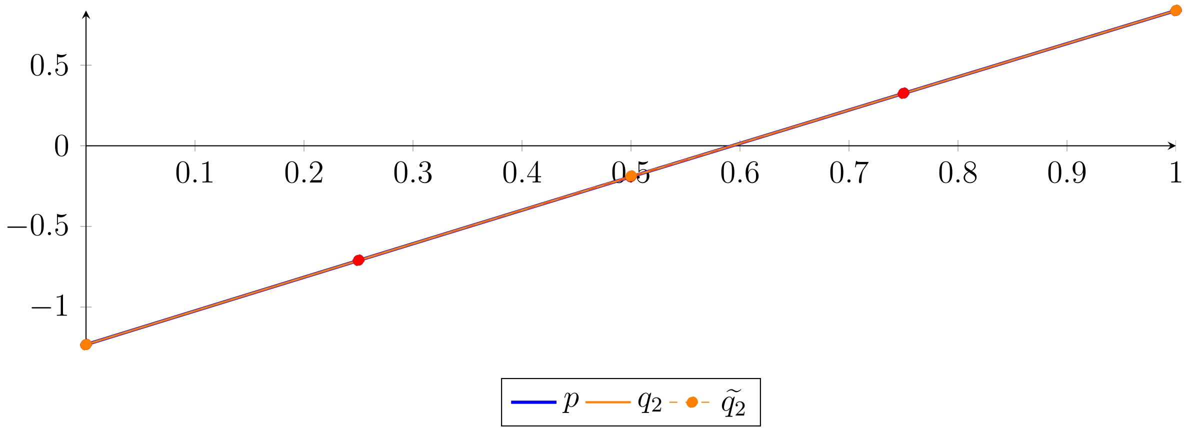
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.45283 \cdot 10^{-17} X^4 + 3.33934 \cdot 10^{-17} X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

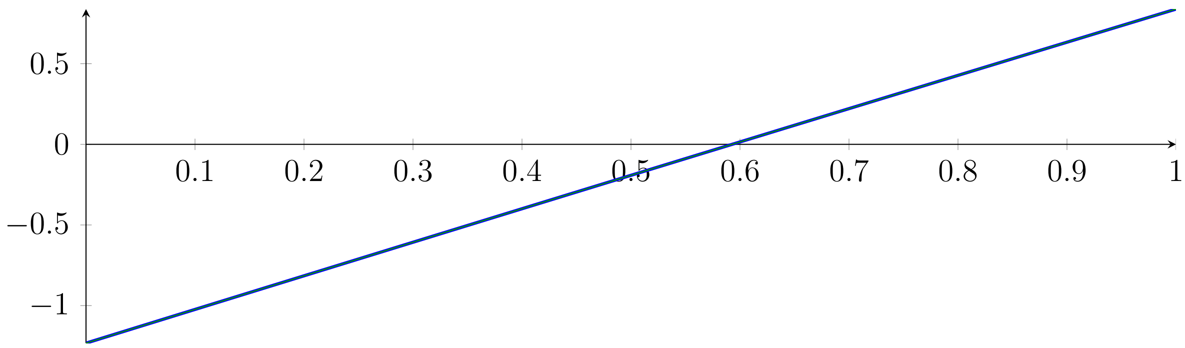
Bounding polynomials M and m:

$$M = -0.020089X^2 + 2.09166X - 1.23274$$
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$[0.592734, 0.592807]$

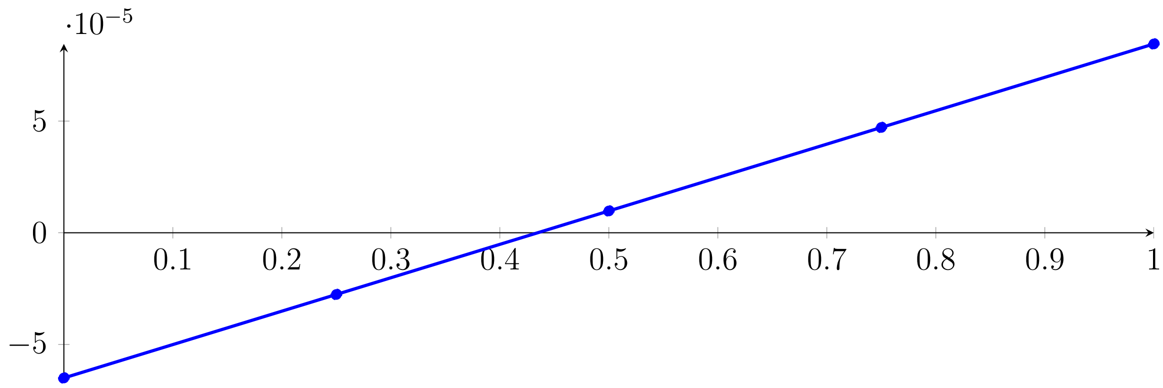
Longest intersection interval: $7.23183 \cdot 10^{-05}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

113.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

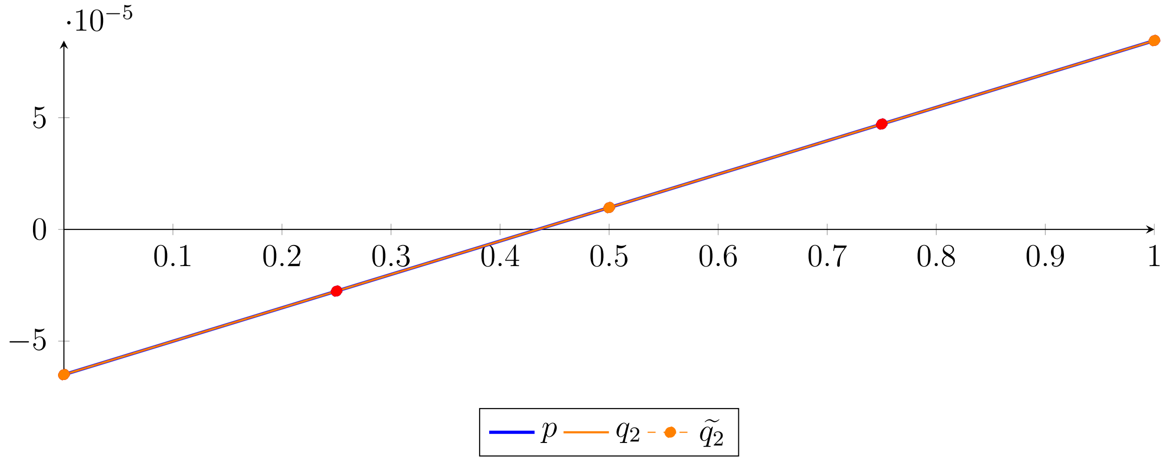
$$p = -1.05879 \cdot 10^{-22} X^4 - 2.82525 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\widetilde{q}_2 = -4.49986 \cdot 10^{-22} X^4 + 3.33519 \cdot 10^{-21} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



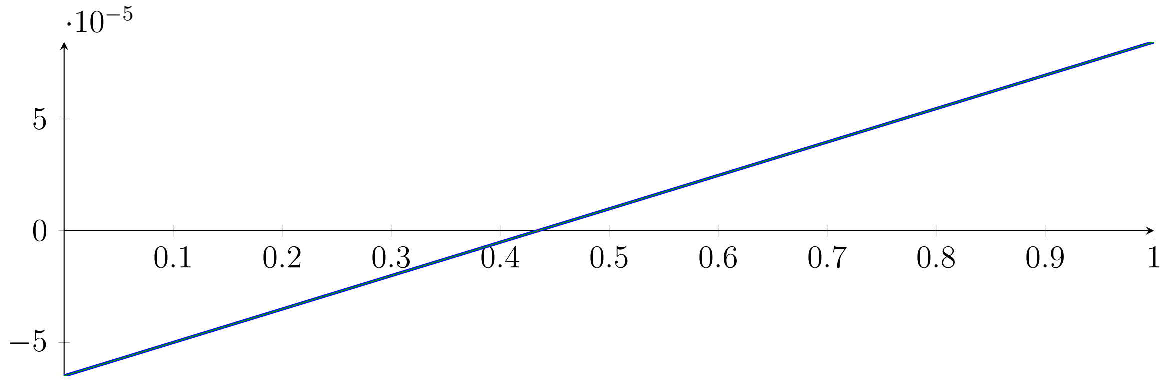
The maximum difference of the Bézier coefficients is $\delta = 2.82529 \cdot 10^{-17}$.
Bounding polynomials M and m:

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.434685, 1.4089 \cdot 10^{6}\} \qquad N(m) = \{0.434685, 1.4089 \cdot 10^{6}\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

Longest intersection interval: $3.74055 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

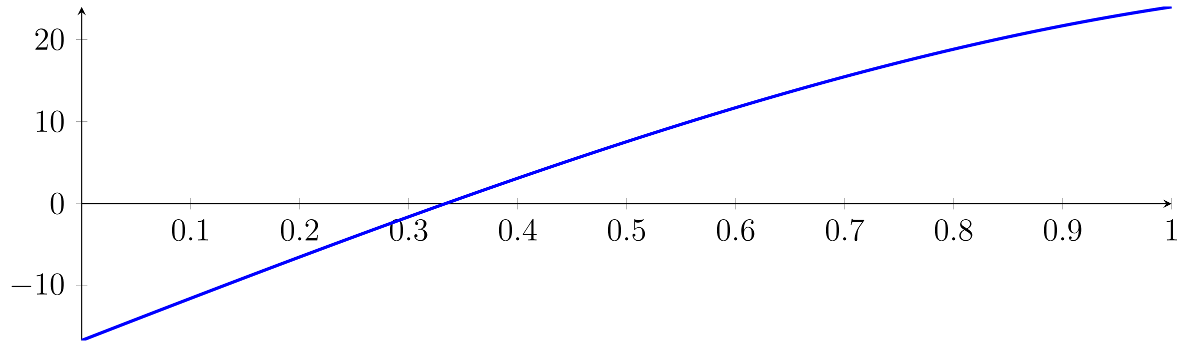
113.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

113.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-08}$.

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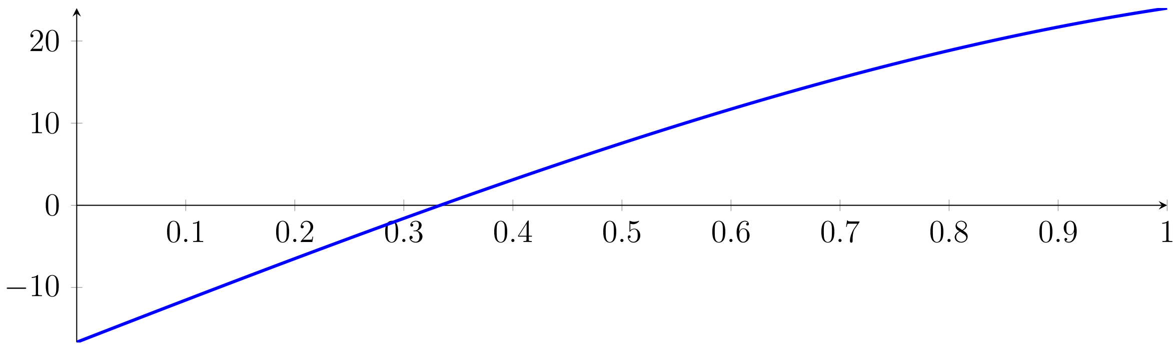
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

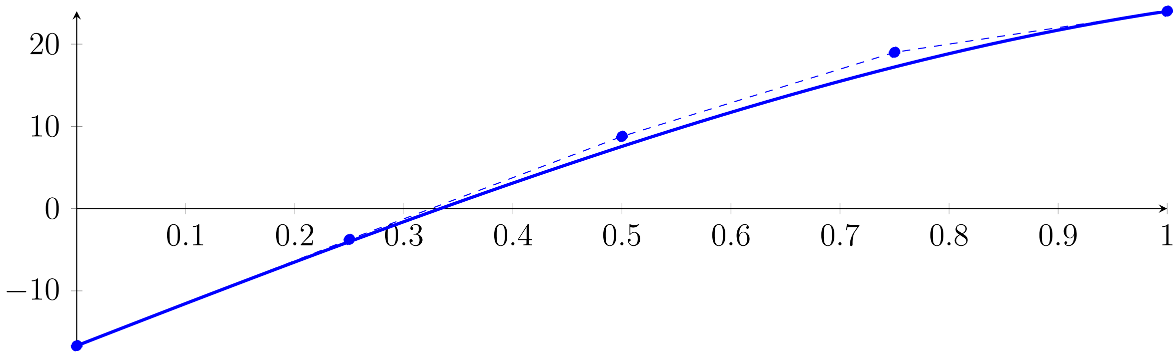
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



114.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

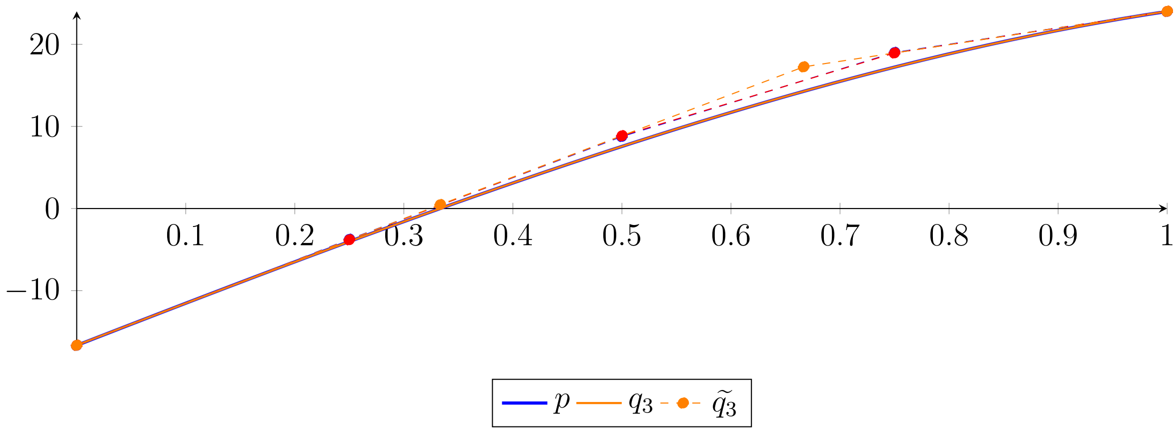
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= 3.34802 \cdot 10^{-16}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



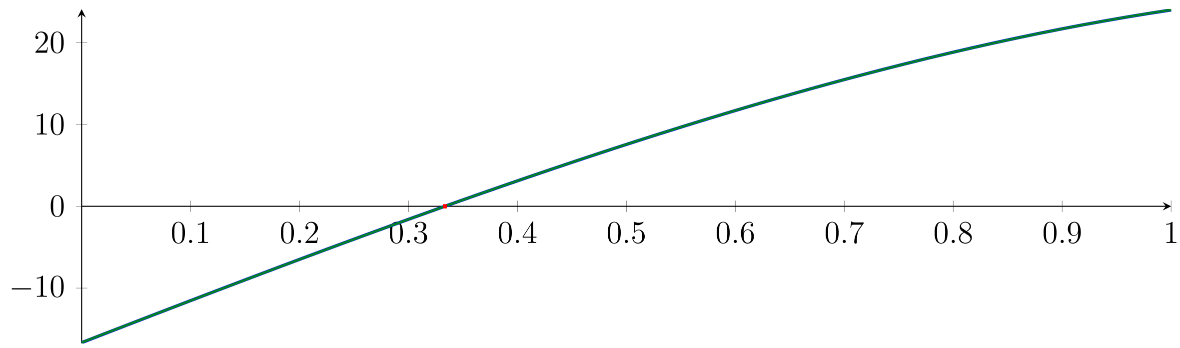
The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



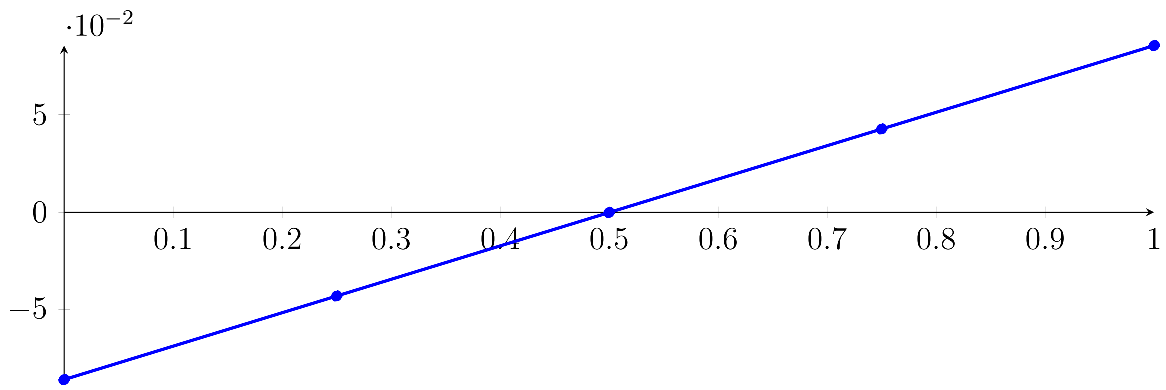
$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

114.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

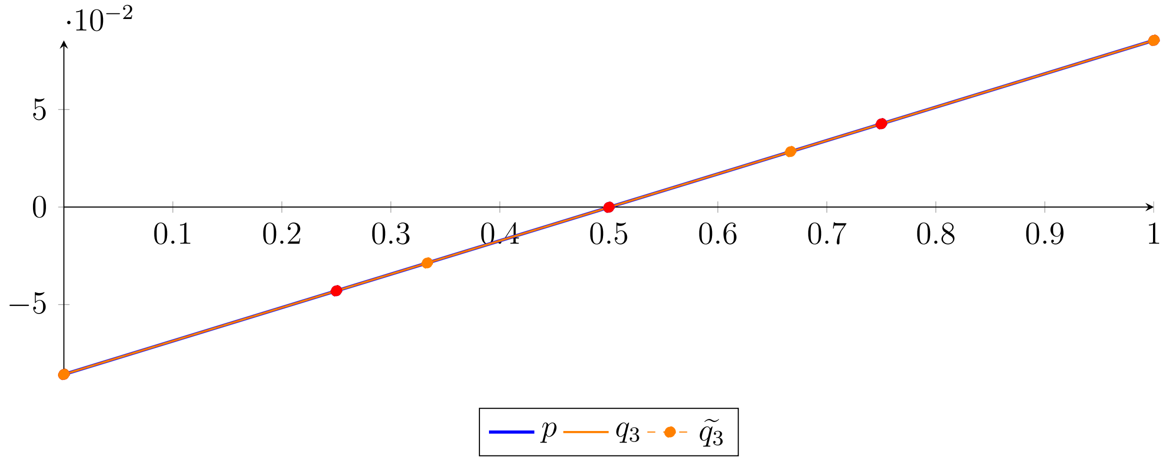
$$\begin{aligned} p &= -1.70219\cdot10^{-10}X^4 - 4.23789\cdot10^{-07}X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4}(X) - 0.0429507B_{1,4}(X) - 0.000129666B_{2,4}(X) \\ &\quad + 0.0426682B_{3,4}(X) + 0.0854427B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413\cdot10^{-07}X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,3} - 0.0286693B_{1,3} + 0.02841B_{2,3} + 0.0854427B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 1.99222\cdot10^{-18}X^4 - 4.2413\cdot10^{-07}X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4} - 0.0429507B_{1,4} - 0.000129666B_{2,4} + 0.0426682B_{3,4} + 0.0854427B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.
Bounding polynomials M and m:

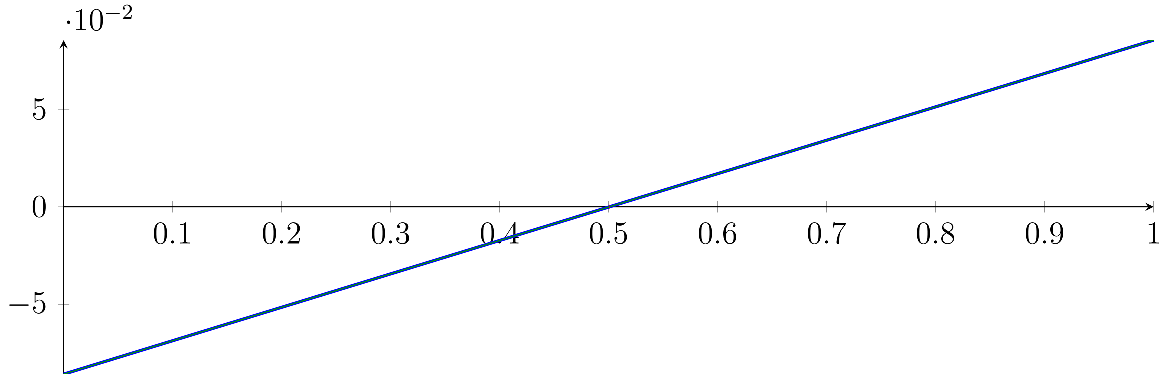
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

Longest intersection interval: $1.7041 \cdot 10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

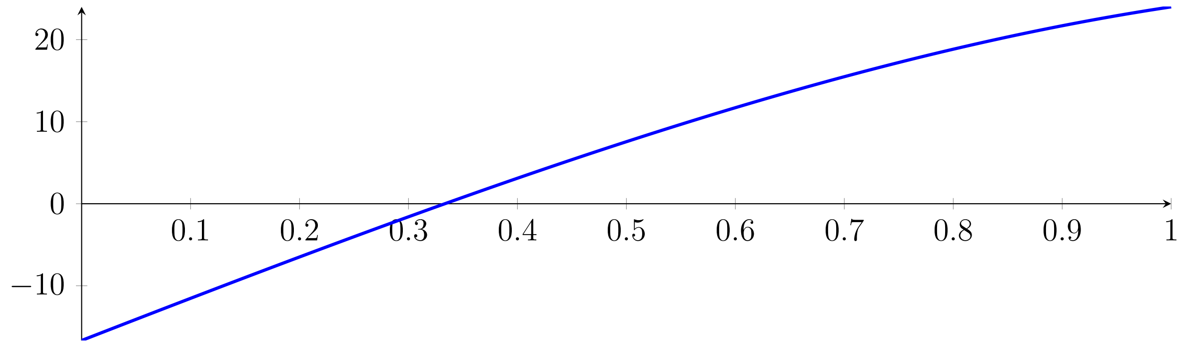
114.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

114.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-08}$.

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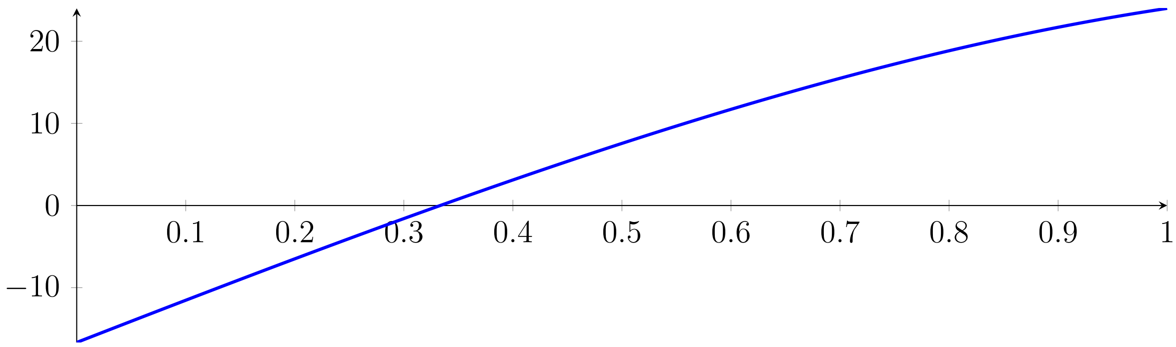
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

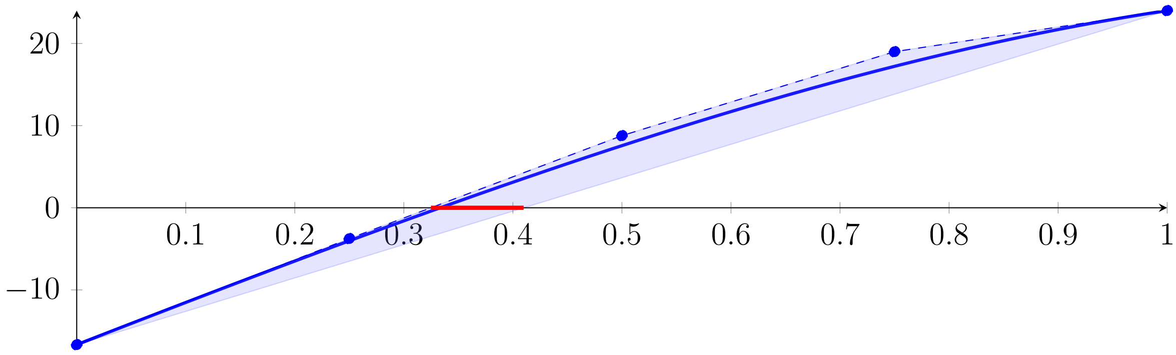
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



115.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

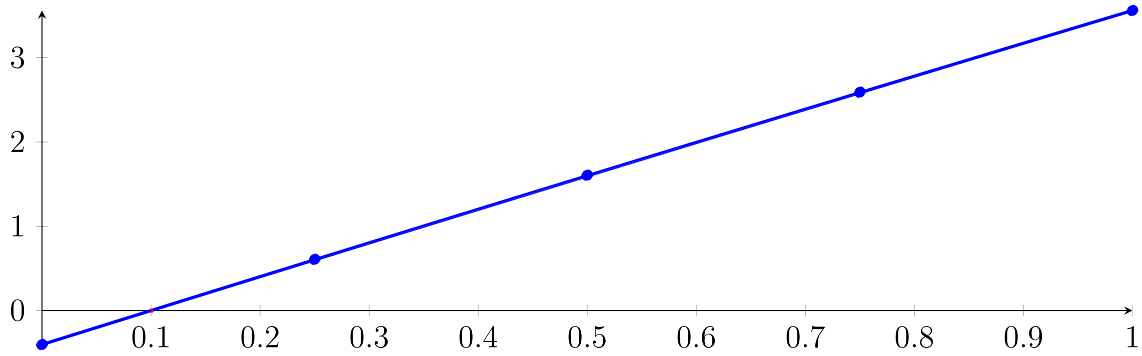
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

115.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064\cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

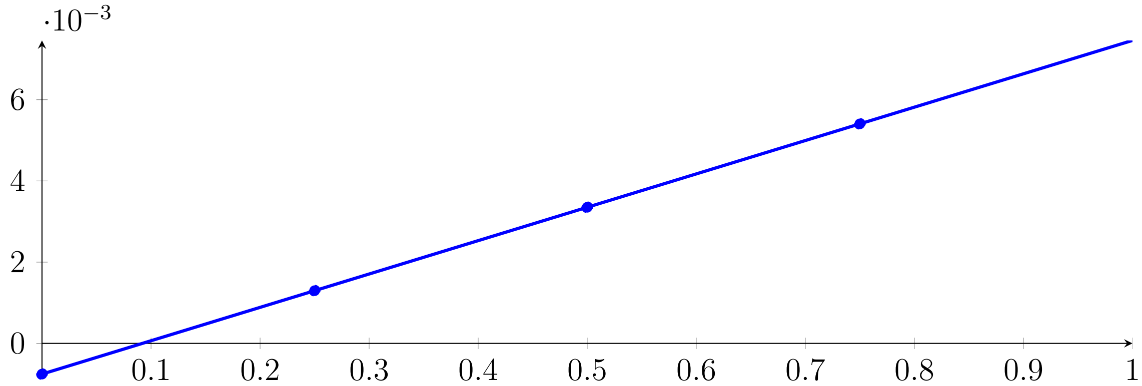
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: [0.333317, 0.333491],

115.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01974 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

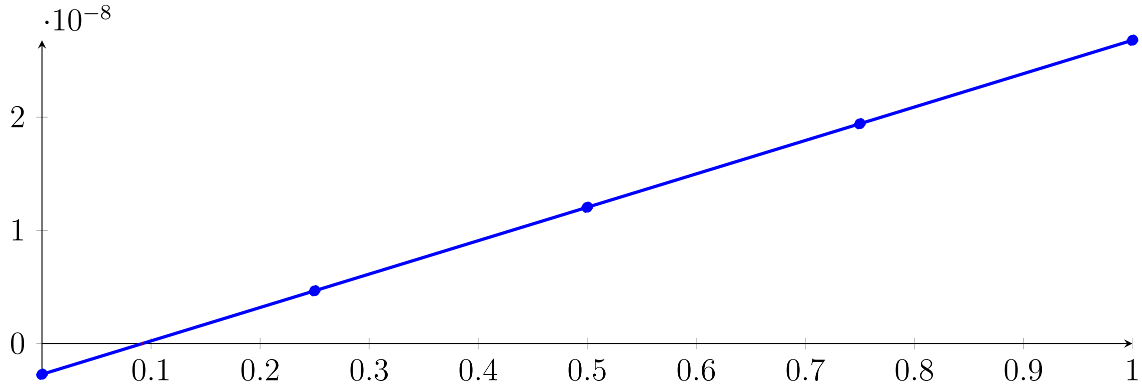
$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: [0.333333, 0.333333],

115.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.61559 \cdot 10^{-27} X^4 - 3.23117 \cdot 10^{-27} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09}$$
$$= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28975 \cdot 10^{-11}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

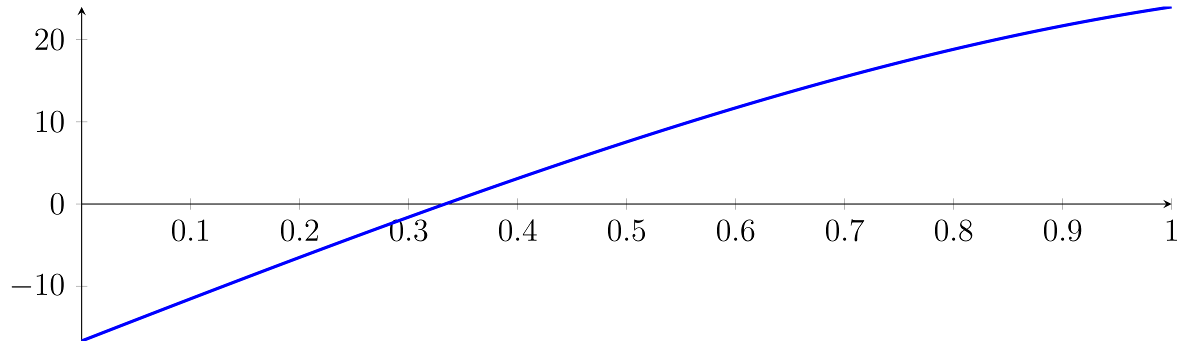
115.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

115.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

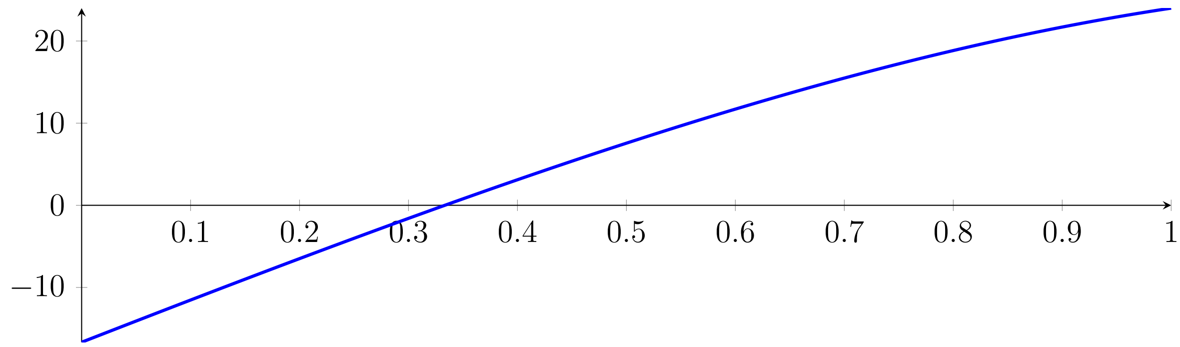
with precision $\varepsilon = 1 \cdot 10^{-16}$.

116 Running QuadClip on f_4 with epsilon 16

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0,1]$:

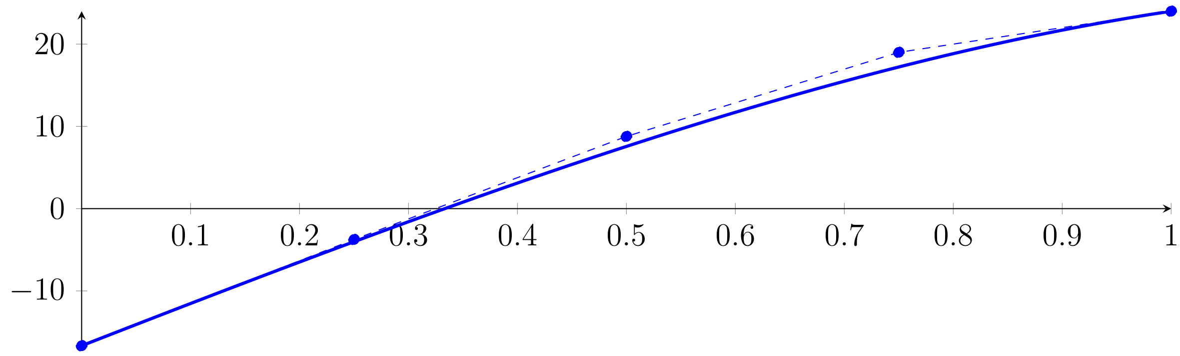
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



116.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

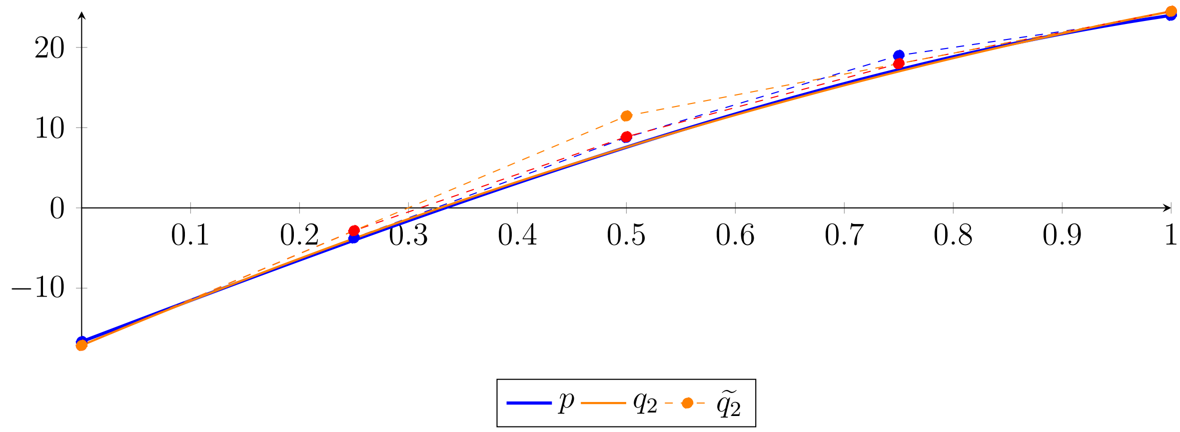
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -1.45023 \cdot 10^{-15}X^4 + 4.00374 \cdot 10^{-15}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

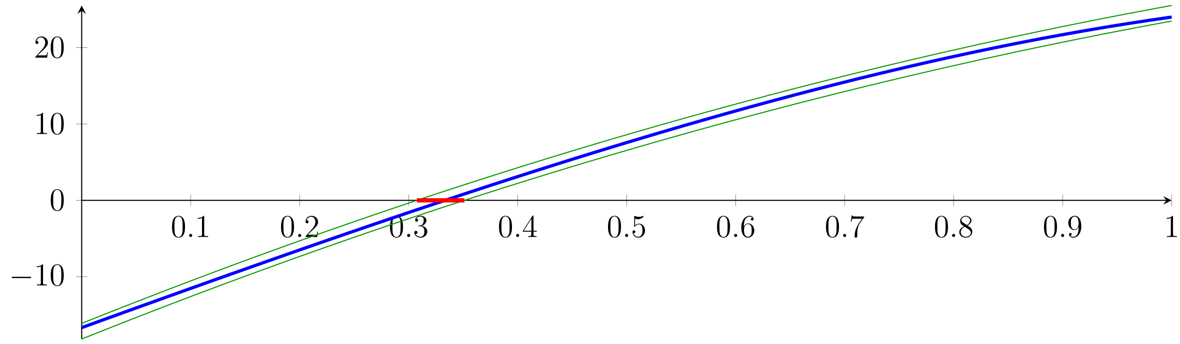
$$M = -15.5476X^2 + 57.181X - 16.1119$$

$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



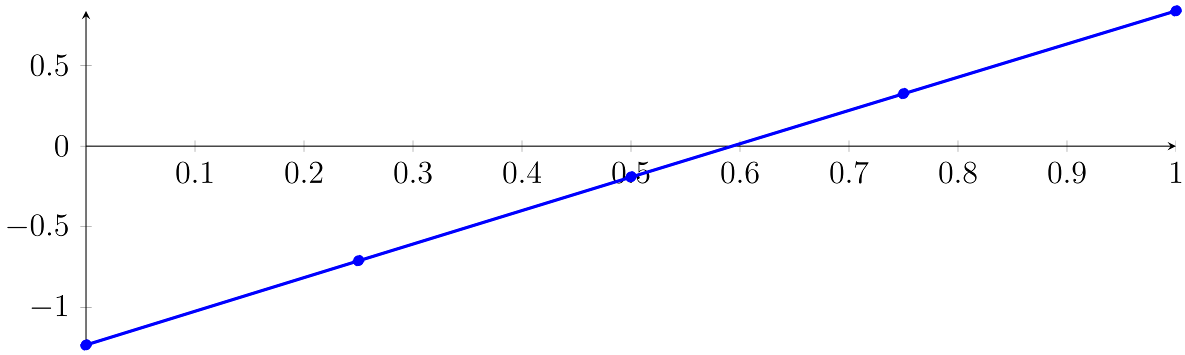
$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

116.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

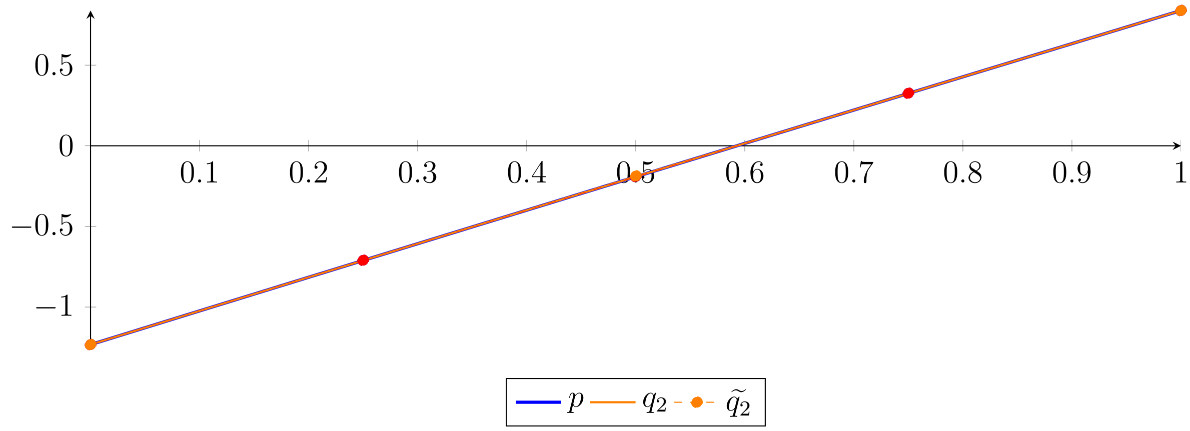
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06}X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.45283 \cdot 10^{-17}X^4 + 3.33934 \cdot 10^{-17}X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

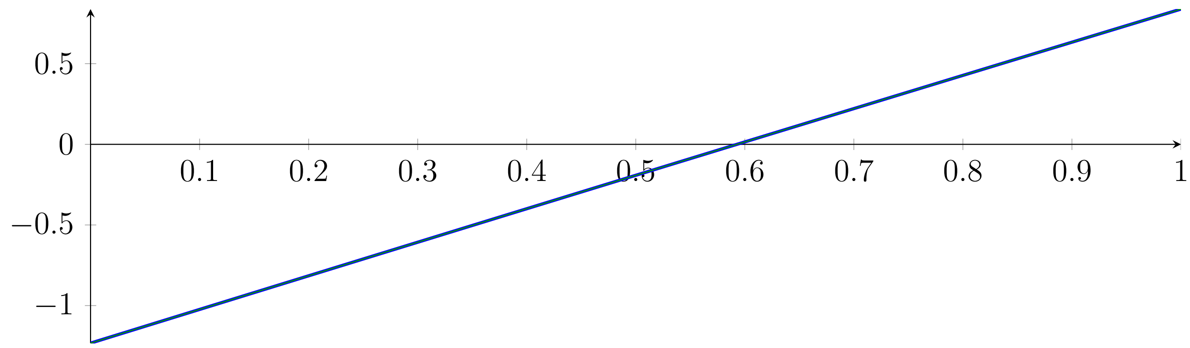
Bounding polynomials M and m:

$$M = -0.020089X^2 + 2.09166X - 1.23274$$
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

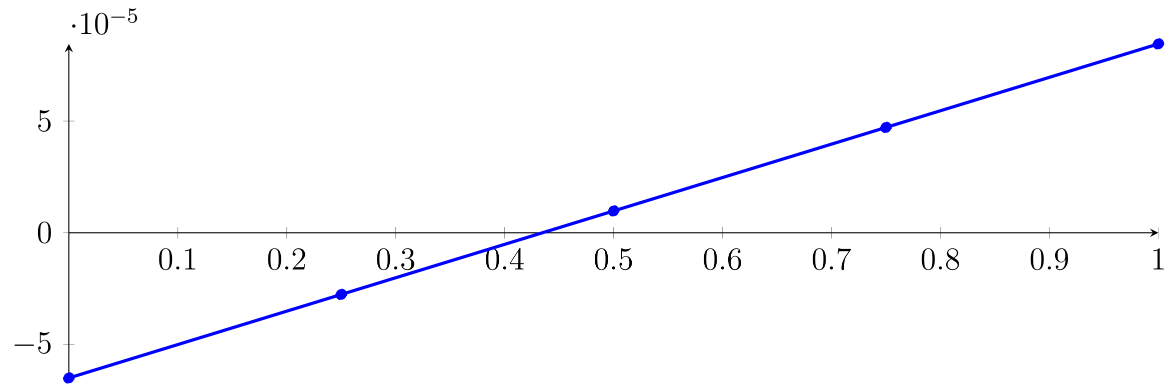
Longest intersection interval: $7.23183 \cdot 10^{-05}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

116.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

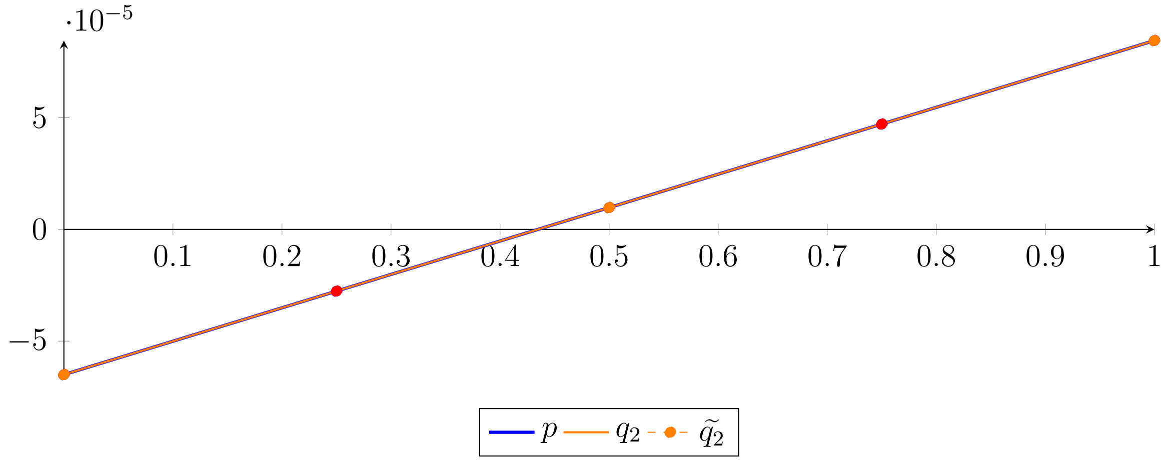
$$p = -1.05879 \cdot 10^{-22} X^4 - 2.82525 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777$$
$$\cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.06147\cdot10^{-10}X^2 + 0.000149549X - 6.50069\cdot10^{-05}$$
$$= -6.50069\cdot10^{-05}B_{0,2} + 9.76779\cdot10^{-06}B_{1,2} + 8.45424\cdot10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = -4.49986\cdot10^{-22}X^4 + 3.33519\cdot10^{-21}X^3 - 1.06147\cdot10^{-10}X^2 + 0.000149549X - 6.50069\cdot10^{-05}$$
$$= -6.50069\cdot10^{-05}B_{0,4} - 2.76196\cdot10^{-05}B_{1,4} + 9.76777\cdot10^{-06}B_{2,4} + 4.71551\cdot10^{-05}B_{3,4} + 8.45424\cdot10^{-05}B_{4,4}$$



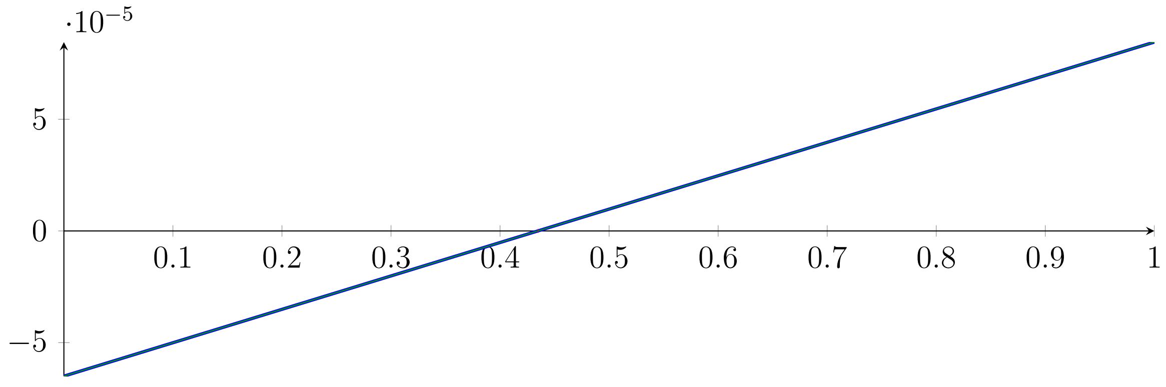
The maximum difference of the Bézier coefficients is $\delta = 2.82529\cdot10^{-17}$.
Bounding polynomials M and m:

$$M = -1.06147\cdot10^{-10}X^2 + 0.000149549X - 6.50069\cdot10^{-05}$$
$$m = -1.06147\cdot10^{-10}X^2 + 0.000149549X - 6.50069\cdot10^{-05}$$

Root of M and m:

$$N(M) = \{0.434685, 1.4089\cdot10^{6}\} \qquad N(m) = \{0.434685, 1.4089\cdot10^{6}\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

Longest intersection interval: $3.74055\cdot10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

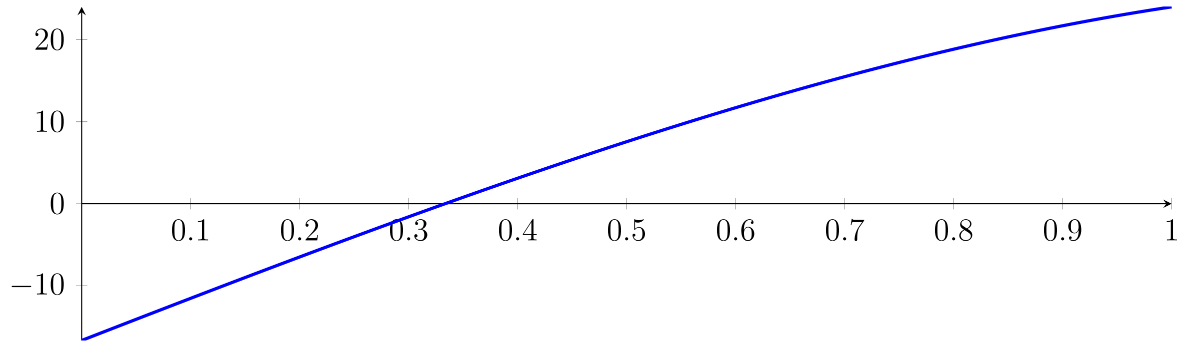
116.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

116.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-16}$.

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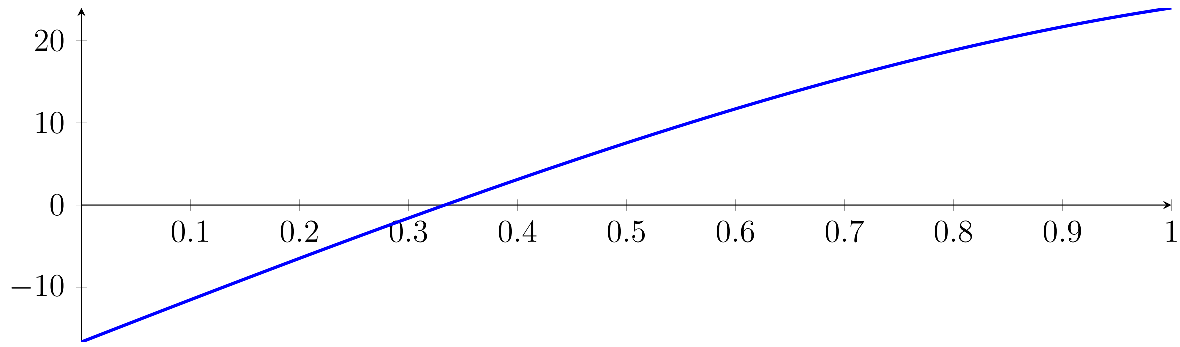
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

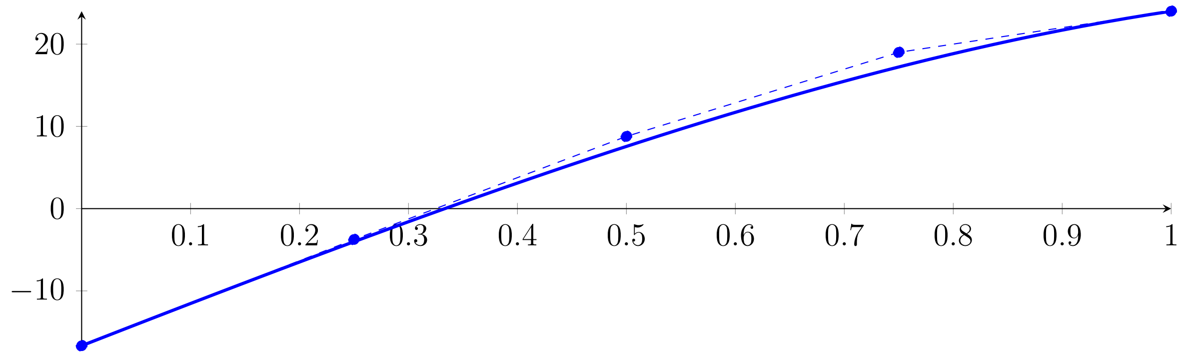
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



117.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

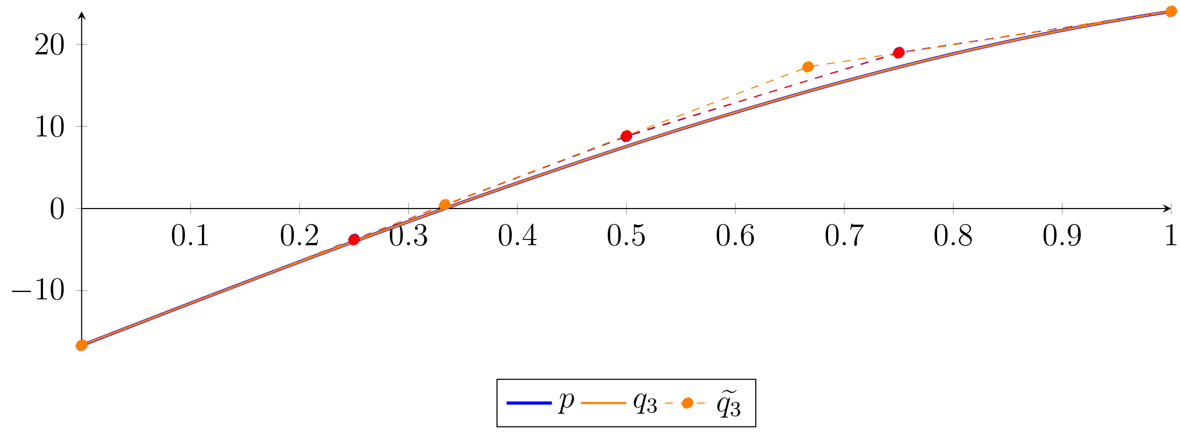
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= 3.34802 \cdot 10^{-16}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m:

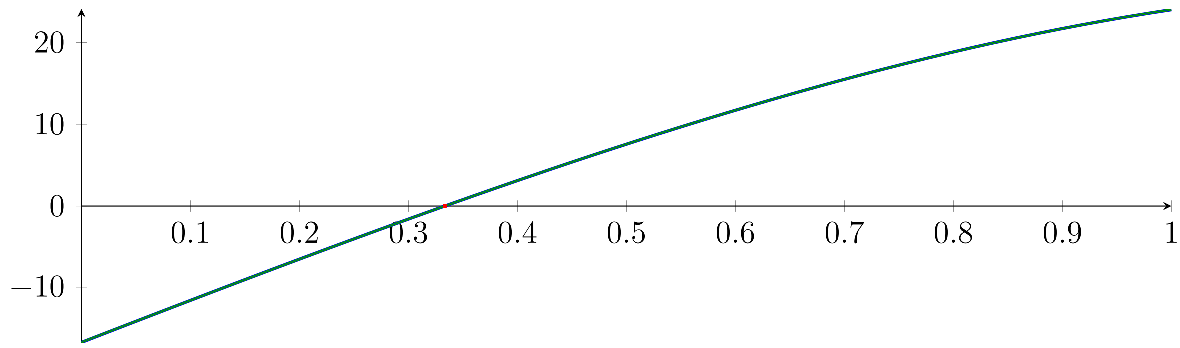
$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

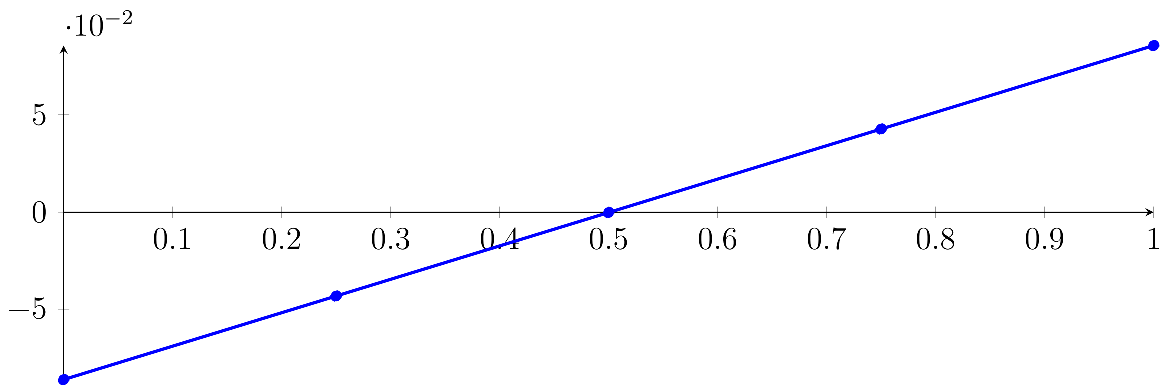
Longest intersection interval: 0.00361204

$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

117.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

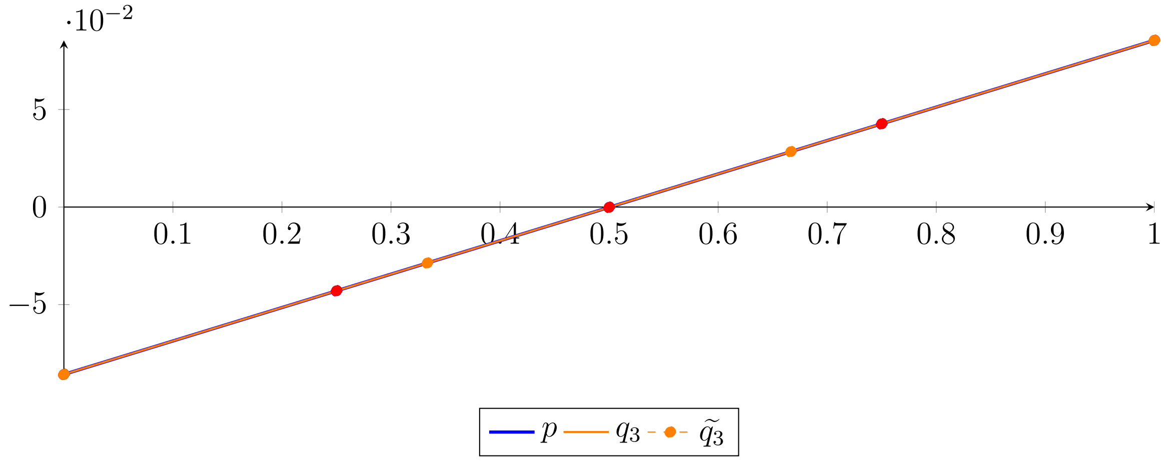
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10}X^4 - 4.23789 \cdot 10^{-07}X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4}(X) - 0.0429507B_{1,4}(X) - 0.000129666B_{2,4}(X) \\ &\quad + 0.0426682B_{3,4}(X) + 0.0854427B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07}X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,3} - 0.0286693B_{1,3} + 0.02841B_{2,3} + 0.0854427B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 1.99222 \cdot 10^{-18}X^4 - 4.2413 \cdot 10^{-07}X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4} - 0.0429507B_{1,4} - 0.000129666B_{2,4} + 0.0426682B_{3,4} + 0.0854427B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m:

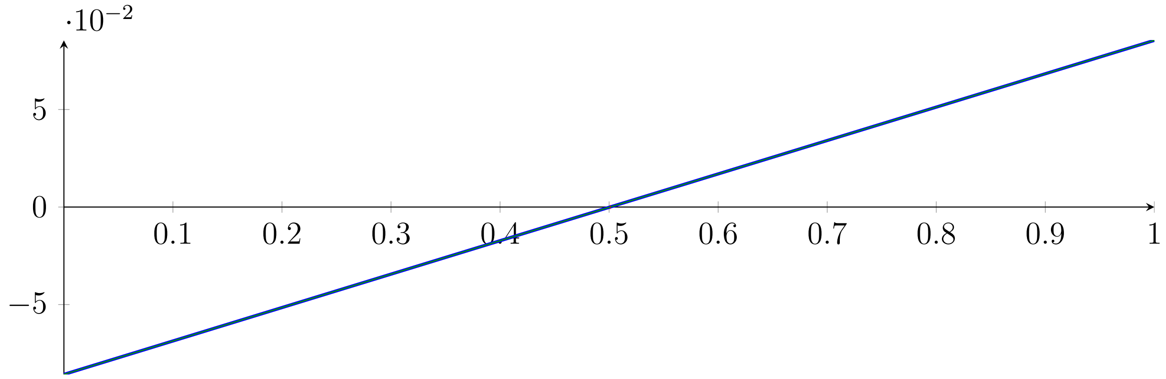
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

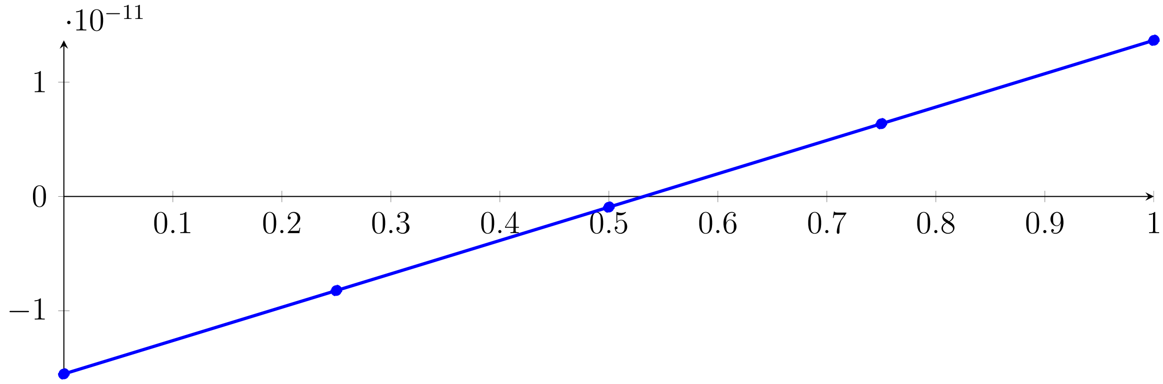
Longest intersection interval: $1.7041 \cdot 10^{-10}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

117.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

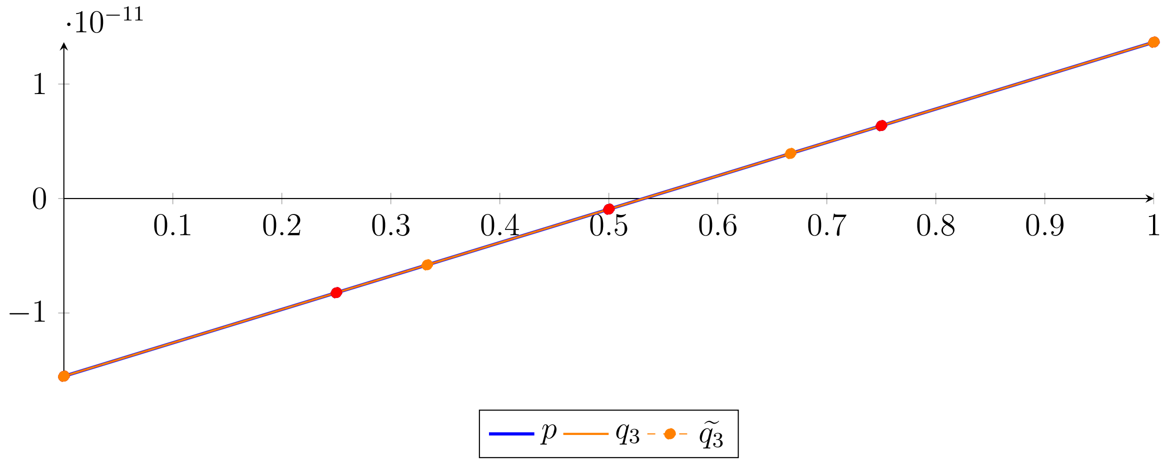
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\ &= -1.55233 \cdot 10^{-11} B_{0,4}(X) - 8.22817 \cdot 10^{-12} B_{1,4}(X) - 9.33054 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36206 \cdot 10^{-12} B_{3,4}(X) + 1.36572 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.87145{\cdot}10^{-28}X^3 - 4.04172{\cdot}10^{-24}X^2 + 2.91805{\cdot}10^{-11}X - 1.55233{\cdot}10^{-11}$$
$$= -1.55233{\cdot}10^{-11}B_{0,3} - 5.79647{\cdot}10^{-12}B_{1,3} + 3.93036{\cdot}10^{-12}B_{2,3} + 1.36572{\cdot}10^{-11}B_{3,3}$$
$$\widetilde{q_3} = 3.84964{\cdot}10^{-28}X^4 - 2.1457{\cdot}10^{-28}X^3 - 4.04147{\cdot}10^{-24}X^2 + 2.91805{\cdot}10^{-11}X - 1.55233{\cdot}10^{-11}$$
$$= -1.55233{\cdot}10^{-11}B_{0,4} - 8.22817{\cdot}10^{-12}B_{1,4} - 9.33054{\cdot}10^{-13}B_{2,4} + 6.36206{\cdot}10^{-12}B_{3,4} + 1.36572{\cdot}10^{-11}B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 1.13596{\cdot}10^{-28}$.

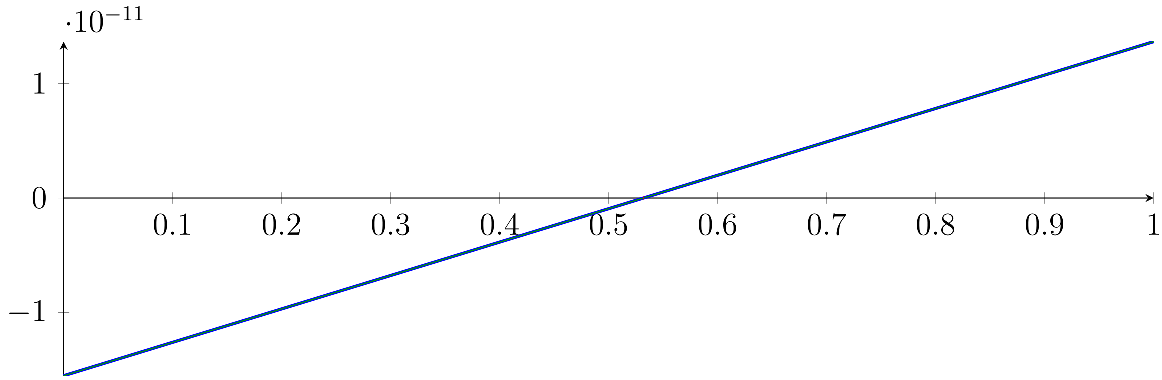
Bounding polynomials M and m:

$$M = 2.87145{\cdot}10^{-28}X^3 - 4.04172{\cdot}10^{-24}X^2 + 2.91805{\cdot}10^{-11}X - 1.55233{\cdot}10^{-11}$$
$$m = 2.87145{\cdot}10^{-28}X^3 - 4.04172{\cdot}10^{-24}X^2 + 2.91805{\cdot}10^{-11}X - 1.55233{\cdot}10^{-11}$$

Root of M and m:

$$N(M) = \{0.531975\} \qquad N(m) = \{0.531975\}$$

Intersection intervals:



$$[0.531975, 0.531975]$$

Longest intersection interval: 0

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

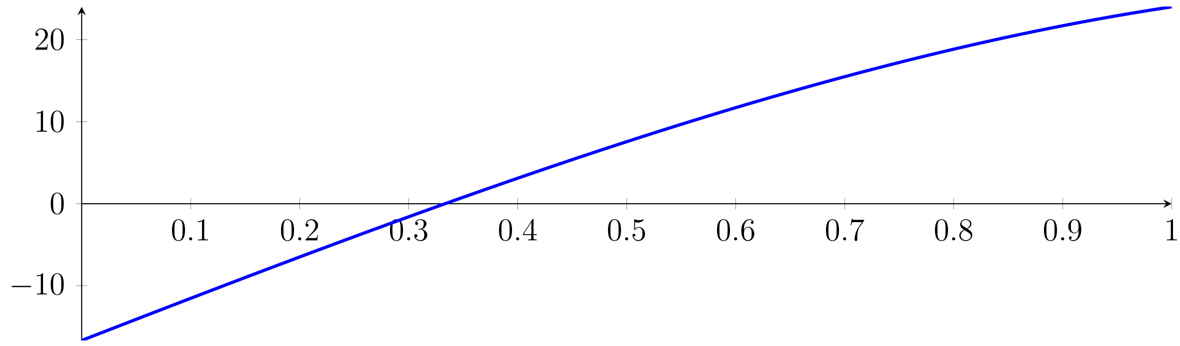
117.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

117.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

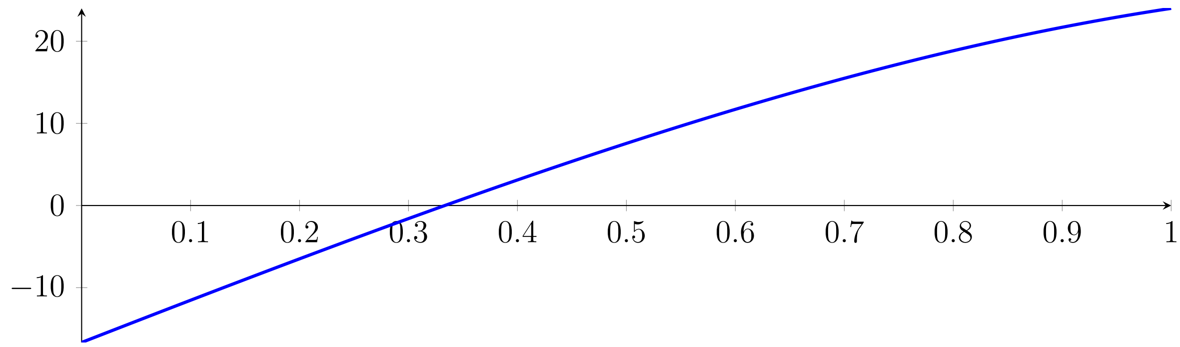
with precision $\varepsilon = 1 \cdot 10^{-16}$.

118 Running BezClip on f_4 with epsilon 32

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0,1]$:

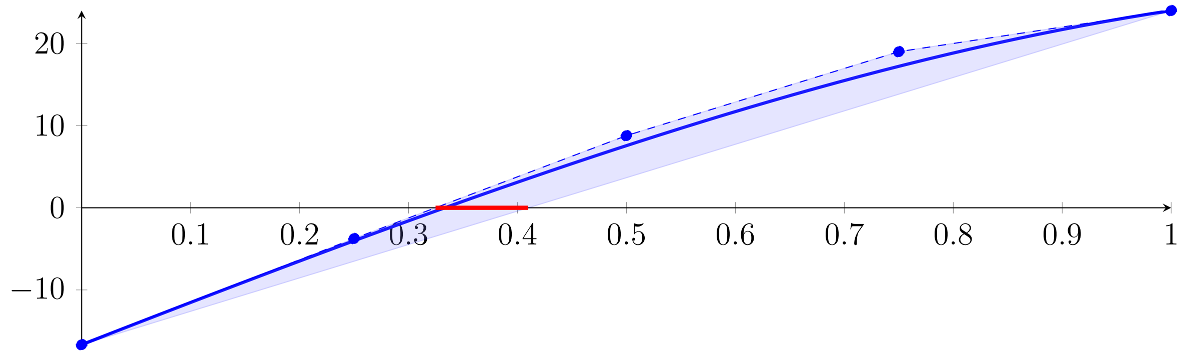
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



118.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

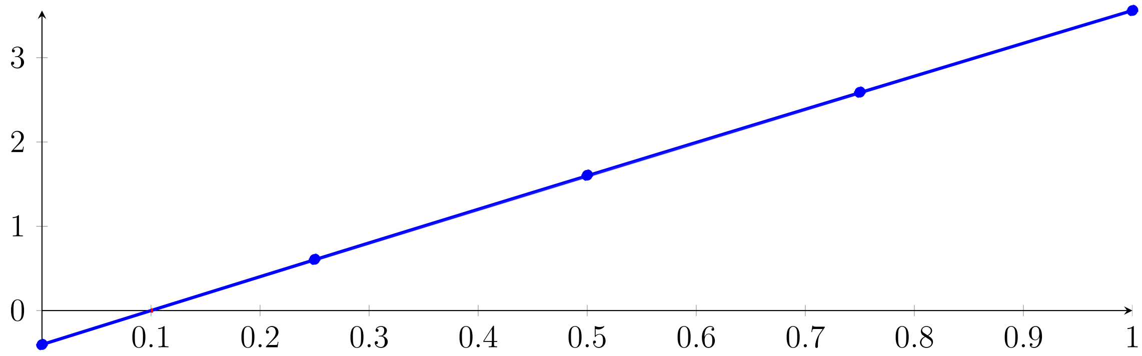
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

118.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

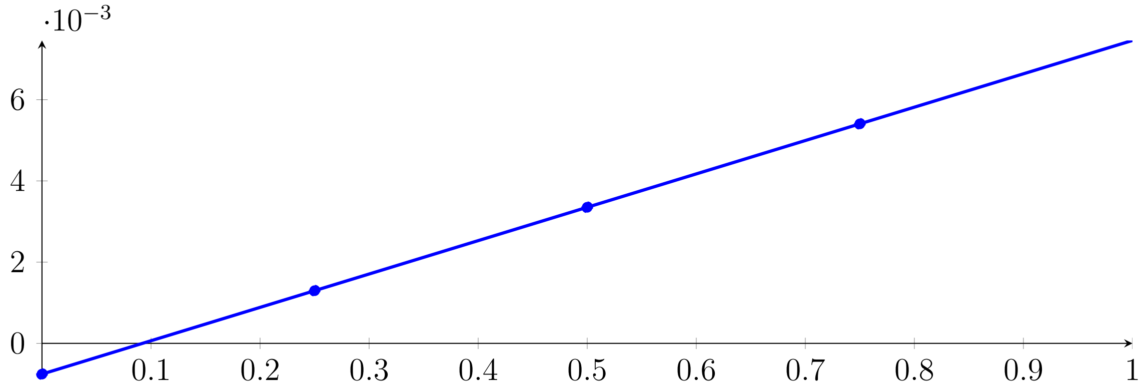
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

118.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -9.01974 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

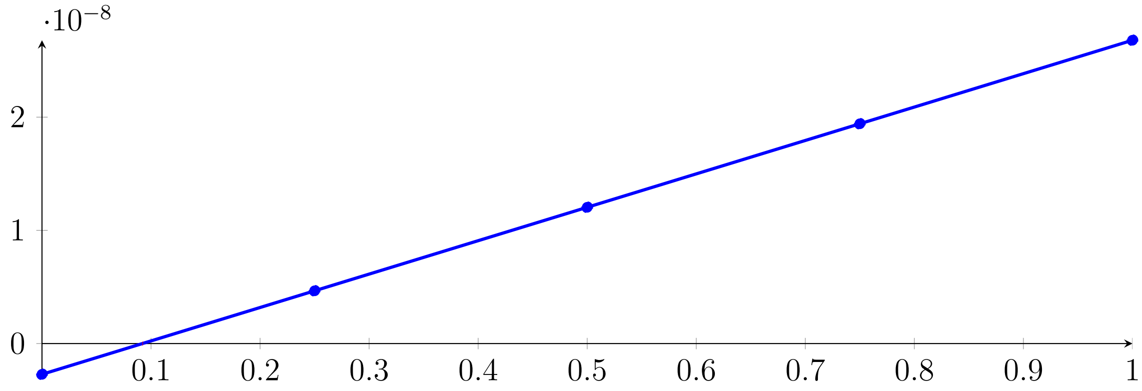
$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

118.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.61559 \cdot 10^{-27} X^4 - 3.23117 \cdot 10^{-27} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09}$$
$$= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28975 \cdot 10^{-11}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

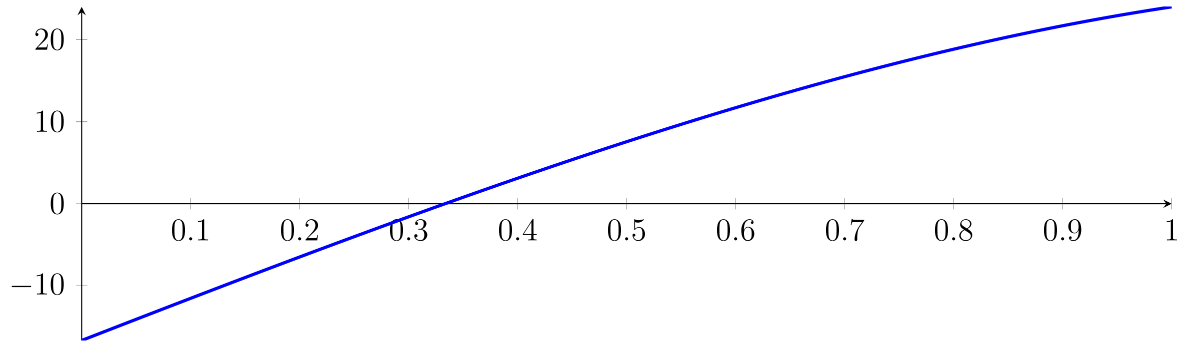
118.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

118.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-32}$.

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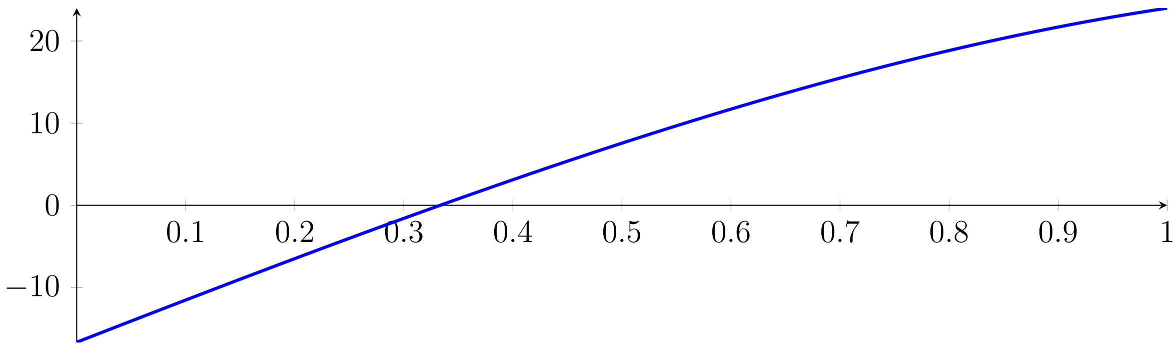
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

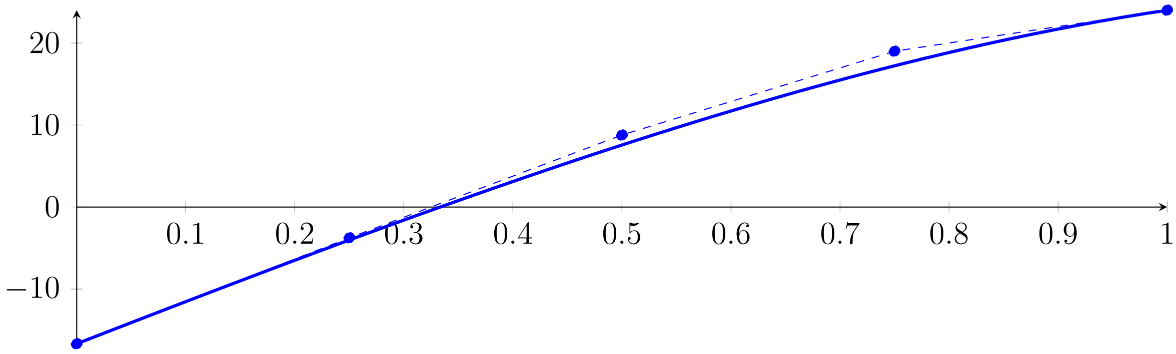
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



119.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

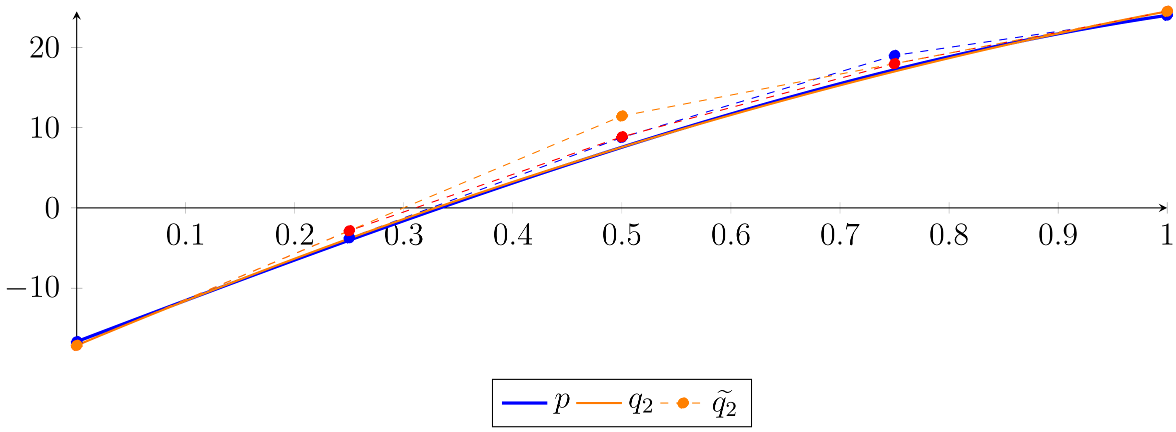
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -1.45023 \cdot 10^{-15}X^4 + 4.00374 \cdot 10^{-15}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m:

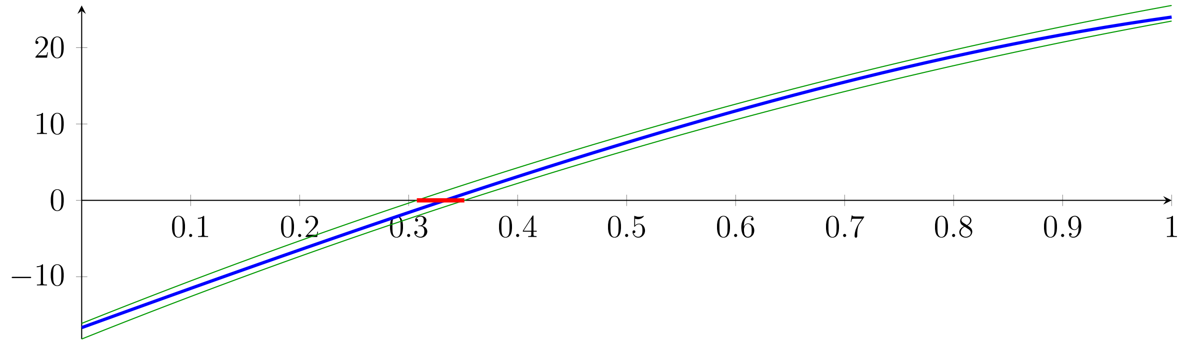
$$M = -15.5476X^2 + 57.181X - 16.1119$$

$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

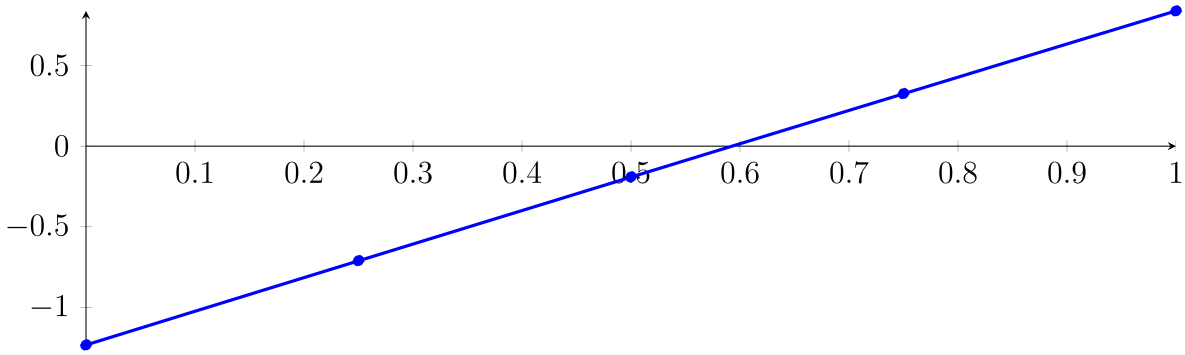
Longest intersection interval: 0.0436205

$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

119.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

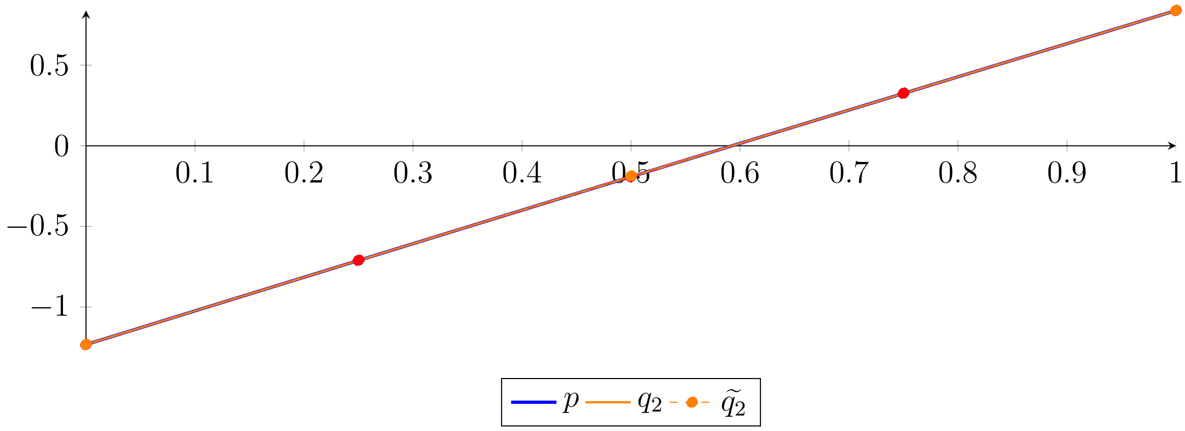
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06}X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.45283 \cdot 10^{-17}X^4 + 3.33934 \cdot 10^{-17}X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



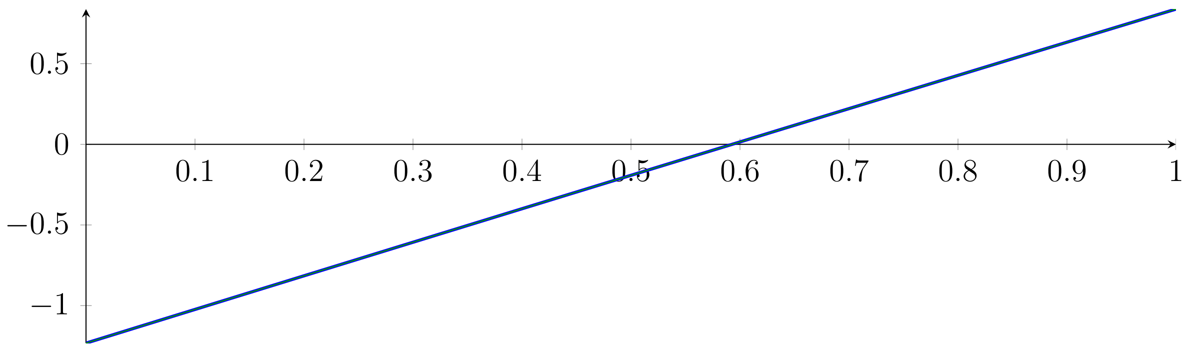
The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.
Bounding polynomials M and m:

$$M = -0.020089X^2 + 2.09166X - 1.23274$$
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



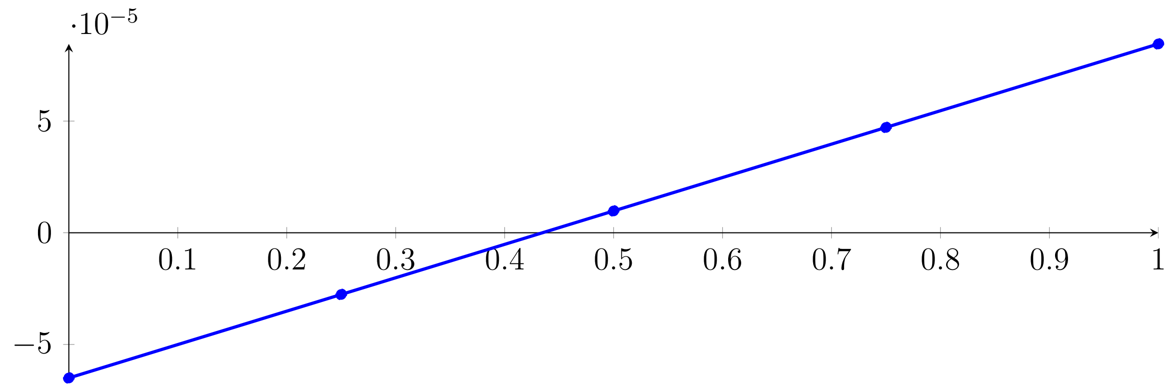
$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

119.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

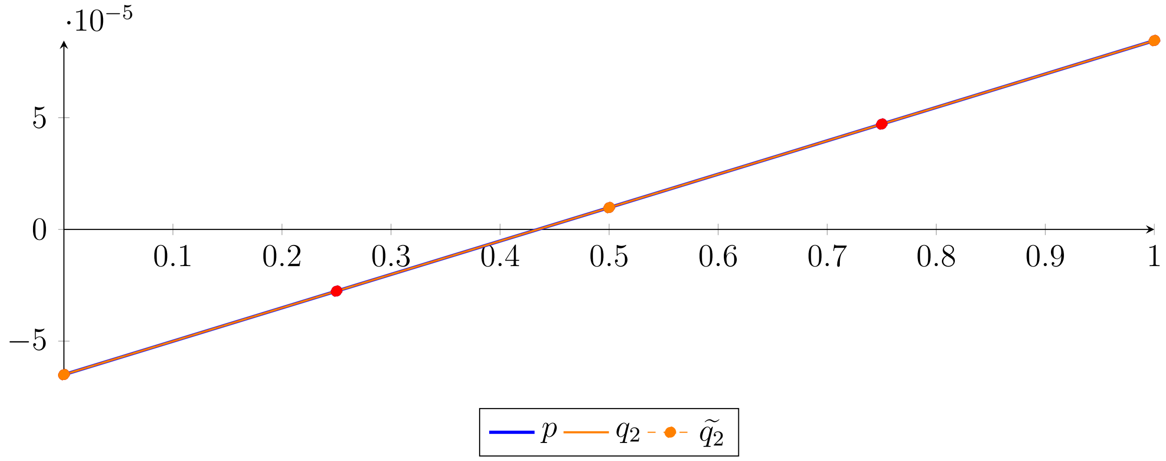
$$p = -1.05879 \cdot 10^{-22} X^4 - 2.82525 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777$$
$$\cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.06147{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$
$$= -6.50069{\cdot}10^{-05}B_{0,2} + 9.76779{\cdot}10^{-06}B_{1,2} + 8.45424{\cdot}10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = -4.49986{\cdot}10^{-22}X^4 + 3.33519{\cdot}10^{-21}X^3 - 1.06147{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$
$$= -6.50069{\cdot}10^{-05}B_{0,4} - 2.76196{\cdot}10^{-05}B_{1,4} + 9.76777{\cdot}10^{-06}B_{2,4} + 4.71551{\cdot}10^{-05}B_{3,4} + 8.45424{\cdot}10^{-05}B_{4,4}$$



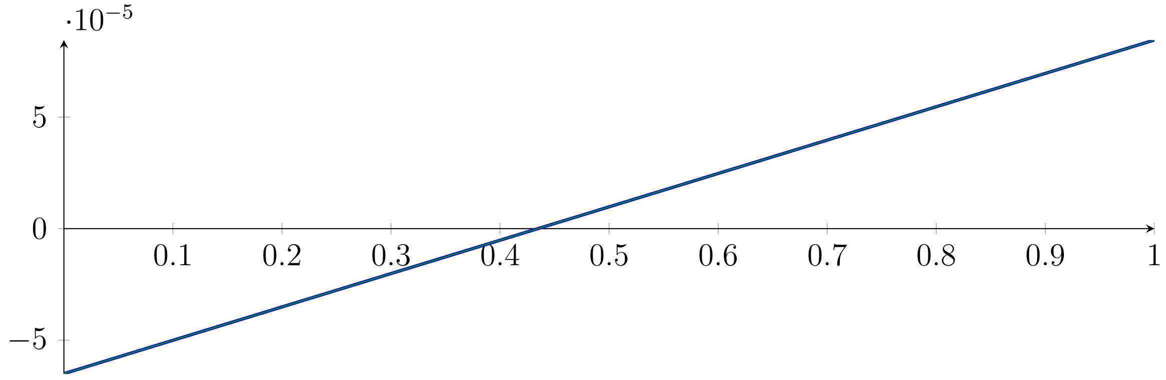
The maximum difference of the Bézier coefficients is $\delta = 2.82529{\cdot}10^{-17}$.
Bounding polynomials M and m:

$$M = -1.06147{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$
$$m = -1.06147{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{0.434685, 1.4089{\cdot}10^{6}\} \qquad N(m) = \{0.434685, 1.4089{\cdot}10^{6}\}$$

Intersection intervals:



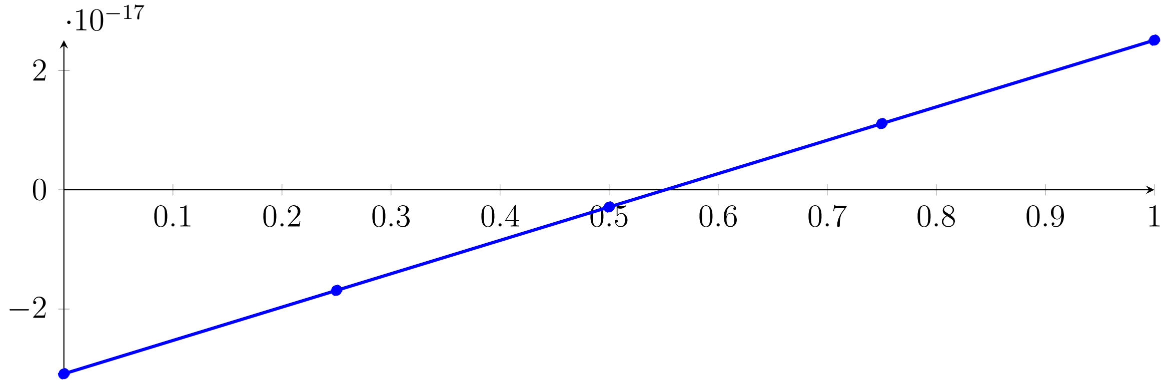
$$[0.434685, 0.434685]$$

Longest intersection interval: $3.74055{\cdot}10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

119.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

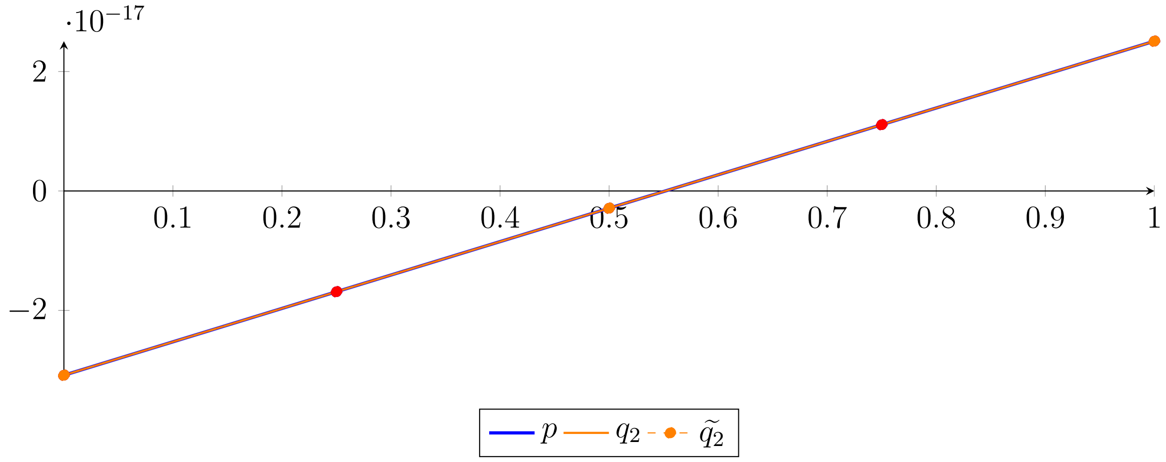
$$p = -1.20371\cdot 10^{-35}X^3 + 5.59397\cdot 10^{-17}X - 3.08561\cdot 10^{-17}$$
$$= -3.08561\cdot 10^{-17}B_{0,4}(X) - 1.68712\cdot 10^{-17}B_{1,4}(X) - 2.88624\cdot 10^{-18}B_{2,4}(X) + 1.10987\cdot 10^{-17}B_{3,4}(X) + 2.50836\cdot 10^{-17}B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = 1.50463\cdot 10^{-35}X^2 + 5.59397\cdot 10^{-17}X - 3.08561\cdot 10^{-17}$$
$$= -3.08561\cdot 10^{-17}B_{0,2} - 2.88624\cdot 10^{-18}B_{1,2} + 2.50836\cdot 10^{-17}B_{2,2}$$

$$\widetilde{q}_2 = -3.03936\cdot 10^{-34}X^4 + 9.14817\cdot 10^{-34}X^3 - 6.31946\cdot 10^{-34}X^2 + 5.59397\cdot 10^{-17}X - 3.08561\cdot 10^{-17}$$
$$= -3.08561\cdot 10^{-17}B_{0,4} - 1.68712\cdot 10^{-17}B_{1,4} - 2.88624\cdot 10^{-18}B_{2,4} + 1.10987\cdot 10^{-17}B_{3,4} + 2.50836\cdot 10^{-17}B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 7.14701\cdot 10^{-35}$.
Bounding polynomials M and m:

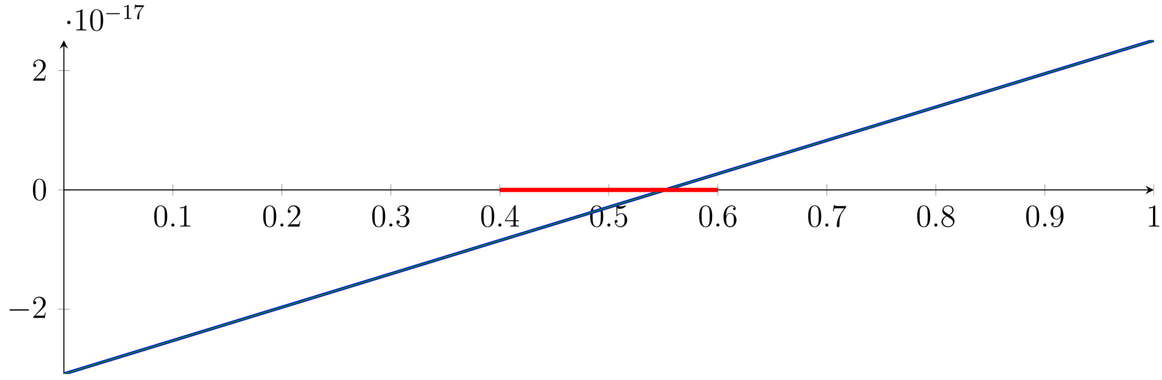
$$M = 1.50463\cdot 10^{-35}X^2 + 5.59397\cdot 10^{-17}X - 3.08561\cdot 10^{-17}$$

$$m = 1.50463\cdot 10^{-35}X^2 + 5.59397\cdot 10^{-17}X - 3.08561\cdot 10^{-17}$$

Root of M and m:

$$N(M) = \{-3.71783\cdot 10^{18}, 0.6\} \qquad N(m) = \{-3.71783\cdot 10^{18}, 0.4\}$$

Intersection intervals:



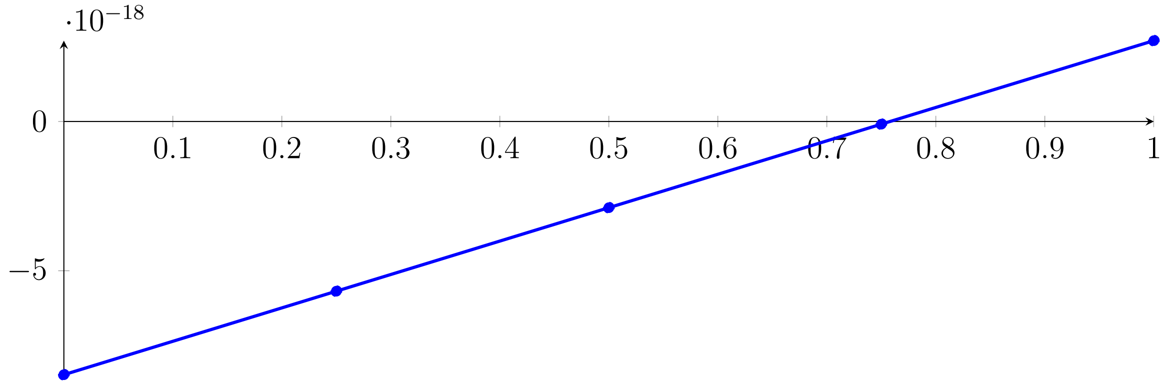
[0.4, 0.6]

Longest intersection interval: 0.2
$\implies$ Selective recursion: interval 1: [0.333333, 0.333333],

119.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

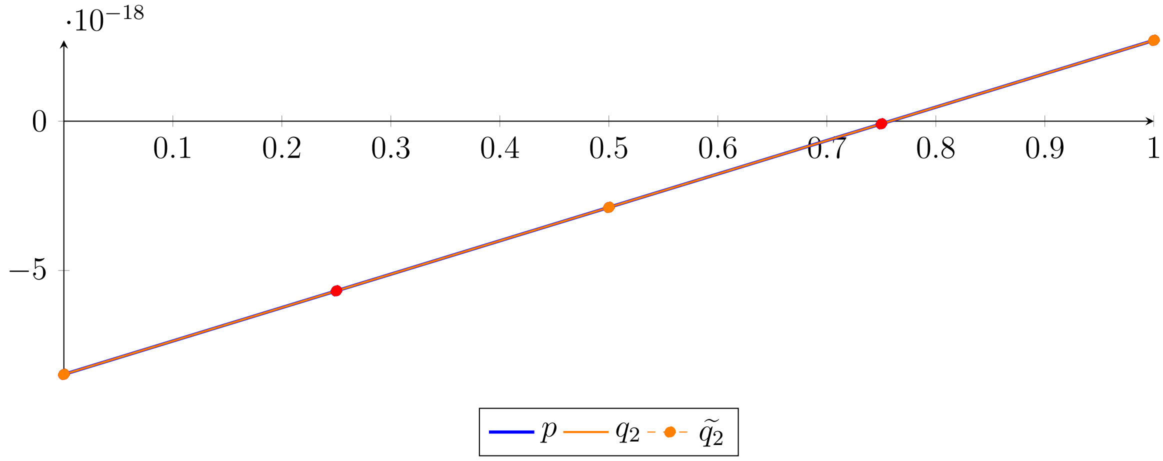
$$p = 1.50463 \cdot 10^{-36} X^4 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$
$$= -8.48022 \cdot 10^{-18} B_{0,4}(X) - 5.68323 \cdot 10^{-18} B_{1,4}(X) - 2.88624 \cdot 10^{-18} B_{2,4}(X) - 8.9255 \cdot 10^{-20} B_{3,4}(X) + 2.70773 \cdot 10^{-18} B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = 6.01853 \cdot 10^{-36} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$
$$= -8.48022 \cdot 10^{-18} B_{0,2} - 2.88624 \cdot 10^{-18} B_{1,2} + 2.70773 \cdot 10^{-18} B_{2,2}$$

$$\widetilde{q_2} = -1.06829 \cdot 10^{-34} X^4 + 6.62038 \cdot 10^{-35} X^3 + 7.67363 \cdot 10^{-35} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$
$$= -8.48022 \cdot 10^{-18} B_{0,4} - 5.68323 \cdot 10^{-18} B_{1,4} - 2.88624 \cdot 10^{-18} B_{2,4} - 8.9255 \cdot 10^{-20} B_{3,4} + 2.70773 \cdot 10^{-18} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.36039 \cdot 10^{-35}$.
Bounding polynomials M and m:

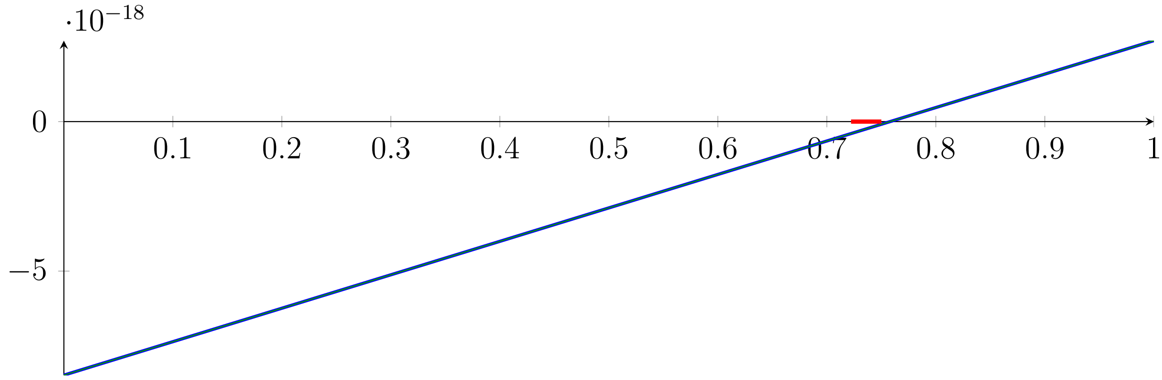
$$M = 6.01853 \cdot 10^{-36} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$

$$m = 6.77085 \cdot 10^{-36} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$

Root of M and m:

$$N(M) = \{-1.85892 \cdot 10^{18}, 0.75\} \qquad N(m) = \{-1.65237 \cdot 10^{18}, 0.722222\}$$

Intersection intervals:



$$[0.722222, 0.75]$$

Longest intersection interval: 0.0277778
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

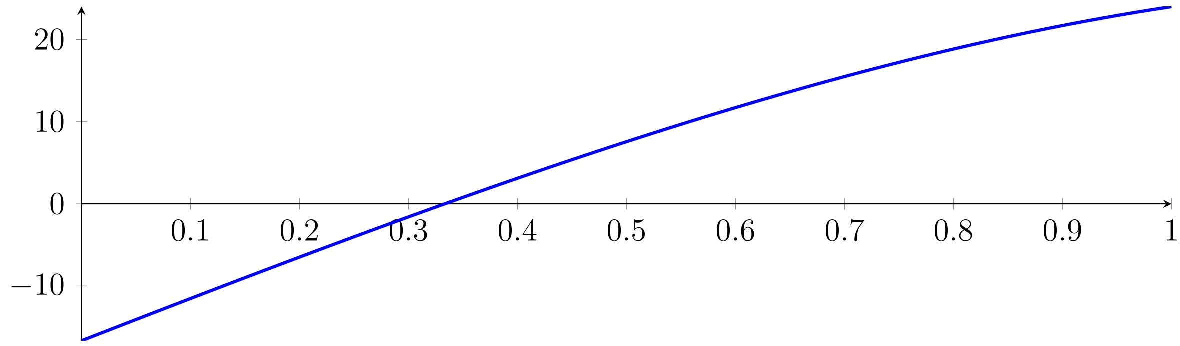
119.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 6!
p(0) = -4.00031e-19 - p(1) -8.9255e-20

119.7 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

with precision $\varepsilon = 1 \cdot 10^{-32}$.

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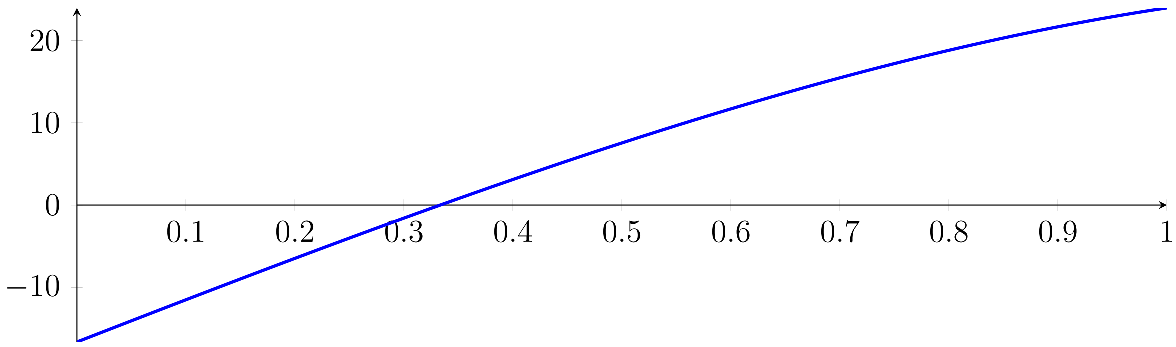
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

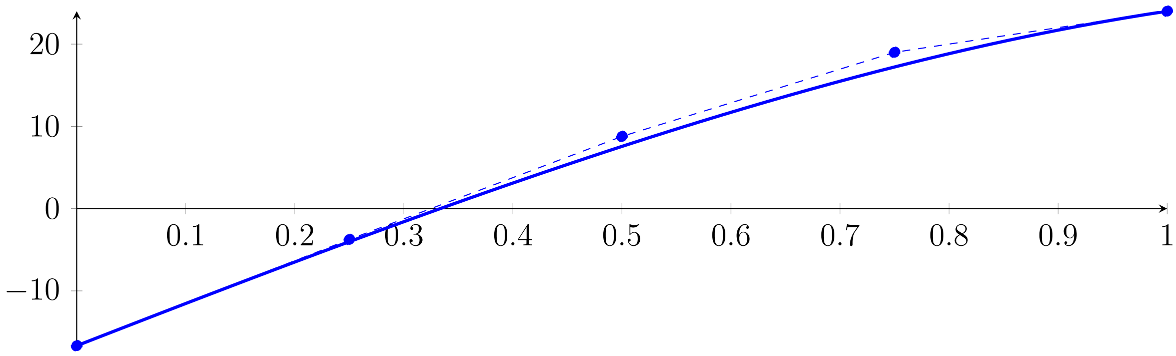
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



120.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

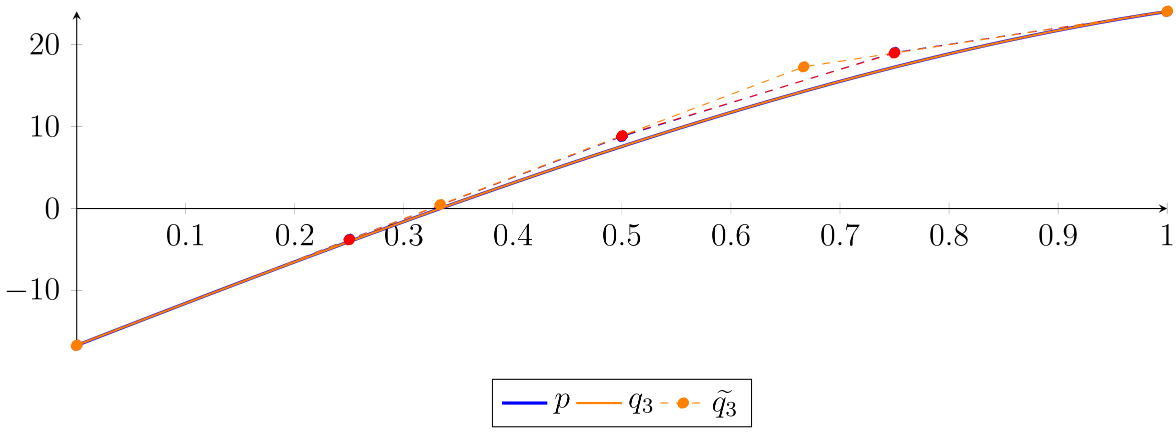
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= 3.34802 \cdot 10^{-16}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



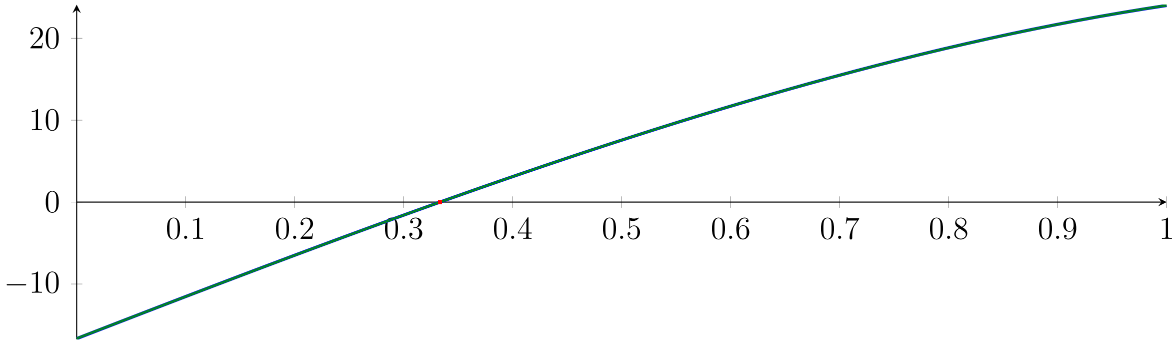
The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



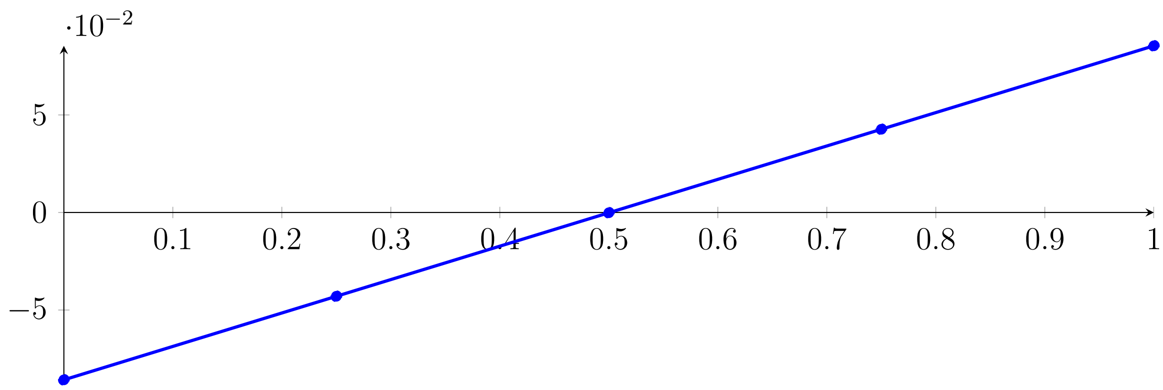
$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

120.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

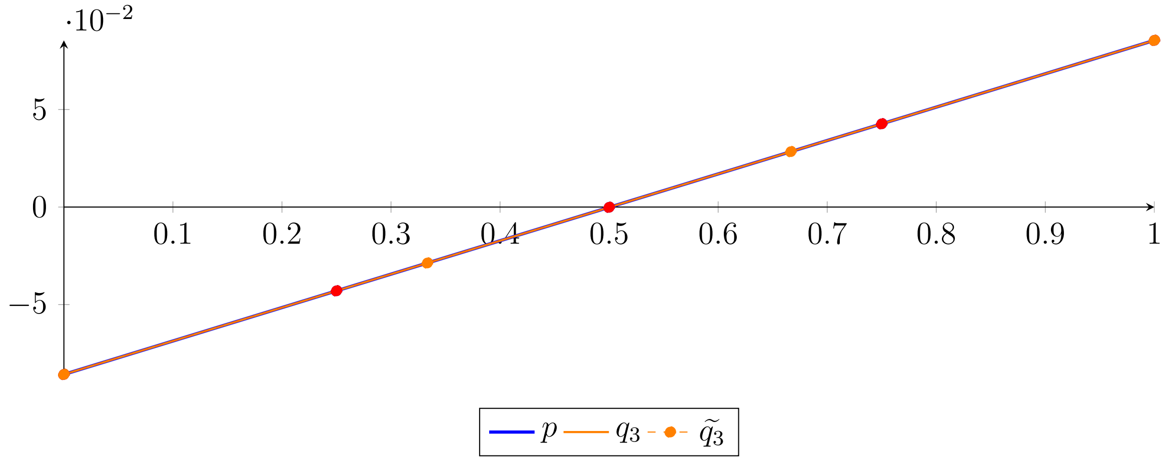
$$p = -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948$$
$$= -0.0857948B_{0,4}(X) - 0.0429507B_{1,4}(X) - 0.000129666B_{2,4}(X)$$
$$+ 0.0426682B_{3,4}(X) + 0.0854427B_{4,4}(X)$$



Degree reduction and raising:

$$q_3 = -4.2413 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948$$
$$= -0.0857948B_{0,3} - 0.0286693B_{1,3} + 0.02841B_{2,3} + 0.0854427B_{3,3}$$

$$\widetilde{q}_3 = 1.99222 \cdot 10^{-18} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948$$
$$= -0.0857948B_{0,4} - 0.0429507B_{1,4} - 0.000129666B_{2,4} + 0.0426682B_{3,4} + 0.0854427B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m:

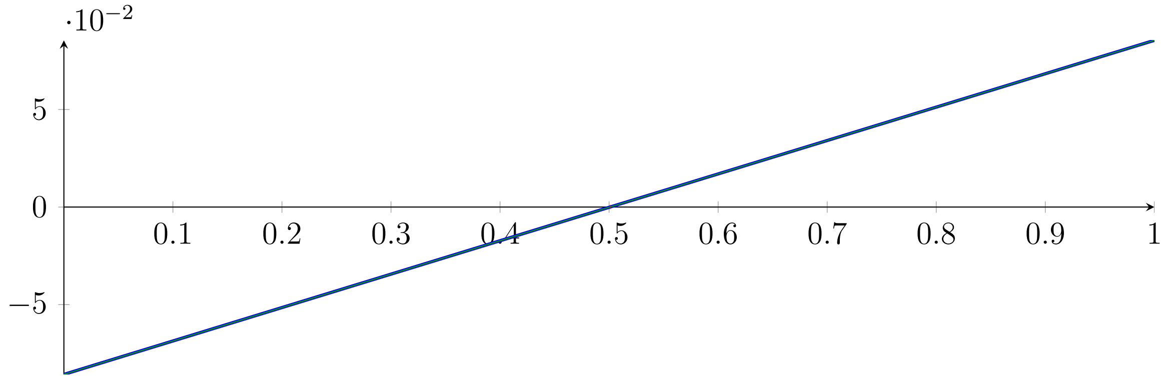
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

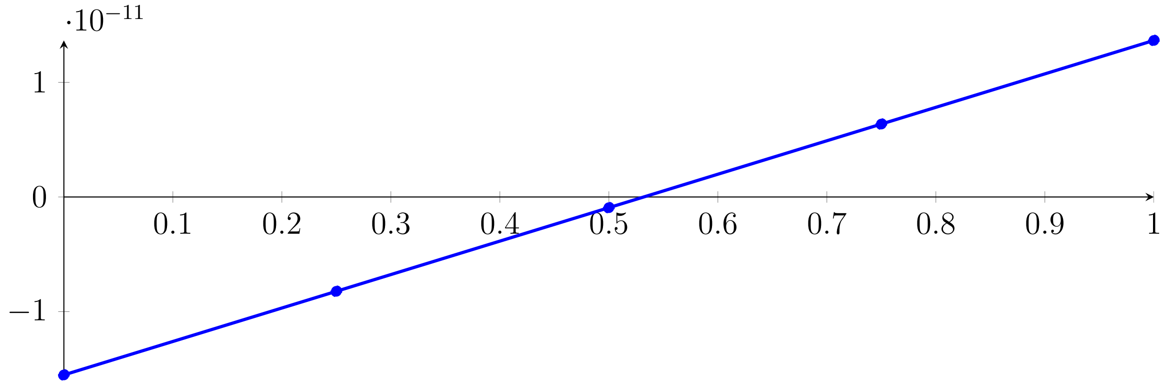
Longest intersection interval: $1.7041 \cdot 10^{-10}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

120.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

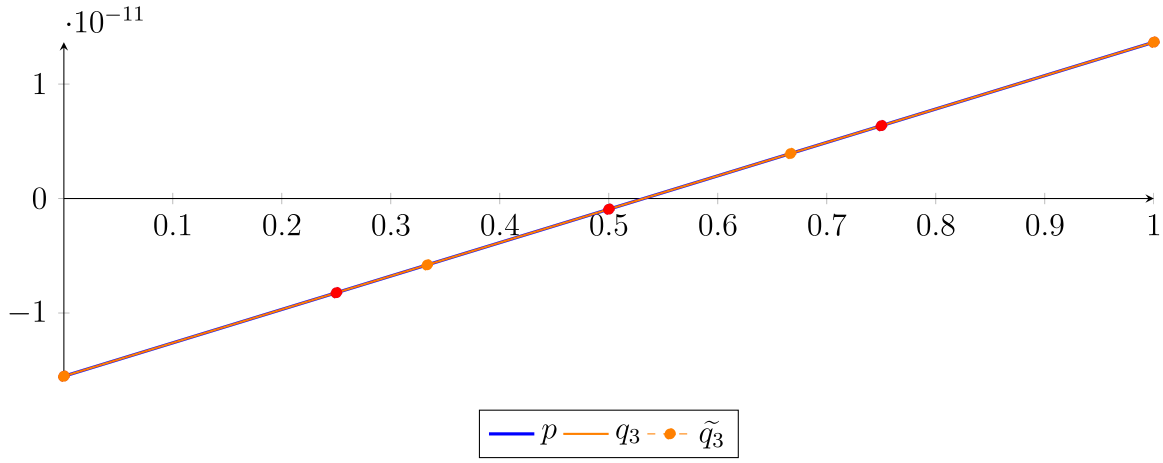
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\ &= -1.55233 \cdot 10^{-11} B_{0,4}(X) - 8.22817 \cdot 10^{-12} B_{1,4}(X) - 9.33054 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36206 \cdot 10^{-12} B_{3,4}(X) + 1.36572 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}$$
$$= -1.55233 \cdot 10^{-11} B_{0,3} - 5.79647 \cdot 10^{-12} B_{1,3} + 3.93036 \cdot 10^{-12} B_{2,3} + 1.36572 \cdot 10^{-11} B_{3,3}$$
$$\widetilde{q_3} = 3.84964 \cdot 10^{-28} X^4 - 2.1457 \cdot 10^{-28} X^3 - 4.04147 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}$$
$$= -1.55233 \cdot 10^{-11} B_{0,4} - 8.22817 \cdot 10^{-12} B_{1,4} - 9.33054 \cdot 10^{-13} B_{2,4} + 6.36206 \cdot 10^{-12} B_{3,4} + 1.36572 \cdot 10^{-11} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 1.13596 \cdot 10^{-28}$.

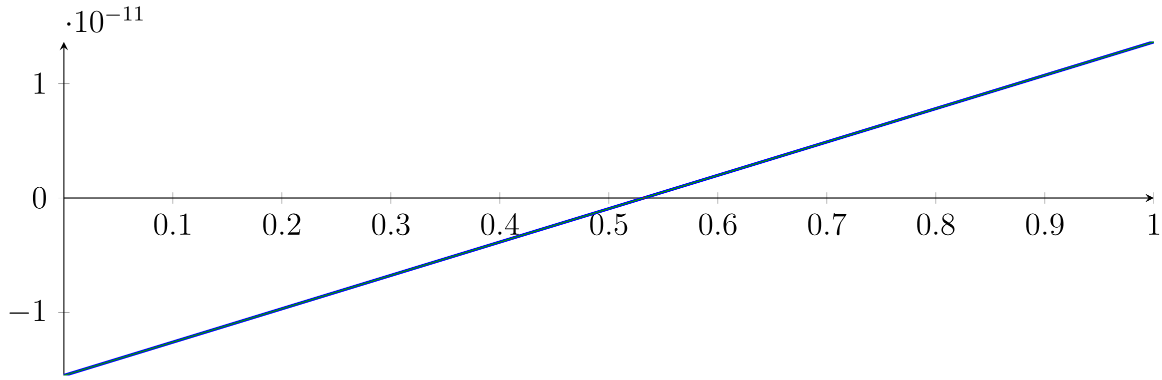
Bounding polynomials M and m:

$$M = 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}$$
$$m = 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}$$

Root of M and m:

$$N(M) = \{0.531975\} \qquad N(m) = \{0.531975\}$$

Intersection intervals:



$$[0.531975, 0.531975]$$

Longest intersection interval: 0

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

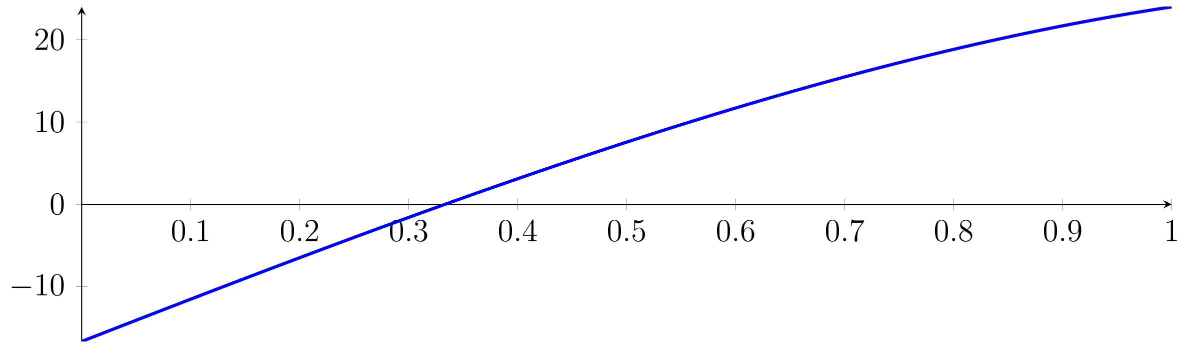
120.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

120.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

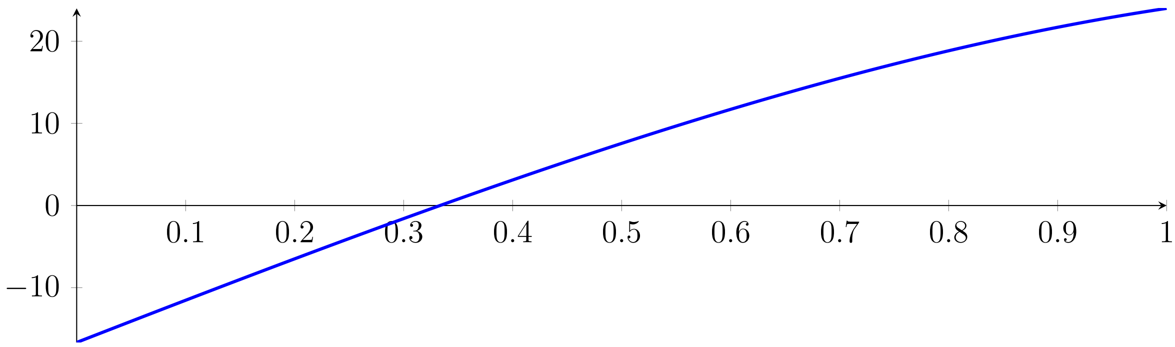
with precision $\varepsilon = 1 \cdot 10^{-32}$.

121 Running `BezClip` on f_4 with epsilon 64

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

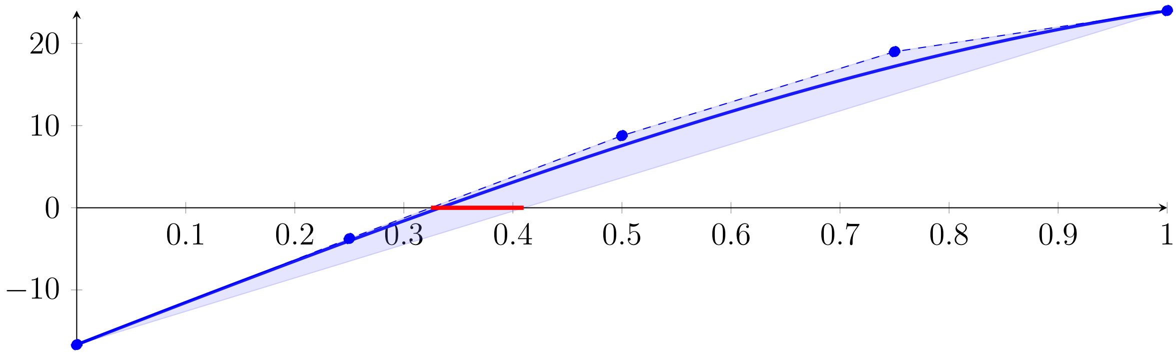
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



121.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$
$$= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

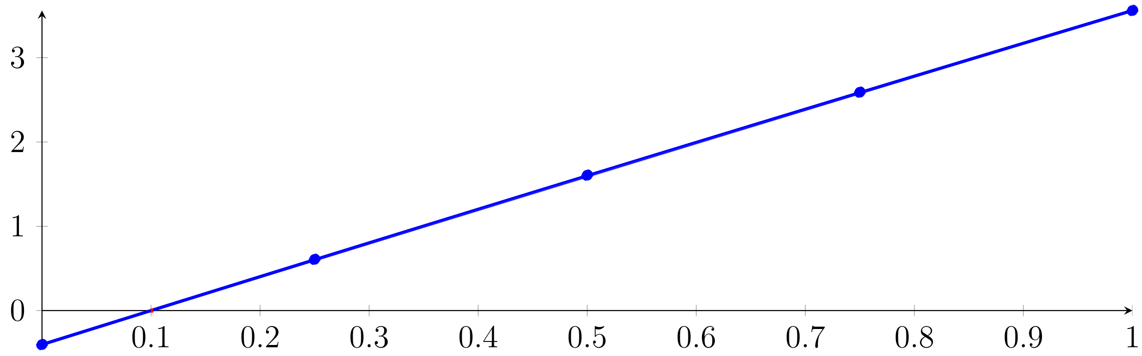
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

121.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711$$
$$= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

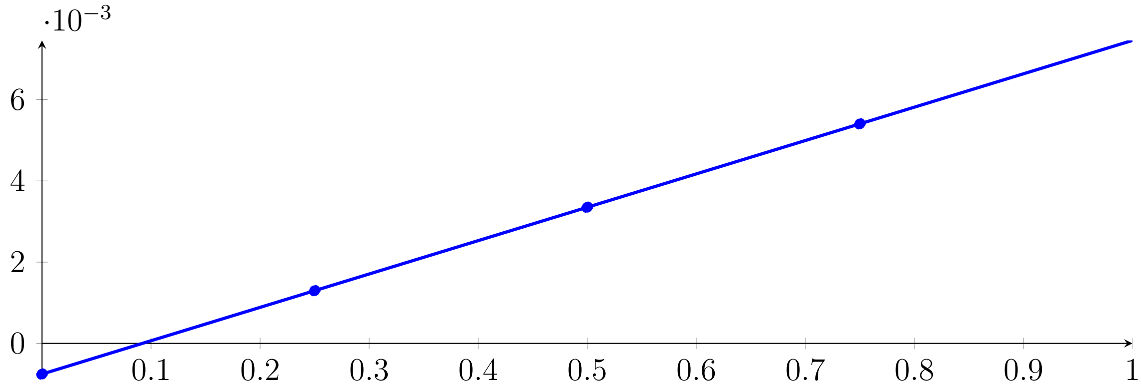
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: [0.333317, 0.333491],

121.3 Recursion Branch 1 1 1 in Interval 1: [0.333317, 0.333491]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01974 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

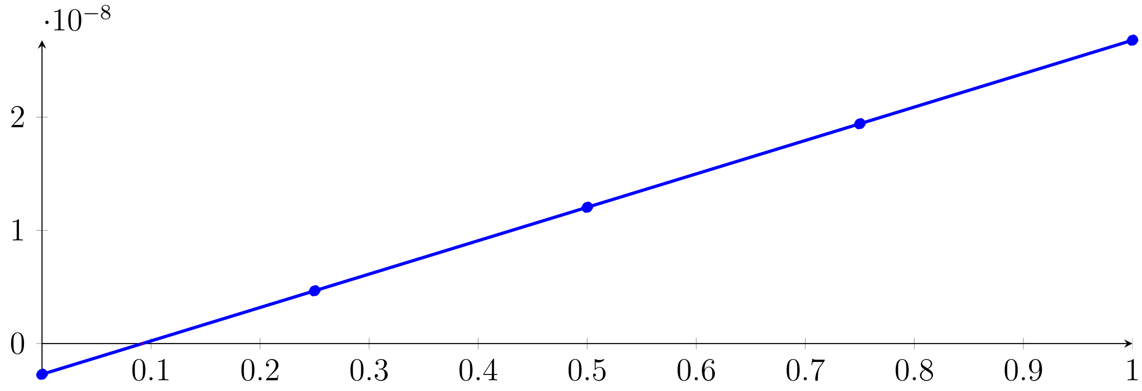
$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: [0.333333, 0.333333],

121.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.61559 \cdot 10^{-27} X^4 - 3.23117 \cdot 10^{-27} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09}$$
$$= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28975 \cdot 10^{-11}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

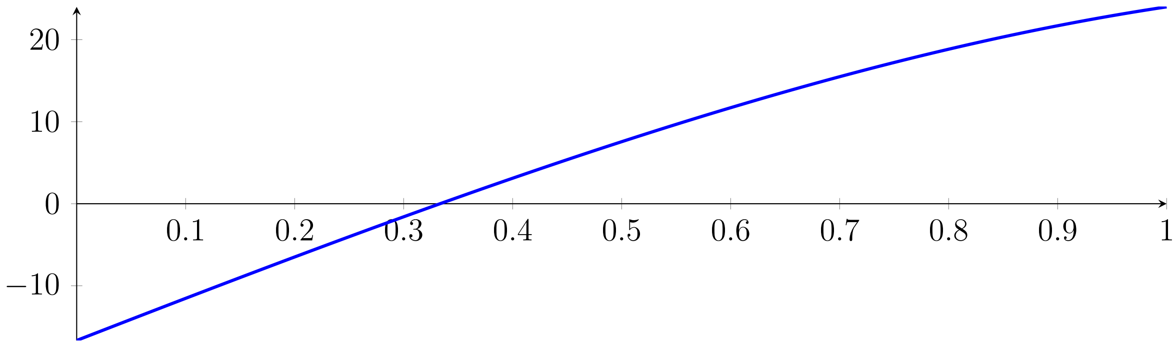
121.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

121.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-64}$.

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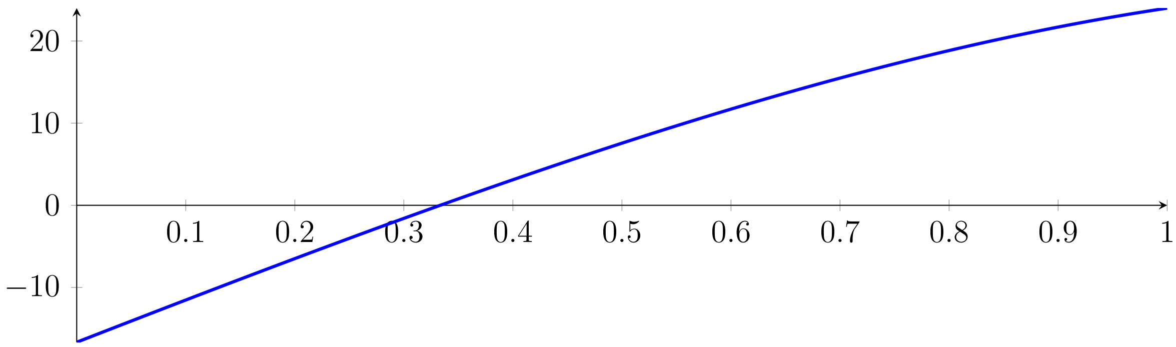
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0,1]$:

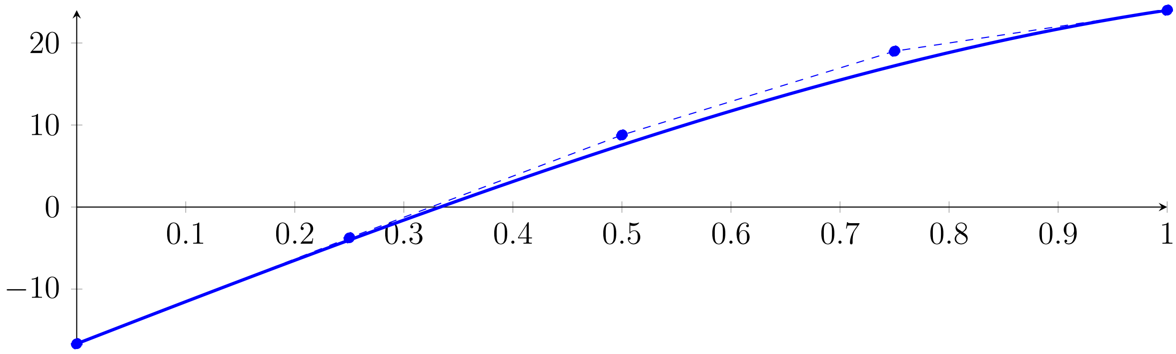
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



122.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

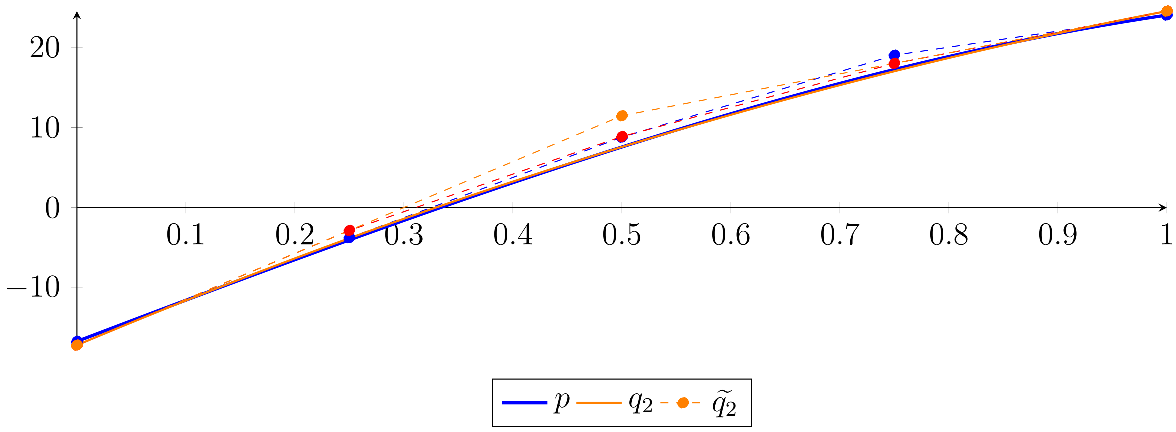
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.45023 \cdot 10^{-15}X^4 + 4.00374 \cdot 10^{-15}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

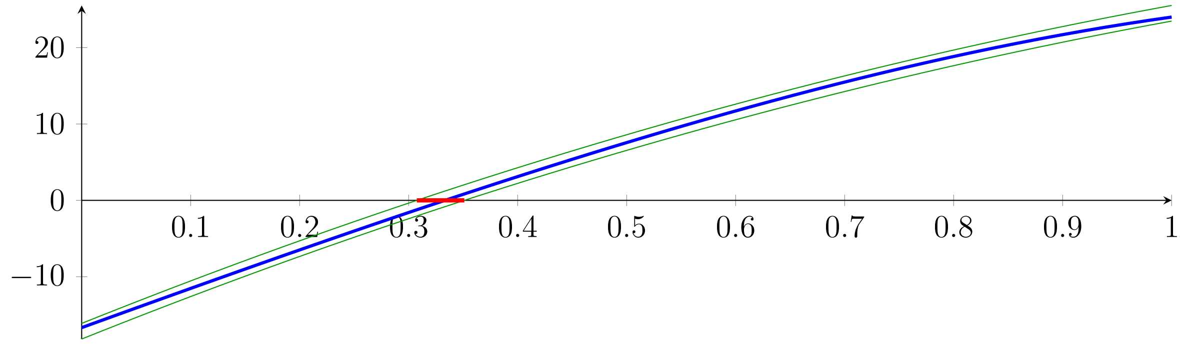
$$M = -15.5476X^2 + 57.181X - 16.1119$$

$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



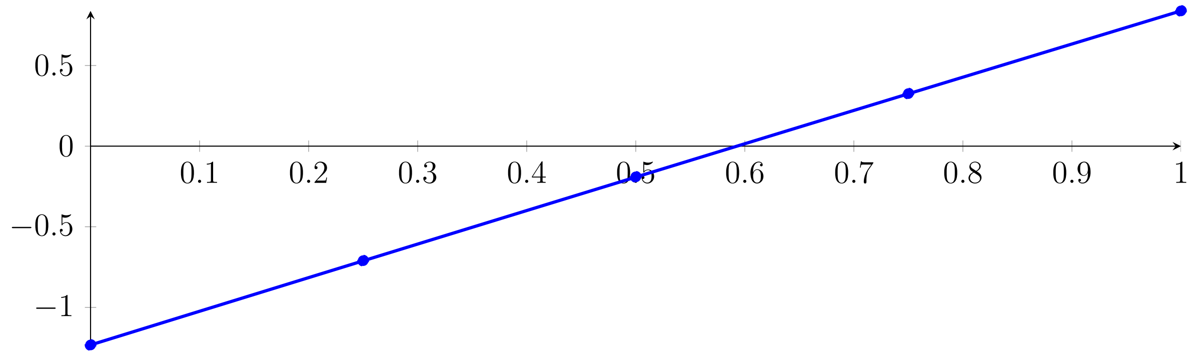
$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

122.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

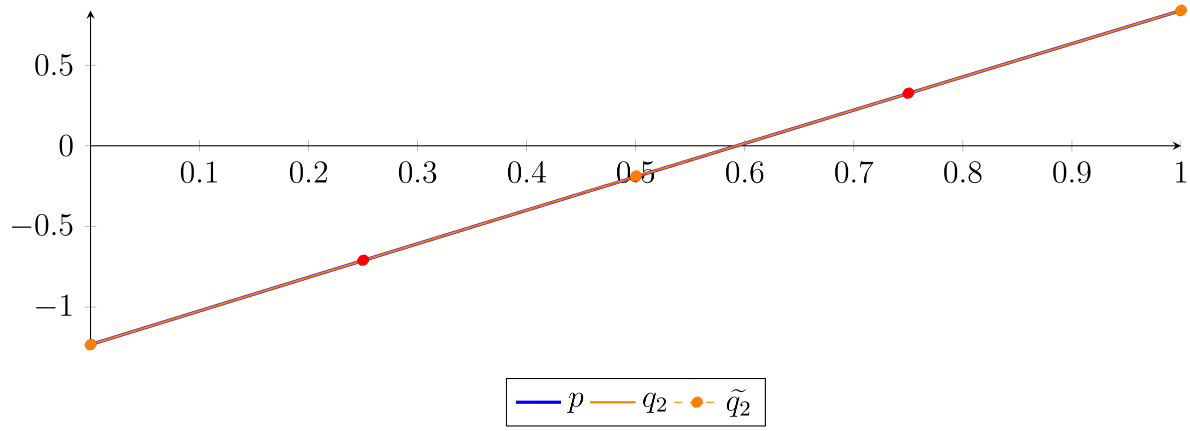
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06}X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.45283 \cdot 10^{-17}X^4 + 3.33934 \cdot 10^{-17}X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



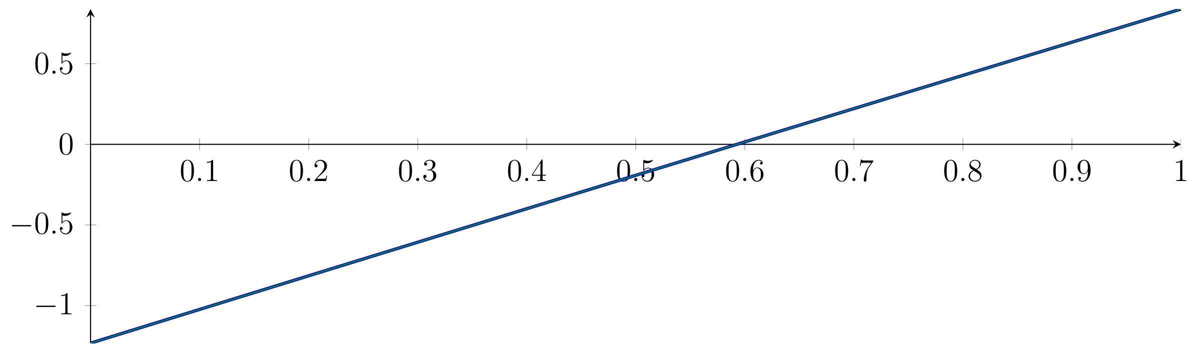
The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.
Bounding polynomials M and m:

$$M = -0.020089X^2 + 2.09166X - 1.23274$$
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



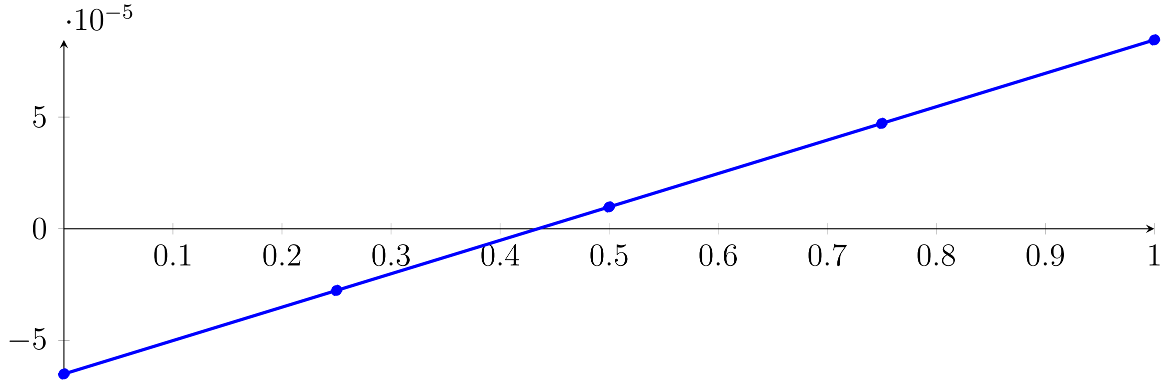
$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

122.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

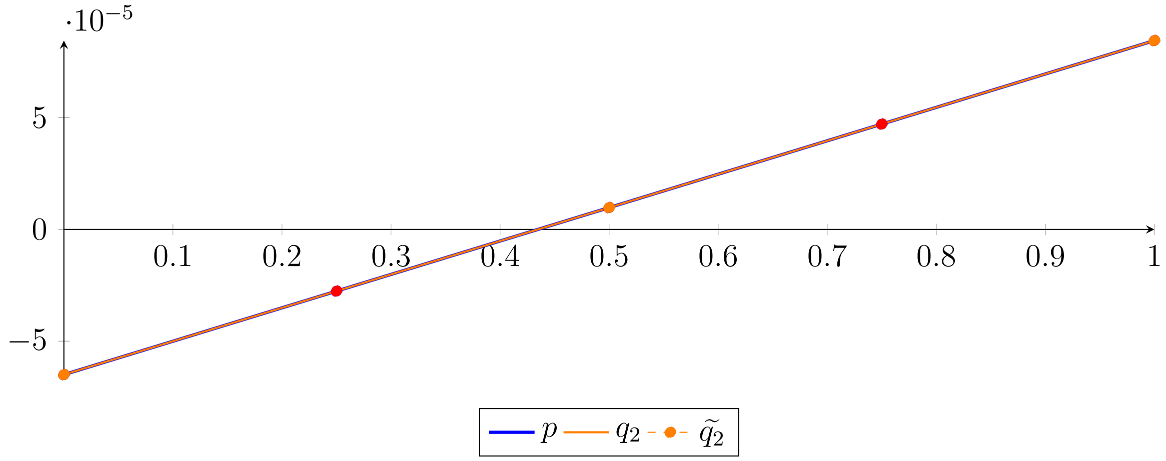
$$p = -1.05879 \cdot 10^{-22} X^4 - 2.82525 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.06147\cdot 10^{-10}X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$
$$= -6.50069\cdot 10^{-05}B_{0,2} + 9.76779\cdot 10^{-06}B_{1,2} + 8.45424\cdot 10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = -4.49986\cdot 10^{-22}X^4 + 3.33519\cdot 10^{-21}X^3 - 1.06147\cdot 10^{-10}X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$
$$= -6.50069\cdot 10^{-05}B_{0,4} - 2.76196\cdot 10^{-05}B_{1,4} + 9.76777\cdot 10^{-06}B_{2,4} + 4.71551\cdot 10^{-05}B_{3,4} + 8.45424\cdot 10^{-05}B_{4,4}$$



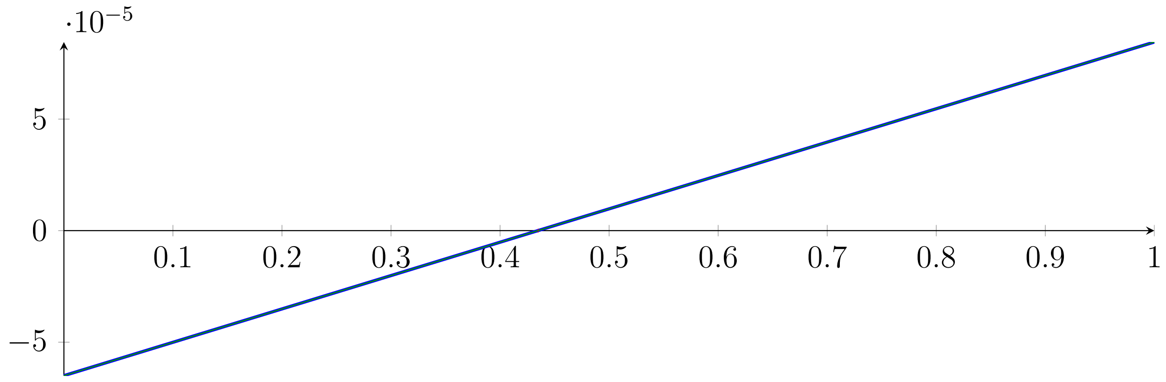
The maximum difference of the Bézier coefficients is $\delta = 2.82529\cdot 10^{-17}$.
Bounding polynomials M and m:

$$M = -1.06147\cdot 10^{-10}X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$
$$m = -1.06147\cdot 10^{-10}X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.434685, 1.4089\cdot 10^{6}\} \qquad N(m) = \{0.434685, 1.4089\cdot 10^{6}\}$$

Intersection intervals:



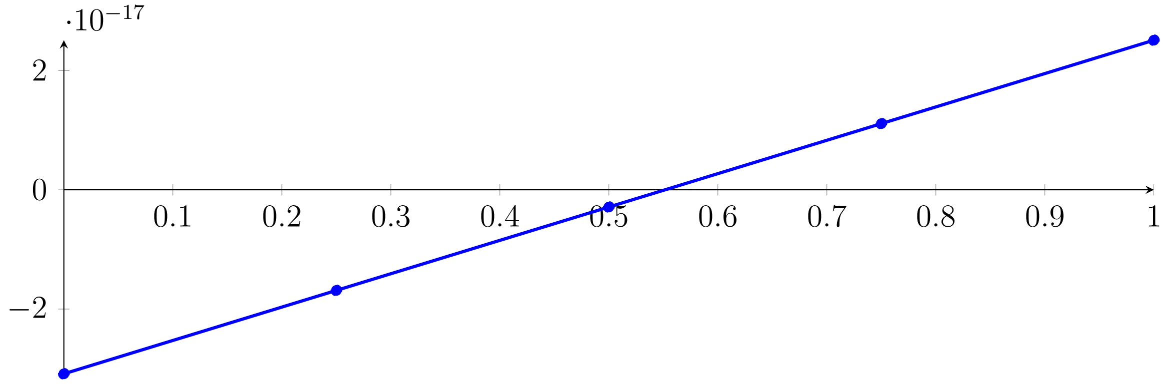
$$[0.434685, 0.434685]$$

Longest intersection interval: $3.74055\cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

122.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

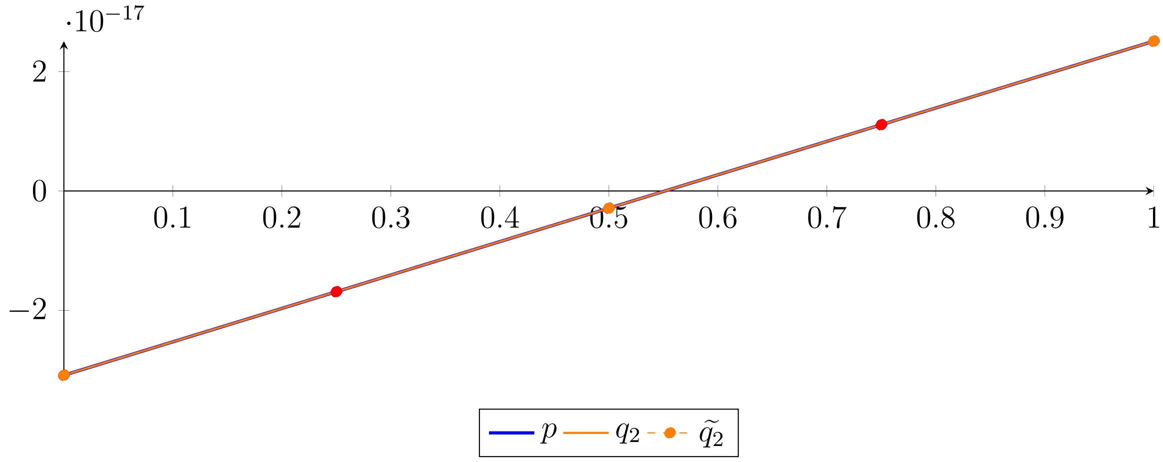
$$p = -1.20371 \cdot 10^{-35} X^3 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17}$$
$$= -3.08561 \cdot 10^{-17} B_{0,4}(X) - 1.68712 \cdot 10^{-17} B_{1,4}(X) - 2.88624 \cdot 10^{-18} B_{2,4}(X) + 1.10987 \cdot 10^{-17} B_{3,4}(X) + 2.50836 \cdot 10^{-17} B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = 1.50463 \cdot 10^{-35} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17}$$
$$= -3.08561 \cdot 10^{-17} B_{0,2} - 2.88624 \cdot 10^{-18} B_{1,2} + 2.50836 \cdot 10^{-17} B_{2,2}$$

$$\widetilde{q}_2 = -3.03936 \cdot 10^{-34} X^4 + 9.14817 \cdot 10^{-34} X^3 - 6.31946 \cdot 10^{-34} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17}$$
$$= -3.08561 \cdot 10^{-17} B_{0,4} - 1.68712 \cdot 10^{-17} B_{1,4} - 2.88624 \cdot 10^{-18} B_{2,4} + 1.10987 \cdot 10^{-17} B_{3,4} + 2.50836 \cdot 10^{-17} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 7.14701 \cdot 10^{-35}$.

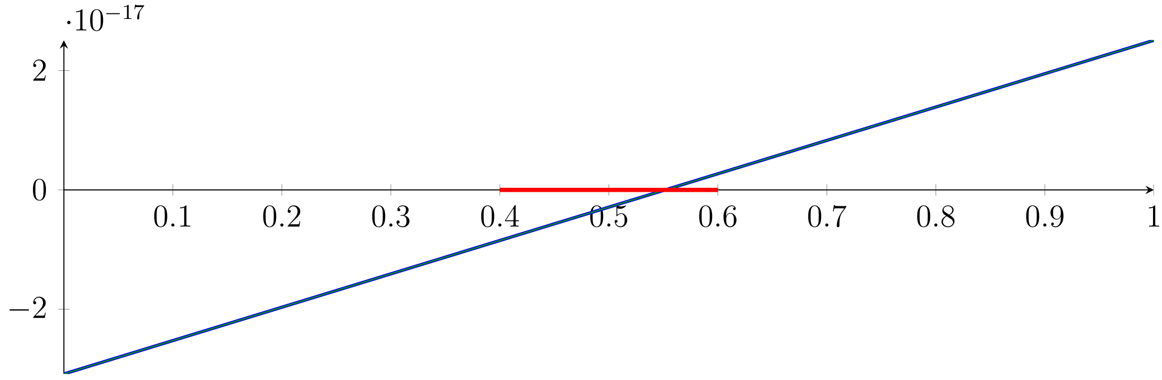
Bounding polynomials M and m:

$$M = 1.50463 \cdot 10^{-35} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17}$$
$$m = 1.50463 \cdot 10^{-35} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17}$$

Root of M and m:

$$N(M) = \{-3.71783 \cdot 10^{18}, 0.6\} \qquad N(m) = \{-3.71783 \cdot 10^{18}, 0.4\}$$

Intersection intervals:



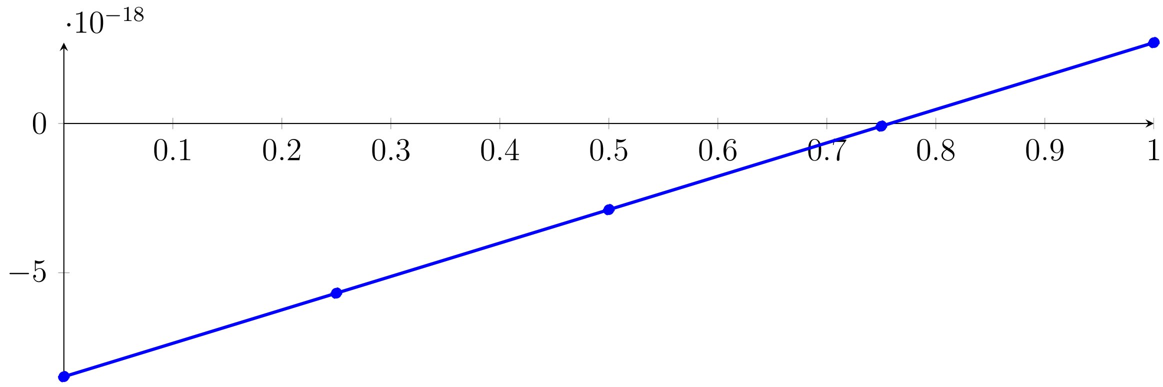
[0.4, 0.6]

Longest intersection interval: 0.2
⟹ Selective recursion: interval 1: [0.333333, 0.333333],

122.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

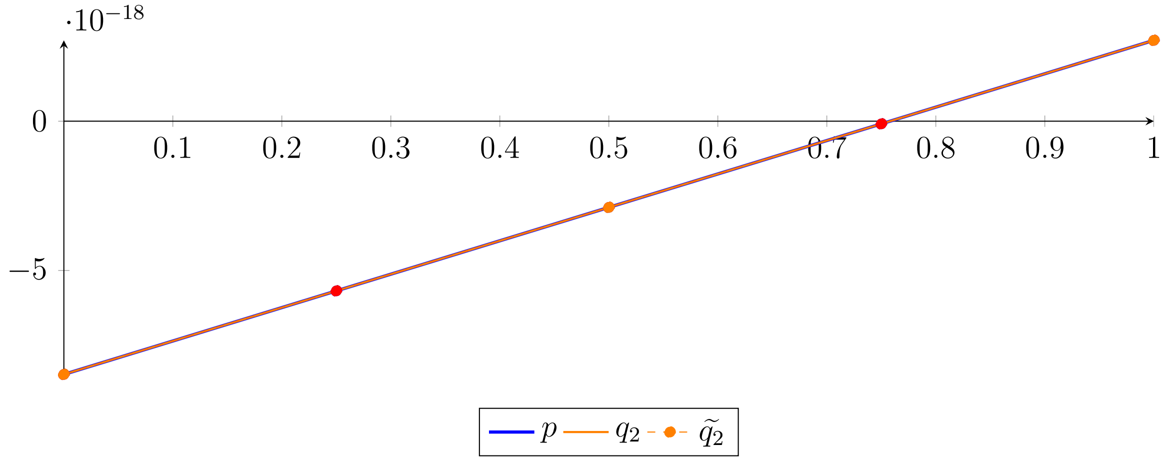
$$p = 1.50463 \cdot 10^{-36} X^4 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$
$$= -8.48022 \cdot 10^{-18} B_{0,4}(X) - 5.68323 \cdot 10^{-18} B_{1,4}(X) - 2.88624 \cdot 10^{-18} B_{2,4}(X) - 8.9255 \cdot 10^{-20} B_{3,4}(X) + 2.70773 \cdot 10^{-18} B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = 6.01853 \cdot 10^{-36} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$
$$= -8.48022 \cdot 10^{-18} B_{0,2} - 2.88624 \cdot 10^{-18} B_{1,2} + 2.70773 \cdot 10^{-18} B_{2,2}$$

$$\widetilde{q}_2 = -1.06829 \cdot 10^{-34} X^4 + 6.62038 \cdot 10^{-35} X^3 + 7.67363 \cdot 10^{-35} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$
$$= -8.48022 \cdot 10^{-18} B_{0,4} - 5.68323 \cdot 10^{-18} B_{1,4} - 2.88624 \cdot 10^{-18} B_{2,4} - 8.9255 \cdot 10^{-20} B_{3,4} + 2.70773 \cdot 10^{-18} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.36039{\cdot}10^{-35}$.
Bounding polynomials M and m:

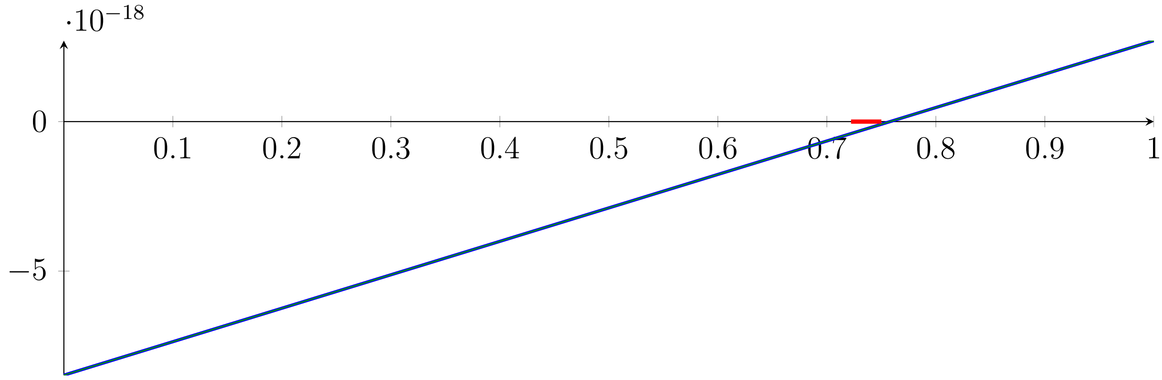
$$M = 6.01853{\cdot}10^{-36}X^2 + 1.11879{\cdot}10^{-17}X - 8.48022{\cdot}10^{-18}$$

$$m = 6.77085{\cdot}10^{-36}X^2 + 1.11879{\cdot}10^{-17}X - 8.48022{\cdot}10^{-18}$$

Root of M and m:

$$N(M) = \{-1.85892{\cdot}10^{18}, 0.75\} \qquad N(m) = \{-1.65237{\cdot}10^{18}, 0.722222\}$$

Intersection intervals:



$$[0.722222, 0.75]$$

Longest intersection interval: 0.0277778
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

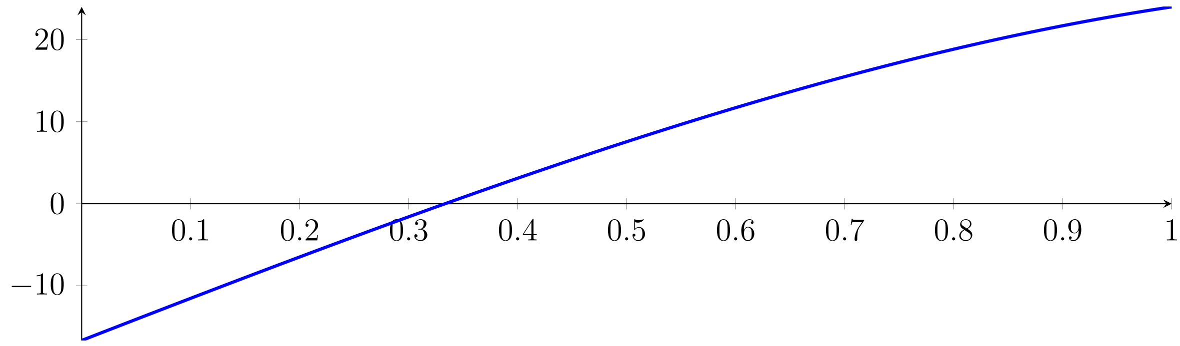
122.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 6!
p(0) = -4.00031e-19 - p(1) -8.9255e-20

122.7 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

with precision $\varepsilon = 1 \cdot 10^{-64}$.

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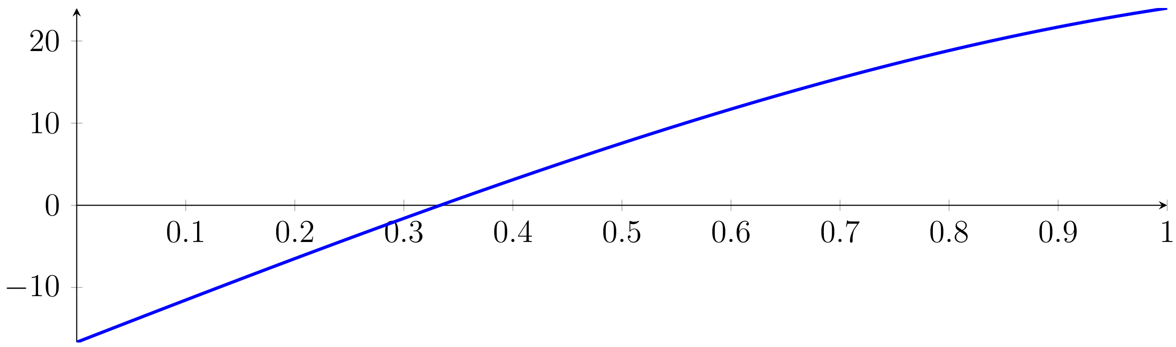
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0,1]$:

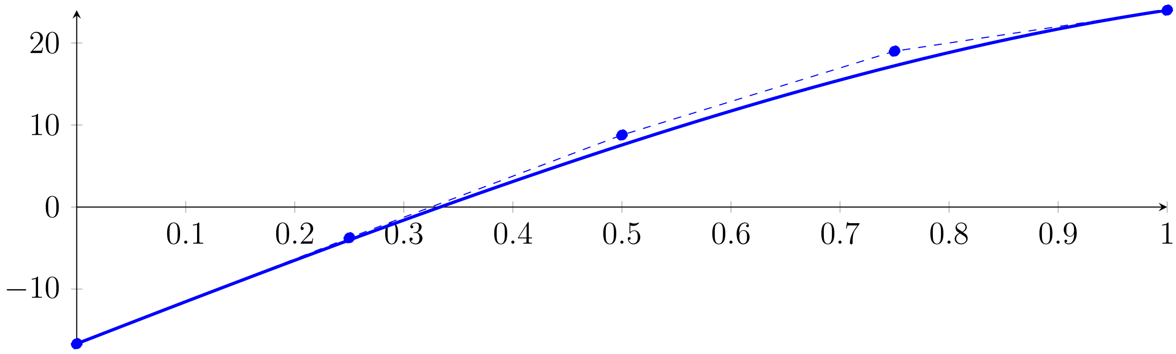
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



123.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

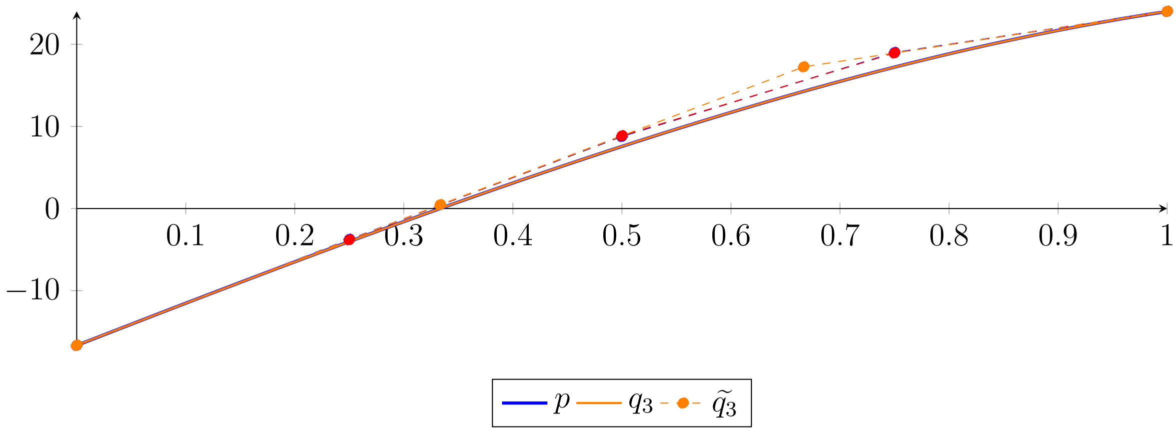
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$
$$= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X)$$



Degree reduction and raising:

$$q_3 = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524$$
$$= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3}$$

$$\tilde{q}_3 = 3.34802 \cdot 10^{-16}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524$$
$$= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4}$$



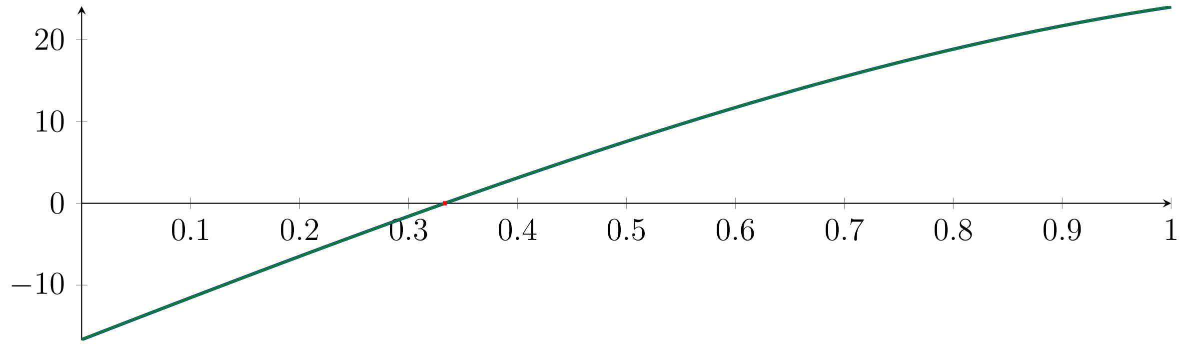
The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



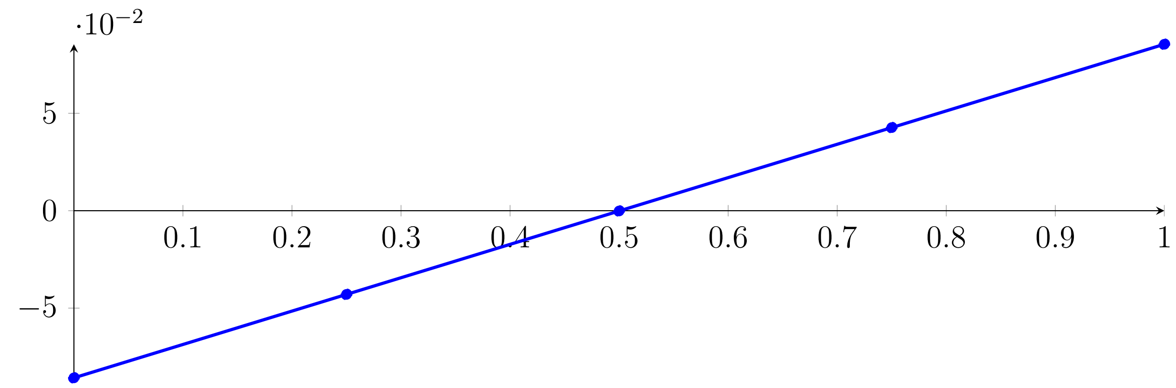
$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

123.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

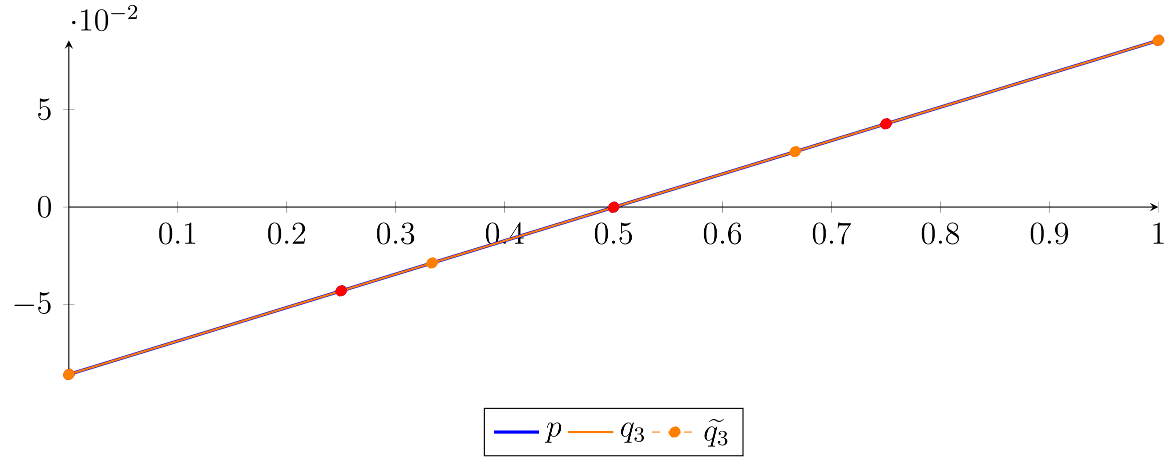
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4}(X) - 0.0429507B_{1,4}(X) - 0.000129666B_{2,4}(X) \\ &\quad + 0.0426682B_{3,4}(X) + 0.0854427B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,3} - 0.0286693B_{1,3} + 0.02841B_{2,3} + 0.0854427B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 1.99222 \cdot 10^{-18} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4} - 0.0429507B_{1,4} - 0.000129666B_{2,4} + 0.0426682B_{3,4} + 0.0854427B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m:

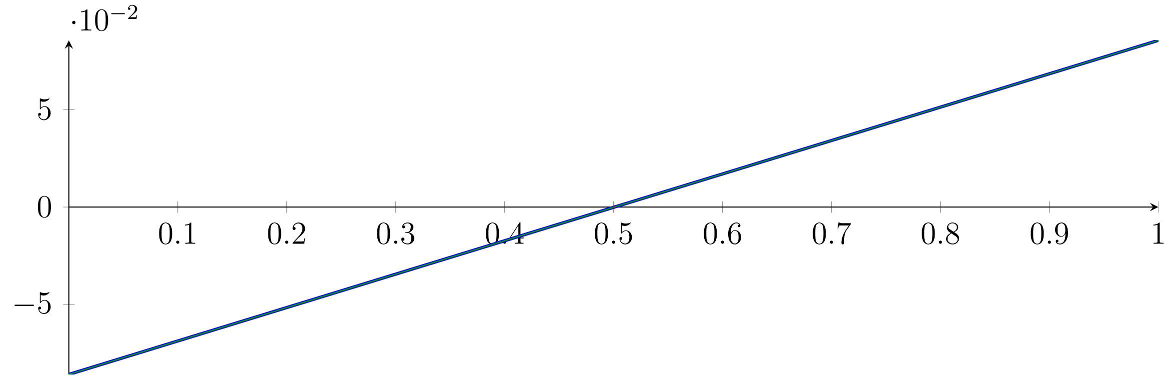
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

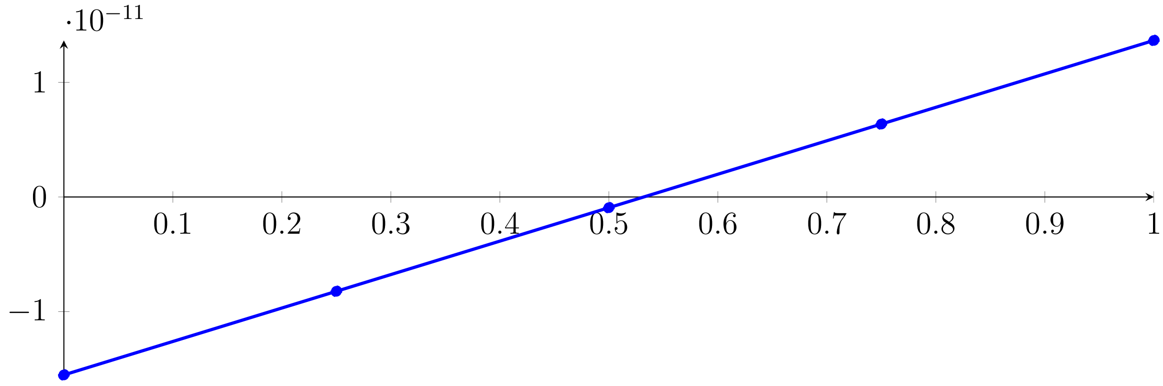
Longest intersection interval: $1.7041 \cdot 10^{-10}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

123.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

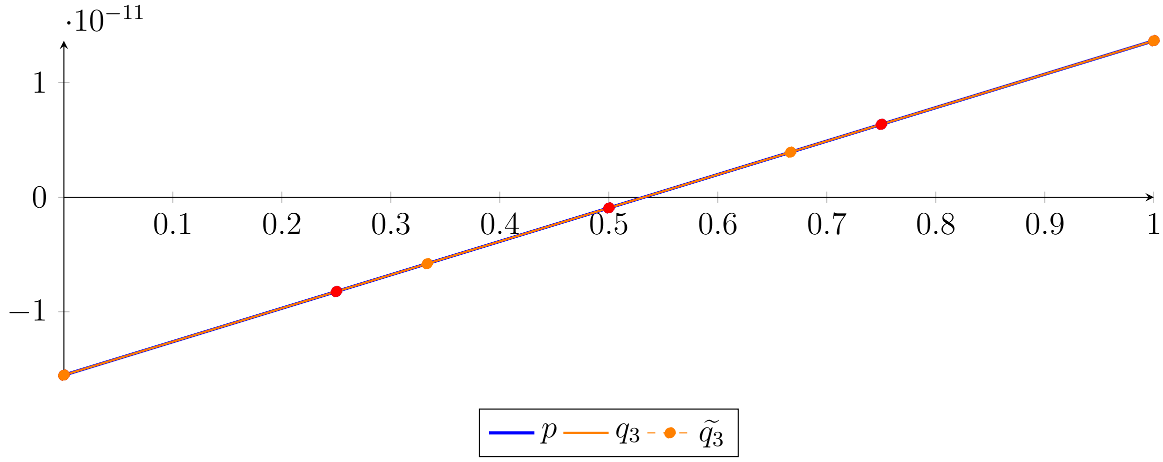
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\ &= -1.55233 \cdot 10^{-11} B_{0,4}(X) - 8.22817 \cdot 10^{-12} B_{1,4}(X) - 9.33054 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36206 \cdot 10^{-12} B_{3,4}(X) + 1.36572 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}$$
$$= -1.55233 \cdot 10^{-11} B_{0,3} - 5.79647 \cdot 10^{-12} B_{1,3} + 3.93036 \cdot 10^{-12} B_{2,3} + 1.36572 \cdot 10^{-11} B_{3,3}$$
$$\widetilde{q}_3 = 3.84964 \cdot 10^{-28} X^4 - 2.1457 \cdot 10^{-28} X^3 - 4.04147 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}$$
$$= -1.55233 \cdot 10^{-11} B_{0,4} - 8.22817 \cdot 10^{-12} B_{1,4} - 9.33054 \cdot 10^{-13} B_{2,4} + 6.36206 \cdot 10^{-12} B_{3,4} + 1.36572 \cdot 10^{-11} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 1.13596 \cdot 10^{-28}$.

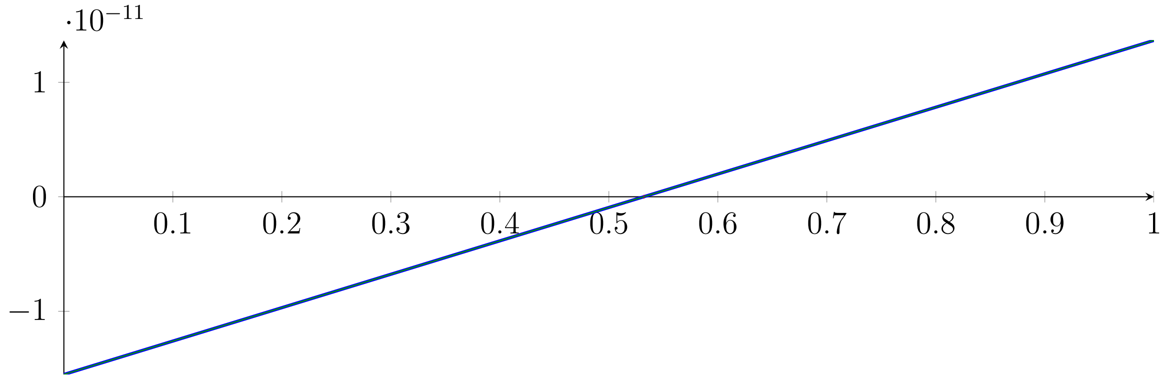
Bounding polynomials M and m:

$$M = 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}$$
$$m = 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}$$

Root of M and m:

$$N(M) = \{0.531975\} \qquad N(m) = \{0.531975\}$$

Intersection intervals:



$$[0.531975, 0.531975]$$

Longest intersection interval: 0

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

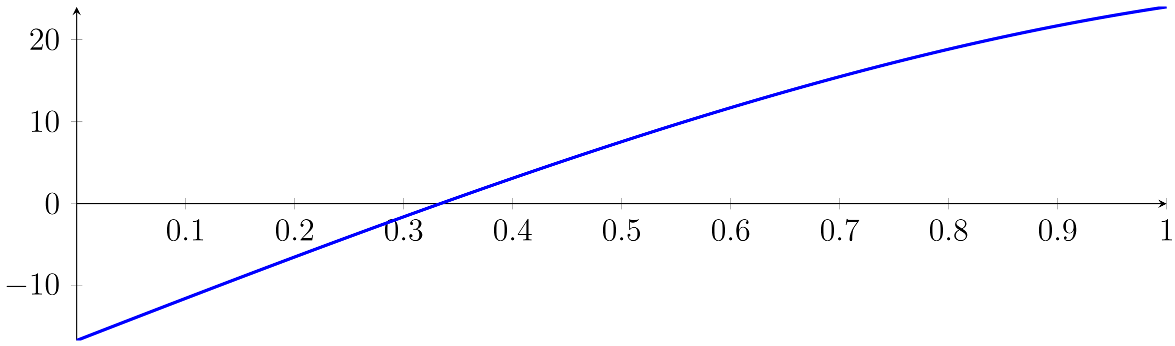
123.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

123.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-64}$.

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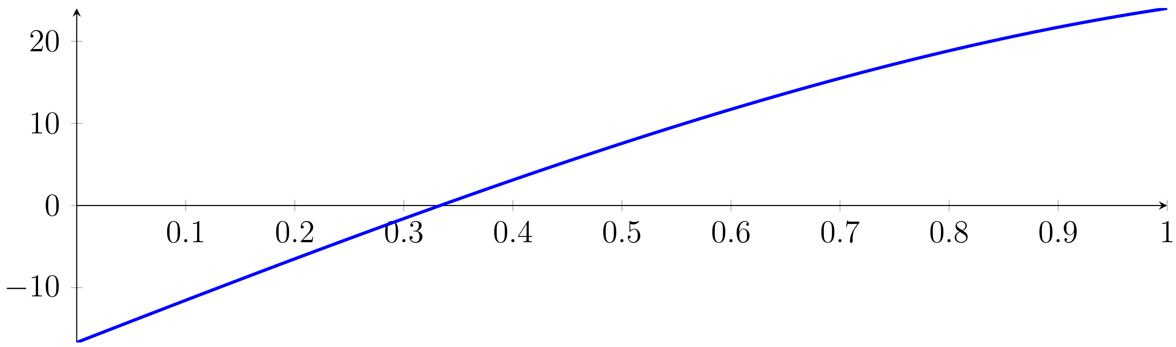
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

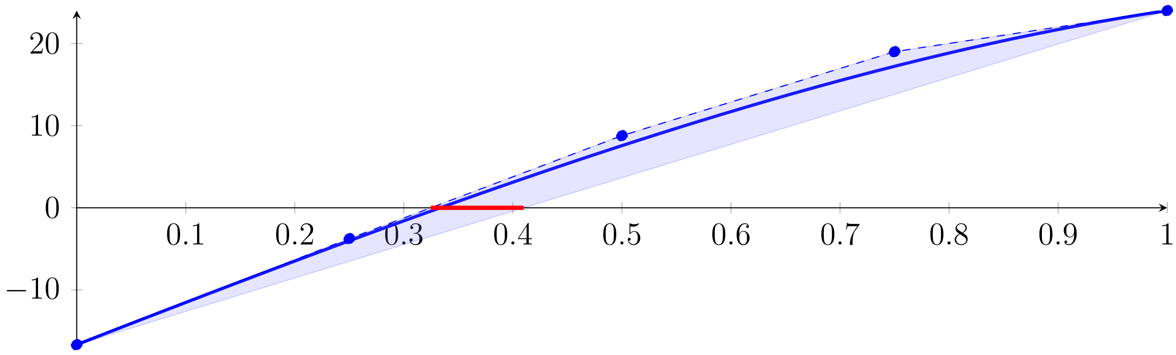
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



124.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

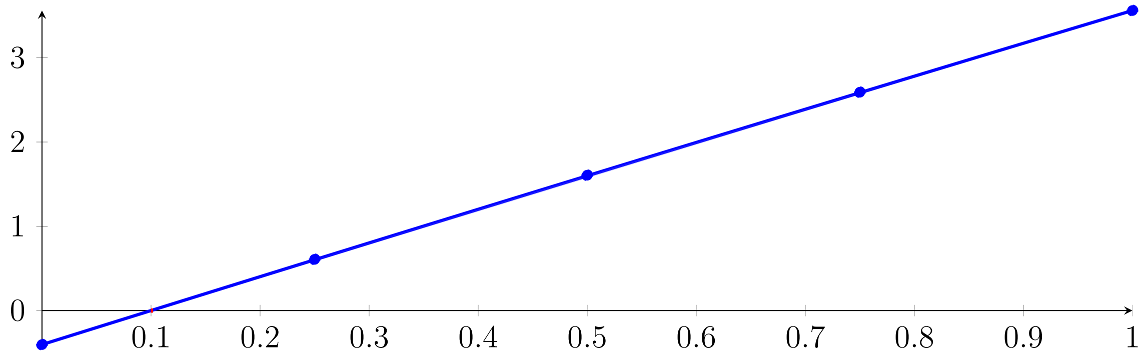
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

124.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

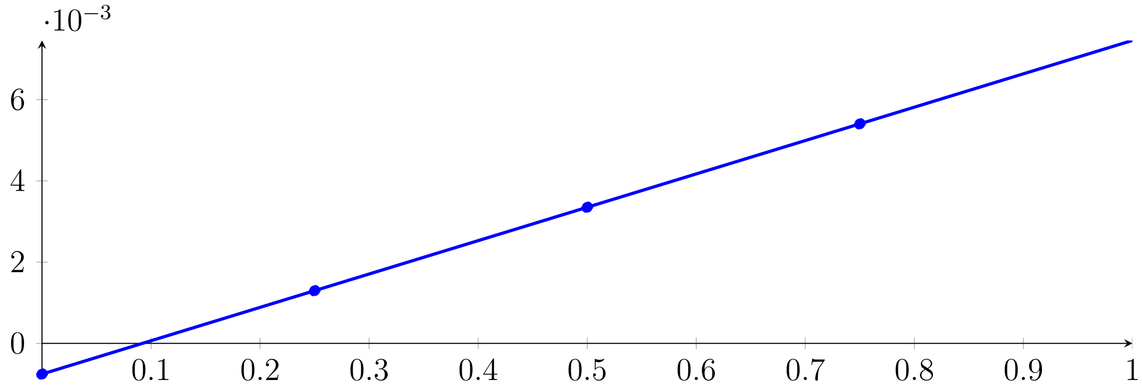
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: [0.333317, 0.333491],

124.3 Recursion Branch 1 1 1 in Interval 1: [0.333317, 0.333491]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -9.01974 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

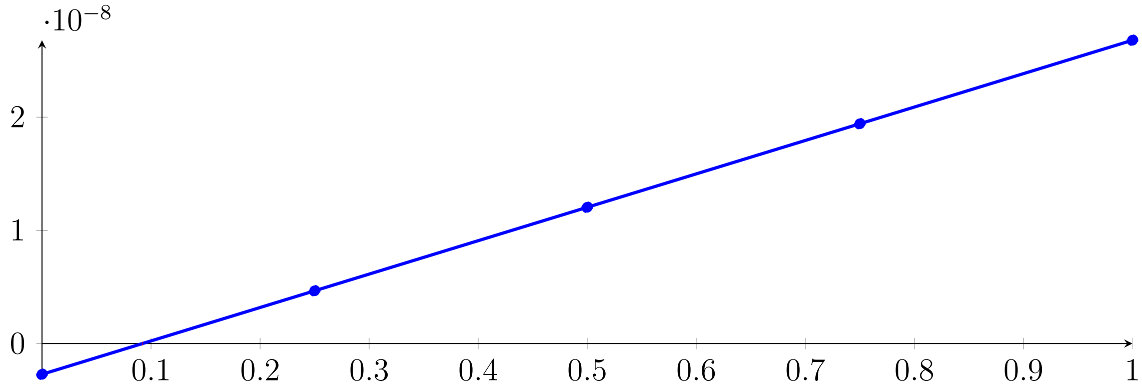
$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: [0.333333, 0.333333],

124.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.61559\cdot10^{-27}X^4 - 3.23117\cdot10^{-27}X^3 - 4.13296\cdot10^{-18}X^2 + 2.95096\cdot10^{-08}X - 2.71749\cdot10^{-09}$$
$$= -2.71749\cdot10^{-09}B_{0,4}(X) + 4.6599\cdot10^{-09}B_{1,4}(X) + 1.20373 \cdot10^{-08}B_{2,4}(X) + 1.94147\cdot10^{-08}B_{3,4}(X) + 2.67921\cdot10^{-08}B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28975\cdot10^{-11}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

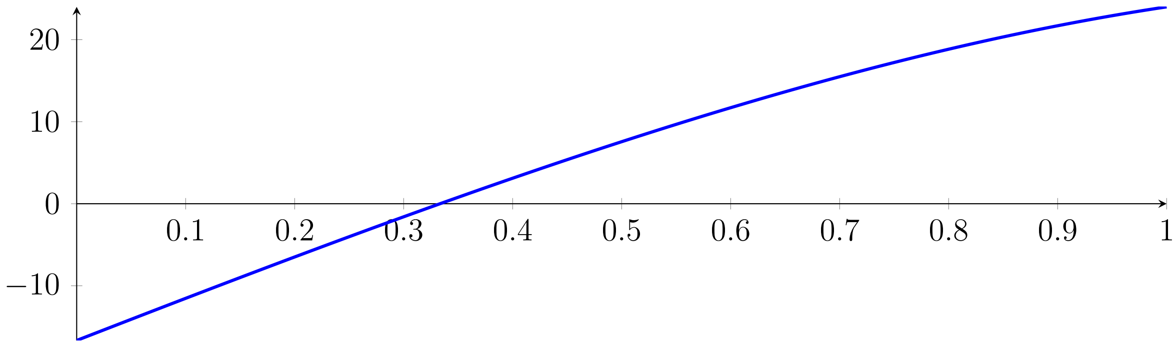
124.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

124.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-128}$.

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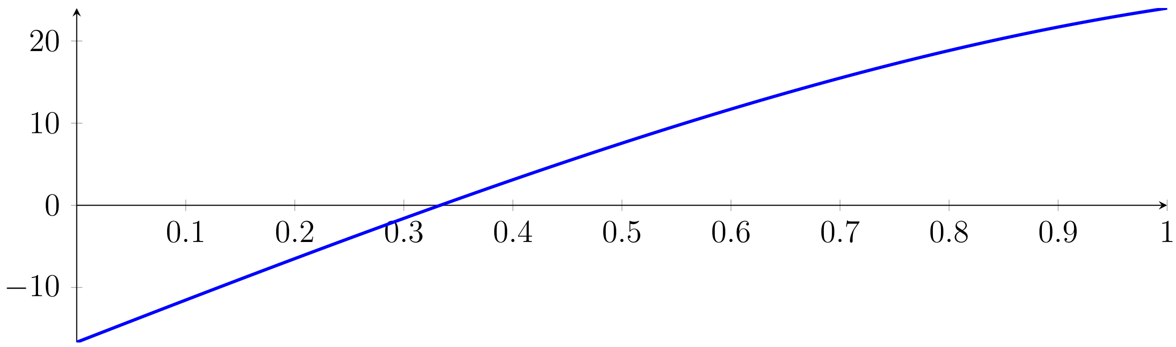
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called `QuadClip` with input polynomial on interval $[0, 1]$:

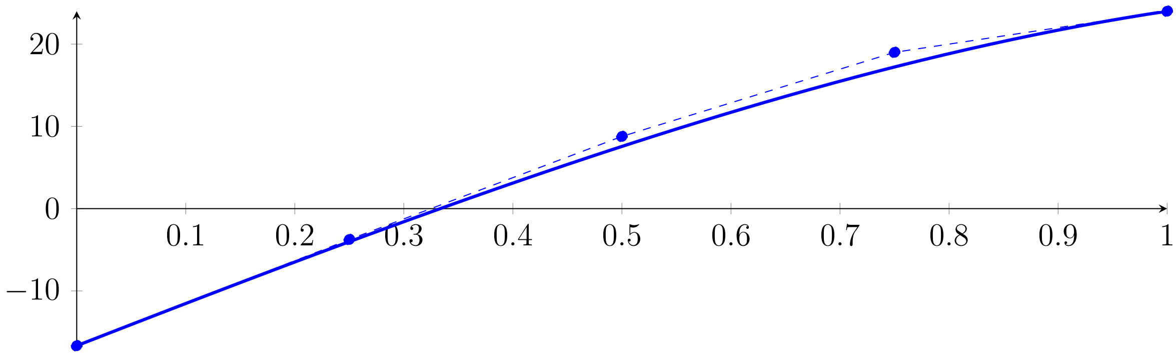
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



125.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

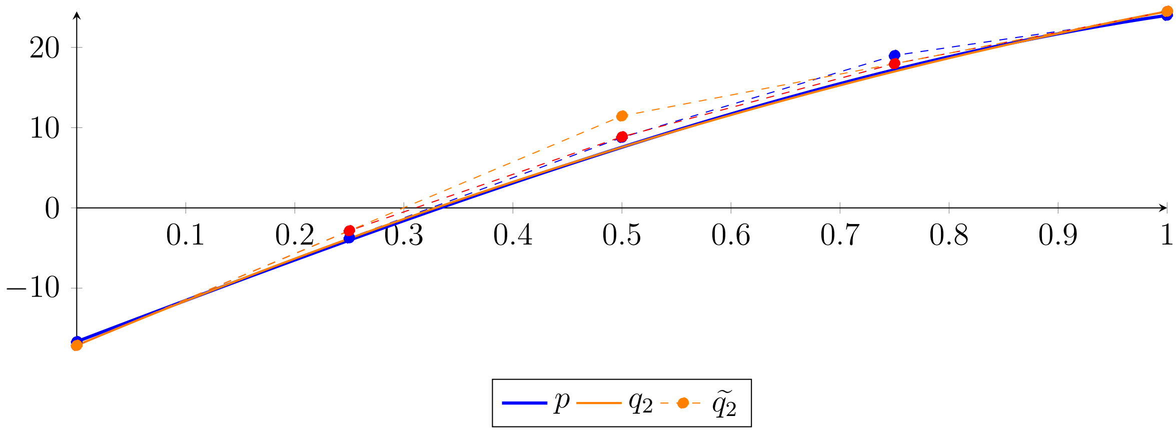
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1.45023\cdot 10^{-15}X^4 + 4.00374\cdot 10^{-15}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

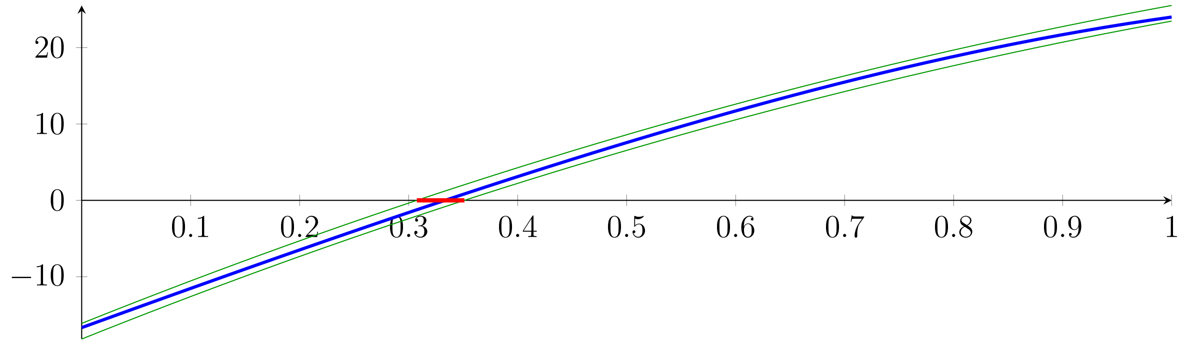
$$M = -15.5476X^2 + 57.181X - 16.1119$$

$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



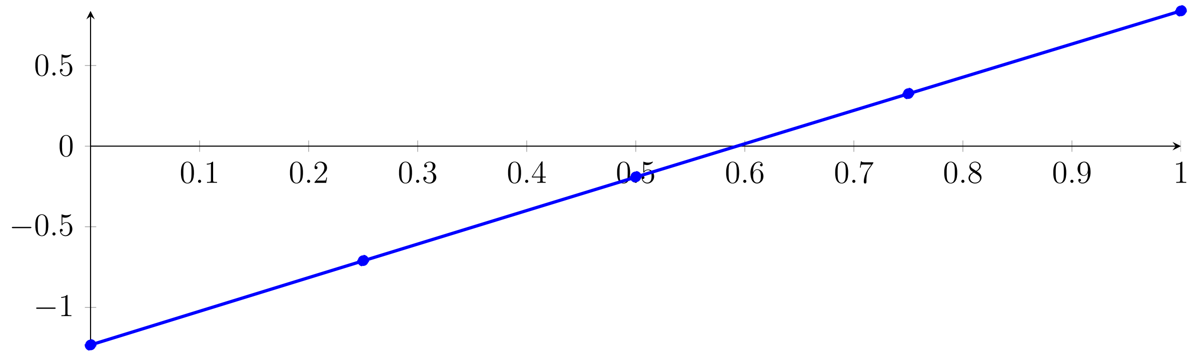
$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

125.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

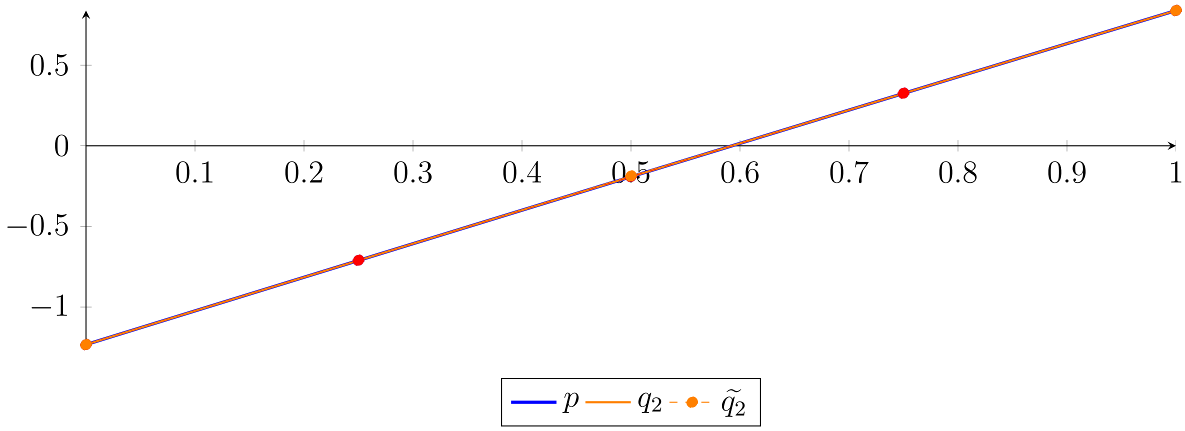
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1.45283 \cdot 10^{-17} X^4 + 3.33934 \cdot 10^{-17} X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

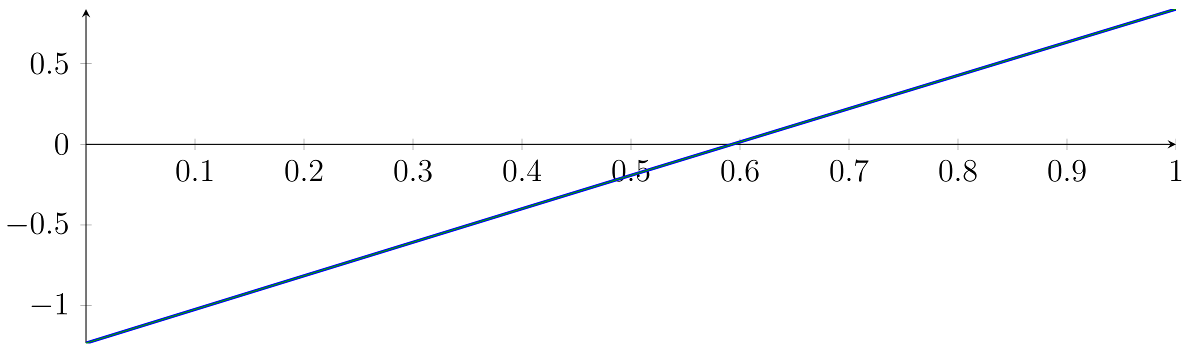
Bounding polynomials M and m:

$$M = -0.020089X^2 + 2.09166X - 1.23274$$
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

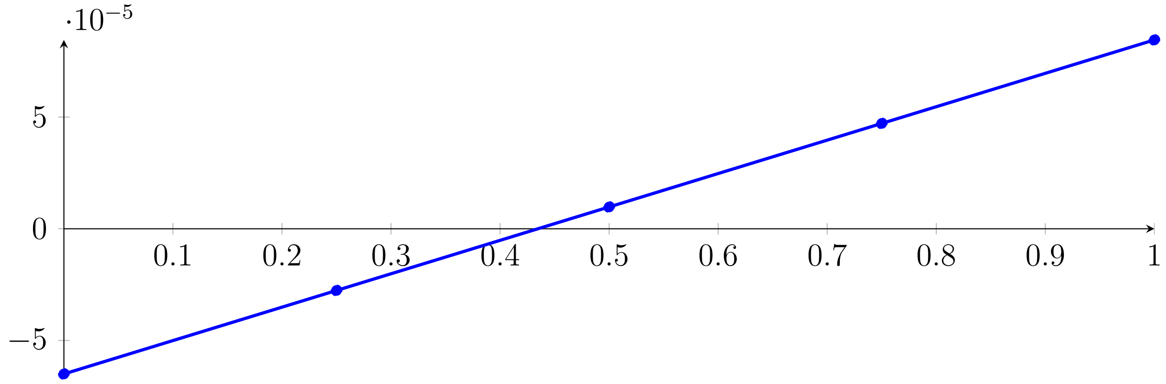
Longest intersection interval: $7.23183 \cdot 10^{-05}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

125.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

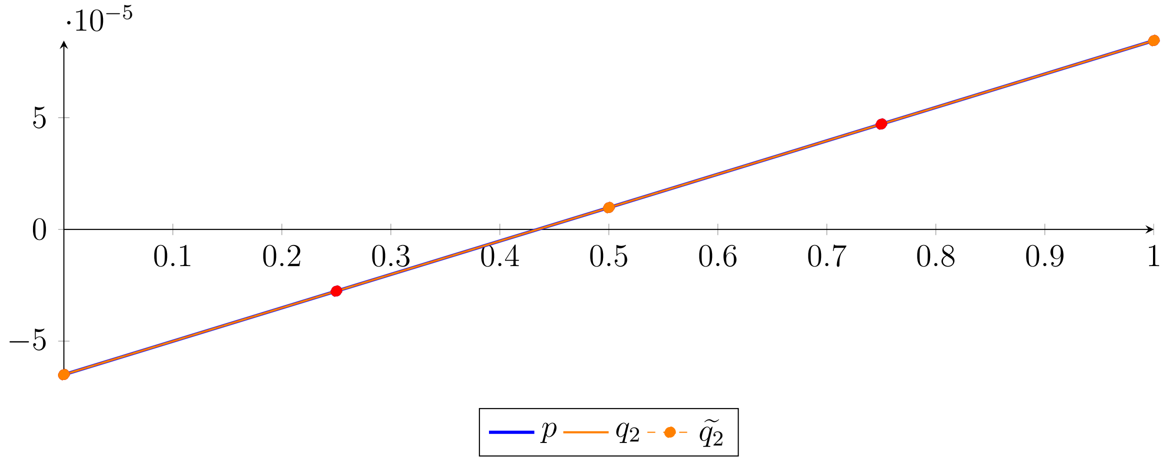
$$p = -1.05879 \cdot 10^{-22}X^4 - 2.82525 \cdot 10^{-16}X^3 - 1.06146 \cdot 10^{-10}X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05}B_{0,4}(X) - 2.76196 \cdot 10^{-05}B_{1,4}(X) + 9.76777$$
$$\cdot 10^{-06}B_{2,4}(X) + 4.71551 \cdot 10^{-05}B_{3,4}(X) + 8.45424 \cdot 10^{-05}B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.06147\cdot 10^{-10}X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$
$$= -6.50069\cdot 10^{-05}B_{0,2} + 9.76779\cdot 10^{-06}B_{1,2} + 8.45424\cdot 10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = -4.49986\cdot 10^{-22}X^4 + 3.33519\cdot 10^{-21}X^3 - 1.06147\cdot 10^{-10}X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$
$$= -6.50069\cdot 10^{-05}B_{0,4} - 2.76196\cdot 10^{-05}B_{1,4} + 9.76777\cdot 10^{-06}B_{2,4} + 4.71551\cdot 10^{-05}B_{3,4} + 8.45424\cdot 10^{-05}B_{4,4}$$



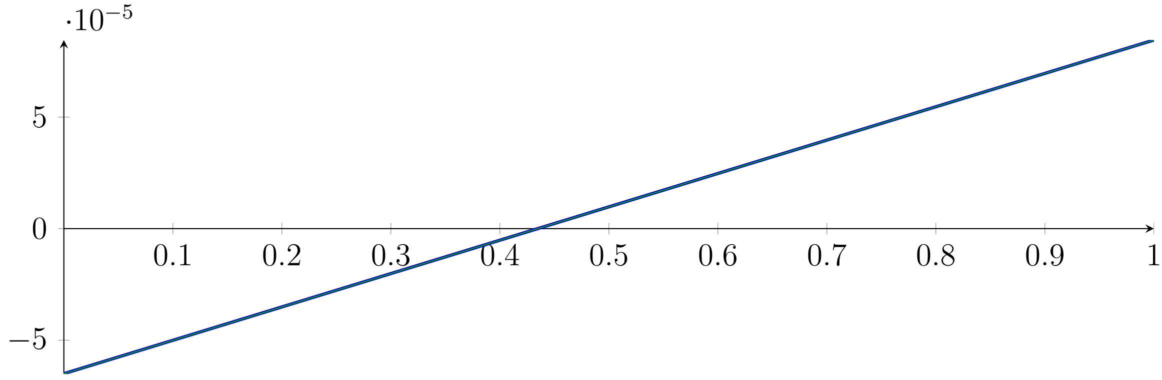
The maximum difference of the Bézier coefficients is $\delta = 2.82529\cdot 10^{-17}$.
Bounding polynomials M and m:

$$M = -1.06147\cdot 10^{-10}X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$
$$m = -1.06147\cdot 10^{-10}X^2 + 0.000149549X - 6.50069\cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.434685, 1.4089\cdot 10^{6}\} \qquad N(m) = \{0.434685, 1.4089\cdot 10^{6}\}$$

Intersection intervals:



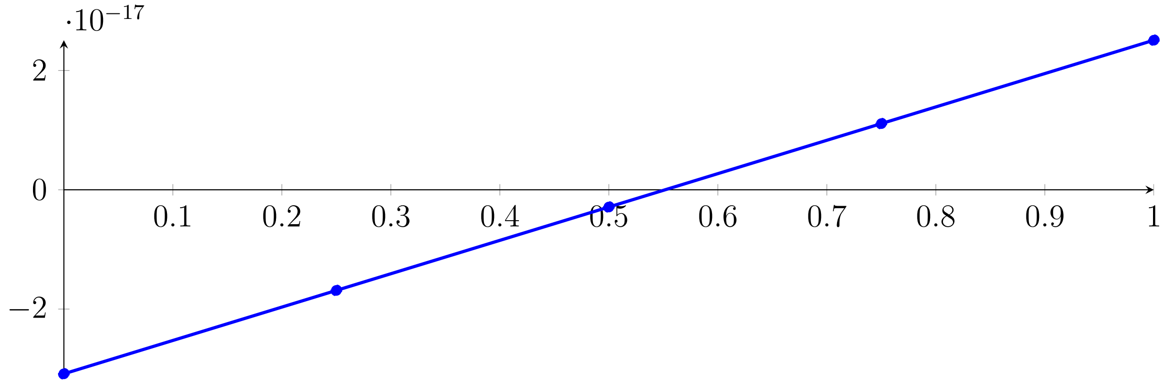
$$[0.434685, 0.434685]$$

Longest intersection interval: $3.74055\cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

125.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

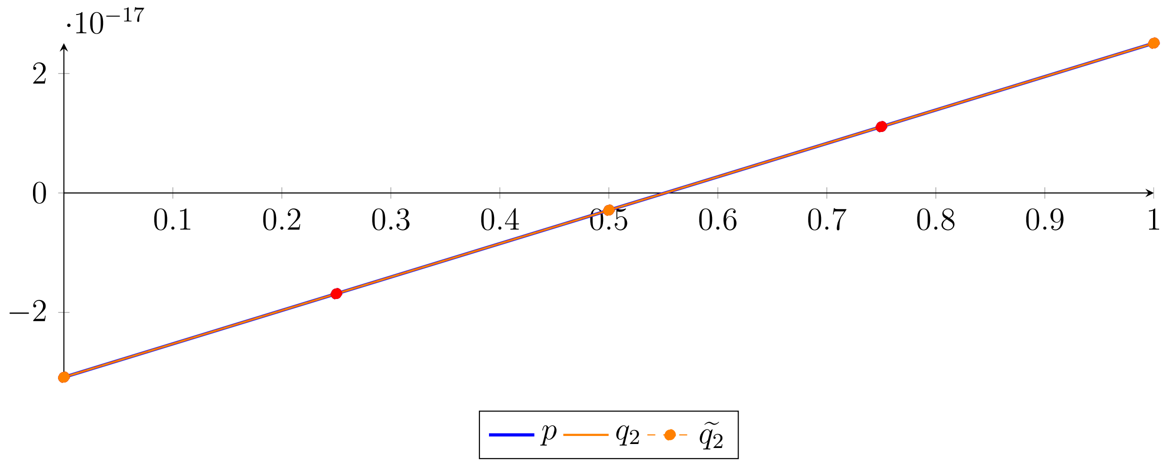
$$p = -1.20371 \cdot 10^{-35} X^3 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17}$$
$$= -3.08561 \cdot 10^{-17} B_{0,4}(X) - 1.68712 \cdot 10^{-17} B_{1,4}(X) - 2.88624 \cdot 10^{-18} B_{2,4}(X) + 1.10987 \cdot 10^{-17} B_{3,4}(X) + 2.50836 \cdot 10^{-17} B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = 1.50463 \cdot 10^{-35} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17}$$
$$= -3.08561 \cdot 10^{-17} B_{0,2} - 2.88624 \cdot 10^{-18} B_{1,2} + 2.50836 \cdot 10^{-17} B_{2,2}$$

$$\widetilde{q}_2 = -3.03936 \cdot 10^{-34} X^4 + 9.14817 \cdot 10^{-34} X^3 - 6.31946 \cdot 10^{-34} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17}$$
$$= -3.08561 \cdot 10^{-17} B_{0,4} - 1.68712 \cdot 10^{-17} B_{1,4} - 2.88624 \cdot 10^{-18} B_{2,4} + 1.10987 \cdot 10^{-17} B_{3,4} + 2.50836 \cdot 10^{-17} B_{4,4}$$



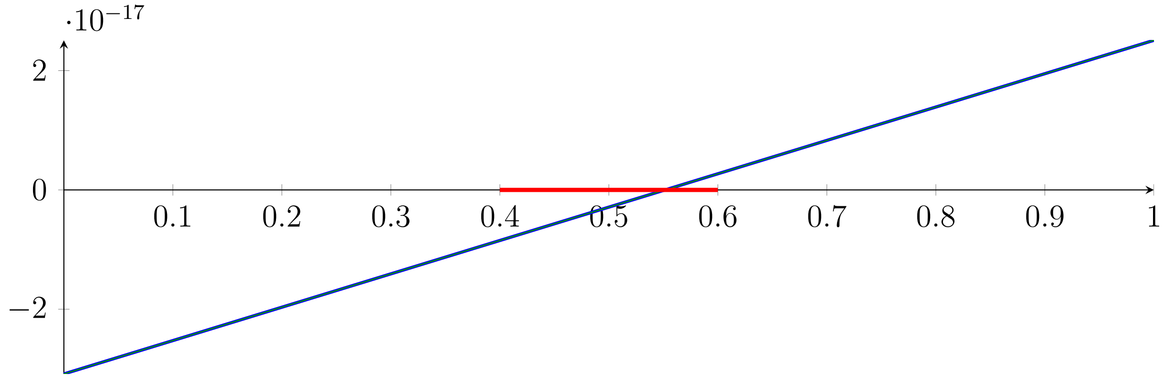
The maximum difference of the Bézier coefficients is $\delta = 7.14701 \cdot 10^{-35}$.
Bounding polynomials M and m:

$$M = 1.50463 \cdot 10^{-35} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17}$$
$$m = 1.50463 \cdot 10^{-35} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17}$$

Root of M and m:

$$N(M) = \{-3.71783 \cdot 10^{18}, 0.6\} \qquad N(m) = \{-3.71783 \cdot 10^{18}, 0.4\}$$

Intersection intervals:



[0.4, 0.6]

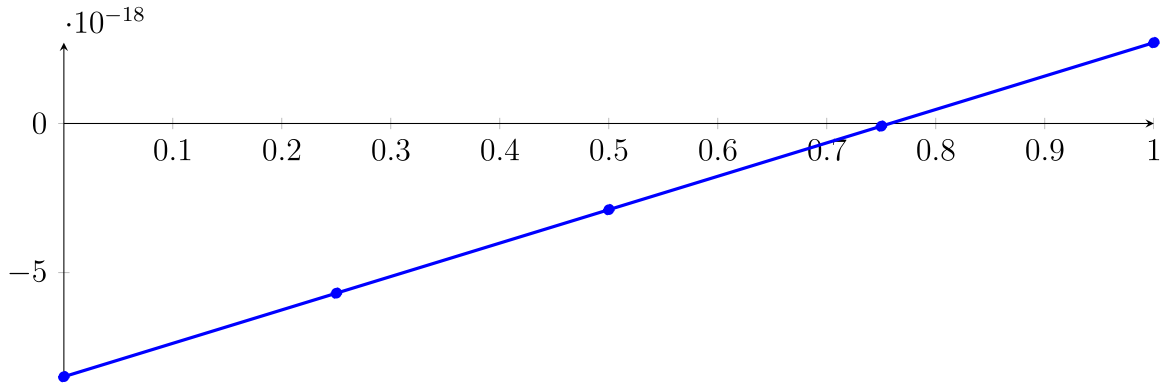
Longest intersection interval: 0.2

⟹ Selective recursion: interval 1: [0.333333, 0.333333],

125.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

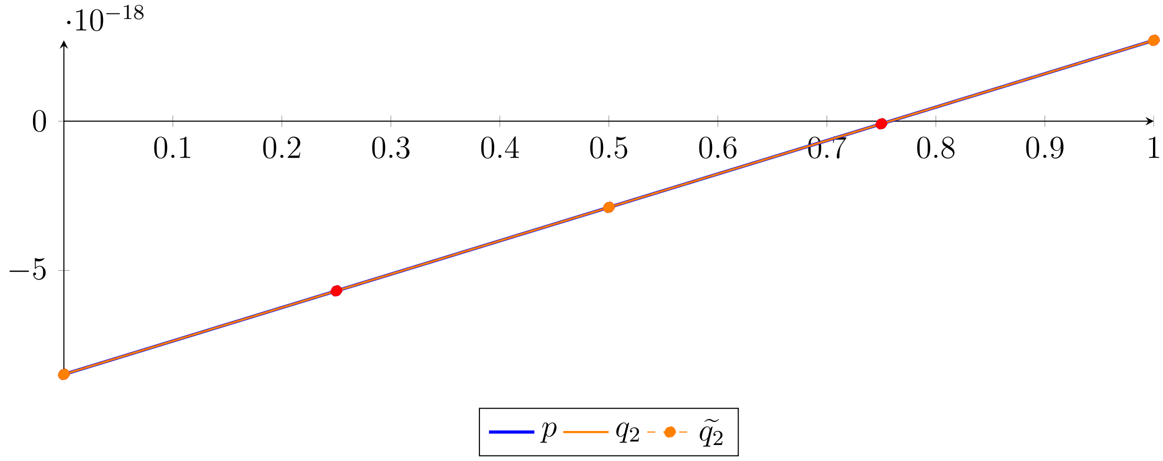
$$p = 1.50463 \cdot 10^{-36} X^4 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$
$$= -8.48022 \cdot 10^{-18} B_{0,4}(X) - 5.68323 \cdot 10^{-18} B_{1,4}(X) - 2.88624 \cdot 10^{-18} B_{2,4}(X) - 8.9255 \cdot 10^{-20} B_{3,4}(X) + 2.70773 \cdot 10^{-18} B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = 6.01853 \cdot 10^{-36} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$
$$= -8.48022 \cdot 10^{-18} B_{0,2} - 2.88624 \cdot 10^{-18} B_{1,2} + 2.70773 \cdot 10^{-18} B_{2,2}$$

$$\widetilde{q}_2 = -1.06829 \cdot 10^{-34} X^4 + 6.62038 \cdot 10^{-35} X^3 + 7.67363 \cdot 10^{-35} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$
$$= -8.48022 \cdot 10^{-18} B_{0,4} - 5.68323 \cdot 10^{-18} B_{1,4} - 2.88624 \cdot 10^{-18} B_{2,4} - 8.9255 \cdot 10^{-20} B_{3,4} + 2.70773 \cdot 10^{-18} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.36039 \cdot 10^{-35}$.
Bounding polynomials M and m:

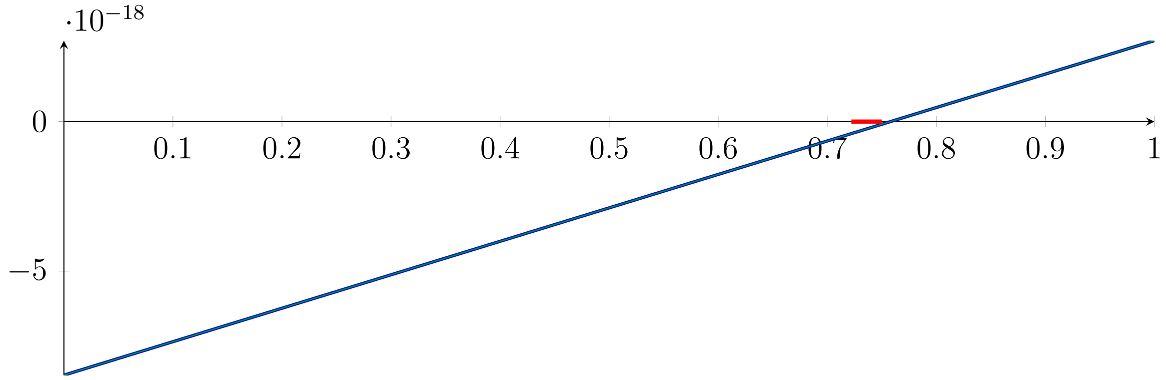
$$M = 6.01853 \cdot 10^{-36} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$

$$m = 6.77085 \cdot 10^{-36} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$

Root of M and m:

$$N(M) = \{-1.85892 \cdot 10^{18}, 0.75\} \qquad N(m) = \{-1.65237 \cdot 10^{18}, 0.722222\}$$

Intersection intervals:



$$[0.722222, 0.75]$$

Longest intersection interval: 0.0277778
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

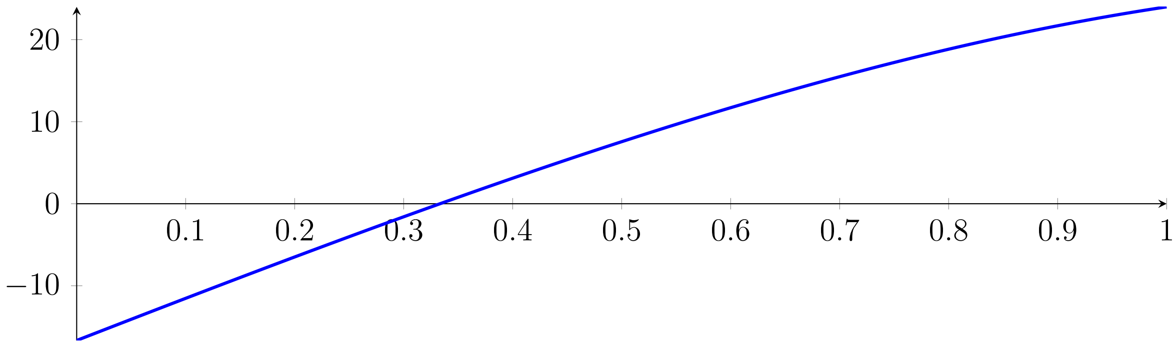
125.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 6!
p(0) = -4.00031e-19 - p(1) -8.9255e-20

125.7 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

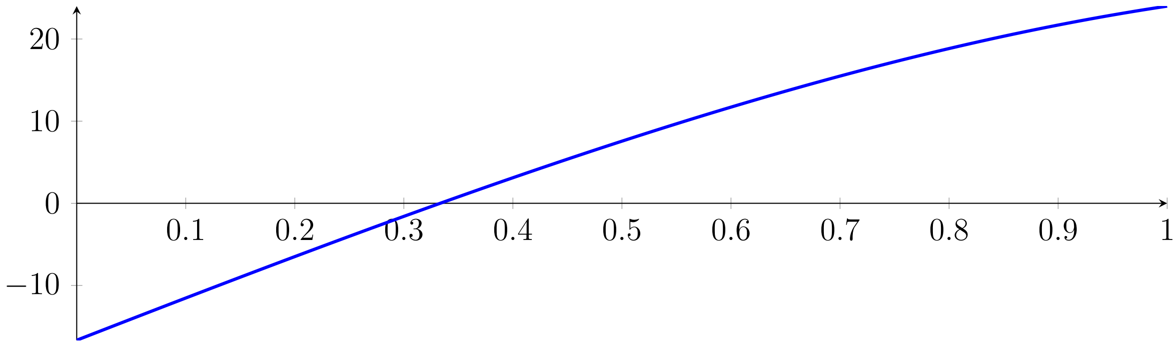
with precision $\varepsilon = 1 \cdot 10^{-128}$.

126 Running CubeClip on f_4 with epsilon 128

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0,1]$:

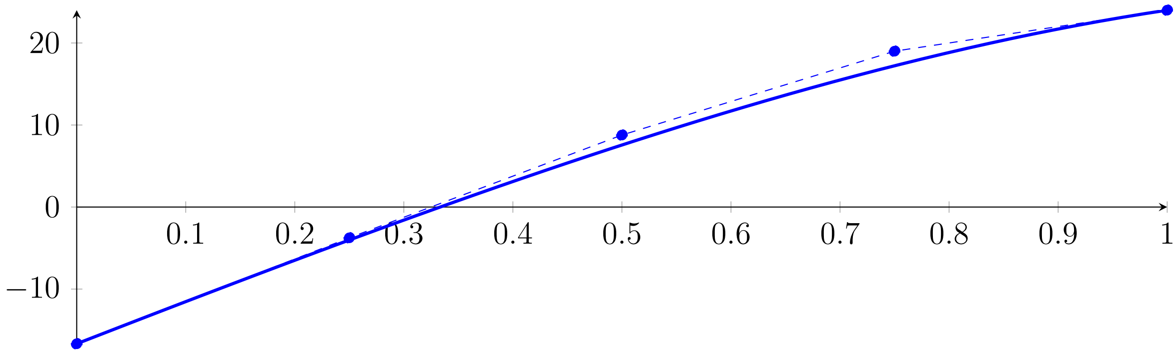
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



126.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

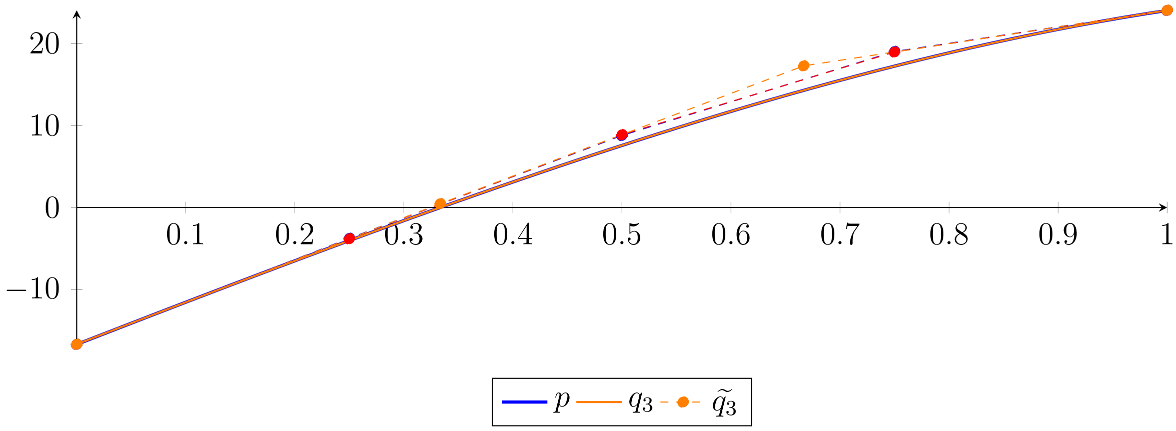
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$
$$= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X)$$



Degree reduction and raising:

$$q_3 = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524$$
$$= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3}$$

$$\widetilde{q_3} = 3.34802 \cdot 10^{-16}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524$$
$$= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4}$$



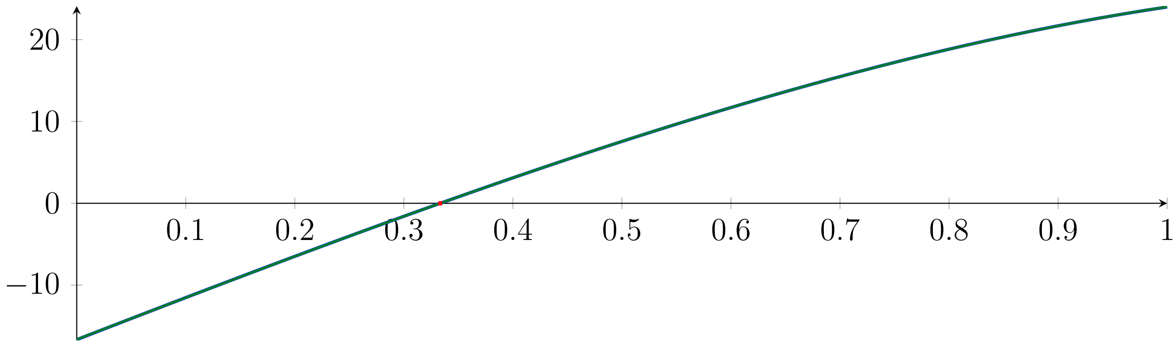
The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



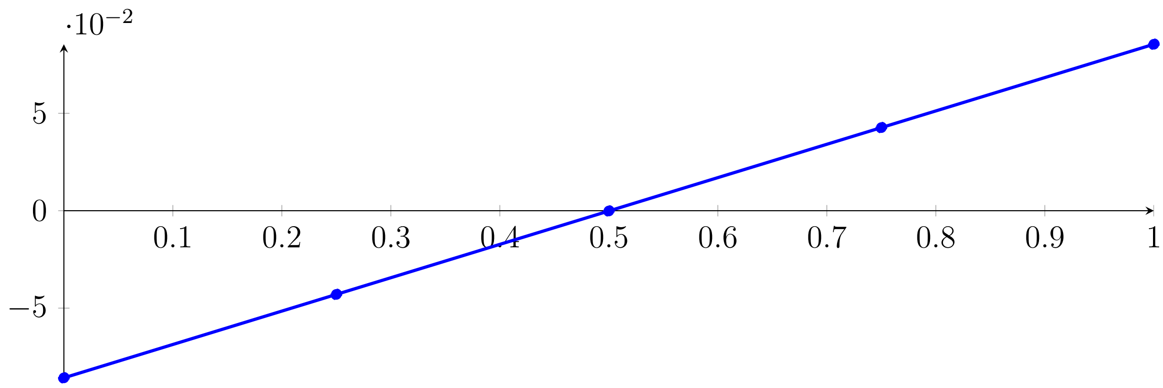
$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

126.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

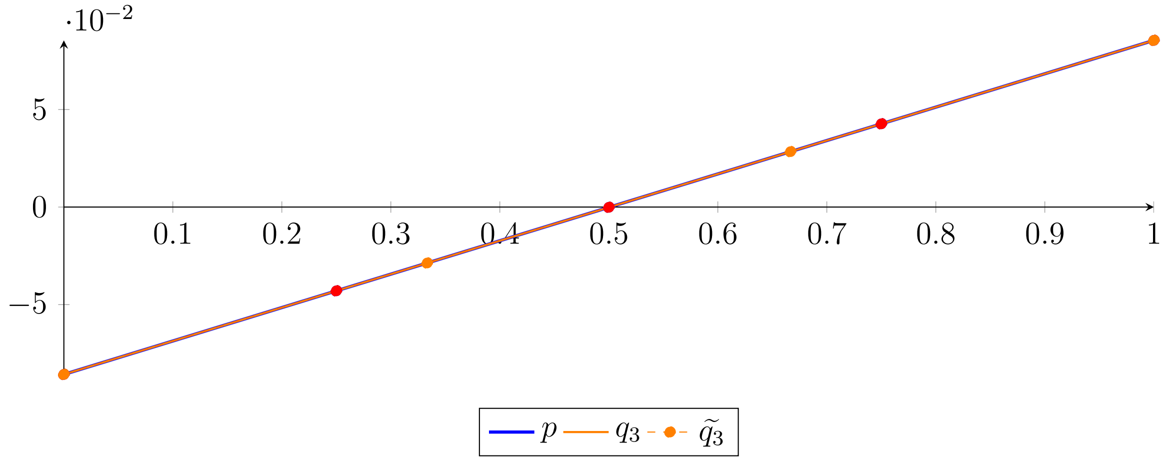
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 1.99222 \cdot 10^{-18} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m:

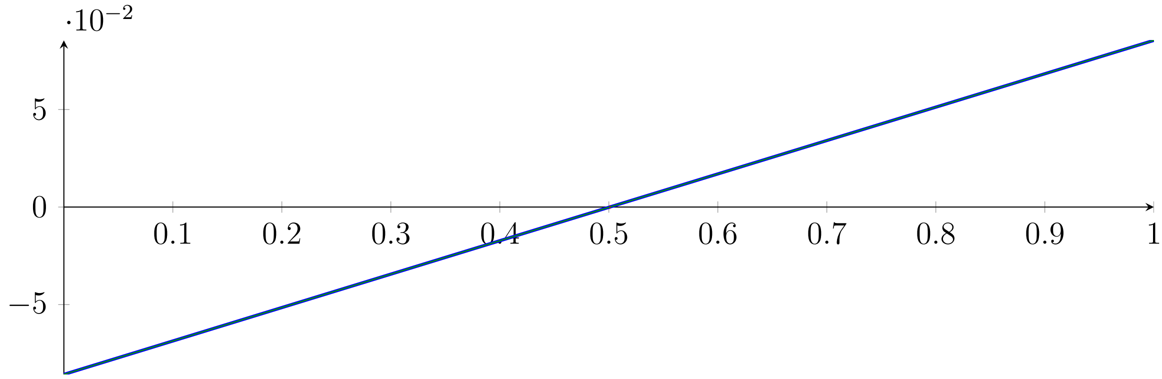
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

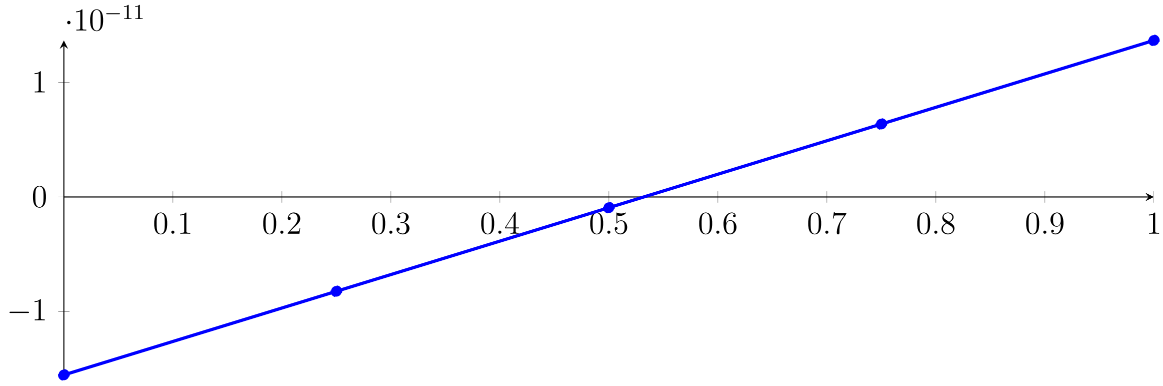
Longest intersection interval: $1.7041 \cdot 10^{-10}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

126.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

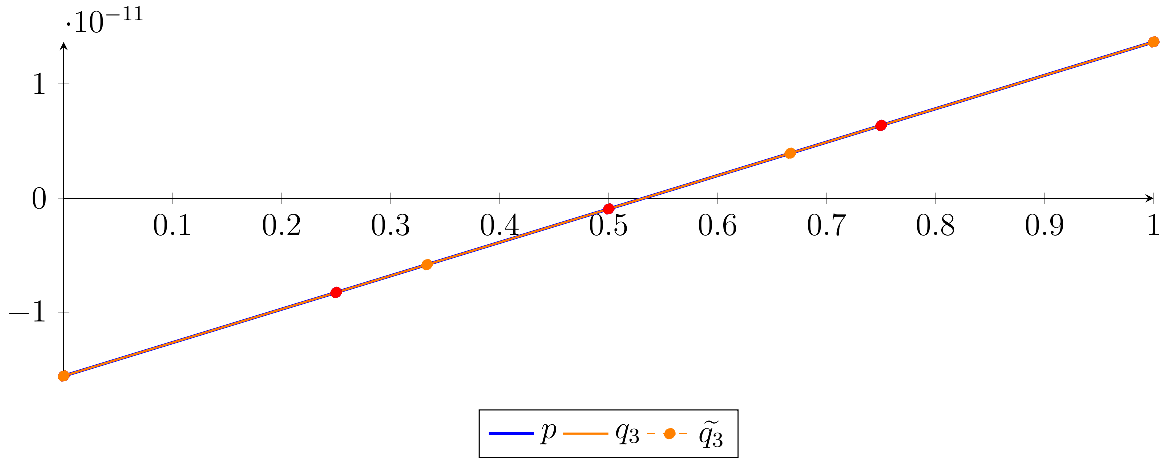
$$\begin{aligned} p &= -4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\ &= -1.55233 \cdot 10^{-11} B_{0,4}(X) - 8.22817 \cdot 10^{-12} B_{1,4}(X) - 9.33054 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36206 \cdot 10^{-12} B_{3,4}(X) + 1.36572 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$q_3 = 2.87145\cdot10^{-28}X^3 - 4.04172\cdot10^{-24}X^2 + 2.91805\cdot10^{-11}X - 1.55233\cdot10^{-11}$
$\quad = -1.55233\cdot10^{-11}B_{0,3} - 5.79647\cdot10^{-12}B_{1,3} + 3.93036\cdot10^{-12}B_{2,3} + 1.36572\cdot10^{-11}B_{3,3}$

$\widetilde{q_3} = 3.84964\cdot10^{-28}X^4 - 2.1457\cdot10^{-28}X^3 - 4.04147\cdot10^{-24}X^2 + 2.91805\cdot10^{-11}X - 1.55233\cdot10^{-11}$
$\quad = -1.55233\cdot10^{-11}B_{0,4} - 8.22817\cdot10^{-12}B_{1,4} - 9.33054\cdot10^{-13}B_{2,4} + 6.36206\cdot10^{-12}B_{3,4} + 1.36572\cdot10^{-11}B_{4,4}$



The maximum difference of the Bézier coefficients is $\delta = 1.13596\cdot10^{-28}$.

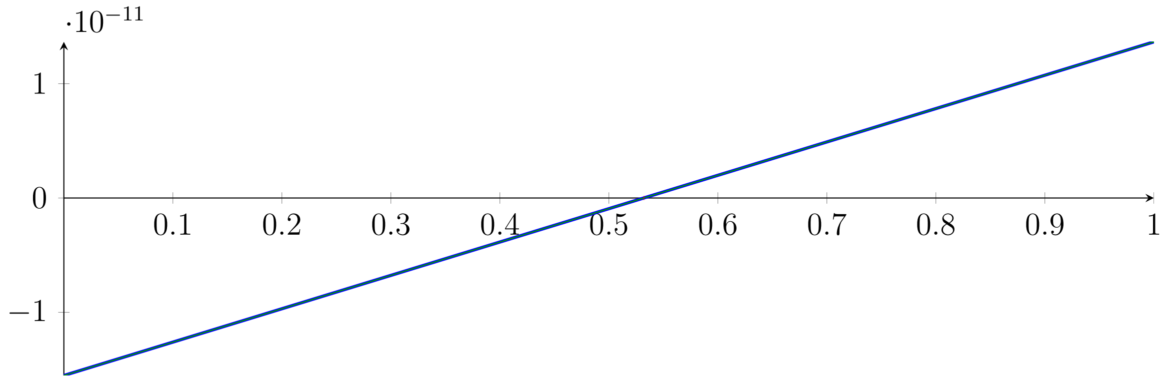
Bounding polynomials M and m:

$$M = 2.87145\cdot10^{-28}X^3 - 4.04172\cdot10^{-24}X^2 + 2.91805\cdot10^{-11}X - 1.55233\cdot10^{-11}$$
$$m = 2.87145\cdot10^{-28}X^3 - 4.04172\cdot10^{-24}X^2 + 2.91805\cdot10^{-11}X - 1.55233\cdot10^{-11}$$

Root of M and m:

$$N(M) = \{0.531975\} \qquad N(m) = \{0.531975\}$$

Intersection intervals:



$$[0.531975, 0.531975]$$

Longest intersection interval: 0

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

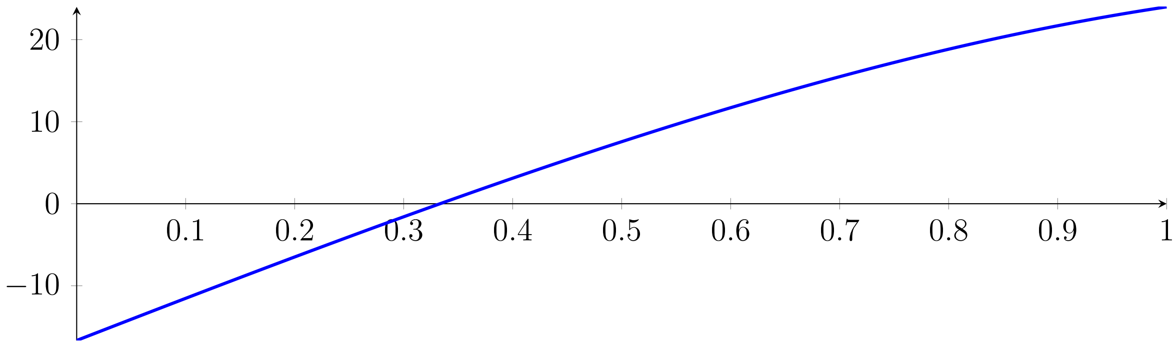
126.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

126.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

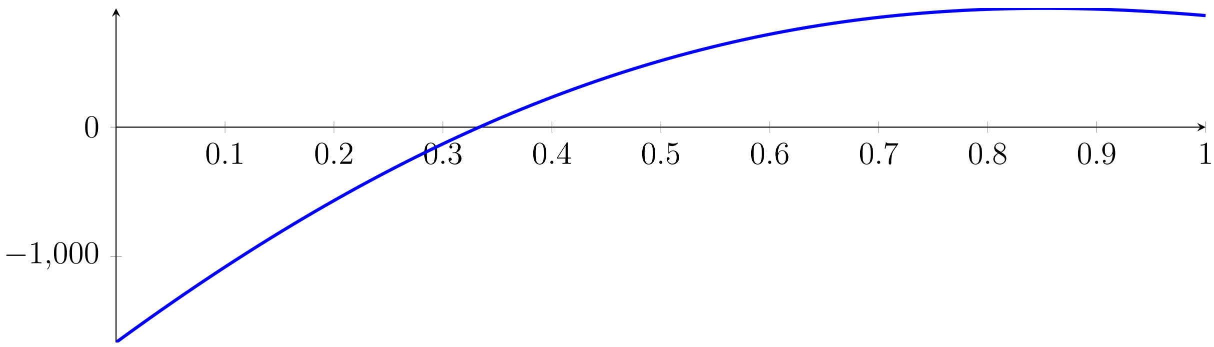
with precision $\varepsilon = 1 \cdot 10^{-128}$.

127 Running `BezClip` on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

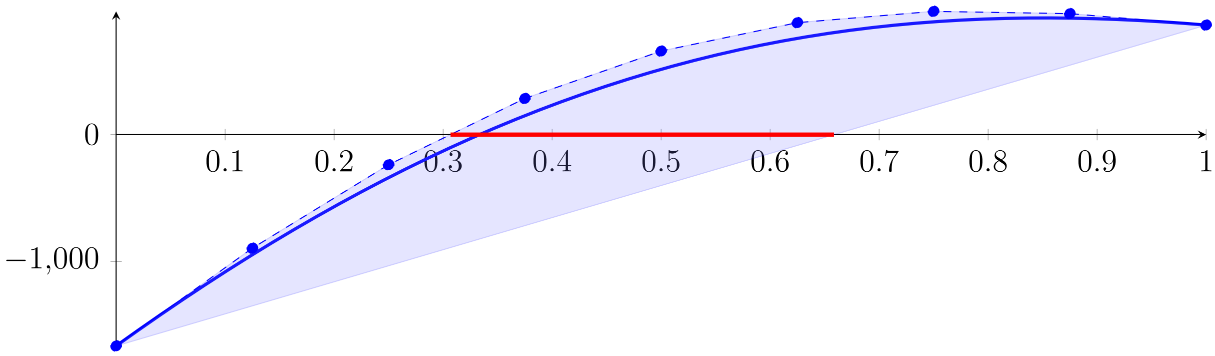
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



127.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

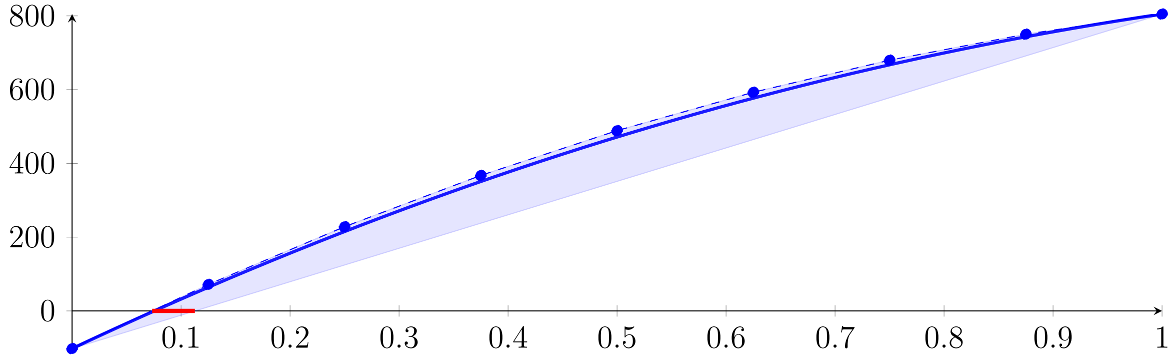
$$[0.306796, 0.658588]$$

Longest intersection interval: 0.351792
$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

127.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\ &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\ &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\ &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

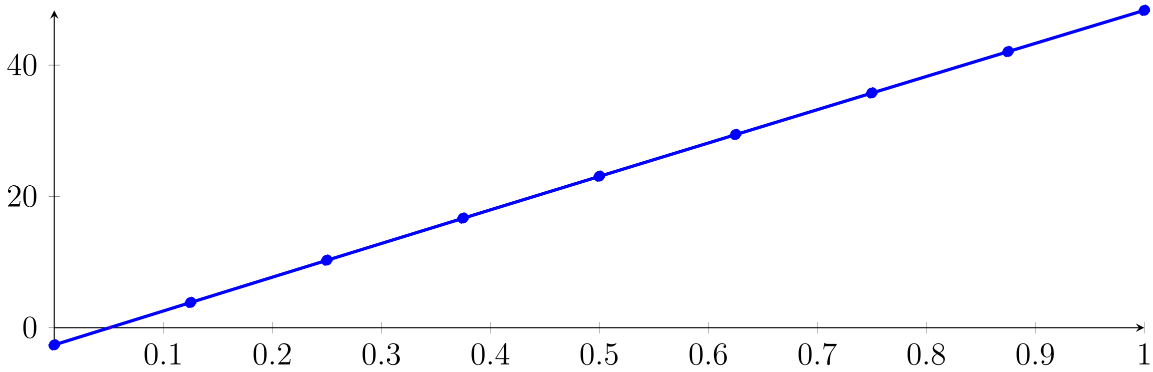
Longest intersection interval: 0.0391855

$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

127.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -3.03577 \cdot 10^{-15}X^8 - 1.54633 \cdot 10^{-12}X^7 - 4.95836 \cdot 10^{-10}X^6 + 3.66751 \cdot 10^{-08}X^5 \\ &\quad + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\ &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\ &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

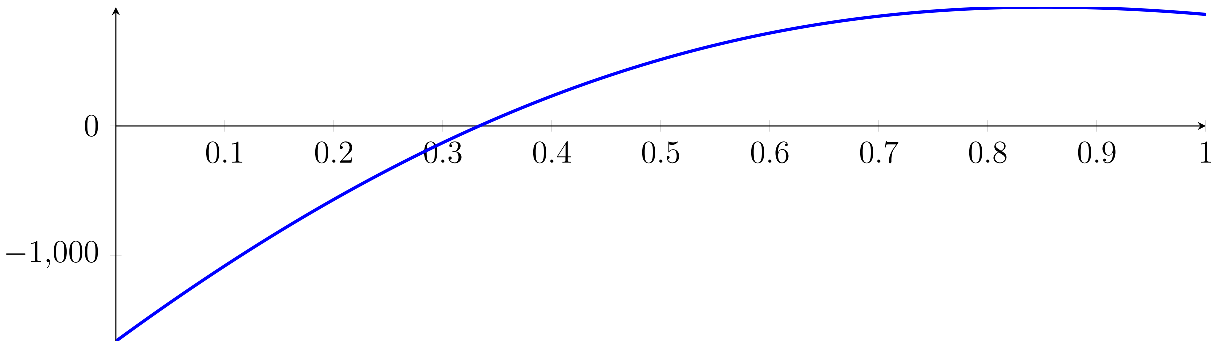
127.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Found root in interval $[0.333333, 0.333343]$ at recursion depth 4!

127.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333343]$$

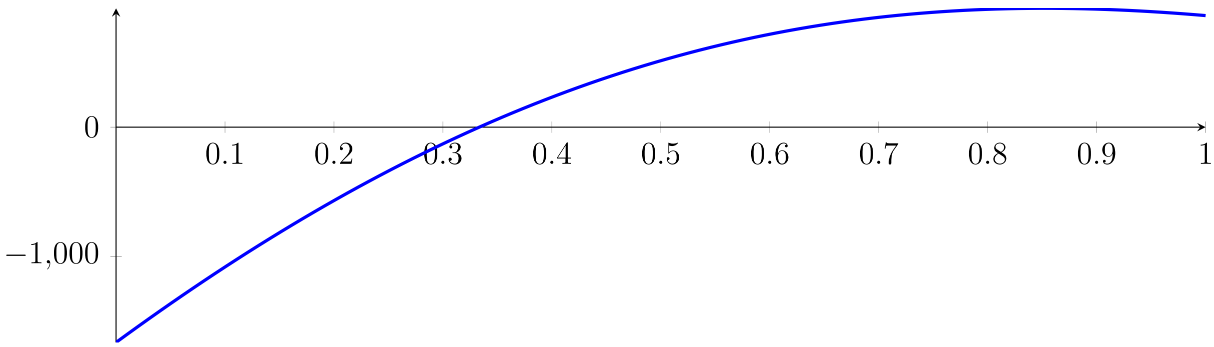
with precision $\varepsilon = 0.01$.

128 Running QuadClip on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

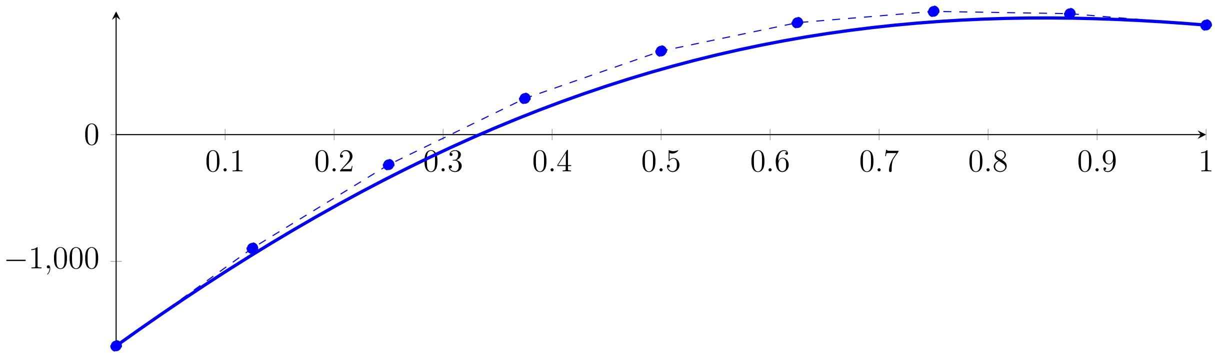
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



128.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

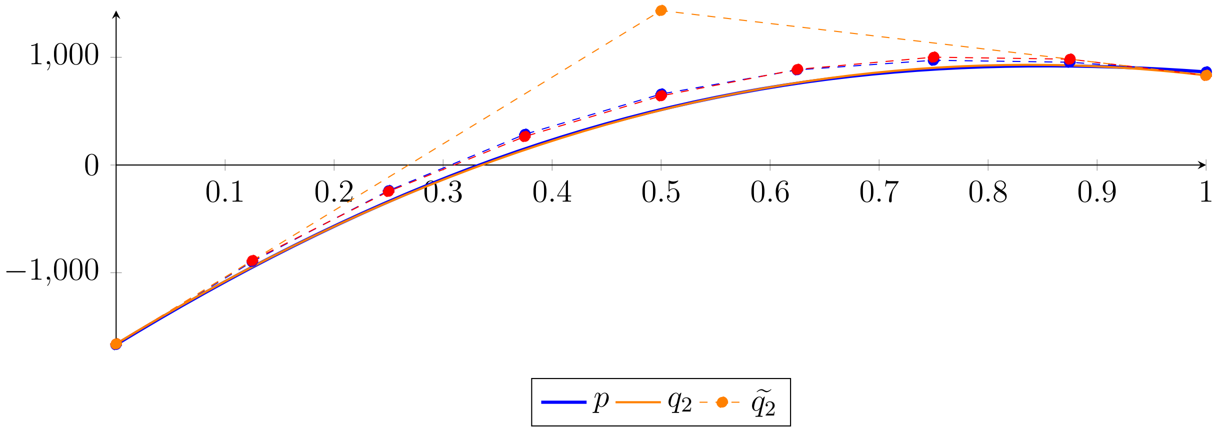
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -5.06304 \cdot 10^{-10}X^8 + 1.93305 \cdot 10^{-09}X^7 - 2.97598 \cdot 10^{-09}X^6 + 2.35116 \cdot 10^{-09}X^5 \\ &\quad - 9.91232 \cdot 10^{-10}X^4 + 2.00705 \cdot 10^{-10}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



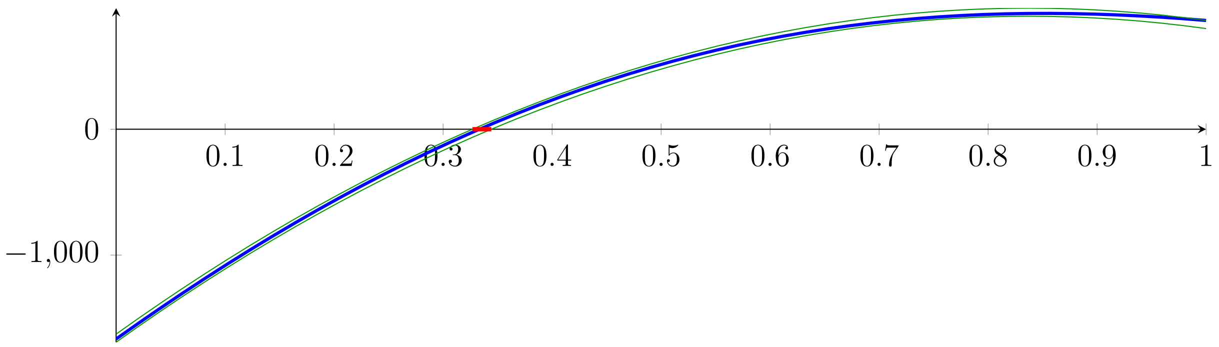
The maximum difference of the Bézier coefficients is $\delta = 32.1356$.
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



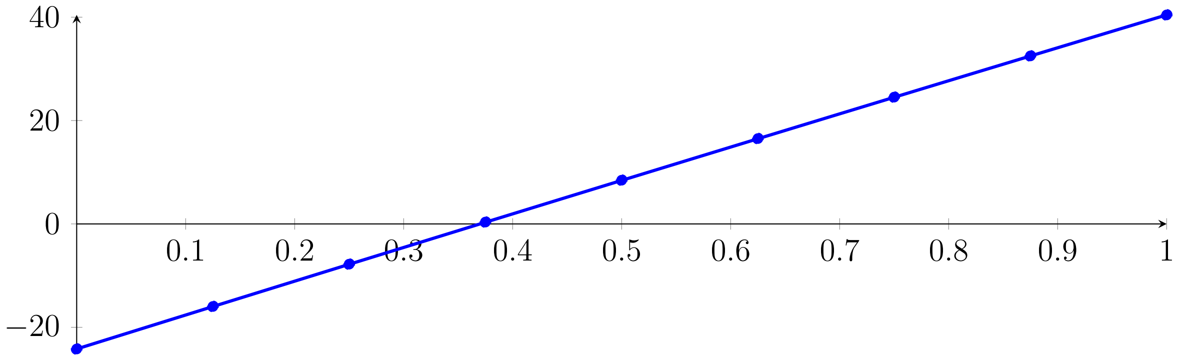
$[0.326917, 0.344255]$

Longest intersection interval: 0.0173372
$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

128.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

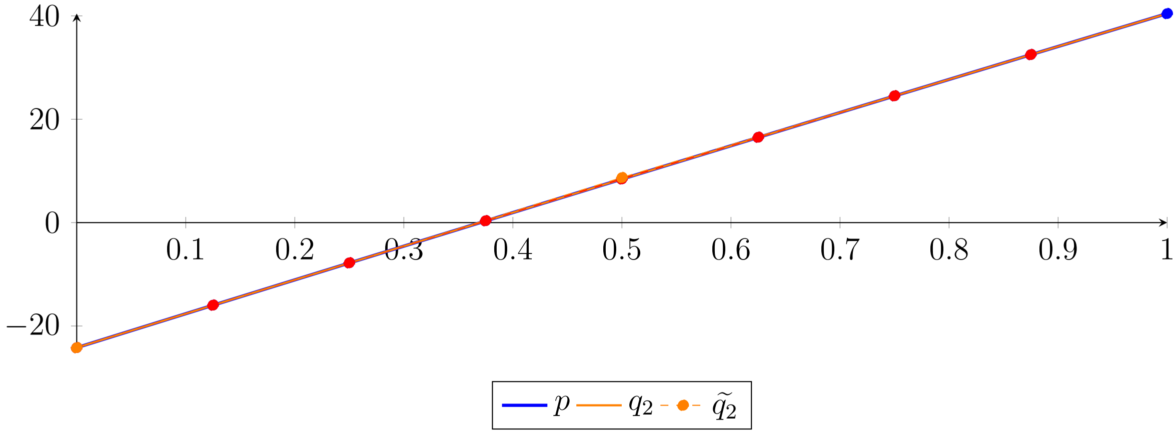
$$\begin{aligned} p = &-8.80546\cdot 10^{-15}X^8 - 7.66587\cdot 10^{-12}X^7 - 1.9444\cdot 10^{-09}X^6 + 1.19263\cdot 10^{-07}X^5 \\ &+ 8.12335\cdot 10^{-05}X^4 - 0.0012733X^3 - 1.18084X^2 + 65.8155X - 24.1945 \\ = &-24.1945B_{0,8}(X) - 15.9676B_{1,8}(X) - 7.78282B_{2,8}(X) + 0.35975B_{3,8}(X) \\ &+ 8.4601B_{4,8}(X) + 16.5182B_{5,8}(X) + 24.5341B_{6,8}(X) + 32.5077B_{7,8}(X) + 40.4389B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -2.59056\cdot 10^{-11}X^8 + 1.00262\cdot 10^{-10}X^7 - 1.57692\cdot 10^{-10}X^6 + 1.29283\cdot 10^{-10}X^5 \\ &\quad - 5.86775\cdot 10^{-11}X^4 + 1.42642\cdot 10^{-11}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894\cdot 10^{-05}$.
Bounding polynomials M and m:

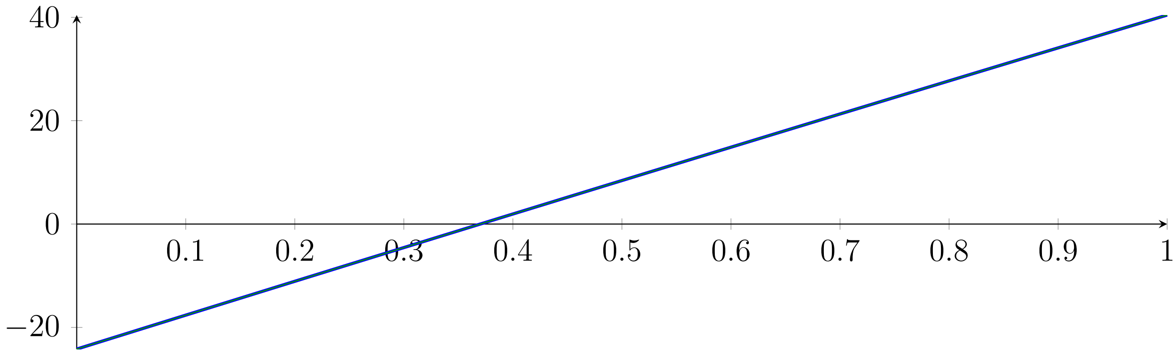
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588\cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

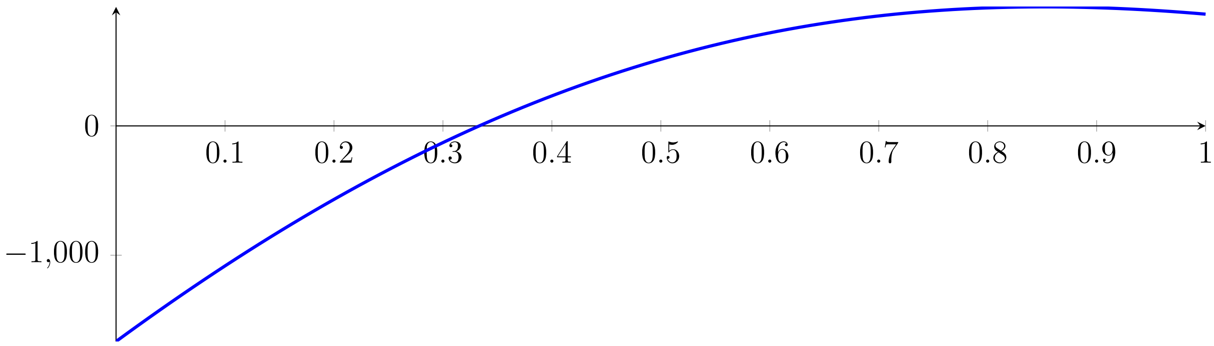
128.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

128.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

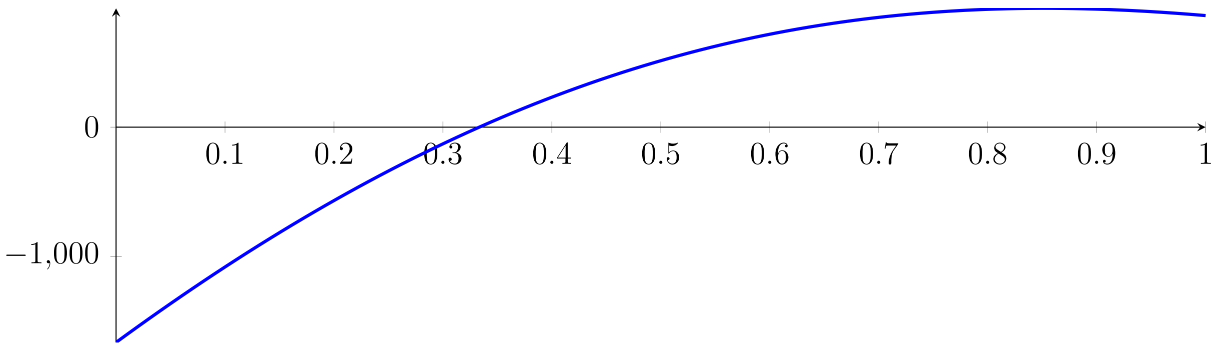
with precision $\varepsilon = 0.01$.

129 Running CubeClip on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

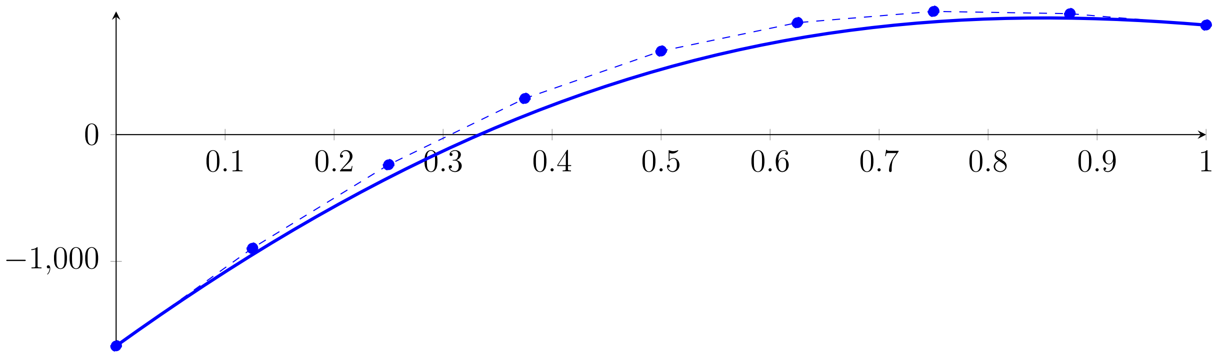
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



129.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

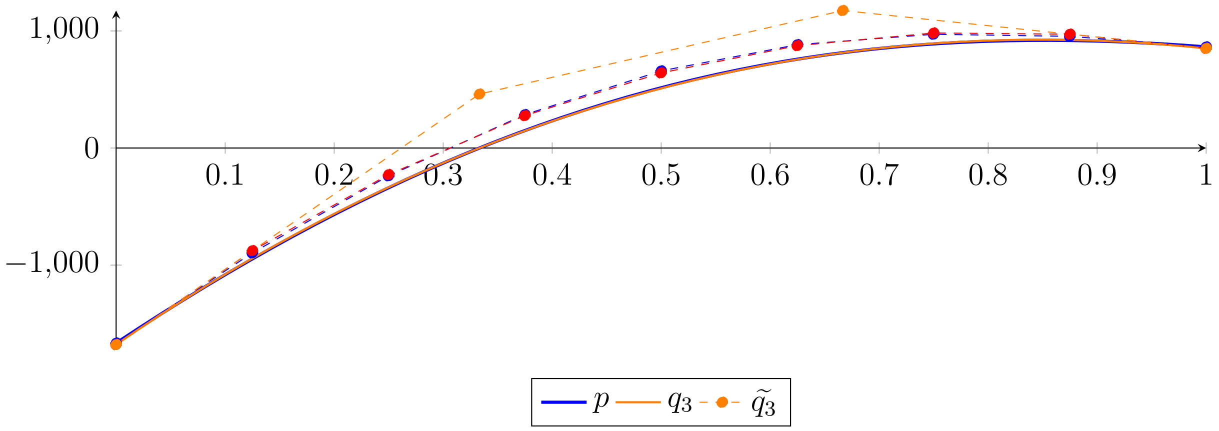
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.39389 \cdot 10^{-09}X^8 + 9.60737 \cdot 10^{-09}X^7 - 1.57477 \cdot 10^{-08}X^6 + 1.35479 \cdot 10^{-08}X^5 \\ &\quad - 6.53397 \cdot 10^{-09}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

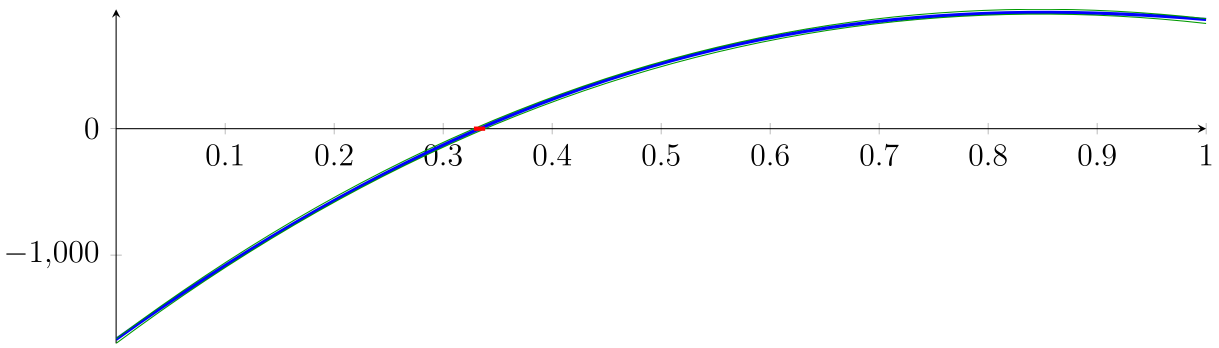
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$[0.328258, 0.338551]$

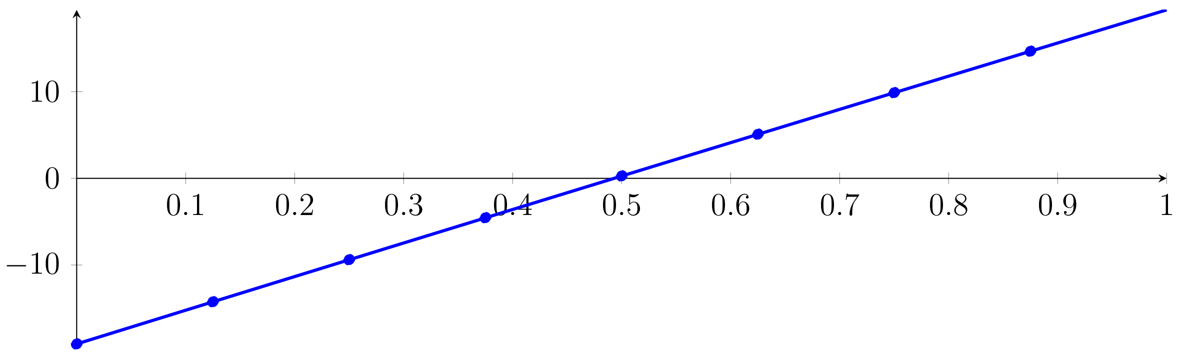
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

129.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

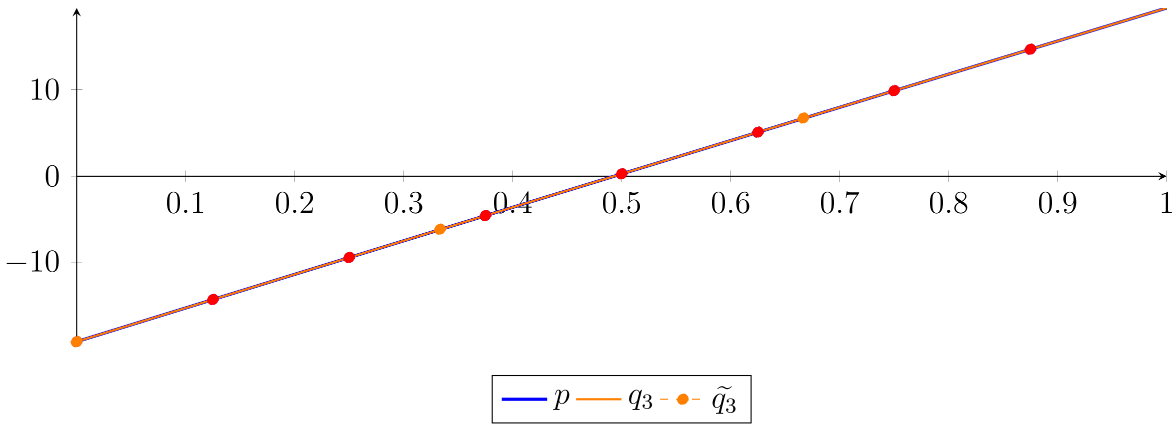
$$\begin{aligned} p = &-1.66533\cdot 10^{-16}X^8 - 1.99007\cdot 10^{-13}X^7 - 8.53059\cdot 10^{-11}X^6 + 8.7284\cdot 10^{-09}X^5 \\ &+ 1.00963\cdot 10^{-05}X^4 - 0.000261161X^3 - 0.416284X^2 + 38.9643X - 19.1124 \\ = &-19.1124B_{0,8}(X) - 14.2419B_{1,8}(X) - 9.38619B_{2,8}(X) - 4.54539B_{3,8}(X) \\ &+ 0.280537B_{4,8}(X) + 5.09158B_{5,8}(X) + 9.88774B_{6,8}(X) + 14.669B_{7,8}(X) + 19.4354B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2.82588 \cdot 10^{-12}X^8 - 1.08529 \cdot 10^{-11}X^7 + 1.68738 \cdot 10^{-11}X^6 - 1.34636 \cdot 10^{-11}X^5 \\ &\quad + 5.72618 \cdot 10^{-12}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m:

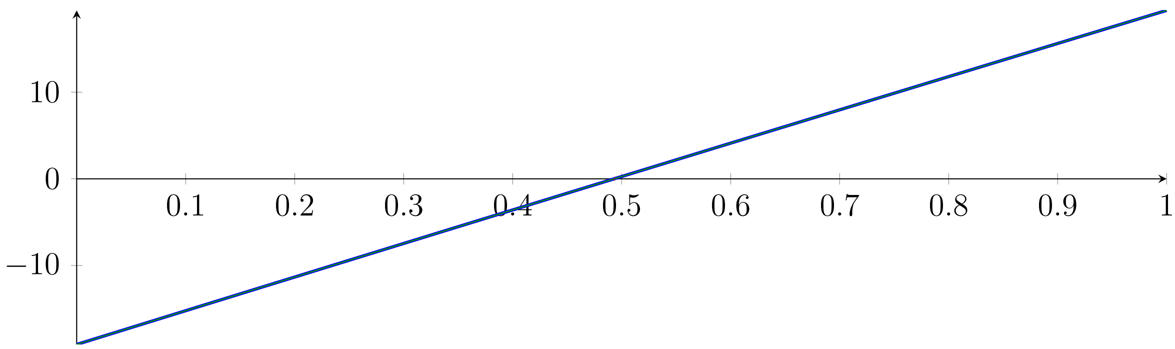
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.1252 \cdot 10^{-08}$

$\implies$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

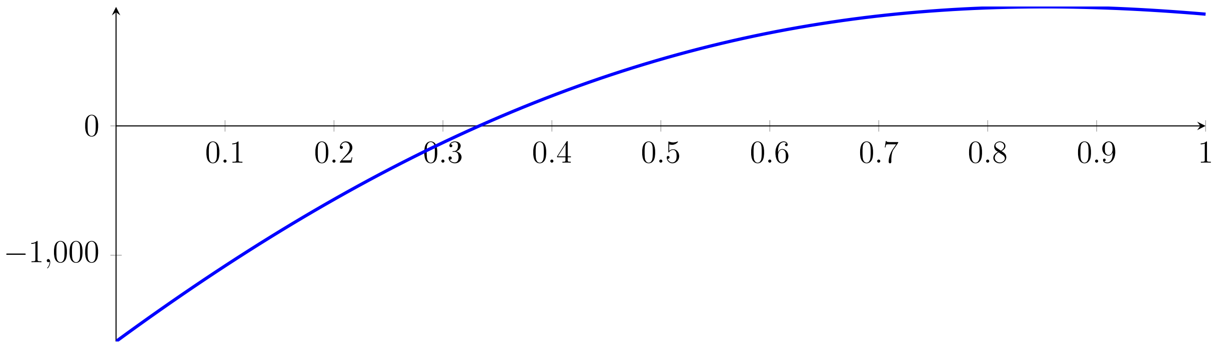
129.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

129.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



Result: Root Intervals

$$[0.333333, 0.333333]$$

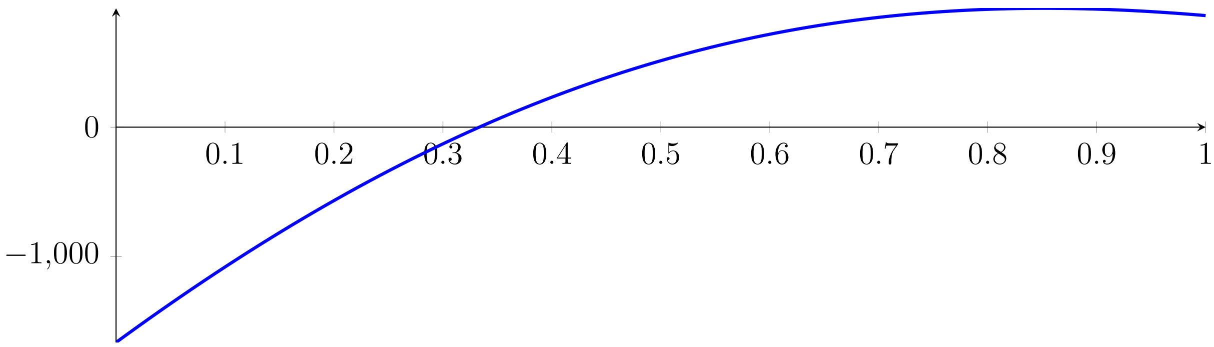
with precision $\varepsilon = 0.01$.

130 Running BezClip on f_8 with epsilon 4

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

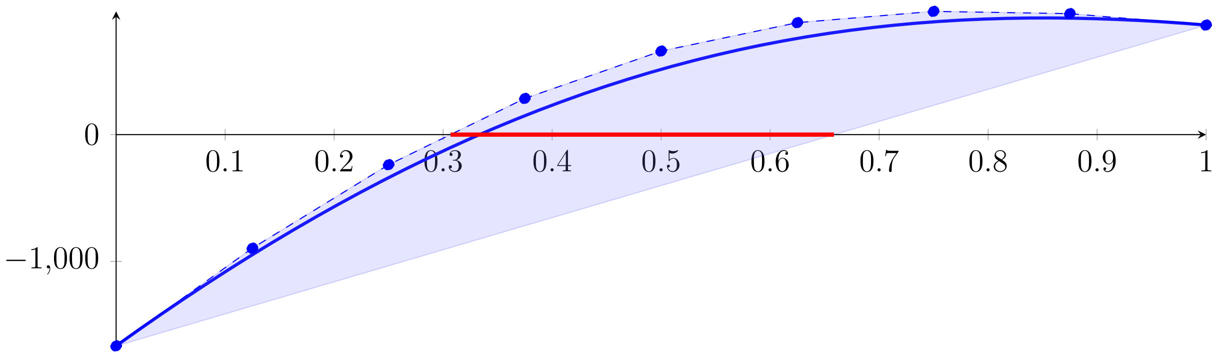
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



130.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

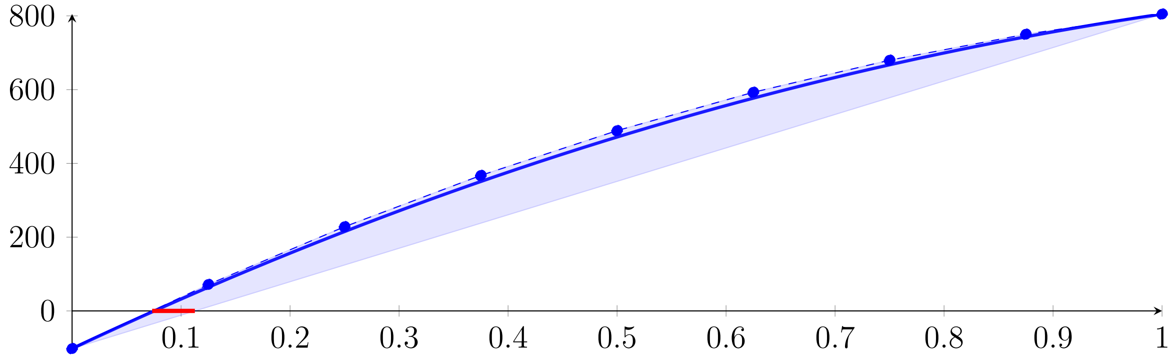
$$[0.306796, 0.658588]$$

Longest intersection interval: 0.351792
$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

130.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\ &+ 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\ = &-102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\ &+ 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

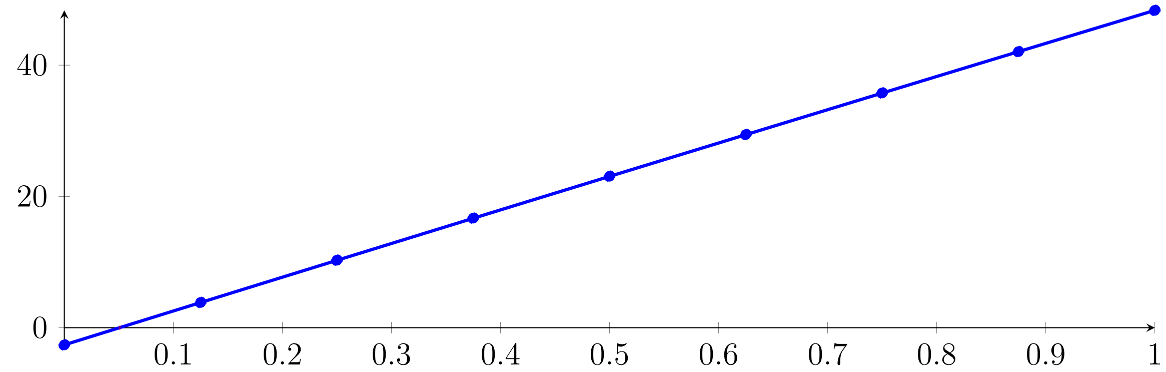
Longest intersection interval: 0.0391855

$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

130.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-3.03577 \cdot 10^{-15}X^8 - 1.54633 \cdot 10^{-12}X^7 - 4.95836 \cdot 10^{-10}X^6 + 3.66751 \cdot 10^{-08}X^5 \\ &+ 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\ = &-2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\ &+ 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

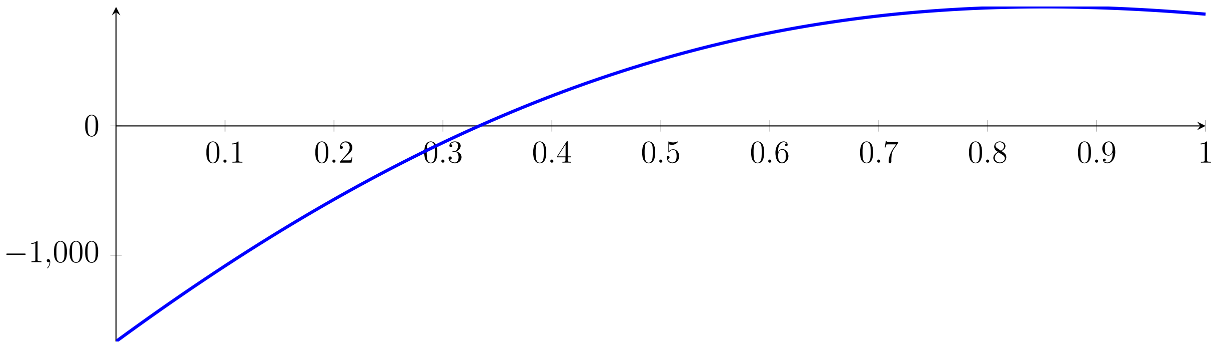
130.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Found root in interval $[0.333333, 0.333343]$ at recursion depth 4!

130.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333343]$$

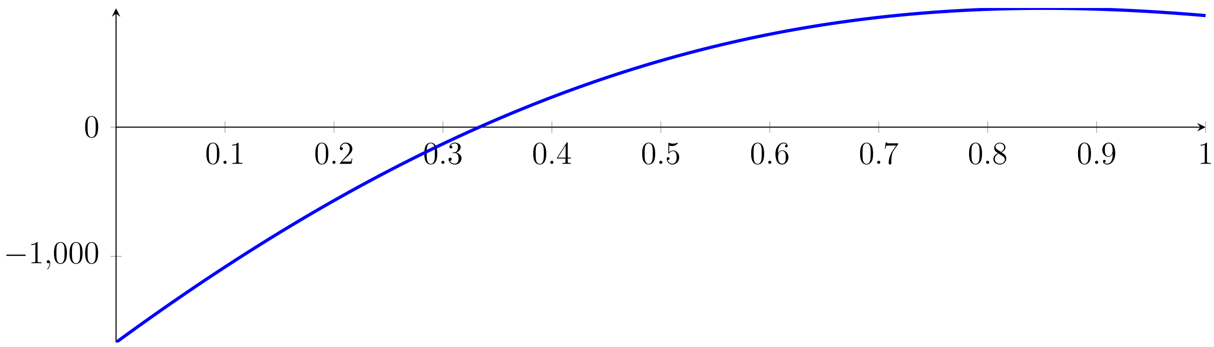
with precision $\varepsilon = 0.0001$.

131 Running QuadClip on f_8 with epsilon 4

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

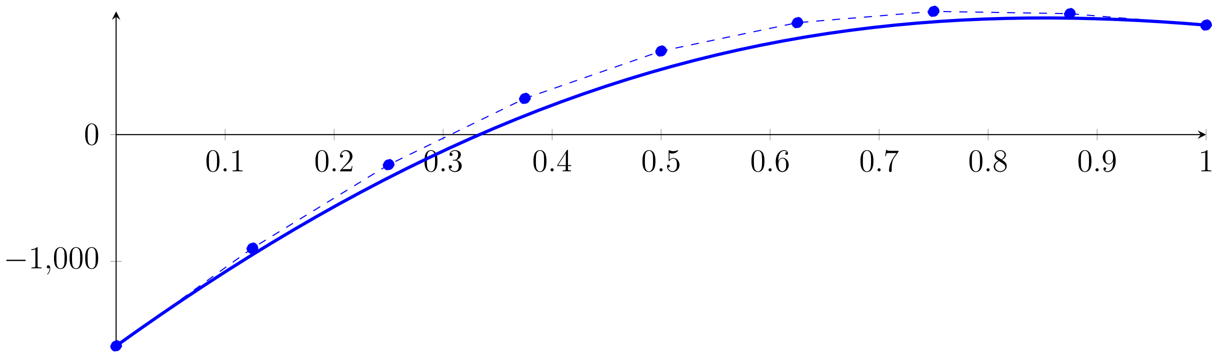
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



131.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

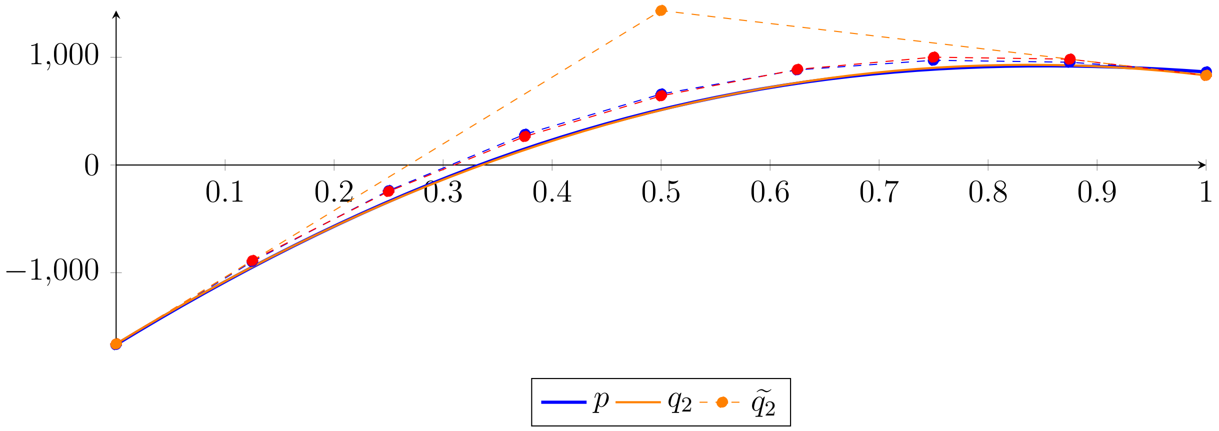
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -5.06304\cdot 10^{-10}X^8 + 1.93305\cdot 10^{-09}X^7 - 2.97598\cdot 10^{-09}X^6 + 2.35116\cdot 10^{-09}X^5 \\ &\quad - 9.91232\cdot 10^{-10}X^4 + 2.00705\cdot 10^{-10}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

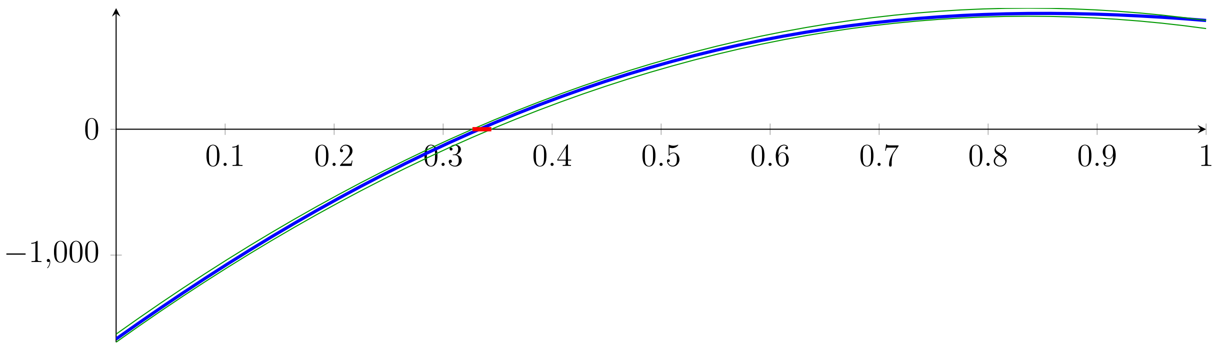
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

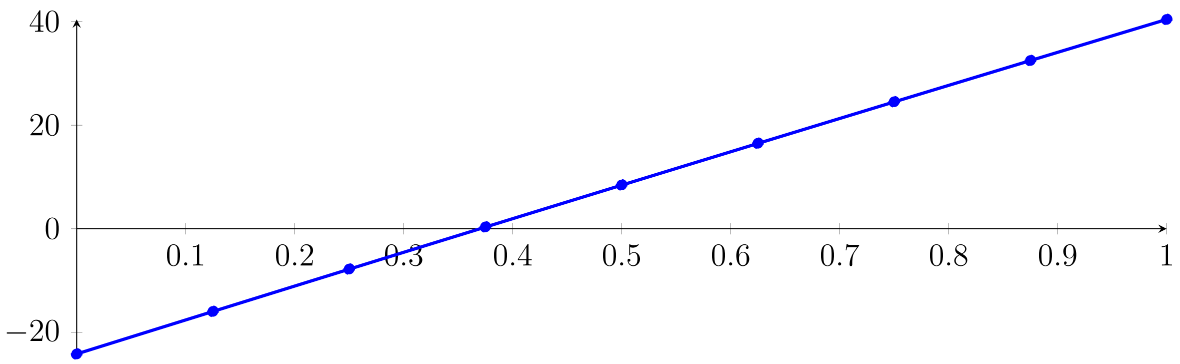
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

131.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

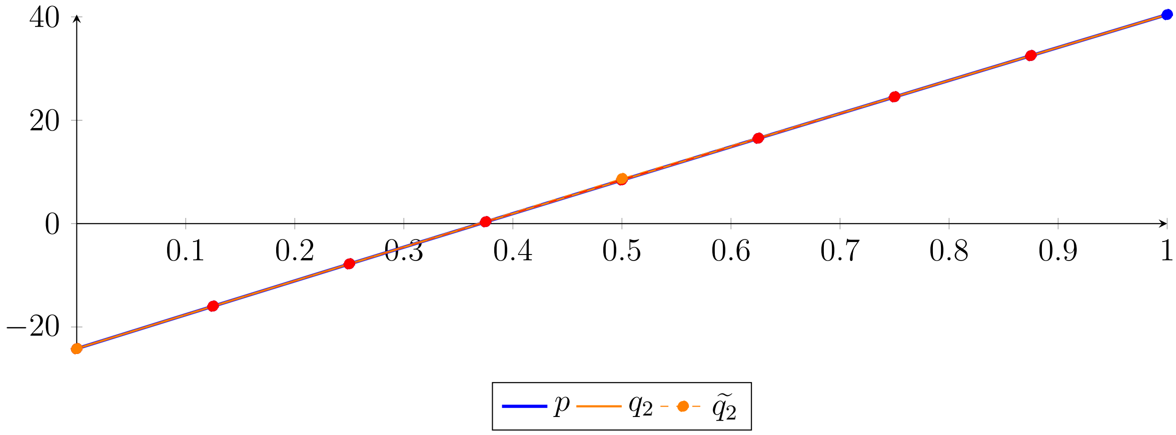
$$\begin{aligned}p = &-8.80546\cdot 10^{-15}X^8 - 7.66587\cdot 10^{-12}X^7 - 1.9444\cdot 10^{-09}X^6 + 1.19263\cdot 10^{-07}X^5 \\ &+ 8.12335\cdot 10^{-05}X^4 - 0.0012733X^3 - 1.18084X^2 + 65.8155X - 24.1945 \\ = &-24.1945B_{0,8}(X) - 15.9676B_{1,8}(X) - 7.78282B_{2,8}(X) + 0.35975B_{3,8}(X) \\ &+ 8.4601B_{4,8}(X) + 16.5182B_{5,8}(X) + 24.5341B_{6,8}(X) + 32.5077B_{7,8}(X) + 40.4389B_{8,8}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -2.59056\cdot 10^{-11}X^8 + 1.00262\cdot 10^{-10}X^7 - 1.57692\cdot 10^{-10}X^6 + 1.29283\cdot 10^{-10}X^5 \\ &\quad - 5.86775\cdot 10^{-11}X^4 + 1.42642\cdot 10^{-11}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894\cdot 10^{-05}$.
Bounding polynomials M and m:

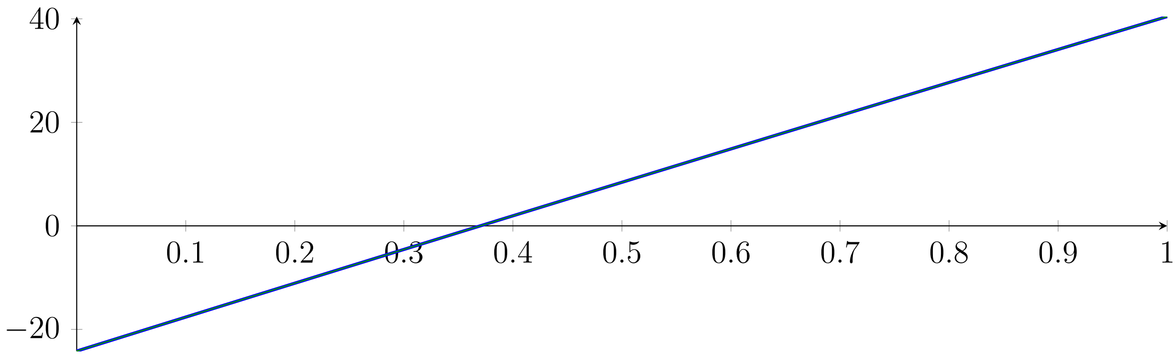
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588\cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

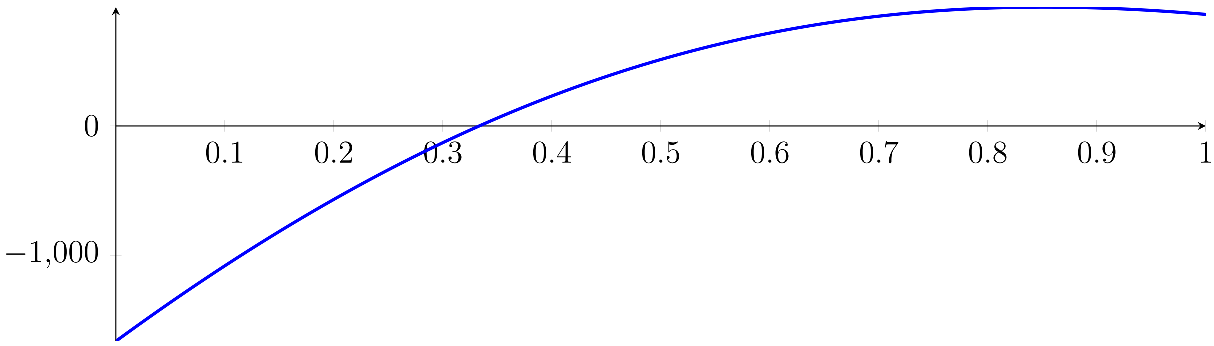
131.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

131.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 0.0001$.

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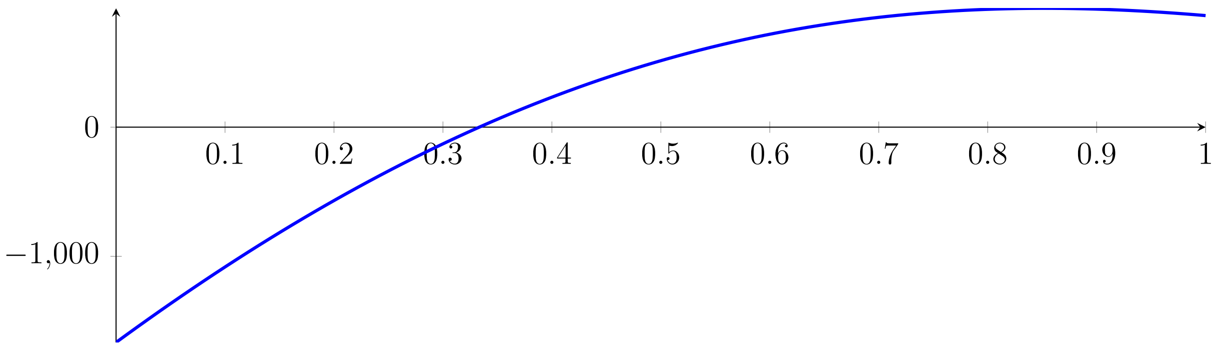
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

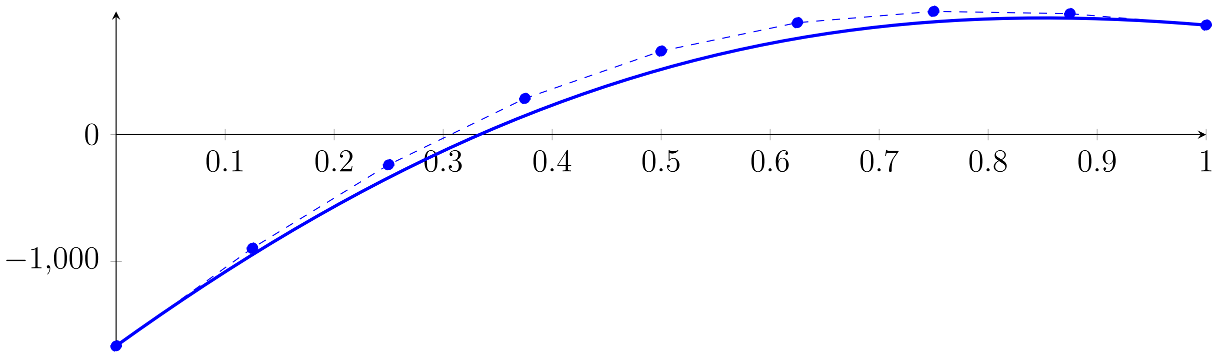
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



132.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

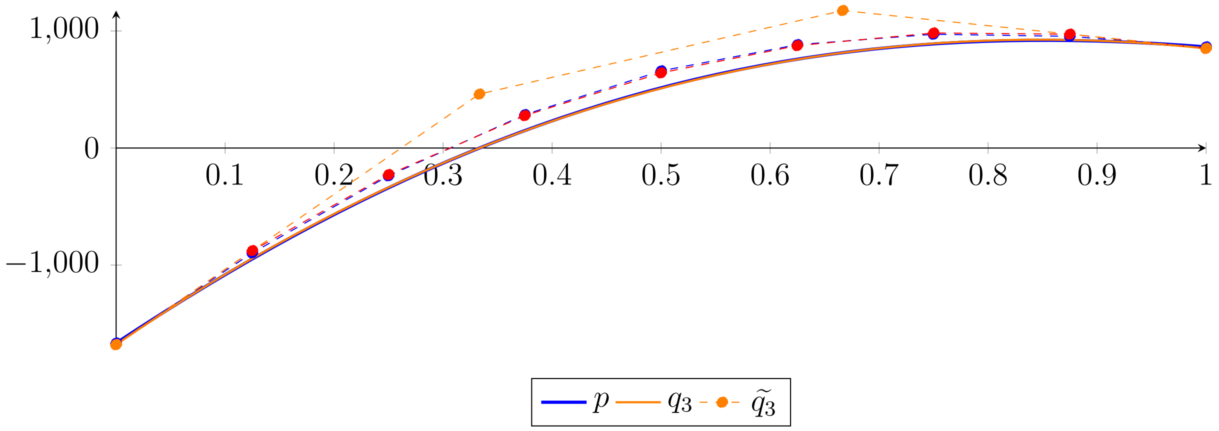
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.39389 \cdot 10^{-09}X^8 + 9.60737 \cdot 10^{-09}X^7 - 1.57477 \cdot 10^{-08}X^6 + 1.35479 \cdot 10^{-08}X^5 \\ &\quad - 6.53397 \cdot 10^{-09}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

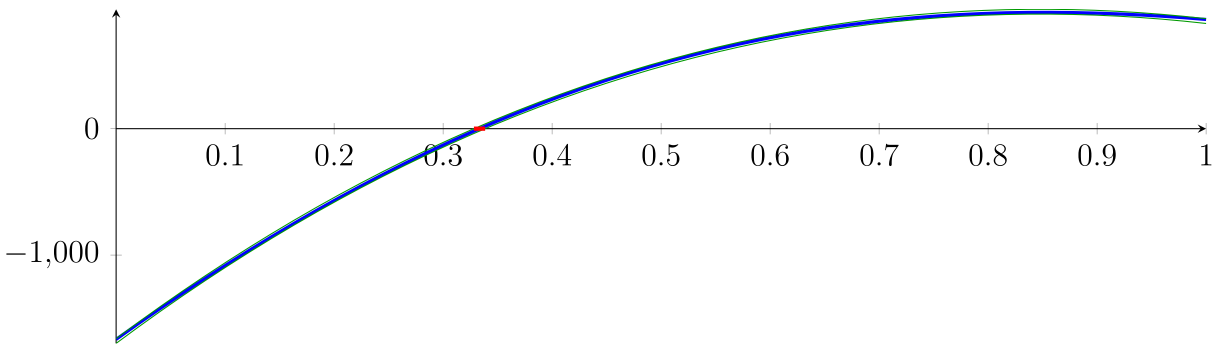
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$[0.328258, 0.338551]$

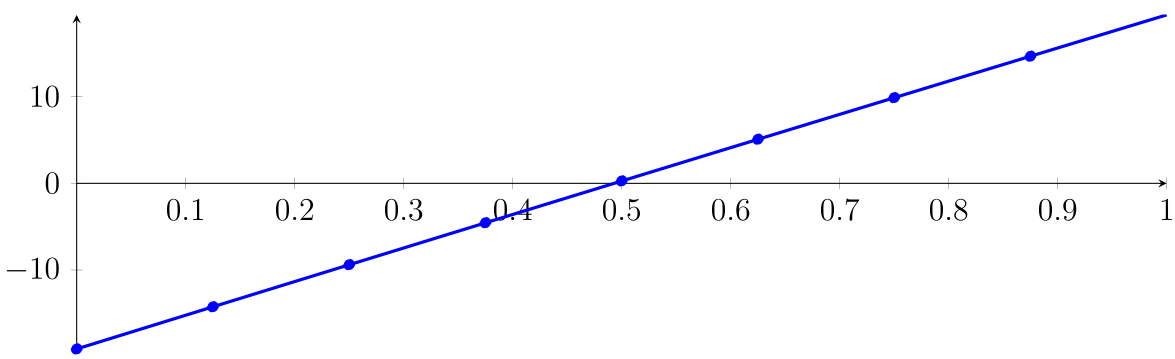
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

132.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

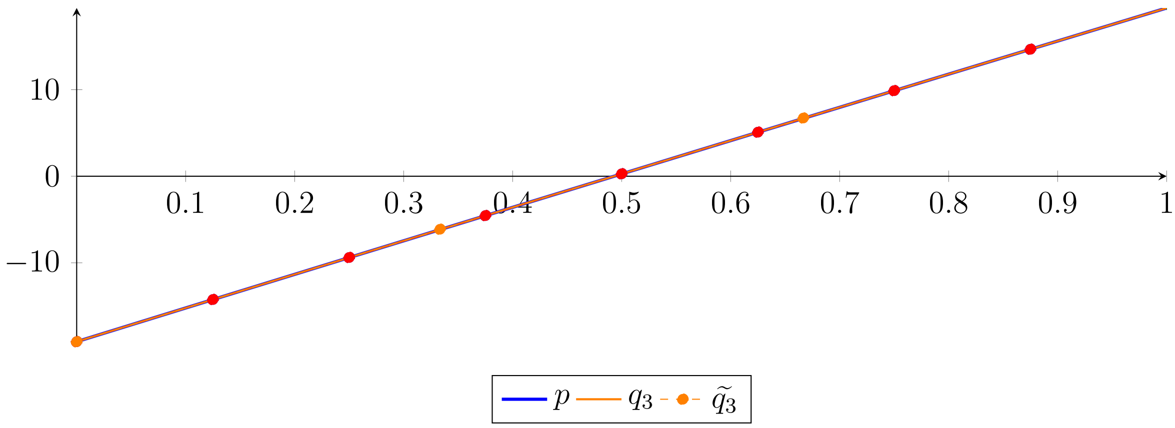
$$\begin{aligned} p = &-1.66533 \cdot 10^{-16}X^8 - 1.99007 \cdot 10^{-13}X^7 - 8.53059 \cdot 10^{-11}X^6 + 8.7284 \cdot 10^{-09}X^5 \\ &+ 1.00963 \cdot 10^{-05}X^4 - 0.000261161X^3 - 0.416284X^2 + 38.9643X - 19.1124 \\ = &-19.1124B_{0,8}(X) - 14.2419B_{1,8}(X) - 9.38619B_{2,8}(X) - 4.54539B_{3,8}(X) \\ &+ 0.280537B_{4,8}(X) + 5.09158B_{5,8}(X) + 9.88774B_{6,8}(X) + 14.669B_{7,8}(X) + 19.4354B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= 2.82588 \cdot 10^{-12}X^8 - 1.08529 \cdot 10^{-11}X^7 + 1.68738 \cdot 10^{-11}X^6 - 1.34636 \cdot 10^{-11}X^5 \\ &\quad + 5.72618 \cdot 10^{-12}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.
Bounding polynomials M and m:

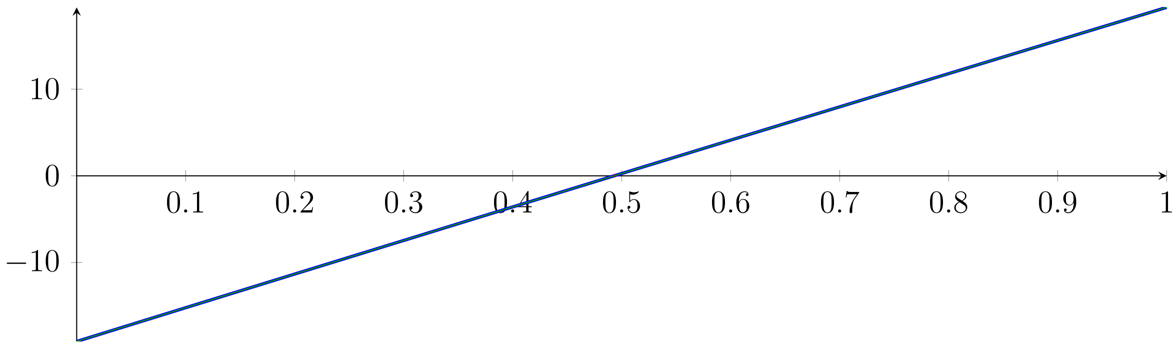
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.1252 \cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

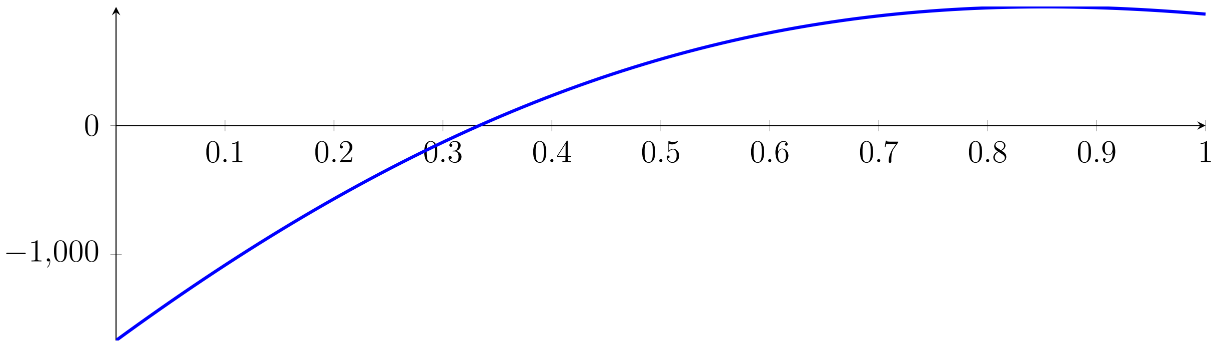
132.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

132.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 0.0001$.

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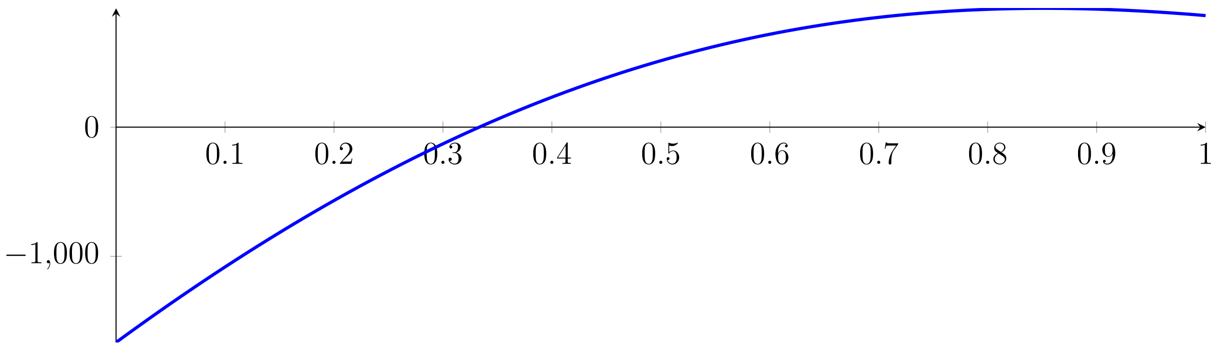
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

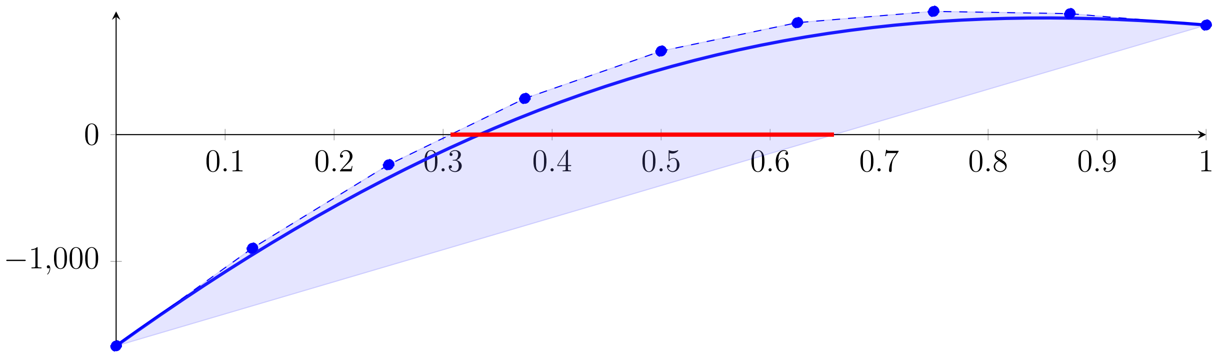
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



133.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

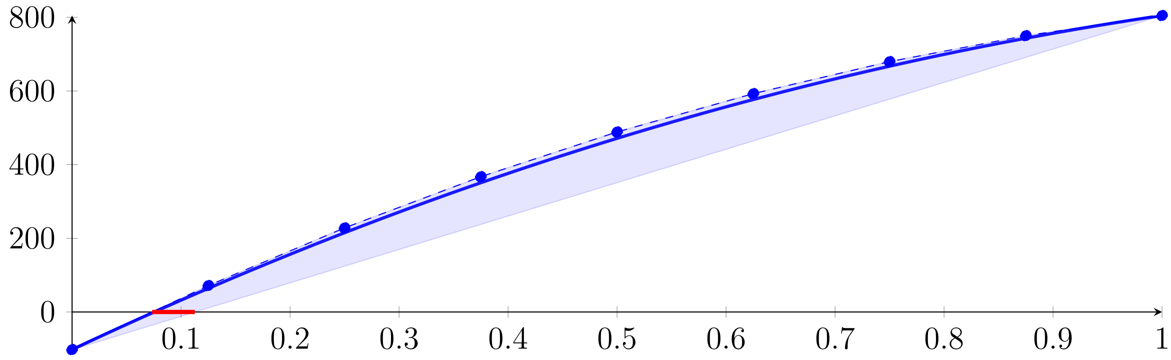
Longest intersection interval: 0.351792

$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

133.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17$$
$$= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

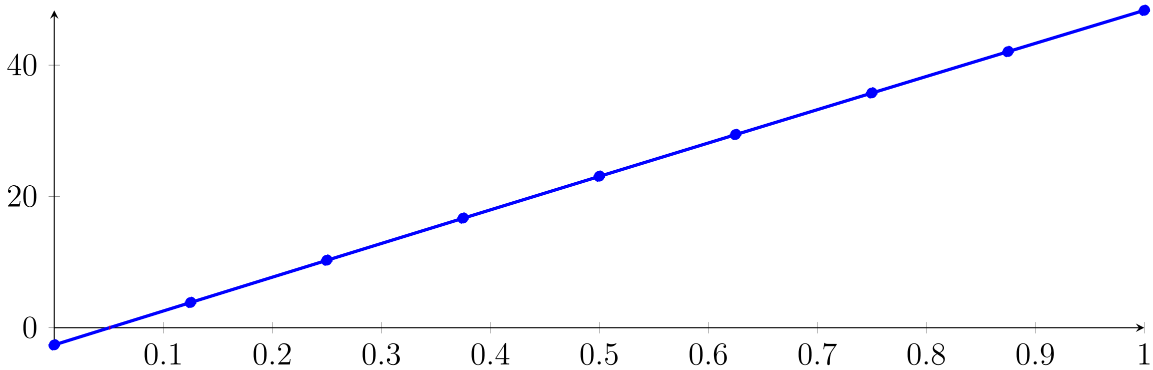
Longest intersection interval: 0.0391855

$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

133.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.03577 \cdot 10^{-15}X^8 - 1.54633 \cdot 10^{-12}X^7 - 4.95836 \cdot 10^{-10}X^6 + 3.66751 \cdot 10^{-08}X^5 + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683$$
$$= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

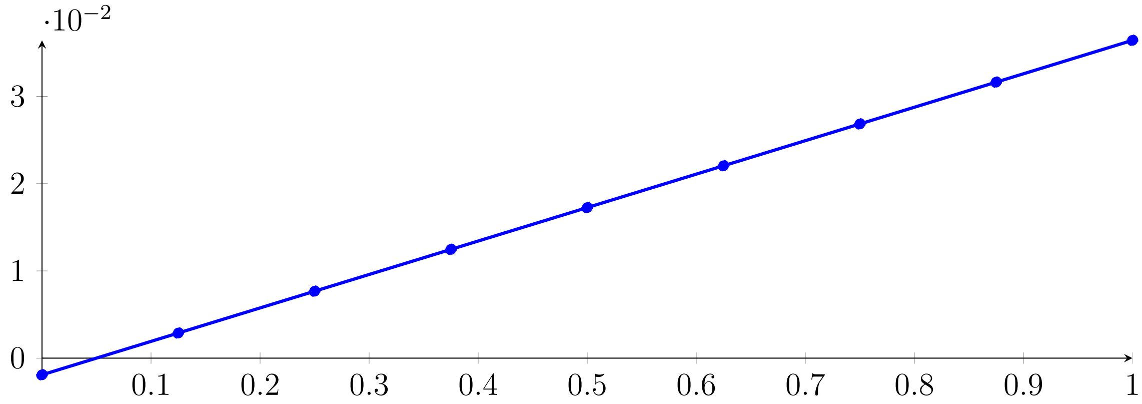
Longest intersection interval: 0.000742589

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

133.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.28918 \cdot 10^{-18} X^8 - 1.32815 \cdot 10^{-18} X^7 - 4.50622 \cdot 10^{-18} X^6 + 2.22939 \cdot 10^{-18} X^5 \\ &\quad + 9.48677 \cdot 10^{-18} X^4 - 2.37322 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\ &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\ &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\ &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

Longest intersection interval: $5.36469 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

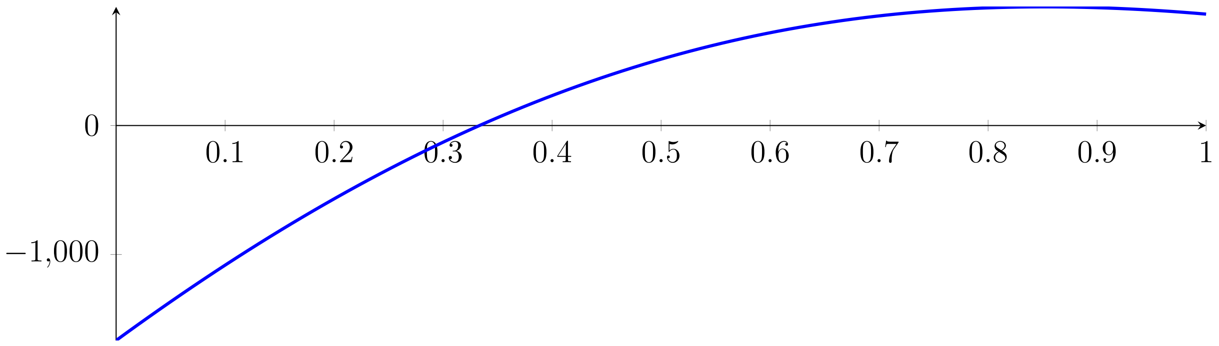
133.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

133.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-08}$.

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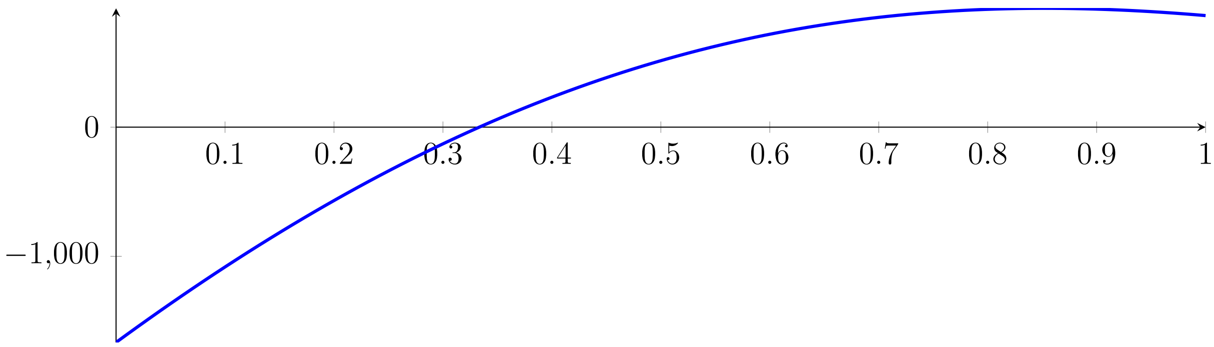
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0,1]$:

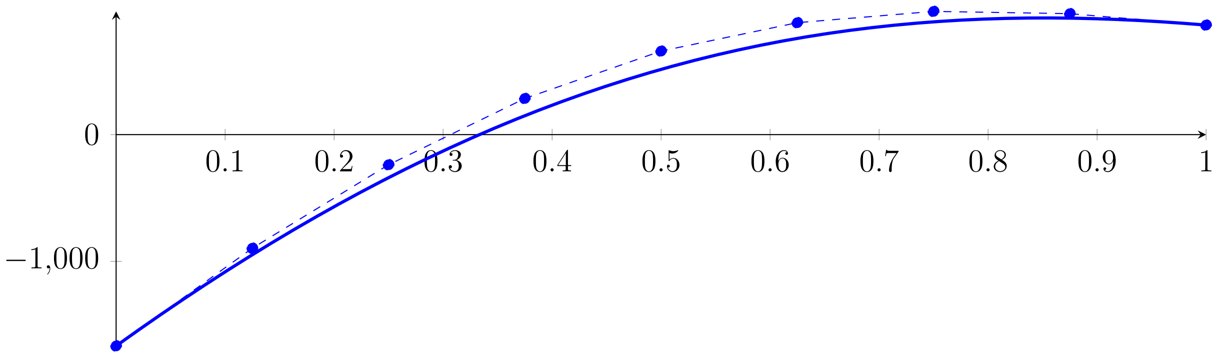
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



134.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

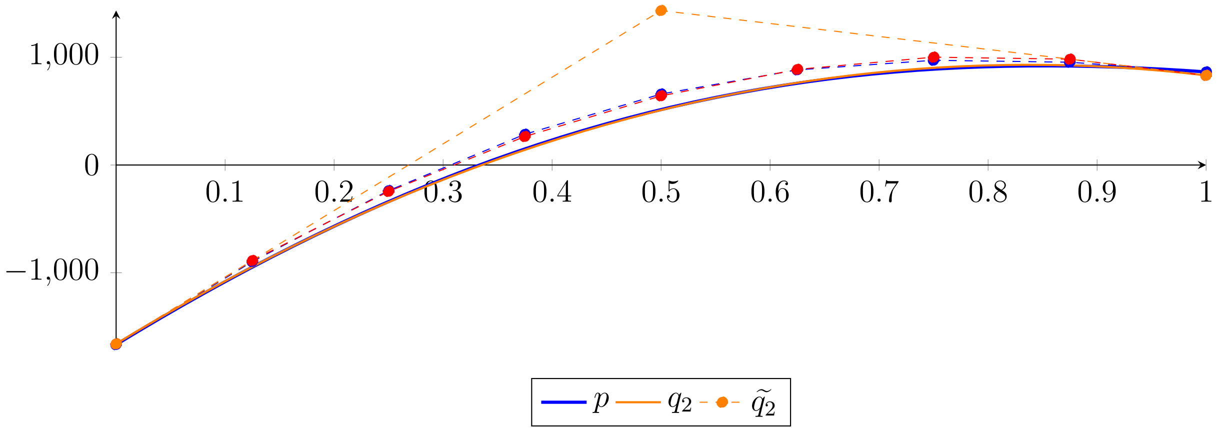
$$\begin{aligned}p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67\\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X)\\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -3695.78X^2 + 6187.64X - 1660\\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_2 &= -5.06304\cdot10^{-10}X^8 + 1.93305\cdot10^{-09}X^7 - 2.97598\cdot10^{-09}X^6 + 2.35116\cdot10^{-09}X^5\\ &\quad - 9.91232\cdot10^{-10}X^4 + 2.00705\cdot10^{-10}X^3 - 3695.78X^2 + 6187.64X - 1660\\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8}\\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8}\end{aligned}$$



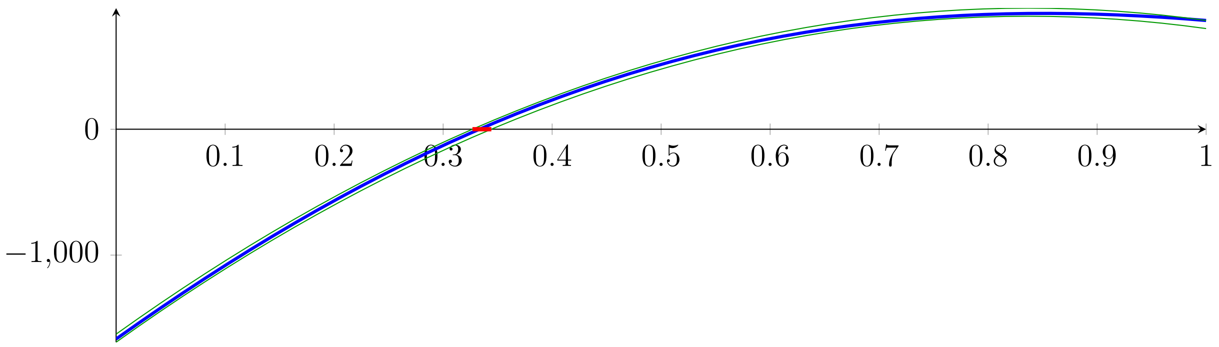
The maximum difference of the Bézier coefficients is $\delta = 32.1356$.
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



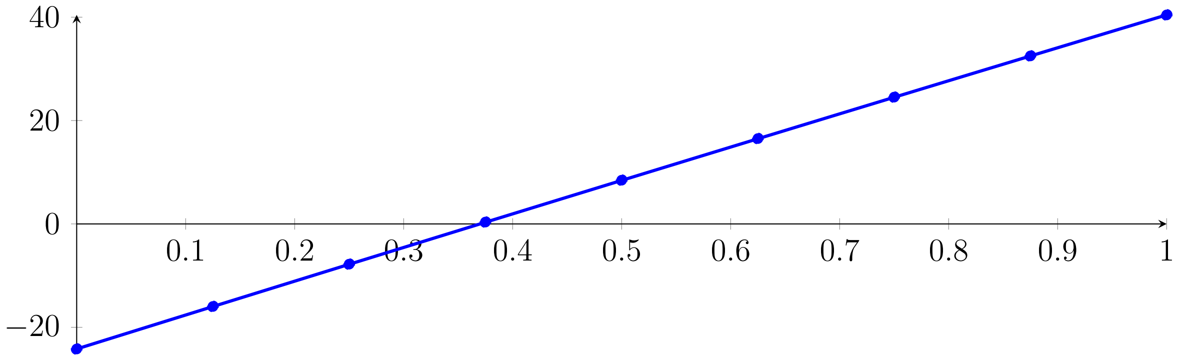
$$[0.326917, 0.344255]$$

Longest intersection interval: 0.0173372
$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

134.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

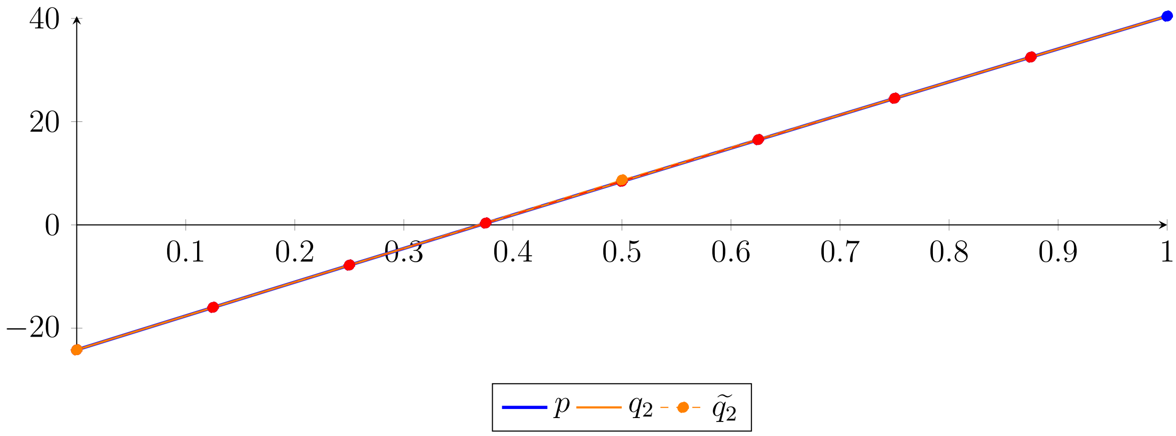
$$\begin{aligned} p = &-8.80546\cdot 10^{-15}X^8 - 7.66587\cdot 10^{-12}X^7 - 1.9444\cdot 10^{-09}X^6 + 1.19263\cdot 10^{-07}X^5 \\ &+ 8.12335\cdot 10^{-05}X^4 - 0.0012733X^3 - 1.18084X^2 + 65.8155X - 24.1945 \\ = &-24.1945B_{0,8}(X) - 15.9676B_{1,8}(X) - 7.78282B_{2,8}(X) + 0.35975B_{3,8}(X) \\ &+ 8.4601B_{4,8}(X) + 16.5182B_{5,8}(X) + 24.5341B_{6,8}(X) + 32.5077B_{7,8}(X) + 40.4389B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -2.59056\cdot 10^{-11}X^8 + 1.00262\cdot 10^{-10}X^7 - 1.57692\cdot 10^{-10}X^6 + 1.29283\cdot 10^{-10}X^5 \\ &\quad - 5.86775\cdot 10^{-11}X^4 + 1.42642\cdot 10^{-11}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894\cdot 10^{-05}$.
Bounding polynomials M and m:

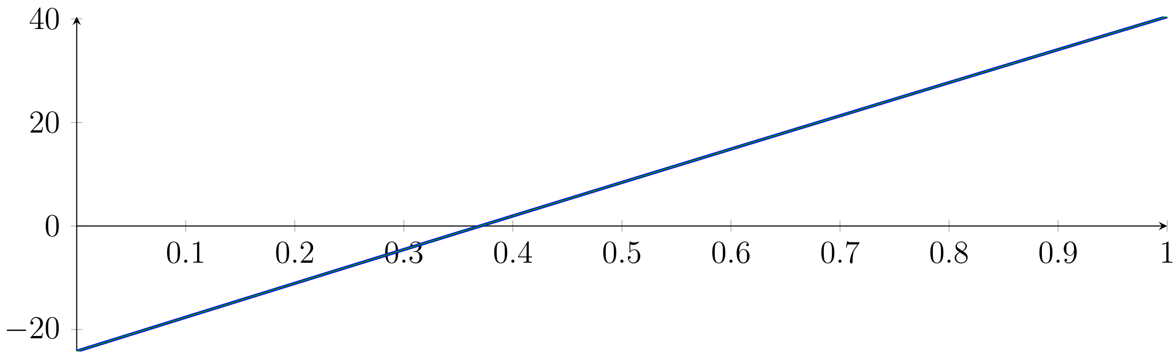
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



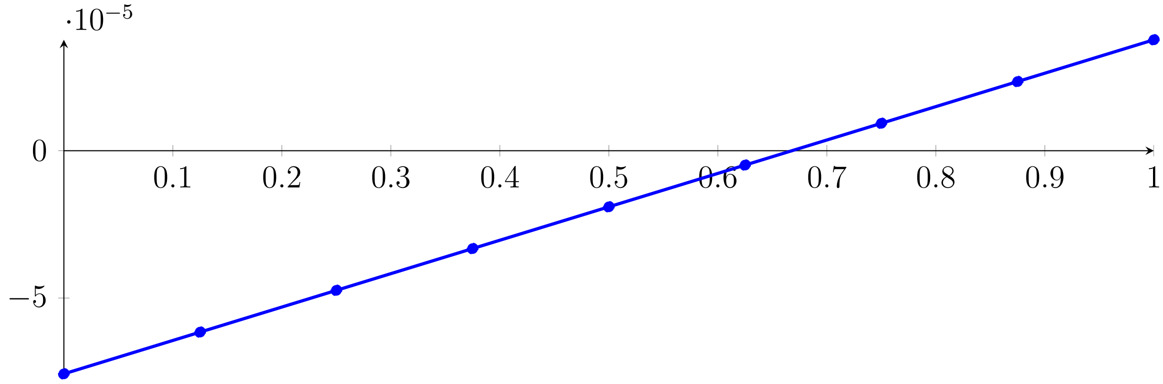
$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588\cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

134.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

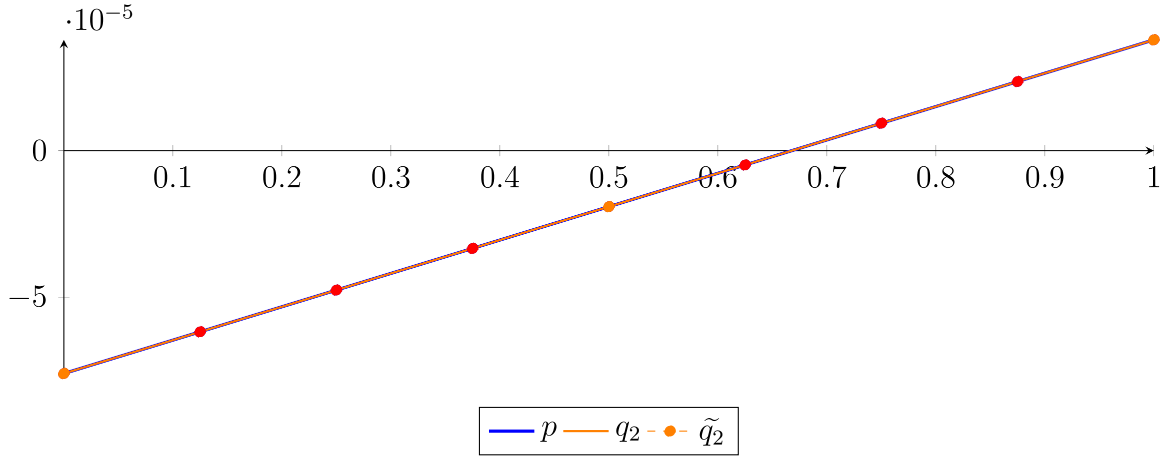
$$p = 1.16467 \cdot 10^{-21} X^8 + 3.17637 \cdot 10^{-21} X^7 + 1.18585 \cdot 10^{-20} X^6 - 1.48231 \cdot 10^{-21} X^5 + 9.26442 \cdot 10^{-22} X^4 - 5.92923 \cdot 10^{-21} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$
$$= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)$$



Degree reduction and raising:

$$q_2 = -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$
$$= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}$$

$$\widetilde{q}_2 = 3.26671 \cdot 10^{-17} X^8 - 1.38104 \cdot 10^{-16} X^7 + 2.39221 \cdot 10^{-16} X^6 - 2.17429 \cdot 10^{-16} X^5 + 1.10046 \cdot 10^{-16} X^4 - 3.0162 \cdot 10^{-17} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$
$$= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 1.84643 \cdot 10^{-19}$.

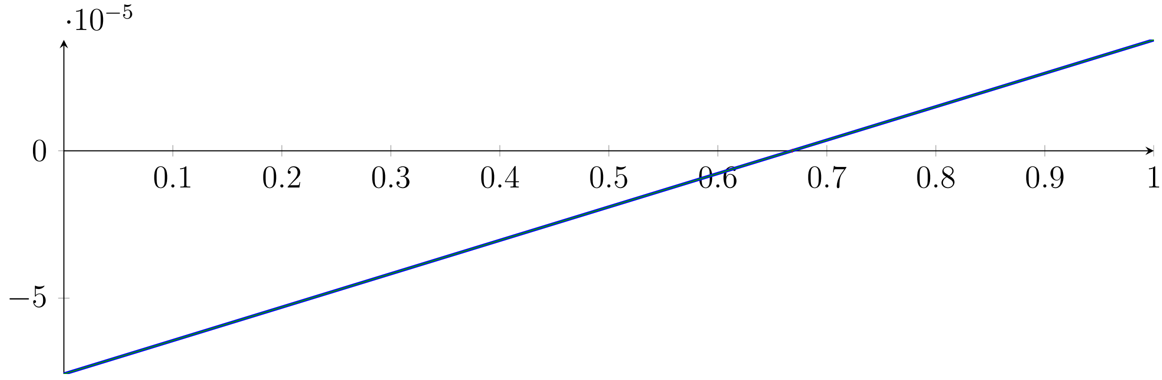
Bounding polynomials M and m:

$$M = -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$
$$m = -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.667955, 3.14644 \cdot 10^{7}\} \qquad N(m) = \{0.667955, 3.14644 \cdot 10^{7}\}$$

Intersection intervals:



$[0.667955, 0.667955]$

Longest intersection interval: $3.08439 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

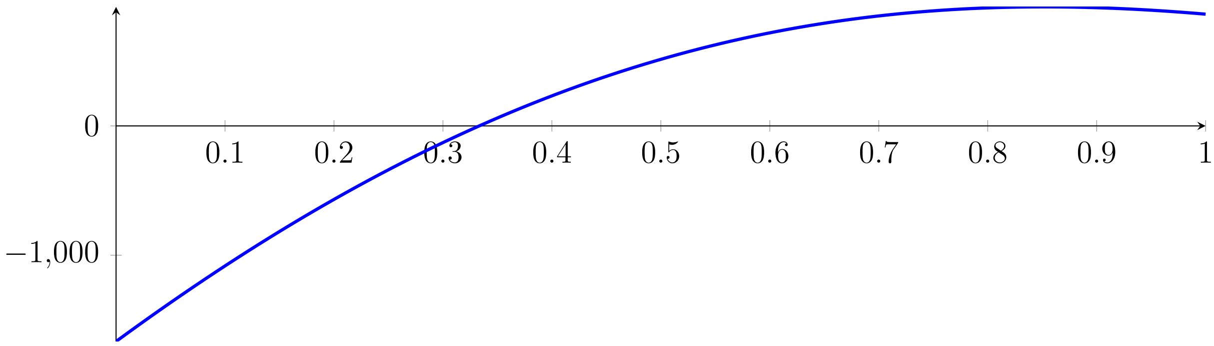
134.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

134.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-08}$.

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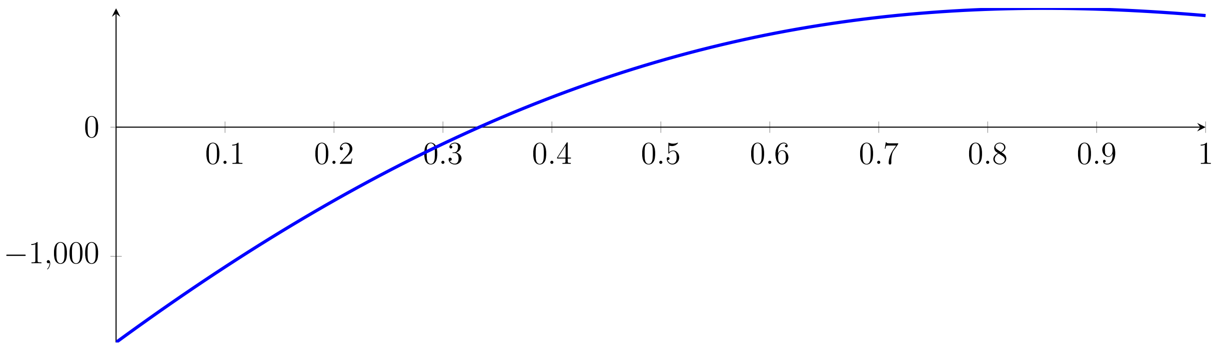
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

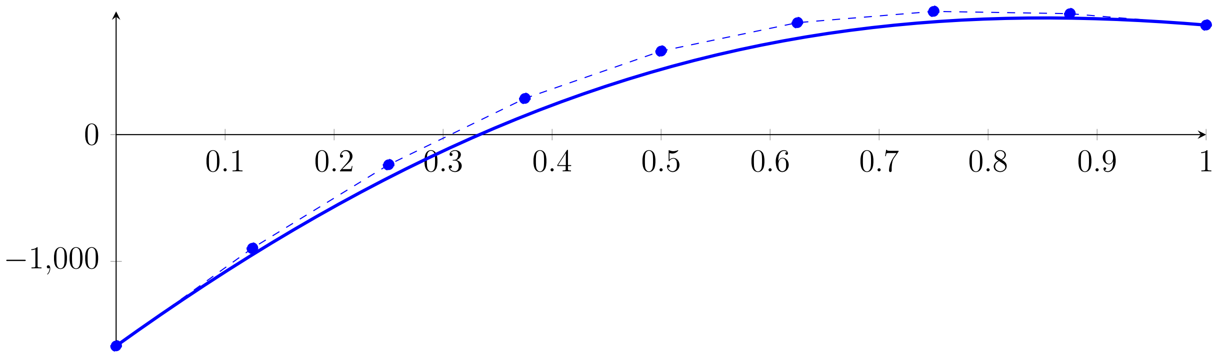
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



135.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

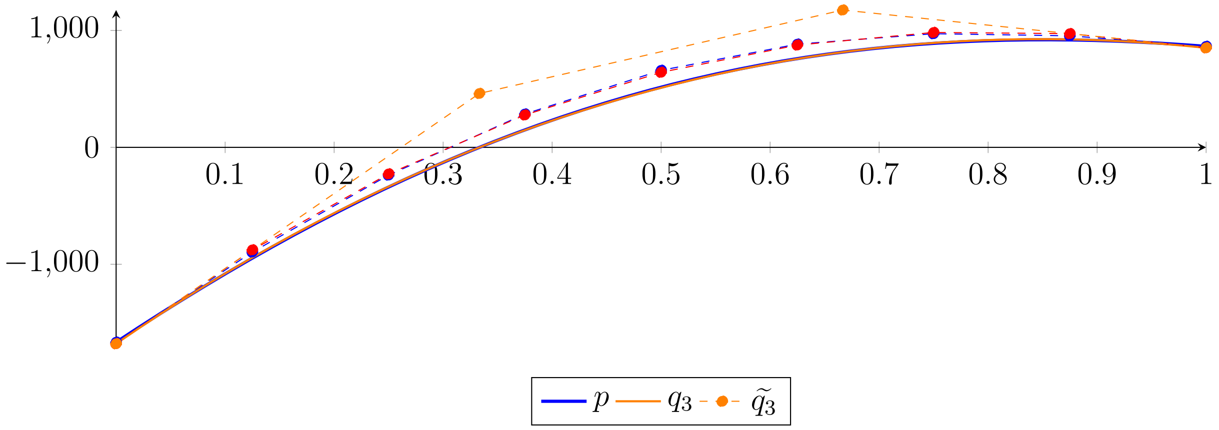
$$\begin{aligned}p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67\\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X)\\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47\\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3}\end{aligned}$$

$$\begin{aligned}\tilde{q}_3 &= -2.39389 \cdot 10^{-09}X^8 + 9.60737 \cdot 10^{-09}X^7 - 1.57477 \cdot 10^{-08}X^6 + 1.35479 \cdot 10^{-08}X^5\\ &\quad - 6.53397 \cdot 10^{-09}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47\\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8}\\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

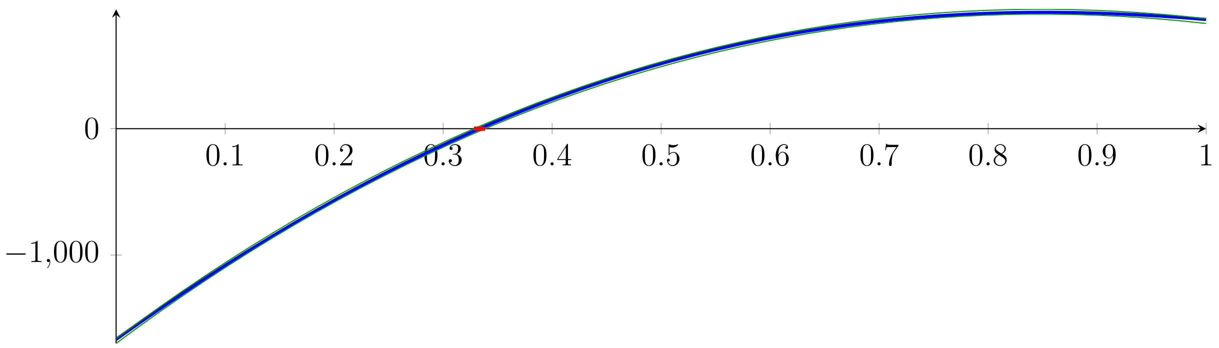
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

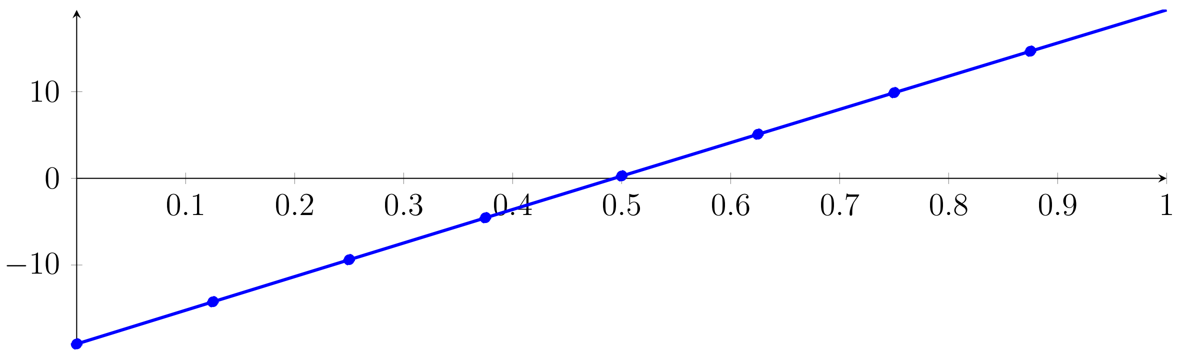
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

135.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

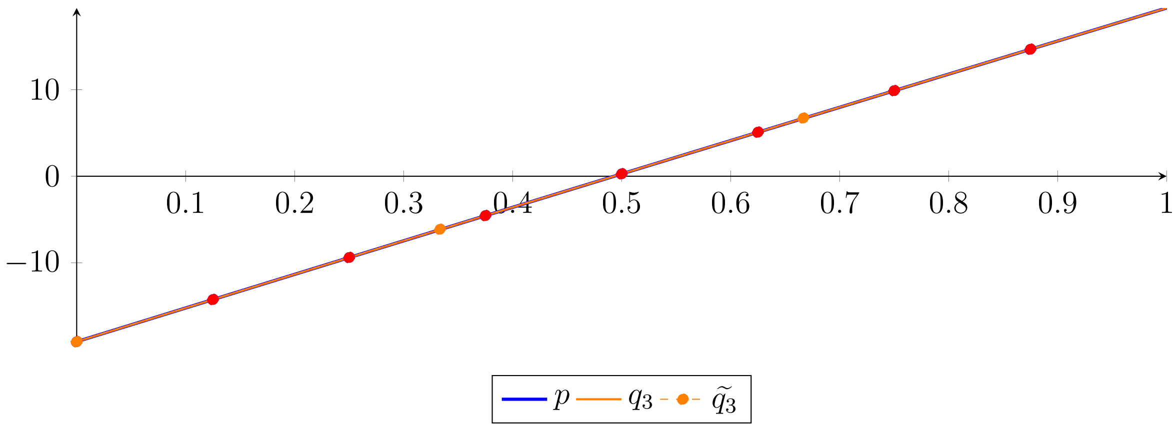
$$\begin{aligned} p = &-1.66533 \cdot 10^{-16} X^8 - 1.99007 \cdot 10^{-13} X^7 - 8.53059 \cdot 10^{-11} X^6 + 8.7284 \cdot 10^{-09} X^5 \\ &+ 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ = &-19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ &+ 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= 2.82588\cdot10^{-12}X^8 - 1.08529\cdot10^{-11}X^7 + 1.68738\cdot10^{-11}X^6 - 1.34636\cdot10^{-11}X^5 \\ &\quad + 5.72618\cdot10^{-12}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902\cdot10^{-07}$.

Bounding polynomials M and m:

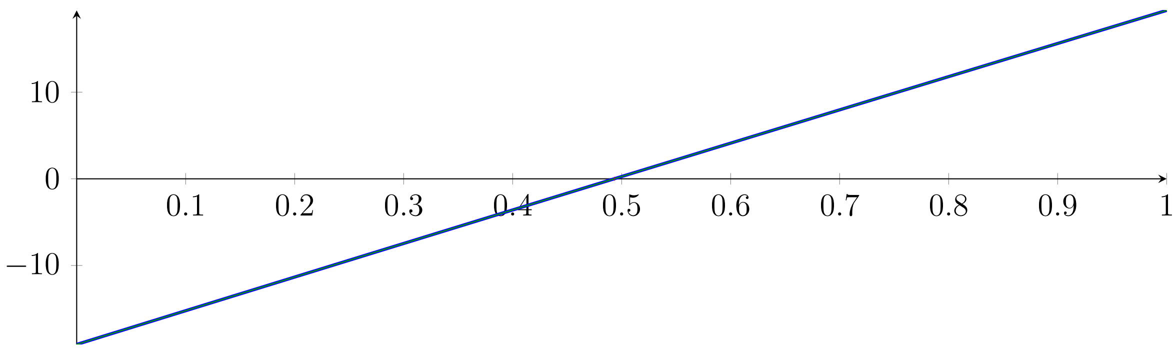
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.1252\cdot10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

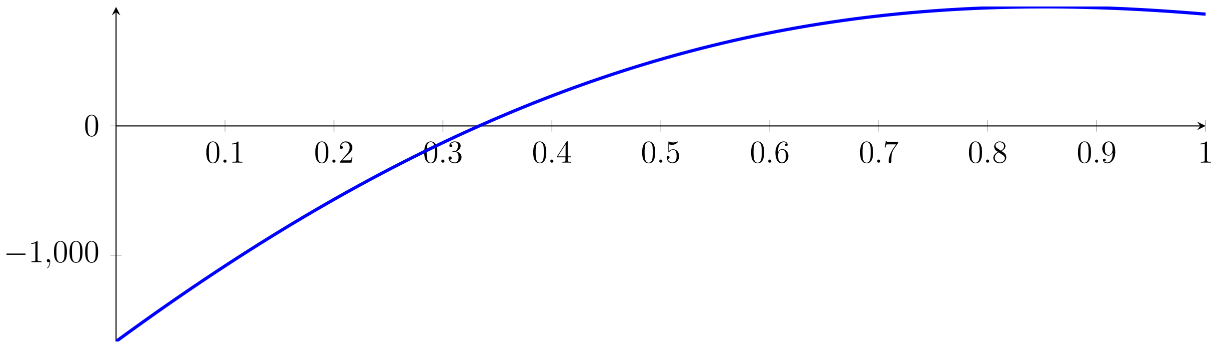
135.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

135.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-08}$.

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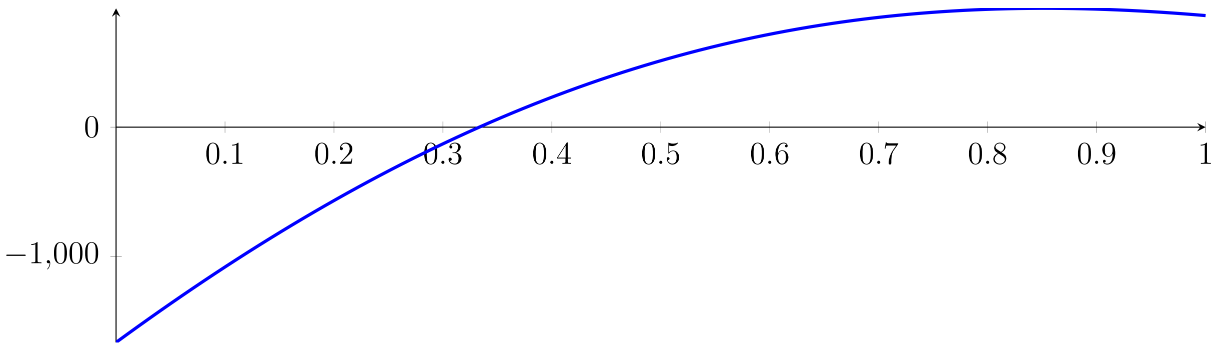
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

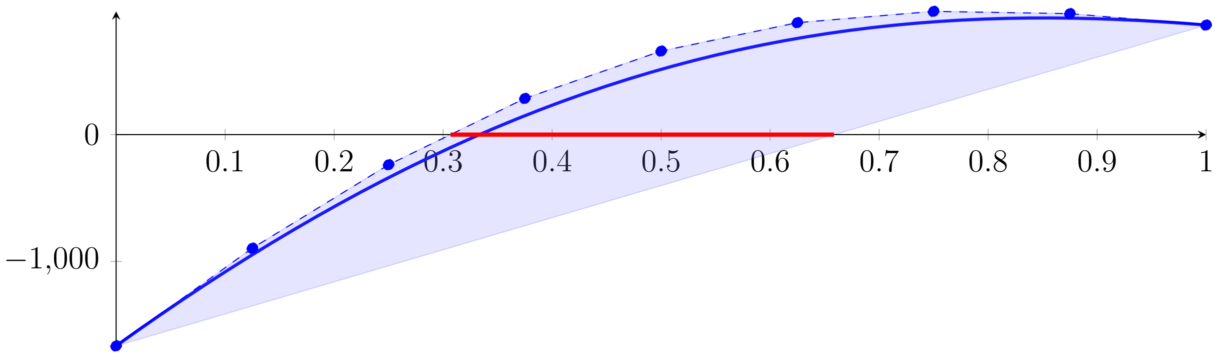
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



136.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

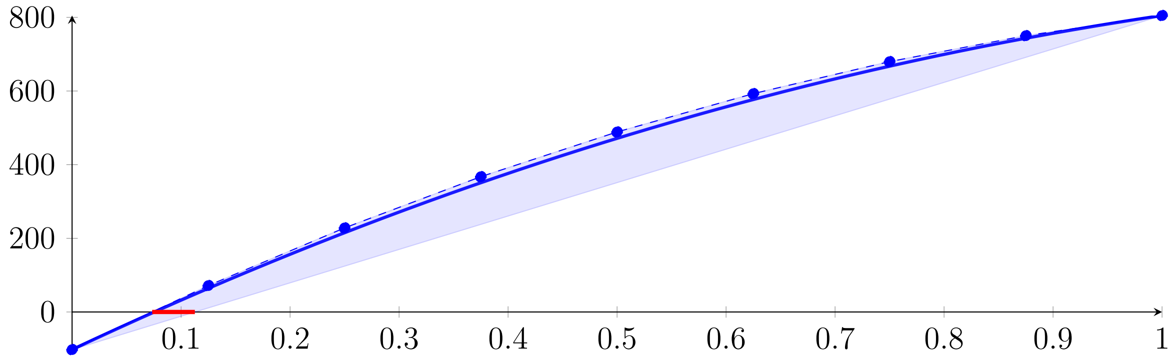
$$[0.306796, 0.658588]$$

Longest intersection interval: 0.351792
$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

136.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\ &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\ &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\ &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

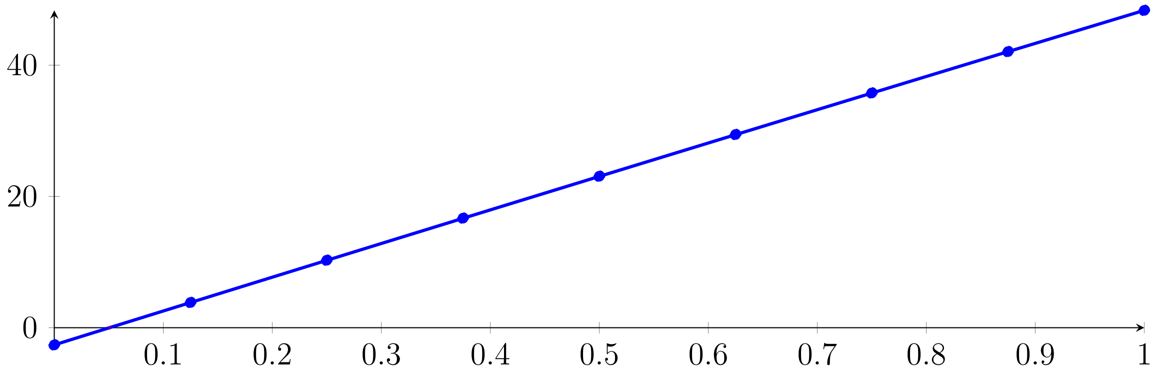
$$[0.0734515, 0.112637]$$

Longest intersection interval: 0.0391855
$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

136.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -3.03577 \cdot 10^{-15}X^8 - 1.54633 \cdot 10^{-12}X^7 - 4.95836 \cdot 10^{-10}X^6 + 3.66751 \cdot 10^{-08}X^5 \\ &\quad + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\ &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\ &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

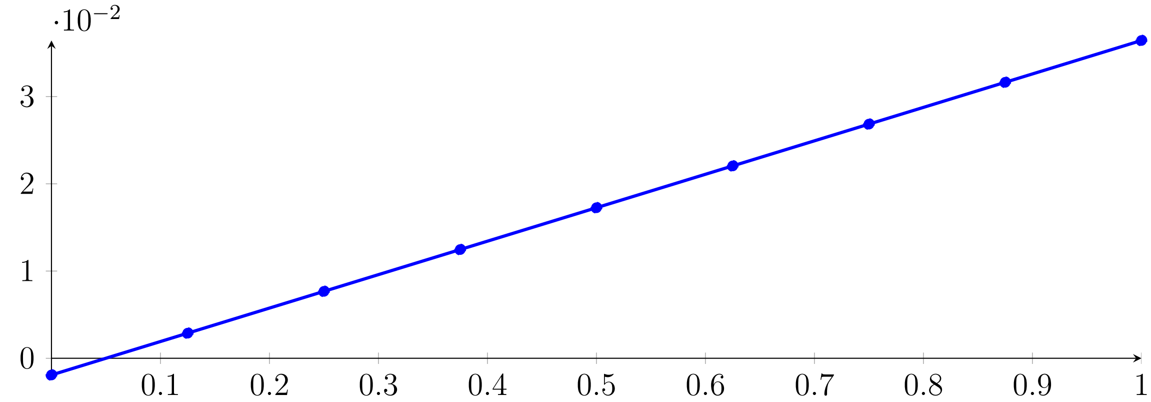
$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

136.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.28918{\cdot}10^{-18}X^8 - 1.32815{\cdot}10^{-18}X^7 - 4.50622{\cdot}10^{-18}X^6 + 2.22939{\cdot}10^{-18}X^5 \\ &\quad + 9.48677{\cdot}10^{-18}X^4 - 2.37322{\cdot}10^{-13}X^3 - 4.12146{\cdot}10^{-07}X^2 + 0.0383444X - 0.00191378 \\ &= -0.00191378B_{0,8}(X) + 0.00287927B_{1,8}(X) + 0.00767231B_{2,8}(X) \\ &\quad + 0.0124653B_{3,8}(X) + 0.0172583B_{4,8}(X) + 0.0220513B_{5,8}(X) \\ &\quad + 0.0268443B_{6,8}(X) + 0.0316373B_{7,8}(X) + 0.0364302B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

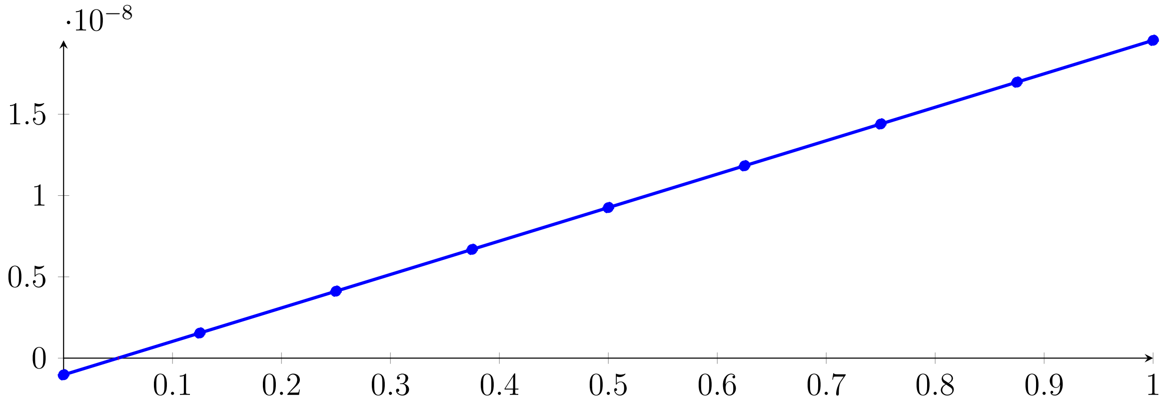
$$[0.0499103, 0.0499109]$$

Longest intersection interval: $5.36469{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

136.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -7.43978{\cdot}10^{-25}X^8 - 4.71751{\cdot}10^{-25}X^7 - 2.488{\cdot}10^{-24}X^6 + 1.04044{\cdot}10^{-24}X^5 \\ &\quad - 2.26182{\cdot}10^{-25}X^4 - 1.18615{\cdot}10^{-19}X^2 + 2.05706{\cdot}10^{-08}X - 1.02667{\cdot}10^{-09} \\ &= -1.02667{\cdot}10^{-09}B_{0,8}(X) + 1.54465{\cdot}10^{-09}B_{1,8}(X) + 4.11597{\cdot}10^{-09}B_{2,8}(X) \\ &\quad + 6.68729{\cdot}10^{-09}B_{3,8}(X) + 9.25862{\cdot}10^{-09}B_{4,8}(X) + 1.18299{\cdot}10^{-08}B_{5,8}(X) \\ &\quad + 1.44013{\cdot}10^{-08}B_{6,8}(X) + 1.69726{\cdot}10^{-08}B_{7,8}(X) + 1.95439{\cdot}10^{-08}B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87793 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

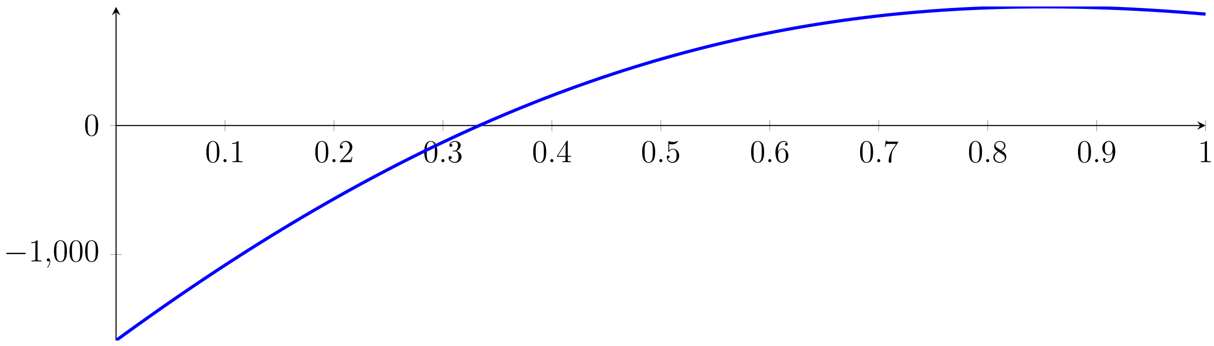
136.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

136.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-16}$.

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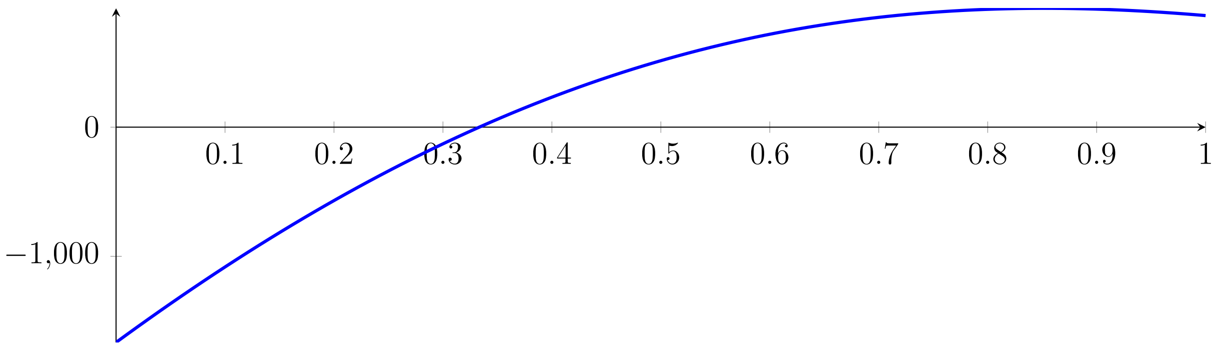
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

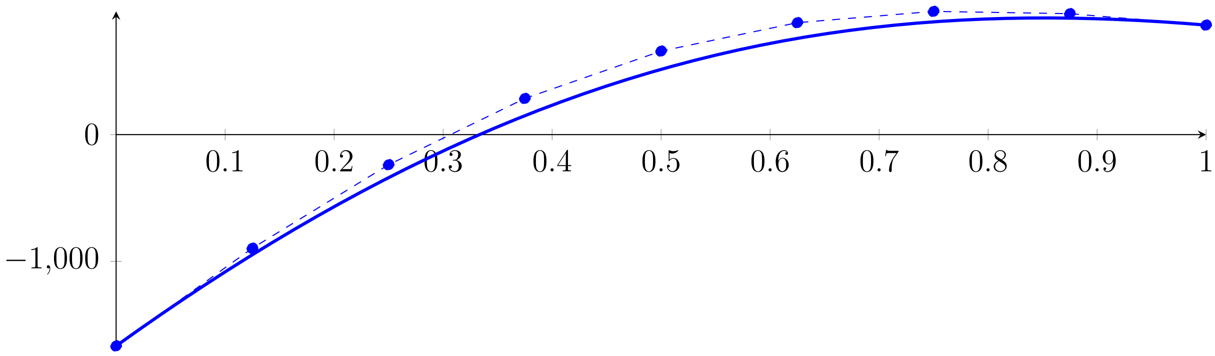
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



137.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

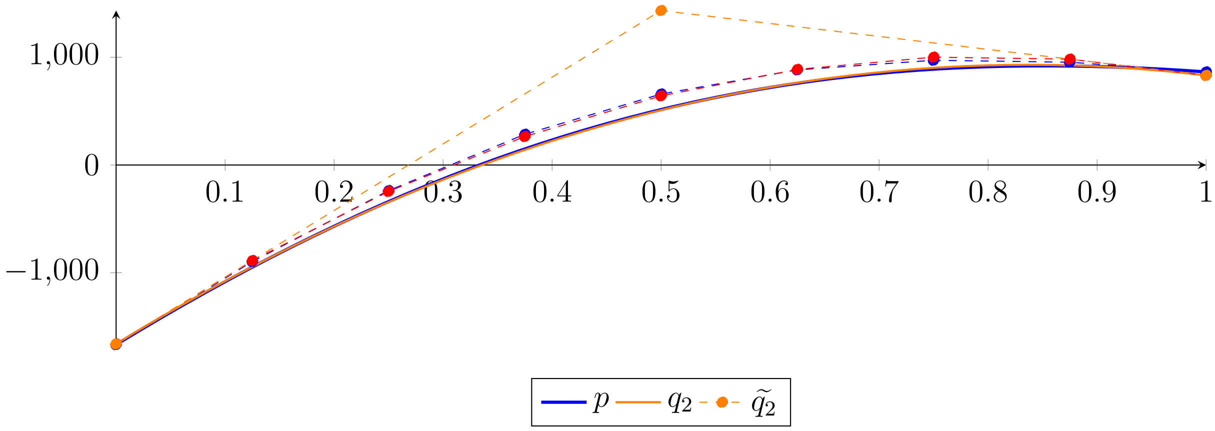
$$\begin{aligned}p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67\\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X)\\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -3695.78X^2 + 6187.64X - 1660\\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q_2} &= -5.06304\cdot10^{-10}X^8 + 1.93305\cdot10^{-09}X^7 - 2.97598\cdot10^{-09}X^6 + 2.35116\cdot10^{-09}X^5\\ &\quad - 9.91232\cdot10^{-10}X^4 + 2.00705\cdot10^{-10}X^3 - 3695.78X^2 + 6187.64X - 1660\\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8}\\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

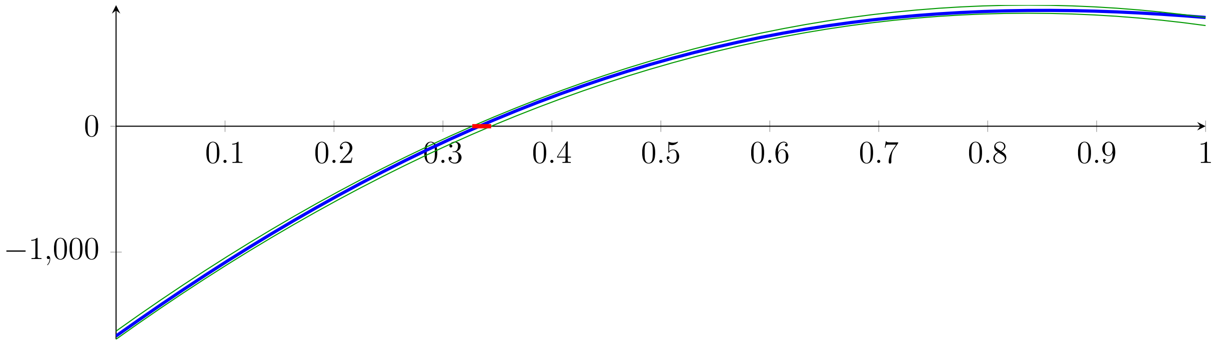
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

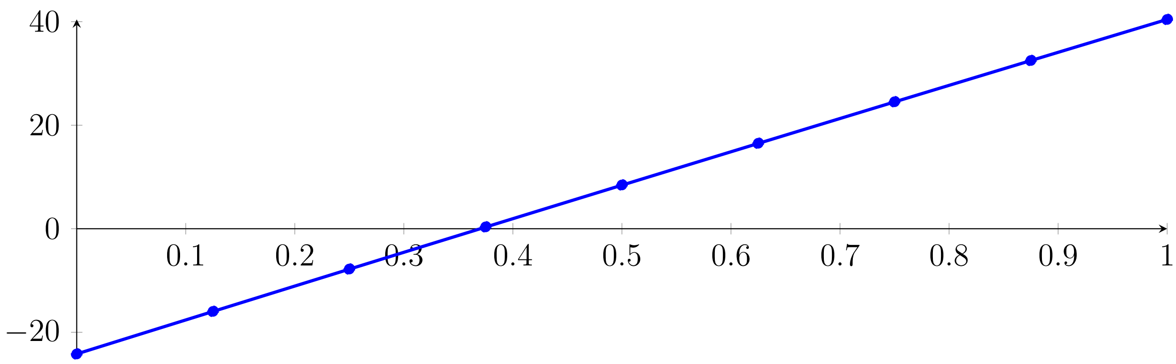
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

137.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

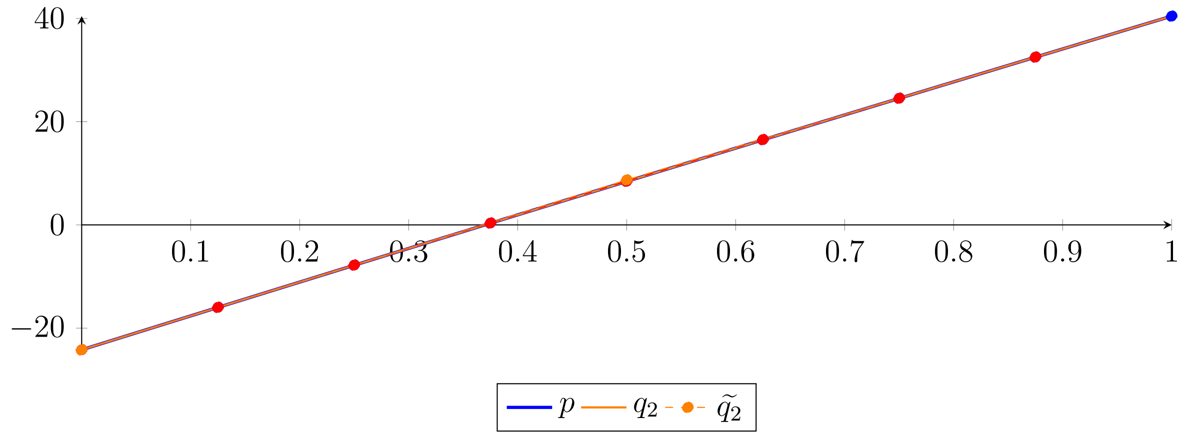
$$\begin{aligned} p = & -8.80546\cdot 10^{-15}X^8 - 7.66587\cdot 10^{-12}X^7 - 1.9444\cdot 10^{-09}X^6 + 1.19263\cdot 10^{-07}X^5 \\ & + 8.12335\cdot 10^{-05}X^4 - 0.0012733X^3 - 1.18084X^2 + 65.8155X - 24.1945 \\ = & -24.1945B_{0,8}(X) - 15.9676B_{1,8}(X) - 7.78282B_{2,8}(X) + 0.35975B_{3,8}(X) \\ & + 8.4601B_{4,8}(X) + 16.5182B_{5,8}(X) + 24.5341B_{6,8}(X) + 32.5077B_{7,8}(X) + 40.4389B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -2.59056\cdot10^{-11}X^8 + 1.00262\cdot10^{-10}X^7 - 1.57692\cdot10^{-10}X^6 + 1.29283\cdot10^{-10}X^5 \\ &\quad - 5.86775\cdot10^{-11}X^4 + 1.42642\cdot10^{-11}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894\cdot10^{-05}$.
Bounding polynomials M and m:

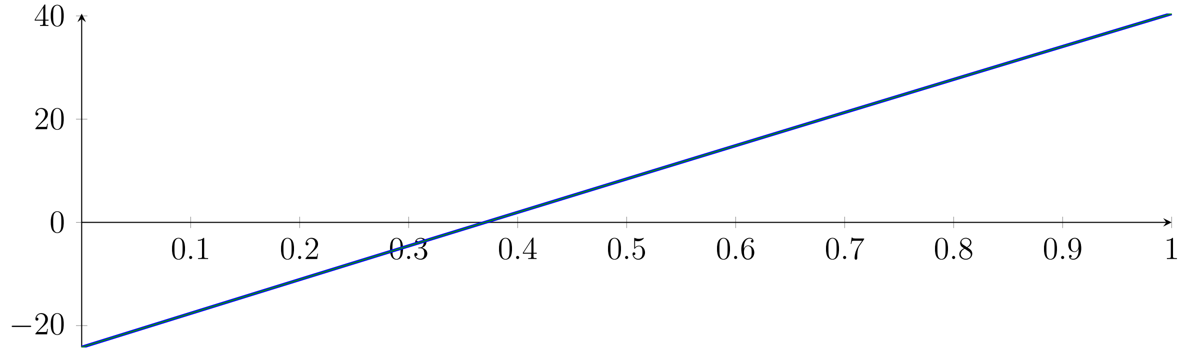
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



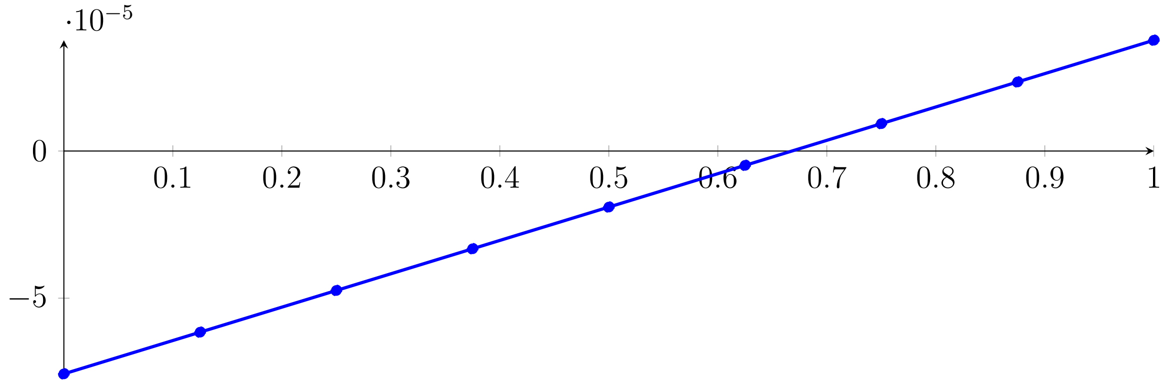
$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588\cdot10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

137.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

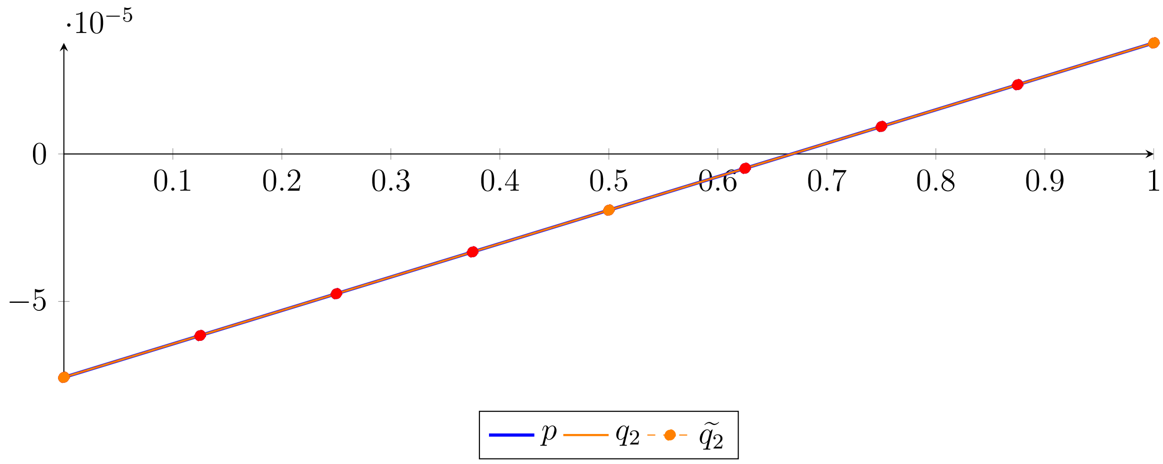
$$p = 1.16467 \cdot 10^{-21} X^8 + 3.17637 \cdot 10^{-21} X^7 + 1.18585 \cdot 10^{-20} X^6 - 1.48231 \cdot 10^{-21} X^5 + 9.26442 \cdot 10^{-22} X^4 - 5.92923 \cdot 10^{-21} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$
$$= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)$$



Degree reduction and raising:

$$q_2 = -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$
$$= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = 3.26671 \cdot 10^{-17} X^8 - 1.38104 \cdot 10^{-16} X^7 + 2.39221 \cdot 10^{-16} X^6 - 2.17429 \cdot 10^{-16} X^5 + 1.10046 \cdot 10^{-16} X^4 - 3.0162 \cdot 10^{-17} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$
$$= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 1.84643 \cdot 10^{-19}$.

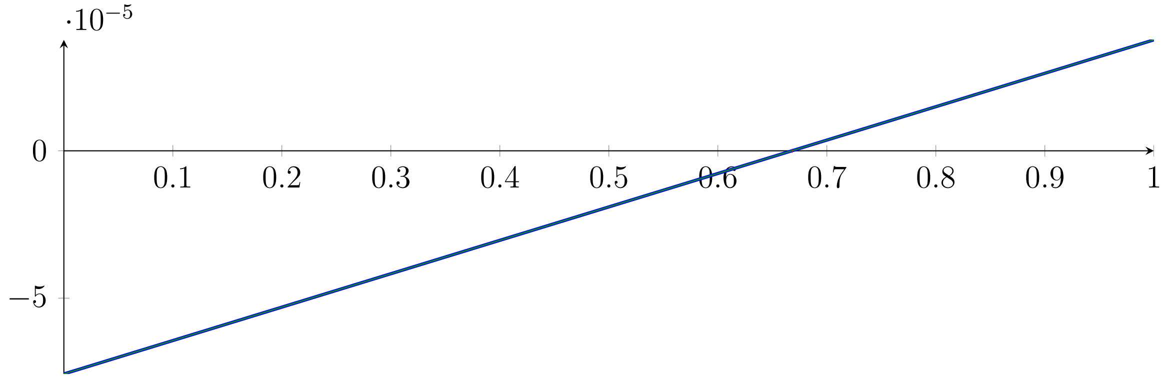
Bounding polynomials M and m:

$$M = -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$
$$m = -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.667955, 3.14644 \cdot 10^{7}\} \qquad N(m) = \{0.667955, 3.14644 \cdot 10^{7}\}$$

Intersection intervals:



$[0.667955, 0.667955]$

Longest intersection interval: $3.08439 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

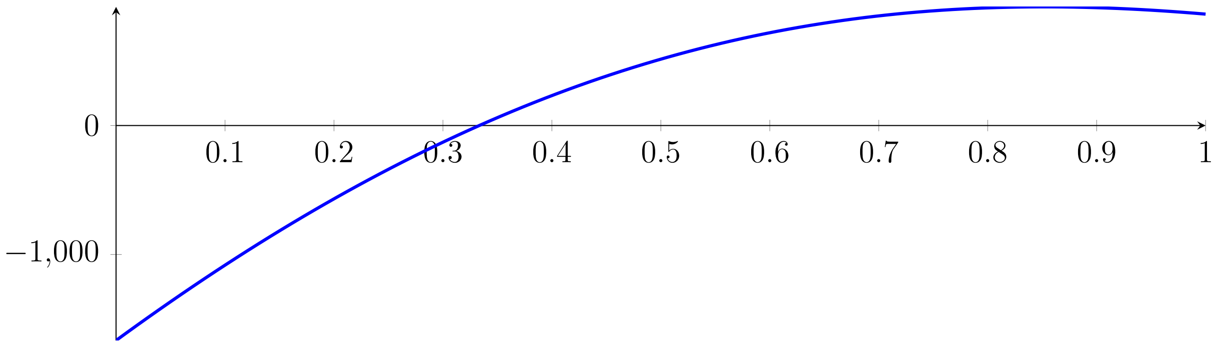
137.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

137.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-16}$.

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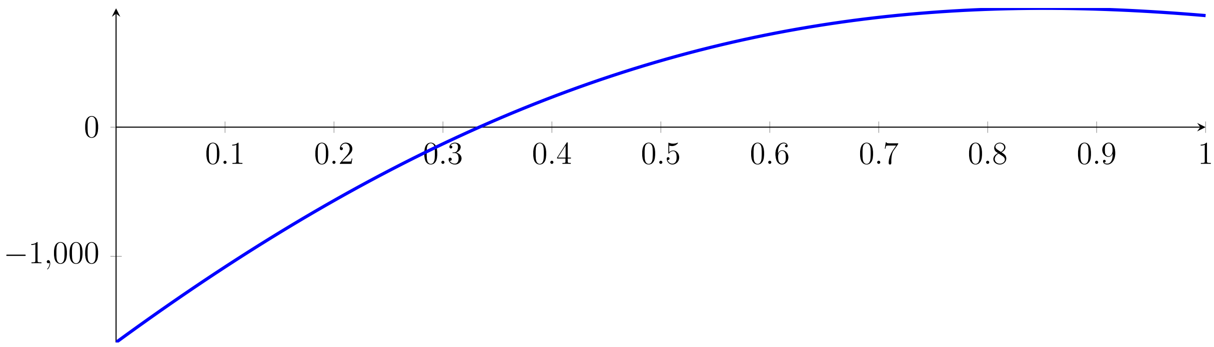
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

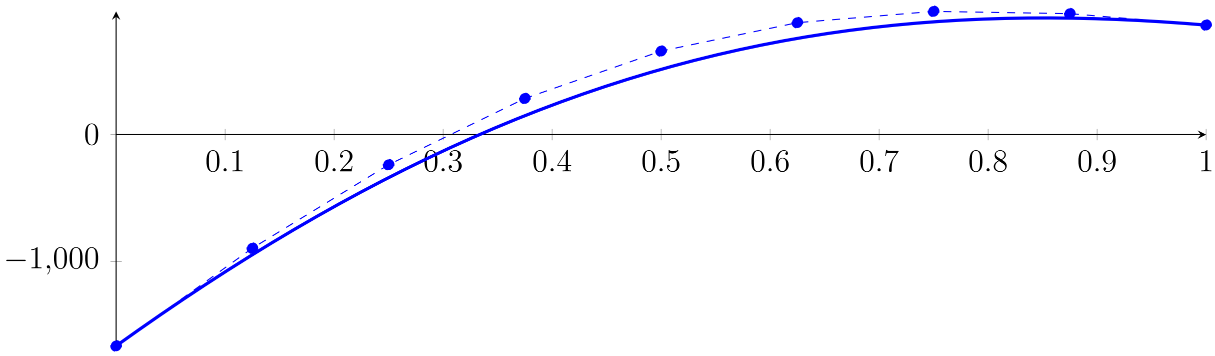
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



138.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

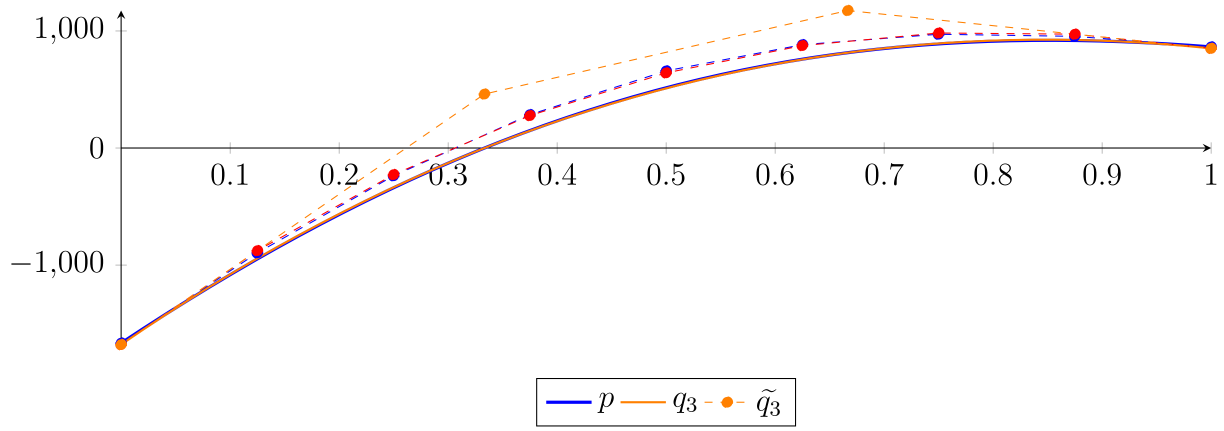
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.39389 \cdot 10^{-09}X^8 + 9.60737 \cdot 10^{-09}X^7 - 1.57477 \cdot 10^{-08}X^6 + 1.35479 \cdot 10^{-08}X^5 \\ &\quad - 6.53397 \cdot 10^{-09}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



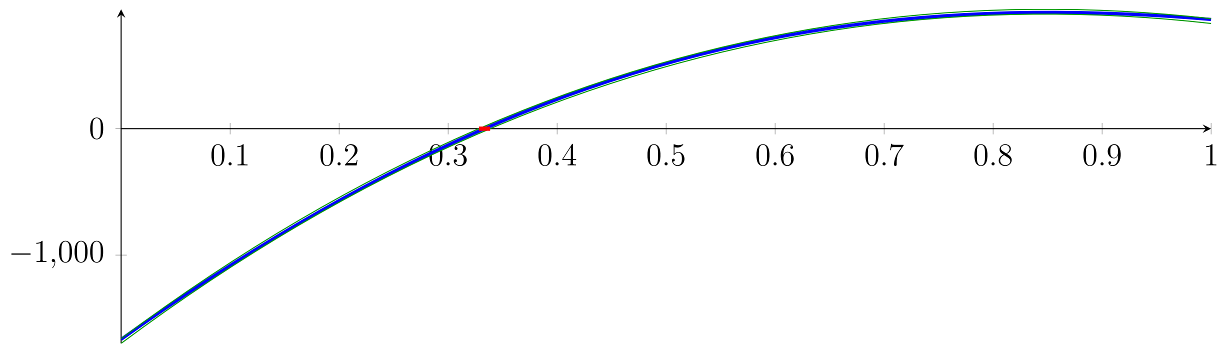
The maximum difference of the Bézier coefficients is $\delta = 19.0273$.
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



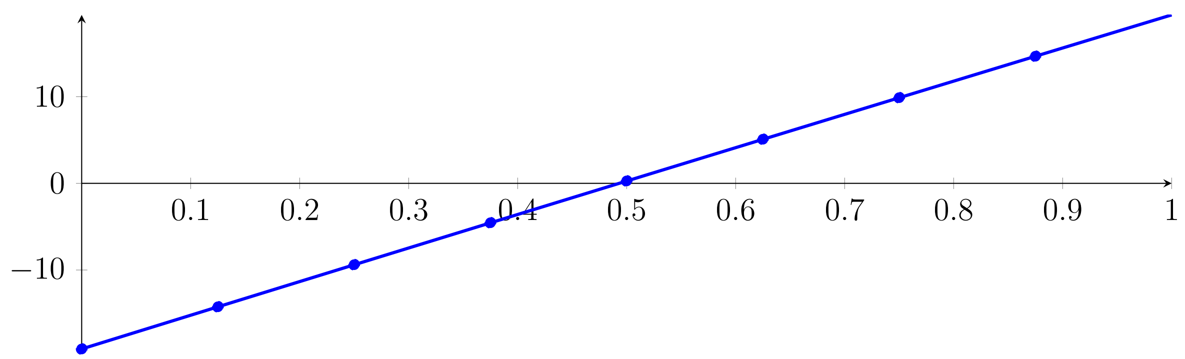
$[0.328258, 0.338551]$

Longest intersection interval: 0.0102926
$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

138.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

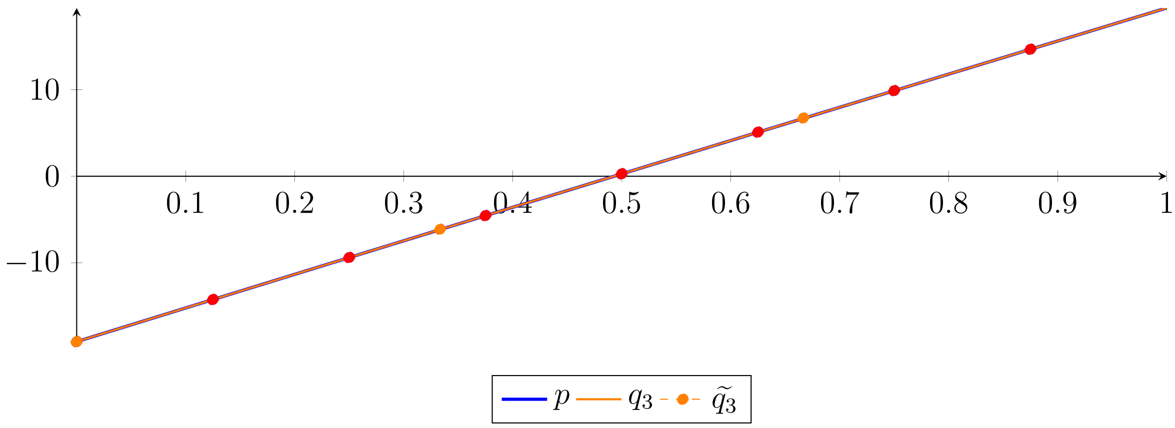
$$\begin{aligned} p = & -1.66533 \cdot 10^{-16} X^8 - 1.99007 \cdot 10^{-13} X^7 - 8.53059 \cdot 10^{-11} X^6 + 8.7284 \cdot 10^{-09} X^5 \\ & + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ = & -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ & + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2.82588{\cdot}10^{-12}X^8 - 1.08529{\cdot}10^{-11}X^7 + 1.68738{\cdot}10^{-11}X^6 - 1.34636{\cdot}10^{-11}X^5 \\ &\quad + 5.72618{\cdot}10^{-12}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902{\cdot}10^{-07}$.

Bounding polynomials M and m:

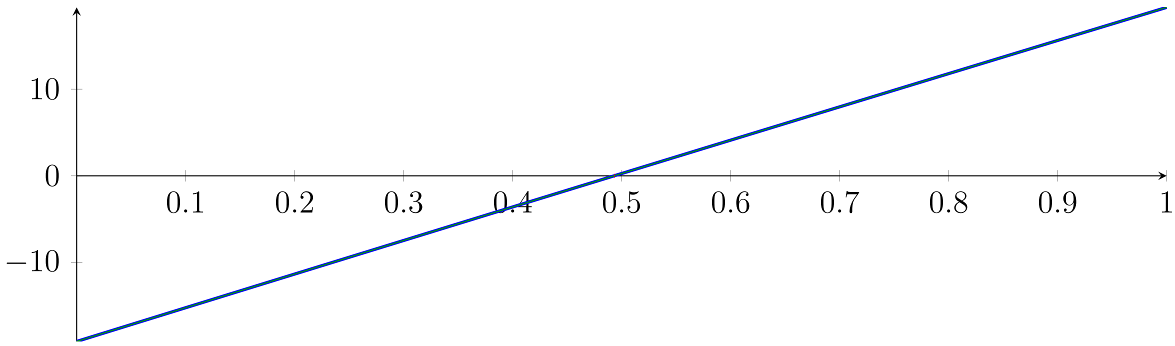
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

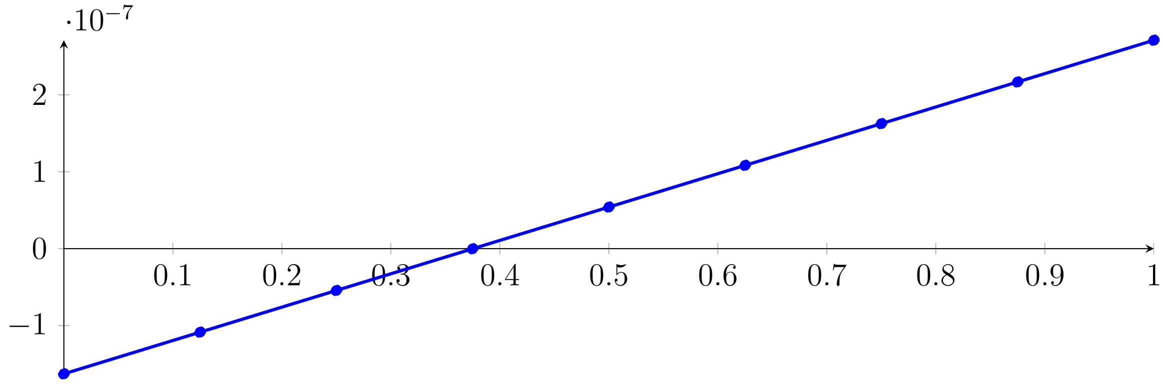
Longest intersection interval: $1.1252{\cdot}10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

138.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

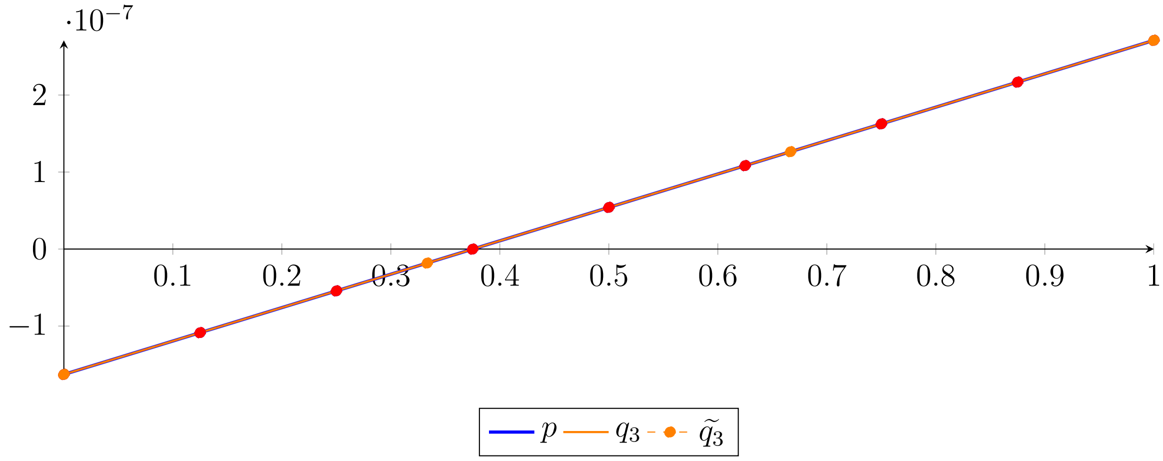
$$p = -4.65289\cdot10^{-24}X^8 - 8.27181\cdot10^{-25}X^7 - 4.3427\cdot10^{-24}X^6 + 8.6854\cdot10^{-24}X^5 + 1.80946\cdot10^{-24}X^4 - 7.23783\cdot10^{-25}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,8}(X) - 1.08585\cdot10^{-07}B_{1,8}(X) - 5.43593\cdot10^{-08}B_{2,8}(X) - 1.33715\cdot10^{-10}B_{3,8}(X) + 5.40918\cdot10^{-08}B_{4,8}(X) + 1.08317\cdot10^{-07}B_{5,8}(X) + 1.62543\cdot10^{-07}B_{6,8}(X) + 2.16768\cdot10^{-07}B_{7,8}(X) + 2.70994\cdot10^{-07}B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = 1.42947\cdot10^{-23}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,3} - 1.82089\cdot10^{-08}B_{1,3} + 1.26393\cdot10^{-07}B_{2,3} + 2.70994\cdot10^{-07}B_{3,3}$$

$$\tilde{q}_3 = -2.97639\cdot10^{-20}X^8 + 1.26259\cdot10^{-19}X^7 - 2.19172\cdot10^{-19}X^6 + 2.00419\cdot10^{-19}X^5 - 1.02758\cdot10^{-19}X^4 + 2.83318\cdot10^{-20}X^3 - 5.27552\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,8} - 1.08585\cdot10^{-07}B_{1,8} - 5.43593\cdot10^{-08}B_{2,8} - 1.33715\cdot10^{-10}B_{3,8} + 5.40918\cdot10^{-08}B_{4,8} + 1.08317\cdot10^{-07}B_{5,8} + 1.62543\cdot10^{-07}B_{6,8} + 2.16768\cdot10^{-07}B_{7,8} + 2.70994\cdot10^{-07}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 1.97535\cdot10^{-22}$.

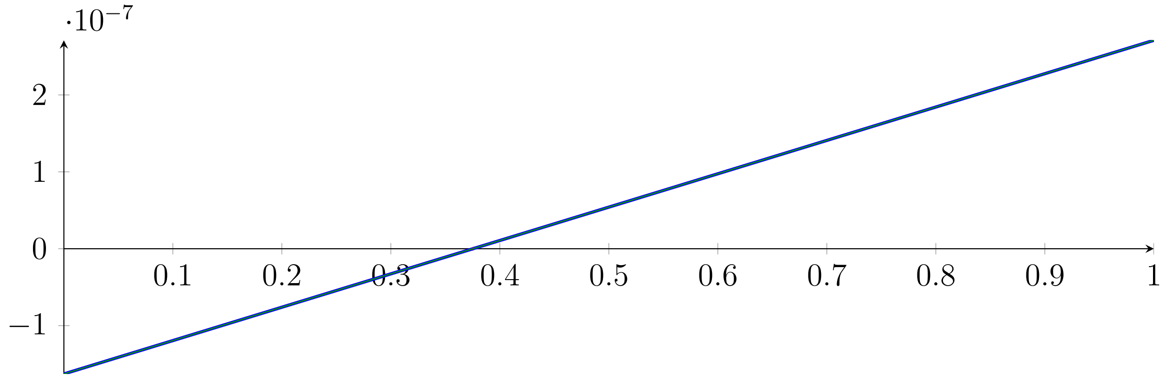
Bounding polynomials M and m:

$$M = 1.42689\cdot10^{-23}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$m = 1.43206\cdot10^{-23}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$

Root of M and m:

$$N(M) = \{0.375308\} \qquad N(m) = \{0.375308\}$$

Intersection intervals:



$[0.375308, 0.375308]$

Longest intersection interval: $1.36424 \cdot 10^{-12}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

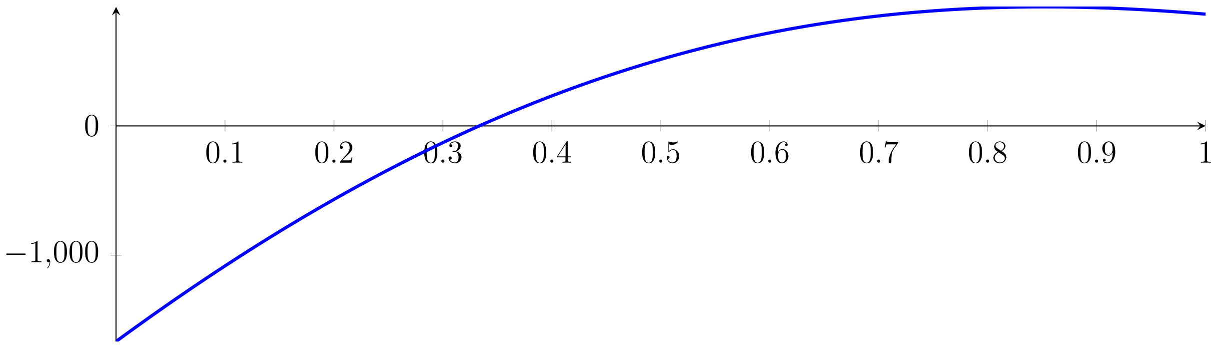
138.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

138.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



Result: Root Intervals

$$[0.333333, 0.333333]$$

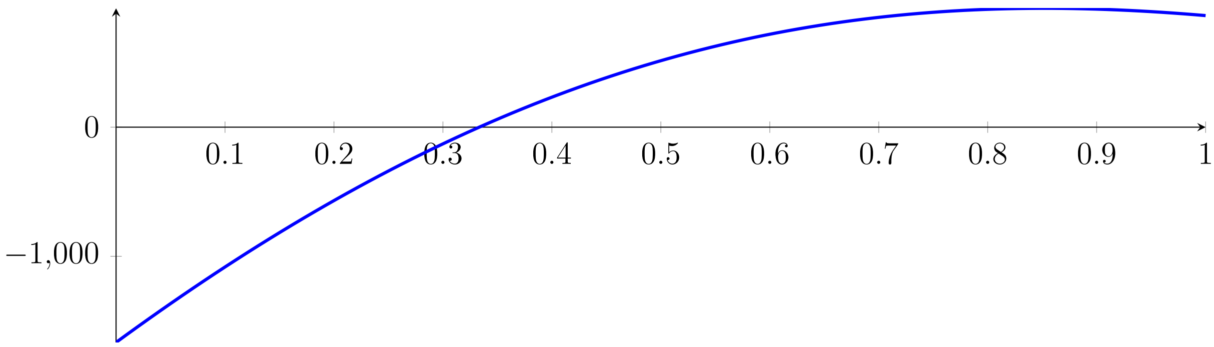
with precision $\varepsilon = 1 \cdot 10^{-16}$.

139 Running `BezClip` on f_8 with epsilon 32

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

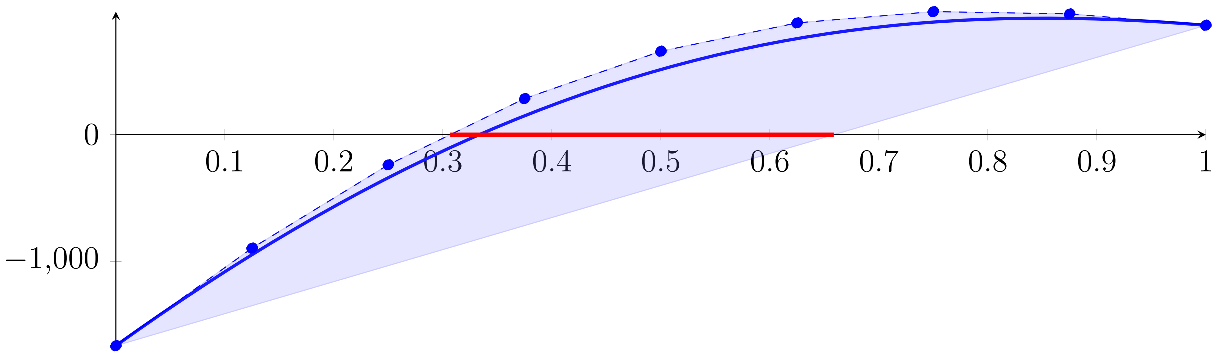
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



139.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

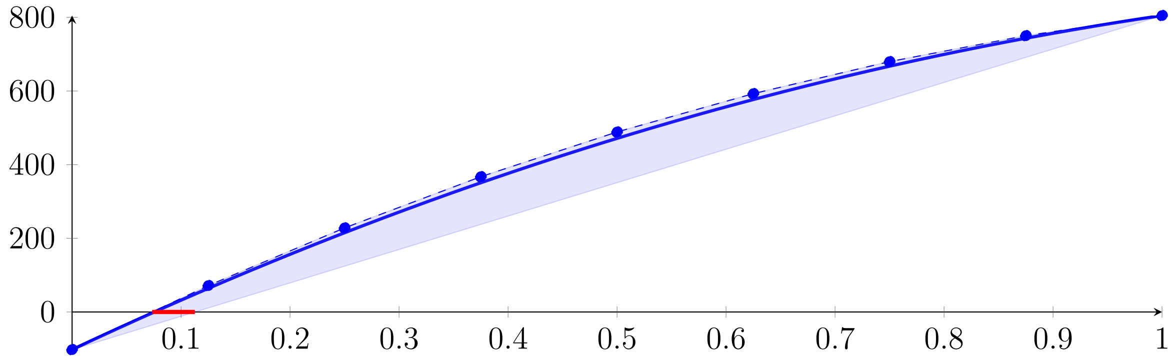
Longest intersection interval: 0.351792

$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

139.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5\\ &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17\\ &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X)\\ &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

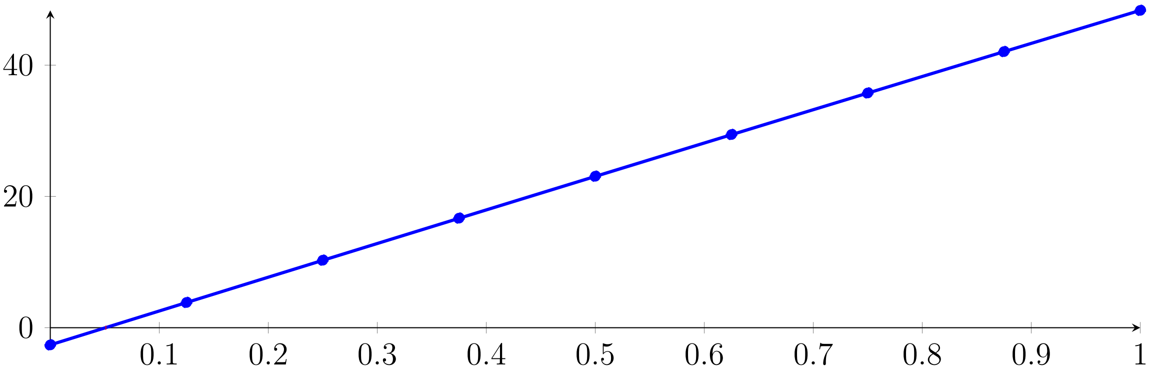
Longest intersection interval: 0.0391855

$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

139.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -3.03577\cdot 10^{-15}X^8 - 1.54633\cdot 10^{-12}X^7 - 4.95836\cdot 10^{-10}X^6 + 3.66751\cdot 10^{-08}X^5\\ &\quad + 3.25466\cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683\\ &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X)\\ &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

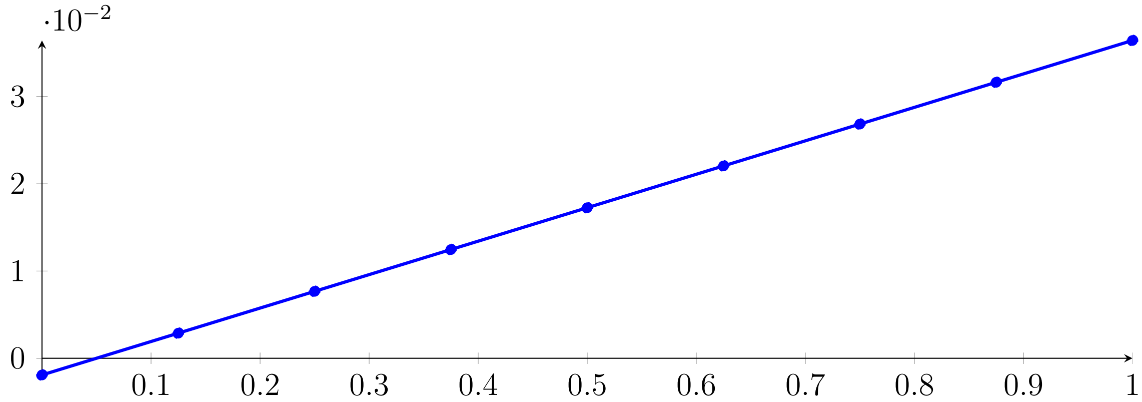
Longest intersection interval: 0.000742589

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

139.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -1.28918{\cdot}10^{-18}X^8 - 1.32815{\cdot}10^{-18}X^7 - 4.50622{\cdot}10^{-18}X^6 + 2.22939{\cdot}10^{-18}X^5 \\ &\quad + 9.48677{\cdot}10^{-18}X^4 - 2.37322{\cdot}10^{-13}X^3 - 4.12146{\cdot}10^{-07}X^2 + 0.0383444X - 0.00191378 \\ &= -0.00191378B_{0,8}(X) + 0.00287927B_{1,8}(X) + 0.00767231B_{2,8}(X) \\ &\quad + 0.0124653B_{3,8}(X) + 0.0172583B_{4,8}(X) + 0.0220513B_{5,8}(X) \\ &\quad + 0.0268443B_{6,8}(X) + 0.0316373B_{7,8}(X) + 0.0364302B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

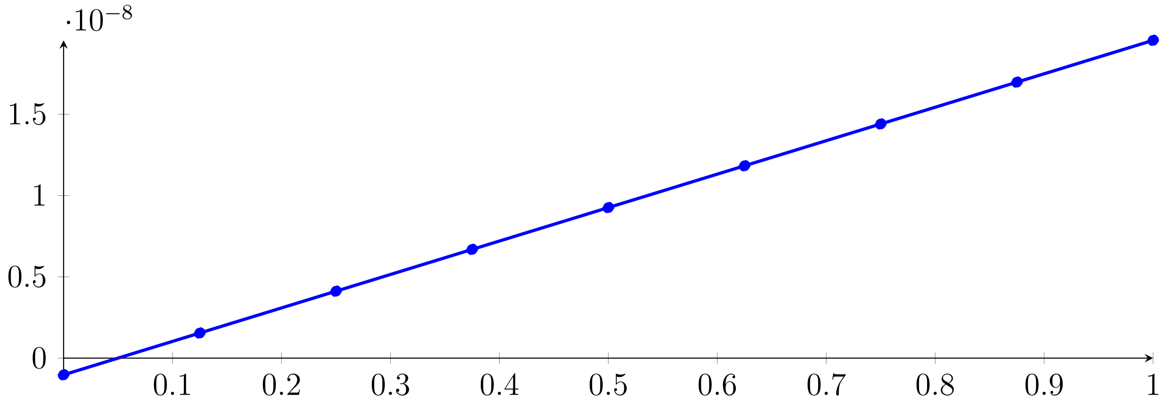
$$[0.0499103, 0.0499109]$$

Longest intersection interval: $5.36469{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

139.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -7.43978{\cdot}10^{-25}X^8 - 4.71751{\cdot}10^{-25}X^7 - 2.488{\cdot}10^{-24}X^6 + 1.04044{\cdot}10^{-24}X^5 \\ &\quad - 2.26182{\cdot}10^{-25}X^4 - 1.18615{\cdot}10^{-19}X^2 + 2.05706{\cdot}10^{-08}X - 1.02667{\cdot}10^{-09} \\ &= -1.02667{\cdot}10^{-09}B_{0,8}(X) + 1.54465{\cdot}10^{-09}B_{1,8}(X) + 4.11597{\cdot}10^{-09}B_{2,8}(X) \\ &\quad + 6.68729{\cdot}10^{-09}B_{3,8}(X) + 9.25862{\cdot}10^{-09}B_{4,8}(X) + 1.18299{\cdot}10^{-08}B_{5,8}(X) \\ &\quad + 1.44013{\cdot}10^{-08}B_{6,8}(X) + 1.69726{\cdot}10^{-08}B_{7,8}(X) + 1.95439{\cdot}10^{-08}B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87793 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

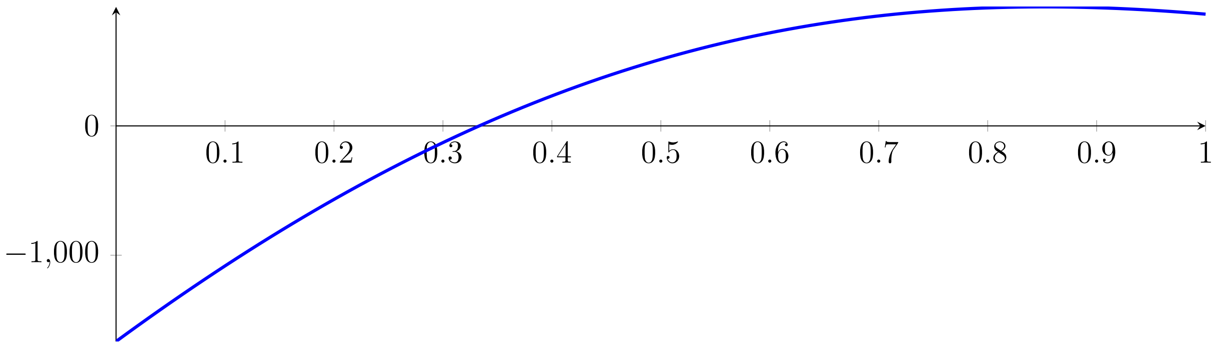
139.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

139.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



Result: Root Intervals

$$[0.333333, 0.333333]$$

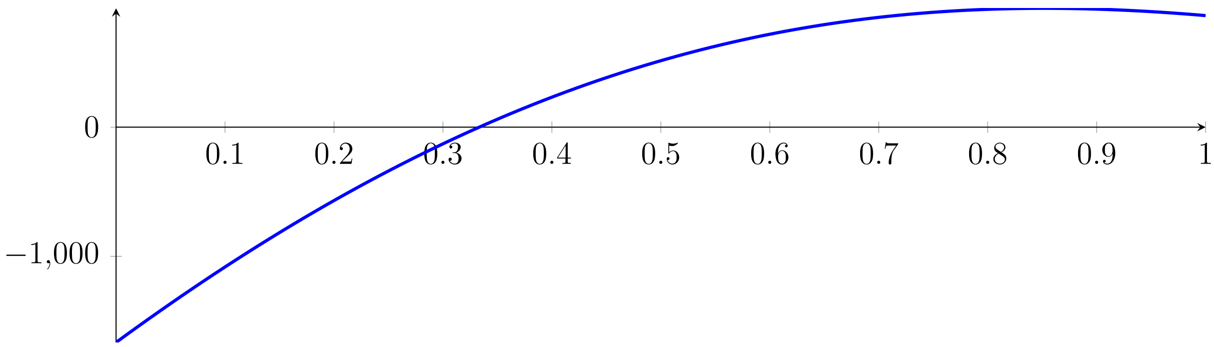
with precision $\varepsilon = 1{\cdot}10^{-32}$.

140 Running QuadClip on f_8 with epsilon 32

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0,1]$:

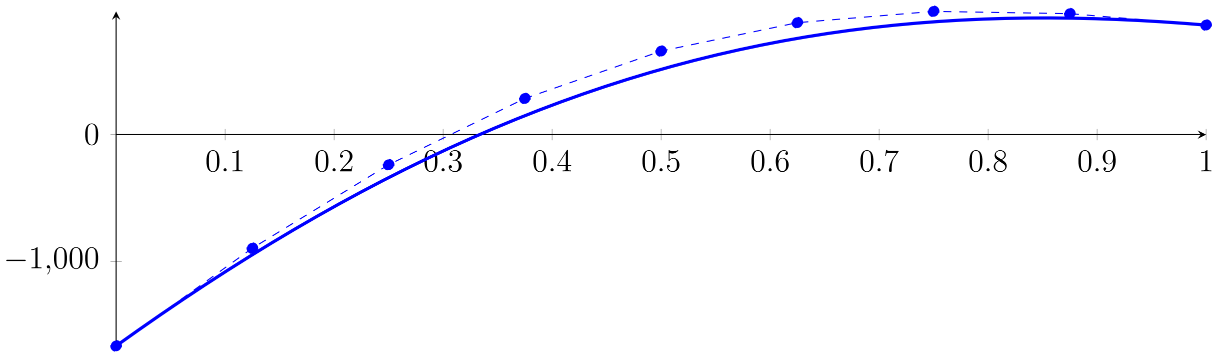
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



140.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

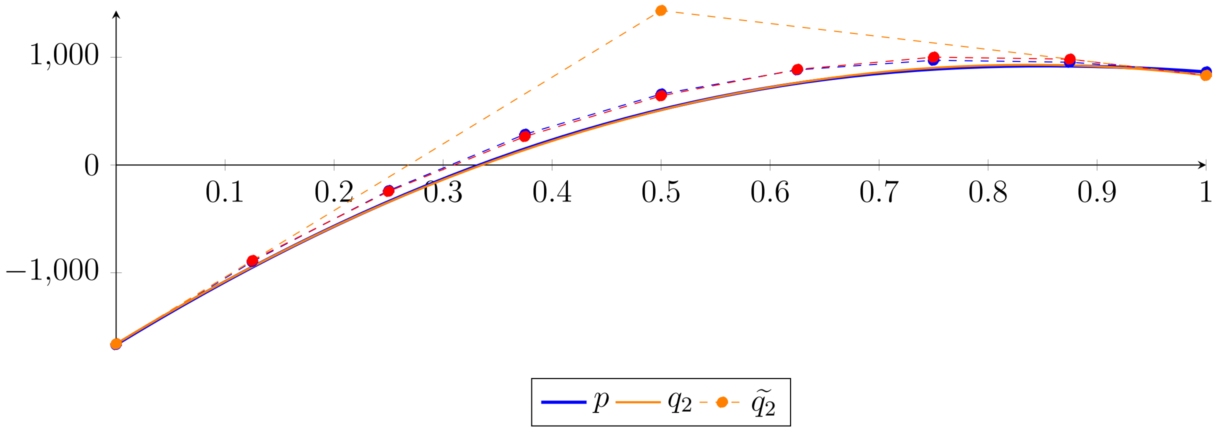
$$\begin{aligned}p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67\\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X)\\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -3695.78X^2 + 6187.64X - 1660\\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q_2} &= -5.06304 \cdot 10^{-10}X^8 + 1.93305 \cdot 10^{-09}X^7 - 2.97598 \cdot 10^{-09}X^6 + 2.35116 \cdot 10^{-09}X^5\\ &\quad - 9.91232 \cdot 10^{-10}X^4 + 2.00705 \cdot 10^{-10}X^3 - 3695.78X^2 + 6187.64X - 1660\\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8}\\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

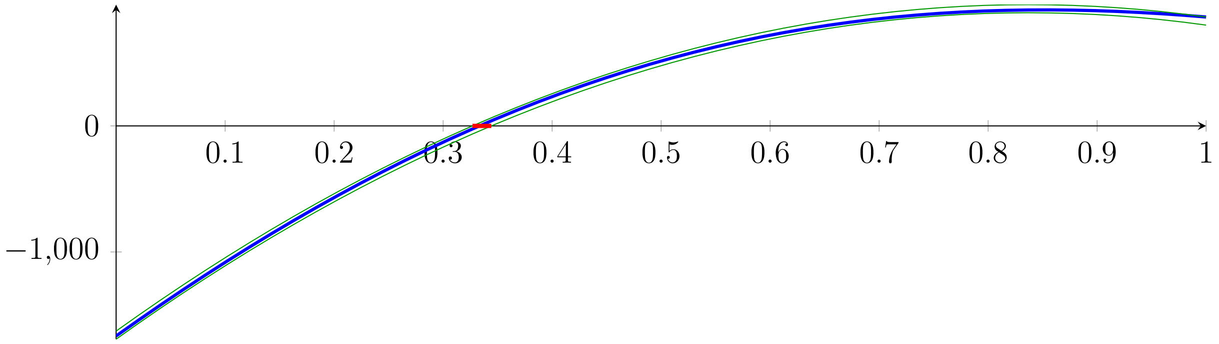
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

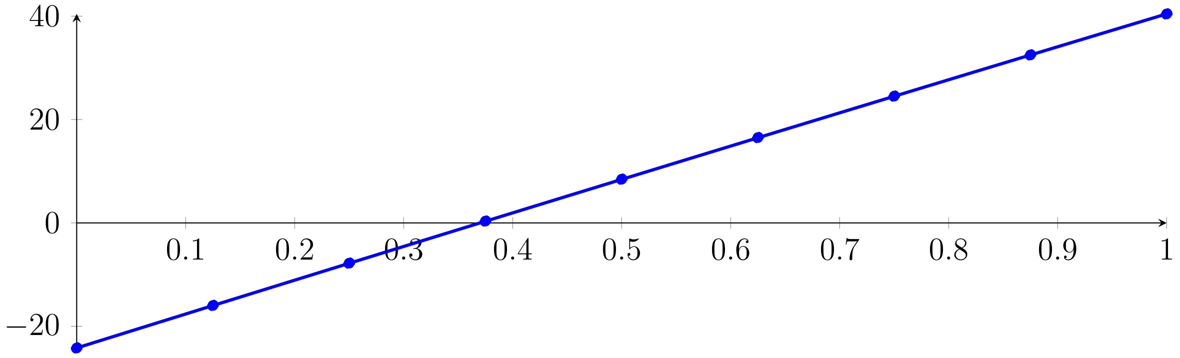
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

140.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

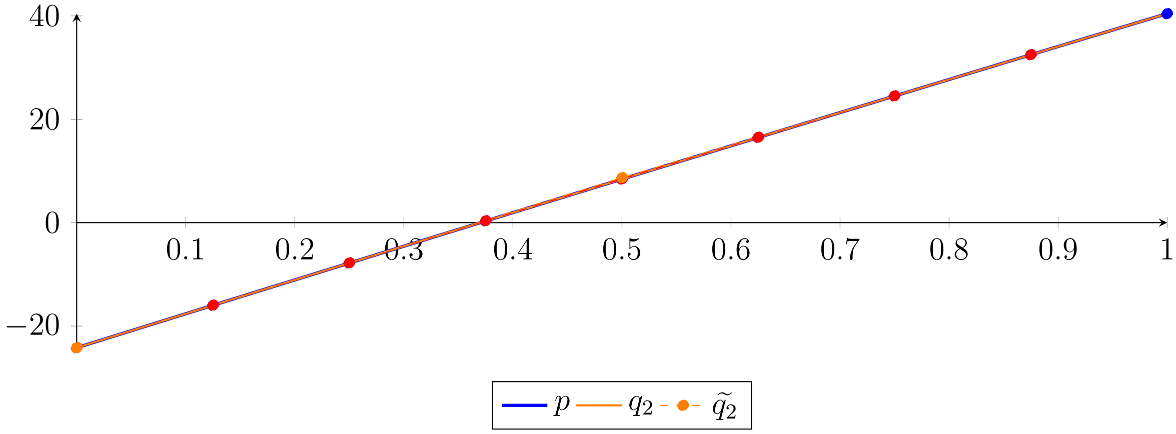
$$\begin{aligned} p = & -8.80546 \cdot 10^{-15} X^8 - 7.66587 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ & + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ = & -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ & + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -2.59056{\cdot}10^{-11}X^8 + 1.00262{\cdot}10^{-10}X^7 - 1.57692{\cdot}10^{-10}X^6 + 1.29283{\cdot}10^{-10}X^5 \\ &\quad - 5.86775{\cdot}10^{-11}X^4 + 1.42642{\cdot}10^{-11}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894{\cdot}10^{-05}$.
Bounding polynomials M and m:

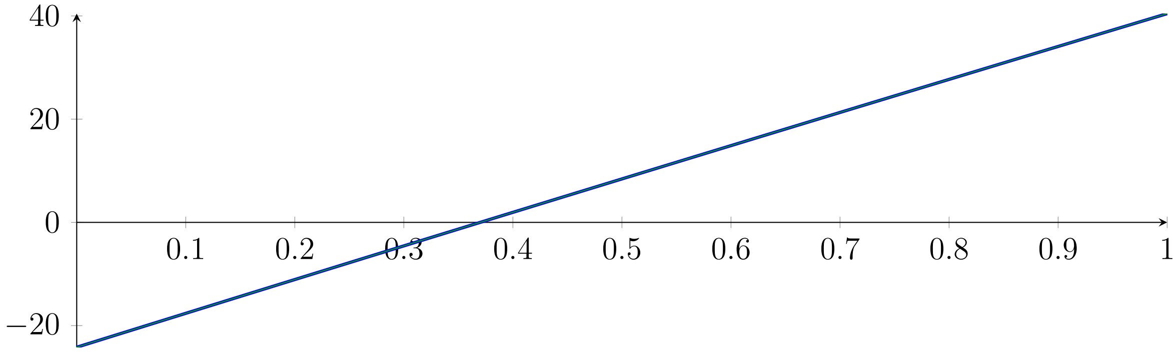
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



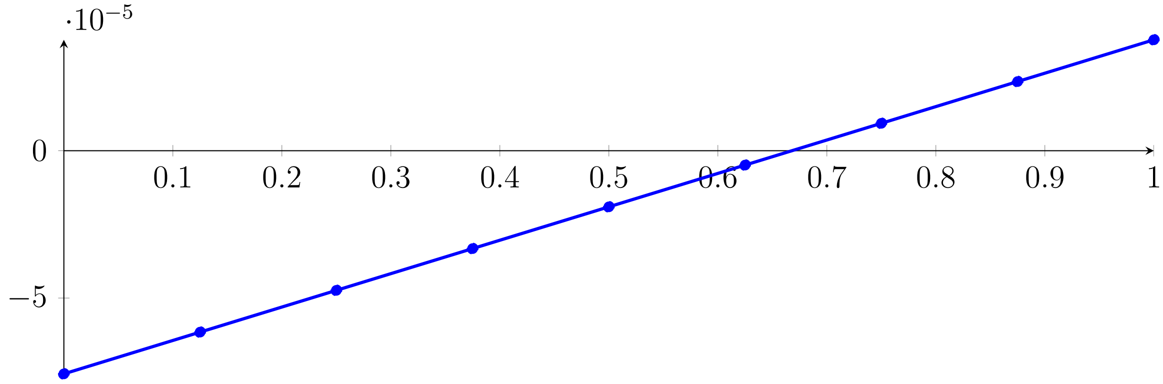
$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588{\cdot}10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

140.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

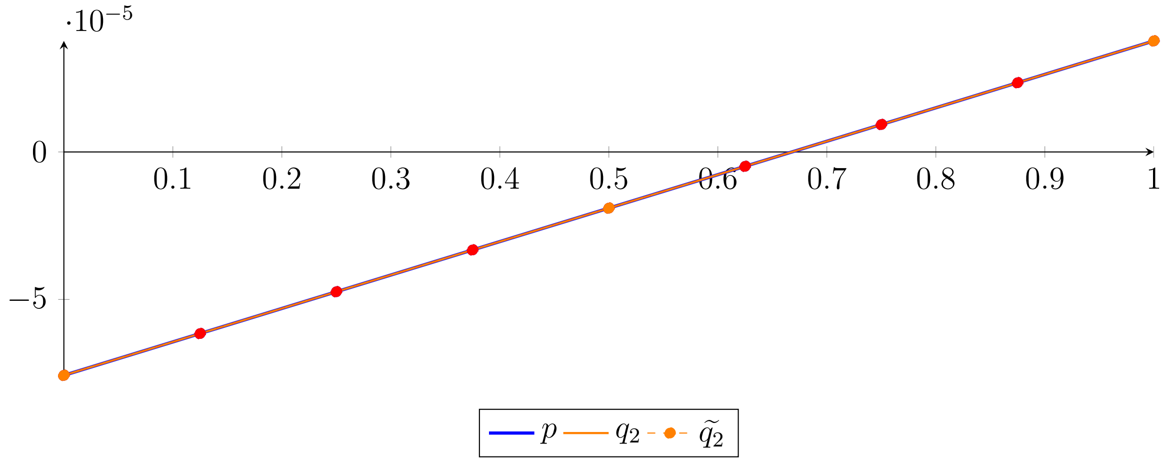
$$p = 1.16467\cdot10^{-21}X^8 + 3.17637\cdot10^{-21}X^7 + 1.18585\cdot10^{-20}X^6 - 1.48231\cdot10^{-21}X^5 + 9.26442\cdot10^{-22}X^4 - 5.92923\cdot10^{-21}X^3 - 3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$= -7.5732\cdot10^{-05}B_{0,8}(X) - 6.15596\cdot10^{-05}B_{1,8}(X) - 4.73873\cdot10^{-05}B_{2,8}(X) - 3.32149\cdot10^{-05}B_{3,8}(X) - 1.90425\cdot10^{-05}B_{4,8}(X) - 4.87016\cdot10^{-06}B_{5,8}(X) + 9.3022\cdot10^{-06}B_{6,8}(X) + 2.34746\cdot10^{-05}B_{7,8}(X) + 3.76469\cdot10^{-05}B_{8,8}(X)$$



Degree reduction and raising:

$$q_2 = -3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$= -7.5732\cdot10^{-05}B_{0,2} - 1.90425\cdot10^{-05}B_{1,2} + 3.76469\cdot10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = 3.26671\cdot10^{-17}X^8 - 1.38104\cdot10^{-16}X^7 + 2.39221\cdot10^{-16}X^6 - 2.17429\cdot10^{-16}X^5 + 1.10046\cdot10^{-16}X^4 - 3.0162\cdot10^{-17}X^3 - 3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$= -7.5732\cdot10^{-05}B_{0,8} - 6.15596\cdot10^{-05}B_{1,8} - 4.73873\cdot10^{-05}B_{2,8} - 3.32149\cdot10^{-05}B_{3,8} - 1.90425\cdot10^{-05}B_{4,8} - 4.87016\cdot10^{-06}B_{5,8} + 9.3022\cdot10^{-06}B_{6,8} + 2.34746\cdot10^{-05}B_{7,8} + 3.76469\cdot10^{-05}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 1.84643\cdot10^{-19}$.

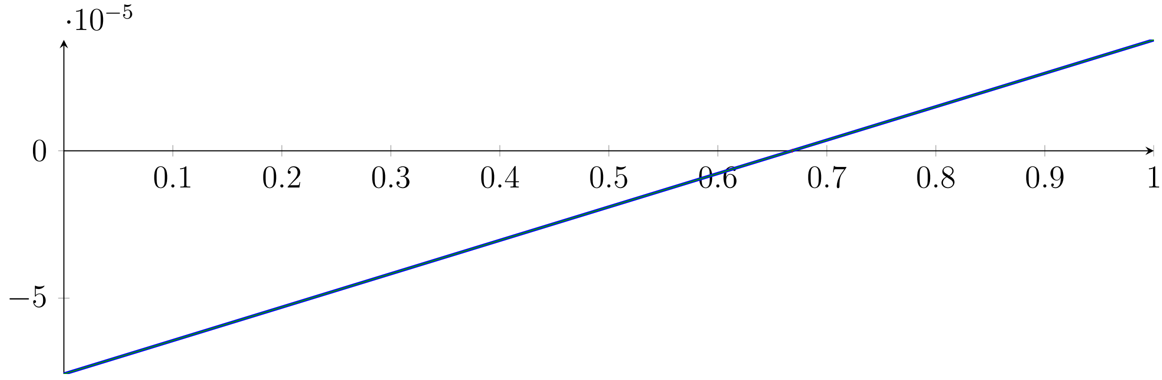
Bounding polynomials M and m:

$$M = -3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$m = -3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$

Root of M and m:

$$N(M) = \{0.667955, 3.14644\cdot10^{7}\} \qquad N(m) = \{0.667955, 3.14644\cdot10^{7}\}$$

Intersection intervals:



$[0.667955, 0.667955]$

Longest intersection interval: $3.08439 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

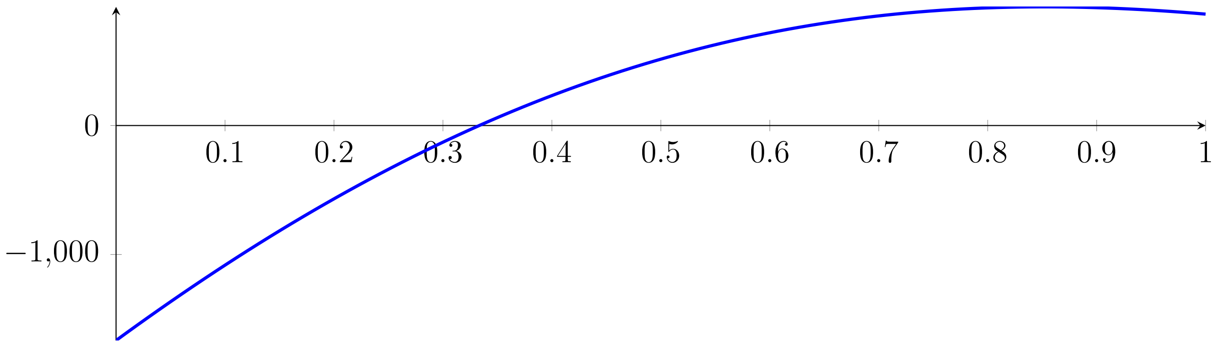
140.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 4!
p(0) = 2.85706e-18 - p(1) 3.78276e-17

140.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

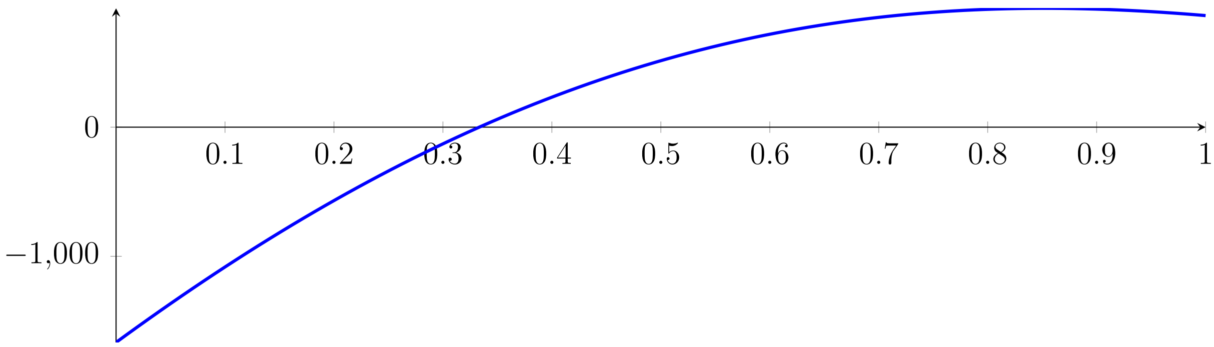
with precision $\varepsilon = 1 \cdot 10^{-32}$.

141 Running CubeClip on f_8 with epsilon 32

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

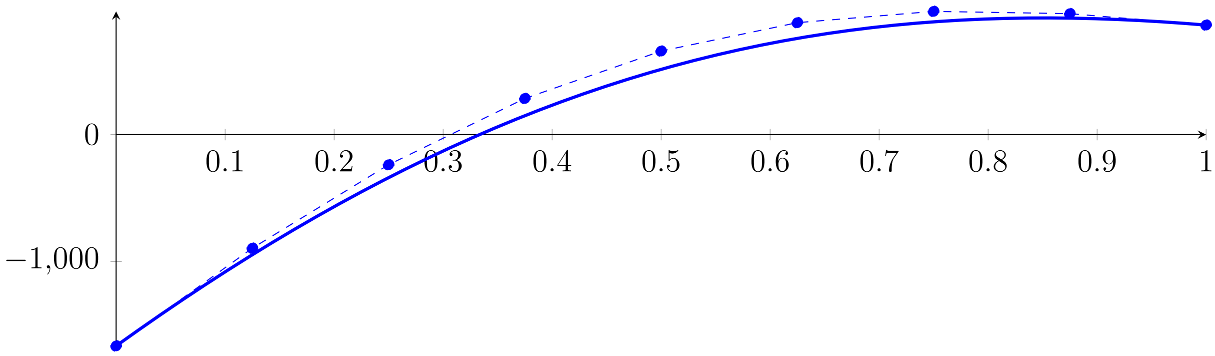
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



141.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

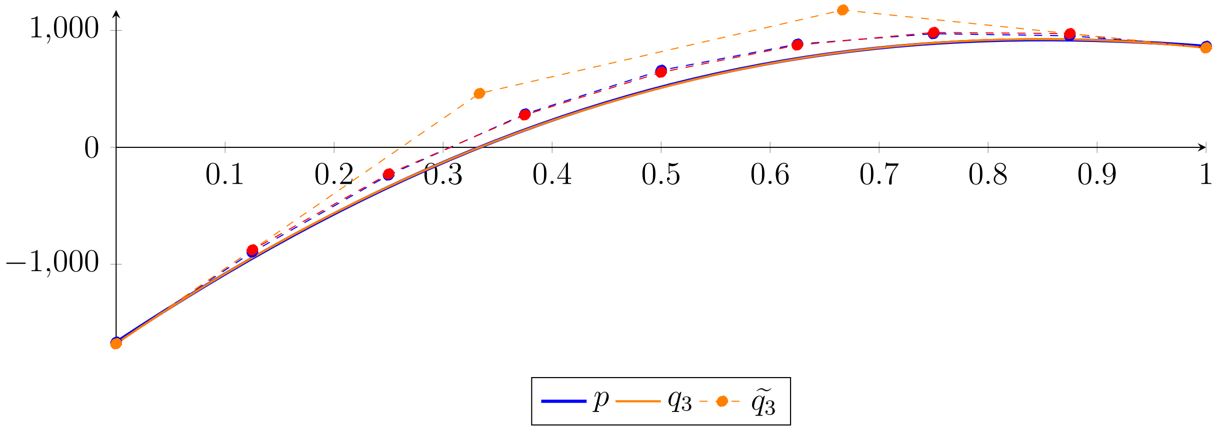
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.39389 \cdot 10^{-09}X^8 + 9.60737 \cdot 10^{-09}X^7 - 1.57477 \cdot 10^{-08}X^6 + 1.35479 \cdot 10^{-08}X^5 \\ &\quad - 6.53397 \cdot 10^{-09}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

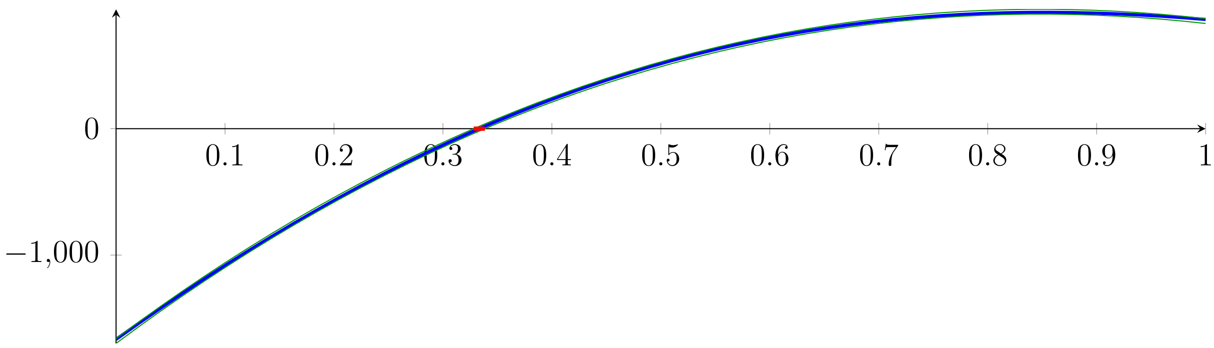
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$[0.328258, 0.338551]$

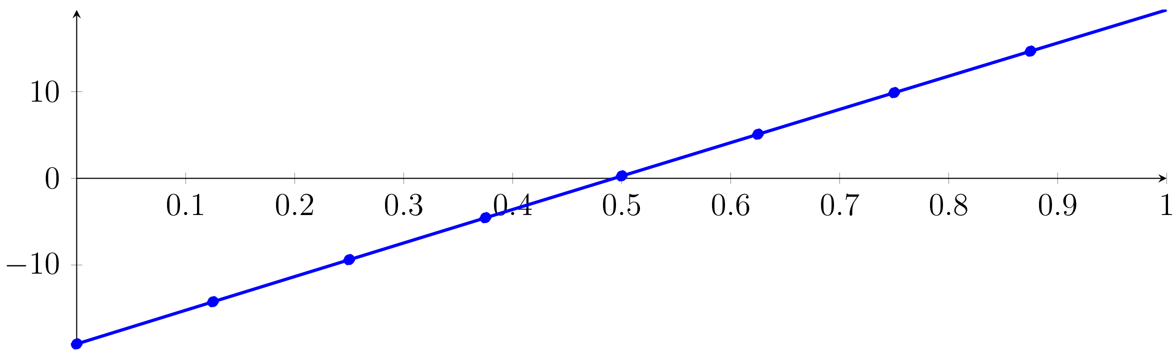
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

141.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

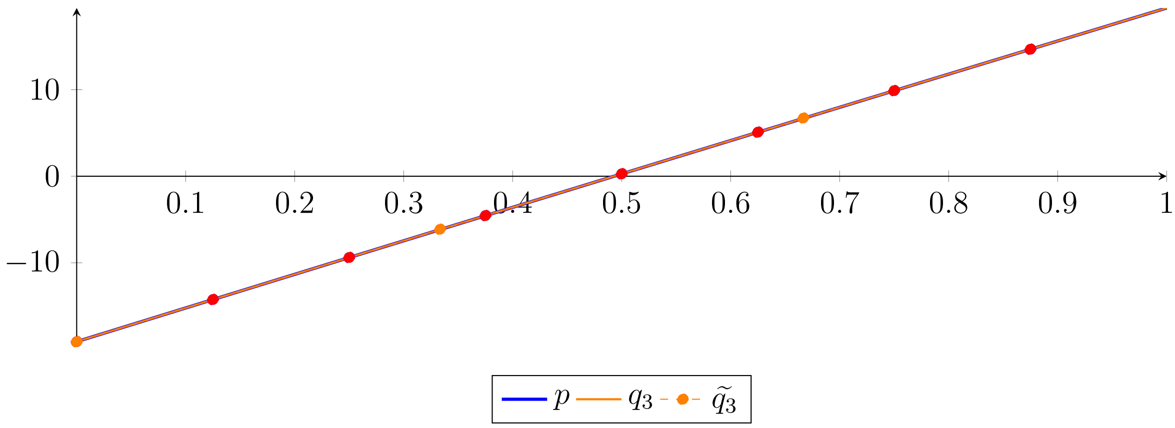
$$\begin{aligned} p = &-1.66533 \cdot 10^{-16} X^8 - 1.99007 \cdot 10^{-13} X^7 - 8.53059 \cdot 10^{-11} X^6 + 8.7284 \cdot 10^{-09} X^5 \\ &+ 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ = &-19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ &+ 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2.82588\cdot10^{-12}X^8 - 1.08529\cdot10^{-11}X^7 + 1.68738\cdot10^{-11}X^6 - 1.34636\cdot10^{-11}X^5 \\ &\quad + 5.72618\cdot10^{-12}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902\cdot10^{-07}$.

Bounding polynomials M and m:

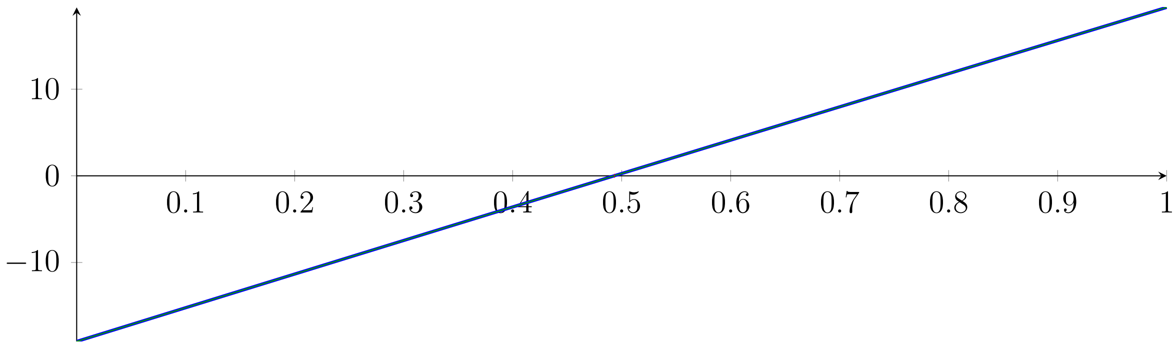
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

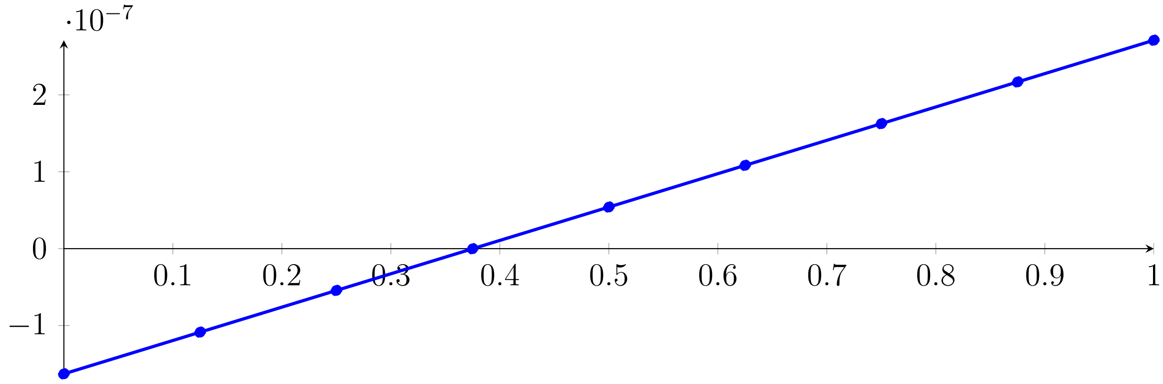
Longest intersection interval: $1.1252\cdot10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

141.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

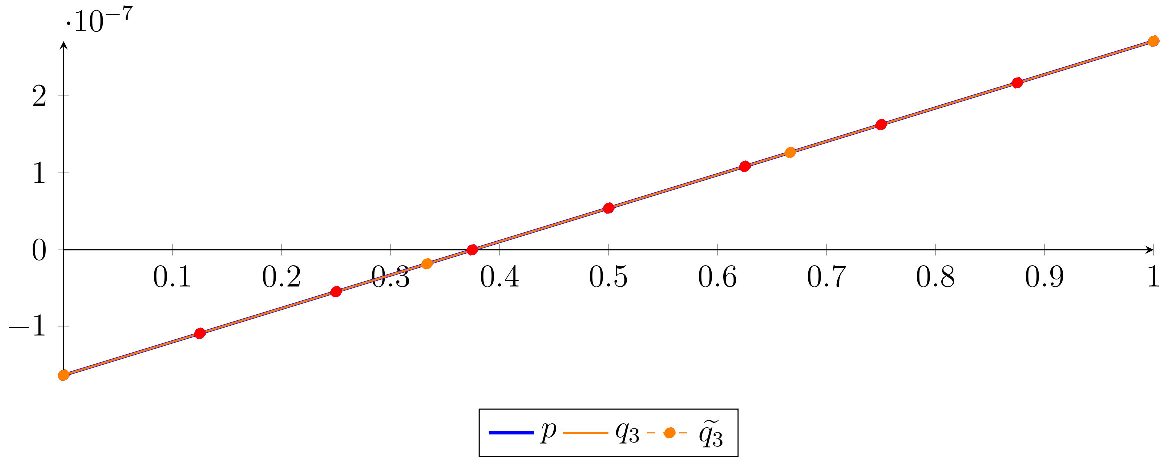
$$p = -4.65289{\cdot}10^{-24}X^8 - 8.27181{\cdot}10^{-25}X^7 - 4.3427{\cdot}10^{-24}X^6 + 8.6854{\cdot}10^{-24}X^5 + 1.80946{\cdot}10^{-24}X^4 - 7.23783{\cdot}10^{-25}X^3 - 5.27516{\cdot}10^{-17}X^2 + 4.33804{\cdot}10^{-07}X - 1.6281{\cdot}10^{-07}$$
$$= -1.6281{\cdot}10^{-07}B_{0,8}(X) - 1.08585{\cdot}10^{-07}B_{1,8}(X) - 5.43593{\cdot}10^{-08}B_{2,8}(X) - 1.33715{\cdot}10^{-10}B_{3,8}(X) + 5.40918{\cdot}10^{-08}B_{4,8}(X) + 1.08317{\cdot}10^{-07}B_{5,8}(X) + 1.62543{\cdot}10^{-07}B_{6,8}(X) + 2.16768{\cdot}10^{-07}B_{7,8}(X) + 2.70994{\cdot}10^{-07}B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = 1.42947{\cdot}10^{-23}X^3 - 5.27516{\cdot}10^{-17}X^2 + 4.33804{\cdot}10^{-07}X - 1.6281{\cdot}10^{-07}$$
$$= -1.6281{\cdot}10^{-07}B_{0,3} - 1.82089{\cdot}10^{-08}B_{1,3} + 1.26393{\cdot}10^{-07}B_{2,3} + 2.70994{\cdot}10^{-07}B_{3,3}$$

$$\widetilde{q}_3 = -2.97639{\cdot}10^{-20}X^8 + 1.26259{\cdot}10^{-19}X^7 - 2.19172{\cdot}10^{-19}X^6 + 2.00419{\cdot}10^{-19}X^5 - 1.02758{\cdot}10^{-19}X^4 + 2.83318{\cdot}10^{-20}X^3 - 5.27552{\cdot}10^{-17}X^2 + 4.33804{\cdot}10^{-07}X - 1.6281{\cdot}10^{-07}$$
$$= -1.6281{\cdot}10^{-07}B_{0,8} - 1.08585{\cdot}10^{-07}B_{1,8} - 5.43593{\cdot}10^{-08}B_{2,8} - 1.33715{\cdot}10^{-10}B_{3,8} + 5.40918{\cdot}10^{-08}B_{4,8} + 1.08317{\cdot}10^{-07}B_{5,8} + 1.62543{\cdot}10^{-07}B_{6,8} + 2.16768{\cdot}10^{-07}B_{7,8} + 2.70994{\cdot}10^{-07}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 1.97535{\cdot}10^{-22}$.

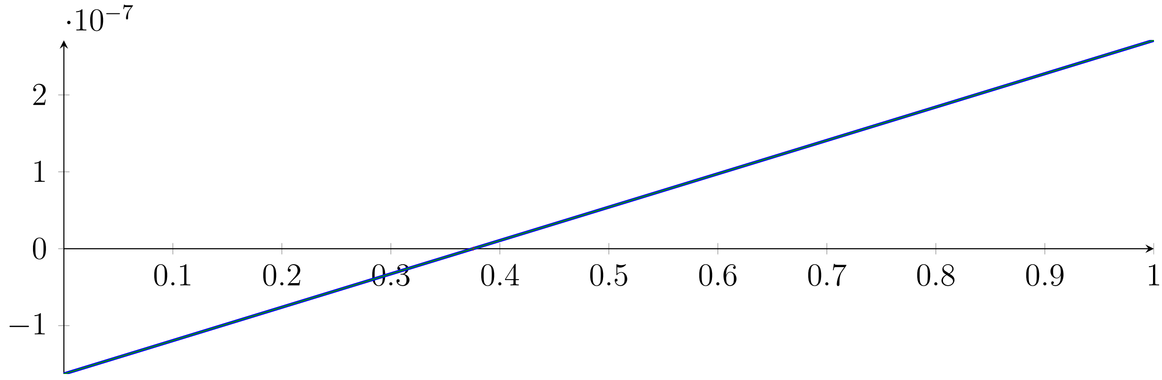
Bounding polynomials M and m:

$$M = 1.42689{\cdot}10^{-23}X^3 - 5.27516{\cdot}10^{-17}X^2 + 4.33804{\cdot}10^{-07}X - 1.6281{\cdot}10^{-07}$$
$$m = 1.43206{\cdot}10^{-23}X^3 - 5.27516{\cdot}10^{-17}X^2 + 4.33804{\cdot}10^{-07}X - 1.6281{\cdot}10^{-07}$$

Root of M and m:

$$N(M) = \{0.375308\} \qquad N(m) = \{0.375308\}$$

Intersection intervals:



$[0.375308, 0.375308]$

Longest intersection interval: $1.36424 \cdot 10^{-12}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

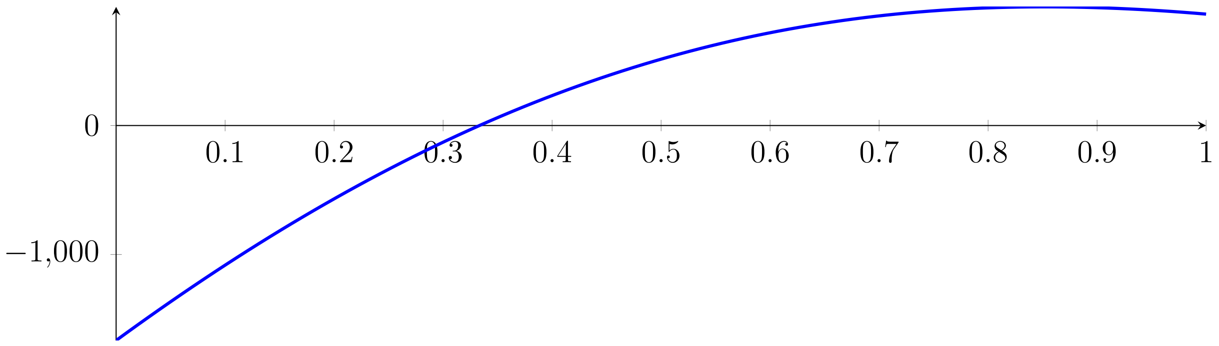
141.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 4!
p(0) = -1.10673e-18 - p(1) -5.14919e-19

141.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

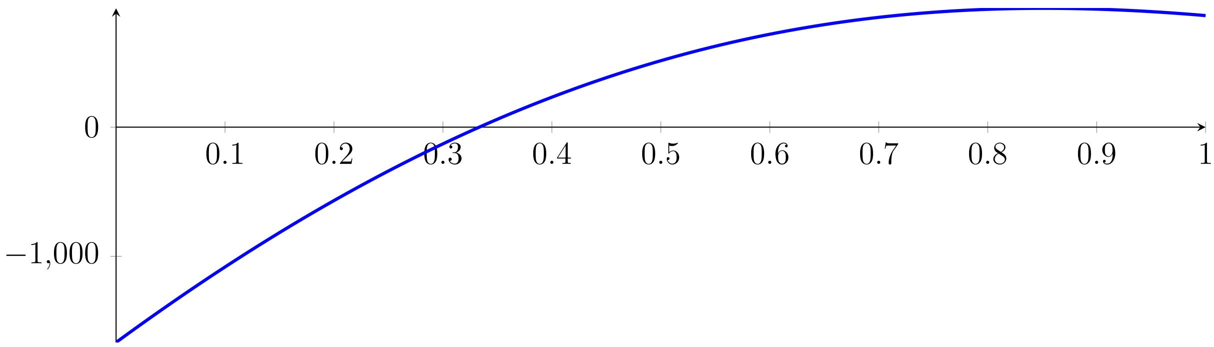
with precision $\varepsilon = 1 \cdot 10^{-32}$.

142 Running `BezClip` on f_8 with epsilon 64

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

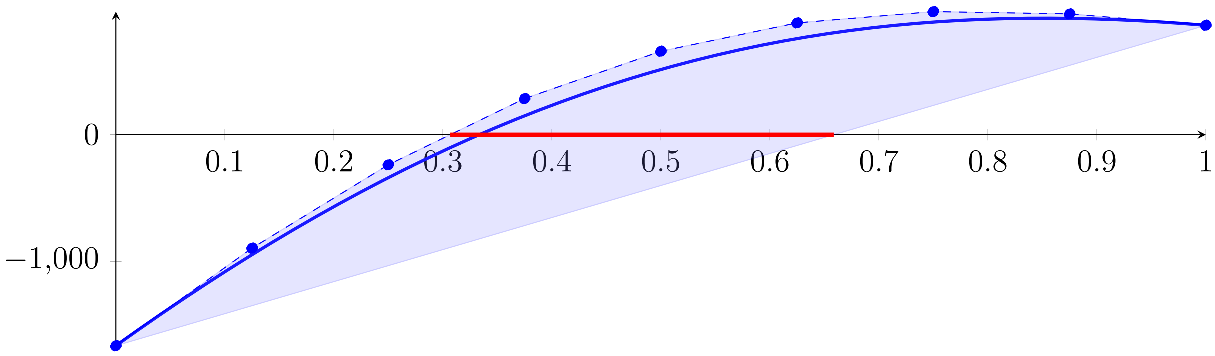
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



142.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67\\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X)\\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

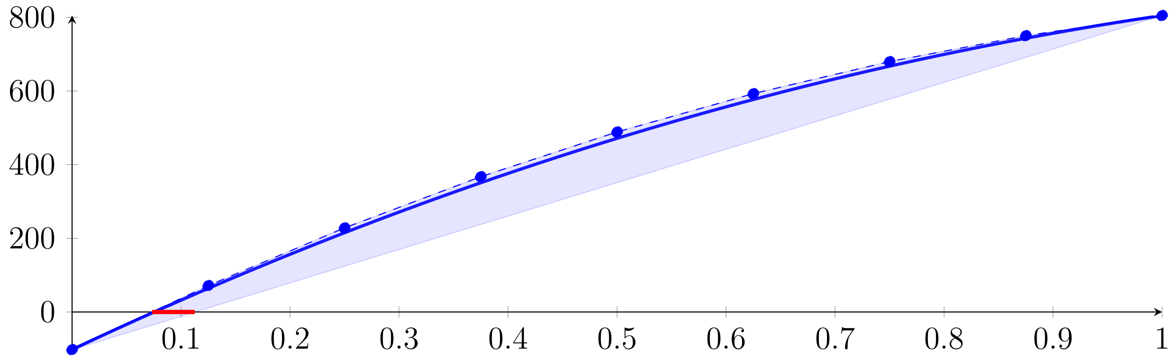
Longest intersection interval: 0.351792

$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

142.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\ &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\ &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\ &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

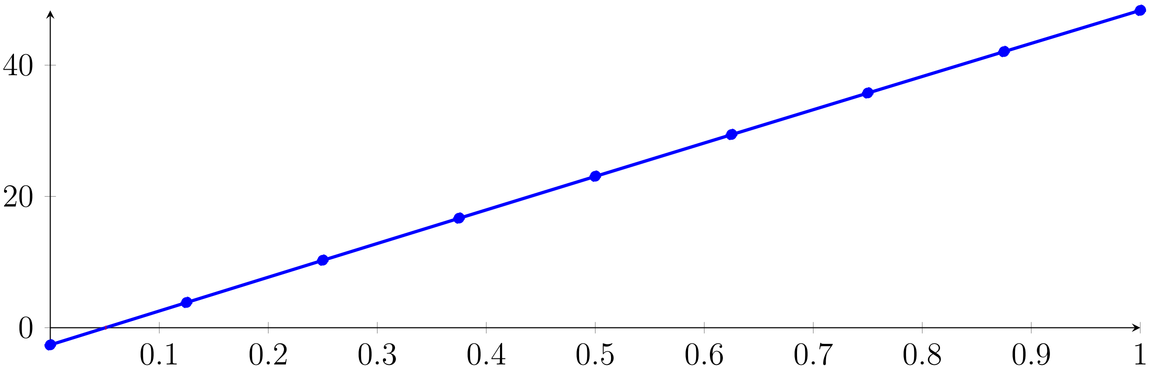
Longest intersection interval: 0.0391855

$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

142.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -3.03577\cdot 10^{-15}X^8 - 1.54633\cdot 10^{-12}X^7 - 4.95836\cdot 10^{-10}X^6 + 3.66751\cdot 10^{-08}X^5 \\ &\quad + 3.25466\cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\ &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\ &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

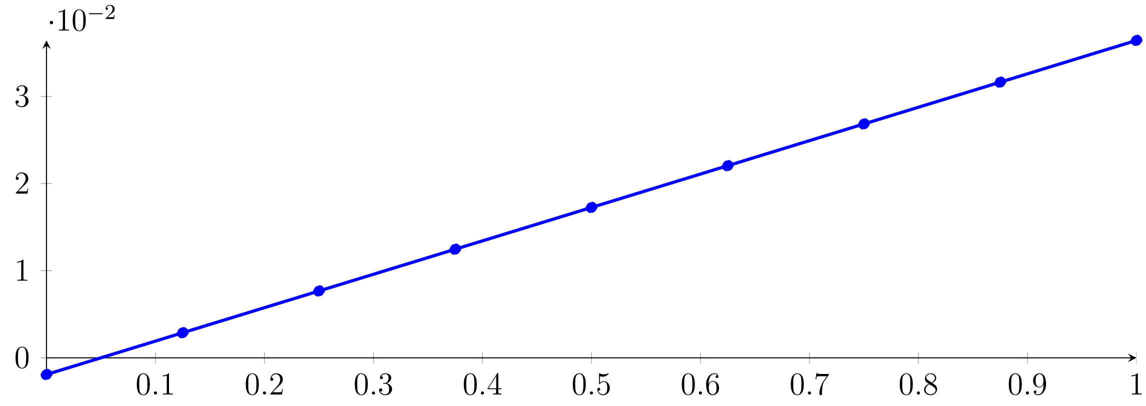
Longest intersection interval: 0.000742589

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

142.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -1.28918\cdot10^{-18}X^8 - 1.32815\cdot10^{-18}X^7 - 4.50622\cdot10^{-18}X^6 + 2.22939\cdot10^{-18}X^5 \\ &\quad + 9.48677\cdot10^{-18}X^4 - 2.37322\cdot10^{-13}X^3 - 4.12146\cdot10^{-07}X^2 + 0.0383444X - 0.00191378 \\ &= -0.00191378B_{0,8}(X) + 0.00287927B_{1,8}(X) + 0.00767231B_{2,8}(X) \\ &\quad + 0.0124653B_{3,8}(X) + 0.0172583B_{4,8}(X) + 0.0220513B_{5,8}(X) \\ &\quad + 0.0268443B_{6,8}(X) + 0.0316373B_{7,8}(X) + 0.0364302B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

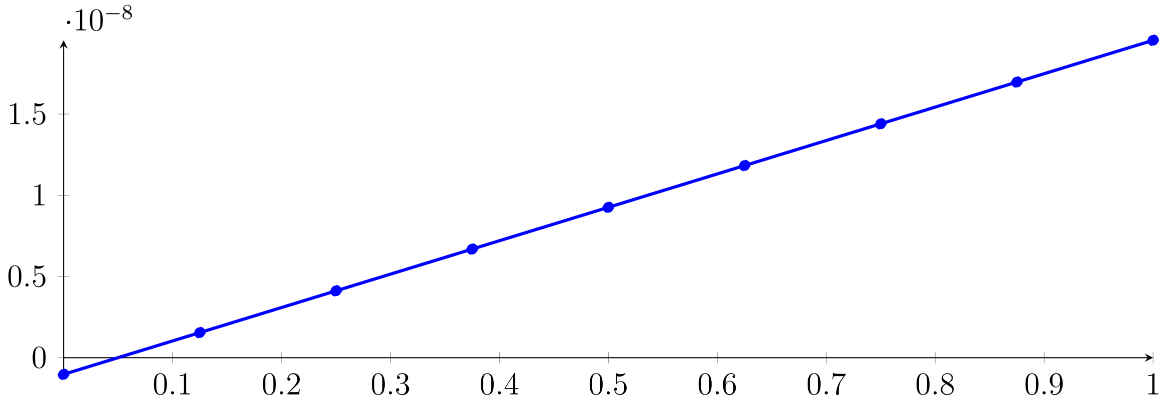
$$[0.0499103, 0.0499109]$$

Longest intersection interval: $5.36469\cdot10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

142.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -7.43978\cdot10^{-25}X^8 - 4.71751\cdot10^{-25}X^7 - 2.488\cdot10^{-24}X^6 + 1.04044\cdot10^{-24}X^5 \\ &\quad - 2.26182\cdot10^{-25}X^4 - 1.18615\cdot10^{-19}X^2 + 2.05706\cdot10^{-08}X - 1.02667\cdot10^{-09} \\ &= -1.02667\cdot10^{-09}B_{0,8}(X) + 1.54465\cdot10^{-09}B_{1,8}(X) + 4.11597\cdot10^{-09}B_{2,8}(X) \\ &\quad + 6.68729\cdot10^{-09}B_{3,8}(X) + 9.25862\cdot10^{-09}B_{4,8}(X) + 1.18299\cdot10^{-08}B_{5,8}(X) \\ &\quad + 1.44013\cdot10^{-08}B_{6,8}(X) + 1.69726\cdot10^{-08}B_{7,8}(X) + 1.95439\cdot10^{-08}B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87793 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

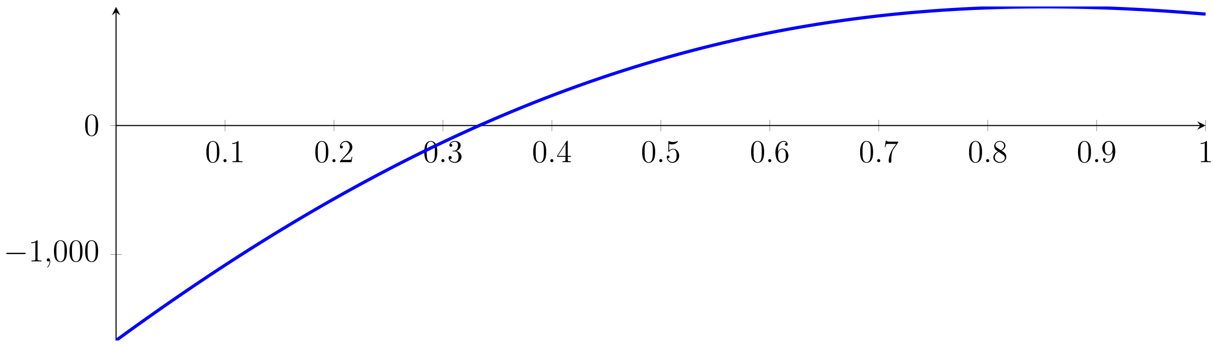
142.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

142.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

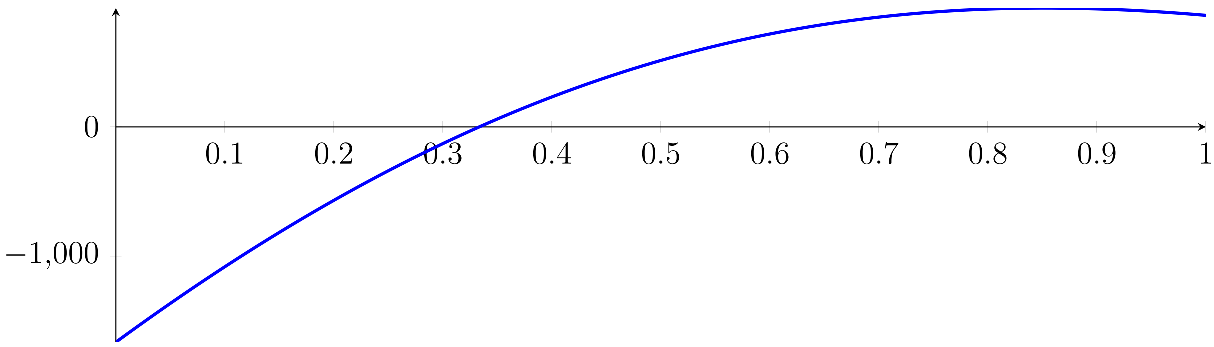
with precision $\varepsilon = 1 \cdot 10^{-64}$.

143 Running QuadClip on f_8 with epsilon 64

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

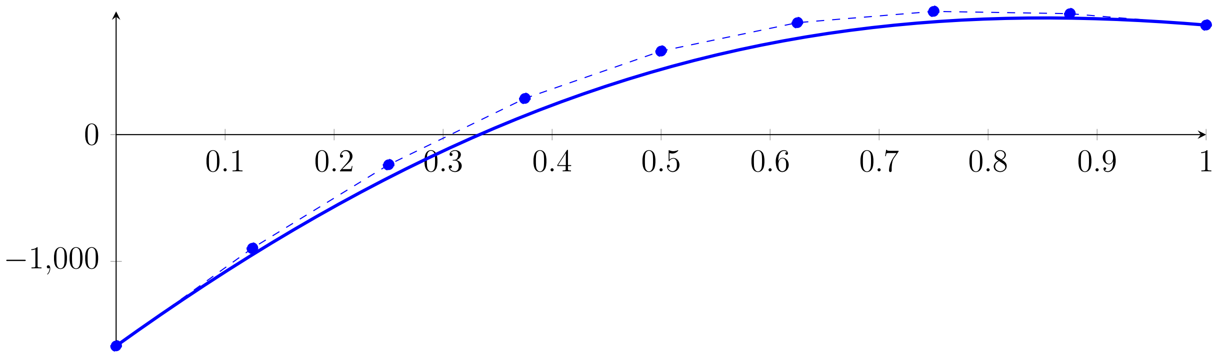
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



143.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

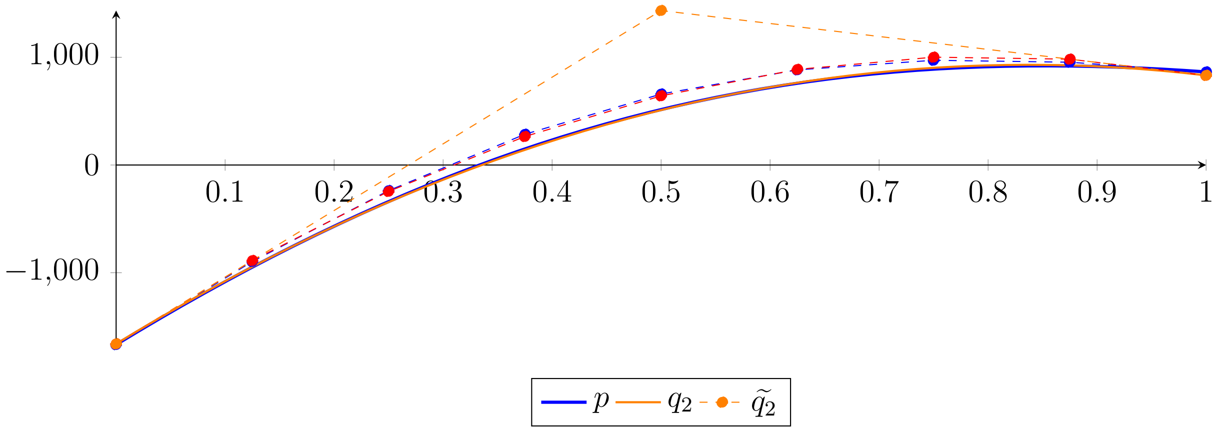
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -5.06304 \cdot 10^{-10}X^8 + 1.93305 \cdot 10^{-09}X^7 - 2.97598 \cdot 10^{-09}X^6 + 2.35116 \cdot 10^{-09}X^5 \\ &\quad - 9.91232 \cdot 10^{-10}X^4 + 2.00705 \cdot 10^{-10}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

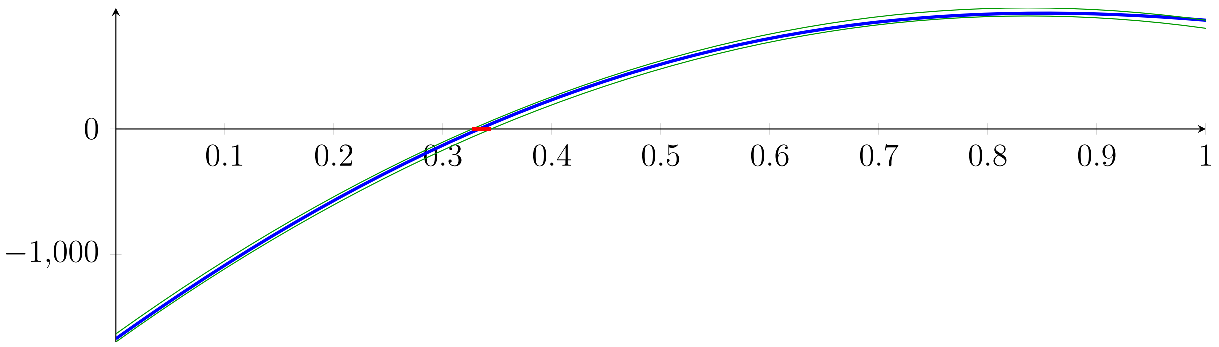
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

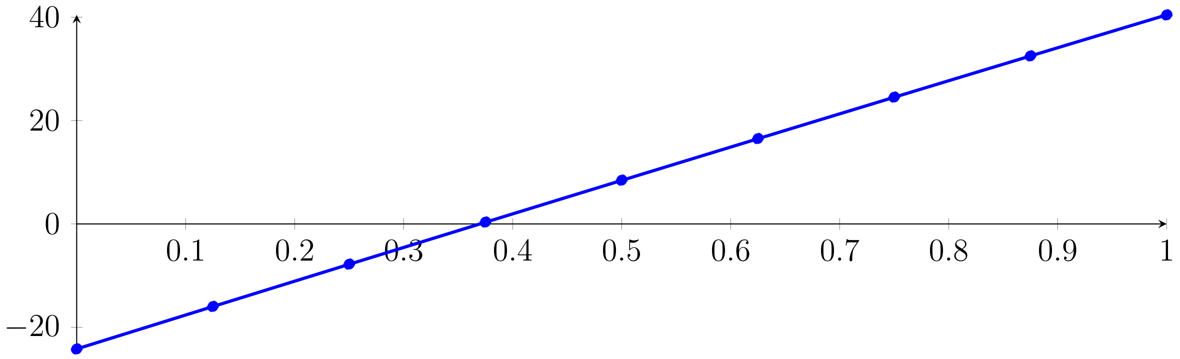
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

143.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

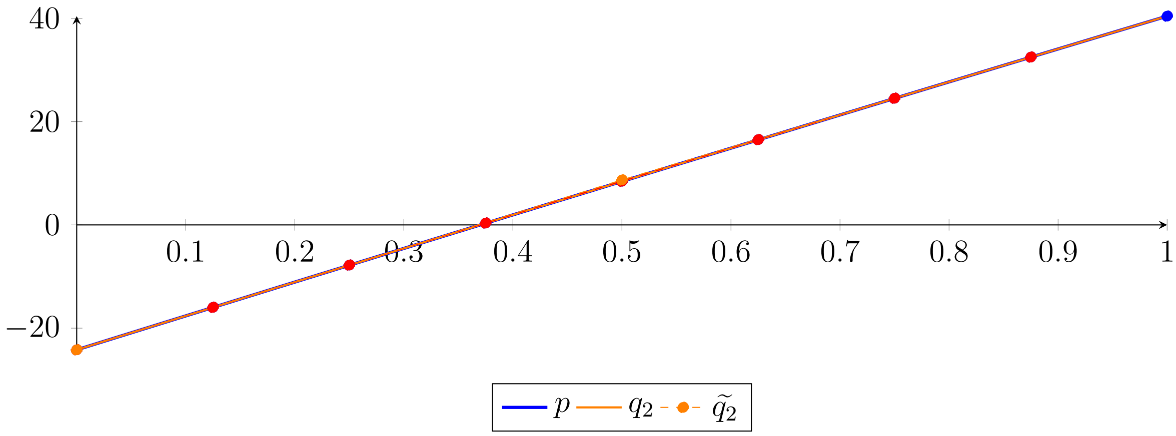
$$\begin{aligned} p = & -8.80546 \cdot 10^{-15} X^8 - 7.66587 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ & + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ = & -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ & + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -2.59056\cdot 10^{-11}X^8 + 1.00262\cdot 10^{-10}X^7 - 1.57692\cdot 10^{-10}X^6 + 1.29283\cdot 10^{-10}X^5 \\ &\quad - 5.86775\cdot 10^{-11}X^4 + 1.42642\cdot 10^{-11}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894\cdot 10^{-05}$.
Bounding polynomials M and m:

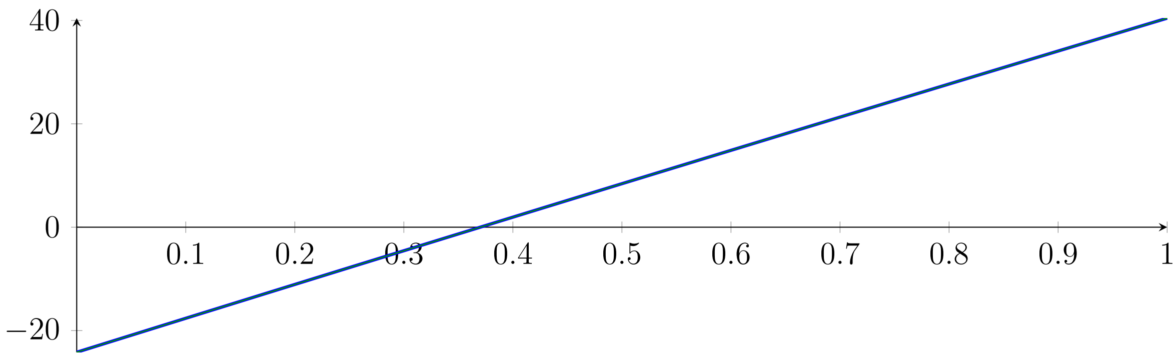
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



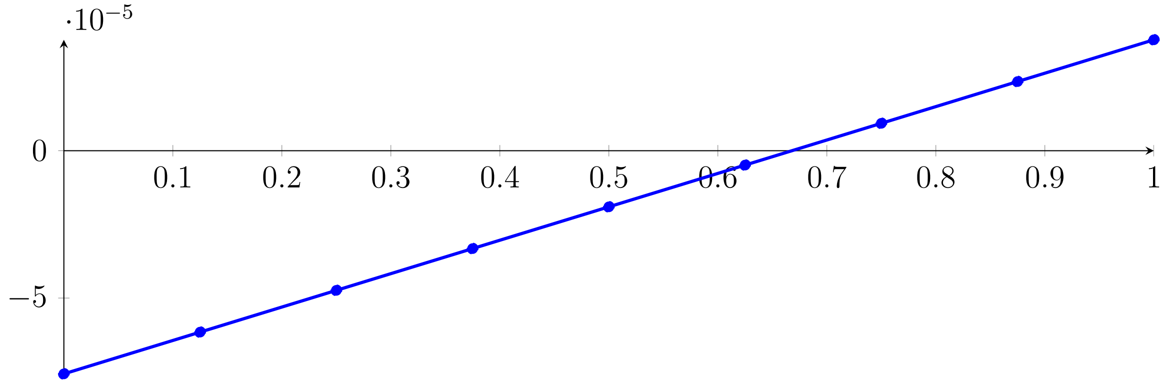
$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588\cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

143.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

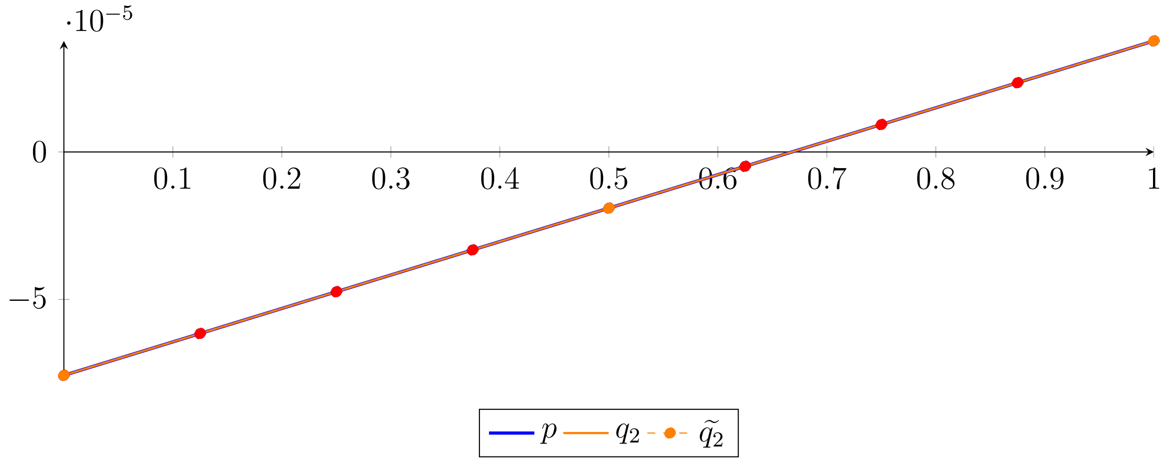
$$p = 1.16467 \cdot 10^{-21} X^8 + 3.17637 \cdot 10^{-21} X^7 + 1.18585 \cdot 10^{-20} X^6 - 1.48231 \cdot 10^{-21} X^5 + 9.26442 \cdot 10^{-22} X^4 - 5.92923 \cdot 10^{-21} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$
$$= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)$$



Degree reduction and raising:

$$q_2 = -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$
$$= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}$$

$$\widetilde{q}_2 = 3.26671 \cdot 10^{-17} X^8 - 1.38104 \cdot 10^{-16} X^7 + 2.39221 \cdot 10^{-16} X^6 - 2.17429 \cdot 10^{-16} X^5 + 1.10046 \cdot 10^{-16} X^4 - 3.0162 \cdot 10^{-17} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$
$$= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 1.84643 \cdot 10^{-19}$.

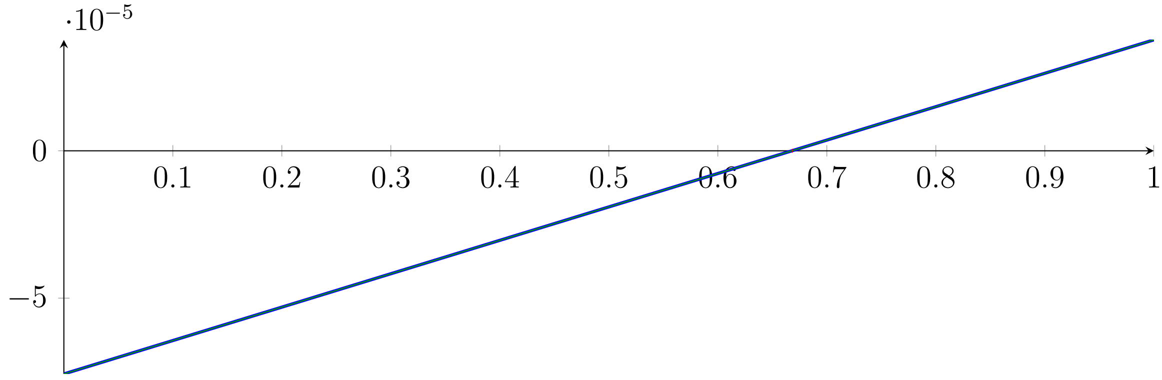
Bounding polynomials M and m:

$$M = -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$
$$m = -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.667955, 3.14644 \cdot 10^{7}\} \qquad N(m) = \{0.667955, 3.14644 \cdot 10^{7}\}$$

Intersection intervals:



$[0.667955, 0.667955]$

Longest intersection interval: $3.08439 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

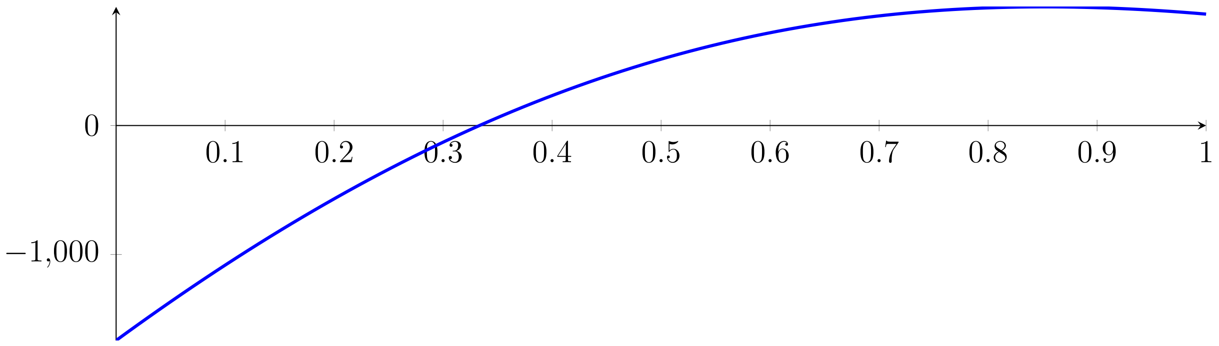
143.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 4!
p(0) = 2.85706e-18 - p(1) 3.78276e-17

143.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

with precision $\varepsilon = 1 \cdot 10^{-64}$.

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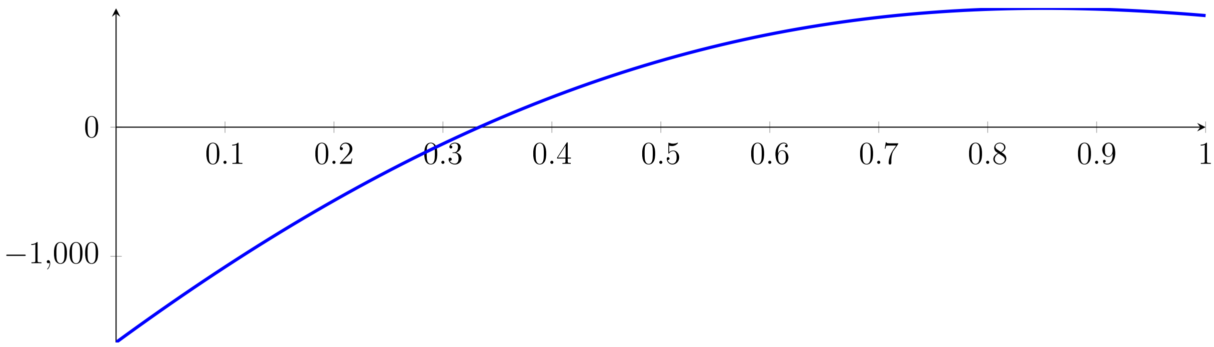
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

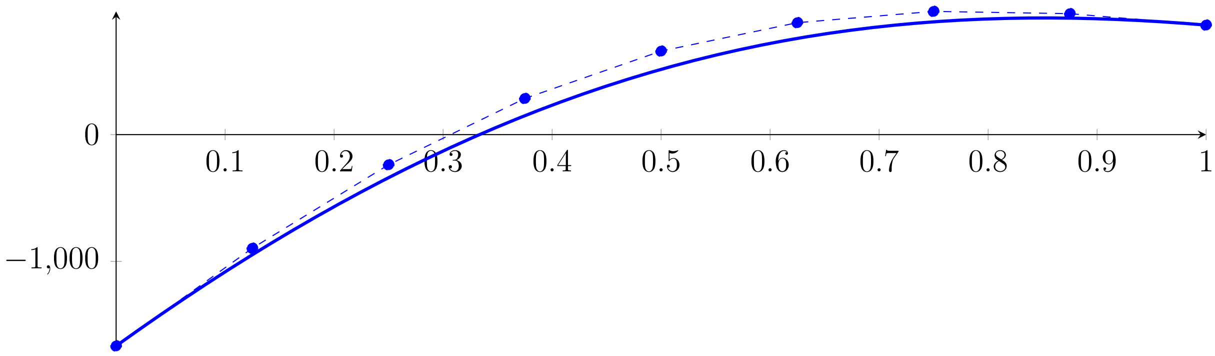
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



144.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

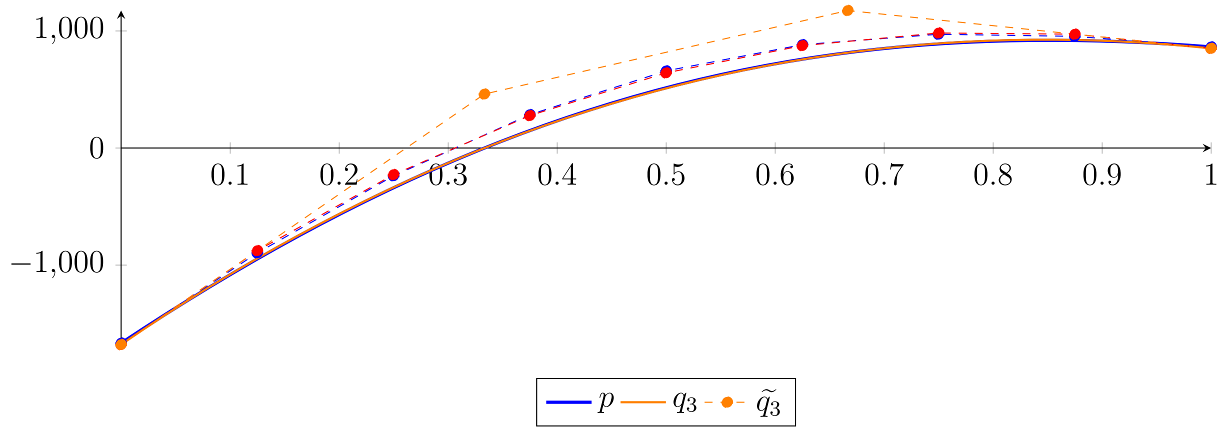
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.39389 \cdot 10^{-09}X^8 + 9.60737 \cdot 10^{-09}X^7 - 1.57477 \cdot 10^{-08}X^6 + 1.35479 \cdot 10^{-08}X^5 \\ &\quad - 6.53397 \cdot 10^{-09}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

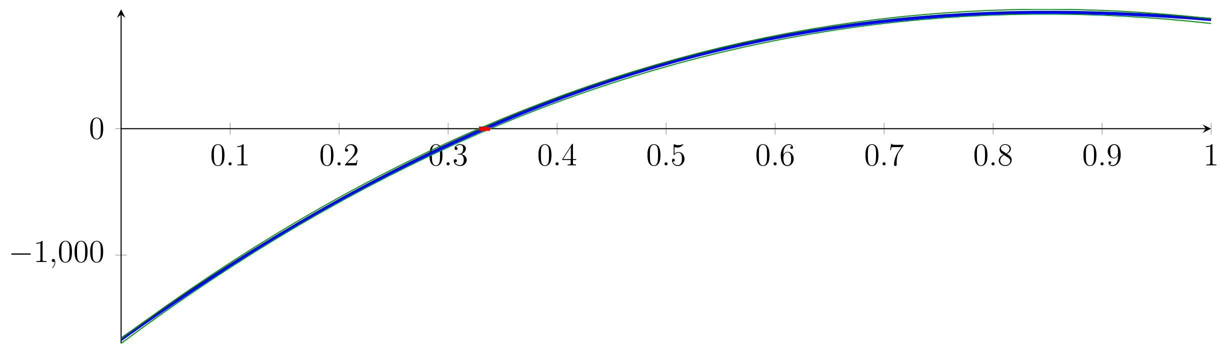
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$[0.328258, 0.338551]$

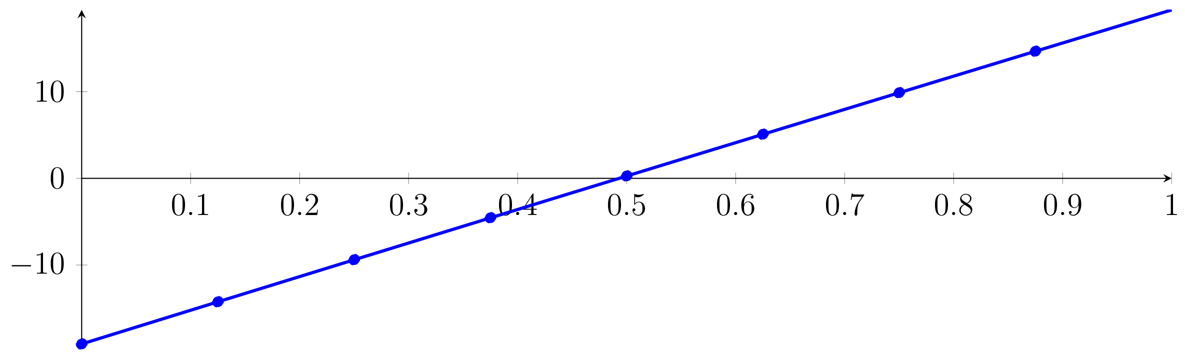
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

144.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

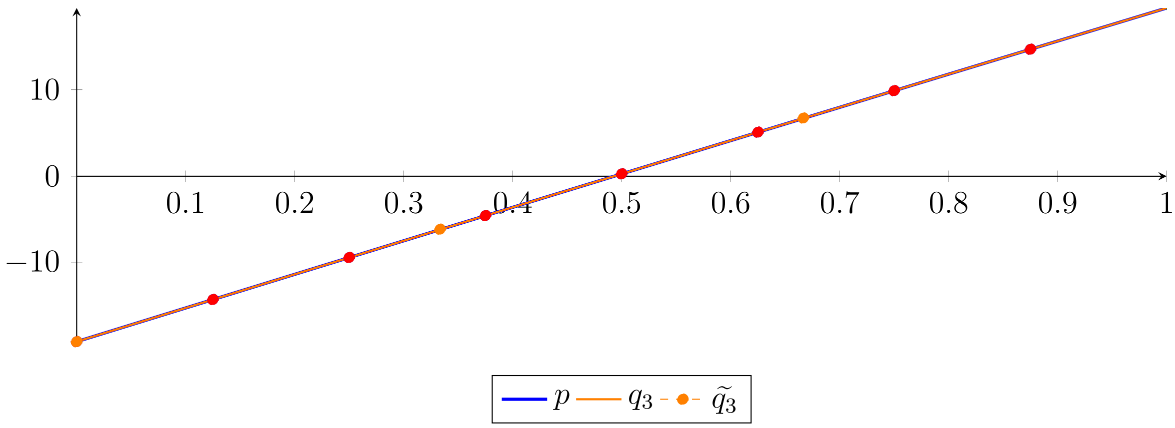
$$\begin{aligned} p = &-1.66533\cdot 10^{-16}X^8 - 1.99007\cdot 10^{-13}X^7 - 8.53059\cdot 10^{-11}X^6 + 8.7284\cdot 10^{-09}X^5 \\ &+ 1.00963\cdot 10^{-05}X^4 - 0.000261161X^3 - 0.416284X^2 + 38.9643X - 19.1124 \\ = &-19.1124B_{0,8}(X) - 14.2419B_{1,8}(X) - 9.38619B_{2,8}(X) - 4.54539B_{3,8}(X) \\ &+ 0.280537B_{4,8}(X) + 5.09158B_{5,8}(X) + 9.88774B_{6,8}(X) + 14.669B_{7,8}(X) + 19.4354B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2.82588 \cdot 10^{-12}X^8 - 1.08529 \cdot 10^{-11}X^7 + 1.68738 \cdot 10^{-11}X^6 - 1.34636 \cdot 10^{-11}X^5 \\ &\quad + 5.72618 \cdot 10^{-12}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m:

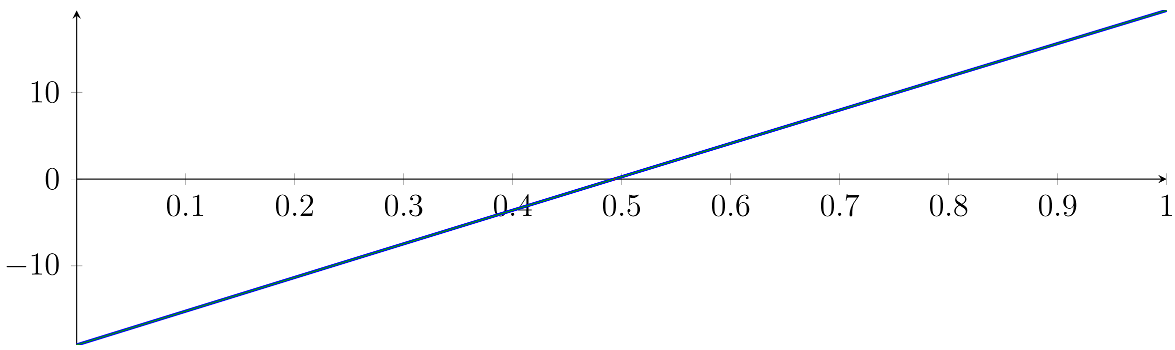
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

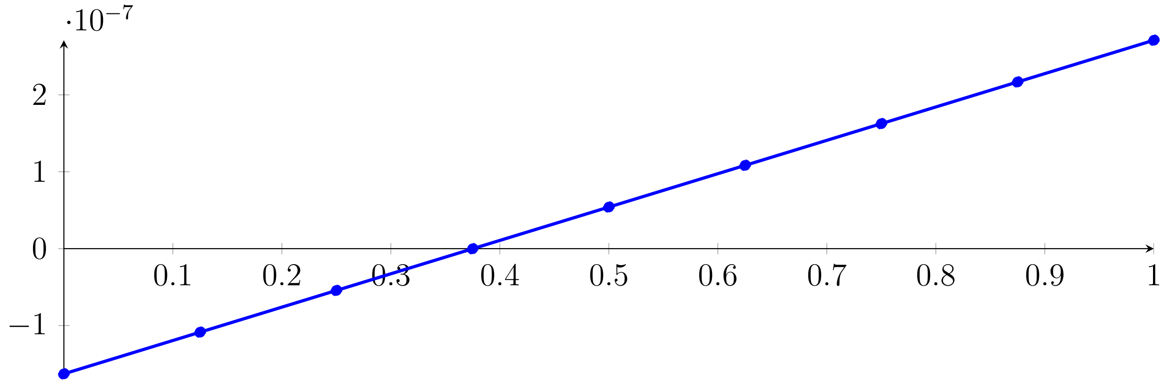
Longest intersection interval: $1.1252 \cdot 10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

144.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

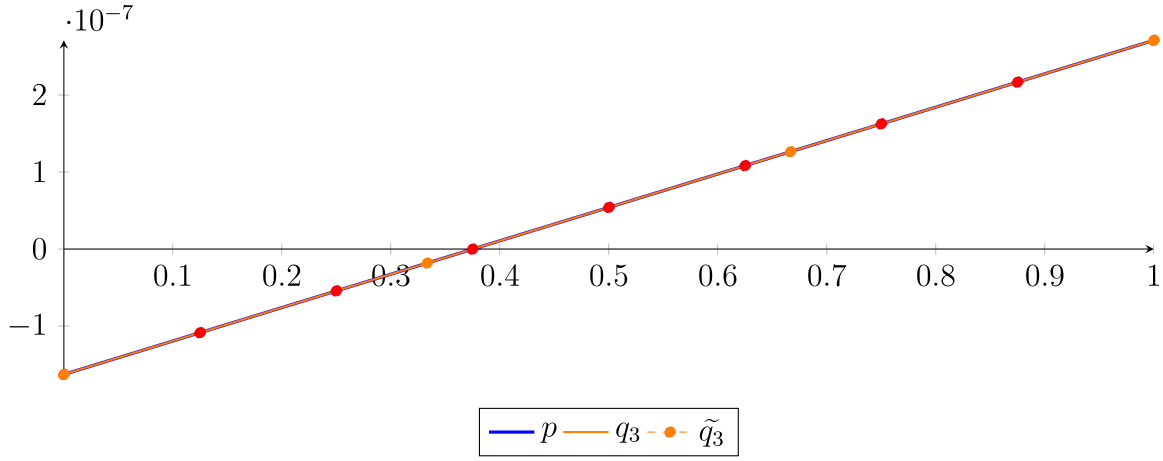
$$p = -4.65289\cdot10^{-24}X^8 - 8.27181\cdot10^{-25}X^7 - 4.3427\cdot10^{-24}X^6 + 8.6854\cdot10^{-24}X^5 + 1.80946 \cdot10^{-24}X^4 - 7.23783\cdot10^{-25}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,8}(X) - 1.08585\cdot10^{-07}B_{1,8}(X) - 5.43593\cdot10^{-08}B_{2,8}(X) - 1.33715\cdot10^{-10}B_{3,8}(X) + 5.40918\cdot10^{-08}B_{4,8}(X) + 1.08317\cdot10^{-07}B_{5,8}(X) + 1.62543\cdot10^{-07}B_{6,8}(X) + 2.16768\cdot10^{-07}B_{7,8}(X) + 2.70994\cdot10^{-07}B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = 1.42947\cdot10^{-23}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,3} - 1.82089\cdot10^{-08}B_{1,3} + 1.26393\cdot10^{-07}B_{2,3} + 2.70994\cdot10^{-07}B_{3,3}$$

$$\tilde{q}_3 = -2.97639\cdot10^{-20}X^8 + 1.26259\cdot10^{-19}X^7 - 2.19172\cdot10^{-19}X^6 + 2.00419\cdot10^{-19}X^5 - 1.02758 \cdot10^{-19}X^4 + 2.83318\cdot10^{-20}X^3 - 5.27552\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,8} - 1.08585\cdot10^{-07}B_{1,8} - 5.43593\cdot10^{-08}B_{2,8} - 1.33715\cdot10^{-10}B_{3,8} + 5.40918 \cdot10^{-08}B_{4,8} + 1.08317\cdot10^{-07}B_{5,8} + 1.62543\cdot10^{-07}B_{6,8} + 2.16768\cdot10^{-07}B_{7,8} + 2.70994\cdot10^{-07}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 1.97535\cdot10^{-22}$.

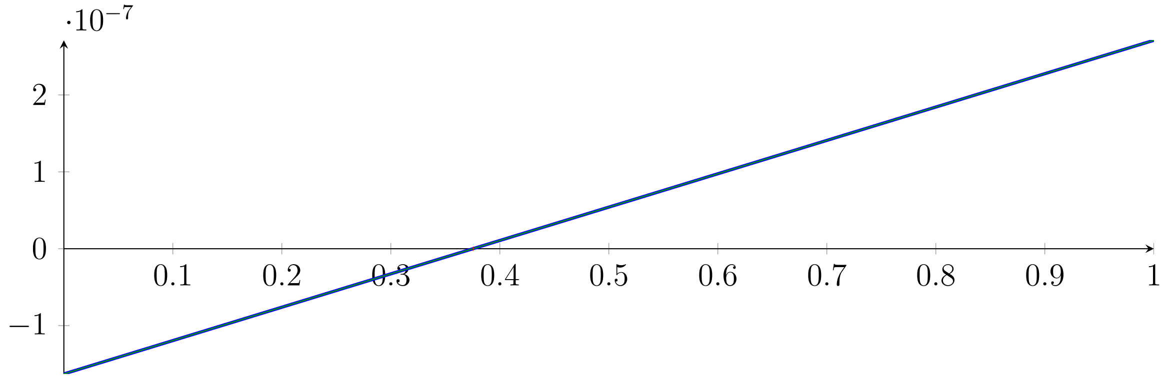
Bounding polynomials M and m:

$$M = 1.42689\cdot10^{-23}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$m = 1.43206\cdot10^{-23}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$

Root of M and m:

$$N(M) = \{0.375308\} \qquad N(m) = \{0.375308\}$$

Intersection intervals:



[0.375308, 0.375308]

Longest intersection interval: $1.36424 \cdot 10^{-12}$
$\Longrightarrow$ Selective recursion: interval 1: [0.333333, 0.333333],

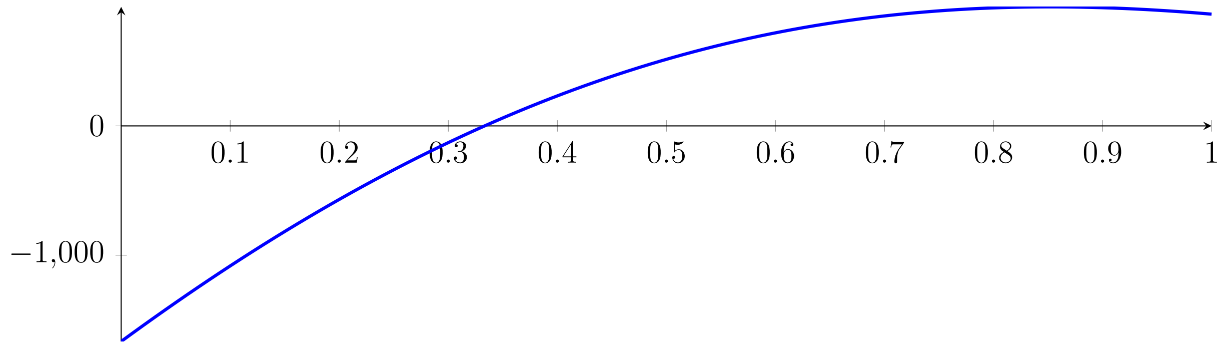
144.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Reached interval [0.333333, 0.333333] **without sign change** at depth 4!
p(0) = -1.10673e-18 - p(1) -5.14919e-19

144.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

with precision $\varepsilon = 1 \cdot 10^{-64}$.

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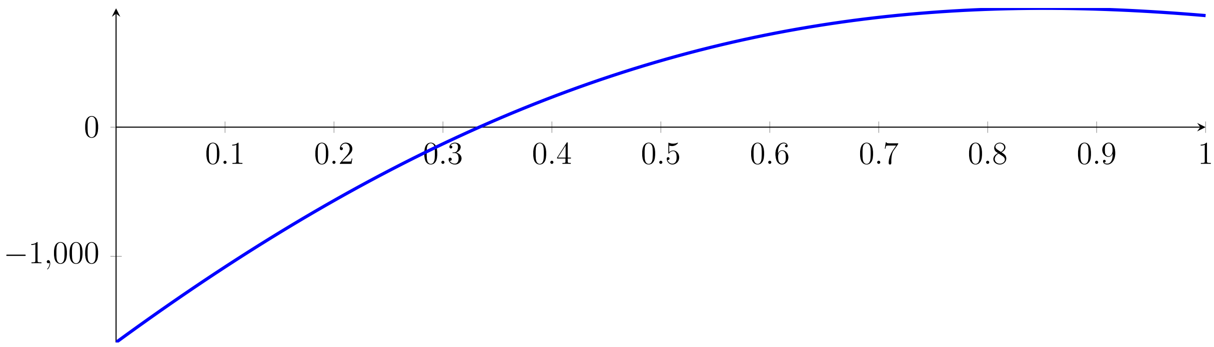
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

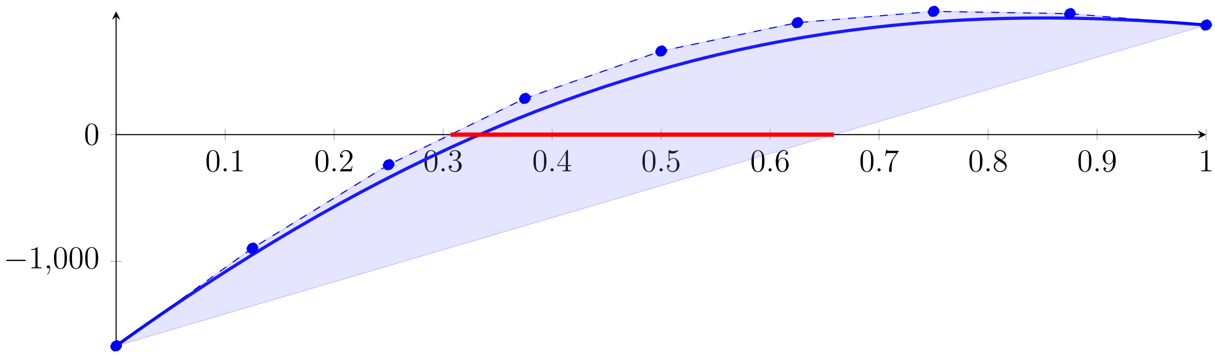
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



145.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

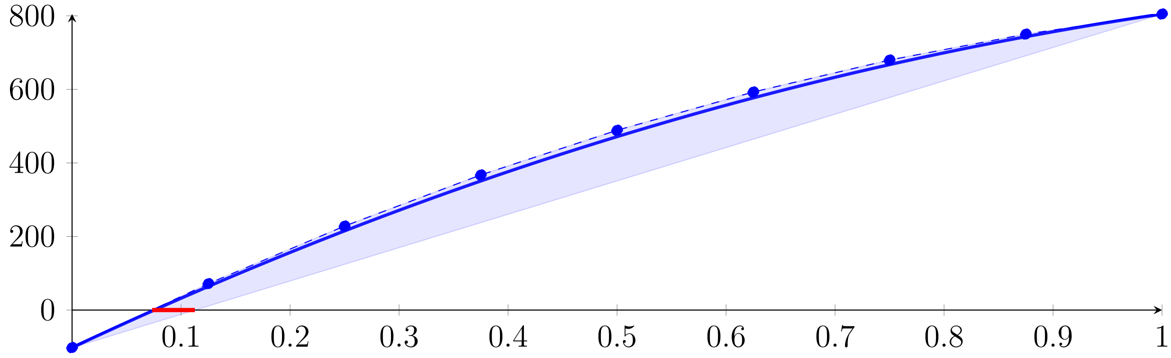
$$[0.306796, 0.658588]$$

Longest intersection interval: 0.351792
$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

145.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\ &+ 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\ = &-102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\ &+ 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

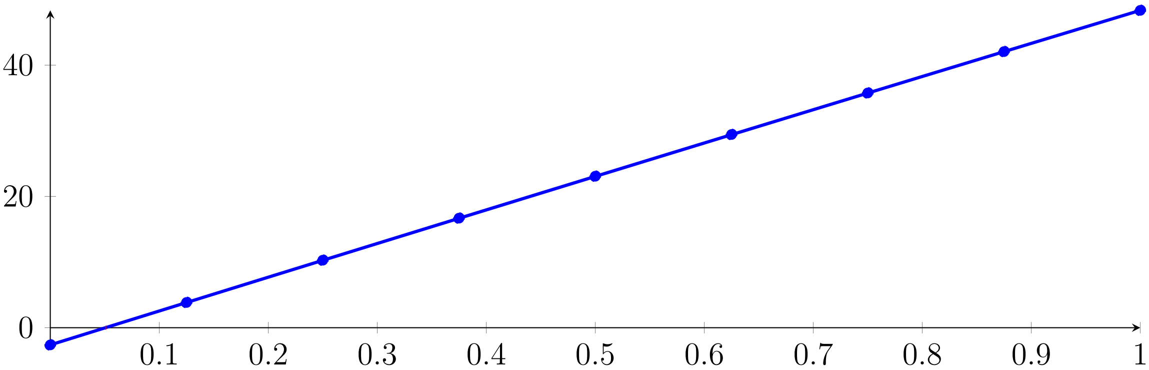
Longest intersection interval: 0.0391855

$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

145.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-3.03577 \cdot 10^{-15}X^8 - 1.54633 \cdot 10^{-12}X^7 - 4.95836 \cdot 10^{-10}X^6 + 3.66751 \cdot 10^{-08}X^5 \\ &+ 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\ = &-2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\ &+ 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

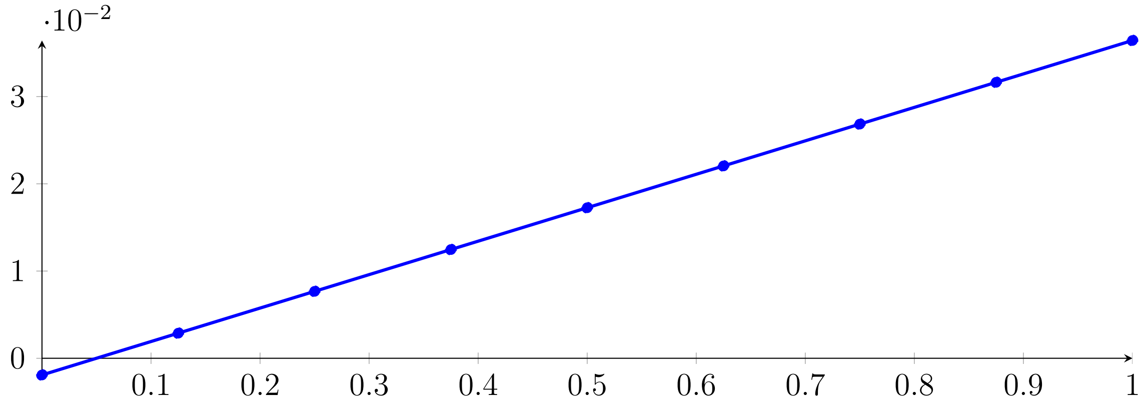
Longest intersection interval: 0.000742589

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

145.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$
\begin{aligned}
p &= -1.28918\cdot 10^{-18}X^8 - 1.32815\cdot 10^{-18}X^7 - 4.50622\cdot 10^{-18}X^6 + 2.22939\cdot 10^{-18}X^5 \\
&\quad + 9.48677\cdot 10^{-18}X^4 - 2.37322\cdot 10^{-13}X^3 - 4.12146\cdot 10^{-07}X^2 + 0.0383444X - 0.00191378 \\
&= -0.00191378B_{0,8}(X) + 0.00287927B_{1,8}(X) + 0.00767231B_{2,8}(X) \\
&\quad + 0.0124653B_{3,8}(X) + 0.0172583B_{4,8}(X) + 0.0220513B_{5,8}(X) \\
&\quad + 0.0268443B_{6,8}(X) + 0.0316373B_{7,8}(X) + 0.0364302B_{8,8}(X)
\end{aligned}
$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

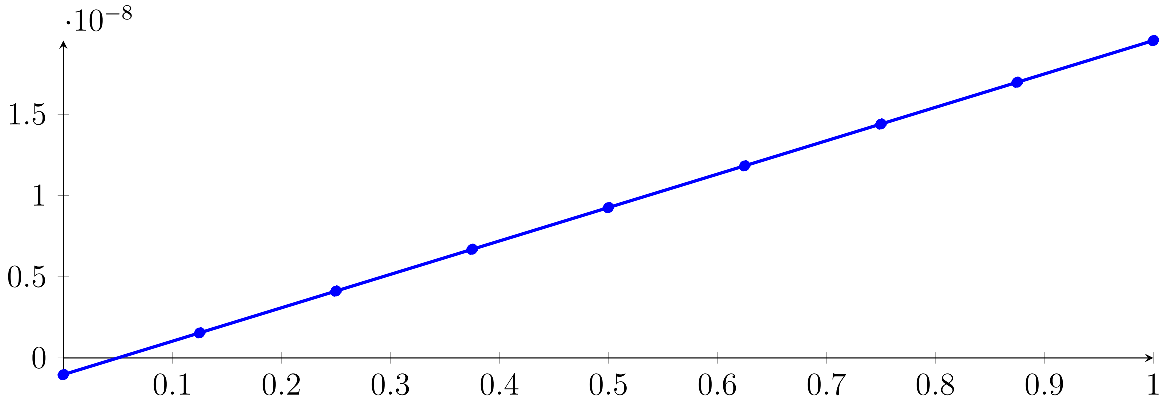
$$[0.0499103, 0.0499109]$$

Longest intersection interval: $5.36469\cdot 10^{-07}$
$\implies$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

145.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$
\begin{aligned}
p &= -7.43978\cdot 10^{-25}X^8 - 4.71751\cdot 10^{-25}X^7 - 2.488\cdot 10^{-24}X^6 + 1.04044\cdot 10^{-24}X^5 \\
&\quad - 2.26182\cdot 10^{-25}X^4 - 1.18615\cdot 10^{-19}X^2 + 2.05706\cdot 10^{-08}X - 1.02667\cdot 10^{-09} \\
&= -1.02667\cdot 10^{-09}B_{0,8}(X) + 1.54465\cdot 10^{-09}B_{1,8}(X) + 4.11597\cdot 10^{-09}B_{2,8}(X) \\
&\quad + 6.68729\cdot 10^{-09}B_{3,8}(X) + 9.25862\cdot 10^{-09}B_{4,8}(X) + 1.18299\cdot 10^{-08}B_{5,8}(X) \\
&\quad + 1.44013\cdot 10^{-08}B_{6,8}(X) + 1.69726\cdot 10^{-08}B_{7,8}(X) + 1.95439\cdot 10^{-08}B_{8,8}(X)
\end{aligned}
$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87793 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

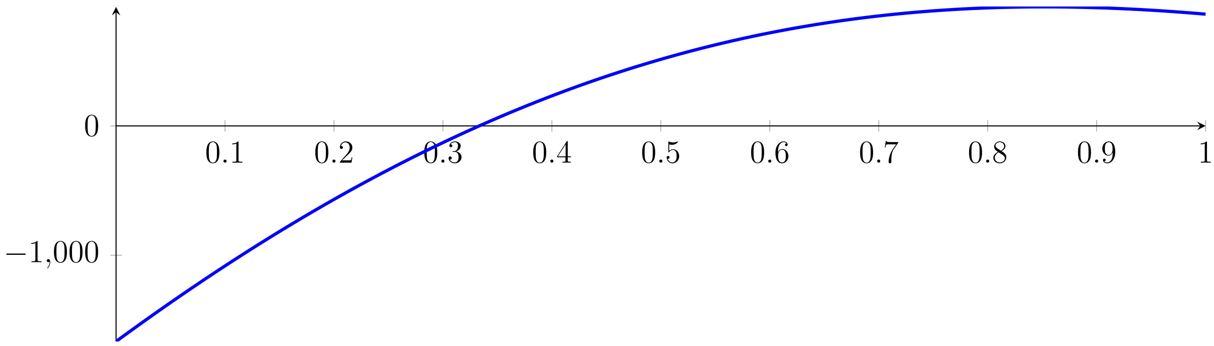
145.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

145.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-128}$.

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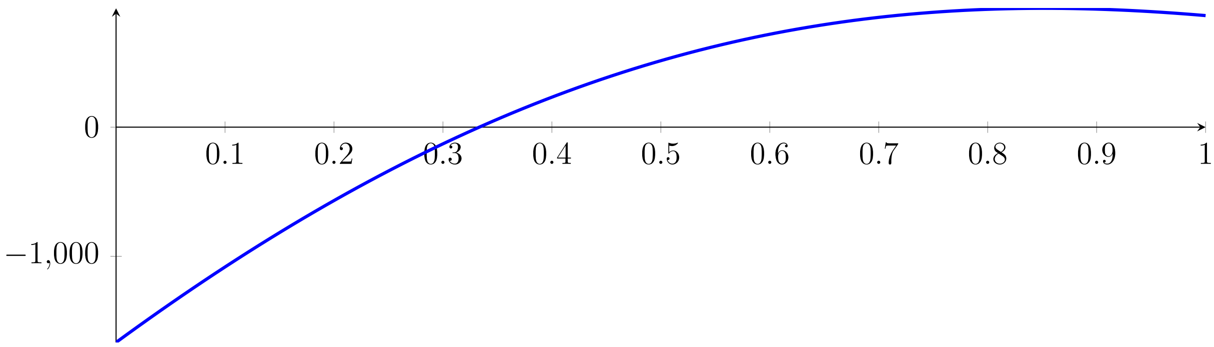
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

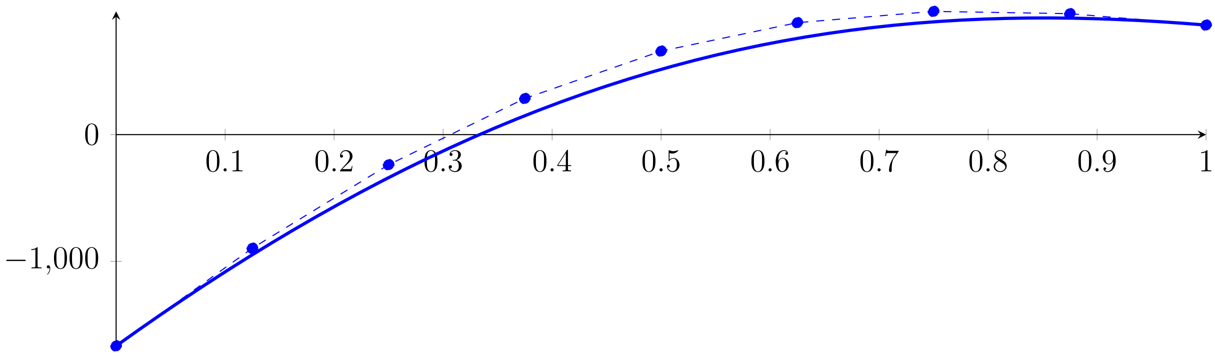
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



146.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

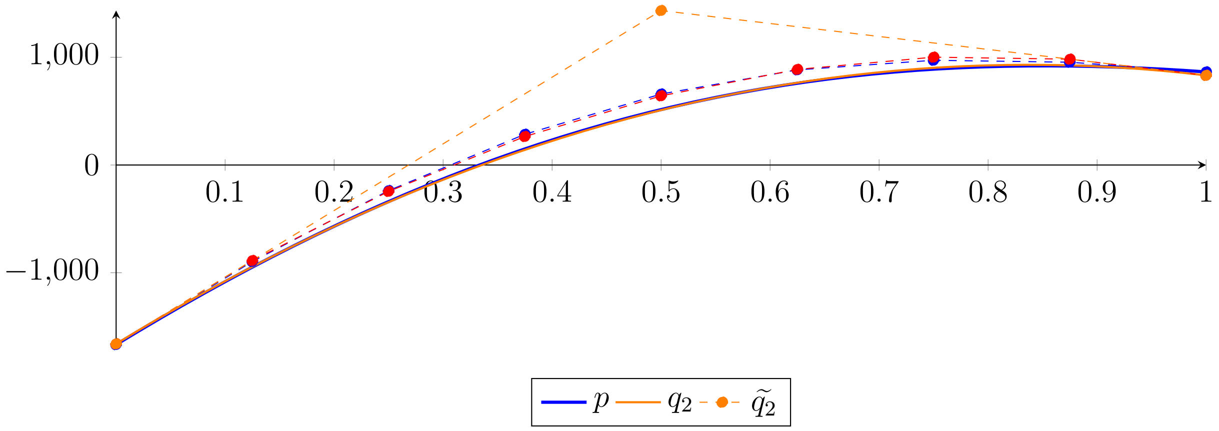
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -5.06304 \cdot 10^{-10}X^8 + 1.93305 \cdot 10^{-09}X^7 - 2.97598 \cdot 10^{-09}X^6 + 2.35116 \cdot 10^{-09}X^5 \\ &\quad - 9.91232 \cdot 10^{-10}X^4 + 2.00705 \cdot 10^{-10}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

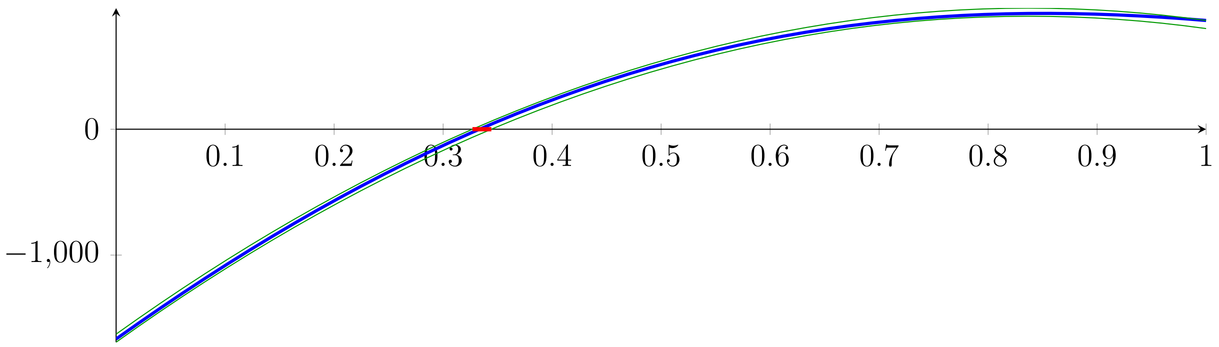
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

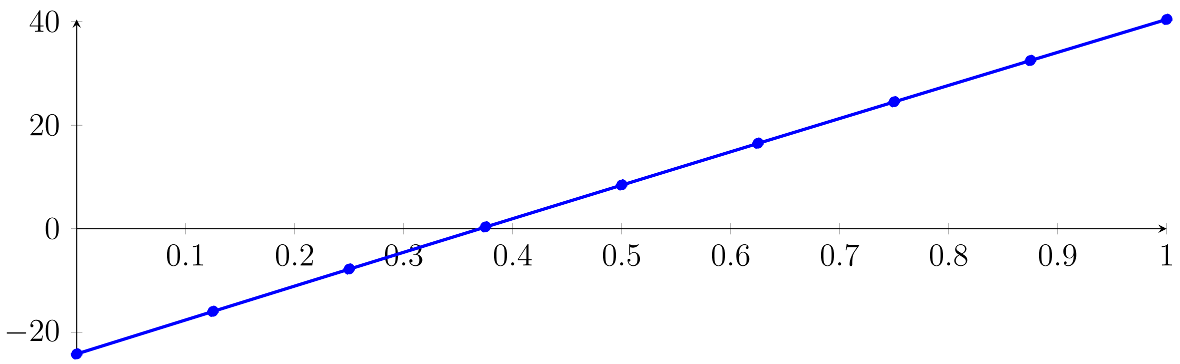
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

146.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

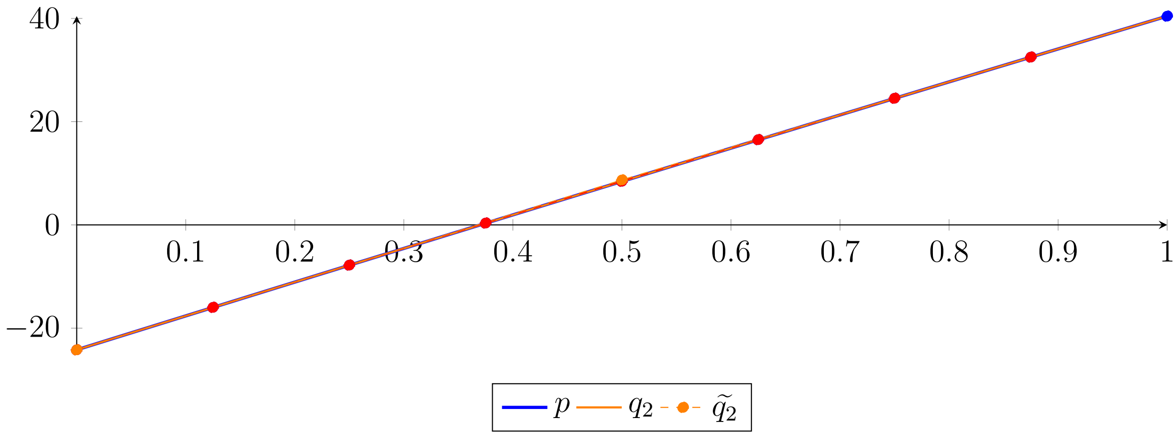
$$\begin{aligned} p = &-8.80546\cdot10^{-15}X^8 - 7.66587\cdot10^{-12}X^7 - 1.9444\cdot10^{-09}X^6 + 1.19263\cdot10^{-07}X^5 \\ &+ 8.12335\cdot10^{-05}X^4 - 0.0012733X^3 - 1.18084X^2 + 65.8155X - 24.1945 \\ = &-24.1945B_{0,8}(X) - 15.9676B_{1,8}(X) - 7.78282B_{2,8}(X) + 0.35975B_{3,8}(X) \\ &+ 8.4601B_{4,8}(X) + 16.5182B_{5,8}(X) + 24.5341B_{6,8}(X) + 32.5077B_{7,8}(X) + 40.4389B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -2.59056\cdot 10^{-11}X^8 + 1.00262\cdot 10^{-10}X^7 - 1.57692\cdot 10^{-10}X^6 + 1.29283\cdot 10^{-10}X^5 \\ &\quad - 5.86775\cdot 10^{-11}X^4 + 1.42642\cdot 10^{-11}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894\cdot 10^{-05}$.
Bounding polynomials M and m:

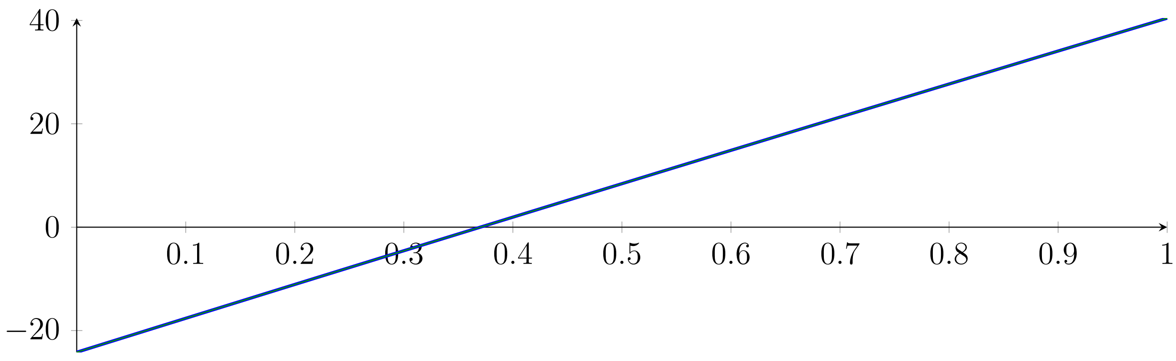
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



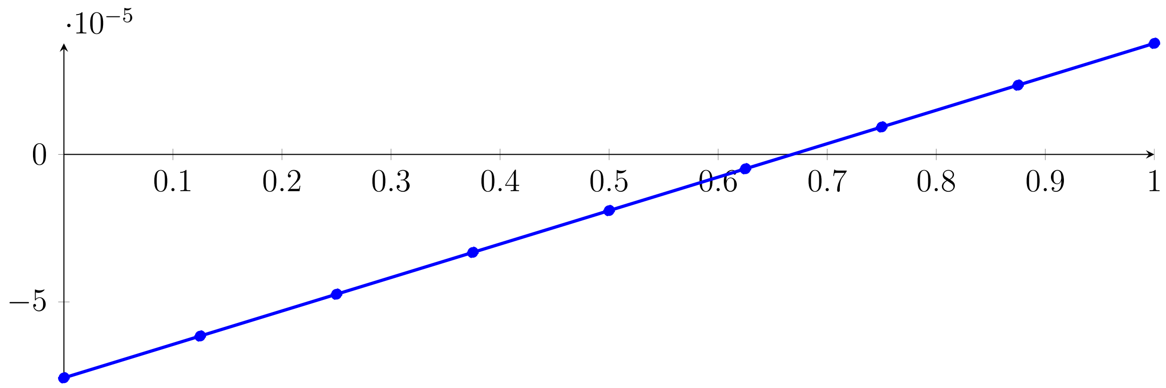
$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588\cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

146.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

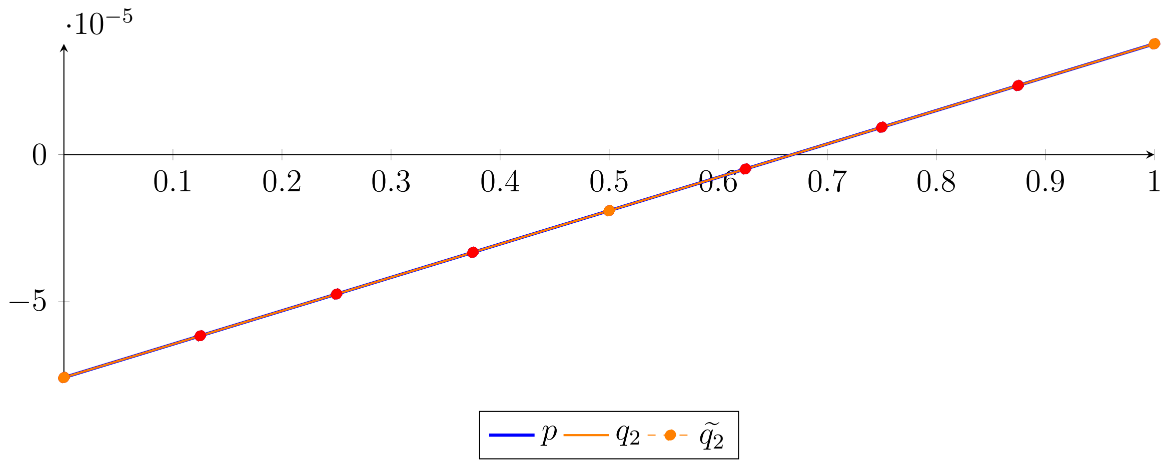
$$p = 1.16467\cdot10^{-21}X^8 + 3.17637\cdot10^{-21}X^7 + 1.18585\cdot10^{-20}X^6 - 1.48231\cdot10^{-21}X^5 + 9.26442\cdot10^{-22}X^4 - 5.92923\cdot10^{-21}X^3 - 3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$= -7.5732\cdot10^{-05}B_{0,8}(X) - 6.15596\cdot10^{-05}B_{1,8}(X) - 4.73873\cdot10^{-05}B_{2,8}(X) - 3.32149\cdot10^{-05}B_{3,8}(X) - 1.90425\cdot10^{-05}B_{4,8}(X) - 4.87016\cdot10^{-06}B_{5,8}(X) + 9.3022\cdot10^{-06}B_{6,8}(X) + 2.34746\cdot10^{-05}B_{7,8}(X) + 3.76469\cdot10^{-05}B_{8,8}(X)$$



Degree reduction and raising:

$$q_2 = -3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$= -7.5732\cdot10^{-05}B_{0,2} - 1.90425\cdot10^{-05}B_{1,2} + 3.76469\cdot10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = 3.26671\cdot10^{-17}X^8 - 1.38104\cdot10^{-16}X^7 + 2.39221\cdot10^{-16}X^6 - 2.17429\cdot10^{-16}X^5 + 1.10046\cdot10^{-16}X^4 - 3.0162\cdot10^{-17}X^3 - 3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$= -7.5732\cdot10^{-05}B_{0,8} - 6.15596\cdot10^{-05}B_{1,8} - 4.73873\cdot10^{-05}B_{2,8} - 3.32149\cdot10^{-05}B_{3,8} - 1.90425\cdot10^{-05}B_{4,8} - 4.87016\cdot10^{-06}B_{5,8} + 9.3022\cdot10^{-06}B_{6,8} + 2.34746\cdot10^{-05}B_{7,8} + 3.76469\cdot10^{-05}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 1.84643\cdot10^{-19}$.

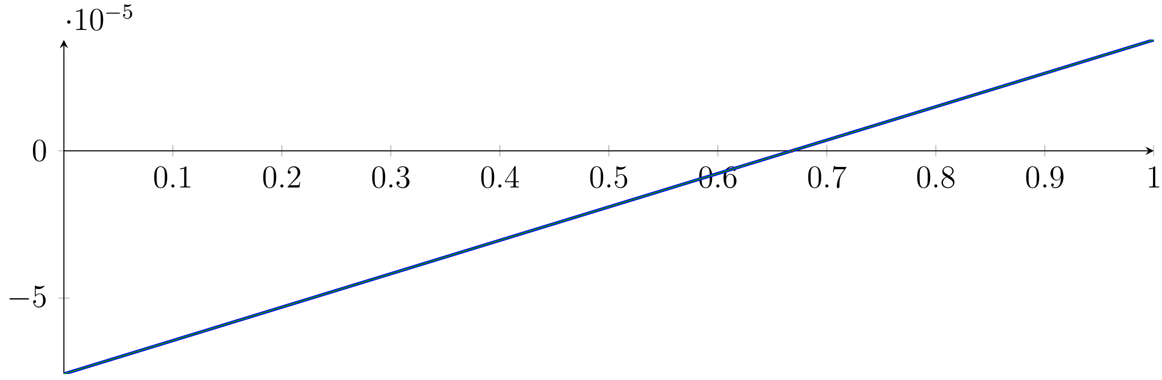
Bounding polynomials M and m:

$$M = -3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$m = -3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$

Root of M and m:

$$N(M) = \{0.667955, 3.14644\cdot10^{7}\} \qquad N(m) = \{0.667955, 3.14644\cdot10^{7}\}$$

Intersection intervals:



$[0.667955, 0.667955]$

Longest intersection interval: $3.08439 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

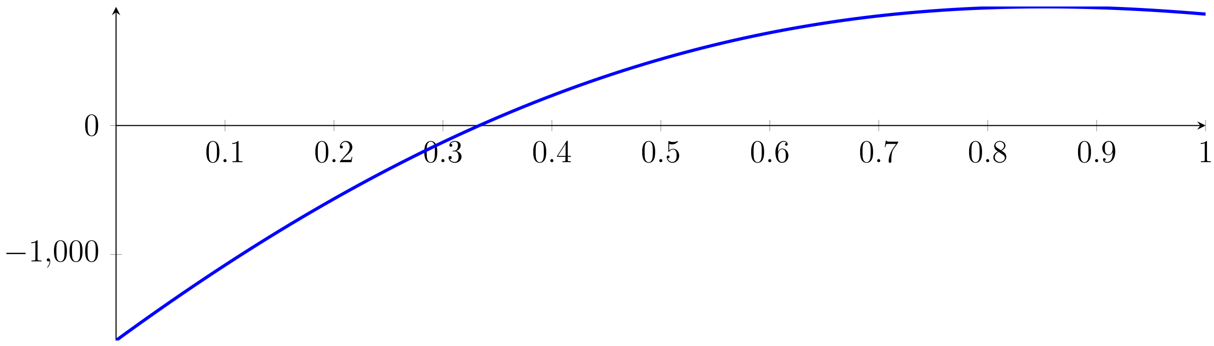
146.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 4!
p(0) = 2.85706e-18 - p(1) 3.78276e-17

146.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

with precision $\varepsilon = 1 \cdot 10^{-128}$.

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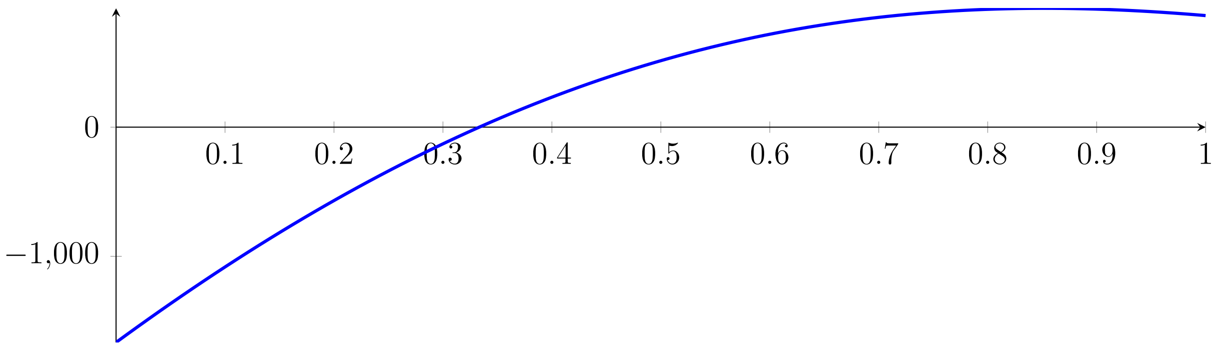
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0,1]$:

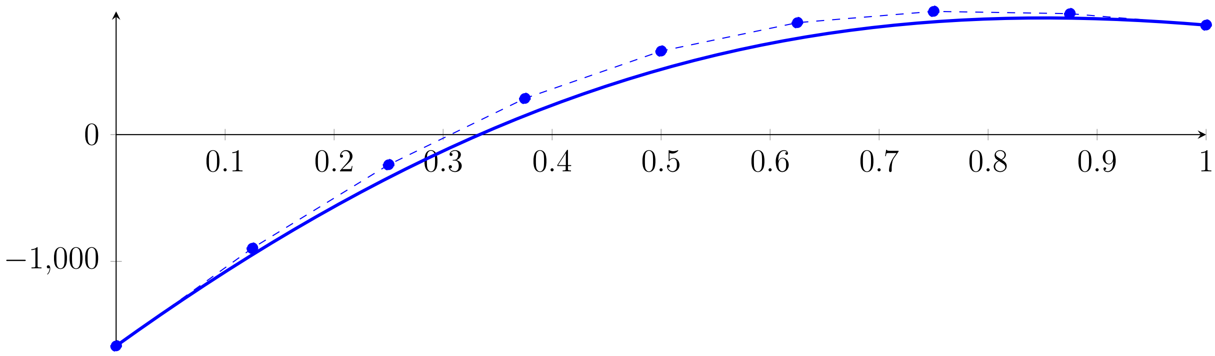
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



147.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

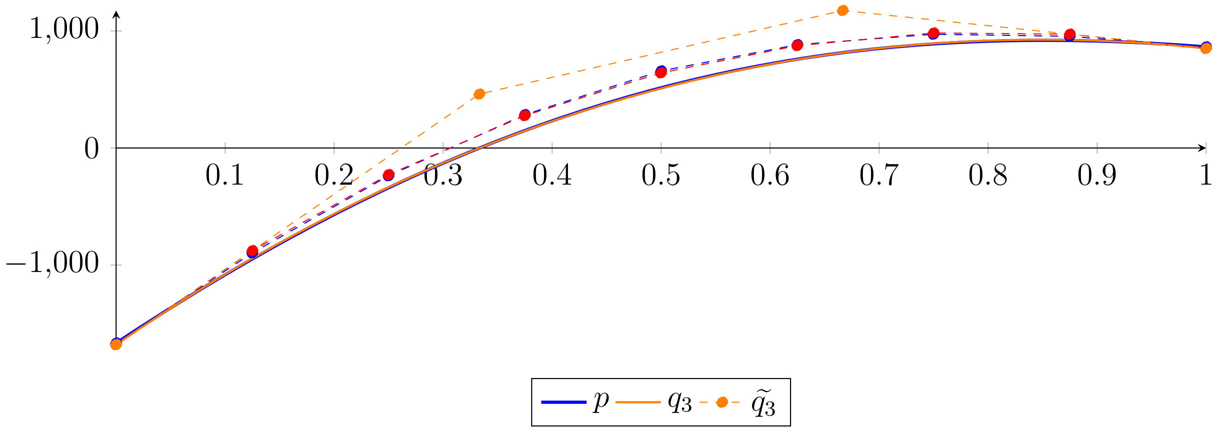
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.39389 \cdot 10^{-09}X^8 + 9.60737 \cdot 10^{-09}X^7 - 1.57477 \cdot 10^{-08}X^6 + 1.35479 \cdot 10^{-08}X^5 \\ &\quad - 6.53397 \cdot 10^{-09}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

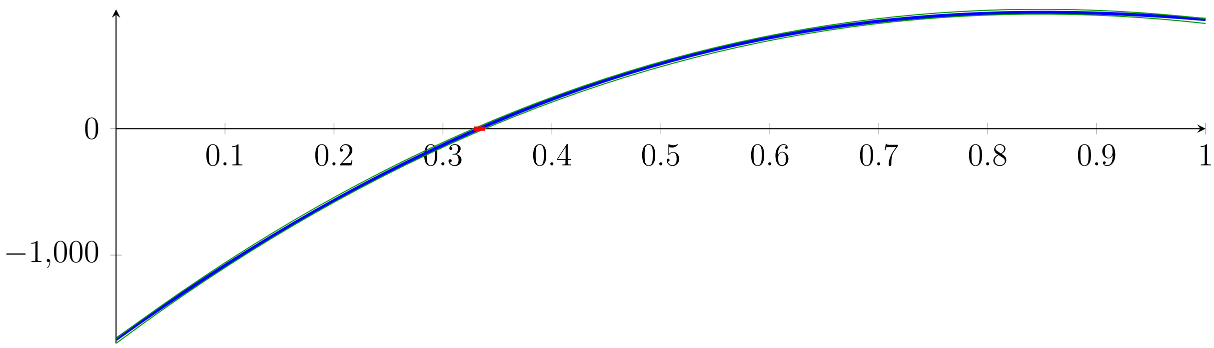
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$[0.328258, 0.338551]$

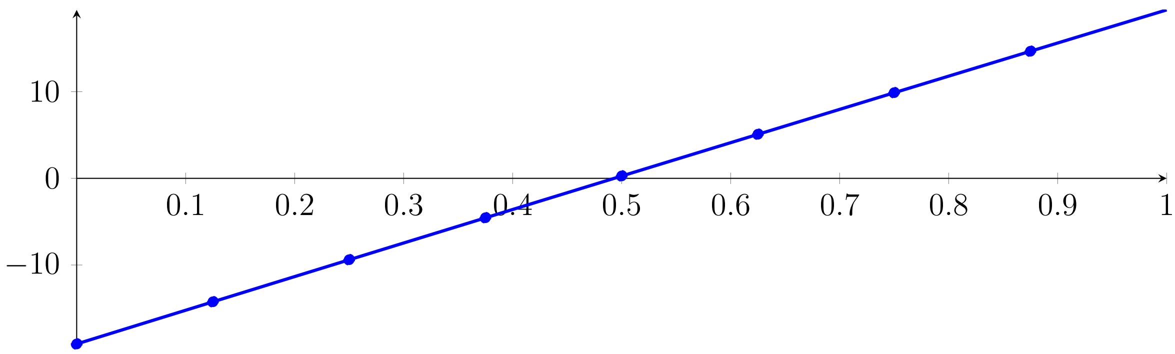
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

147.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

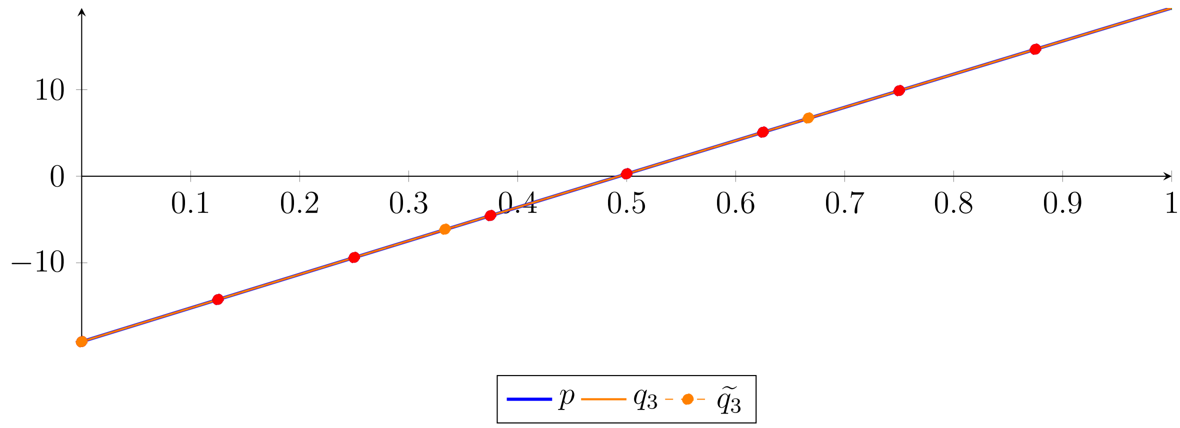
$$\begin{aligned} p = &-1.66533 \cdot 10^{-16} X^8 - 1.99007 \cdot 10^{-13} X^7 - 8.53059 \cdot 10^{-11} X^6 + 8.7284 \cdot 10^{-09} X^5 \\ &+ 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ = &-19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ &+ 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2.82588{\cdot}10^{-12}X^8 - 1.08529{\cdot}10^{-11}X^7 + 1.68738{\cdot}10^{-11}X^6 - 1.34636{\cdot}10^{-11}X^5 \\ &\quad + 5.72618{\cdot}10^{-12}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902{\cdot}10^{-07}$.
Bounding polynomials M and m:

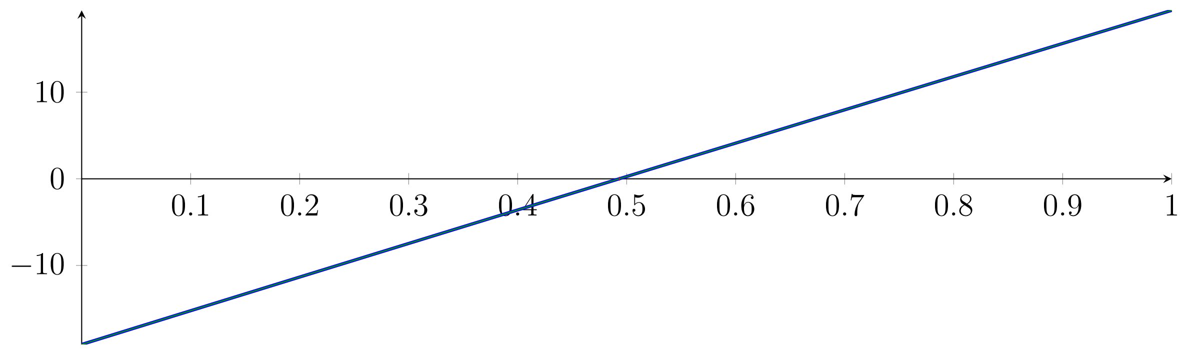
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$N(M) = \{-1816.81, 0.493109, 88.5414\}$ $\qquad$ $N(m) = \{-1816.81, 0.493109, 88.5414\}$

Intersection intervals:



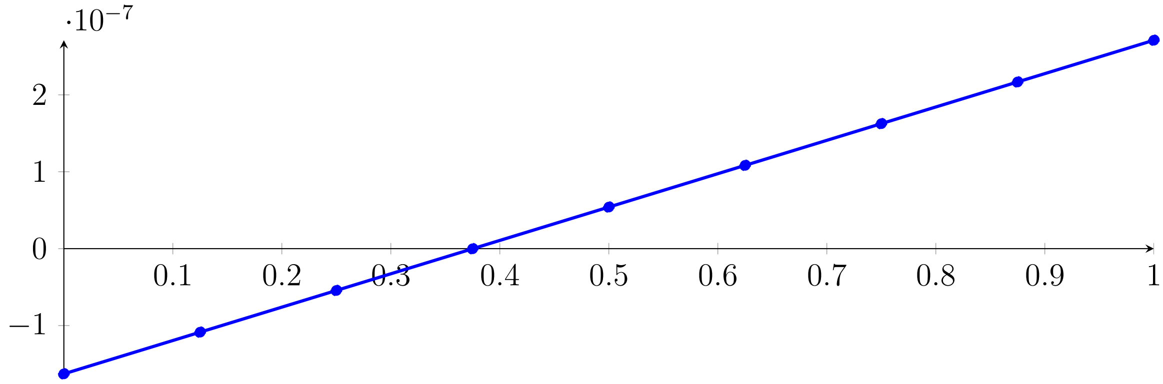
$[0.493109, 0.493109]$

Longest intersection interval: $1.1252{\cdot}10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

147.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

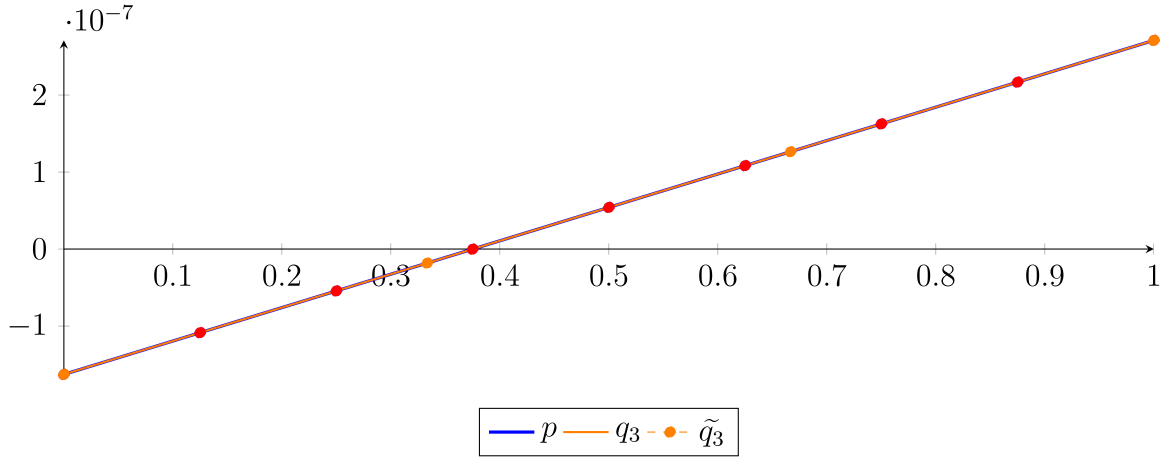
$$p = -4.65289\cdot10^{-24}X^8 - 8.27181\cdot10^{-25}X^7 - 4.3427\cdot10^{-24}X^6 + 8.6854\cdot10^{-24}X^5 + 1.80946\cdot10^{-24}X^4 - 7.23783\cdot10^{-25}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,8}(X) - 1.08585\cdot10^{-07}B_{1,8}(X) - 5.43593\cdot10^{-08}B_{2,8}(X) - 1.33715\cdot10^{-10}B_{3,8}(X) + 5.40918\cdot10^{-08}B_{4,8}(X) + 1.08317\cdot10^{-07}B_{5,8}(X) + 1.62543\cdot10^{-07}B_{6,8}(X) + 2.16768\cdot10^{-07}B_{7,8}(X) + 2.70994\cdot10^{-07}B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = 1.42947\cdot10^{-23}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,3} - 1.82089\cdot10^{-08}B_{1,3} + 1.26393\cdot10^{-07}B_{2,3} + 2.70994\cdot10^{-07}B_{3,3}$$

$$\tilde{q}_3 = -2.97639\cdot10^{-20}X^8 + 1.26259\cdot10^{-19}X^7 - 2.19172\cdot10^{-19}X^6 + 2.00419\cdot10^{-19}X^5 - 1.02758\cdot10^{-19}X^4 + 2.83318\cdot10^{-20}X^3 - 5.27552\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,8} - 1.08585\cdot10^{-07}B_{1,8} - 5.43593\cdot10^{-08}B_{2,8} - 1.33715\cdot10^{-10}B_{3,8} + 5.40918\cdot10^{-08}B_{4,8} + 1.08317\cdot10^{-07}B_{5,8} + 1.62543\cdot10^{-07}B_{6,8} + 2.16768\cdot10^{-07}B_{7,8} + 2.70994\cdot10^{-07}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 1.97535\cdot10^{-22}$.

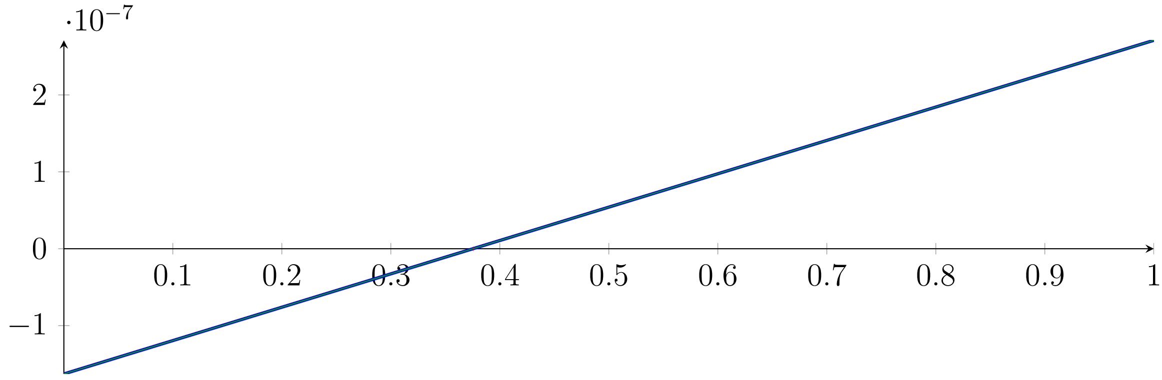
Bounding polynomials M and m:

$$M = 1.42689\cdot10^{-23}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$m = 1.43206\cdot10^{-23}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$

Root of M and m:

$$N(M) = \{0.375308\} \qquad N(m) = \{0.375308\}$$

Intersection intervals:



$[0.375308, 0.375308]$

Longest intersection interval: $1.36424 \cdot 10^{-12}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

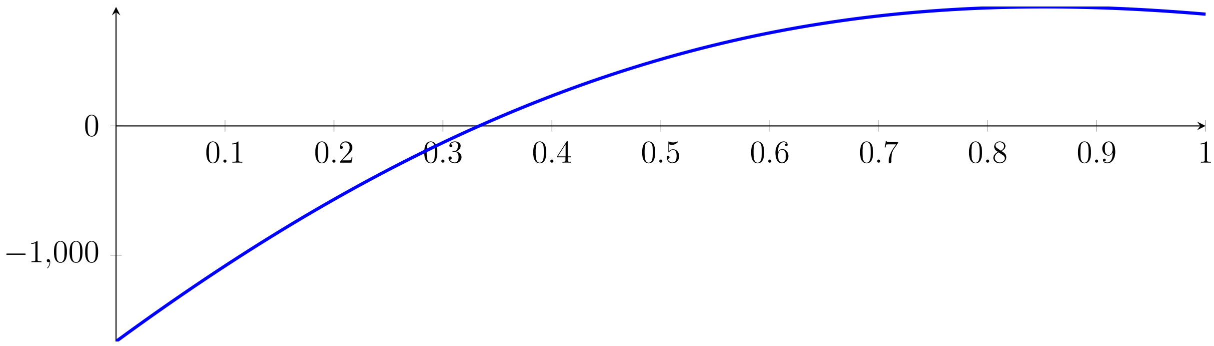
147.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 4!
p(0) = -1.10673e-18 - p(1) -5.14919e-19

147.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

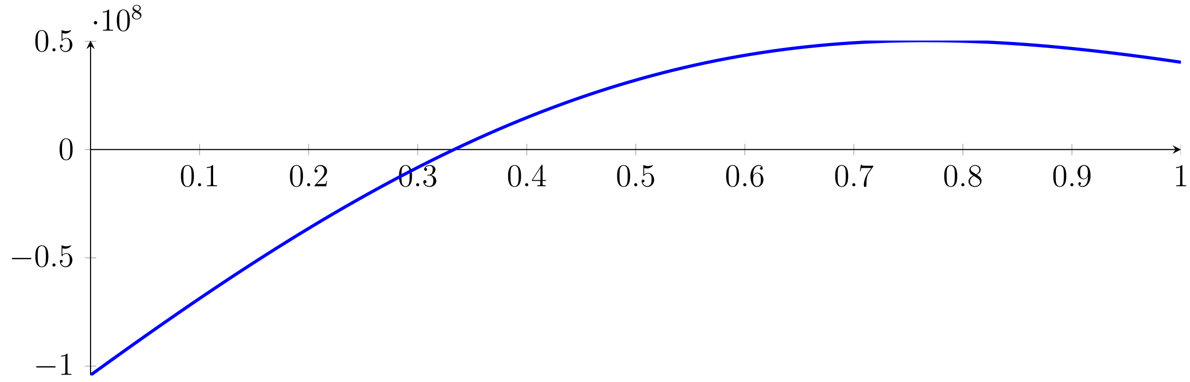
with precision $\varepsilon = 1 \cdot 10^{-128}$.

148 Running BezClip on f_{16} with epsilon 2

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^6X^9 - 2.52262\cdot 10^6X^8 - 1.43506\cdot 10^7X^7 + 138542X^6 + 7.92823\cdot 10^7X^5 + 3.97396\cdot 10^7X^4 - 2.40625\cdot 10^8X^3 - 8.33333\cdot 10^7X^2 + 3.64583\cdot 10^8X - 1.04167\cdot 10^8$$

Called BezClip with input polynomial on interval $[0, 1]$:

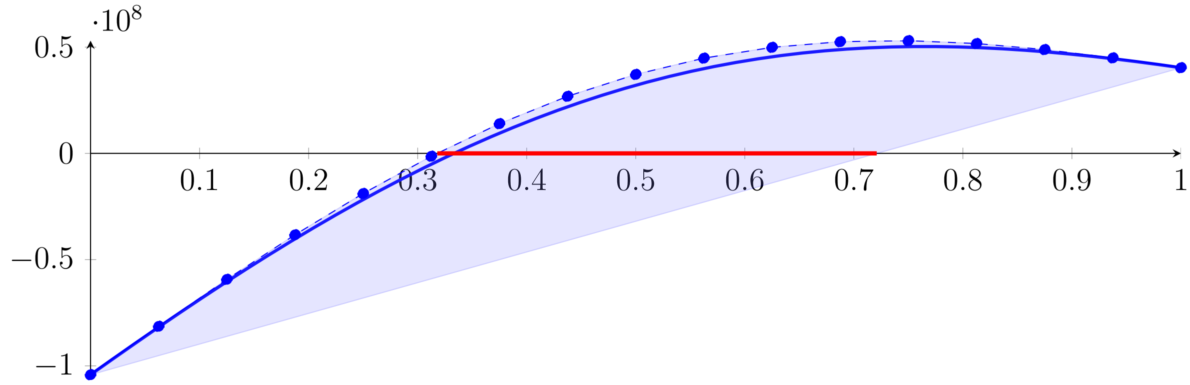
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^6X^9 - 2.52262\cdot 10^6X^8 - 1.43506\cdot 10^7X^7 + 138542X^6 + 7.92823\cdot 10^7X^5 + 3.97396\cdot 10^7X^4 - 2.40625\cdot 10^8X^3 - 8.33333\cdot 10^7X^2 + 3.64583\cdot 10^8X - 1.04167\cdot 10^8$$



148.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot 10^6X^9 - 2.52262\cdot 10^6X^8 - 1.43506\cdot 10^7X^7 + 138542X^6 + 7.92823 \\ &\quad \cdot 10^7X^5 + 3.97396\cdot 10^7X^4 - 2.40625\cdot 10^8X^3 - 8.33333\cdot 10^7X^2 + 3.64583\cdot 10^8X - 1.04167\cdot 10^8 \\ &= -1.04167\cdot 10^8B_{0,16}(X) - 8.13802\cdot 10^7B_{1,16}(X) - 5.92882\cdot 10^7B_{2,16}(X) - 3.83203 \\ &\quad \cdot 10^7B_{3,16}(X) - 1.88844\cdot 10^7B_{4,16}(X) - 1.34837\cdot 10^6B_{5,16}(X) + 1.39781\cdot 10^7B_{6,16}(X) \\ &\quad + 2.68604\cdot 10^7B_{7,16}(X) + 3.71532\cdot 10^7B_{8,16}(X) + 4.48105\cdot 10^7B_{9,16}(X) + 4.98901 \\ &\quad \cdot 10^7B_{10,16}(X) + 5.25504\cdot 10^7B_{11,16}(X) + 5.30407\cdot 10^7B_{12,16}(X) + 5.16832\cdot 10^7B_{13,16}(X) \\ &\quad + 4.88488\cdot 10^7B_{14,16}(X) + 4.49297\cdot 10^7B_{15,16}(X) + 4.03108\cdot 10^7B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

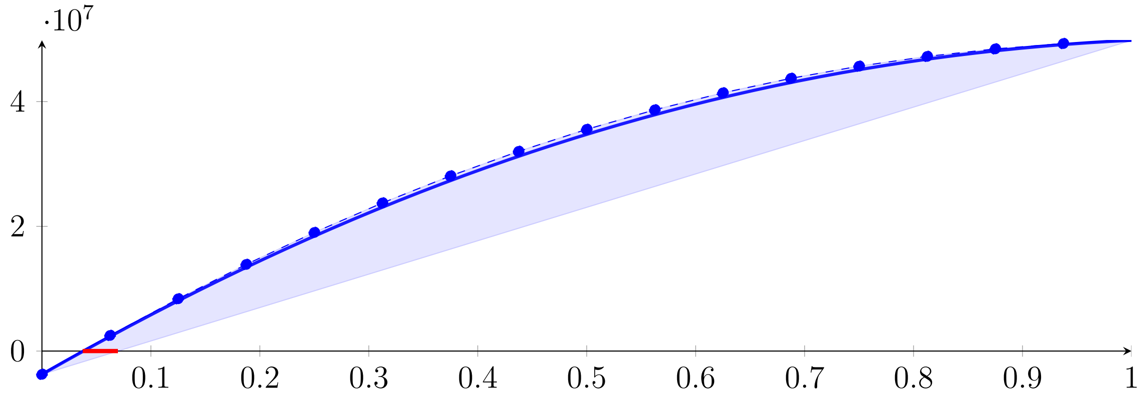
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

148.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
p &= 1.59825 \cdot 10^{-06} X^{16} - 5.93153 \cdot 10^{-05} X^{15} - 0.00248867 X^{14} - 0.0650056 X^{13} - 0.909142 X^{12} \\
&\quad - 5.03931 X^{11} + 36.4549 X^{10} + 692.921 X^{9} + 1886.97 X^{8} - 25792 X^{7} - 149671 X^{6} + 492605 X^{5} \\
&\quad + 3.91945 \cdot 10^{6} X^{4} - 7.54561 \cdot 10^{6} X^{3} - 4.29225 \cdot 10^{7} X^{2} + 9.97982 \cdot 10^{7} X - 3.73501 \cdot 10^{6} \\
&= -3.73501 \cdot 10^{6} B_{0,16}(X) + 2.50237 \cdot 10^{6} B_{1,16}(X) + 8.38207 \cdot 10^{6} B_{2,16}(X) + 1.38906 \\
&\quad \cdot 10^{7} B_{3,16}(X) + 1.90167 \cdot 10^{7} B_{4,16}(X) + 2.37512 \cdot 10^{7} B_{5,16}(X) + 2.80875 \cdot 10^{7} B_{6,16}(X) \\
&\quad + 3.20213 \cdot 10^{7} B_{7,16}(X) + 3.55508 \cdot 10^{7} B_{8,16}(X) + 3.86768 \cdot 10^{7} B_{9,16}(X) + 4.14025 \\
&\quad \cdot 10^{7} B_{10,16}(X) + 4.37336 \cdot 10^{7} B_{11,16}(X) + 4.56783 \cdot 10^{7} B_{12,16}(X) + 4.7247 \cdot 10^{7} B_{13,16}(X) \\
&\quad + 4.84524 \cdot 10^{7} B_{14,16}(X) + 4.93093 \cdot 10^{7} B_{15,16}(X) + 4.98343 \cdot 10^{7} B_{16,16}(X)
\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

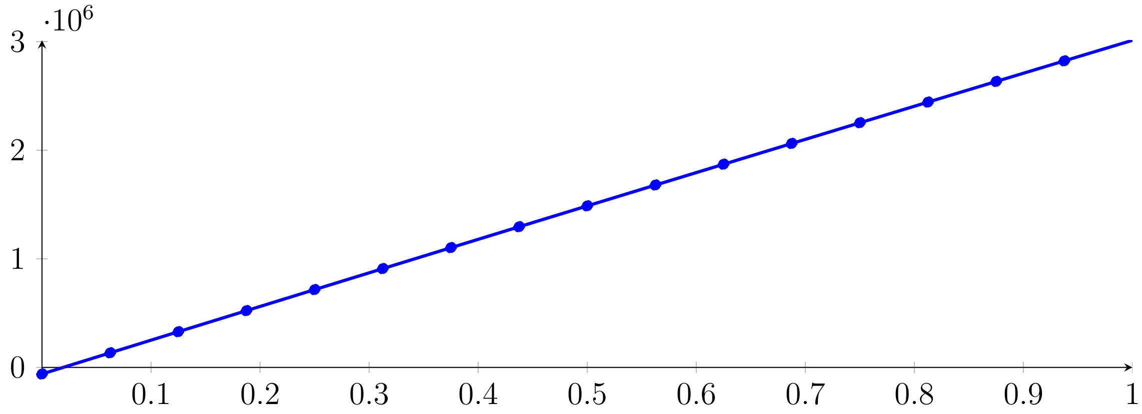
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

148.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
p &= 9.01396 \cdot 10^{-08} X^{16} - 2.65848 \cdot 10^{-07} X^{15} + 2.13948 \cdot 10^{-06} X^{14} - 1.33627 \cdot 10^{-06} X^{13} + 2.46973 \cdot 10^{-06} X^{12} \\
&\quad - 2.45524 \cdot 10^{-06} X^{11} + 5.50112 \cdot 10^{-07} X^{10} - 1.64198 \cdot 10^{-07} X^{9} - 7.35598 \cdot 10^{-07} X^{8} - 1.00892 \cdot 10^{-06} X^{7} \\
&\quad - 0.000177509 X^{6} + 0.0161038 X^{5} + 4.36155 X^{4} - 234.216 X^{3} - 45622.2 X^{2} + 3.11845 \cdot 10^{6} X - 60508.5 \\
&= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\
&\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^{6} B_{6,16}(X) + 1.29582 \cdot 10^{6} B_{7,16}(X) + 1.48805 \cdot 10^{6} B_{8,16}(X) \\
&\quad + 1.6799 \cdot 10^{6} B_{9,16}(X) + 1.87137 \cdot 10^{6} B_{10,16}(X) + 2.06245 \cdot 10^{6} B_{11,16}(X) + 2.25315 \cdot 10^{6} B_{12,16}(X) \\
&\quad + 2.44346 \cdot 10^{6} B_{13,16}(X) + 2.63339 \cdot 10^{6} B_{14,16}(X) + 2.82293 \cdot 10^{6} B_{15,16}(X) + 3.01209 \cdot 10^{6} B_{16,16}(X)
\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\implies$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

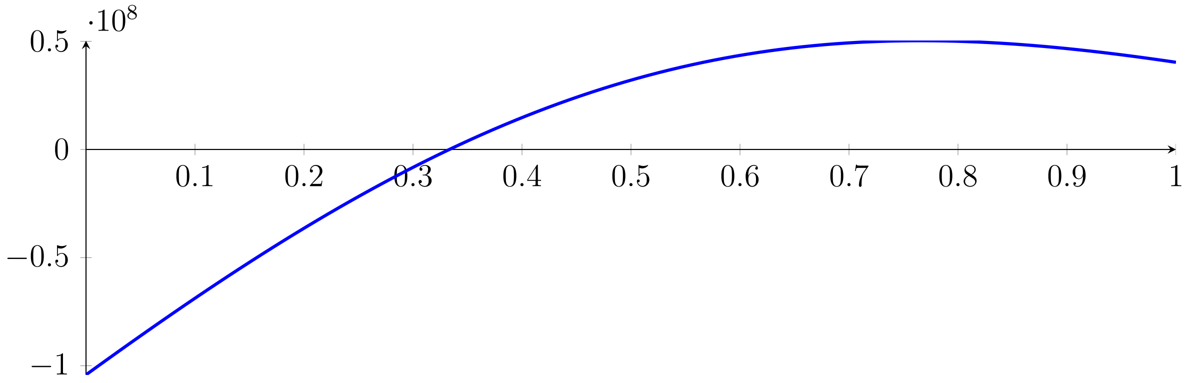
148.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Found root in interval $[0.333333, 0.333337]$ at recursion depth 4!

148.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333337]$$

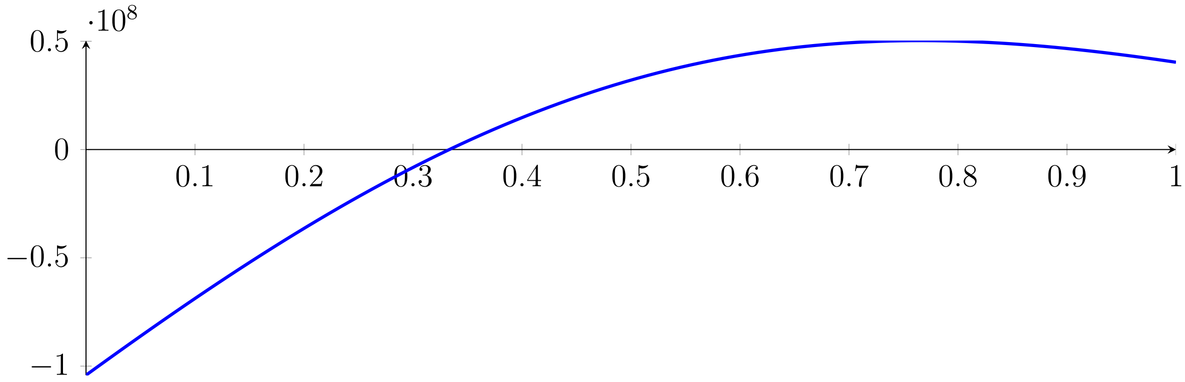
with precision $\varepsilon = 0.01$.

149 Running QuadClip on f_{16} with epsilon 2

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called QuadClip with input polynomial on interval $[0,1]$:

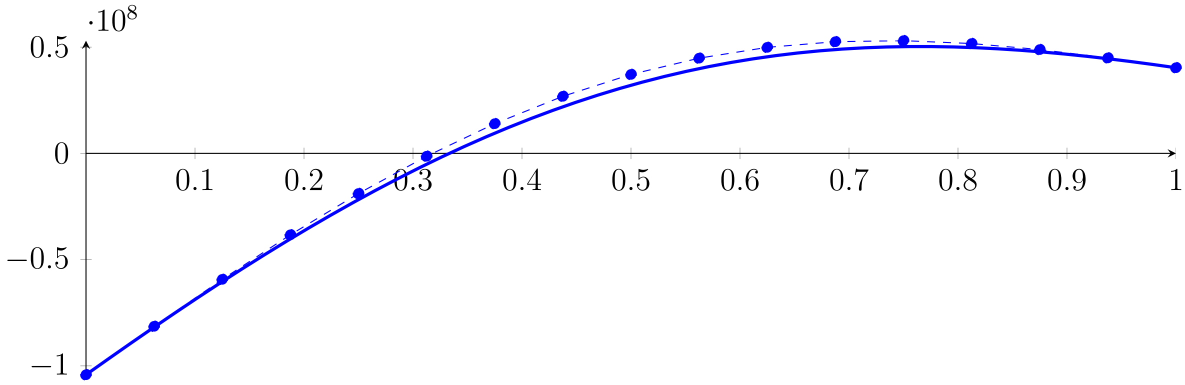
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} \\ &+ 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \end{aligned}$$



149.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

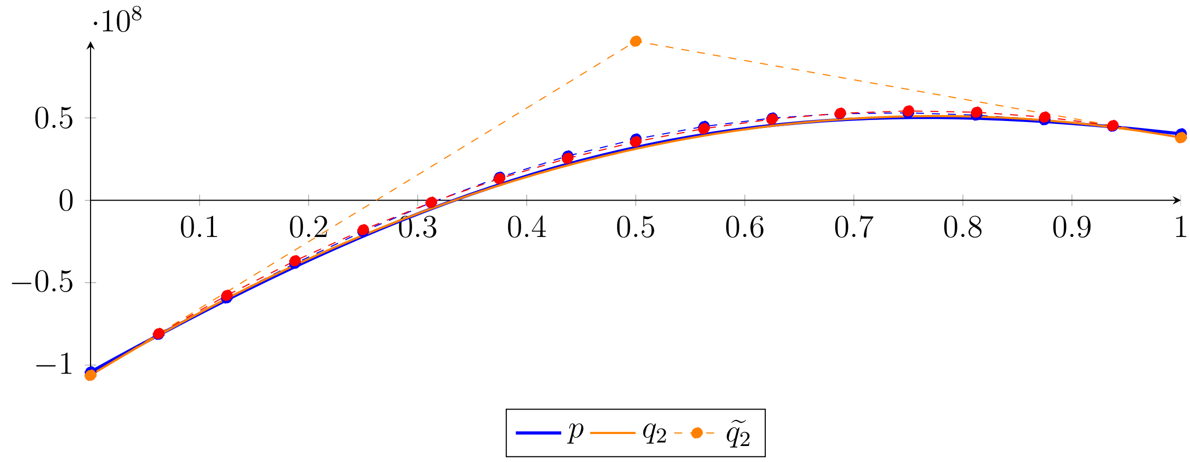
$$\begin{aligned} p = &-1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &+ 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ = &-1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ &\cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ &+ 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ &\cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ &+ 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8 \\ &= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 6049.18 X^{16} - 48305.2 X^{15} + 174971 X^{14} - 380294 X^{13} + 552846 X^{12} - 567203 X^{11} \\ &\quad + 422303 X^{10} - 231038 X^9 + 93003.6 X^8 - 27320.1 X^7 + 5752.57 X^6 - 843.63 X^5 \\ &\quad + 82.5145 X^4 - 5.01388 X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8 \\ &= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017 \\ &\quad \cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16} \\ &\quad + 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16} \\ &\quad + 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m:

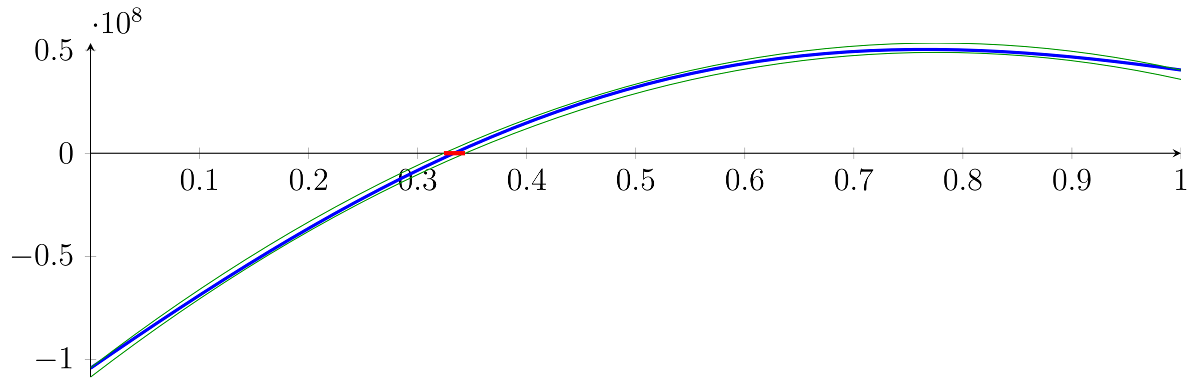
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

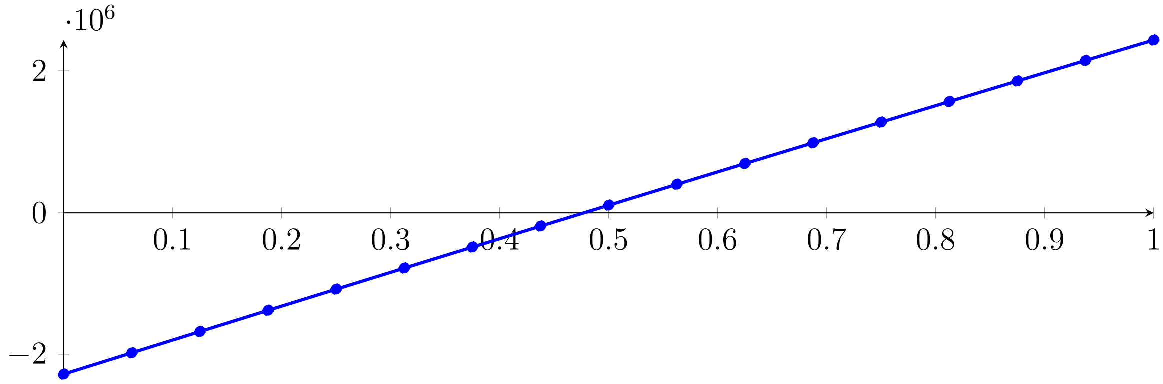
Longest intersection interval: 0.0196686

$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

149.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

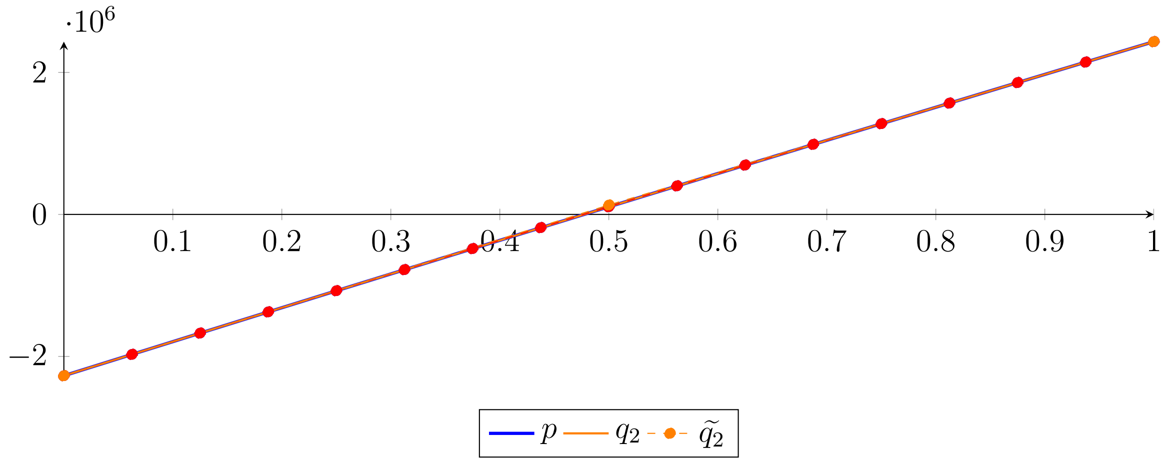
$$p = -1.54841\cdot10^{-10}X^{16} - 1.66965\cdot10^{-07}X^{15} - 2.92739\cdot10^{-07}X^{14} - 1.77943\cdot10^{-06}X^{13} - 1.17235\cdot10^{-06}X^{12} - 2.42234\cdot10^{-06}X^{11} - 6.86445\cdot10^{-07}X^{10} - 1.39162\cdot10^{-06}X^{9} + 1.07395\cdot10^{-06}X^{8} - 1.67072\cdot10^{-05}X^{7} - 0.00205879X^{6} + 0.132721X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874\cdot10^{6}X - 2.2715\cdot10^{6}$$
$$= -2.2715\cdot10^{6}B_{0,16}(X) - 1.97096\cdot10^{6}B_{1,16}(X) - 1.67127\cdot10^{6}B_{2,16}(X) - 1.37244\cdot10^{6}B_{3,16}(X) - 1.07448\cdot10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806\cdot10^{6}B_{12,16}(X) + 1.5682\cdot10^{6}B_{13,16}(X) + 1.85747\cdot10^{6}B_{14,16}(X) + 2.14586\cdot10^{6}B_{15,16}(X) + 2.43338\cdot10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265X^{2} + 4.80923\cdot10^{6}X - 2.27154\cdot10^{6}$$
$$= -2.27154\cdot10^{6}B_{0,2} + 133069B_{1,2} + 2.43342\cdot10^{6}B_{2,2}$$

$$\widetilde{q}_2 = 22.7036X^{16} - 178.964X^{15} + 638.976X^{14} - 1366.64X^{13} + 1951.1X^{12} - 1960.92X^{11} + 1425.35X^{10} - 757.631X^{9} + 294.177X^{8} - 82.4368X^{7} + 16.2856X^{6} - 2.18949X^{5} + 0.191237X^{4} - 0.0101048X^{3} - 104265X^{2} + 4.80923\cdot10^{6}X - 2.27154\cdot10^{6}$$
$$= -2.27154\cdot10^{6}B_{0,16} - 1.97097\cdot10^{6}B_{1,16} - 1.67126\cdot10^{6}B_{2,16} - 1.37242\cdot10^{6}B_{3,16} - 1.07445\cdot10^{6}B_{4,16} - 777350B_{5,16} - 481118B_{6,16} - 185754B_{7,16} + 108740B_{8,16} + 402366B_{9,16} + 695123B_{10,16} + 987011B_{11,16} + 1.27803\cdot10^{6}B_{12,16} + 1.56818\cdot10^{6}B_{13,16} + 1.85746\cdot10^{6}B_{14,16} + 2.14587\cdot10^{6}B_{15,16} + 2.43342\cdot10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m:

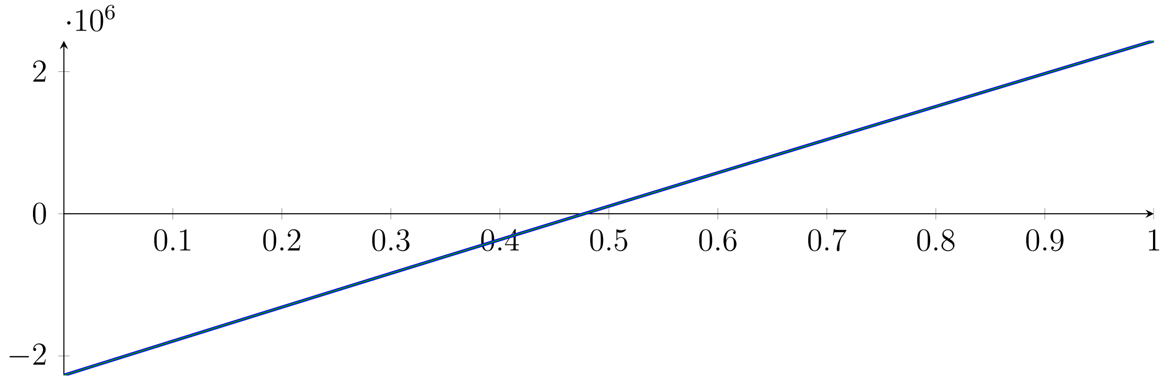
$$M = -104265X^{2} + 4.80923\cdot10^{6}X - 2.2715\cdot10^{6}$$

$$m = -104265X^2 + 4.80923{\cdot}10^6X - 2.27159{\cdot}10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301{\cdot}10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

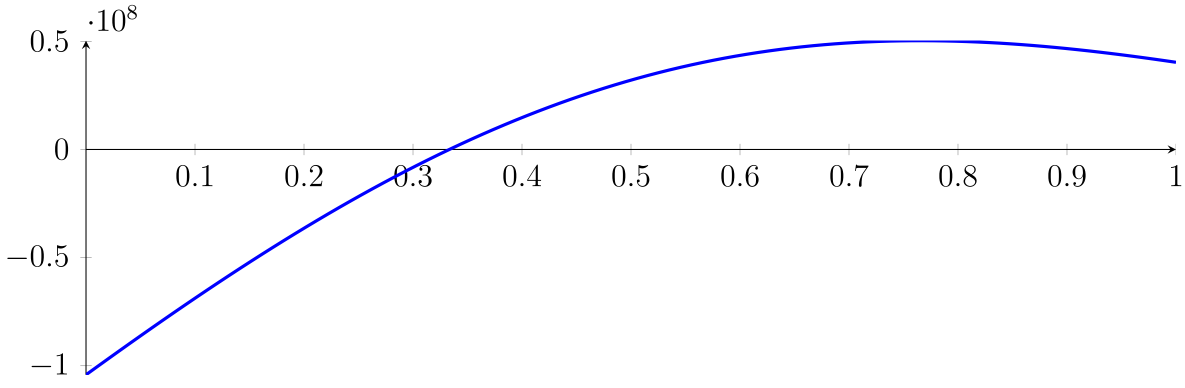
149.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

149.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

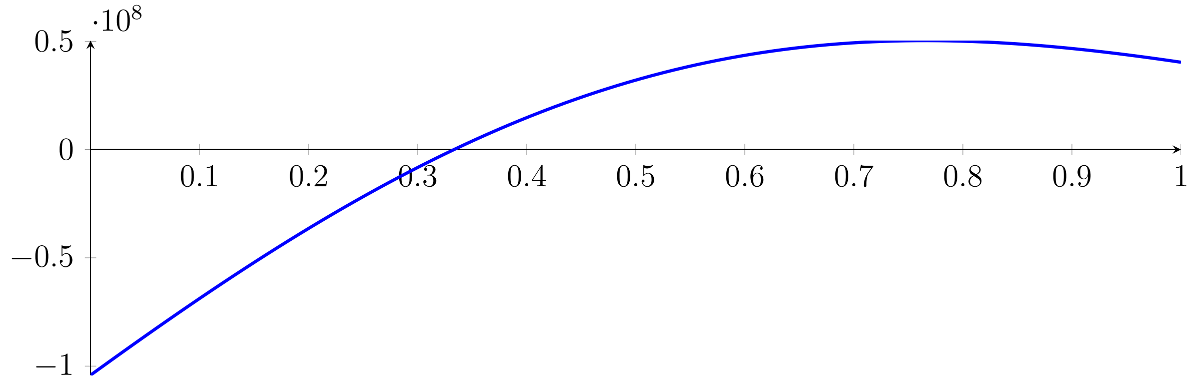
with precision $\varepsilon = 0.01$.

150 Running CubeClip on f_{16} with epsilon 2

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^6X^9 - 2.52262\cdot 10^6X^8 - 1.43506\cdot 10^7X^7 + 138542X^6 + 7.92823\cdot 10^7X^5 + 3.97396\cdot 10^7X^4 - 2.40625\cdot 10^8X^3 - 8.33333\cdot 10^7X^2 + 3.64583\cdot 10^8X - 1.04167\cdot 10^8$$

Called CubeClip with input polynomial on interval $[0, 1]$:

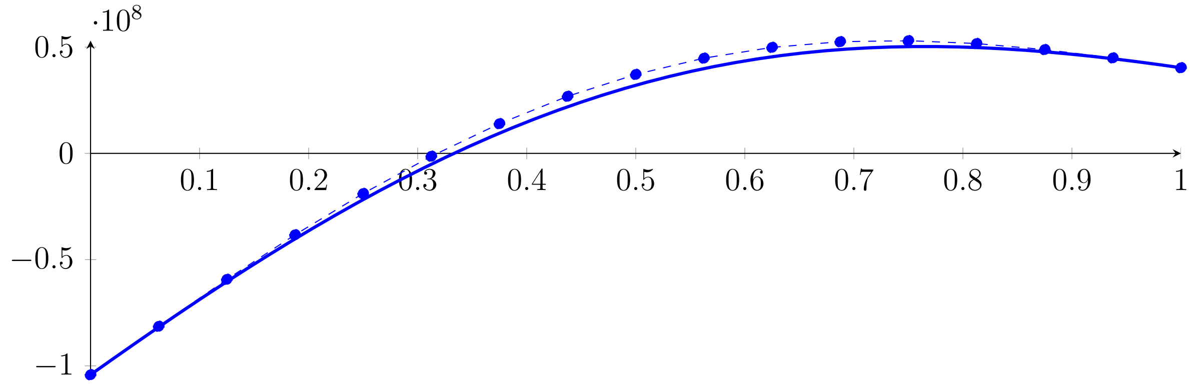
$$\begin{aligned}p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10}\\ &+ 1.12172\cdot 10^6X^9 - 2.52262\cdot 10^6X^8 - 1.43506\cdot 10^7X^7 + 138542X^6 + 7.92823\cdot 10^7X^5\\ &+ 3.97396\cdot 10^7X^4 - 2.40625\cdot 10^8X^3 - 8.33333\cdot 10^7X^2 + 3.64583\cdot 10^8X - 1.04167\cdot 10^8\end{aligned}$$



150.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

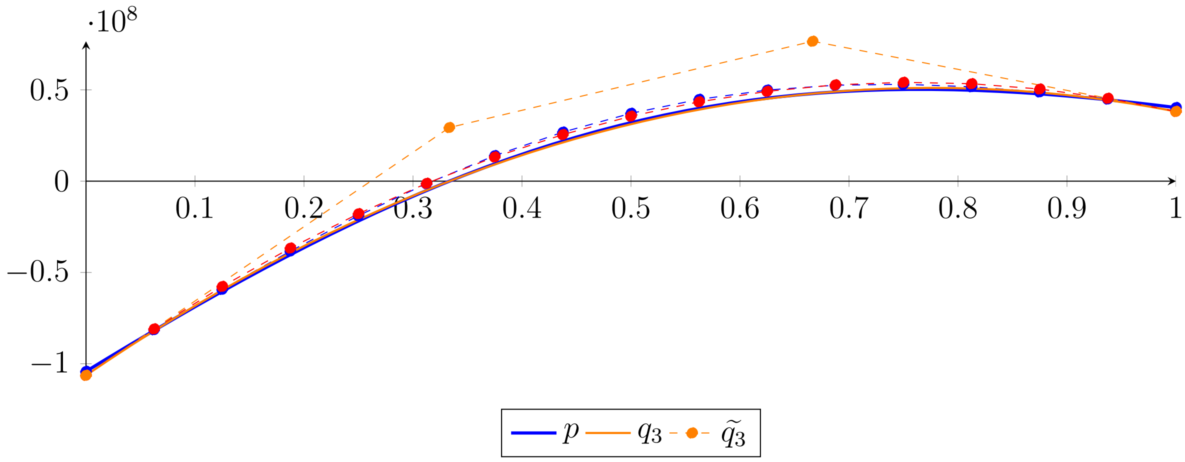
$$\begin{aligned}p = &-1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11}\\ &+ 451673X^{10} + 1.12172\cdot 10^6X^9 - 2.52262\cdot 10^6X^8 - 1.43506\cdot 10^7X^7 + 138542X^6 + 7.92823\\ &\cdot 10^7X^5 + 3.97396\cdot 10^7X^4 - 2.40625\cdot 10^8X^3 - 8.33333\cdot 10^7X^2 + 3.64583\cdot 10^8X - 1.04167\cdot 10^8\\ = &-1.04167\cdot 10^8B_{0,16}(X) - 8.13802\cdot 10^7B_{1,16}(X) - 5.92882\cdot 10^7B_{2,16}(X) - 3.83203\\ &\cdot 10^7B_{3,16}(X) - 1.88844\cdot 10^7B_{4,16}(X) - 1.34837\cdot 10^6B_{5,16}(X) + 1.39781\cdot 10^7B_{6,16}(X)\\ &+ 2.68604\cdot 10^7B_{7,16}(X) + 3.71532\cdot 10^7B_{8,16}(X) + 4.48105\cdot 10^7B_{9,16}(X) + 4.98901\\ &\cdot 10^7B_{10,16}(X) + 5.25504\cdot 10^7B_{11,16}(X) + 5.30407\cdot 10^7B_{12,16}(X) + 5.16832\cdot 10^7B_{13,16}(X)\\ &+ 4.88488\cdot 10^7B_{14,16}(X) + 4.49297\cdot 10^7B_{15,16}(X) + 4.03108\cdot 10^7B_{16,16}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.06335\cdot 10^8 \\ &= -1.06335\cdot 10^8 B_{0,3} + 2.93558\cdot 10^7 B_{1,3} + 7.66068\cdot 10^7 B_{2,3} + 3.81764\cdot 10^7 B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2461.93X^{16} - 19614.9X^{15} + 70879.5X^{14} - 153661X^{13} + 222746X^{12} - 227755X^{11} \\ &\quad + 168826X^{10} - 91798.7X^9 + 36630.3X^8 - 10627.3X^7 + 2200.54X^6 - 316.059X^5 \\ &\quad + 30.1958X^4 + 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.06335\cdot 10^8 \\ &= -1.06335\cdot 10^8 B_{0,16} - 8.08928\cdot 10^7 B_{1,16} - 5.76618\cdot 10^7 B_{2,16} - 3.66369\cdot 10^7 B_{3,16} - 1.78131 \\ &\quad \cdot 10^7 B_{4,16} - 1.18551\cdot 10^6 B_{5,16} + 1.32508\cdot 10^7 B_{6,16} + 2.55007\cdot 10^7 B_{7,16} + 3.55692\cdot 10^7 B_{8,16} \\ &\quad + 4.34611\cdot 10^7 B_{9,16} + 4.91815\cdot 10^7 B_{10,16} + 5.27353\cdot 10^7 B_{11,16} + 5.41273\cdot 10^7 B_{12,16} \\ &\quad + 5.33624\cdot 10^7 B_{13,16} + 5.04457\cdot 10^7 B_{14,16} + 4.53821\cdot 10^7 B_{15,16} + 3.81764\cdot 10^7 B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806\cdot 10^6$.
Bounding polynomials M and m:

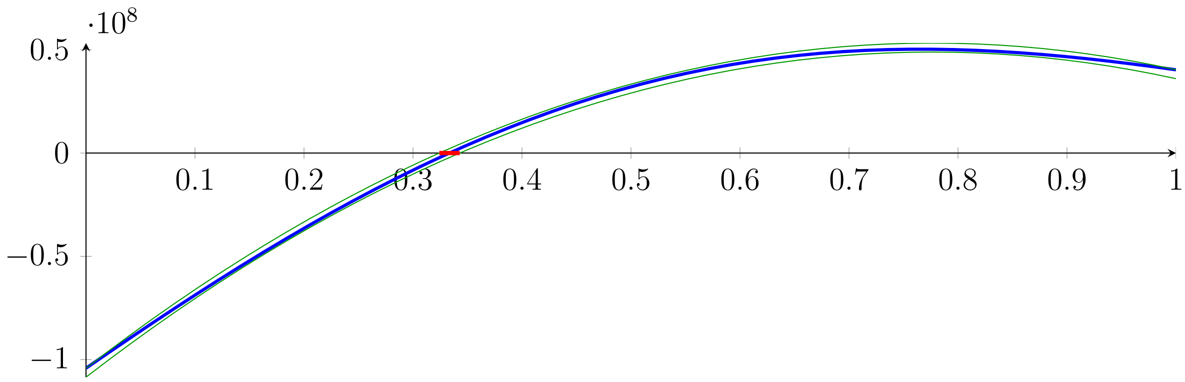
$$M = 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.04167\cdot 10^8$$

$$m = 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.08503\cdot 10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



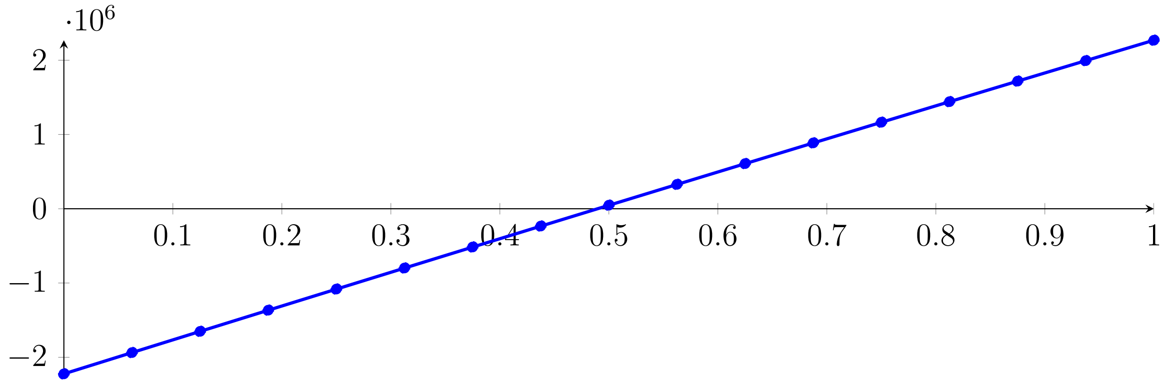
$$[0.324143, 0.342913]$$

Longest intersection interval: 0.0187703
$\Longrightarrow$ Selective recursion: interval 1: [0.324143, 0.342913],

150.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

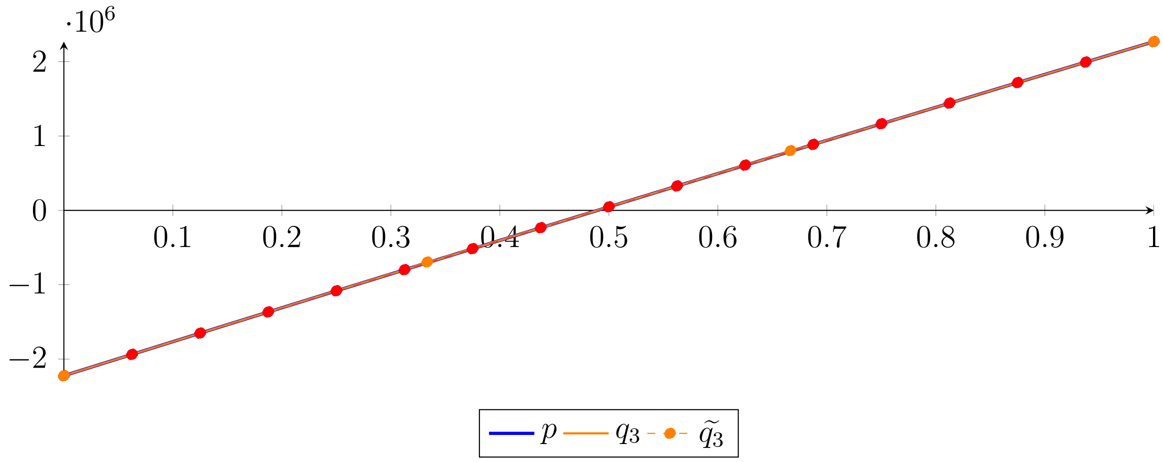
$$p = -3.66617\cdot10^{-09}X^{16} - 1.53217\cdot10^{-07}X^{15} - 3.62234\cdot10^{-07}X^{14} - 1.65579\cdot10^{-06}X^{13} - 1.15373\cdot10^{-06}X^{12} - 2.3399\cdot10^{-06}X^{11} - 5.02543\cdot10^{-07}X^{10} - 1.38381\cdot10^{-06}X^{9} + 1.1237\cdot10^{-06}X^{8} - 1.19653\cdot10^{-05}X^{7} - 0.00155608X^{6} + 0.10496X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,16}(X) - 1.93666\cdot10^{6}B_{1,16}(X) - 1.65074\cdot10^{6}B_{2,16}(X) - 1.36561\cdot10^{6}B_{3,16}(X) - 1.08127\cdot10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X) + 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot10^{6}B_{12,16}(X) + 1.44233\cdot10^{6}B_{13,16}(X) + 1.71876\cdot10^{6}B_{14,16}(X) + 1.99439\cdot10^{6}B_{15,16}(X) + 2.26922\cdot10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot10^{6}B_{3,3}$$

$$\widetilde{q}_3 = 16.4956X^{16} - 129.161X^{15} + 457.83X^{14} - 971.671X^{13} + 1375.95X^{12} - 1370.96X^{11} + 987.265X^{10} - 519.476X^{9} + 199.587X^{8} - 55.434X^{7} + 10.9237X^{6} - 1.48019X^{5} + 0.129516X^{4} - 700.679X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,16} - 1.93666\cdot10^{6}B_{1,16} - 1.65074\cdot10^{6}B_{2,16} - 1.36561\cdot10^{6}B_{3,16} - 1.08126\cdot10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.4B_{8,16} + 328650B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot10^{6}B_{12,16} + 1.44233\cdot10^{6}B_{13,16} + 1.71876\cdot10^{6}B_{14,16} + 1.99439\cdot10^{6}B_{15,16} + 2.26922\cdot10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m:

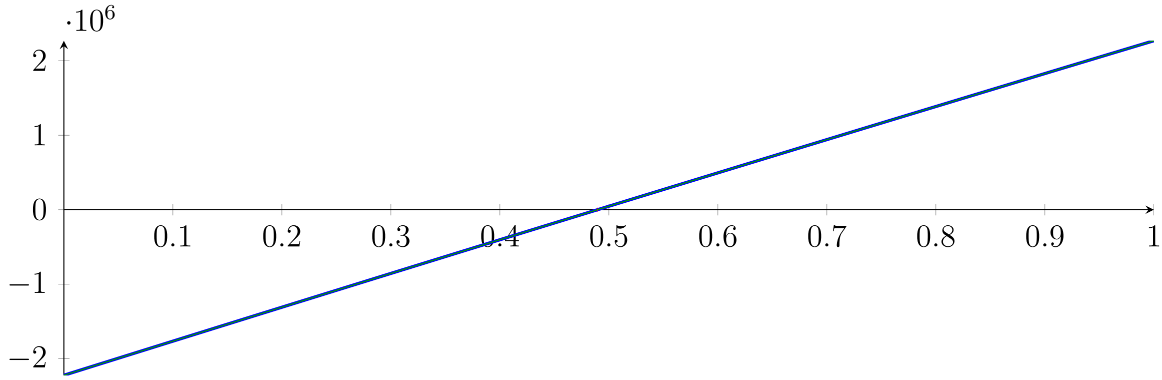
$$M = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715{\cdot}10^6X - 2.22335{\cdot}10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

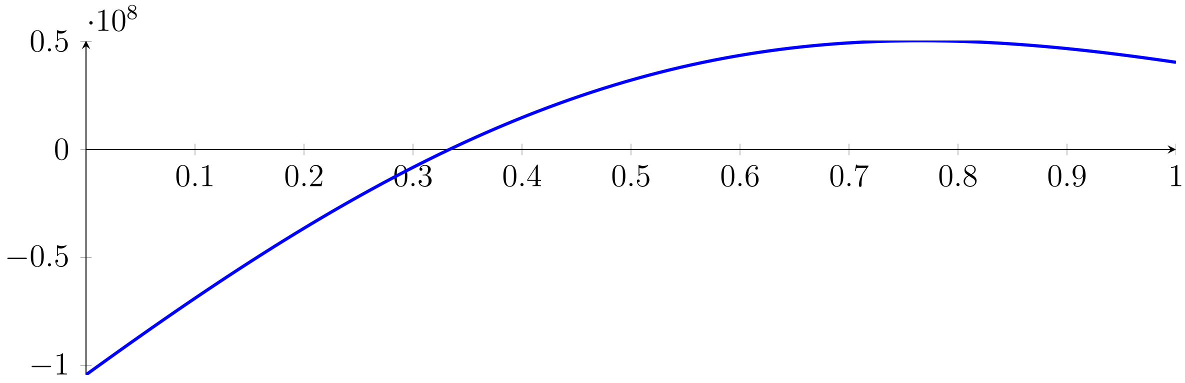
150.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

150.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

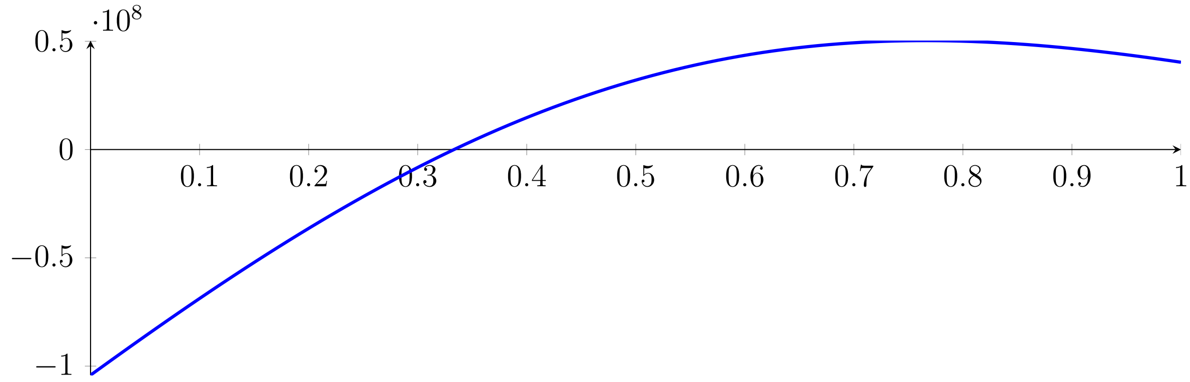
with precision $\varepsilon = 0.01$.

151 Running BezClip on f_{16} with epsilon 4

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called BezClip with input polynomial on interval $[0, 1]$:

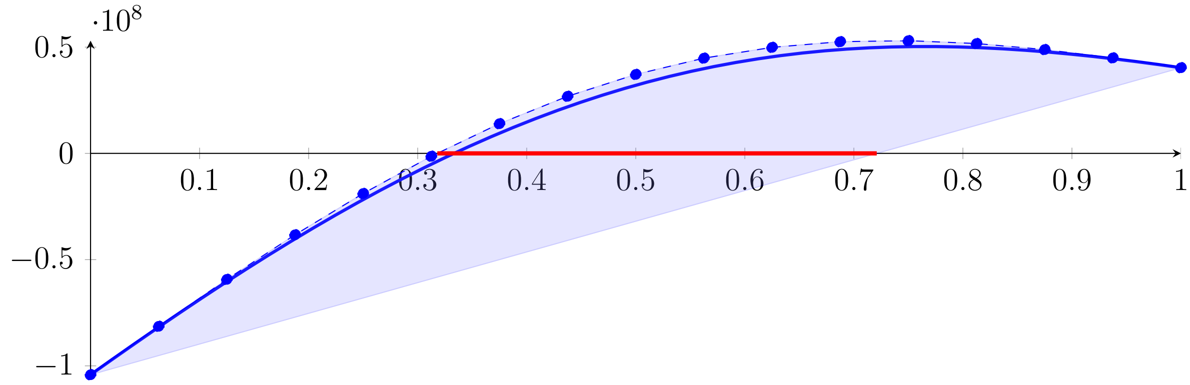
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



151.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\quad \cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \\ &= -1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

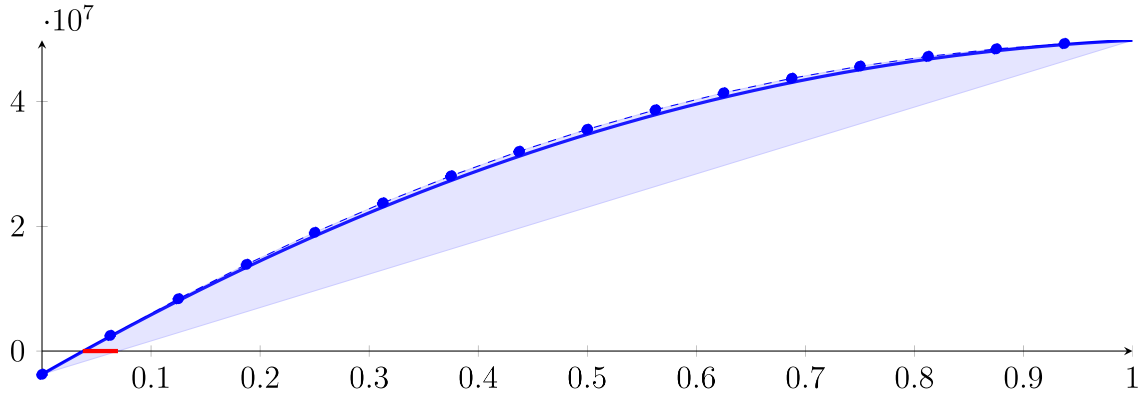
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

151.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.59825\cdot 10^{-06}X^{16} - 5.93153\cdot 10^{-05}X^{15} - 0.00248867X^{14} - 0.0650056X^{13} - 0.909142X^{12} \\ &\quad - 5.03931X^{11} + 36.4549X^{10} + 692.921X^{9} + 1886.97X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5} \\ &\quad + 3.91945\cdot 10^{6}X^{4} - 7.54561\cdot 10^{6}X^{3} - 4.29225\cdot 10^{7}X^{2} + 9.97982\cdot 10^{7}X - 3.73501\cdot 10^{6} \\ &= -3.73501\cdot 10^{6}B_{0,16}(X) + 2.50237\cdot 10^{6}B_{1,16}(X) + 8.38207\cdot 10^{6}B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^{7}B_{3,16}(X) + 1.90167\cdot 10^{7}B_{4,16}(X) + 2.37512\cdot 10^{7}B_{5,16}(X) + 2.80875\cdot 10^{7}B_{6,16}(X) \\ &\quad + 3.20213\cdot 10^{7}B_{7,16}(X) + 3.55508\cdot 10^{7}B_{8,16}(X) + 3.86768\cdot 10^{7}B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^{7}B_{10,16}(X) + 4.37336\cdot 10^{7}B_{11,16}(X) + 4.56783\cdot 10^{7}B_{12,16}(X) + 4.7247\cdot 10^{7}B_{13,16}(X) \\ &\quad + 4.84524\cdot 10^{7}B_{14,16}(X) + 4.93093\cdot 10^{7}B_{15,16}(X) + 4.98343\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

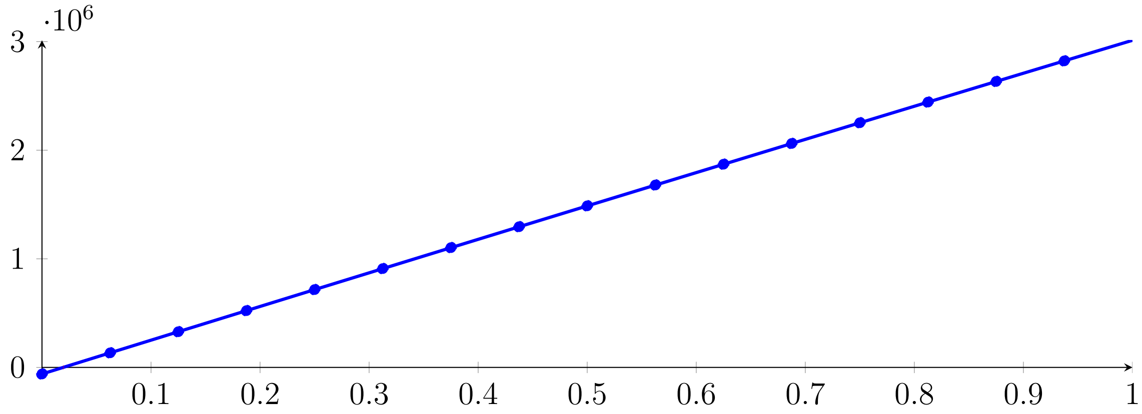
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

151.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 9.01396\cdot 10^{-08}X^{16} - 2.65848\cdot 10^{-07}X^{15} + 2.13948\cdot 10^{-06}X^{14} - 1.33627\cdot 10^{-06}X^{13} + 2.46973\cdot 10^{-06}X^{12} \\ &\quad - 2.45524\cdot 10^{-06}X^{11} + 5.50112\cdot 10^{-07}X^{10} - 1.64198\cdot 10^{-07}X^{9} - 7.35598\cdot 10^{-07}X^{8} - 1.00892\cdot 10^{-06}X^{7} \\ &\quad - 0.000177509X^{6} + 0.0161038X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot 10^{6}X - 60508.5 \\ &= -60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X) \\ &\quad + 910202B_{5,16}(X) + 1.1032\cdot 10^{6}B_{6,16}(X) + 1.29582\cdot 10^{6}B_{7,16}(X) + 1.48805\cdot 10^{6}B_{8,16}(X) \\ &\quad + 1.6799\cdot 10^{6}B_{9,16}(X) + 1.87137\cdot 10^{6}B_{10,16}(X) + 2.06245\cdot 10^{6}B_{11,16}(X) + 2.25315\cdot 10^{6}B_{12,16}(X) \\ &\quad + 2.44346\cdot 10^{6}B_{13,16}(X) + 2.63339\cdot 10^{6}B_{14,16}(X) + 2.82293\cdot 10^{6}B_{15,16}(X) + 3.01209\cdot 10^{6}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

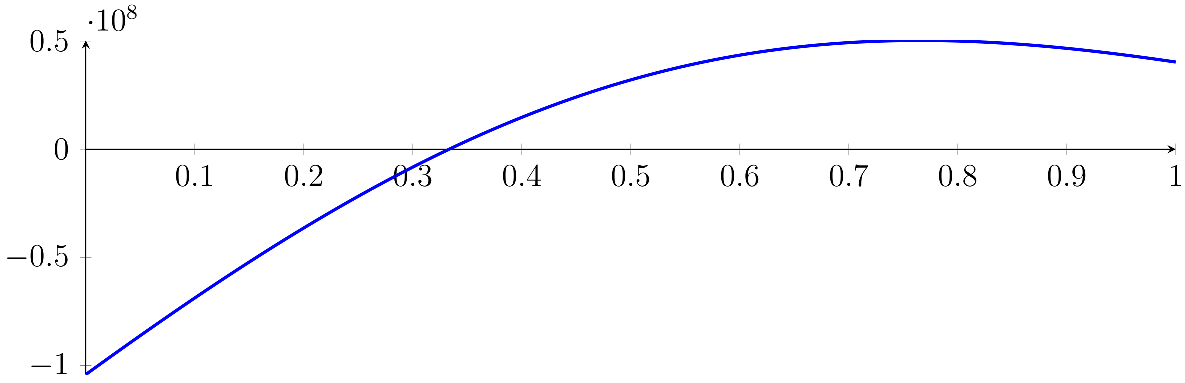
151.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Found root in interval $[0.333333, 0.333337]$ at recursion depth 4!

151.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333337]$$

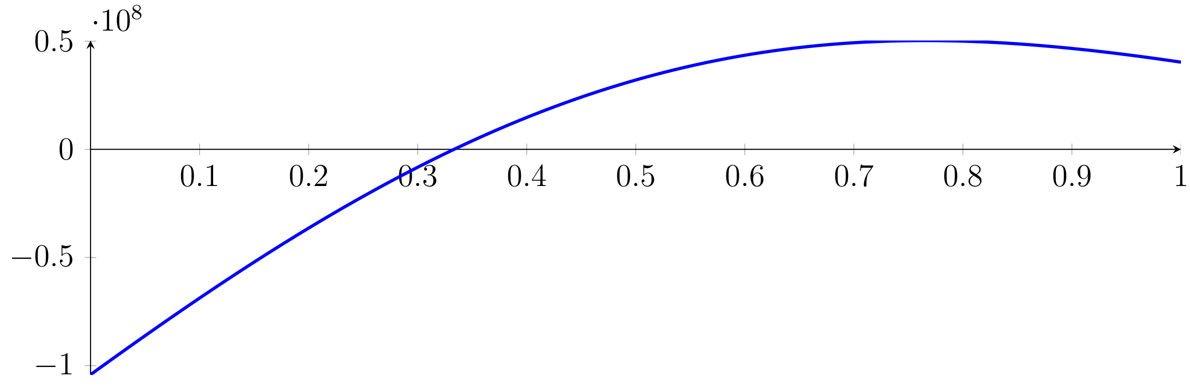
with precision $\varepsilon = 0.0001$.

152 Running QuadClip on f_{16} with epsilon 4

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

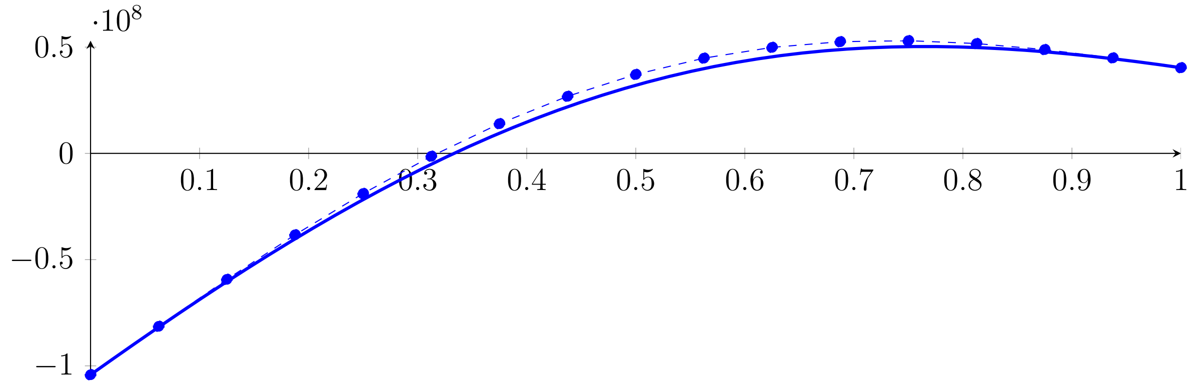
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



152.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

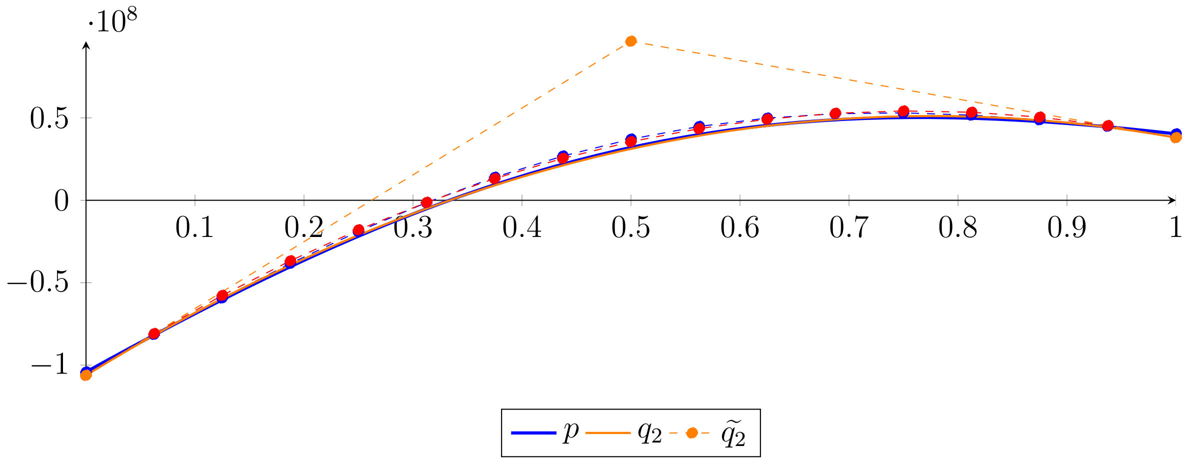
$$\begin{aligned} p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\quad \cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ &= -1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.06197{\cdot}10^8 \\ &= -1.06197{\cdot}10^8 B_{0,2} + 9.65115{\cdot}10^7 B_{1,2} + 3.80384{\cdot}10^7 B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 6049.18X^{16} - 48305.2X^{15} + 174971X^{14} - 380294X^{13} + 552846X^{12} - 567203X^{11} \\ &\quad + 422303X^{10} - 231038X^9 + 93003.6X^8 - 27320.1X^7 + 5752.57X^6 - 843.63X^5 \\ &\quad + 82.5145X^4 - 5.01388X^3 - 2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.06197{\cdot}10^8 \\ &= -1.06197{\cdot}10^8 B_{0,16} - 8.08583{\cdot}10^7 B_{1,16} - 5.76963{\cdot}10^7 B_{2,16} - 3.67107{\cdot}10^7 B_{3,16} - 1.79017 \\ &\quad {\cdot}10^7 B_{4,16} - 1.26924{\cdot}10^6 B_{5,16} + 1.31867{\cdot}10^7 B_{6,16} + 2.54662{\cdot}10^7 B_{7,16} + 3.55692{\cdot}10^7 B_{8,16} \\ &\quad + 4.34956{\cdot}10^7 B_{9,16} + 4.92456{\cdot}10^7 B_{10,16} + 5.2819{\cdot}10^7 B_{11,16} + 5.42159{\cdot}10^7 B_{12,16} \\ &\quad + 5.34363{\cdot}10^7 B_{13,16} + 5.04802{\cdot}10^7 B_{14,16} + 4.53476{\cdot}10^7 B_{15,16} + 3.80384{\cdot}10^7 B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233{\cdot}10^6$.
Bounding polynomials M and m:

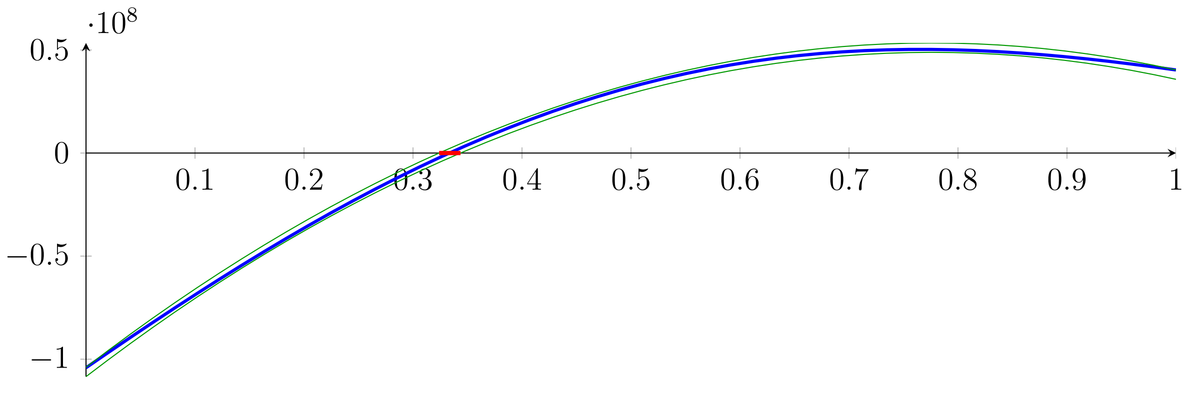
$$M = -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.03924{\cdot}10^8$$

$$m = -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.08469{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



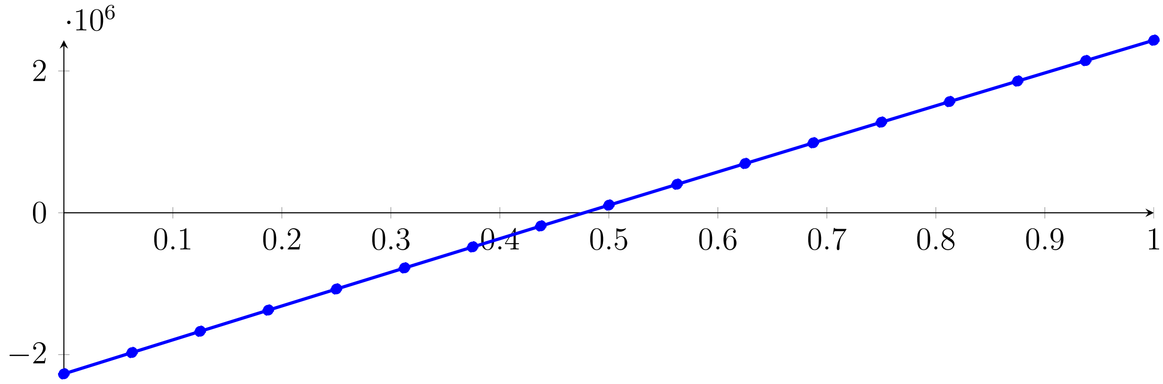
$$[0.323946, 0.343615]$$

Longest intersection interval: 0.0196686
$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

152.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

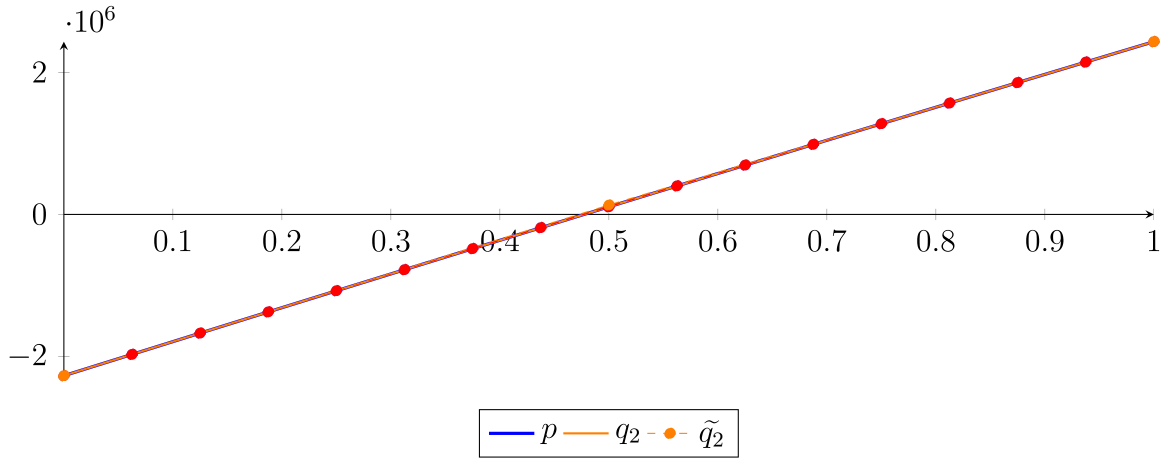
$$p = -1.54841\cdot 10^{-10} X^{16} - 1.66965\cdot 10^{-07} X^{15} - 2.92739\cdot 10^{-07} X^{14} - 1.77943\cdot 10^{-06} X^{13} - 1.17235\cdot 10^{-06} X^{12} - 2.42234\cdot 10^{-06} X^{11} - 6.86445\cdot 10^{-07} X^{10} - 1.39162\cdot 10^{-06} X^{9} + 1.07395\cdot 10^{-06} X^{8} - 1.67072\cdot 10^{-05} X^{7} - 0.00205879 X^{6} + 0.132721 X^{5} + 22.4437 X^{4} - 850.239 X^{3} - 103028 X^{2} + 4.80874\cdot 10^{6} X - 2.2715\cdot 10^{6}$$
$$= -2.2715\cdot 10^{6} B_{0,16}(X) - 1.97096\cdot 10^{6} B_{1,16}(X) - 1.67127\cdot 10^{6} B_{2,16}(X) - 1.37244\cdot 10^{6} B_{3,16}(X) - 1.07448\cdot 10^{6} B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806\cdot 10^{6} B_{12,16}(X) + 1.5682 \cdot 10^{6} B_{13,16}(X) + 1.85747\cdot 10^{6} B_{14,16}(X) + 2.14586\cdot 10^{6} B_{15,16}(X) + 2.43338\cdot 10^{6} B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265 X^{2} + 4.80923\cdot 10^{6} X - 2.27154\cdot 10^{6}$$
$$= -2.27154\cdot 10^{6} B_{0,2} + 133069 B_{1,2} + 2.43342\cdot 10^{6} B_{2,2}$$

$$\widetilde{q}_2 = 22.7036 X^{16} - 178.964 X^{15} + 638.976 X^{14} - 1366.64 X^{13} + 1951.1 X^{12} - 1960.92 X^{11} + 1425.35 X^{10} - 757.631 X^{9} + 294.177 X^{8} - 82.4368 X^{7} + 16.2856 X^{6} - 2.18949 X^{5} + 0.191237 X^{4} - 0.0101048 X^{3} - 104265 X^{2} + 4.80923\cdot 10^{6} X - 2.27154\cdot 10^{6}$$
$$= -2.27154\cdot 10^{6} B_{0,16} - 1.97097\cdot 10^{6} B_{1,16} - 1.67126\cdot 10^{6} B_{2,16} - 1.37242\cdot 10^{6} B_{3,16} - 1.07445\cdot 10^{6} B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803\cdot 10^{6} B_{12,16} + 1.56818 \cdot 10^{6} B_{13,16} + 1.85746\cdot 10^{6} B_{14,16} + 2.14587\cdot 10^{6} B_{15,16} + 2.43342\cdot 10^{6} B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m:

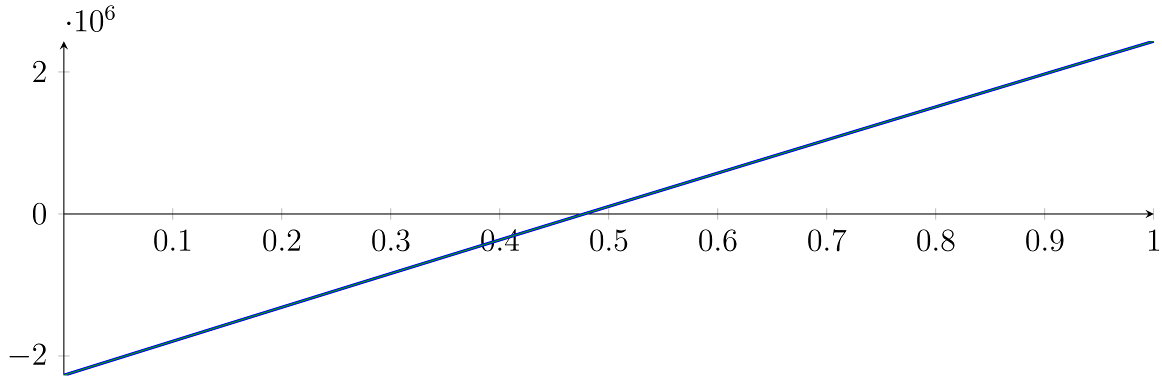
$$M = -104265 X^{2} + 4.80923\cdot 10^{6} X - 2.2715\cdot 10^{6}$$

$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

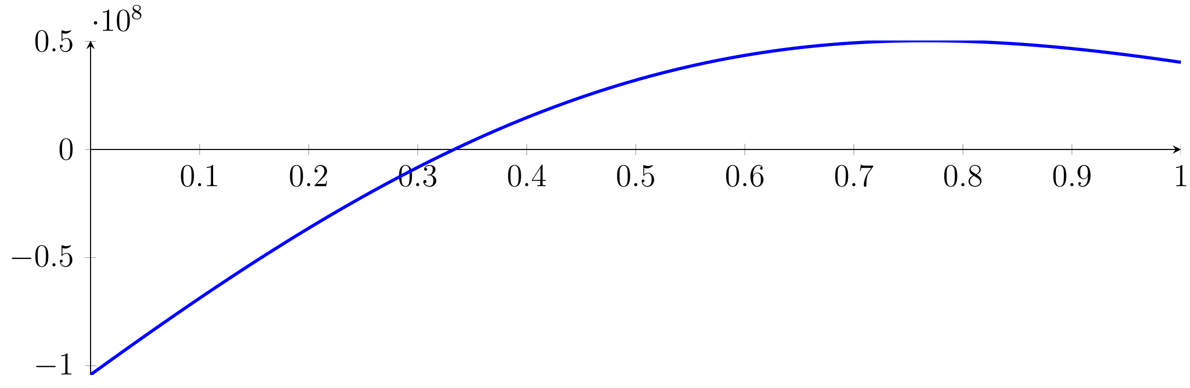
152.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

152.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

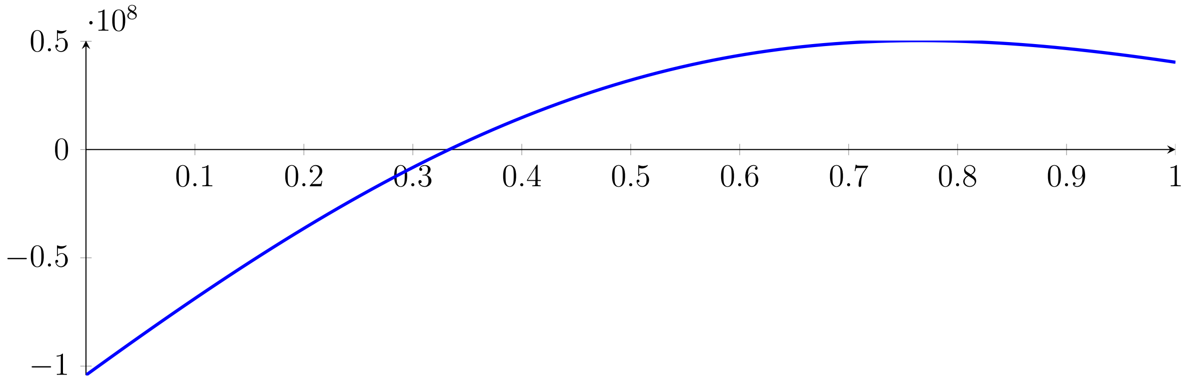
with precision $\varepsilon = 0.0001$.

153 Running CubeClip on f_{16} with epsilon 4

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

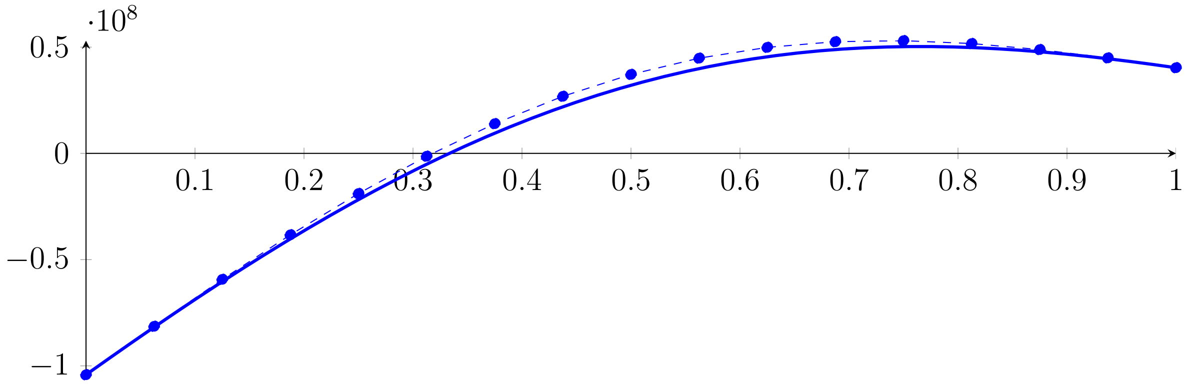
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



153.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

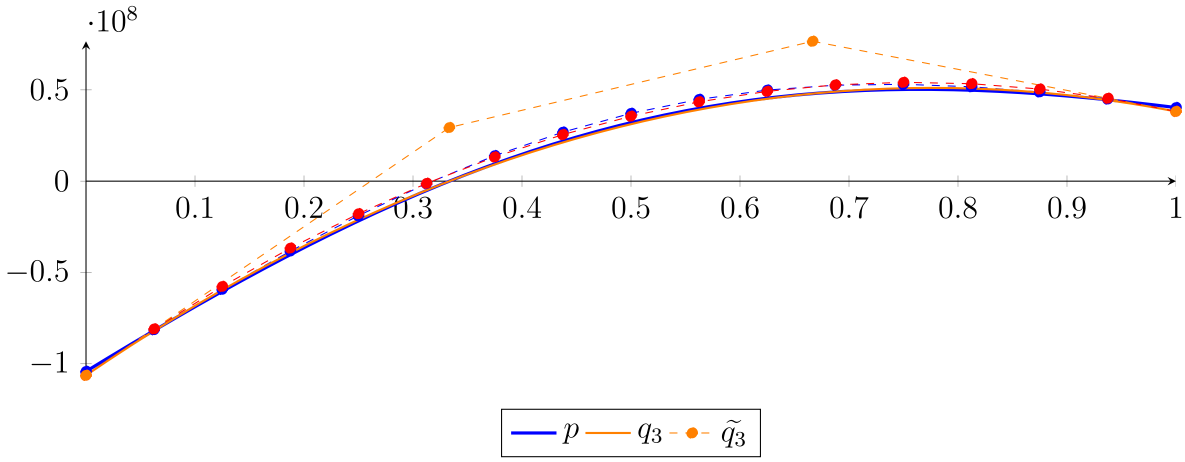
$$\begin{aligned} p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\quad \cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \\ &= -1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.06335{\cdot}10^8$$
$$= -1.06335{\cdot}10^8B_{0,3} + 2.93558{\cdot}10^7B_{1,3} + 7.66068{\cdot}10^7B_{2,3} + 3.81764{\cdot}10^7B_{3,3}$$

$$\widetilde{q_3} = 2461.93X^{16} - 19614.9X^{15} + 70879.5X^{14} - 153661X^{13} + 222746X^{12} - 227755X^{11} + 168826X^{10} - 91798.7X^9 + 36630.3X^8 - 10627.3X^7 + 2200.54X^6 - 316.059X^5 + 30.1958X^4 + 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.06335{\cdot}10^8$$
$$= -1.06335{\cdot}10^8B_{0,16} - 8.08928{\cdot}10^7B_{1,16} - 5.76618{\cdot}10^7B_{2,16} - 3.66369{\cdot}10^7B_{3,16} - 1.78131{\cdot}10^7B_{4,16} - 1.18551{\cdot}10^6B_{5,16} + 1.32508{\cdot}10^7B_{6,16} + 2.55007{\cdot}10^7B_{7,16} + 3.55692{\cdot}10^7B_{8,16} + 4.34611{\cdot}10^7B_{9,16} + 4.91815{\cdot}10^7B_{10,16} + 5.27353{\cdot}10^7B_{11,16} + 5.41273{\cdot}10^7B_{12,16} + 5.33624{\cdot}10^7B_{13,16} + 5.04457{\cdot}10^7B_{14,16} + 4.53821{\cdot}10^7B_{15,16} + 3.81764{\cdot}10^7B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806{\cdot}10^6$.

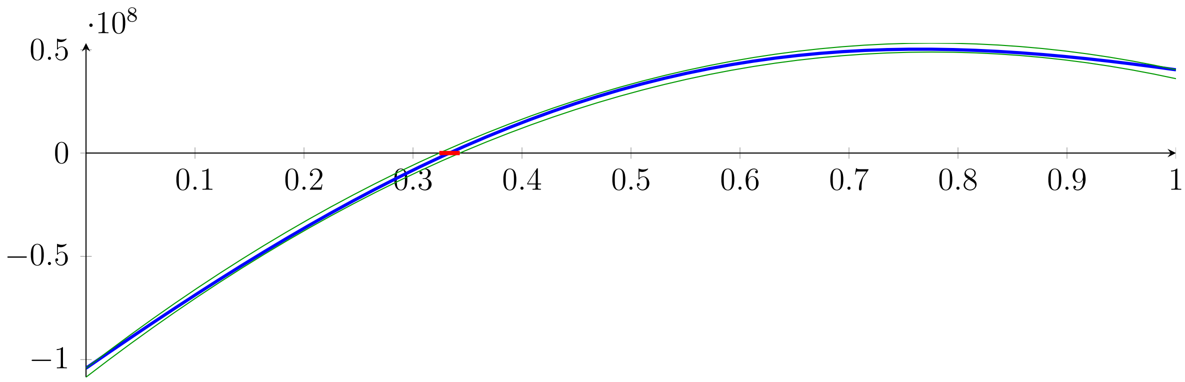
Bounding polynomials M and m:

$$M = 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.04167{\cdot}10^8$$
$$m = 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.08503{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

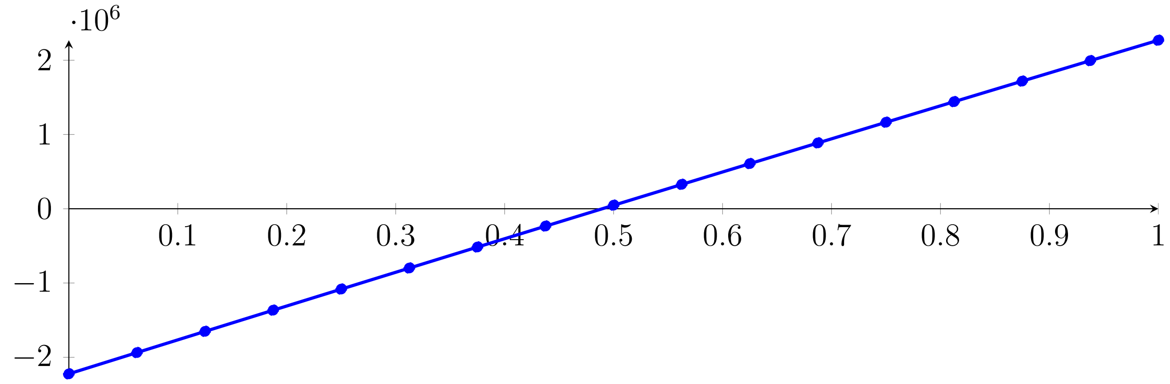
Longest intersection interval: 0.0187703

$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

153.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

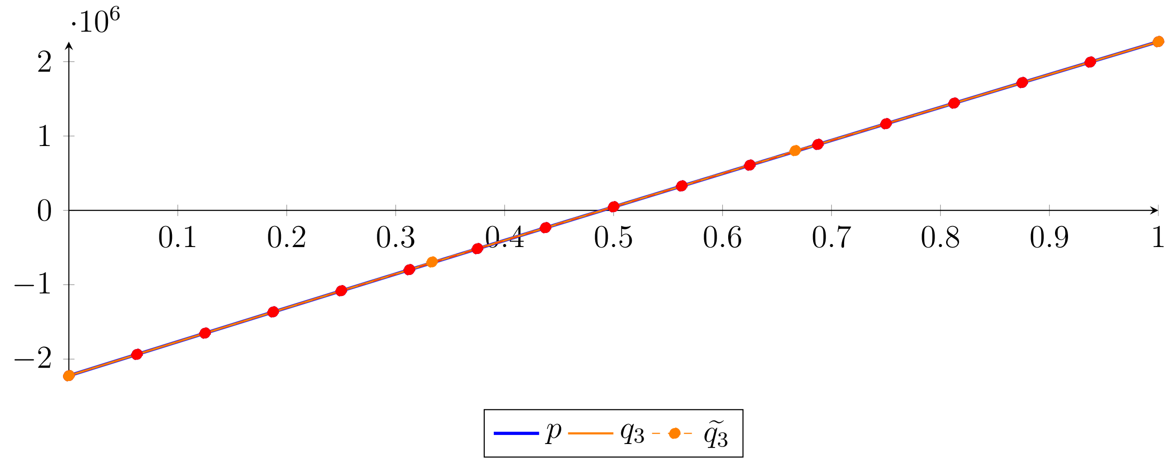
$$\begin{aligned}p = &-3.66617\cdot10^{-09}X^{16} - 1.53217\cdot10^{-07}X^{15} - 3.62234\cdot10^{-07}X^{14} - 1.65579\cdot10^{-06}X^{13} - 1.15373\cdot10^{-06}X^{12}\\ &- 2.3399\cdot10^{-06}X^{11} - 5.02543\cdot10^{-07}X^{10} - 1.38381\cdot10^{-06}X^{9} + 1.1237\cdot10^{-06}X^{8} - 1.19653\cdot10^{-05}X^{7}\\ &- 0.00155608X^{6} + 0.10496X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}\\ = &-2.22335\cdot10^{6}B_{0,16}(X) - 1.93666\cdot10^{6}B_{1,16}(X) - 1.65074\cdot10^{6}B_{2,16}(X) - 1.36561\cdot10^{6}B_{3,16}(X)\\ &- 1.08127\cdot10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X)\\ &+ 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot10^{6}B_{12,16}(X) + 1.44233\\ &\cdot10^{6}B_{13,16}(X) + 1.71876\cdot10^{6}B_{14,16}(X) + 1.99439\cdot10^{6}B_{15,16}(X) + 2.26922\cdot10^{6}B_{16,16}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}\\ &= -2.22335\cdot10^{6}B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot10^{6}B_{3,3}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= 16.4956X^{16} - 129.161X^{15} + 457.83X^{14} - 971.671X^{13} + 1375.95X^{12} - 1370.96X^{11}\\ &+ 987.265X^{10} - 519.476X^{9} + 199.587X^{8} - 55.434X^{7} + 10.9237X^{6} - 1.48019X^{5}\\ &+ 0.129516X^{4} - 700.679X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}\\ &= -2.22335\cdot10^{6}B_{0,16} - 1.93666\cdot10^{6}B_{1,16} - 1.65074\cdot10^{6}B_{2,16} - 1.36561\cdot10^{6}B_{3,16}\\ &- 1.08126\cdot10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.4B_{8,16}\\ &+ 328650B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot10^{6}B_{12,16} + 1.44233\\ &\cdot10^{6}B_{13,16} + 1.71876\cdot10^{6}B_{14,16} + 1.99439\cdot10^{6}B_{15,16} + 2.26922\cdot10^{6}B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m:

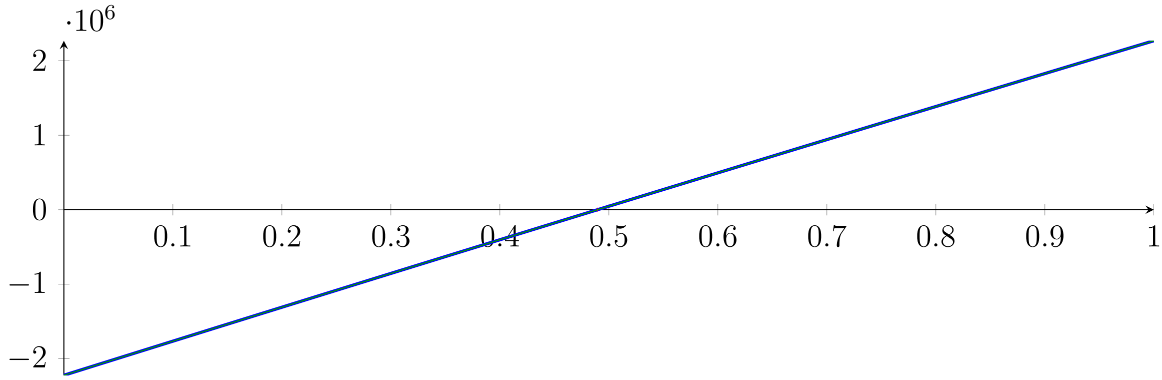
$$M = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

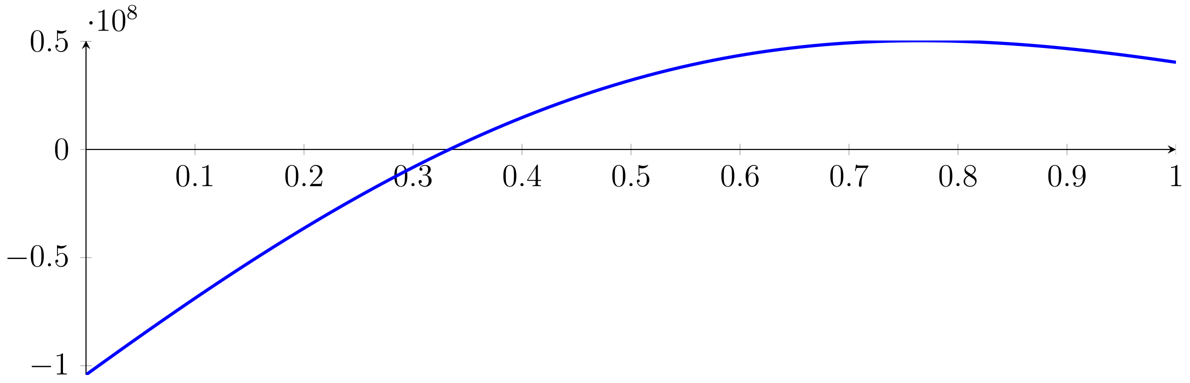
153.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

153.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

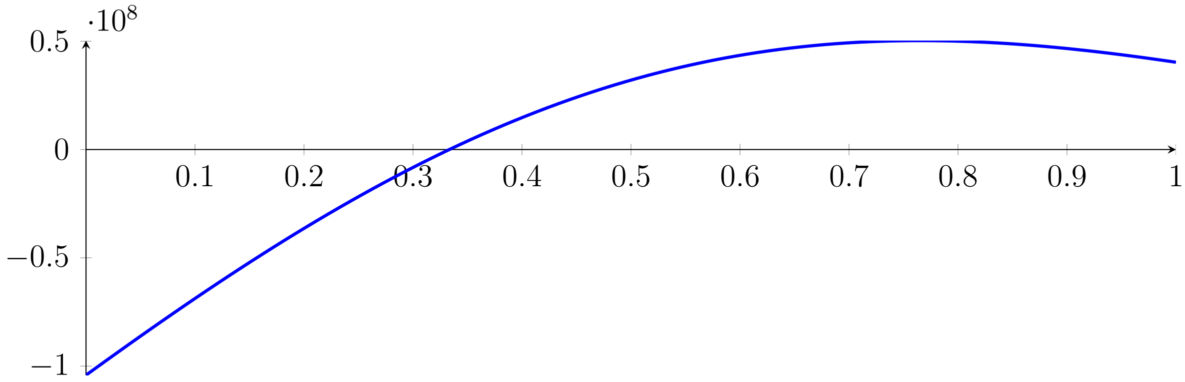
with precision $\varepsilon = 0.0001$.

154 Running BezClip on f_{16} with epsilon 8

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^6X^9 - 2.52262\cdot10^6X^8 - 1.43506\cdot10^7X^7 + 138542X^6 + 7.92823\cdot10^7X^5 + 3.97396\cdot10^7X^4 - 2.40625\cdot10^8X^3 - 8.33333\cdot10^7X^2 + 3.64583\cdot10^8X - 1.04167\cdot10^8$$

Called BezClip with input polynomial on interval $[0, 1]$:

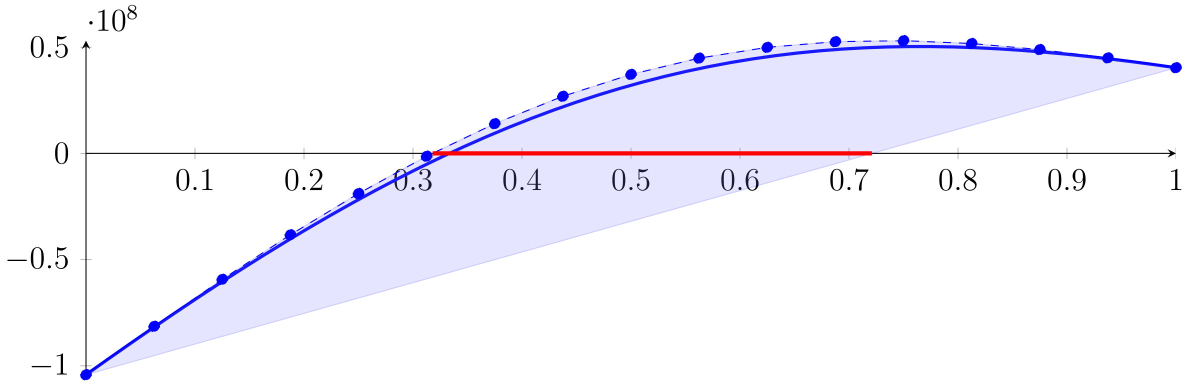
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^6X^9 - 2.52262\cdot10^6X^8 - 1.43506\cdot10^7X^7 + 138542X^6 + 7.92823\cdot10^7X^5 + 3.97396\cdot10^7X^4 - 2.40625\cdot10^8X^3 - 8.33333\cdot10^7X^2 + 3.64583\cdot10^8X - 1.04167\cdot10^8$$



154.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot10^6X^9 - 2.52262\cdot10^6X^8 - 1.43506\cdot10^7X^7 + 138542X^6 + 7.92823 \\ &\quad \cdot10^7X^5 + 3.97396\cdot10^7X^4 - 2.40625\cdot10^8X^3 - 8.33333\cdot10^7X^2 + 3.64583\cdot10^8X - 1.04167\cdot10^8 \\ &= -1.04167\cdot10^8B_{0,16}(X) - 8.13802\cdot10^7B_{1,16}(X) - 5.92882\cdot10^7B_{2,16}(X) - 3.83203 \\ &\quad \cdot10^7B_{3,16}(X) - 1.88844\cdot10^7B_{4,16}(X) - 1.34837\cdot10^6B_{5,16}(X) + 1.39781\cdot10^7B_{6,16}(X) \\ &\quad + 2.68604\cdot10^7B_{7,16}(X) + 3.71532\cdot10^7B_{8,16}(X) + 4.48105\cdot10^7B_{9,16}(X) + 4.98901 \\ &\quad \cdot10^7B_{10,16}(X) + 5.25504\cdot10^7B_{11,16}(X) + 5.30407\cdot10^7B_{12,16}(X) + 5.16832\cdot10^7B_{13,16}(X) \\ &\quad + 4.88488\cdot10^7B_{14,16}(X) + 4.49297\cdot10^7B_{15,16}(X) + 4.03108\cdot10^7B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

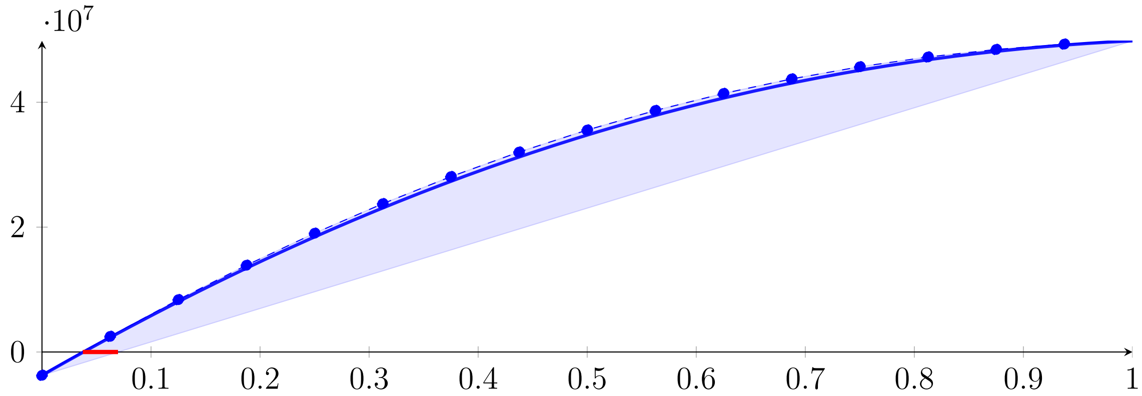
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

154.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 1.59825\cdot 10^{-06}X^{16} - 5.93153\cdot 10^{-05}X^{15} - 0.00248867X^{14} - 0.0650056X^{13} - 0.909142X^{12}\\ &\quad - 5.03931X^{11} + 36.4549X^{10} + 692.921X^{9} + 1886.97X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5}\\ &\quad + 3.91945\cdot 10^{6}X^{4} - 7.54561\cdot 10^{6}X^{3} - 4.29225\cdot 10^{7}X^{2} + 9.97982\cdot 10^{7}X - 3.73501\cdot 10^{6}\\ &= -3.73501\cdot 10^{6}B_{0,16}(X) + 2.50237\cdot 10^{6}B_{1,16}(X) + 8.38207\cdot 10^{6}B_{2,16}(X) + 1.38906\\ &\quad \cdot 10^{7}B_{3,16}(X) + 1.90167\cdot 10^{7}B_{4,16}(X) + 2.37512\cdot 10^{7}B_{5,16}(X) + 2.80875\cdot 10^{7}B_{6,16}(X)\\ &\quad + 3.20213\cdot 10^{7}B_{7,16}(X) + 3.55508\cdot 10^{7}B_{8,16}(X) + 3.86768\cdot 10^{7}B_{9,16}(X) + 4.14025\\ &\quad \cdot 10^{7}B_{10,16}(X) + 4.37336\cdot 10^{7}B_{11,16}(X) + 4.56783\cdot 10^{7}B_{12,16}(X) + 4.7247\cdot 10^{7}B_{13,16}(X)\\ &\quad + 4.84524\cdot 10^{7}B_{14,16}(X) + 4.93093\cdot 10^{7}B_{15,16}(X) + 4.98343\cdot 10^{7}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

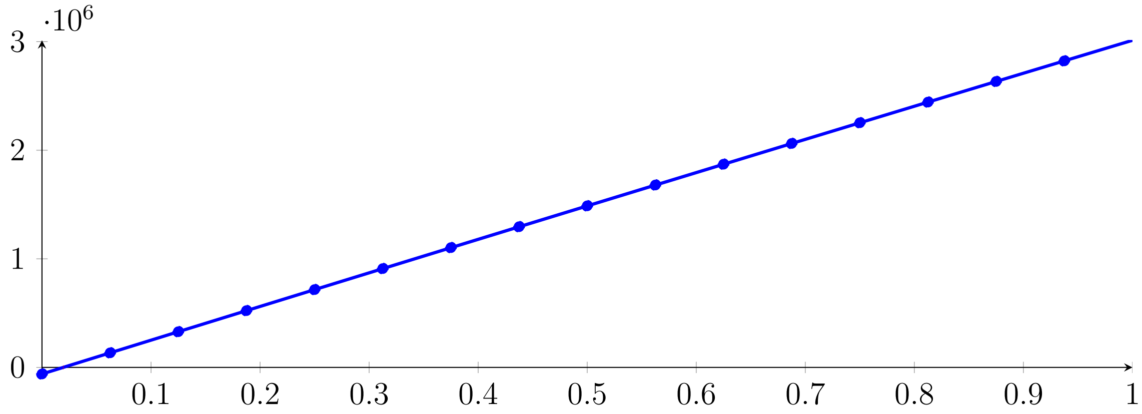
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

154.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 9.01396\cdot 10^{-08}X^{16} - 2.65848\cdot 10^{-07}X^{15} + 2.13948\cdot 10^{-06}X^{14} - 1.33627\cdot 10^{-06}X^{13} + 2.46973\cdot 10^{-06}X^{12}\\ &\quad - 2.45524\cdot 10^{-06}X^{11} + 5.50112\cdot 10^{-07}X^{10} - 1.64198\cdot 10^{-07}X^{9} - 7.35598\cdot 10^{-07}X^{8} - 1.00892\cdot 10^{-06}X^{7}\\ &\quad - 0.000177509X^{6} + 0.0161038X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot 10^{6}X - 60508.5\\ &= -60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X)\\ &\quad + 910202B_{5,16}(X) + 1.1032\cdot 10^{6}B_{6,16}(X) + 1.29582\cdot 10^{6}B_{7,16}(X) + 1.48805\cdot 10^{6}B_{8,16}(X)\\ &\quad + 1.6799\cdot 10^{6}B_{9,16}(X) + 1.87137\cdot 10^{6}B_{10,16}(X) + 2.06245\cdot 10^{6}B_{11,16}(X) + 2.25315\cdot 10^{6}B_{12,16}(X)\\ &\quad + 2.44346\cdot 10^{6}B_{13,16}(X) + 2.63339\cdot 10^{6}B_{14,16}(X) + 2.82293\cdot 10^{6}B_{15,16}(X) + 3.01209\cdot 10^{6}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

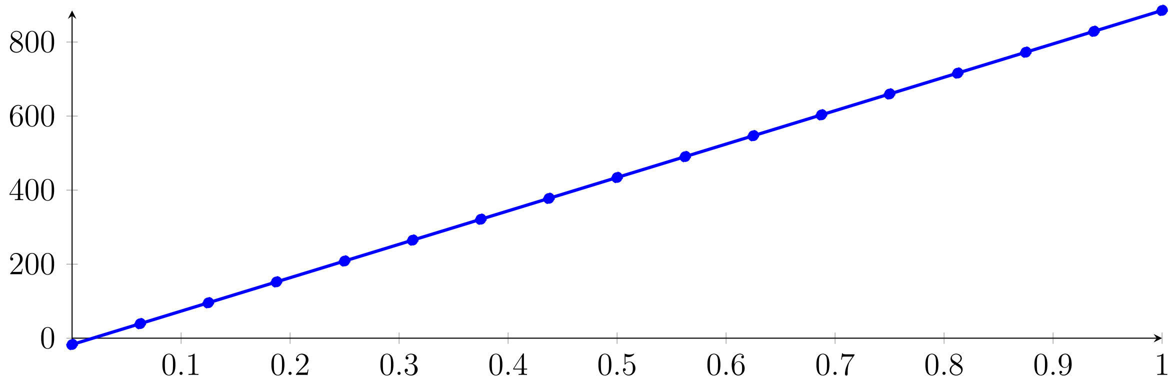
$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

154.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = {} & 2.55372\cdot 10^{-11}X^{16} - 7.21263\cdot 10^{-11}X^{15} + 6.24141\cdot 10^{-10}X^{14} - 4.11162\cdot 10^{-10}X^{13} \\ & + 6.82359\cdot 10^{-10}X^{12} - 7.09475\cdot 10^{-10}X^{11} + 9.71305\cdot 10^{-11}X^{10} - 3.46101\cdot 10^{-11}X^{9} \\ & - 2.13971\cdot 10^{-10}X^{8} - 1.46061\cdot 10^{-11}X^{7} - 1.63366\cdot 10^{-11}X^{6} + 1.87916\cdot 10^{-12}X^{5} \\ & + 2.52576\cdot 10^{-14}X^{4} - 5.67777\cdot 10^{-09}X^{3} - 0.00382618X^{2} + 902.448X - 17.178 \\ = {} & -17.178B_{0,16}(X) + 39.225B_{1,16}(X) + 95.6279B_{2,16}(X) + 152.031B_{3,16}(X) + 208.434B_{4,16}(X) \\ & + 264.837B_{5,16}(X) + 321.24B_{6,16}(X) + 377.642B_{7,16}(X) + 434.045B_{8,16}(X) \\ & + 490.448B_{9,16}(X) + 546.851B_{10,16}(X) + 603.253B_{11,16}(X) + 659.656B_{12,16}(X) \\ & + 716.059B_{13,16}(X) + 772.461B_{14,16}(X) + 828.864B_{15,16}(X) + 885.266B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045\cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

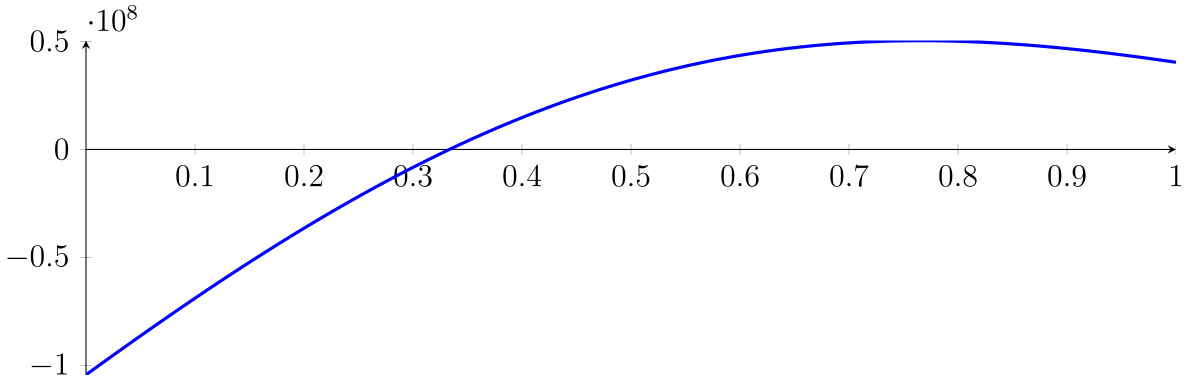
154.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

154.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

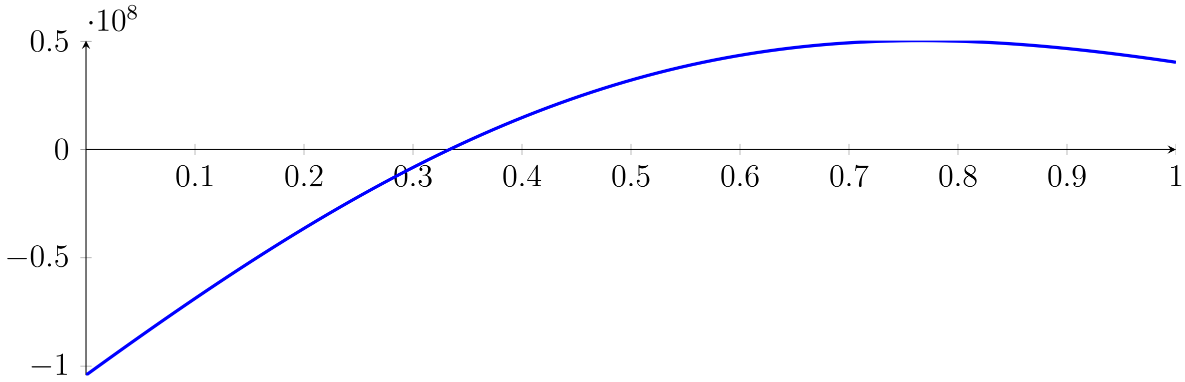
with precision $\varepsilon = 1\cdot10^{-08}$.

155 Running QuadClip on f_{16} with epsilon 8

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

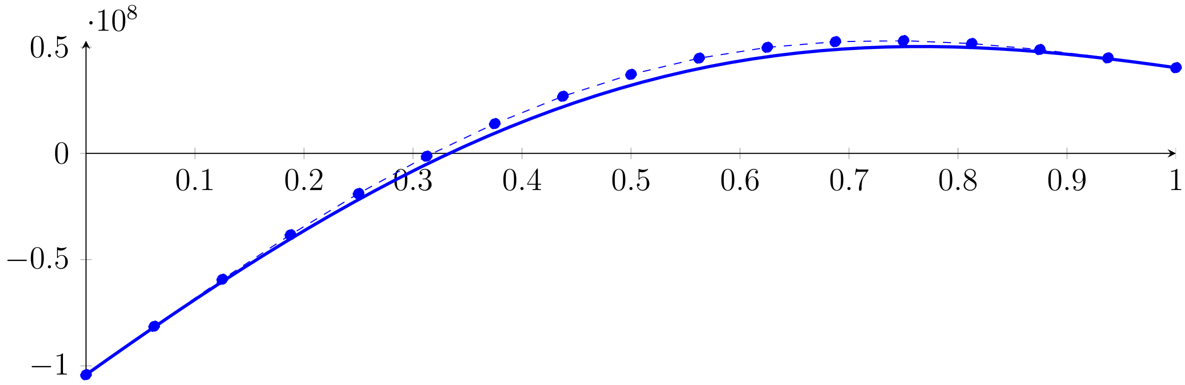
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



155.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

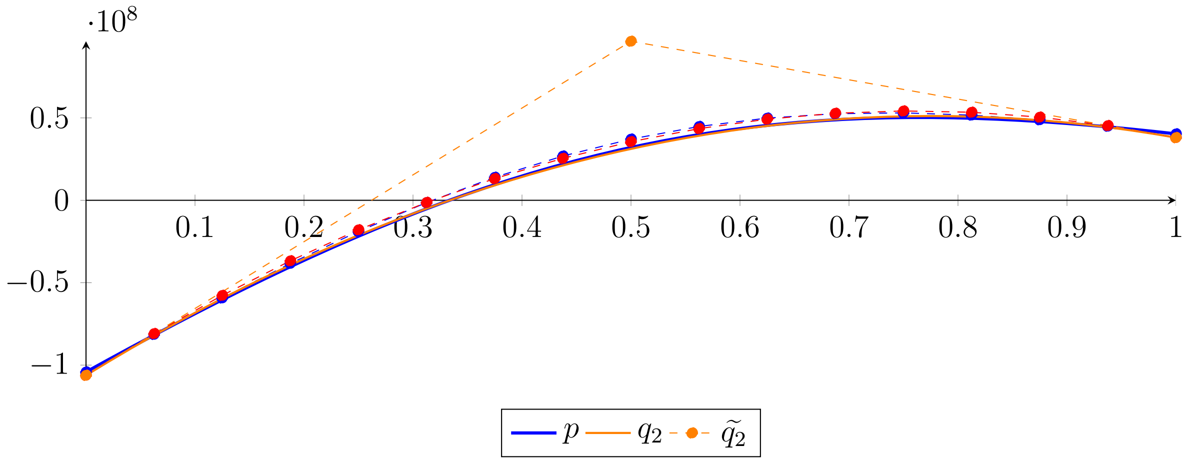
$$\begin{aligned}p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\quad \cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \\ &= -1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -2.61181\cdot10^8X^2 + 4.05417\cdot10^8X - 1.06197\cdot10^8 \\ &= -1.06197\cdot10^8B_{0,2} + 9.65115\cdot10^7B_{1,2} + 3.80384\cdot10^7B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 6049.18X^{16} - 48305.2X^{15} + 174971X^{14} - 380294X^{13} + 552846X^{12} - 567203X^{11} \\ &\quad + 422303X^{10} - 231038X^9 + 93003.6X^8 - 27320.1X^7 + 5752.57X^6 - 843.63X^5 \\ &\quad + 82.5145X^4 - 5.01388X^3 - 2.61181\cdot10^8X^2 + 4.05417\cdot10^8X - 1.06197\cdot10^8 \\ &= -1.06197\cdot10^8B_{0,16} - 8.08583\cdot10^7B_{1,16} - 5.76963\cdot10^7B_{2,16} - 3.67107\cdot10^7B_{3,16} - 1.79017 \\ &\quad \cdot10^7B_{4,16} - 1.26924\cdot10^6B_{5,16} + 1.31867\cdot10^7B_{6,16} + 2.54662\cdot10^7B_{7,16} + 3.55692\cdot10^7B_{8,16} \\ &\quad + 4.34956\cdot10^7B_{9,16} + 4.92456\cdot10^7B_{10,16} + 5.2819\cdot10^7B_{11,16} + 5.42159\cdot10^7B_{12,16} \\ &\quad + 5.34363\cdot10^7B_{13,16} + 5.04802\cdot10^7B_{14,16} + 4.53476\cdot10^7B_{15,16} + 3.80384\cdot10^7B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233\cdot10^6$.

Bounding polynomials M and m:

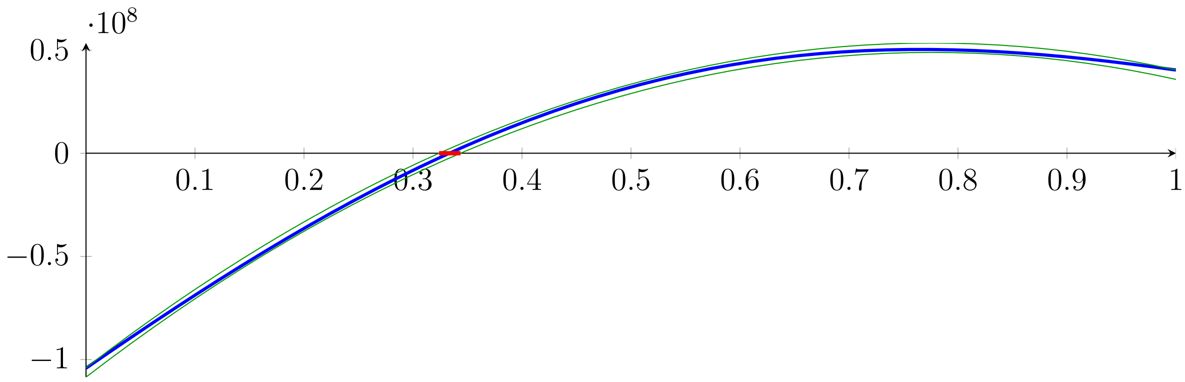
$$M = -2.61181\cdot10^8X^2 + 4.05417\cdot10^8X - 1.03924\cdot10^8$$

$$m = -2.61181\cdot10^8X^2 + 4.05417\cdot10^8X - 1.08469\cdot10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

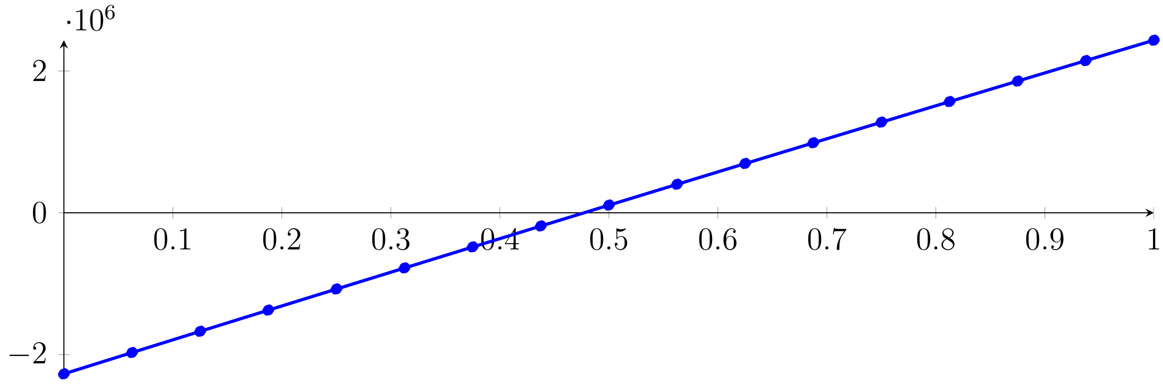
Longest intersection interval: 0.0196686

$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

155.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

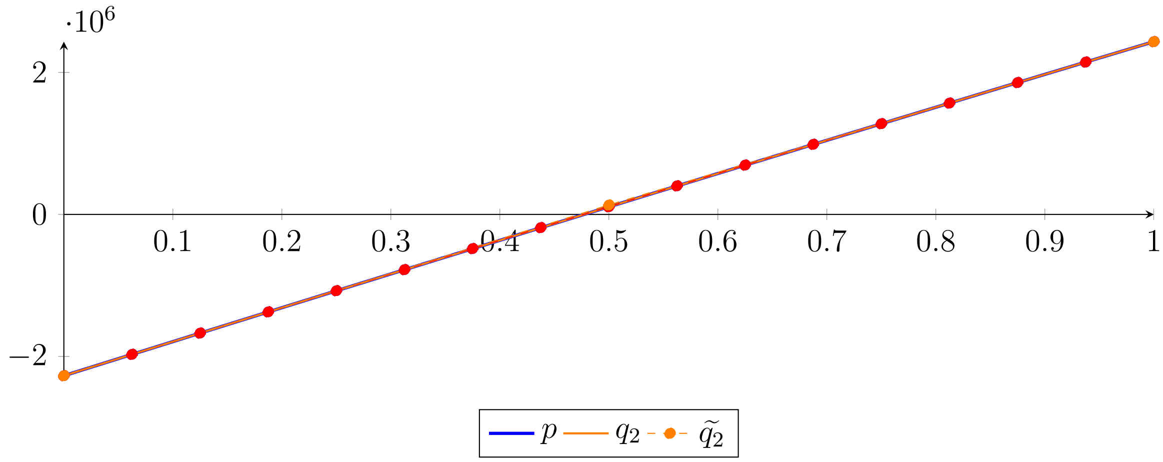
$$p = -1.54841 \cdot 10^{-10} X^{16} - 1.66965 \cdot 10^{-07} X^{15} - 2.92739 \cdot 10^{-07} X^{14} - 1.77943 \cdot 10^{-06} X^{13} - 1.17235 \cdot 10^{-06} X^{12} - 2.42234 \cdot 10^{-06} X^{11} - 6.86445 \cdot 10^{-07} X^{10} - 1.39162 \cdot 10^{-06} X^{9} + 1.07395 \cdot 10^{-06} X^{8} - 1.67072 \cdot 10^{-05} X^{7} - 0.00205879 X^{6} + 0.132721 X^{5} + 22.4437 X^{4} - 850.239 X^{3} - 103028 X^{2} + 4.80874 \cdot 10^{6} X - 2.2715 \cdot 10^{6}$$
$$= -2.2715 \cdot 10^{6} B_{0,16}(X) - 1.97096 \cdot 10^{6} B_{1,16}(X) - 1.67127 \cdot 10^{6} B_{2,16}(X) - 1.37244 \cdot 10^{6} B_{3,16}(X) - 1.07448 \cdot 10^{6} B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^{6} B_{12,16}(X) + 1.5682 \cdot 10^{6} B_{13,16}(X) + 1.85747 \cdot 10^{6} B_{14,16}(X) + 2.14586 \cdot 10^{6} B_{15,16}(X) + 2.43338 \cdot 10^{6} B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265 X^{2} + 4.80923 \cdot 10^{6} X - 2.27154 \cdot 10^{6}$$
$$= -2.27154 \cdot 10^{6} B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^{6} B_{2,2}$$

$$\widetilde{q}_2 = 22.7036 X^{16} - 178.964 X^{15} + 638.976 X^{14} - 1366.64 X^{13} + 1951.1 X^{12} - 1960.92 X^{11} + 1425.35 X^{10} - 757.631 X^{9} + 294.177 X^{8} - 82.4368 X^{7} + 16.2856 X^{6} - 2.18949 X^{5} + 0.191237 X^{4} - 0.0101048 X^{3} - 104265 X^{2} + 4.80923 \cdot 10^{6} X - 2.27154 \cdot 10^{6}$$
$$= -2.27154 \cdot 10^{6} B_{0,16} - 1.97097 \cdot 10^{6} B_{1,16} - 1.67126 \cdot 10^{6} B_{2,16} - 1.37242 \cdot 10^{6} B_{3,16} - 1.07445 \cdot 10^{6} B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803 \cdot 10^{6} B_{12,16} + 1.56818 \cdot 10^{6} B_{13,16} + 1.85746 \cdot 10^{6} B_{14,16} + 2.14587 \cdot 10^{6} B_{15,16} + 2.43342 \cdot 10^{6} B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m:

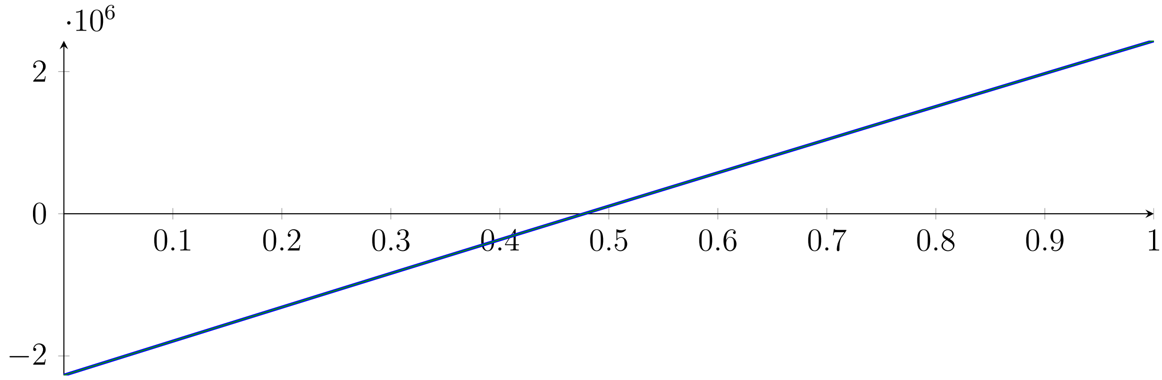
$$M = -104265 X^{2} + 4.80923 \cdot 10^{6} X - 2.2715 \cdot 10^{6}$$

$$m = -104265X^2 + 4.80923{\cdot}10^6 X - 2.27159{\cdot}10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



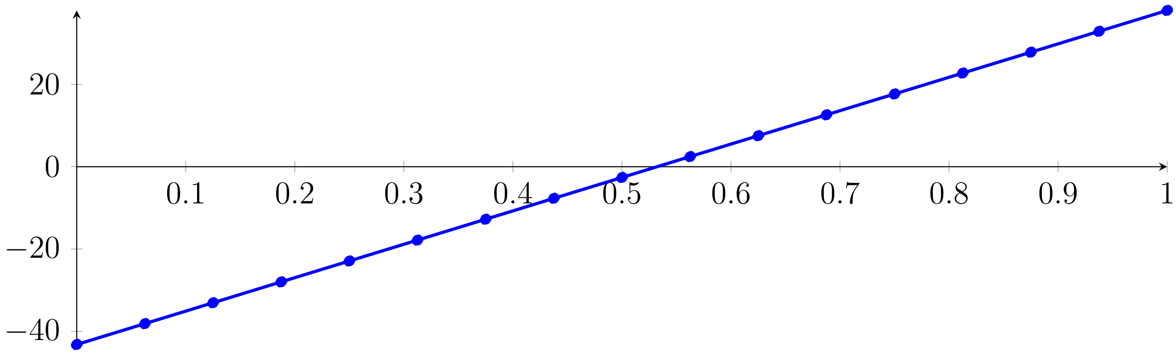
$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301{\cdot}10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

155.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

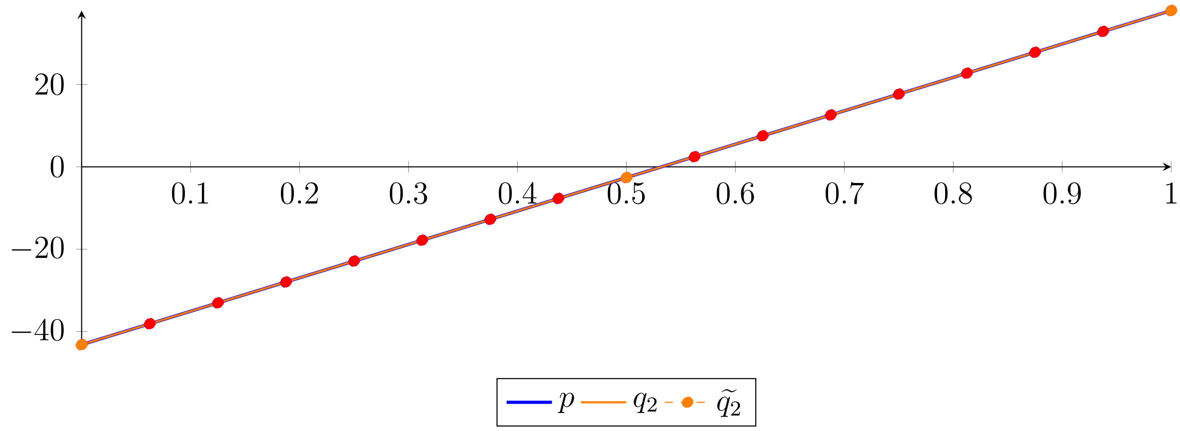
$$\begin{aligned} p = & -2.76723{\cdot}10^{-13}X^{16} - 2.40874{\cdot}10^{-12}X^{15} - 1.25233{\cdot}10^{-11}X^{14} - 3.02935{\cdot}10^{-11}X^{13} \\ & - 3.05617{\cdot}10^{-11}X^{12} - 3.83107{\cdot}10^{-11}X^{11} - 1.26692{\cdot}10^{-11}X^{10} - 2.6672{\cdot}10^{-11}X^{9} \\ & + 2.0004{\cdot}10^{-11}X^{8} + 4.12781{\cdot}10^{-12}X^{7} + 2.44493{\cdot}10^{-12}X^{6} - 1.21236{\cdot}10^{-13}X^{5} \\ & + 1.26288{\cdot}10^{-14}X^{4} - 4.1267{\cdot}10^{-12}X^{3} - 3.09388{\cdot}10^{-05}X^{2} + 81.1506X - 43.1911 \\ = & -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\ & - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X) \\ & + 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X) \\ & + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$\begin{aligned}\tilde{q}_2 &= 5.96265\cdot 10^{-05}X^{16} - 0.000436042X^{15} + 0.00141812X^{14} - 0.00269475X^{13} \\ &\quad + 0.00329809X^{12} - 0.00268757X^{11} + 0.00143268X^{10} - 0.000439599X^{9} \\ &\quad + 1.98418\cdot 10^{-05}X^{8} + 4.87608\cdot 10^{-05}X^{7} - 2.46333\cdot 10^{-05}X^{6} + 6.35808\cdot 10^{-06}X^{5} \\ &\quad - 9.62755\cdot 10^{-07}X^{4} + 8.21372\cdot 10^{-08}X^{3} - 3.09429\cdot 10^{-05}X^{2} + 81.1506X - 43.1911 \\ &= -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} \\ &\quad - 12.7597B_{6,16} - 7.68778B_{7,16} - 2.61587B_{8,16} + 2.45604B_{9,16} + 7.52795B_{10,16} + 12.5999B_{11,16} \\ &\quad + 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.5947\cdot 10^{-09}$.

Bounding polynomials M and m:

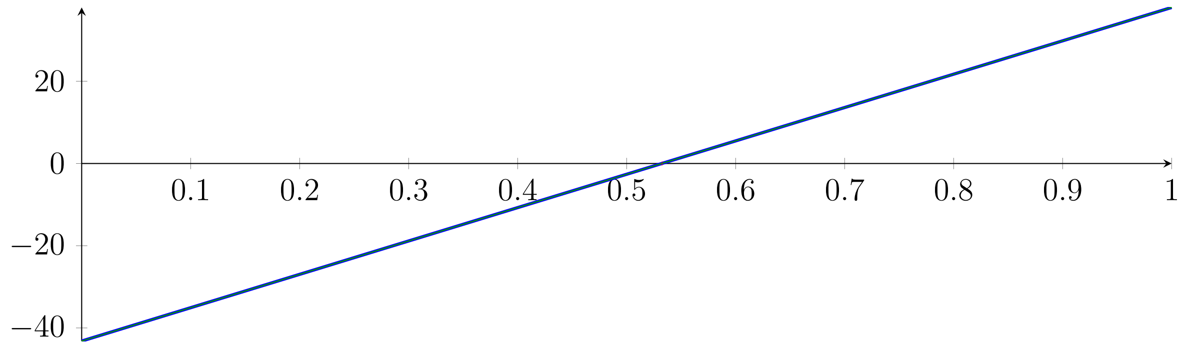
$$M = -3.09389\cdot 10^{-05}X^2 + 81.1506X - 43.1911$$

$$m = -3.09389\cdot 10^{-05}X^2 + 81.1506X - 43.1911$$

Root of M and m:

$$N(M) = \{0.532235, 2.62293\cdot 10^{6}\} \qquad N(m) = \{0.532235, 2.62293\cdot 10^{6}\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

Longest intersection interval: $3.93535\cdot 10^{-11}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

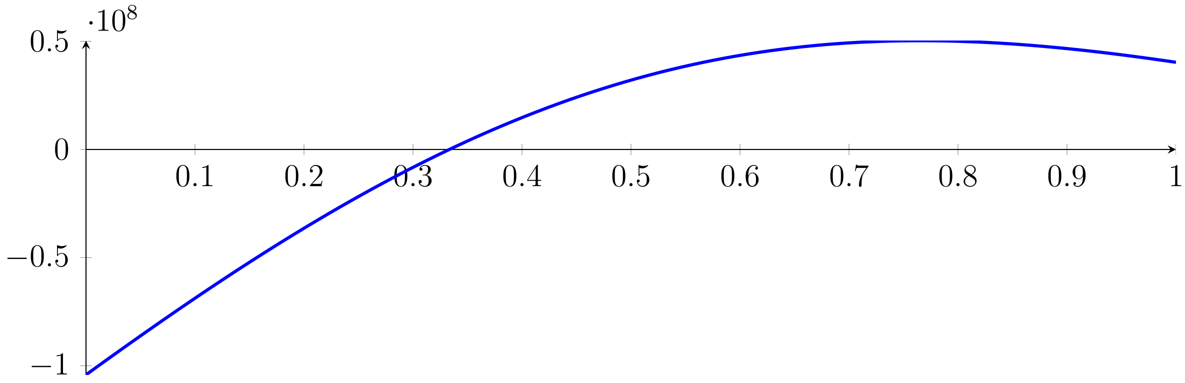
155.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

155.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^{6} X^{9} - 2.52262 \cdot 10^{6} X^{8} - 1.43506 \cdot 10^{7} X^{7} + 138542X^{6} + 7.92823 \cdot 10^{7} X^{5} + 3.97396 \cdot 10^{7} X^{4} - 2.40625 \cdot 10^{8} X^{3} - 8.33333 \cdot 10^{7} X^{2} + 3.64583 \cdot 10^{8} X - 1.04167 \cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-08}$.

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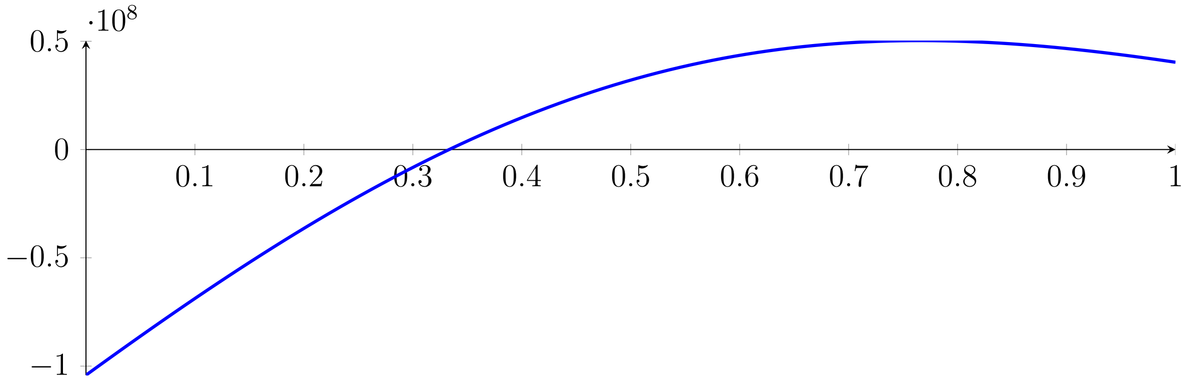
$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^6 X^9 - 2.52262\cdot 10^6 X^8 - 1.43506\cdot 10^7 X^7 + 138542X^6 + 7.92823\cdot 10^7 X^5 + 3.97396\cdot 10^7 X^4 - 2.40625\cdot 10^8 X^3 - 8.33333\cdot 10^7 X^2 + 3.64583\cdot 10^8 X - 1.04167\cdot 10^8$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

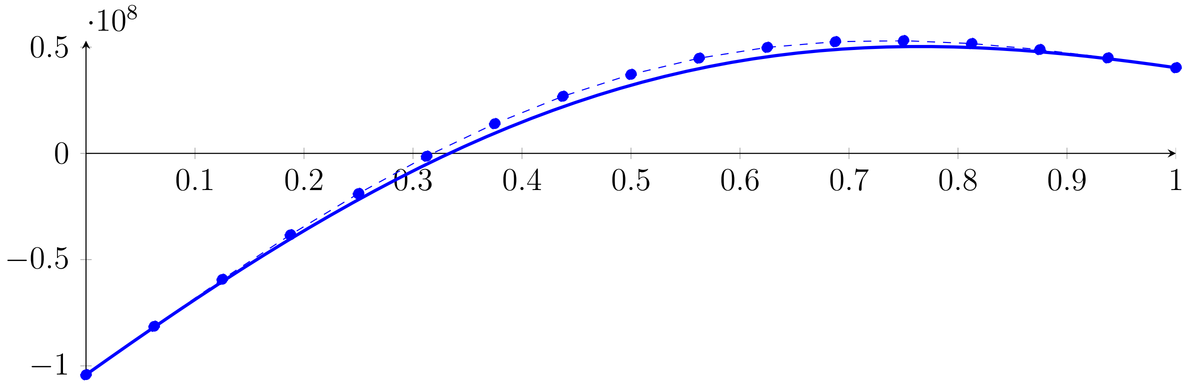
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^6 X^9 - 2.52262\cdot 10^6 X^8 - 1.43506\cdot 10^7 X^7 + 138542X^6 + 7.92823\cdot 10^7 X^5 + 3.97396\cdot 10^7 X^4 - 2.40625\cdot 10^8 X^3 - 8.33333\cdot 10^7 X^2 + 3.64583\cdot 10^8 X - 1.04167\cdot 10^8$$



156.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

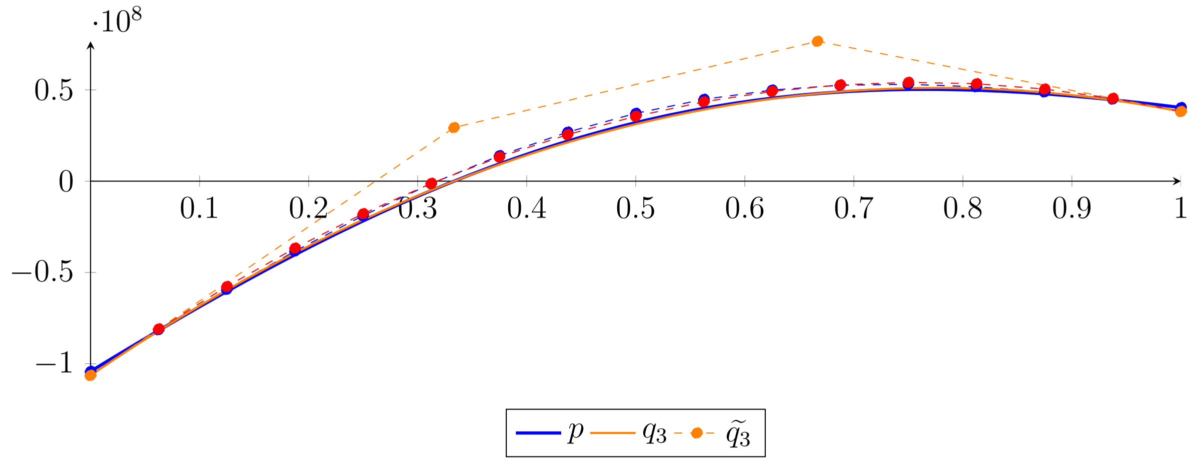
$$\begin{aligned} p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot 10^6 X^9 - 2.52262\cdot 10^6 X^8 - 1.43506\cdot 10^7 X^7 + 138542X^6 + 7.92823 \\ &\quad \cdot 10^7 X^5 + 3.97396\cdot 10^7 X^4 - 2.40625\cdot 10^8 X^3 - 8.33333\cdot 10^7 X^2 + 3.64583\cdot 10^8 X - 1.04167\cdot 10^8 \\ &= -1.04167\cdot 10^8 B_{0,16}(X) - 8.13802\cdot 10^7 B_{1,16}(X) - 5.92882\cdot 10^7 B_{2,16}(X) - 3.83203 \\ &\quad \cdot 10^7 B_{3,16}(X) - 1.88844\cdot 10^7 B_{4,16}(X) - 1.34837\cdot 10^6 B_{5,16}(X) + 1.39781\cdot 10^7 B_{6,16}(X) \\ &\quad + 2.68604\cdot 10^7 B_{7,16}(X) + 3.71532\cdot 10^7 B_{8,16}(X) + 4.48105\cdot 10^7 B_{9,16}(X) + 4.98901 \\ &\quad \cdot 10^7 B_{10,16}(X) + 5.25504\cdot 10^7 B_{11,16}(X) + 5.30407\cdot 10^7 B_{12,16}(X) + 5.16832\cdot 10^7 B_{13,16}(X) \\ &\quad + 4.88488\cdot 10^7 B_{14,16}(X) + 4.49297\cdot 10^7 B_{15,16}(X) + 4.03108\cdot 10^7 B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.06335{\cdot}10^8 \\ &= -1.06335{\cdot}10^8B_{0,3} + 2.93558{\cdot}10^7B_{1,3} + 7.66068{\cdot}10^7B_{2,3} + 3.81764{\cdot}10^7B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2461.93X^{16} - 19614.9X^{15} + 70879.5X^{14} - 153661X^{13} + 222746X^{12} - 227755X^{11} \\ &\quad + 168826X^{10} - 91798.7X^9 + 36630.3X^8 - 10627.3X^7 + 2200.54X^6 - 316.059X^5 \\ &\quad + 30.1958X^4 + 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.06335{\cdot}10^8 \\ &= -1.06335{\cdot}10^8B_{0,16} - 8.08928{\cdot}10^7B_{1,16} - 5.76618{\cdot}10^7B_{2,16} - 3.66369{\cdot}10^7B_{3,16} - 1.78131 \\ &\quad {\cdot}10^7B_{4,16} - 1.18551{\cdot}10^6B_{5,16} + 1.32508{\cdot}10^7B_{6,16} + 2.55007{\cdot}10^7B_{7,16} + 3.55692{\cdot}10^7B_{8,16} \\ &\quad + 4.34611{\cdot}10^7B_{9,16} + 4.91815{\cdot}10^7B_{10,16} + 5.27353{\cdot}10^7B_{11,16} + 5.41273{\cdot}10^7B_{12,16} \\ &\quad + 5.33624{\cdot}10^7B_{13,16} + 5.04457{\cdot}10^7B_{14,16} + 4.53821{\cdot}10^7B_{15,16} + 3.81764{\cdot}10^7B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806{\cdot}10^6$.

Bounding polynomials M and m:

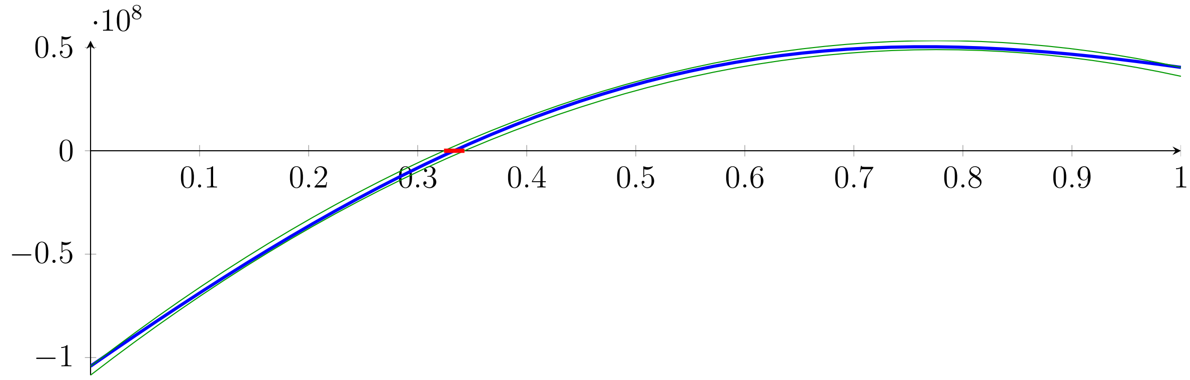
$$M = 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.04167{\cdot}10^8$$

$$m = 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.08503{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

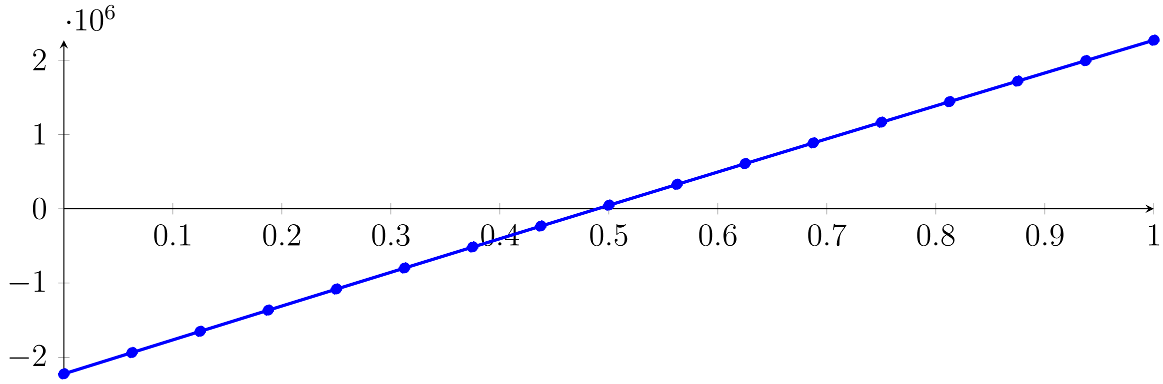
Longest intersection interval: 0.0187703

$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

156.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

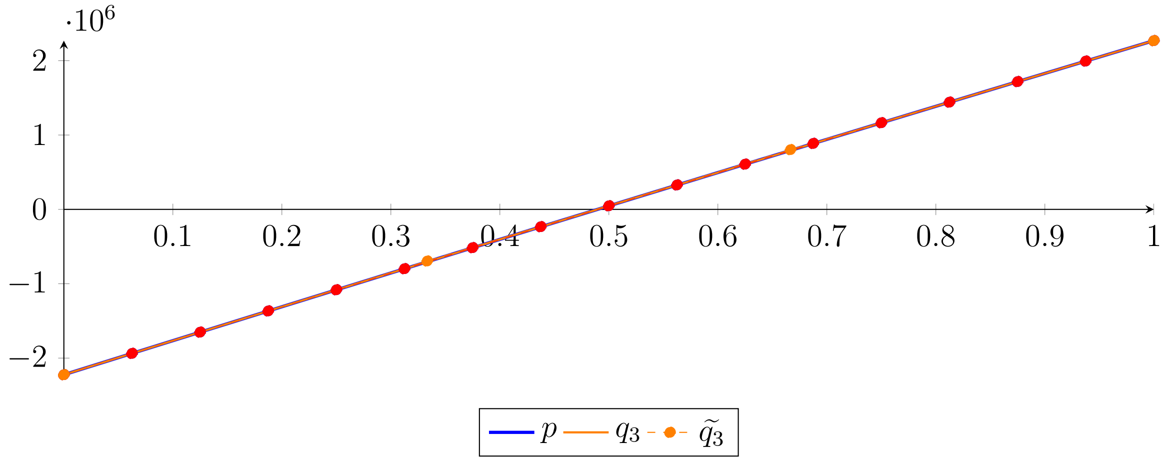
$$p = -3.66617 \cdot 10^{-09} X^{16} - 1.53217 \cdot 10^{-07} X^{15} - 3.62234 \cdot 10^{-07} X^{14} - 1.65579 \cdot 10^{-06} X^{13} - 1.15373 \cdot 10^{-06} X^{12} - 2.3399 \cdot 10^{-06} X^{11} - 5.02543 \cdot 10^{-07} X^{10} - 1.38381 \cdot 10^{-06} X^{9} + 1.1237 \cdot 10^{-06} X^{8} - 1.19653 \cdot 10^{-05} X^{7} - 0.00155608 X^{6} + 0.10496 X^{5} + 18.6215 X^{4} - 738.202 X^{3} - 93855.7 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$
$$= -2.22335 \cdot 10^{6} B_{0,16}(X) - 1.93666 \cdot 10^{6} B_{1,16}(X) - 1.65074 \cdot 10^{6} B_{2,16}(X) - 1.36561 \cdot 10^{6} B_{3,16}(X) - 1.08127 \cdot 10^{6} B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^{6} B_{12,16}(X) + 1.44233 \cdot 10^{6} B_{13,16}(X) + 1.71876 \cdot 10^{6} B_{14,16}(X) + 1.99439 \cdot 10^{6} B_{15,16}(X) + 2.26922 \cdot 10^{6} B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -700.673 X^{3} - 93879.9 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$
$$= -2.22335 \cdot 10^{6} B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^{6} B_{3,3}$$

$$\tilde{q}_3 = 16.4956 X^{16} - 129.161 X^{15} + 457.83 X^{14} - 971.671 X^{13} + 1375.95 X^{12} - 1370.96 X^{11} + 987.265 X^{10} - 519.476 X^{9} + 199.587 X^{8} - 55.434 X^{7} + 10.9237 X^{6} - 1.48019 X^{5} + 0.129516 X^{4} - 700.679 X^{3} - 93879.9 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$
$$= -2.22335 \cdot 10^{6} B_{0,16} - 1.93666 \cdot 10^{6} B_{1,16} - 1.65074 \cdot 10^{6} B_{2,16} - 1.36561 \cdot 10^{6} B_{3,16} - 1.08126 \cdot 10^{6} B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^{6} B_{12,16} + 1.44233 \cdot 10^{6} B_{13,16} + 1.71876 \cdot 10^{6} B_{14,16} + 1.99439 \cdot 10^{6} B_{15,16} + 2.26922 \cdot 10^{6} B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m:

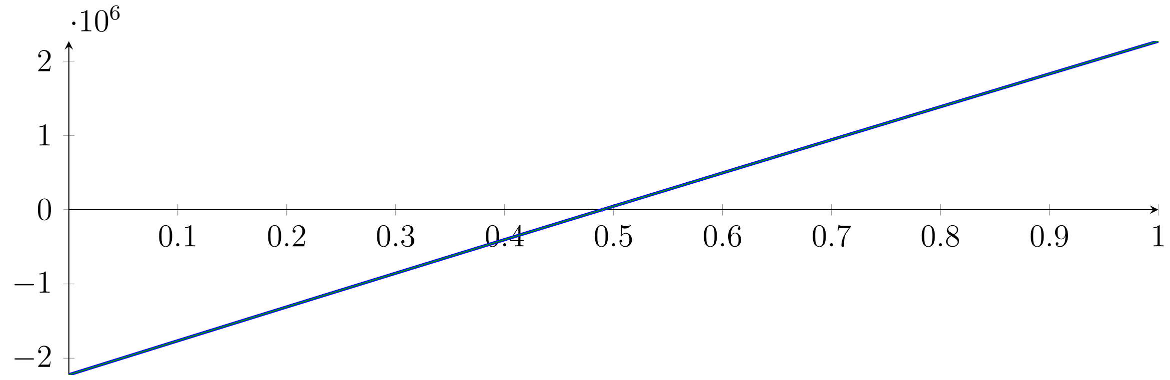
$$M = -700.673 X^{3} - 93879.9 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715{\cdot}10^6X - 2.22335{\cdot}10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

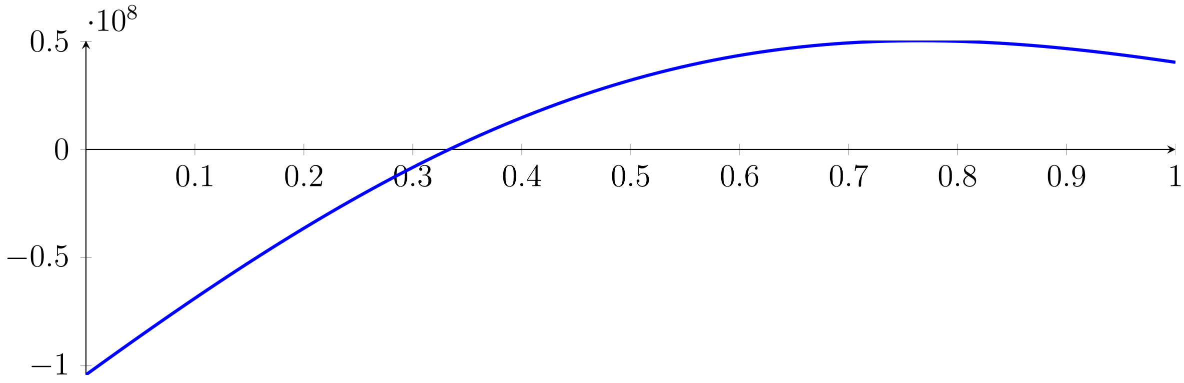
156.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

156.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1\cdot 10^{-08}$.

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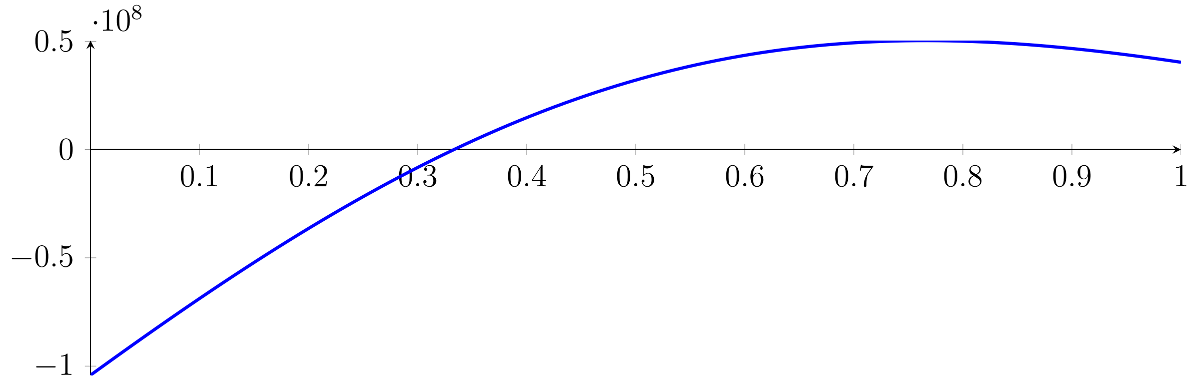
$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called BezClip with input polynomial on interval $[0, 1]$:

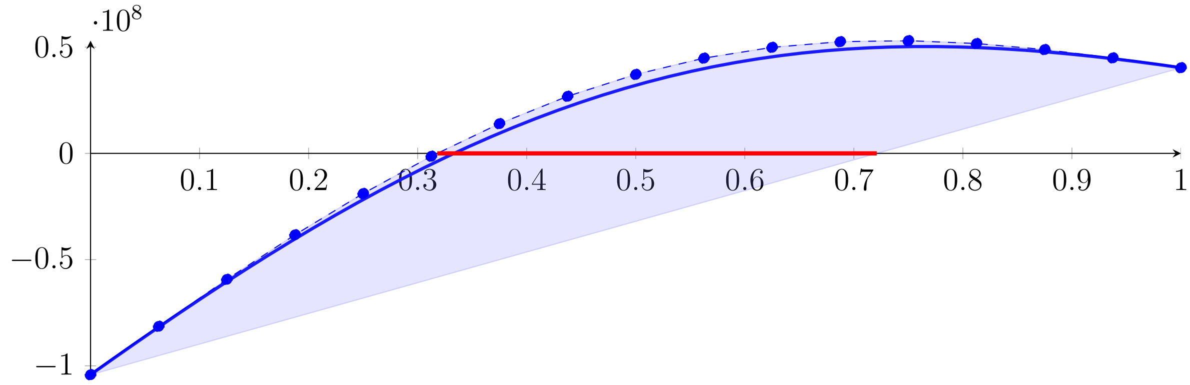
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



157.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\quad \cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ &= -1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

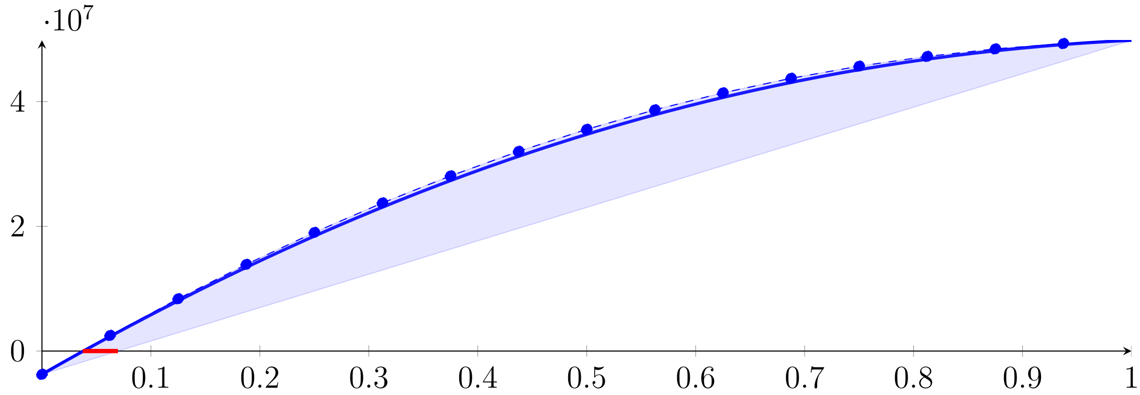
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

157.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 1.59825\cdot 10^{-06}X^{16} - 5.93153\cdot 10^{-05}X^{15} - 0.00248867X^{14} - 0.0650056X^{13} - 0.909142X^{12} \\ &\quad - 5.03931X^{11} + 36.4549X^{10} + 692.921X^{9} + 1886.97X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5} \\ &\quad + 3.91945\cdot 10^{6}X^{4} - 7.54561\cdot 10^{6}X^{3} - 4.29225\cdot 10^{7}X^{2} + 9.97982\cdot 10^{7}X - 3.73501\cdot 10^{6} \\ &= -3.73501\cdot 10^{6}B_{0,16}(X) + 2.50237\cdot 10^{6}B_{1,16}(X) + 8.38207\cdot 10^{6}B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^{7}B_{3,16}(X) + 1.90167\cdot 10^{7}B_{4,16}(X) + 2.37512\cdot 10^{7}B_{5,16}(X) + 2.80875\cdot 10^{7}B_{6,16}(X) \\ &\quad + 3.20213\cdot 10^{7}B_{7,16}(X) + 3.55508\cdot 10^{7}B_{8,16}(X) + 3.86768\cdot 10^{7}B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^{7}B_{10,16}(X) + 4.37336\cdot 10^{7}B_{11,16}(X) + 4.56783\cdot 10^{7}B_{12,16}(X) + 4.7247\cdot 10^{7}B_{13,16}(X) \\ &\quad + 4.84524\cdot 10^{7}B_{14,16}(X) + 4.93093\cdot 10^{7}B_{15,16}(X) + 4.98343\cdot 10^{7}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

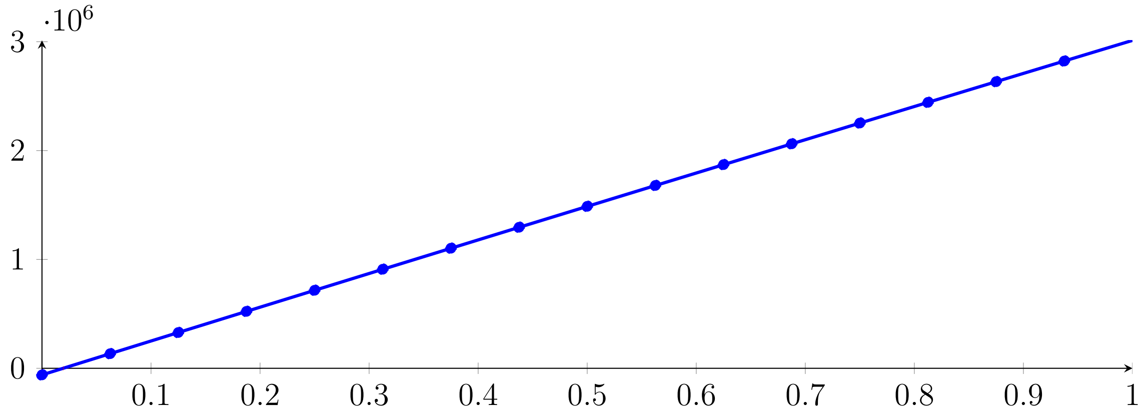
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

157.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 9.01396\cdot 10^{-08}X^{16} - 2.65848\cdot 10^{-07}X^{15} + 2.13948\cdot 10^{-06}X^{14} - 1.33627\cdot 10^{-06}X^{13} + 2.46973\cdot 10^{-06}X^{12} \\ &\quad - 2.45524\cdot 10^{-06}X^{11} + 5.50112\cdot 10^{-07}X^{10} - 1.64198\cdot 10^{-07}X^{9} - 7.35598\cdot 10^{-07}X^{8} - 1.00892\cdot 10^{-06}X^{7} \\ &\quad - 0.000177509X^{6} + 0.0161038X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot 10^{6}X - 60508.5 \\ &= -60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X) \\ &\quad + 910202B_{5,16}(X) + 1.1032\cdot 10^{6}B_{6,16}(X) + 1.29582\cdot 10^{6}B_{7,16}(X) + 1.48805\cdot 10^{6}B_{8,16}(X) \\ &\quad + 1.6799\cdot 10^{6}B_{9,16}(X) + 1.87137\cdot 10^{6}B_{10,16}(X) + 2.06245\cdot 10^{6}B_{11,16}(X) + 2.25315\cdot 10^{6}B_{12,16}(X) \\ &\quad + 2.44346\cdot 10^{6}B_{13,16}(X) + 2.63339\cdot 10^{6}B_{14,16}(X) + 2.82293\cdot 10^{6}B_{15,16}(X) + 3.01209\cdot 10^{6}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

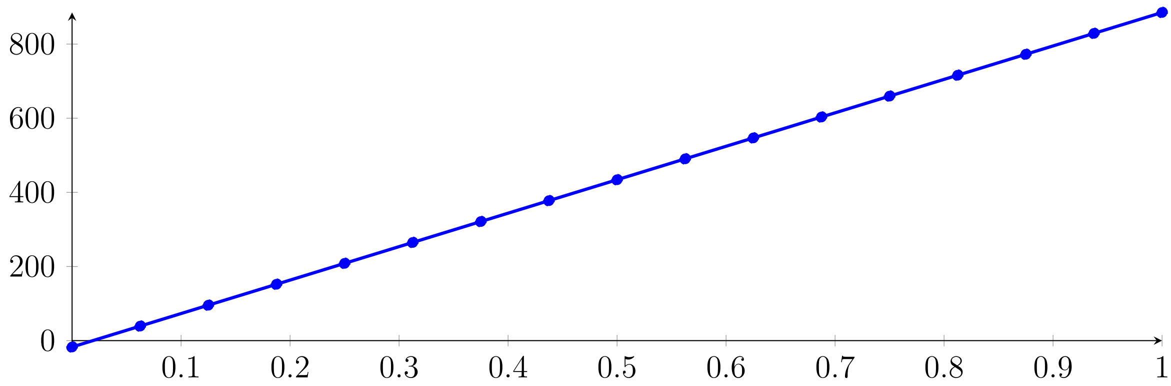
$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

157.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
p &= 2.55372 \cdot 10^{-11} X^{16} - 7.21263 \cdot 10^{-11} X^{15} + 6.24141 \cdot 10^{-10} X^{14} - 4.11162 \cdot 10^{-10} X^{13} \\
&\quad + 6.82359 \cdot 10^{-10} X^{12} - 7.09475 \cdot 10^{-10} X^{11} + 9.71305 \cdot 10^{-11} X^{10} - 3.46101 \cdot 10^{-11} X^{9} \\
&\quad - 2.13971 \cdot 10^{-10} X^{8} - 1.46061 \cdot 10^{-11} X^{7} - 1.63366 \cdot 10^{-11} X^{6} + 1.87916 \cdot 10^{-12} X^{5} \\
&\quad + 2.52576 \cdot 10^{-14} X^{4} - 5.67777 \cdot 10^{-09} X^{3} - 0.00382618 X^{2} + 902.448 X - 17.178 \\
&= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
&\quad + 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
&\quad + 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
&\quad + 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

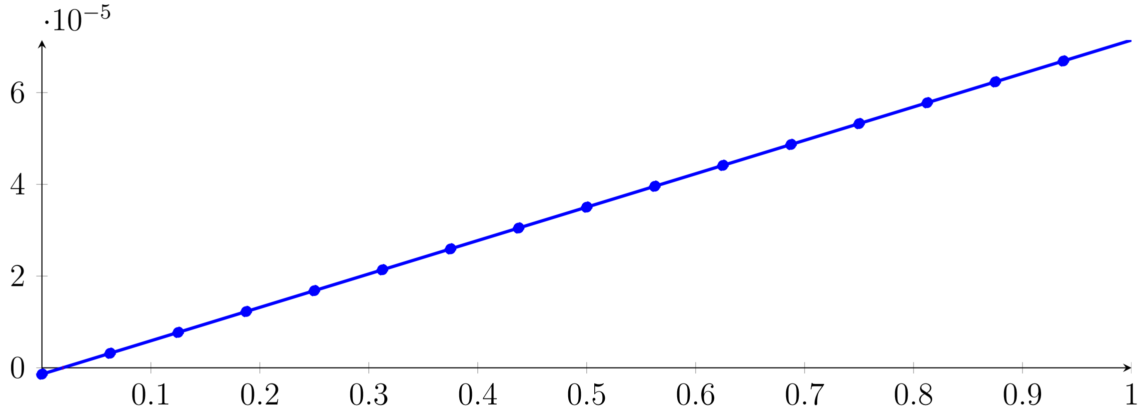
$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045 \cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

157.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 2.14261\cdot10^{-18}X^{16} - 6.28573\cdot10^{-18}X^{15} + 5.28612\cdot10^{-17}X^{14} - 3.67279\cdot10^{-17}X^{13} + 6.10136\cdot10^{-17}X^{12} - 6.60335\cdot10^{-17}X^{11} + 1.66661\cdot10^{-17}X^{10} - 8.36524\cdot10^{-18}X^{9} - 1.56919\cdot10^{-17}X^{8} - 1.85474\cdot10^{-18}X^{7} - 1.4308\cdot10^{-18}X^{6} + 1.1562\cdot10^{-19}X^{5} - 1.20437\cdot10^{-20}X^{4} - 4.63221\cdot10^{-22}X^{3} - 2.49207\cdot10^{-17}X^{2} + 7.28316\cdot10^{-05}X - 1.38634\cdot10^{-06}$$
$$= -1.38634\cdot10^{-06}B_{0,16}(X) + 3.16564\cdot10^{-06}B_{1,16}(X) + 7.71761\cdot10^{-06}B_{2,16}(X) + 1.22696\cdot10^{-05}B_{3,16}(X) + 1.68216\cdot10^{-05}B_{4,16}(X) + 2.13735\cdot10^{-05}B_{5,16}(X) + 2.59255\cdot10^{-05}B_{6,16}(X) + 3.04775\cdot10^{-05}B_{7,16}(X) + 3.50295\cdot10^{-05}B_{8,16}(X) + 3.95814\cdot10^{-05}B_{9,16}(X) + 4.41334\cdot10^{-05}B_{10,16}(X) + 4.86854\cdot10^{-05}B_{11,16}(X) + 5.32374\cdot10^{-05}B_{12,16}(X) + 5.77893\cdot10^{-05}B_{13,16}(X) + 6.23413\cdot10^{-05}B_{14,16}(X) + 6.68933\cdot10^{-05}B_{15,16}(X) + 7.14453\cdot10^{-05}B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.51313\cdot10^{-15}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

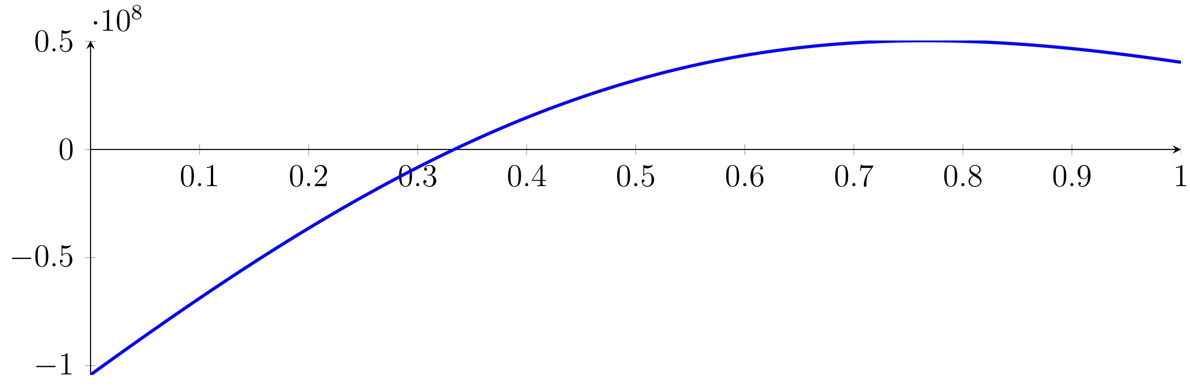
157.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

157.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

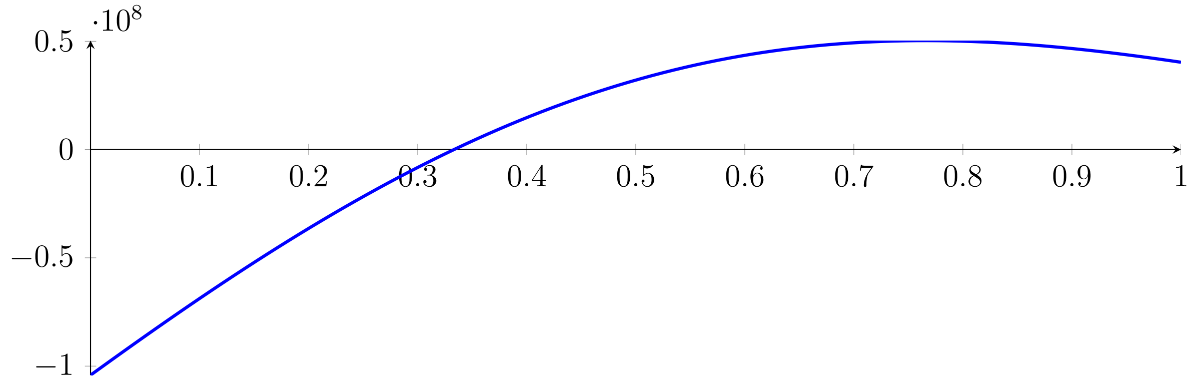
with precision $\varepsilon = 1\cdot 10^{-16}$.

158 Running `QuadClip` on f_{16} with epsilon 16

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called `QuadClip` with input polynomial on interval $[0, 1]$:

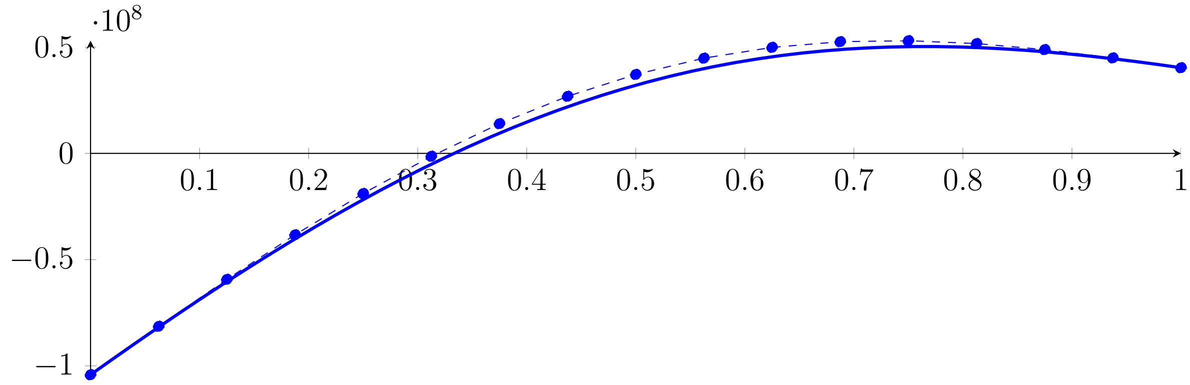
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



158.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

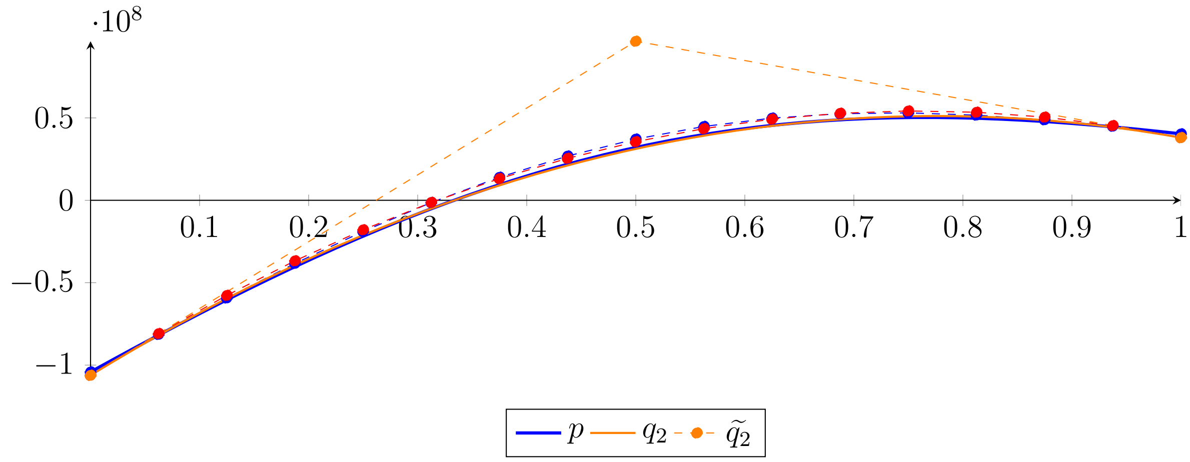
$$\begin{aligned} p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\quad \cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ &= -1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -2.61181{\cdot}10^8X^2 + 4.05417{\cdot}10^8X - 1.06197{\cdot}10^8 \\ &= -1.06197{\cdot}10^8B_{0,2} + 9.65115{\cdot}10^7B_{1,2} + 3.80384{\cdot}10^7B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 6049.18X^{16} - 48305.2X^{15} + 174971X^{14} - 380294X^{13} + 552846X^{12} - 567203X^{11} \\ &\quad + 422303X^{10} - 231038X^9 + 93003.6X^8 - 27320.1X^7 + 5752.57X^6 - 843.63X^5 \\ &\quad + 82.5145X^4 - 5.01388X^3 - 2.61181{\cdot}10^8X^2 + 4.05417{\cdot}10^8X - 1.06197{\cdot}10^8 \\ &= -1.06197{\cdot}10^8B_{0,16} - 8.08583{\cdot}10^7B_{1,16} - 5.76963{\cdot}10^7B_{2,16} - 3.67107{\cdot}10^7B_{3,16} - 1.79017 \\ &\quad {\cdot}10^7B_{4,16} - 1.26924{\cdot}10^6B_{5,16} + 1.31867{\cdot}10^7B_{6,16} + 2.54662{\cdot}10^7B_{7,16} + 3.55692{\cdot}10^7B_{8,16} \\ &\quad + 4.34956{\cdot}10^7B_{9,16} + 4.92456{\cdot}10^7B_{10,16} + 5.2819{\cdot}10^7B_{11,16} + 5.42159{\cdot}10^7B_{12,16} \\ &\quad + 5.34363{\cdot}10^7B_{13,16} + 5.04802{\cdot}10^7B_{14,16} + 4.53476{\cdot}10^7B_{15,16} + 3.80384{\cdot}10^7B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233{\cdot}10^6$.

Bounding polynomials M and m:

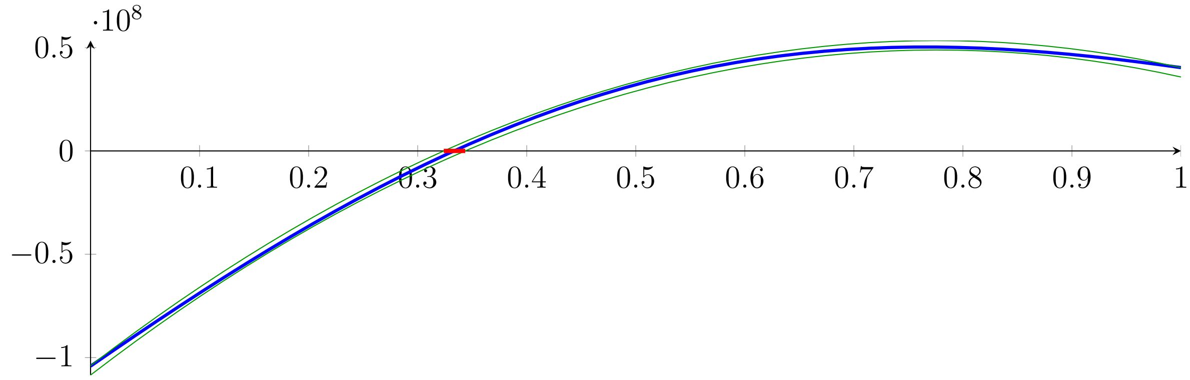
$$M = -2.61181{\cdot}10^8X^2 + 4.05417{\cdot}10^8X - 1.03924{\cdot}10^8$$

$$m = -2.61181{\cdot}10^8X^2 + 4.05417{\cdot}10^8X - 1.08469{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

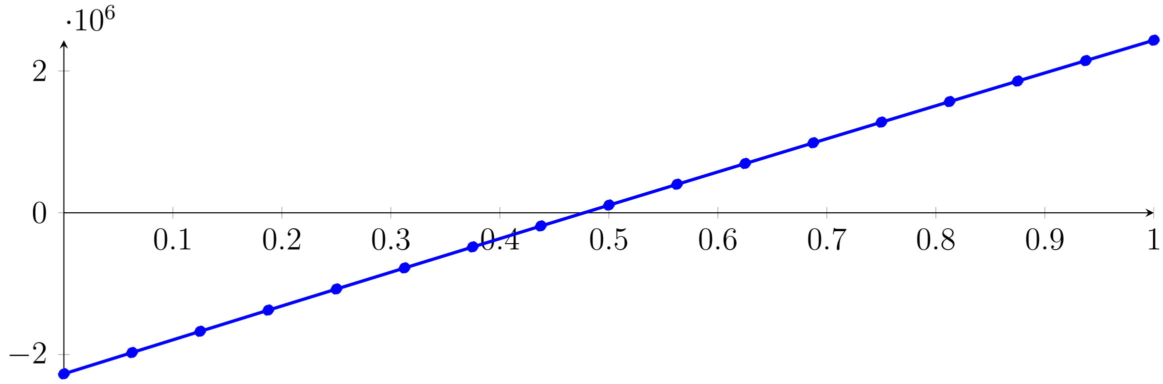
Longest intersection interval: 0.0196686

$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

158.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

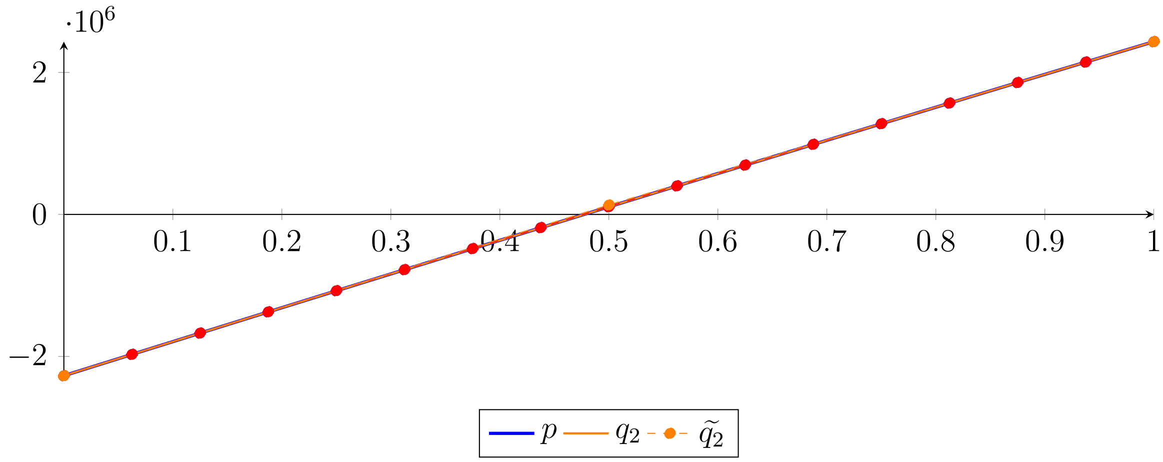
$$p = -1.54841 \cdot 10^{-10} X^{16} - 1.66965 \cdot 10^{-07} X^{15} - 2.92739 \cdot 10^{-07} X^{14} - 1.77943 \cdot 10^{-06} X^{13} - 1.17235 \cdot 10^{-06} X^{12} - 2.42234 \cdot 10^{-06} X^{11} - 6.86445 \cdot 10^{-07} X^{10} - 1.39162 \cdot 10^{-06} X^{9} + 1.07395 \cdot 10^{-06} X^{8} - 1.67072 \cdot 10^{-05} X^{7} - 0.00205879 X^{6} + 0.132721 X^{5} + 22.4437 X^{4} - 850.239 X^{3} - 103028 X^{2} + 4.80874 \cdot 10^{6} X - 2.2715 \cdot 10^{6}$$
$$= -2.2715 \cdot 10^{6} B_{0,16}(X) - 1.97096 \cdot 10^{6} B_{1,16}(X) - 1.67127 \cdot 10^{6} B_{2,16}(X) - 1.37244 \cdot 10^{6} B_{3,16}(X) - 1.07448 \cdot 10^{6} B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^{6} B_{12,16}(X) + 1.5682 \cdot 10^{6} B_{13,16}(X) + 1.85747 \cdot 10^{6} B_{14,16}(X) + 2.14586 \cdot 10^{6} B_{15,16}(X) + 2.43338 \cdot 10^{6} B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265 X^{2} + 4.80923 \cdot 10^{6} X - 2.27154 \cdot 10^{6}$$
$$= -2.27154 \cdot 10^{6} B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^{6} B_{2,2}$$

$$\tilde{q}_2 = 22.7036 X^{16} - 178.964 X^{15} + 638.976 X^{14} - 1366.64 X^{13} + 1951.1 X^{12} - 1960.92 X^{11} + 1425.35 X^{10} - 757.631 X^{9} + 294.177 X^{8} - 82.4368 X^{7} + 16.2856 X^{6} - 2.18949 X^{5} + 0.191237 X^{4} - 0.0101048 X^{3} - 104265 X^{2} + 4.80923 \cdot 10^{6} X - 2.27154 \cdot 10^{6}$$
$$= -2.27154 \cdot 10^{6} B_{0,16} - 1.97097 \cdot 10^{6} B_{1,16} - 1.67126 \cdot 10^{6} B_{2,16} - 1.37242 \cdot 10^{6} B_{3,16} - 1.07445 \cdot 10^{6} B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803 \cdot 10^{6} B_{12,16} + 1.56818 \cdot 10^{6} B_{13,16} + 1.85746 \cdot 10^{6} B_{14,16} + 2.14587 \cdot 10^{6} B_{15,16} + 2.43342 \cdot 10^{6} B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m:

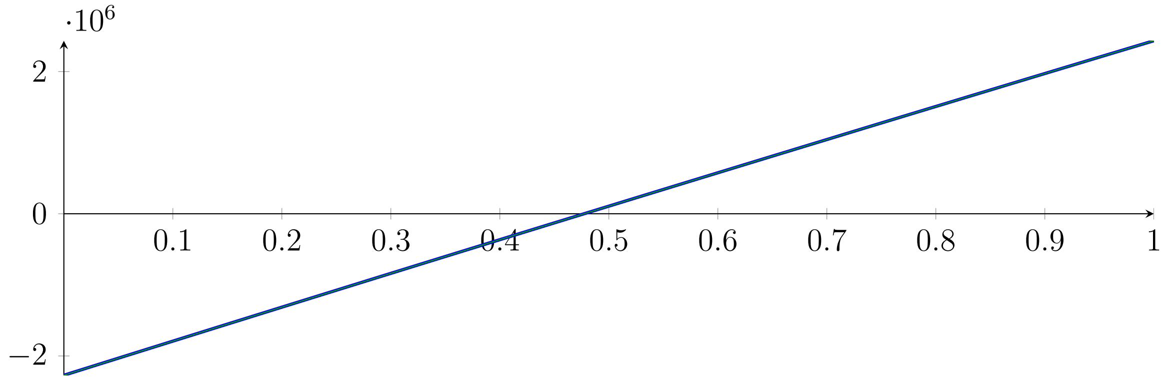
$$M = -104265 X^{2} + 4.80923 \cdot 10^{6} X - 2.2715 \cdot 10^{6}$$

$$m = -104265X^2 + 4.80923{\cdot}10^6 X - 2.27159{\cdot}10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



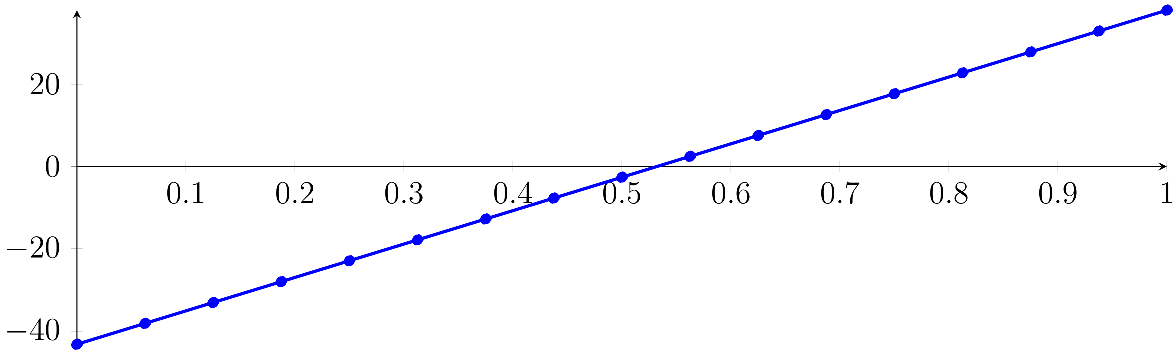
$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301{\cdot}10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

158.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

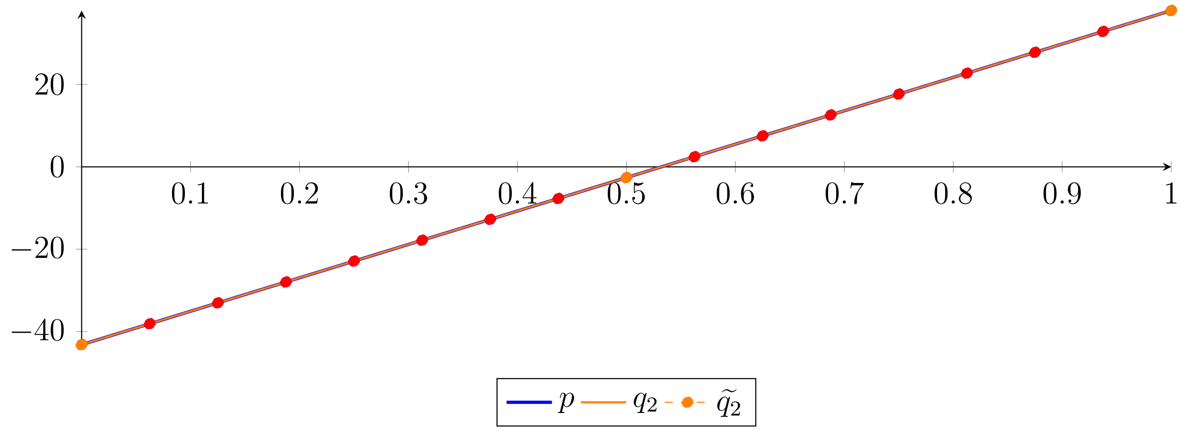
$$\begin{aligned} p = & -2.76723{\cdot}10^{-13}X^{16} - 2.40874{\cdot}10^{-12}X^{15} - 1.25233{\cdot}10^{-11}X^{14} - 3.02935{\cdot}10^{-11}X^{13} \\ & - 3.05617{\cdot}10^{-11}X^{12} - 3.83107{\cdot}10^{-11}X^{11} - 1.26692{\cdot}10^{-11}X^{10} - 2.6672{\cdot}10^{-11}X^{9} \\ & + 2.0004{\cdot}10^{-11}X^{8} + 4.12781{\cdot}10^{-12}X^{7} + 2.44493{\cdot}10^{-12}X^{6} - 1.21236{\cdot}10^{-13}X^{5} \\ & + 1.26288{\cdot}10^{-14}X^{4} - 4.1267{\cdot}10^{-12}X^{3} - 3.09388{\cdot}10^{-05}X^{2} + 81.1506X - 43.1911 \\ = & -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\ & - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X) \\ & + 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X) \\ & + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$
\begin{aligned}
\tilde{q}_2 &= 5.96265 \cdot 10^{-05} X^{16} - 0.000436042 X^{15} + 0.00141812 X^{14} - 0.00269475 X^{13} \\
&\quad + 0.00329809 X^{12} - 0.00268757 X^{11} + 0.00143268 X^{10} - 0.000439599 X^{9} \\
&\quad + 1.98418 \cdot 10^{-05} X^{8} + 4.87608 \cdot 10^{-05} X^{7} - 2.46333 \cdot 10^{-05} X^{6} + 6.35808 \cdot 10^{-06} X^{5} \\
&\quad - 9.62755 \cdot 10^{-07} X^{4} + 8.21372 \cdot 10^{-08} X^{3} - 3.09429 \cdot 10^{-05} X^{2} + 81.1506 X - 43.1911 \\
&= -43.1911 B_{0,16} - 38.1192 B_{1,16} - 33.0473 B_{2,16} - 27.9754 B_{3,16} - 22.9035 B_{4,16} - 17.8316 B_{5,16} \\
&\quad - 12.7597 B_{6,16} - 7.68778 B_{7,16} - 2.61587 B_{8,16} + 2.45604 B_{9,16} + 7.52795 B_{10,16} + 12.5999 B_{11,16} \\
&\quad + 17.6718 B_{12,16} + 22.7437 B_{13,16} + 27.8156 B_{14,16} + 32.8875 B_{15,16} + 37.9594 B_{16,16}
\end{aligned}
$$



The maximum difference of the Bézier coefficients is $\delta = 1.5947 \cdot 10^{-09}$.

Bounding polynomials M and m:

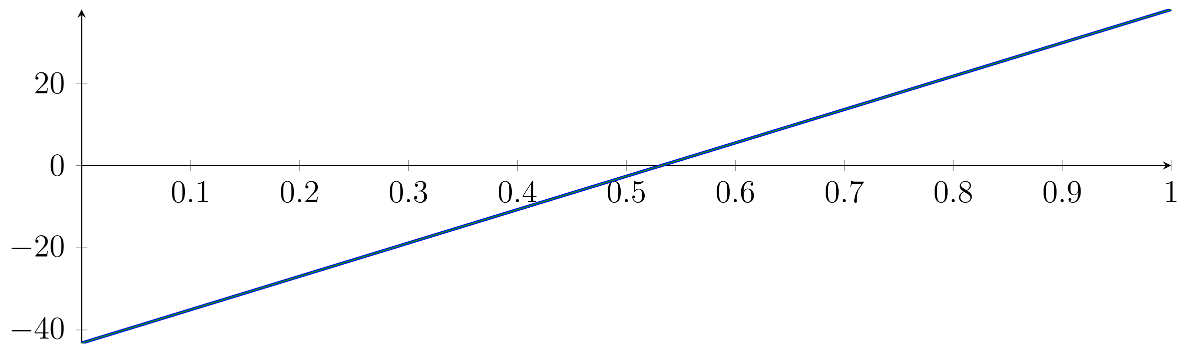
$$M = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

$$m = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

Root of M and m:

$$N(M) = \{0.532235, 2.62293 \cdot 10^{6}\} \qquad N(m) = \{0.532235, 2.62293 \cdot 10^{6}\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

Longest intersection interval: $3.93535 \cdot 10^{-11}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

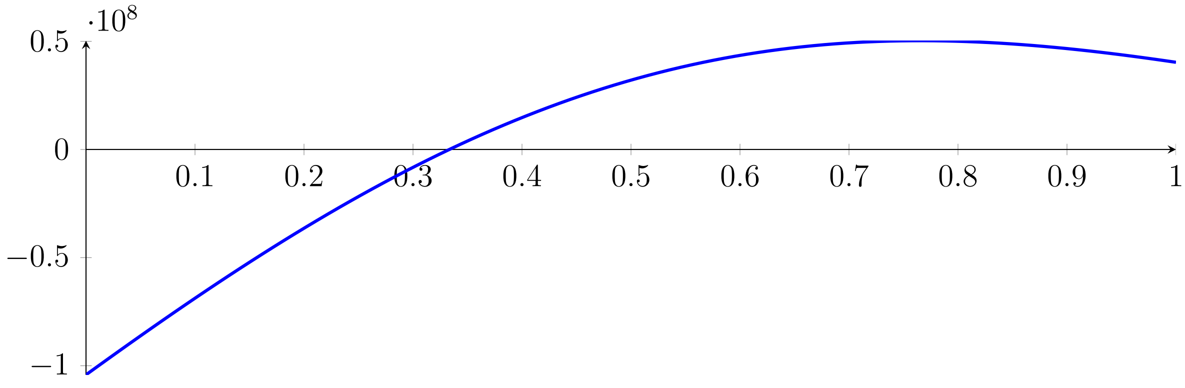
158.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

158.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

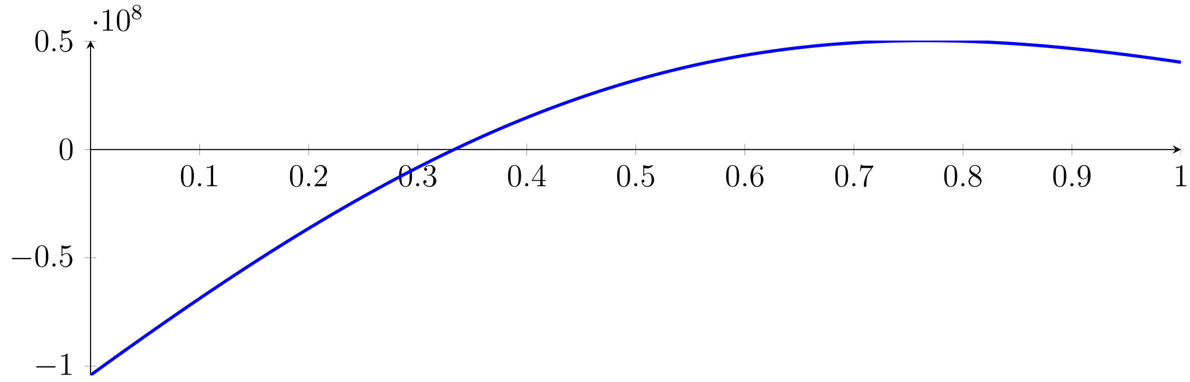
with precision $\varepsilon = 1\cdot10^{-16}$.

159 Running CubeClip on f_{16} with epsilon 16

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^6X^9 - 2.52262\cdot 10^6X^8 - 1.43506\cdot 10^7X^7 + 138542X^6 + 7.92823\cdot 10^7X^5 + 3.97396\cdot 10^7X^4 - 2.40625\cdot 10^8X^3 - 8.33333\cdot 10^7X^2 + 3.64583\cdot 10^8X - 1.04167\cdot 10^8$$

Called CubeClip with input polynomial on interval $[0, 1]$:

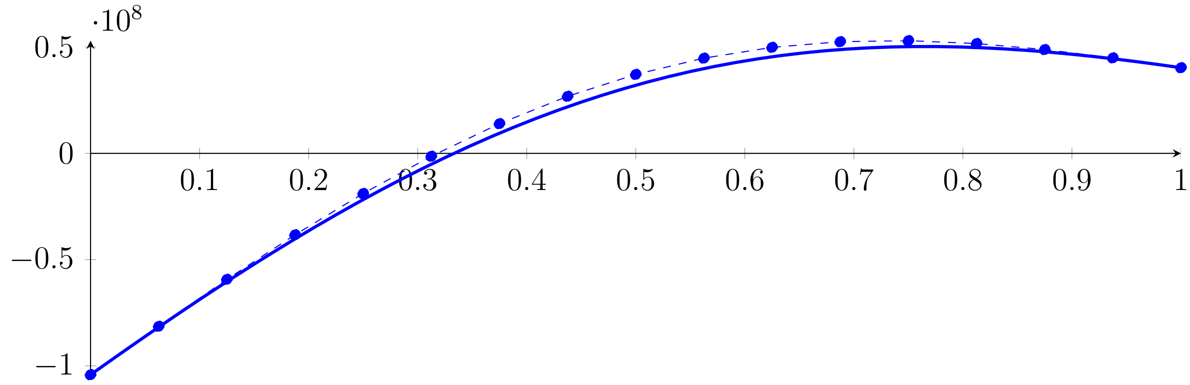
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^6X^9 - 2.52262\cdot 10^6X^8 - 1.43506\cdot 10^7X^7 + 138542X^6 + 7.92823\cdot 10^7X^5 + 3.97396\cdot 10^7X^4 - 2.40625\cdot 10^8X^3 - 8.33333\cdot 10^7X^2 + 3.64583\cdot 10^8X - 1.04167\cdot 10^8$$



159.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

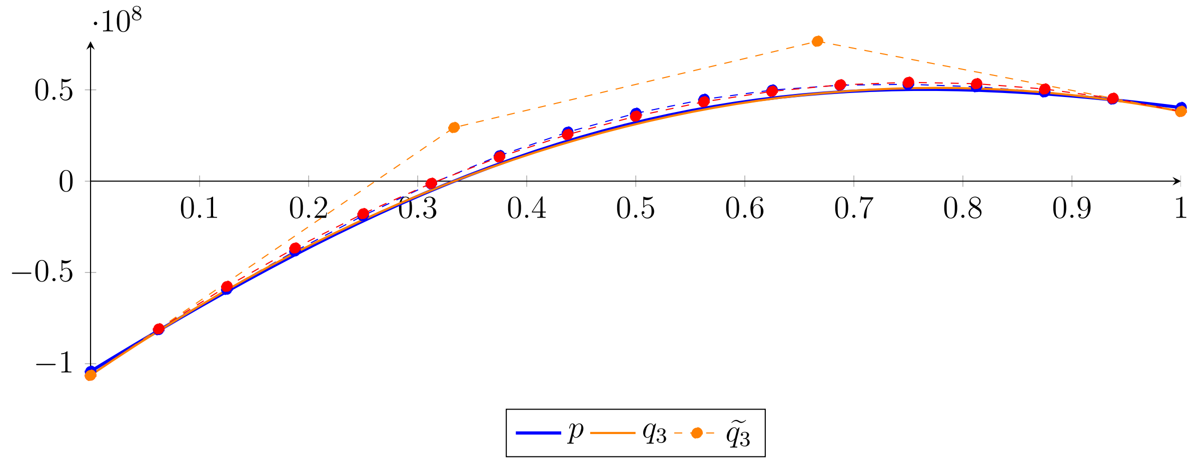
$$\begin{aligned} p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot 10^6X^9 - 2.52262\cdot 10^6X^8 - 1.43506\cdot 10^7X^7 + 138542X^6 + 7.92823 \\ &\quad \cdot 10^7X^5 + 3.97396\cdot 10^7X^4 - 2.40625\cdot 10^8X^3 - 8.33333\cdot 10^7X^2 + 3.64583\cdot 10^8X - 1.04167\cdot 10^8 \\ &= -1.04167\cdot 10^8B_{0,16}(X) - 8.13802\cdot 10^7B_{1,16}(X) - 5.92882\cdot 10^7B_{2,16}(X) - 3.83203 \\ &\quad \cdot 10^7B_{3,16}(X) - 1.88844\cdot 10^7B_{4,16}(X) - 1.34837\cdot 10^6B_{5,16}(X) + 1.39781\cdot 10^7B_{6,16}(X) \\ &\quad + 2.68604\cdot 10^7B_{7,16}(X) + 3.71532\cdot 10^7B_{8,16}(X) + 4.48105\cdot 10^7B_{9,16}(X) + 4.98901 \\ &\quad \cdot 10^7B_{10,16}(X) + 5.25504\cdot 10^7B_{11,16}(X) + 5.30407\cdot 10^7B_{12,16}(X) + 5.16832\cdot 10^7B_{13,16}(X) \\ &\quad + 4.88488\cdot 10^7B_{14,16}(X) + 4.49297\cdot 10^7B_{15,16}(X) + 4.03108\cdot 10^7B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.06335{\cdot}10^8 \\ &= -1.06335{\cdot}10^8B_{0,3} + 2.93558{\cdot}10^7B_{1,3} + 7.66068{\cdot}10^7B_{2,3} + 3.81764{\cdot}10^7B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2461.93X^{16} - 19614.9X^{15} + 70879.5X^{14} - 153661X^{13} + 222746X^{12} - 227755X^{11} \\ &\quad + 168826X^{10} - 91798.7X^9 + 36630.3X^8 - 10627.3X^7 + 2200.54X^6 - 316.059X^5 \\ &\quad + 30.1958X^4 + 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.06335{\cdot}10^8 \\ &= -1.06335{\cdot}10^8B_{0,16} - 8.08928{\cdot}10^7B_{1,16} - 5.76618{\cdot}10^7B_{2,16} - 3.66369{\cdot}10^7B_{3,16} - 1.78131 \\ &\quad {\cdot}10^7B_{4,16} - 1.18551{\cdot}10^6B_{5,16} + 1.32508{\cdot}10^7B_{6,16} + 2.55007{\cdot}10^7B_{7,16} + 3.55692{\cdot}10^7B_{8,16} \\ &\quad + 4.34611{\cdot}10^7B_{9,16} + 4.91815{\cdot}10^7B_{10,16} + 5.27353{\cdot}10^7B_{11,16} + 5.41273{\cdot}10^7B_{12,16} \\ &\quad + 5.33624{\cdot}10^7B_{13,16} + 5.04457{\cdot}10^7B_{14,16} + 4.53821{\cdot}10^7B_{15,16} + 3.81764{\cdot}10^7B_{16,16} \end{aligned}$$



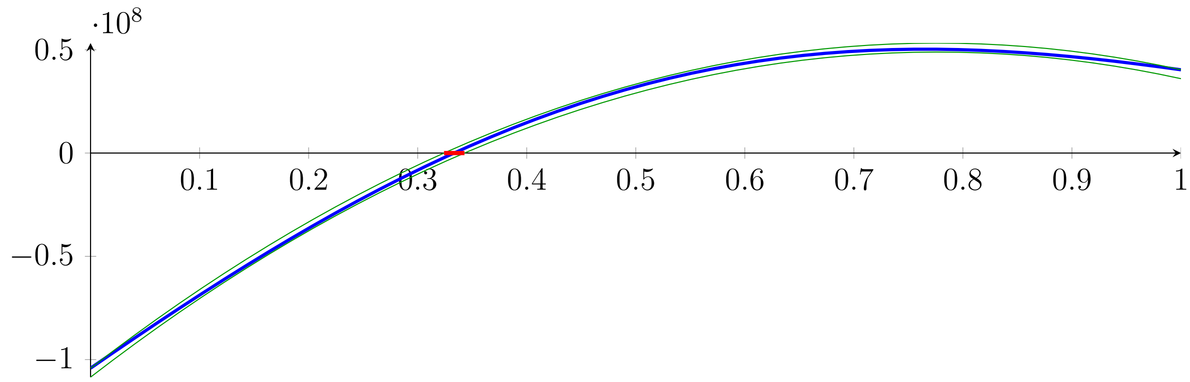
The maximum difference of the Bézier coefficients is $\delta = 2.16806{\cdot}10^6$.
Bounding polynomials M and m:

$$M = 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.04167{\cdot}10^8$$
$$m = 2.75812{\cdot}10^6X^3 - 2.65319{\cdot}10^8X^2 + 4.07072{\cdot}10^8X - 1.08503{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



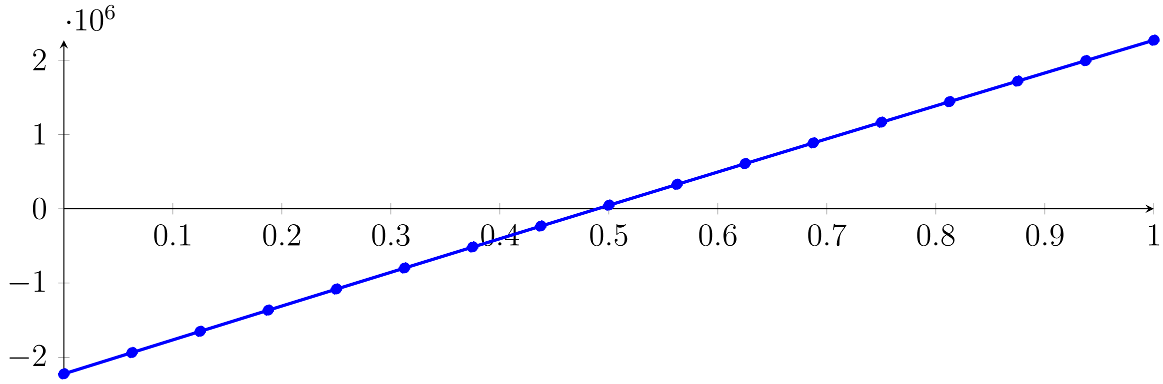
$$[0.324143, 0.342913]$$

Longest intersection interval: 0.0187703
$\Longrightarrow$ Selective recursion: interval 1: [0.324143, 0.342913],

159.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

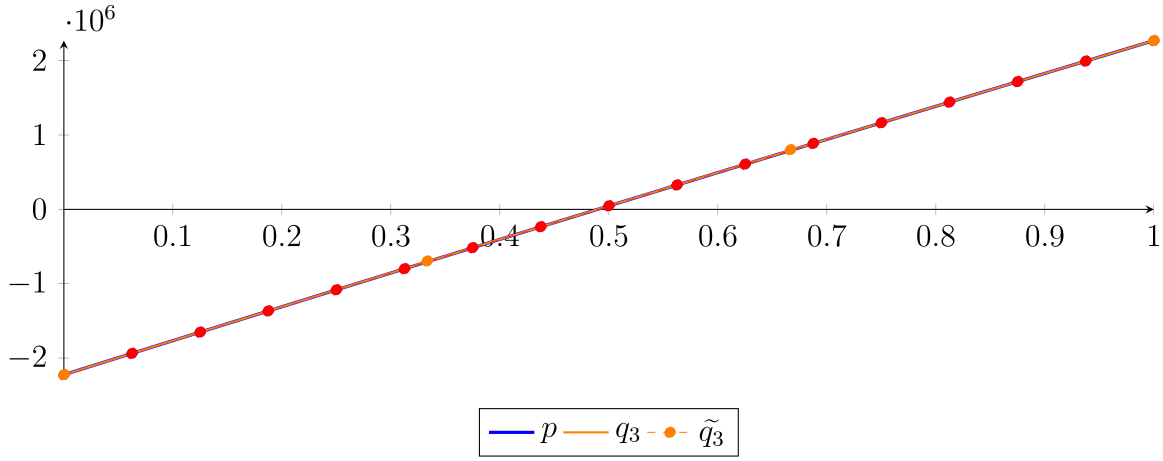
$$p = -3.66617\cdot10^{-09}X^{16} - 1.53217\cdot10^{-07}X^{15} - 3.62234\cdot10^{-07}X^{14} - 1.65579\cdot10^{-06}X^{13} - 1.15373\cdot10^{-06}X^{12} - 2.3399\cdot10^{-06}X^{11} - 5.02543\cdot10^{-07}X^{10} - 1.38381\cdot10^{-06}X^{9} + 1.1237\cdot10^{-06}X^{8} - 1.19653\cdot10^{-05}X^{7} - 0.00155608X^{6} + 0.10496X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,16}(X) - 1.93666\cdot10^{6}B_{1,16}(X) - 1.65074\cdot10^{6}B_{2,16}(X) - 1.36561\cdot10^{6}B_{3,16}(X) - 1.08127\cdot10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X) + 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot10^{6}B_{12,16}(X) + 1.44233\cdot10^{6}B_{13,16}(X) + 1.71876\cdot10^{6}B_{14,16}(X) + 1.99439\cdot10^{6}B_{15,16}(X) + 2.26922\cdot10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot10^{6}B_{3,3}$$

$$\tilde{q}_3 = 16.4956X^{16} - 129.161X^{15} + 457.83X^{14} - 971.671X^{13} + 1375.95X^{12} - 1370.96X^{11} + 987.265X^{10} - 519.476X^{9} + 199.587X^{8} - 55.434X^{7} + 10.9237X^{6} - 1.48019X^{5} + 0.129516X^{4} - 700.679X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,16} - 1.93666\cdot10^{6}B_{1,16} - 1.65074\cdot10^{6}B_{2,16} - 1.36561\cdot10^{6}B_{3,16} - 1.08126\cdot10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.4B_{8,16} + 328650B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot10^{6}B_{12,16} + 1.44233\cdot10^{6}B_{13,16} + 1.71876\cdot10^{6}B_{14,16} + 1.99439\cdot10^{6}B_{15,16} + 2.26922\cdot10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m:

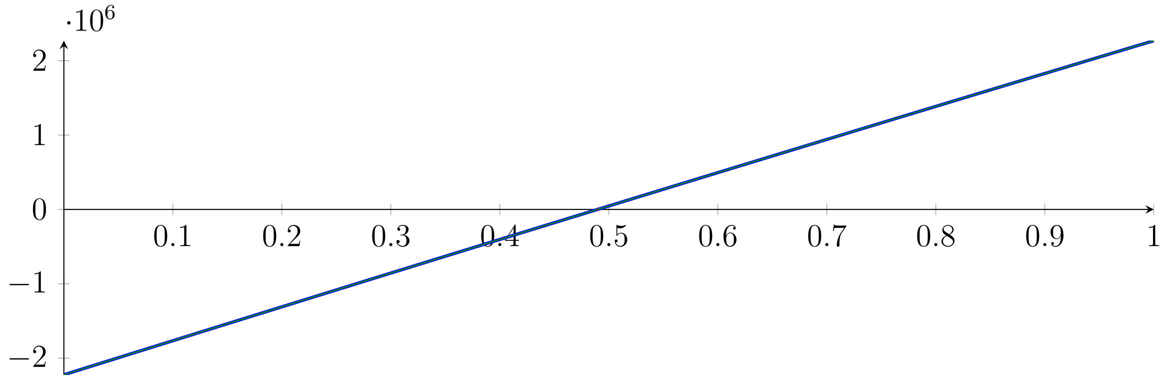
$$M = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715\cdot10^6X - 2.22335\cdot10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



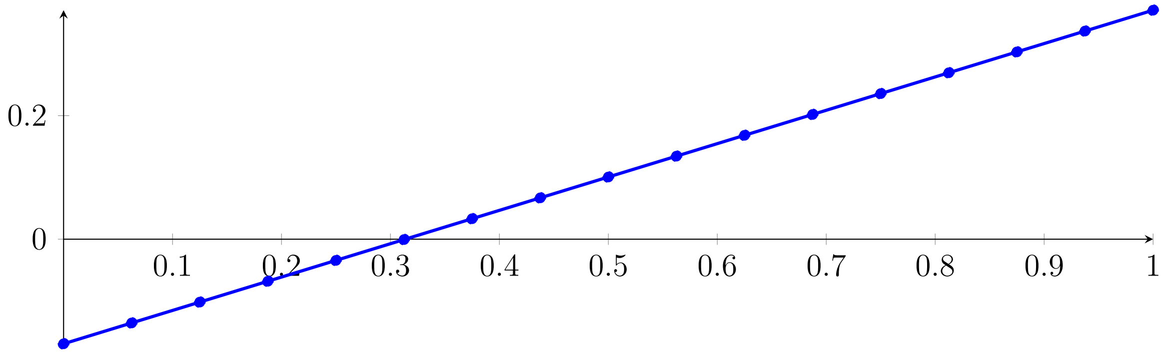
$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174\cdot10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

159.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

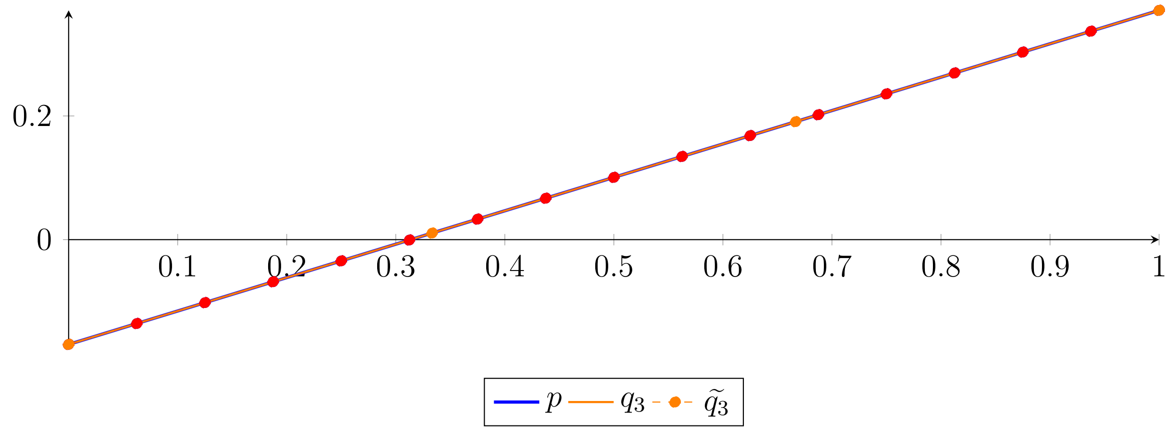
$$\begin{aligned} p = &5.55524\cdot10^{-15}X^{16} - 2.94313\cdot10^{-14}X^{15} + 1.19384\cdot10^{-13}X^{14} - 2.17482\cdot10^{-13}X^{13} + 7.26155\cdot10^{-14}X^{12} \\ &- 3.44766\cdot10^{-13}X^{11} - 3.47292\cdot10^{-15}X^{10} - 1.1287\cdot10^{-13}X^9 + 2.93027\cdot10^{-14}X^8 + 8.06213\cdot10^{-15}X^7 \\ &+ 5.64349\cdot10^{-15}X^6 + 4.93312\cdot10^{-17}X^4 + 1.51788\cdot10^{-17}X^3 - 1.37072\cdot10^{-09}X^2 + 0.540148X - 0.169396 \\ = &-0.169396B_{0,16}(X) - 0.135637B_{1,16}(X) - 0.101877B_{2,16}(X) - 0.068118B_{3,16}(X) \\ &- 0.0343587B_{4,16}(X) - 0.000599476B_{5,16}(X) + 0.0331598B_{6,16}(X) \\ &+ 0.0669191B_{7,16}(X) + 0.100678B_{8,16}(X) + 0.134438B_{9,16}(X) + 0.168197B_{10,16}(X) \\ &+ 0.201956B_{11,16}(X) + 0.235715B_{12,16}(X) + 0.269475B_{13,16}(X) \\ &+ 0.303234B_{14,16}(X) + 0.336993B_{15,16}(X) + 0.370752B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07065\cdot10^{-17}X^3 - 1.37072\cdot10^{-09}X^2 + 0.540148X - 0.169396 \\ &= -0.169396B_{0,3} + 0.0106536B_{1,3} + 0.190703B_{2,3} + 0.370752B_{3,3} \end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 = & 8.59095{\cdot}10^{-06}X^{16} - 6.82648{\cdot}10^{-05}X^{15} + 0.000245968X^{14} - 0.000531568X^{13}\\ & + 0.000767923X^{12} - 0.000782231X^{11} + 0.0005774X^{10} - 0.000312464X^{9}\\ & + 0.000123994X^{8} - 3.57388{\cdot}10^{-05}X^{7} + 7.34249{\cdot}10^{-06}X^{6} - 1.04474{\cdot}10^{-06}X^{5}\\ & + 9.86739{\cdot}10^{-08}X^{4} - 5.7553{\cdot}10^{-09}X^{3} - 1.19186{\cdot}10^{-09}X^{2} + 0.540148X - 0.169396\\ = & -0.169396B_{0,16} - 0.135637B_{1,16} - 0.101877B_{2,16} - 0.068118B_{3,16} - 0.0343587B_{4,16}\\ & - 0.000599476B_{5,16} + 0.0331598B_{6,16} + 0.0669191B_{7,16} + 0.100678B_{8,16}\\ & + 0.134438B_{9,16} + 0.168197B_{10,16} + 0.201956B_{11,16} + 0.235715B_{12,16}\\ & + 0.269475B_{13,16} + 0.303234B_{14,16} + 0.336993B_{15,16} + 0.370752B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.81206{\cdot}10^{-10}$.

Bounding polynomials M and m:

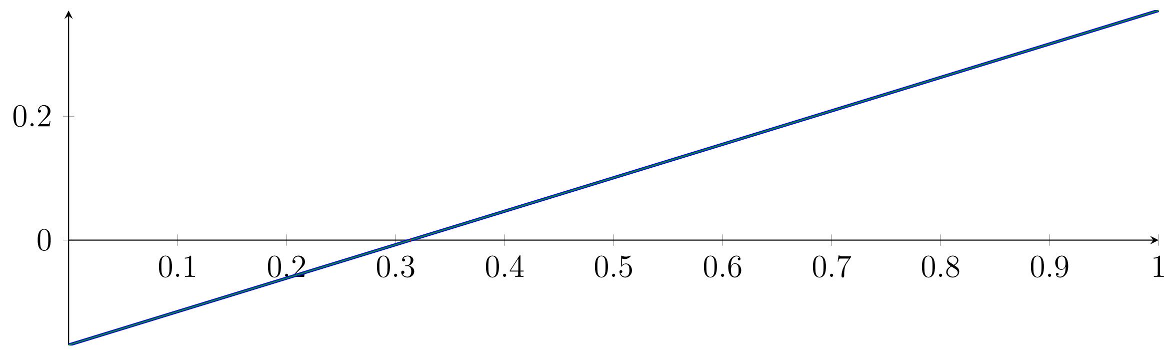
$$M = -1.07065{\cdot}10^{-17}X^{3} - 1.37072{\cdot}10^{-09}X^{2} + 0.540148X - 0.169396$$

$$m = -1.07065{\cdot}10^{-17}X^{3} - 1.37072{\cdot}10^{-09}X^{2} + 0.540148X - 0.169396$$

Root of M and m:

$$N(M) = \{-2.97569{\cdot}10^{8}, 0.31361, 1.69542{\cdot}10^{8}\} \quad N(m) = \{-2.97569{\cdot}10^{8}, 0.31361, 1.69542{\cdot}10^{8}\}$$

Intersection intervals:



$$[0.31361, 0.31361]$$

Longest intersection interval: $7.85803{\cdot}10^{-10}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

159.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

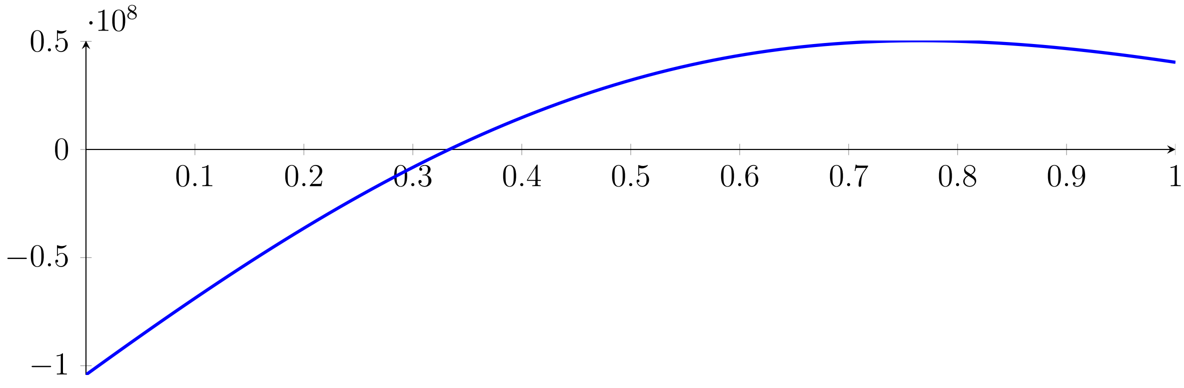
Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 4!

p(0) = -2.39831e-08 - p(1) -2.35587e-08

159.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

with precision $\varepsilon = 1\cdot10^{-16}$.

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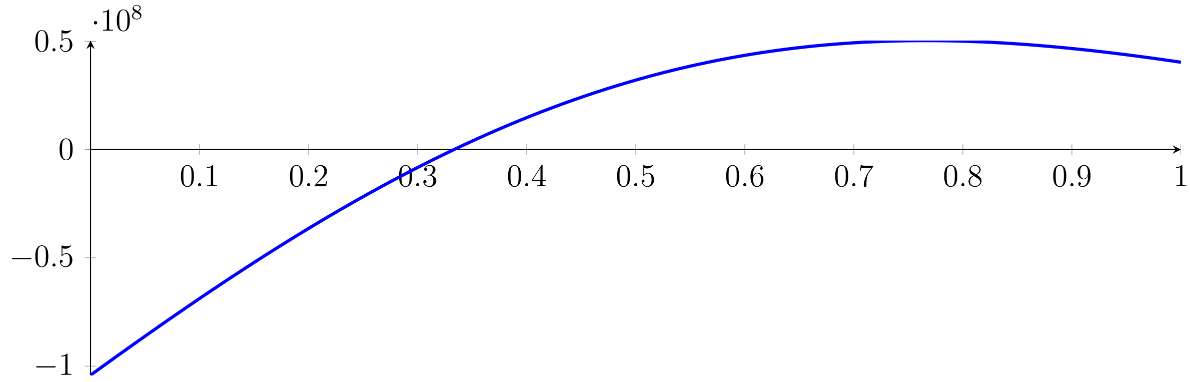
$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8$$

Called BezClip with input polynomial on interval $[0,1]$:

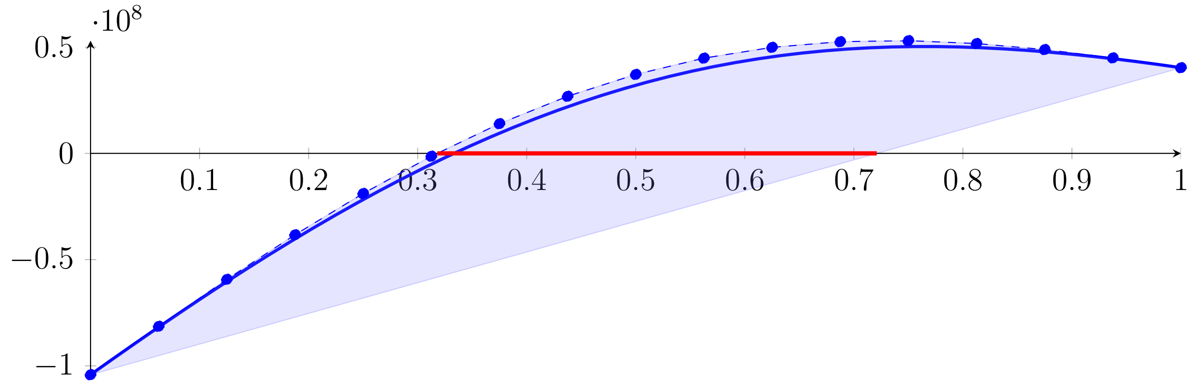
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8$$



160.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\ &\quad \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\ &= -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\ &\quad \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\ &\quad + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\ &\quad \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

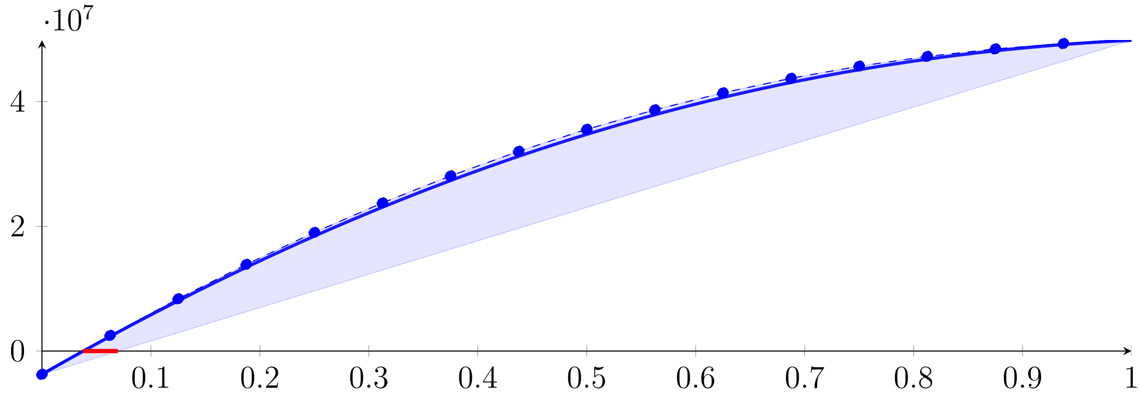
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

160.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 1.59825 \cdot 10^{-06} X^{16} - 5.93153 \cdot 10^{-05} X^{15} - 0.00248867 X^{14} - 0.0650056 X^{13} - 0.909142 X^{12} \\ &\quad - 5.03931 X^{11} + 36.4549 X^{10} + 692.921 X^{9} + 1886.97 X^{8} - 25792 X^{7} - 149671 X^{6} + 492605 X^{5} \\ &\quad + 3.91945 \cdot 10^{6} X^{4} - 7.54561 \cdot 10^{6} X^{3} - 4.29225 \cdot 10^{7} X^{2} + 9.97982 \cdot 10^{7} X - 3.73501 \cdot 10^{6} \\ &= -3.73501 \cdot 10^{6} B_{0,16}(X) + 2.50237 \cdot 10^{6} B_{1,16}(X) + 8.38207 \cdot 10^{6} B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^{7} B_{3,16}(X) + 1.90167 \cdot 10^{7} B_{4,16}(X) + 2.37512 \cdot 10^{7} B_{5,16}(X) + 2.80875 \cdot 10^{7} B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^{7} B_{7,16}(X) + 3.55508 \cdot 10^{7} B_{8,16}(X) + 3.86768 \cdot 10^{7} B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^{7} B_{10,16}(X) + 4.37336 \cdot 10^{7} B_{11,16}(X) + 4.56783 \cdot 10^{7} B_{12,16}(X) + 4.7247 \cdot 10^{7} B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^{7} B_{14,16}(X) + 4.93093 \cdot 10^{7} B_{15,16}(X) + 4.98343 \cdot 10^{7} B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

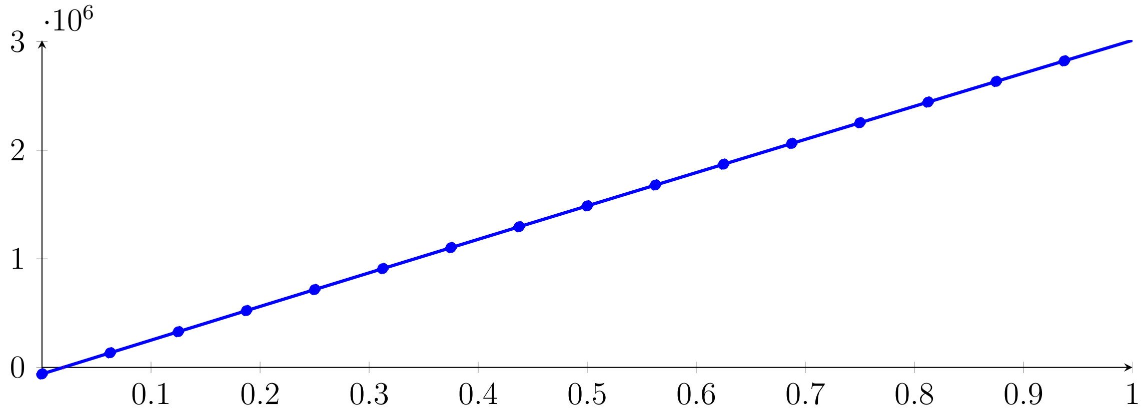
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

160.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 9.01396 \cdot 10^{-08} X^{16} - 2.65848 \cdot 10^{-07} X^{15} + 2.13948 \cdot 10^{-06} X^{14} - 1.33627 \cdot 10^{-06} X^{13} + 2.46973 \cdot 10^{-06} X^{12} \\ &\quad - 2.45524 \cdot 10^{-06} X^{11} + 5.50112 \cdot 10^{-07} X^{10} - 1.64198 \cdot 10^{-07} X^{9} - 7.35598 \cdot 10^{-07} X^{8} - 1.00892 \cdot 10^{-06} X^{7} \\ &\quad - 0.000177509 X^{6} + 0.0161038 X^{5} + 4.36155 X^{4} - 234.216 X^{3} - 45622.2 X^{2} + 3.11845 \cdot 10^{6} X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^{6} B_{6,16}(X) + 1.29582 \cdot 10^{6} B_{7,16}(X) + 1.48805 \cdot 10^{6} B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^{6} B_{9,16}(X) + 1.87137 \cdot 10^{6} B_{10,16}(X) + 2.06245 \cdot 10^{6} B_{11,16}(X) + 2.25315 \cdot 10^{6} B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^{6} B_{13,16}(X) + 2.63339 \cdot 10^{6} B_{14,16}(X) + 2.82293 \cdot 10^{6} B_{15,16}(X) + 3.01209 \cdot 10^{6} B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

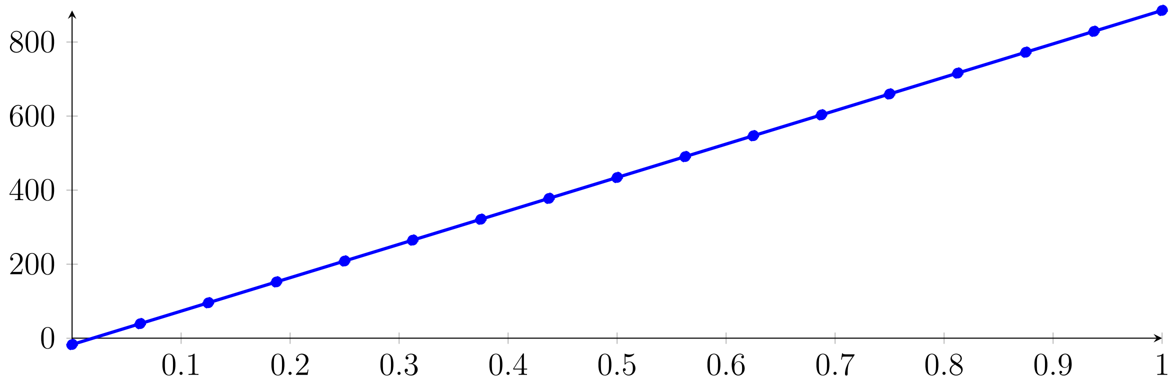
$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

160.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
p &= 2.55372 \cdot 10^{-11} X^{16} - 7.21263 \cdot 10^{-11} X^{15} + 6.24141 \cdot 10^{-10} X^{14} - 4.11162 \cdot 10^{-10} X^{13} \\
&\quad + 6.82359 \cdot 10^{-10} X^{12} - 7.09475 \cdot 10^{-10} X^{11} + 9.71305 \cdot 10^{-11} X^{10} - 3.46101 \cdot 10^{-11} X^{9} \\
&\quad - 2.13971 \cdot 10^{-10} X^{8} - 1.46061 \cdot 10^{-11} X^{7} - 1.63366 \cdot 10^{-11} X^{6} + 1.87916 \cdot 10^{-12} X^{5} \\
&\quad + 2.52576 \cdot 10^{-14} X^{4} - 5.67777 \cdot 10^{-09} X^{3} - 0.00382618 X^{2} + 902.448 X - 17.178 \\
&= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
&\quad + 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
&\quad + 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
&\quad + 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

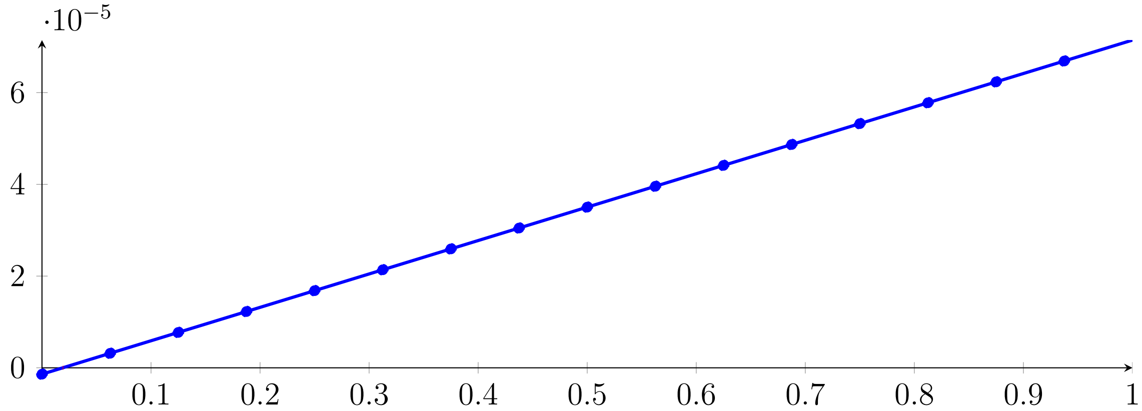
$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045 \cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

160.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 2.14261\cdot 10^{-18}X^{16} - 6.28573\cdot 10^{-18}X^{15} + 5.28612\cdot 10^{-17}X^{14} - 3.67279\cdot 10^{-17}X^{13} \\ &\quad + 6.10136\cdot 10^{-17}X^{12} - 6.60335\cdot 10^{-17}X^{11} + 1.66661\cdot 10^{-17}X^{10} - 8.36524\cdot 10^{-18}X^{9} \\ &\quad - 1.56919\cdot 10^{-17}X^{8} - 1.85474\cdot 10^{-18}X^{7} - 1.4308\cdot 10^{-18}X^{6} + 1.1562\cdot 10^{-19}X^{5} - 1.20437 \\ &\quad \cdot 10^{-20}X^{4} - 4.63221\cdot 10^{-22}X^{3} - 2.49207\cdot 10^{-17}X^{2} + 7.28316\cdot 10^{-05}X - 1.38634\cdot 10^{-06} \\ &= -1.38634\cdot 10^{-06}B_{0,16}(X) + 3.16564\cdot 10^{-06}B_{1,16}(X) + 7.71761\cdot 10^{-06}B_{2,16}(X) + 1.22696 \\ &\quad \cdot 10^{-05}B_{3,16}(X) + 1.68216\cdot 10^{-05}B_{4,16}(X) + 2.13735\cdot 10^{-05}B_{5,16}(X) + 2.59255\cdot 10^{-05}B_{6,16}(X) \\ &\quad + 3.04775\cdot 10^{-05}B_{7,16}(X) + 3.50295\cdot 10^{-05}B_{8,16}(X) + 3.95814\cdot 10^{-05}B_{9,16}(X) + 4.41334 \\ &\quad \cdot 10^{-05}B_{10,16}(X) + 4.86854\cdot 10^{-05}B_{11,16}(X) + 5.32374\cdot 10^{-05}B_{12,16}(X) + 5.77893\cdot 10^{-05}B_{13,16}(X) \\ &\quad + 6.23413\cdot 10^{-05}B_{14,16}(X) + 6.68933\cdot 10^{-05}B_{15,16}(X) + 7.14453\cdot 10^{-05}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.51313\cdot 10^{-15}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

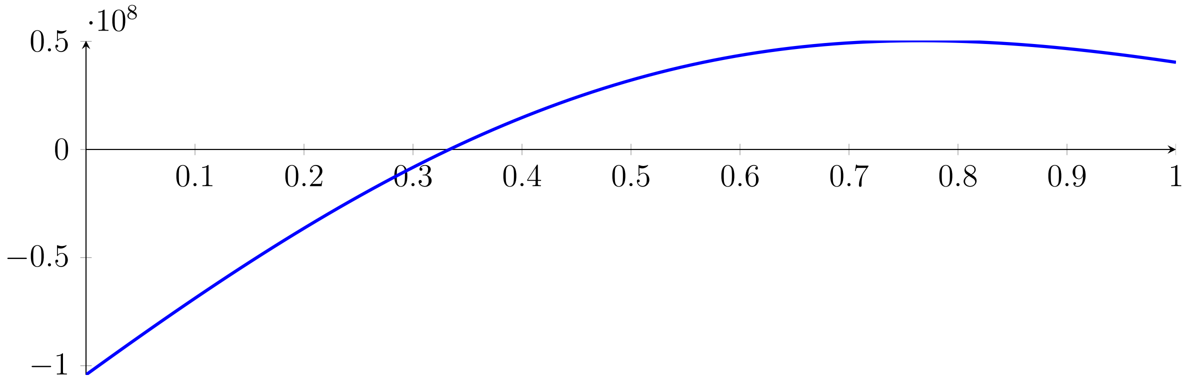
160.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

160.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1\cdot10^{-32}$.

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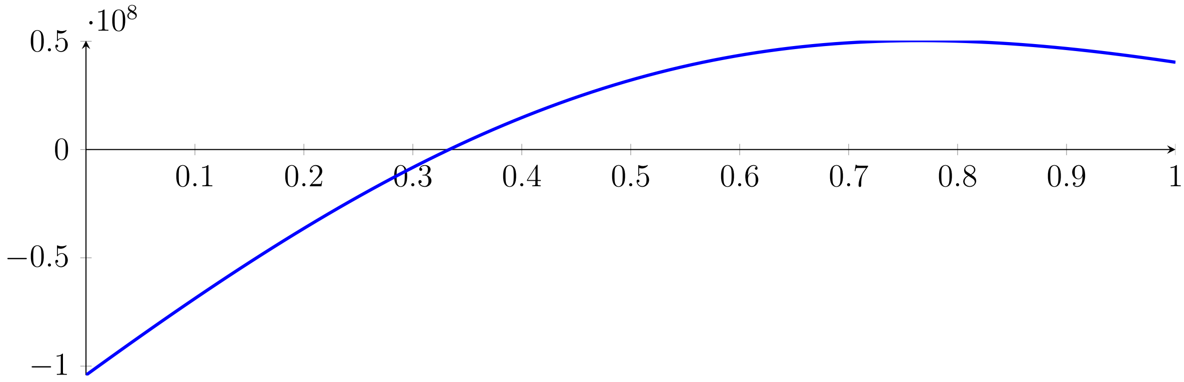
$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called QuadClip with input polynomial on interval $[0,1]$:

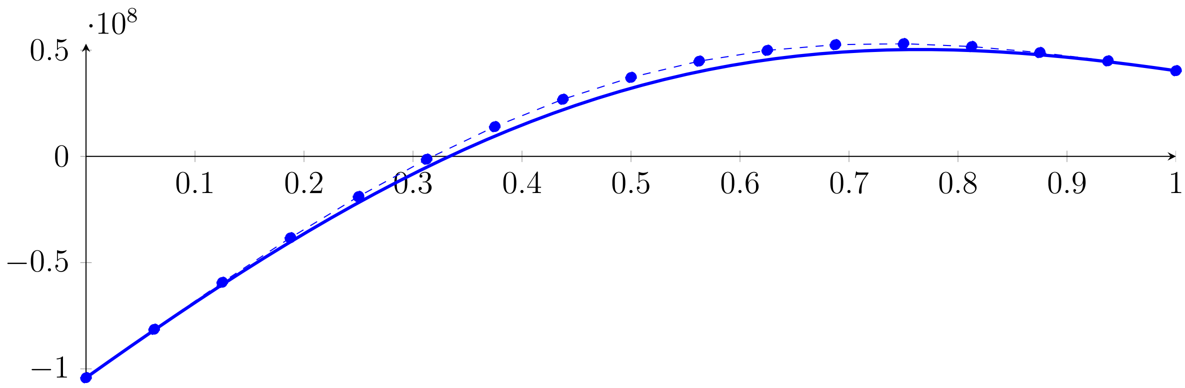
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



161.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

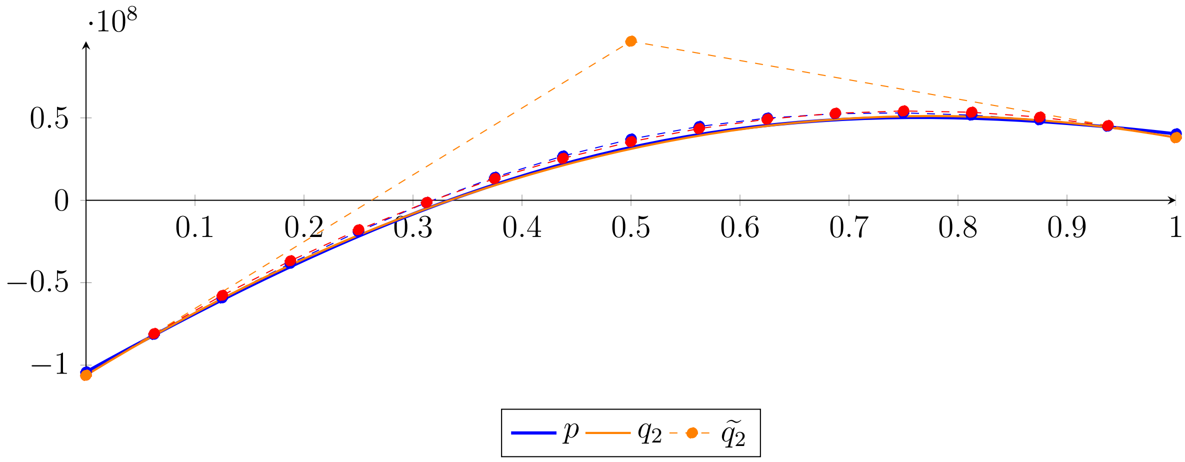
$$\begin{aligned} p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\quad \cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \\ &= -1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -2.61181\cdot10^8X^2 + 4.05417\cdot10^8X - 1.06197\cdot10^8 \\ &= -1.06197\cdot10^8B_{0,2} + 9.65115\cdot10^7B_{1,2} + 3.80384\cdot10^7B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 = {} & 6049.18X^{16} - 48305.2X^{15} + 174971X^{14} - 380294X^{13} + 552846X^{12} - 567203X^{11} \\ & + 422303X^{10} - 231038X^9 + 93003.6X^8 - 27320.1X^7 + 5752.57X^6 - 843.63X^5 \\ & + 82.5145X^4 - 5.01388X^3 - 2.61181\cdot10^8X^2 + 4.05417\cdot10^8X - 1.06197\cdot10^8 \\ = {} & -1.06197\cdot10^8B_{0,16} - 8.08583\cdot10^7B_{1,16} - 5.76963\cdot10^7B_{2,16} - 3.67107\cdot10^7B_{3,16} - 1.79017 \\ & \cdot10^7B_{4,16} - 1.26924\cdot10^6B_{5,16} + 1.31867\cdot10^7B_{6,16} + 2.54662\cdot10^7B_{7,16} + 3.55692\cdot10^7B_{8,16} \\ & + 4.34956\cdot10^7B_{9,16} + 4.92456\cdot10^7B_{10,16} + 5.2819\cdot10^7B_{11,16} + 5.42159\cdot10^7B_{12,16} \\ & + 5.34363\cdot10^7B_{13,16} + 5.04802\cdot10^7B_{14,16} + 4.53476\cdot10^7B_{15,16} + 3.80384\cdot10^7B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233\cdot10^6$.
Bounding polynomials M and m:

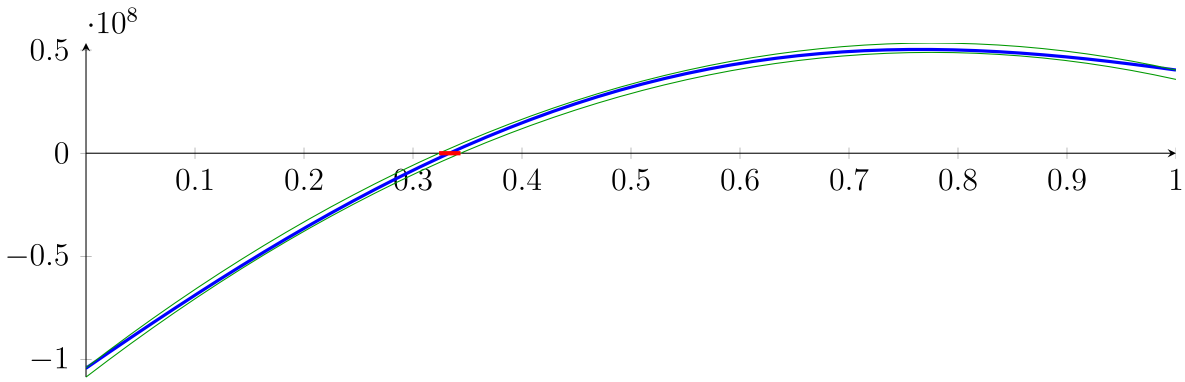
$$M = -2.61181\cdot10^8X^2 + 4.05417\cdot10^8X - 1.03924\cdot10^8$$

$$m = -2.61181\cdot10^8X^2 + 4.05417\cdot10^8X - 1.08469\cdot10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



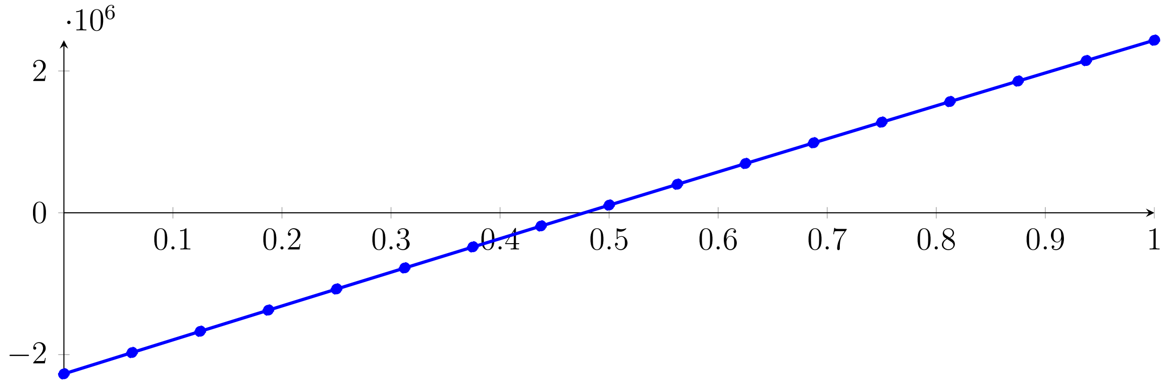
$$[0.323946, 0.343615]$$

Longest intersection interval: 0.0196686
$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

161.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

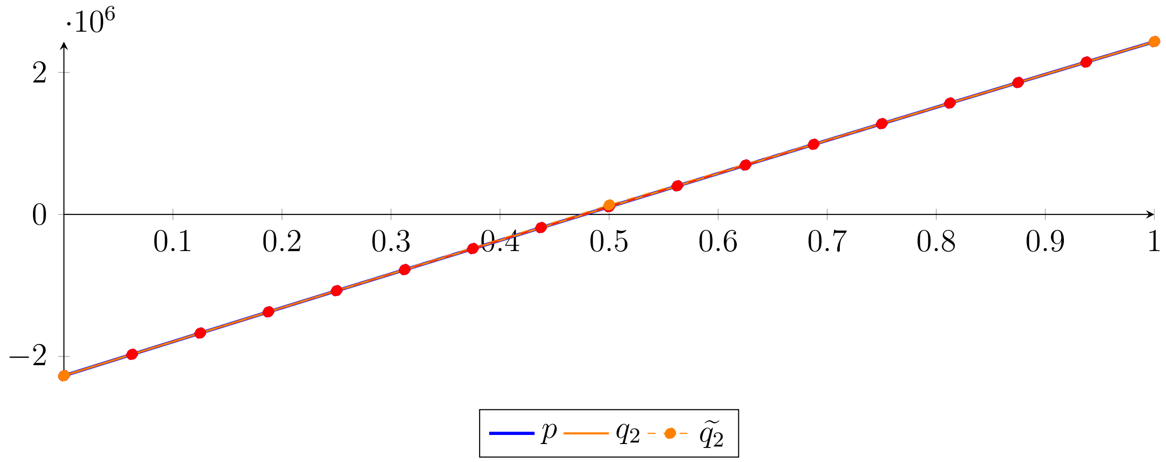
$$p = -1.54841\cdot 10^{-10}X^{16} - 1.66965\cdot 10^{-07}X^{15} - 2.92739\cdot 10^{-07}X^{14} - 1.77943\cdot 10^{-06}X^{13} - 1.17235\cdot 10^{-06}X^{12} - 2.42234\cdot 10^{-06}X^{11} - 6.86445\cdot 10^{-07}X^{10} - 1.39162\cdot 10^{-06}X^{9} + 1.07395\cdot 10^{-06}X^{8} - 1.67072\cdot 10^{-05}X^{7} - 0.00205879X^{6} + 0.132721X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874\cdot 10^{6}X - 2.2715\cdot 10^{6}$$
$$= -2.2715\cdot 10^{6}B_{0,16}(X) - 1.97096\cdot 10^{6}B_{1,16}(X) - 1.67127\cdot 10^{6}B_{2,16}(X) - 1.37244\cdot 10^{6}B_{3,16}(X) - 1.07448\cdot 10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806\cdot 10^{6}B_{12,16}(X) + 1.5682\cdot 10^{6}B_{13,16}(X) + 1.85747\cdot 10^{6}B_{14,16}(X) + 2.14586\cdot 10^{6}B_{15,16}(X) + 2.43338\cdot 10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265X^{2} + 4.80923\cdot 10^{6}X - 2.27154\cdot 10^{6}$$
$$= -2.27154\cdot 10^{6}B_{0,2} + 133069B_{1,2} + 2.43342\cdot 10^{6}B_{2,2}$$

$$\tilde{q}_2 = 22.7036X^{16} - 178.964X^{15} + 638.976X^{14} - 1366.64X^{13} + 1951.1X^{12} - 1960.92X^{11} + 1425.35X^{10} - 757.631X^{9} + 294.177X^{8} - 82.4368X^{7} + 16.2856X^{6} - 2.18949X^{5} + 0.191237X^{4} - 0.0101048X^{3} - 104265X^{2} + 4.80923\cdot 10^{6}X - 2.27154\cdot 10^{6}$$
$$= -2.27154\cdot 10^{6}B_{0,16} - 1.97097\cdot 10^{6}B_{1,16} - 1.67126\cdot 10^{6}B_{2,16} - 1.37242\cdot 10^{6}B_{3,16} - 1.07445\cdot 10^{6}B_{4,16} - 777350B_{5,16} - 481118B_{6,16} - 185754B_{7,16} + 108740B_{8,16} + 402366B_{9,16} + 695123B_{10,16} + 987011B_{11,16} + 1.27803\cdot 10^{6}B_{12,16} + 1.56818\cdot 10^{6}B_{13,16} + 1.85746\cdot 10^{6}B_{14,16} + 2.14587\cdot 10^{6}B_{15,16} + 2.43342\cdot 10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m:

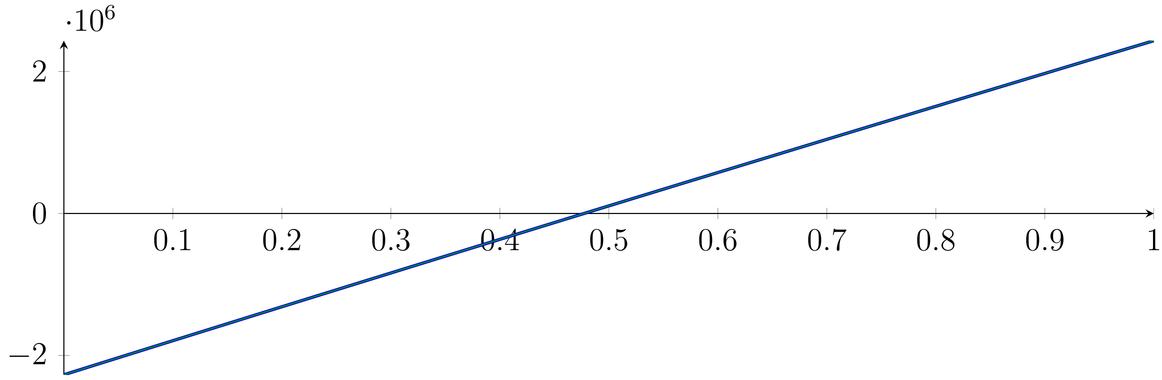
$$M = -104265X^{2} + 4.80923\cdot 10^{6}X - 2.2715\cdot 10^{6}$$

$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



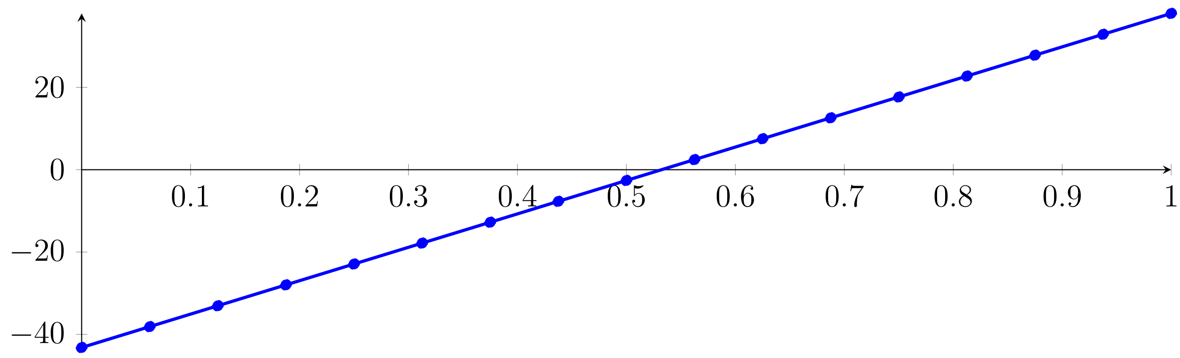
$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

161.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

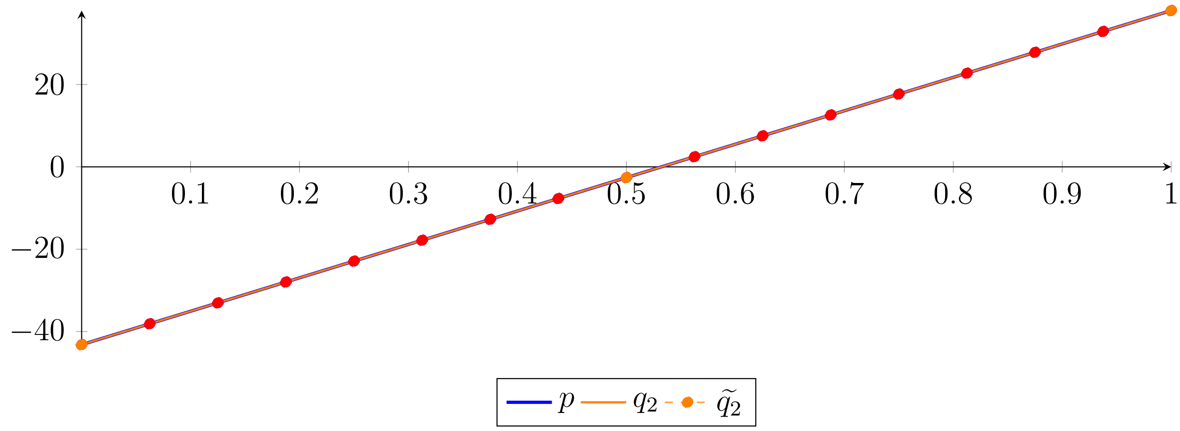
$$\begin{aligned} p = & -2.76723 \cdot 10^{-13} X^{16} - 2.40874 \cdot 10^{-12} X^{15} - 1.25233 \cdot 10^{-11} X^{14} - 3.02935 \cdot 10^{-11} X^{13} \\ & - 3.05617 \cdot 10^{-11} X^{12} - 3.83107 \cdot 10^{-11} X^{11} - 1.26692 \cdot 10^{-11} X^{10} - 2.6672 \cdot 10^{-11} X^9 \\ & + 2.0004 \cdot 10^{-11} X^8 + 4.12781 \cdot 10^{-12} X^7 + 2.44493 \cdot 10^{-12} X^6 - 1.21236 \cdot 10^{-13} X^5 \\ & + 1.26288 \cdot 10^{-14} X^4 - 4.1267 \cdot 10^{-12} X^3 - 3.09388 \cdot 10^{-05} X^2 + 81.1506X - 43.1911 \\ = & -43.1911 B_{0,16}(X) - 38.1192 B_{1,16}(X) - 33.0473 B_{2,16}(X) - 27.9754 B_{3,16}(X) - 22.9035 B_{4,16}(X) \\ & - 17.8316 B_{5,16}(X) - 12.7597 B_{6,16}(X) - 7.68778 B_{7,16}(X) - 2.61587 B_{8,16}(X) \\ & + 2.45604 B_{9,16}(X) + 7.52795 B_{10,16}(X) + 12.5999 B_{11,16}(X) + 17.6718 B_{12,16}(X) \\ & + 22.7437 B_{13,16}(X) + 27.8156 B_{14,16}(X) + 32.8875 B_{15,16}(X) + 37.9594 B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911 \\ &= -43.1911 B_{0,2} - 2.61586 B_{1,2} + 37.9594 B_{2,2} \end{aligned}$$

$$\begin{aligned}\tilde{q}_2 &= 5.96265{\cdot}10^{-05}X^{16} - 0.000436042X^{15} + 0.00141812X^{14} - 0.00269475X^{13}\\ &\quad + 0.00329809X^{12} - 0.00268757X^{11} + 0.00143268X^{10} - 0.000439599X^{9}\\ &\quad + 1.98418{\cdot}10^{-05}X^{8} + 4.87608{\cdot}10^{-05}X^{7} - 2.46333{\cdot}10^{-05}X^{6} + 6.35808{\cdot}10^{-06}X^{5}\\ &\quad - 9.62755{\cdot}10^{-07}X^{4} + 8.21372{\cdot}10^{-08}X^{3} - 3.09429{\cdot}10^{-05}X^{2} + 81.1506X - 43.1911\\ &= -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16}\\ &\quad - 12.7597B_{6,16} - 7.68778B_{7,16} - 2.61587B_{8,16} + 2.45604B_{9,16} + 7.52795B_{10,16} + 12.5999B_{11,16}\\ &\quad + 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.5947{\cdot}10^{-09}$.

Bounding polynomials M and m:

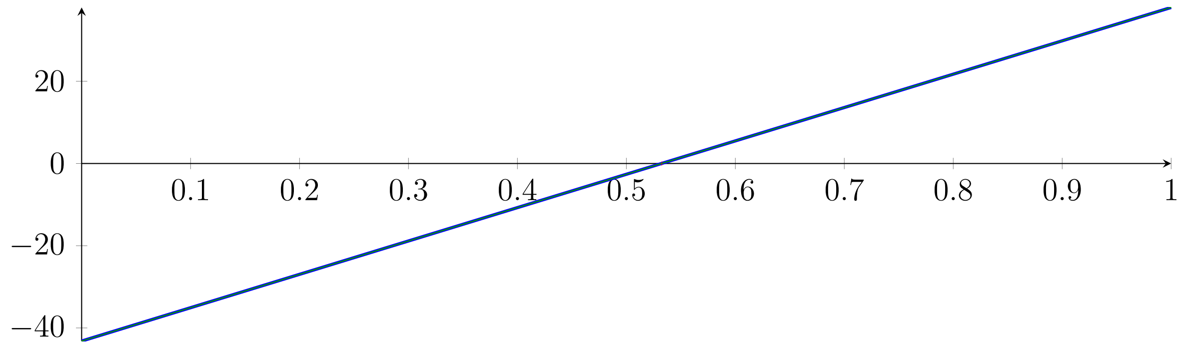
$$M = -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911$$

$$m = -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911$$

Root of M and m:

$$N(M) = \{0.532235, 2.62293{\cdot}10^{6}\} \qquad N(m) = \{0.532235, 2.62293{\cdot}10^{6}\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

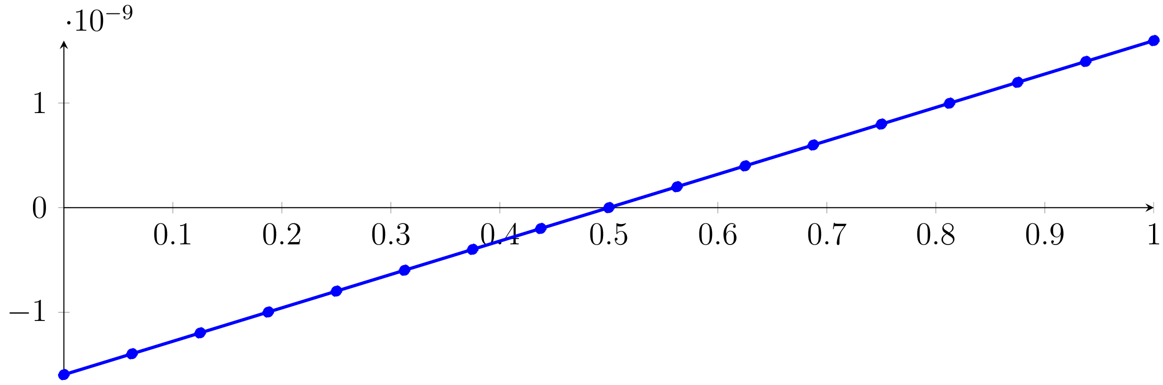
Longest intersection interval: $3.93535{\cdot}10^{-11}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

161.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

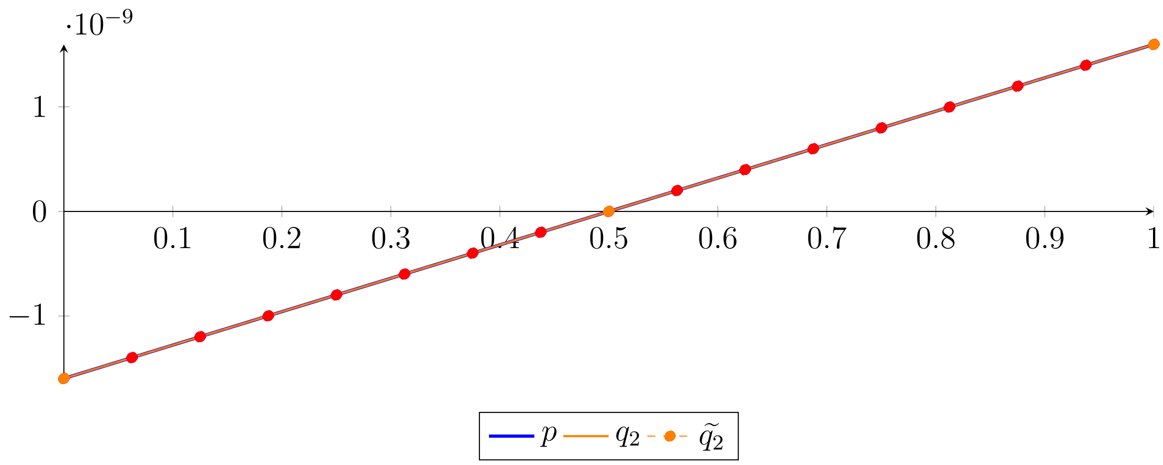
$$p = -4.89361 \cdot 10^{-24} X^{16} - 1.05466 \cdot 10^{-22} X^{15} - 3.09805 \cdot 10^{-22} X^{14} - 1.16981 \cdot 10^{-21} X^{13} - 9.76202 \cdot 10^{-22} X^{12} - 1.69365 \cdot 10^{-21} X^{11} - 5.95131 \cdot 10^{-22} X^{10} - 1.05349 \cdot 10^{-21} X^{9} + 7.5893 \cdot 10^{-22} X^{8} + 1.47859 \cdot 10^{-22} X^{7} + 9.70322 \cdot 10^{-23} X^{6} - 7.05688 \cdot 10^{-24} X^{5} + 1.47018 \cdot 10^{-24} X^{4} - 4.84676 \cdot 10^{-26} X^{2} + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$
$$= -1.59674 \cdot 10^{-09} B_{0,16}(X) - 1.39715 \cdot 10^{-09} B_{1,16}(X) - 1.19755 \cdot 10^{-09} B_{2,16}(X) - 9.97951 \cdot 10^{-10} B_{3,16}(X) - 7.98353 \cdot 10^{-10} B_{4,16}(X) - 5.98756 \cdot 10^{-10} B_{5,16}(X) - 3.99159 \cdot 10^{-10} B_{6,16}(X) - 1.99561 \cdot 10^{-10} B_{7,16}(X) + 3.6039 \cdot 10^{-14} B_{8,16}(X) + 1.99633 \cdot 10^{-10} B_{9,16}(X) + 3.99231 \cdot 10^{-10} B_{10,16}(X) + 5.98828 \cdot 10^{-10} B_{11,16}(X) + 7.98425 \cdot 10^{-10} B_{12,16}(X) + 9.98023 \cdot 10^{-10} B_{13,16}(X) + 1.19762 \cdot 10^{-09} B_{14,16}(X) + 1.39722 \cdot 10^{-09} B_{15,16}(X) + 1.59681 \cdot 10^{-09} B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -4.83666 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$
$$= -1.59674 \cdot 10^{-09} B_{0,2} + 3.6039 \cdot 10^{-14} B_{1,2} + 1.59681 \cdot 10^{-09} B_{2,2}$$

$$\widetilde{q}_2 = 9.45798 \cdot 10^{-15} X^{16} - 7.39324 \cdot 10^{-14} X^{15} + 2.61437 \cdot 10^{-13} X^{14} - 5.52989 \cdot 10^{-13} X^{13} + 7.79462 \cdot 10^{-13} X^{12} - 7.71893 \cdot 10^{-13} X^{11} + 5.51461 \cdot 10^{-13} X^{10} - 2.8712 \cdot 10^{-13} X^{9} + 1.08634 \cdot 10^{-13} X^{8} - 2.94042 \cdot 10^{-14} X^{7} + 5.52081 \cdot 10^{-15} X^{6} - 6.84058 \cdot 10^{-16} X^{5} + 5.20623 \cdot 10^{-17} X^{4} - 2.16513 \cdot 10^{-18} X^{3} + 1.74369 \cdot 10^{-20} X^{2} + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$
$$= -1.59674 \cdot 10^{-09} B_{0,16} - 1.39715 \cdot 10^{-09} B_{1,16} - 1.19755 \cdot 10^{-09} B_{2,16} - 9.97951 \cdot 10^{-10} B_{3,16} - 7.98353 \cdot 10^{-10} B_{4,16} - 5.98756 \cdot 10^{-10} B_{5,16} - 3.99159 \cdot 10^{-10} B_{6,16} - 1.99561 \cdot 10^{-10} B_{7,16} + 3.60393 \cdot 10^{-14} B_{8,16} + 1.99633 \cdot 10^{-10} B_{9,16} + 3.99231 \cdot 10^{-10} B_{10,16} + 5.98828 \cdot 10^{-10} B_{11,16} + 7.98425 \cdot 10^{-10} B_{12,16} + 9.98023 \cdot 10^{-10} B_{13,16} + 1.19762 \cdot 10^{-09} B_{14,16} + 1.39722 \cdot 10^{-09} B_{15,16} + 1.59681 \cdot 10^{-09} B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.02367 \cdot 10^{-19}$.
Bounding polynomials M and m:

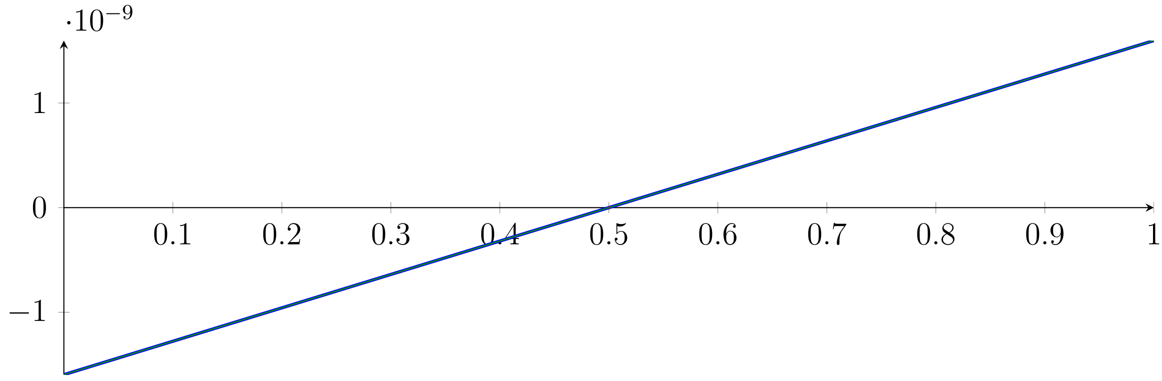
$$M = -4.82657 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$

$$m = -4.84676 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$

Root of M and m:

$$N(M) = \{0.5, 6.61662 \cdot 10^{16}\} \qquad N(m) = \{0.5, 6.58905 \cdot 10^{16}\}$$

Intersection intervals:



$[0.5, 0.5]$

Longest intersection interval: 0
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

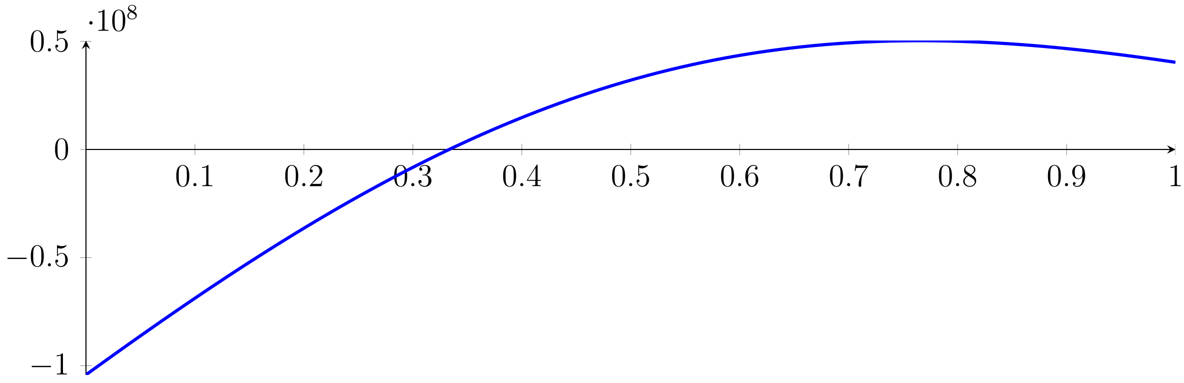
161.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

161.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^{6}X^{9} - 2.52262 \cdot 10^{6}X^{8} - 1.43506 \cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \cdot 10^{7}X^{5} + 3.97396 \cdot 10^{7}X^{4} - 2.40625 \cdot 10^{8}X^{3} - 8.33333 \cdot 10^{7}X^{2} + 3.64583 \cdot 10^{8}X - 1.04167 \cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

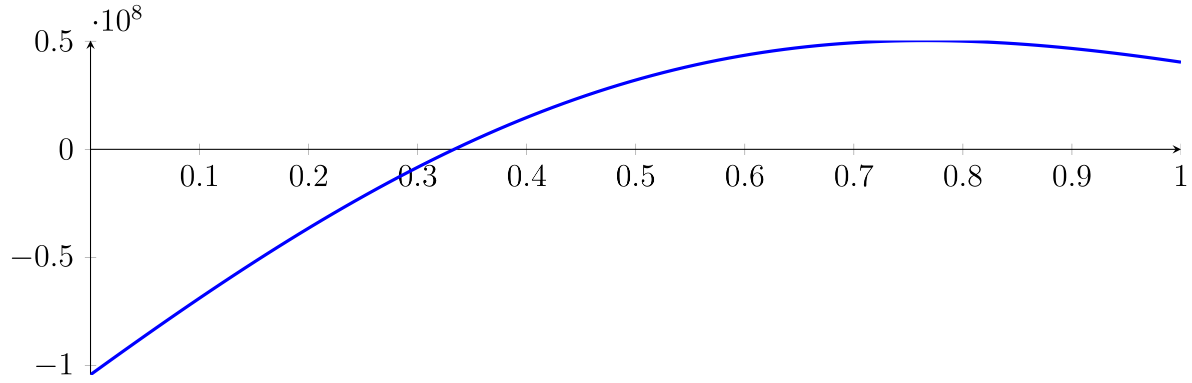
with precision $\varepsilon = 1 \cdot 10^{-32}$.

162 Running CubeClip on f_{16} with epsilon 32

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

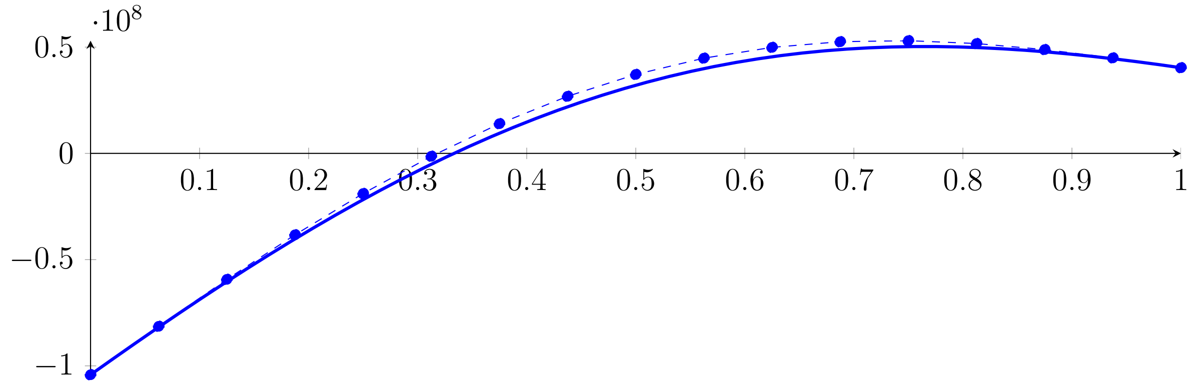
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



162.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

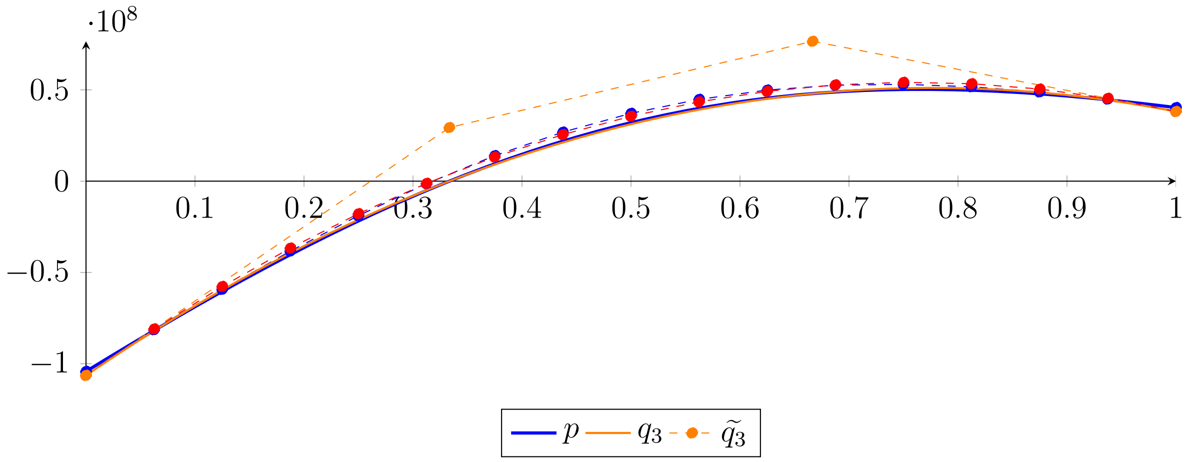
$$\begin{aligned} p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\quad \cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \\ &= -1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.06335{\cdot}10^8 \\ &= -1.06335{\cdot}10^8 B_{0,3} + 2.93558{\cdot}10^7 B_{1,3} + 7.66068{\cdot}10^7 B_{2,3} + 3.81764{\cdot}10^7 B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2461.93X^{16} - 19614.9X^{15} + 70879.5X^{14} - 153661X^{13} + 222746X^{12} - 227755X^{11} \\ &\quad + 168826X^{10} - 91798.7X^9 + 36630.3X^8 - 10627.3X^7 + 2200.54X^6 - 316.059X^5 \\ &\quad + 30.1958X^4 + 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.06335{\cdot}10^8 \\ &= -1.06335{\cdot}10^8 B_{0,16} - 8.08928{\cdot}10^7 B_{1,16} - 5.76618{\cdot}10^7 B_{2,16} - 3.66369{\cdot}10^7 B_{3,16} - 1.78131 \\ &\quad {\cdot}10^7 B_{4,16} - 1.18551{\cdot}10^6 B_{5,16} + 1.32508{\cdot}10^7 B_{6,16} + 2.55007{\cdot}10^7 B_{7,16} + 3.55692{\cdot}10^7 B_{8,16} \\ &\quad + 4.34611{\cdot}10^7 B_{9,16} + 4.91815{\cdot}10^7 B_{10,16} + 5.27353{\cdot}10^7 B_{11,16} + 5.41273{\cdot}10^7 B_{12,16} \\ &\quad + 5.33624{\cdot}10^7 B_{13,16} + 5.04457{\cdot}10^7 B_{14,16} + 4.53821{\cdot}10^7 B_{15,16} + 3.81764{\cdot}10^7 B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806{\cdot}10^6$.
Bounding polynomials M and m:

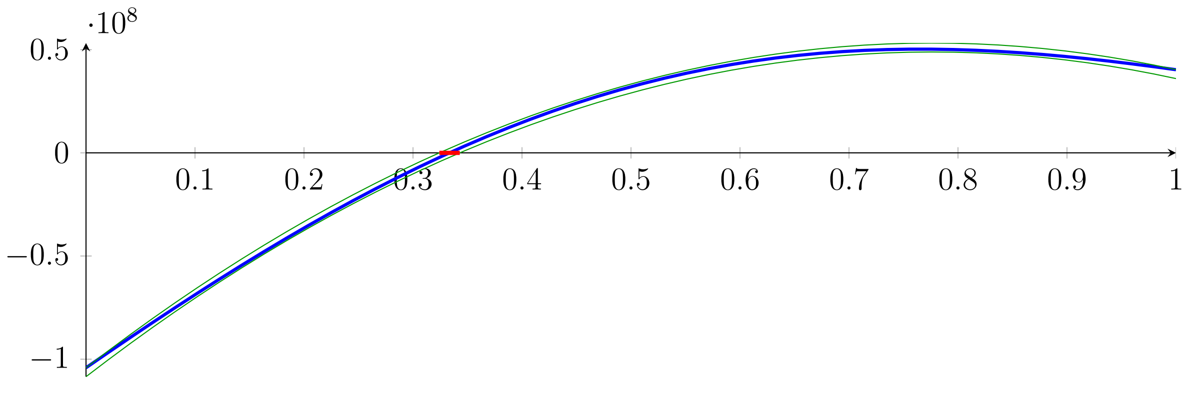
$$M = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.04167{\cdot}10^8$$

$$m = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.08503{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



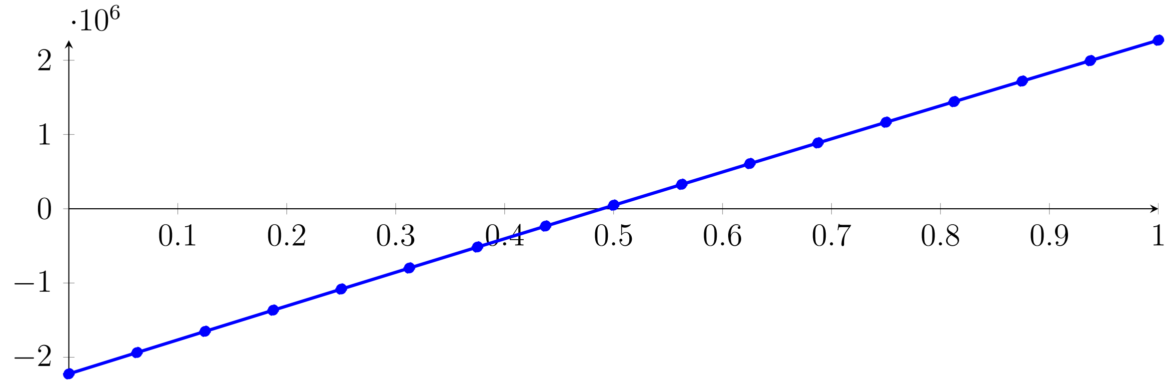
$$[0.324143, 0.342913]$$

Longest intersection interval: 0.0187703
$\Longrightarrow$ Selective recursion: interval 1: [0.324143, 0.342913],

162.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

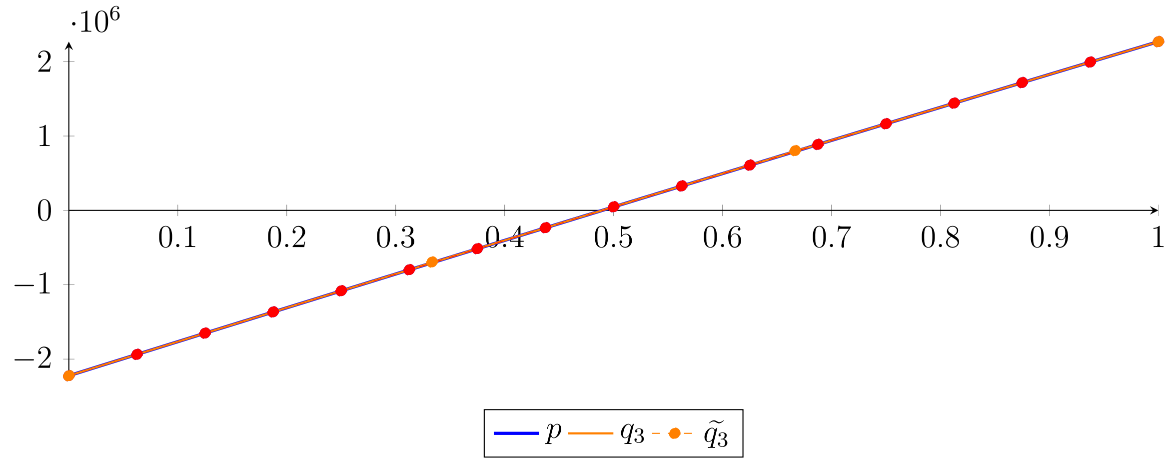
$$p = -3.66617\cdot10^{-09}X^{16} - 1.53217\cdot10^{-07}X^{15} - 3.62234\cdot10^{-07}X^{14} - 1.65579\cdot10^{-06}X^{13} - 1.15373\cdot10^{-06}X^{12} - 2.3399\cdot10^{-06}X^{11} - 5.02543\cdot10^{-07}X^{10} - 1.38381\cdot10^{-06}X^{9} + 1.1237\cdot10^{-06}X^{8} - 1.19653\cdot10^{-05}X^{7} - 0.00155608X^{6} + 0.10496X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,16}(X) - 1.93666\cdot10^{6}B_{1,16}(X) - 1.65074\cdot10^{6}B_{2,16}(X) - 1.36561\cdot10^{6}B_{3,16}(X) - 1.08127\cdot10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X) + 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot10^{6}B_{12,16}(X) + 1.44233\cdot10^{6}B_{13,16}(X) + 1.71876\cdot10^{6}B_{14,16}(X) + 1.99439\cdot10^{6}B_{15,16}(X) + 2.26922\cdot10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot10^{6}B_{3,3}$$

$$\tilde{q}_3 = 16.4956X^{16} - 129.161X^{15} + 457.83X^{14} - 971.671X^{13} + 1375.95X^{12} - 1370.96X^{11} + 987.265X^{10} - 519.476X^{9} + 199.587X^{8} - 55.434X^{7} + 10.9237X^{6} - 1.48019X^{5} + 0.129516X^{4} - 700.679X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,16} - 1.93666\cdot10^{6}B_{1,16} - 1.65074\cdot10^{6}B_{2,16} - 1.36561\cdot10^{6}B_{3,16} - 1.08126\cdot10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.4B_{8,16} + 328650B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot10^{6}B_{12,16} + 1.44233\cdot10^{6}B_{13,16} + 1.71876\cdot10^{6}B_{14,16} + 1.99439\cdot10^{6}B_{15,16} + 2.26922\cdot10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m:

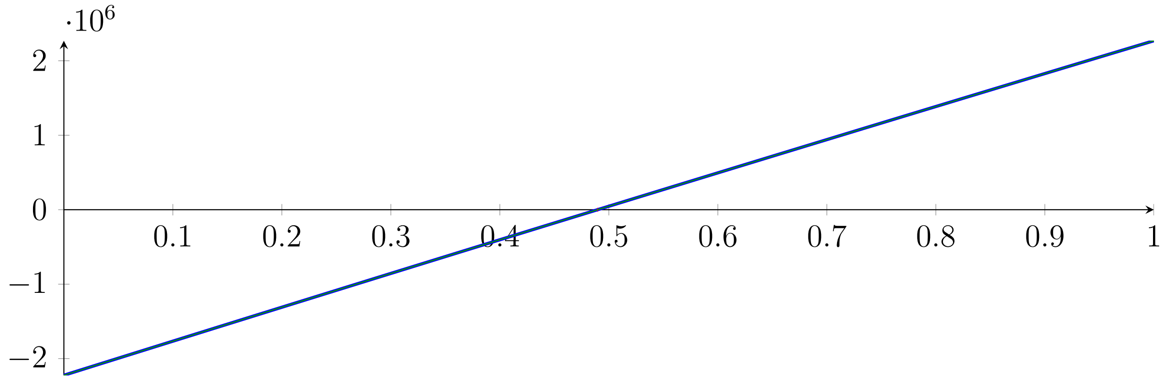
$$M = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715{\cdot}10^6X - 2.22335{\cdot}10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



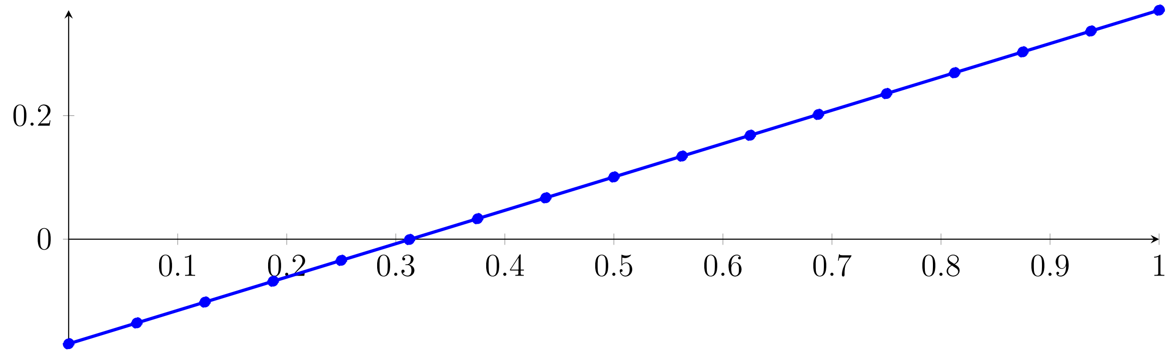
$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

162.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

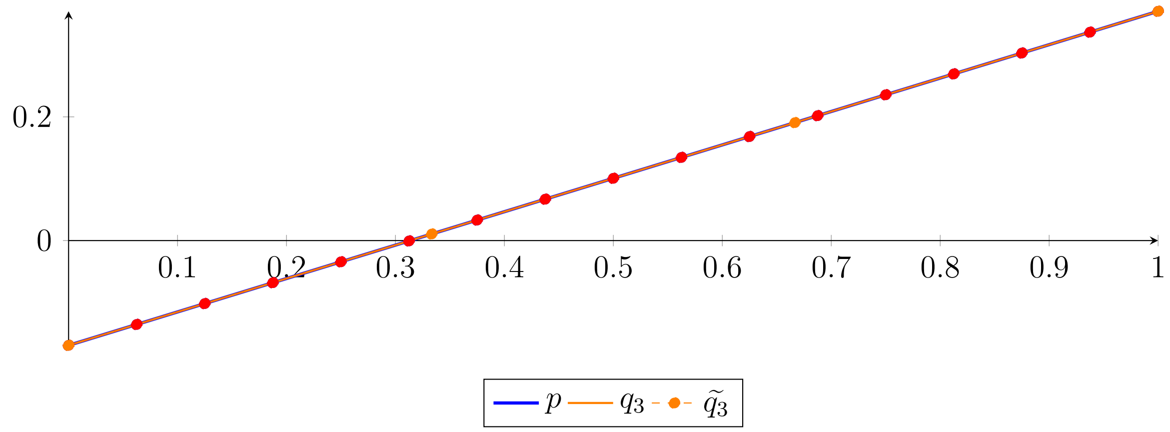
$$\begin{aligned} p = {} & 5.55524{\cdot}10^{-15}X^{16} - 2.94313{\cdot}10^{-14}X^{15} + 1.19384{\cdot}10^{-13}X^{14} - 2.17482{\cdot}10^{-13}X^{13} + 7.26155{\cdot}10^{-14}X^{12} \\ & - 3.44766{\cdot}10^{-13}X^{11} - 3.47292{\cdot}10^{-15}X^{10} - 1.1287{\cdot}10^{-13}X^{9} + 2.93027{\cdot}10^{-14}X^{8} + 8.06213{\cdot}10^{-15}X^{7} \\ & + 5.64349{\cdot}10^{-15}X^{6} + 4.93312{\cdot}10^{-17}X^{4} + 1.51788{\cdot}10^{-17}X^{3} - 1.37072{\cdot}10^{-09}X^{2} + 0.540148X - 0.169396 \\ = {} & -0.169396B_{0,16}(X) - 0.135637B_{1,16}(X) - 0.101877B_{2,16}(X) - 0.068118B_{3,16}(X) \\ & - 0.0343587B_{4,16}(X) - 0.000599476B_{5,16}(X) + 0.0331598B_{6,16}(X) \\ & + 0.0669191B_{7,16}(X) + 0.100678B_{8,16}(X) + 0.134438B_{9,16}(X) + 0.168197B_{10,16}(X) \\ & + 0.201956B_{11,16}(X) + 0.235715B_{12,16}(X) + 0.269475B_{13,16}(X) \\ & + 0.303234B_{14,16}(X) + 0.336993B_{15,16}(X) + 0.370752B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07065{\cdot}10^{-17}X^3 - 1.37072{\cdot}10^{-09}X^2 + 0.540148X - 0.169396 \\ &= -0.169396B_{0,3} + 0.0106536B_{1,3} + 0.190703B_{2,3} + 0.370752B_{3,3} \end{aligned}$$

$$\begin{aligned}\tilde{q}_3 &= 8.59095\cdot10^{-06}X^{16} - 6.82648\cdot10^{-05}X^{15} + 0.000245968X^{14} - 0.000531568X^{13} \\ &\quad + 0.000767923X^{12} - 0.000782231X^{11} + 0.0005774X^{10} - 0.000312464X^{9} \\ &\quad + 0.000123994X^{8} - 3.57388\cdot10^{-05}X^{7} + 7.34249\cdot10^{-06}X^{6} - 1.04474\cdot10^{-06}X^{5} \\ &\quad + 9.86739\cdot10^{-08}X^{4} - 5.7553\cdot10^{-09}X^{3} - 1.19186\cdot10^{-09}X^{2} + 0.540148X - 0.169396 \\ &= -0.169396B_{0,16} - 0.135637B_{1,16} - 0.101877B_{2,16} - 0.068118B_{3,16} - 0.0343587B_{4,16} \\ &\quad - 0.000599476B_{5,16} + 0.0331598B_{6,16} + 0.0669191B_{7,16} + 0.100678B_{8,16} \\ &\quad + 0.134438B_{9,16} + 0.168197B_{10,16} + 0.201956B_{11,16} + 0.235715B_{12,16} \\ &\quad + 0.269475B_{13,16} + 0.303234B_{14,16} + 0.336993B_{15,16} + 0.370752B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.81206\cdot10^{-10}$.

Bounding polynomials M and m:

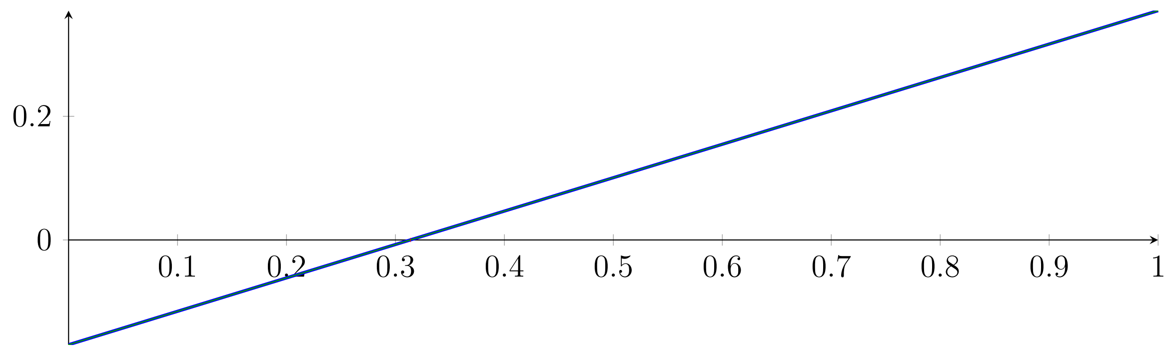
$$M = -1.07065\cdot10^{-17}X^3 - 1.37072\cdot10^{-09}X^2 + 0.540148X - 0.169396$$

$$m = -1.07065\cdot10^{-17}X^3 - 1.37072\cdot10^{-09}X^2 + 0.540148X - 0.169396$$

Root of M and m:

$$N(M) = \{-2.97569\cdot10^{8}, 0.31361, 1.69542\cdot10^{8}\} \quad N(m) = \{-2.97569\cdot10^{8}, 0.31361, 1.69542\cdot10^{8}\}$$

Intersection intervals:



$[0.31361, 0.31361]$

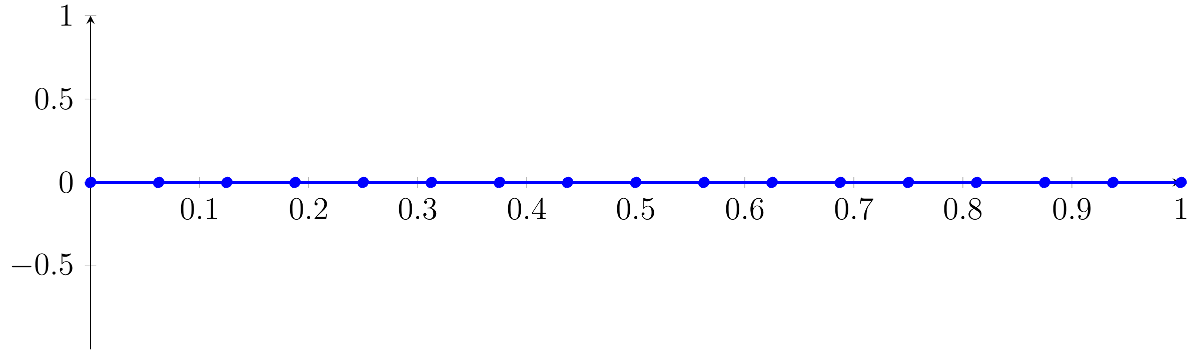
Longest intersection interval: $7.85803\cdot10^{-10}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

162.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

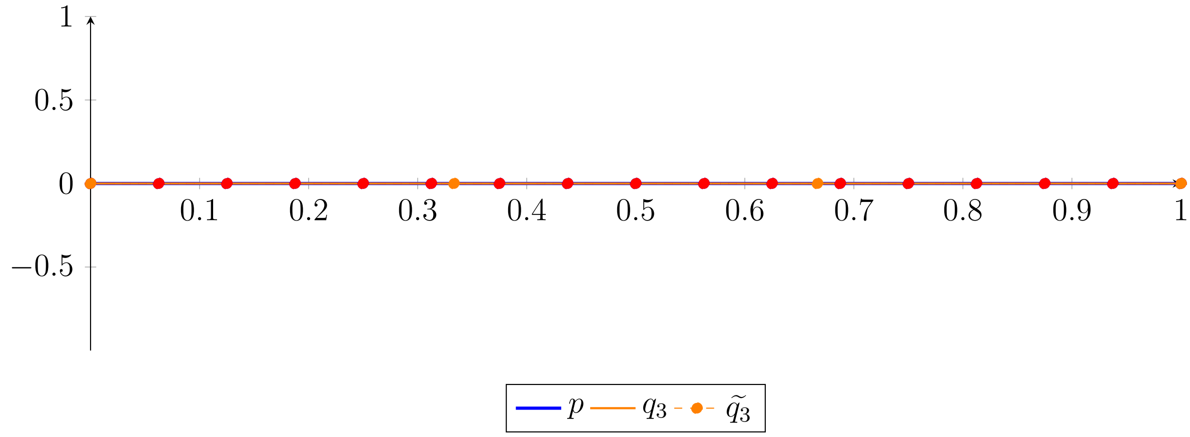
$$p = -1.51576{\cdot}10^{-21}X^{16} + 2.62009{\cdot}10^{-21}X^{15} - 3.98039{\cdot}10^{-20}X^{14} + 3.2136{\cdot}10^{-21}X^{13} - 5.16564{\cdot}10^{-20}X^{12} + 1.52429{\cdot}10^{-20}X^{11} - 1.44901{\cdot}10^{-20}X^{10} - 1.40466{\cdot}10^{-20}X^{9} + 2.34541{\cdot}10^{-20}X^{8} + 3.25289{\cdot}10^{-21}X^{7} + 2.38052{\cdot}10^{-21}X^{6} - 2.2582{\cdot}10^{-22}X^{5} + 3.52844{\cdot}10^{-23}X^{4} + 4.2445{\cdot}10^{-10}X - 2.39831{\cdot}10^{-08}$$
$$= -2.39831{\cdot}10^{-08}B_{0,16}(X) - 2.39566{\cdot}10^{-08}B_{1,16}(X) - 2.39301{\cdot}10^{-08}B_{2,16}(X) - 2.39036{\cdot}10^{-08}B_{3,16}(X) - 2.3877{\cdot}10^{-08}B_{4,16}(X) - 2.38505{\cdot}10^{-08}B_{5,16}(X) - 2.3824{\cdot}10^{-08}B_{6,16}(X) - 2.37974{\cdot}10^{-08}B_{7,16}(X) - 2.37709{\cdot}10^{-08}B_{8,16}(X) - 2.37444{\cdot}10^{-08}B_{9,16}(X) - 2.37179{\cdot}10^{-08}B_{10,16}(X) - 2.36913{\cdot}10^{-08}B_{11,16}(X) - 2.36648{\cdot}10^{-08}B_{12,16}(X) - 2.36383{\cdot}10^{-08}B_{13,16}(X) - 2.36118{\cdot}10^{-08}B_{14,16}(X) - 2.35852{\cdot}10^{-08}B_{15,16}(X) - 2.35587{\cdot}10^{-08}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -3.5866{\cdot}10^{-25}X^3 + 8.33643{\cdot}10^{-25}X^2 + 4.2445{\cdot}10^{-10}X - 2.39831{\cdot}10^{-08}$$
$$= -2.39831{\cdot}10^{-08}B_{0,3} - 2.38417{\cdot}10^{-08}B_{1,3} - 2.37002{\cdot}10^{-08}B_{2,3} - 2.35587{\cdot}10^{-08}B_{3,3}$$

$$\widetilde{q}_3 = -1.64958{\cdot}10^{-12}X^{16} + 1.3166{\cdot}10^{-11}X^{15} - 4.76688{\cdot}10^{-11}X^{14} + 1.03558{\cdot}10^{-10}X^{13} - 1.50448{\cdot}10^{-10}X^{12} + 1.54183{\cdot}10^{-10}X^{11} - 1.1456{\cdot}10^{-10}X^{10} + 6.24452{\cdot}10^{-11}X^{9} - 2.49793{\cdot}10^{-11}X^{8} + 7.26358{\cdot}10^{-12}X^{7} - 1.50649{\cdot}10^{-12}X^{6} + 2.16616{\cdot}10^{-13}X^{5} - 2.07725{\cdot}10^{-14}X^{4} + 1.24748{\cdot}10^{-15}X^{3} - 4.0727{\cdot}10^{-17}X^{2} + 4.2445{\cdot}10^{-10}X - 2.39831{\cdot}10^{-08}$$
$$= -2.39831{\cdot}10^{-08}B_{0,16} - 2.39566{\cdot}10^{-08}B_{1,16} - 2.39301{\cdot}10^{-08}B_{2,16} - 2.39036{\cdot}10^{-08}B_{3,16} - 2.3877{\cdot}10^{-08}B_{4,16} - 2.38505{\cdot}10^{-08}B_{5,16} - 2.3824{\cdot}10^{-08}B_{6,16} - 2.37974{\cdot}10^{-08}B_{7,16} - 2.37709{\cdot}10^{-08}B_{8,16} - 2.37444{\cdot}10^{-08}B_{9,16} - 2.37179{\cdot}10^{-08}B_{10,16} - 2.36913{\cdot}10^{-08}B_{11,16} - 2.36648{\cdot}10^{-08}B_{12,16} - 2.36383{\cdot}10^{-08}B_{13,16} - 2.36118{\cdot}10^{-08}B_{14,16} - 2.35852{\cdot}10^{-08}B_{15,16} - 2.35587{\cdot}10^{-08}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 3.51589{\cdot}10^{-17}$.

Bounding polynomials M and m:

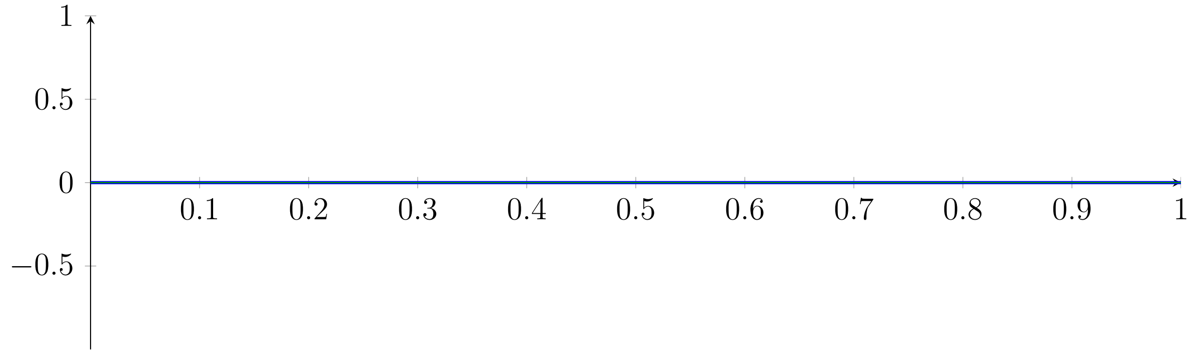
$$M = -3.5866{\cdot}10^{-25}X^3 + 8.33643{\cdot}10^{-25}X^2 + 4.2445{\cdot}10^{-10}X - 2.39831{\cdot}10^{-08}$$

$$m = -3.5866{\cdot}10^{-25}X^3 + 8.33643{\cdot}10^{-25}X^2 + 4.2445{\cdot}10^{-10}X - 2.39831{\cdot}10^{-08}$$

Root of M and m:

$$N(M) = \{-3.44011{\cdot}10^7, 56.504, 3.4401{\cdot}10^7\} \qquad N(m) = \{-3.44011{\cdot}10^7, 56.504, 3.4401{\cdot}10^7\}$$

Intersection intervals:

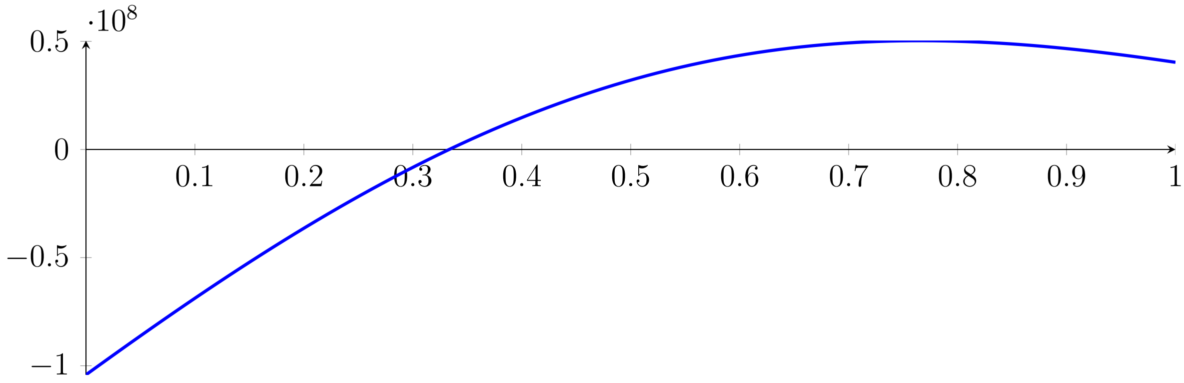


No intersection intervals with the x axis.

162.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

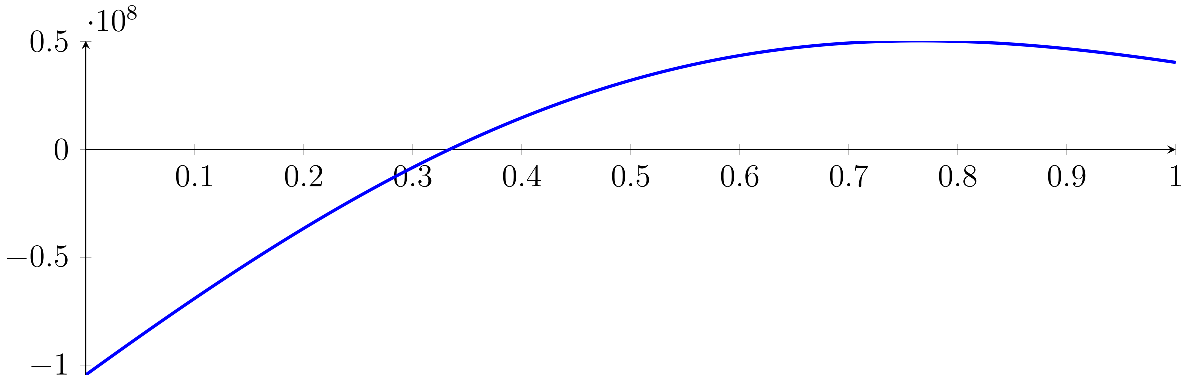
with precision $\varepsilon = 1\cdot10^{-32}$.

163 Running BezClip on f_{16} with epsilon 64

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^6X^9 - 2.52262\cdot10^6X^8 - 1.43506\cdot10^7X^7 + 138542X^6 + 7.92823\cdot10^7X^5 + 3.97396\cdot10^7X^4 - 2.40625\cdot10^8X^3 - 8.33333\cdot10^7X^2 + 3.64583\cdot10^8X - 1.04167\cdot10^8$$

Called BezClip with input polynomial on interval $[0,1]$:

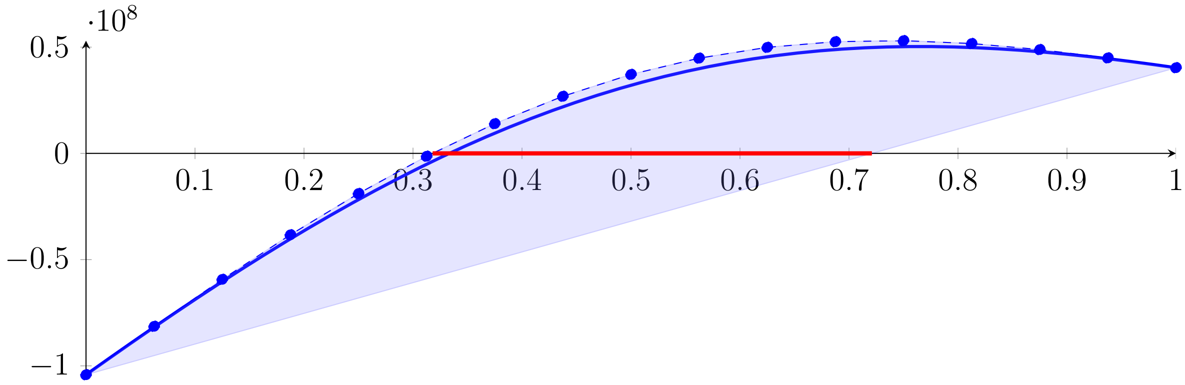
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^6X^9 - 2.52262\cdot10^6X^8 - 1.43506\cdot10^7X^7 + 138542X^6 + 7.92823\cdot10^7X^5 + 3.97396\cdot10^7X^4 - 2.40625\cdot10^8X^3 - 8.33333\cdot10^7X^2 + 3.64583\cdot10^8X - 1.04167\cdot10^8$$



163.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot10^6X^9 - 2.52262\cdot10^6X^8 - 1.43506\cdot10^7X^7 + 138542X^6 + 7.92823 \\ &\quad \cdot10^7X^5 + 3.97396\cdot10^7X^4 - 2.40625\cdot10^8X^3 - 8.33333\cdot10^7X^2 + 3.64583\cdot10^8X - 1.04167\cdot10^8 \\ &= -1.04167\cdot10^8B_{0,16}(X) - 8.13802\cdot10^7B_{1,16}(X) - 5.92882\cdot10^7B_{2,16}(X) - 3.83203 \\ &\quad \cdot10^7B_{3,16}(X) - 1.88844\cdot10^7B_{4,16}(X) - 1.34837\cdot10^6B_{5,16}(X) + 1.39781\cdot10^7B_{6,16}(X) \\ &\quad + 2.68604\cdot10^7B_{7,16}(X) + 3.71532\cdot10^7B_{8,16}(X) + 4.48105\cdot10^7B_{9,16}(X) + 4.98901 \\ &\quad \cdot10^7B_{10,16}(X) + 5.25504\cdot10^7B_{11,16}(X) + 5.30407\cdot10^7B_{12,16}(X) + 5.16832\cdot10^7B_{13,16}(X) \\ &\quad + 4.88488\cdot10^7B_{14,16}(X) + 4.49297\cdot10^7B_{15,16}(X) + 4.03108\cdot10^7B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

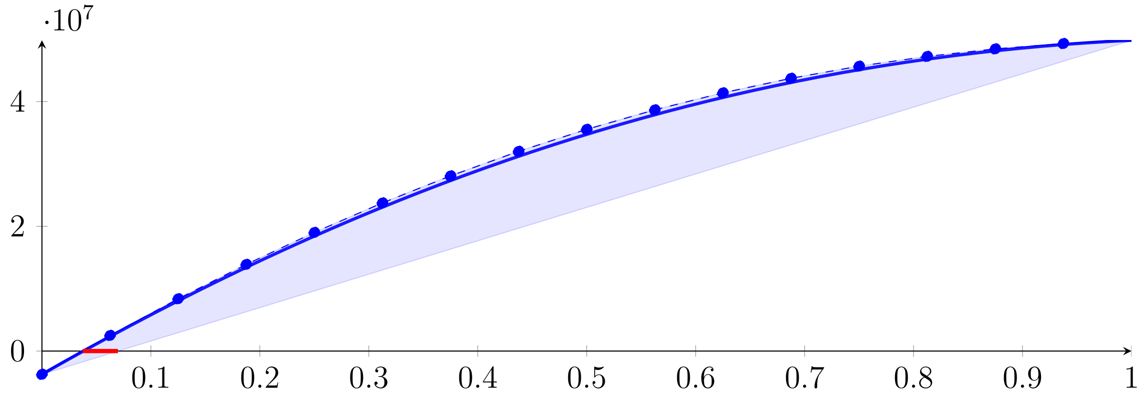
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

163.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 1.59825\cdot 10^{-06}X^{16} - 5.93153\cdot 10^{-05}X^{15} - 0.00248867X^{14} - 0.0650056X^{13} - 0.909142X^{12}\\ &\quad - 5.03931X^{11} + 36.4549X^{10} + 692.921X^{9} + 1886.97X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5}\\ &\quad + 3.91945\cdot 10^{6}X^{4} - 7.54561\cdot 10^{6}X^{3} - 4.29225\cdot 10^{7}X^{2} + 9.97982\cdot 10^{7}X - 3.73501\cdot 10^{6}\\ &= -3.73501\cdot 10^{6}B_{0,16}(X) + 2.50237\cdot 10^{6}B_{1,16}(X) + 8.38207\cdot 10^{6}B_{2,16}(X) + 1.38906\\ &\quad \cdot 10^{7}B_{3,16}(X) + 1.90167\cdot 10^{7}B_{4,16}(X) + 2.37512\cdot 10^{7}B_{5,16}(X) + 2.80875\cdot 10^{7}B_{6,16}(X)\\ &\quad + 3.20213\cdot 10^{7}B_{7,16}(X) + 3.55508\cdot 10^{7}B_{8,16}(X) + 3.86768\cdot 10^{7}B_{9,16}(X) + 4.14025\\ &\quad \cdot 10^{7}B_{10,16}(X) + 4.37336\cdot 10^{7}B_{11,16}(X) + 4.56783\cdot 10^{7}B_{12,16}(X) + 4.7247\cdot 10^{7}B_{13,16}(X)\\ &\quad + 4.84524\cdot 10^{7}B_{14,16}(X) + 4.93093\cdot 10^{7}B_{15,16}(X) + 4.98343\cdot 10^{7}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

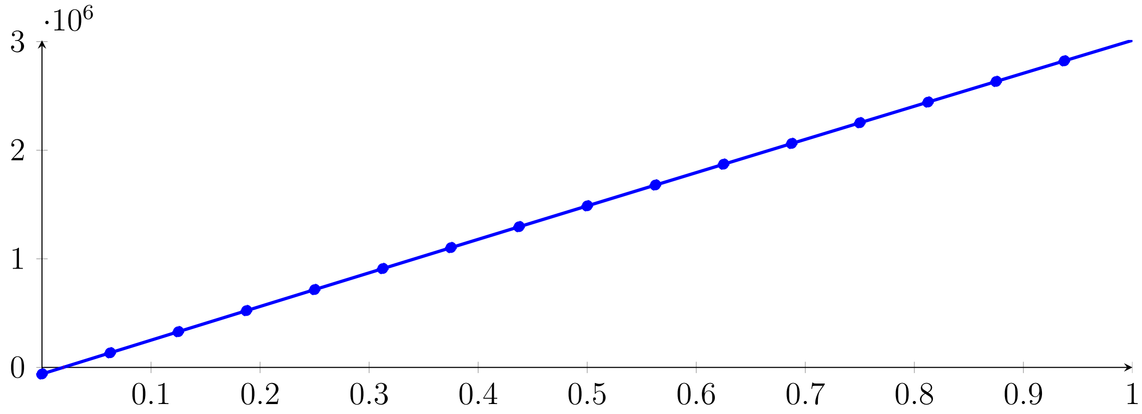
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

163.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 9.01396\cdot 10^{-08}X^{16} - 2.65848\cdot 10^{-07}X^{15} + 2.13948\cdot 10^{-06}X^{14} - 1.33627\cdot 10^{-06}X^{13} + 2.46973\cdot 10^{-06}X^{12}\\ &\quad - 2.45524\cdot 10^{-06}X^{11} + 5.50112\cdot 10^{-07}X^{10} - 1.64198\cdot 10^{-07}X^{9} - 7.35598\cdot 10^{-07}X^{8} - 1.00892\cdot 10^{-06}X^{7}\\ &\quad - 0.000177509X^{6} + 0.0161038X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot 10^{6}X - 60508.5\\ &= -60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X)\\ &\quad + 910202B_{5,16}(X) + 1.1032\cdot 10^{6}B_{6,16}(X) + 1.29582\cdot 10^{6}B_{7,16}(X) + 1.48805\cdot 10^{6}B_{8,16}(X)\\ &\quad + 1.6799\cdot 10^{6}B_{9,16}(X) + 1.87137\cdot 10^{6}B_{10,16}(X) + 2.06245\cdot 10^{6}B_{11,16}(X) + 2.25315\cdot 10^{6}B_{12,16}(X)\\ &\quad + 2.44346\cdot 10^{6}B_{13,16}(X) + 2.63339\cdot 10^{6}B_{14,16}(X) + 2.82293\cdot 10^{6}B_{15,16}(X) + 3.01209\cdot 10^{6}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

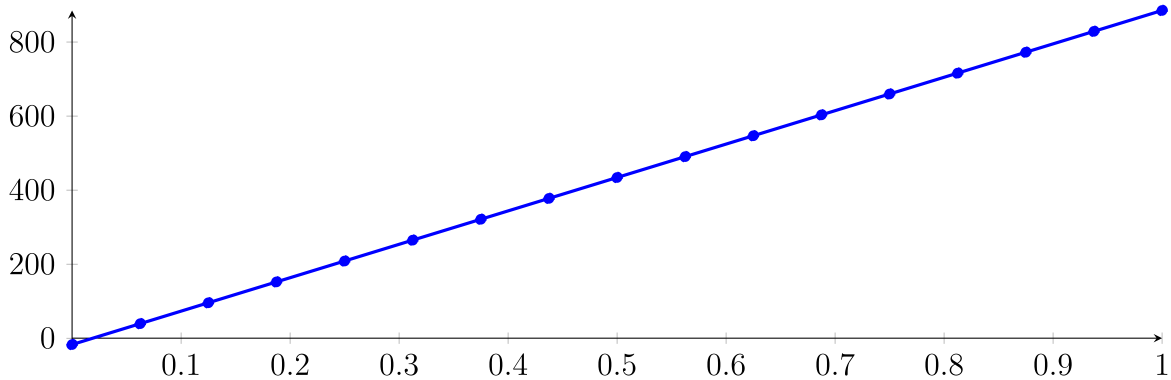
$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

163.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= 2.55372\cdot 10^{-11}X^{16} - 7.21263\cdot 10^{-11}X^{15} + 6.24141\cdot 10^{-10}X^{14} - 4.11162\cdot 10^{-10}X^{13}\\ &\quad + 6.82359\cdot 10^{-10}X^{12} - 7.09475\cdot 10^{-10}X^{11} + 9.71305\cdot 10^{-11}X^{10} - 3.46101\cdot 10^{-11}X^{9}\\ &\quad - 2.13971\cdot 10^{-10}X^{8} - 1.46061\cdot 10^{-11}X^{7} - 1.63366\cdot 10^{-11}X^{6} + 1.87916\cdot 10^{-12}X^{5}\\ &\quad + 2.52576\cdot 10^{-14}X^{4} - 5.67777\cdot 10^{-09}X^{3} - 0.00382618X^{2} + 902.448X - 17.178\\ &= -17.178B_{0,16}(X) + 39.225B_{1,16}(X) + 95.6279B_{2,16}(X) + 152.031B_{3,16}(X) + 208.434B_{4,16}(X)\\ &\quad + 264.837B_{5,16}(X) + 321.24B_{6,16}(X) + 377.642B_{7,16}(X) + 434.045B_{8,16}(X)\\ &\quad + 490.448B_{9,16}(X) + 546.851B_{10,16}(X) + 603.253B_{11,16}(X) + 659.656B_{12,16}(X)\\ &\quad + 716.059B_{13,16}(X) + 772.461B_{14,16}(X) + 828.864B_{15,16}(X) + 885.266B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

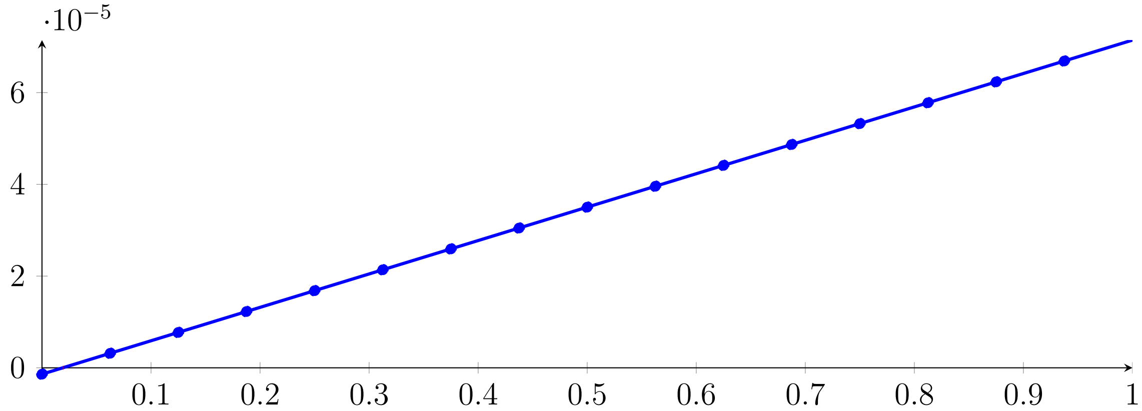
$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045\cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

163.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 2.14261\cdot10^{-18}X^{16} - 6.28573\cdot10^{-18}X^{15} + 5.28612\cdot10^{-17}X^{14} - 3.67279\cdot10^{-17}X^{13} + 6.10136\cdot10^{-17}X^{12} - 6.60335\cdot10^{-17}X^{11} + 1.66661\cdot10^{-17}X^{10} - 8.36524\cdot10^{-18}X^{9} - 1.56919\cdot10^{-17}X^{8} - 1.85474\cdot10^{-18}X^{7} - 1.4308\cdot10^{-18}X^{6} + 1.1562\cdot10^{-19}X^{5} - 1.20437\cdot10^{-20}X^{4} - 4.63221\cdot10^{-22}X^{3} - 2.49207\cdot10^{-17}X^{2} + 7.28316\cdot10^{-05}X - 1.38634\cdot10^{-06}$$
$$= -1.38634\cdot10^{-06}B_{0,16}(X) + 3.16564\cdot10^{-06}B_{1,16}(X) + 7.71761\cdot10^{-06}B_{2,16}(X) + 1.22696\cdot10^{-05}B_{3,16}(X) + 1.68216\cdot10^{-05}B_{4,16}(X) + 2.13735\cdot10^{-05}B_{5,16}(X) + 2.59255\cdot10^{-05}B_{6,16}(X) + 3.04775\cdot10^{-05}B_{7,16}(X) + 3.50295\cdot10^{-05}B_{8,16}(X) + 3.95814\cdot10^{-05}B_{9,16}(X) + 4.41334\cdot10^{-05}B_{10,16}(X) + 4.86854\cdot10^{-05}B_{11,16}(X) + 5.32374\cdot10^{-05}B_{12,16}(X) + 5.77893\cdot10^{-05}B_{13,16}(X) + 6.23413\cdot10^{-05}B_{14,16}(X) + 6.68933\cdot10^{-05}B_{15,16}(X) + 7.14453\cdot10^{-05}B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.51313\cdot10^{-15}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

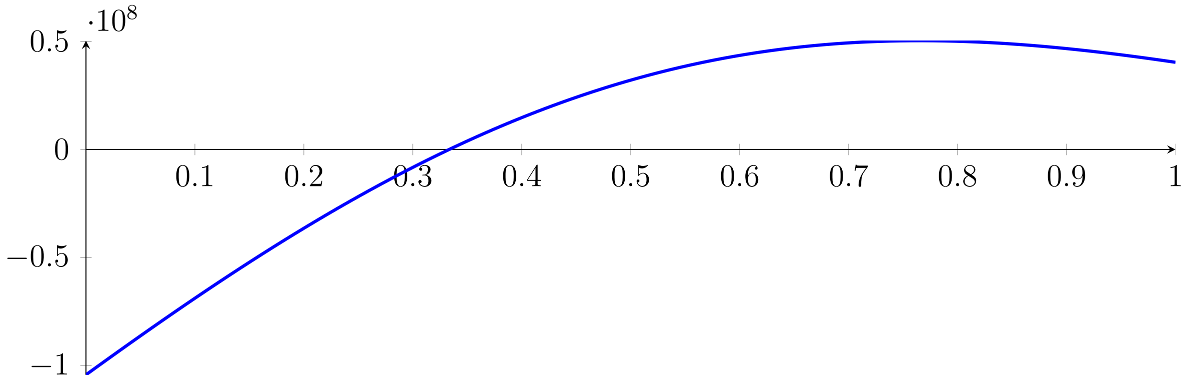
163.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

163.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^{6} X^{9} - 2.52262 \cdot 10^{6} X^{8} - 1.43506 \cdot 10^{7} X^{7} + 138542X^{6} + 7.92823 \cdot 10^{7} X^{5} + 3.97396 \cdot 10^{7} X^{4} - 2.40625 \cdot 10^{8} X^{3} - 8.33333 \cdot 10^{7} X^{2} + 3.64583 \cdot 10^{8} X - 1.04167 \cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

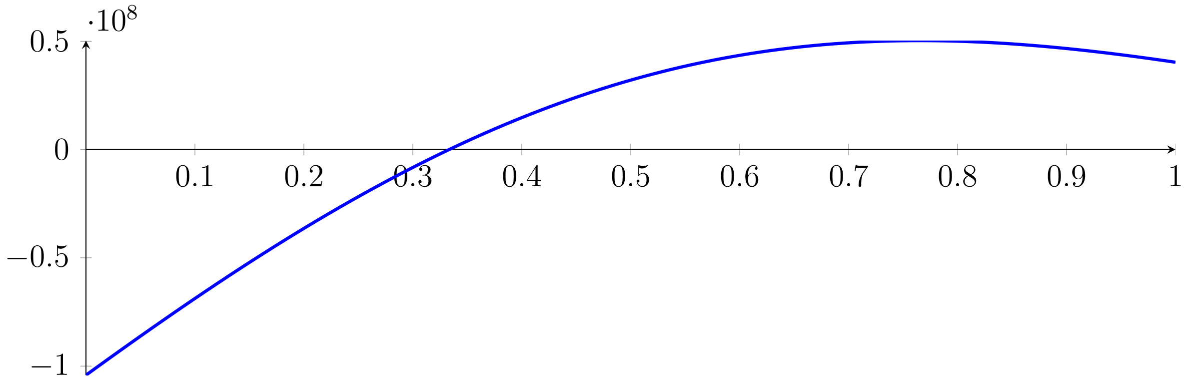
with precision $\varepsilon = 1 \cdot 10^{-64}$.

164 Running QuadClip on f_{16} with epsilon 64

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

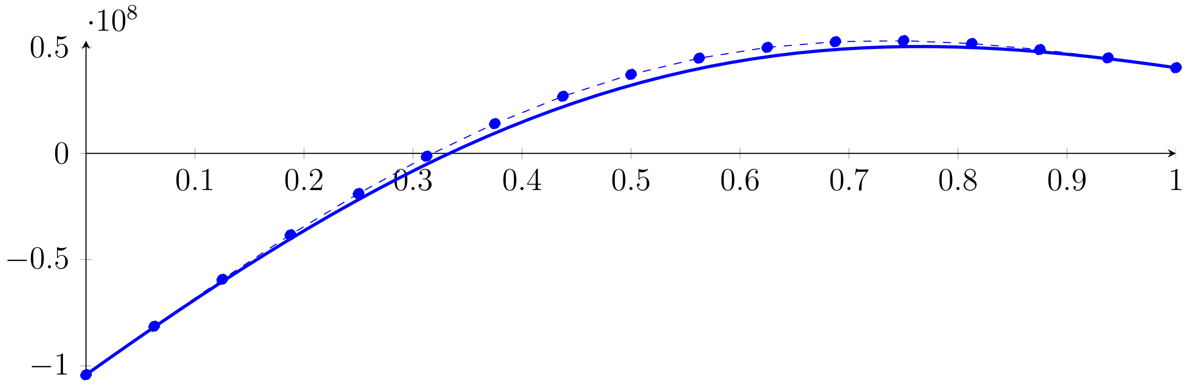
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



164.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

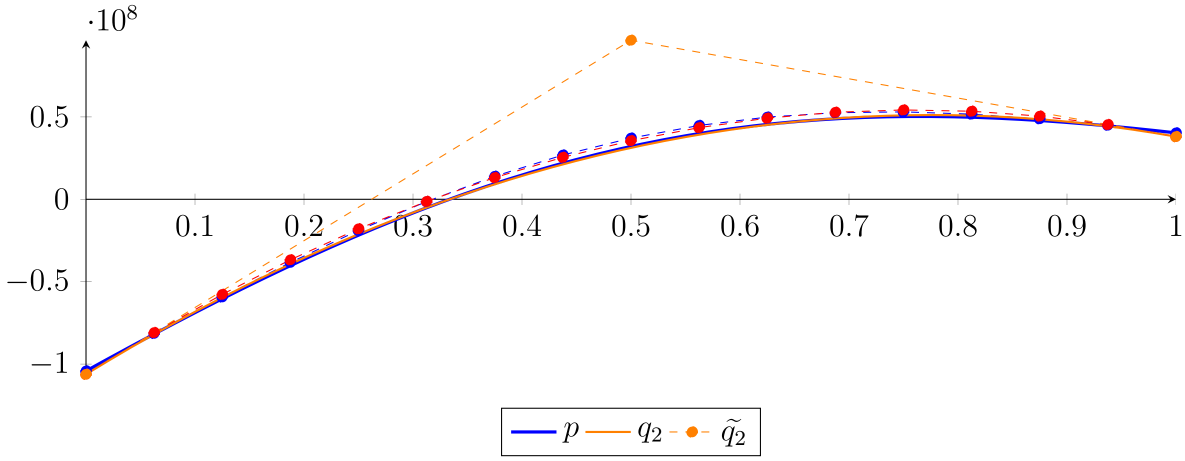
$$p = -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$
$$= -1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203\cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) + 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901\cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) + 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X)$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -2.61181\cdot 10^8 X^2 + 4.05417\cdot 10^8 X - 1.06197\cdot 10^8 \\ &= -1.06197\cdot 10^8 B_{0,2} + 9.65115\cdot 10^7 B_{1,2} + 3.80384\cdot 10^7 B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 6049.18X^{16} - 48305.2X^{15} + 174971X^{14} - 380294X^{13} + 552846X^{12} - 567203X^{11} \\ &\quad + 422303X^{10} - 231038X^9 + 93003.6X^8 - 27320.1X^7 + 5752.57X^6 - 843.63X^5 \\ &\quad + 82.5145X^4 - 5.01388X^3 - 2.61181\cdot 10^8 X^2 + 4.05417\cdot 10^8 X - 1.06197\cdot 10^8 \\ &= -1.06197\cdot 10^8 B_{0,16} - 8.08583\cdot 10^7 B_{1,16} - 5.76963\cdot 10^7 B_{2,16} - 3.67107\cdot 10^7 B_{3,16} - 1.79017 \\ &\quad \cdot 10^7 B_{4,16} - 1.26924\cdot 10^6 B_{5,16} + 1.31867\cdot 10^7 B_{6,16} + 2.54662\cdot 10^7 B_{7,16} + 3.55692\cdot 10^7 B_{8,16} \\ &\quad + 4.34956\cdot 10^7 B_{9,16} + 4.92456\cdot 10^7 B_{10,16} + 5.2819\cdot 10^7 B_{11,16} + 5.42159\cdot 10^7 B_{12,16} \\ &\quad + 5.34363\cdot 10^7 B_{13,16} + 5.04802\cdot 10^7 B_{14,16} + 4.53476\cdot 10^7 B_{15,16} + 3.80384\cdot 10^7 B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233\cdot 10^6$.

Bounding polynomials M and m:

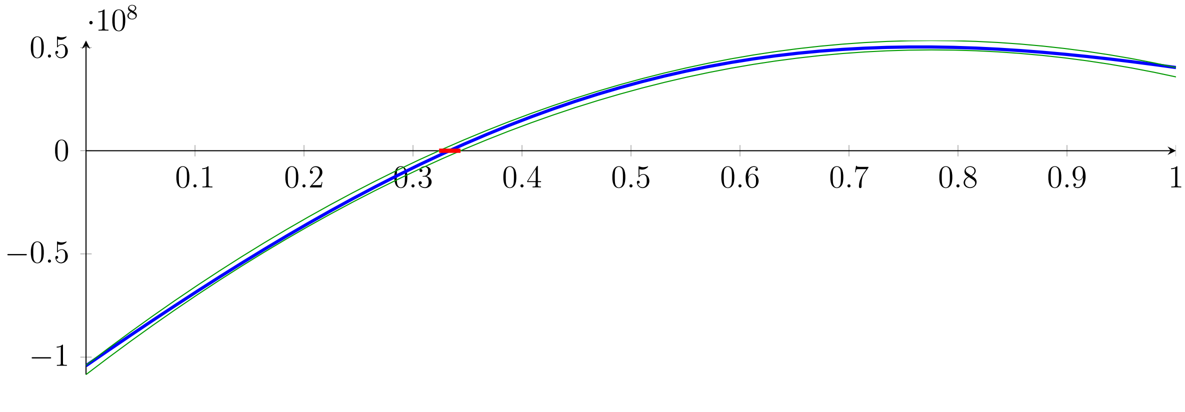
$$M = -2.61181\cdot 10^8 X^2 + 4.05417\cdot 10^8 X - 1.03924\cdot 10^8$$

$$m = -2.61181\cdot 10^8 X^2 + 4.05417\cdot 10^8 X - 1.08469\cdot 10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

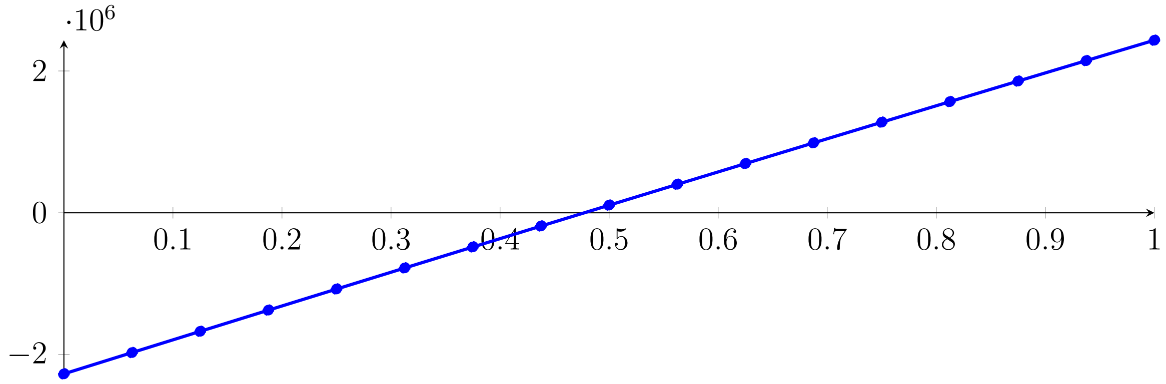
Longest intersection interval: 0.0196686

$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

164.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

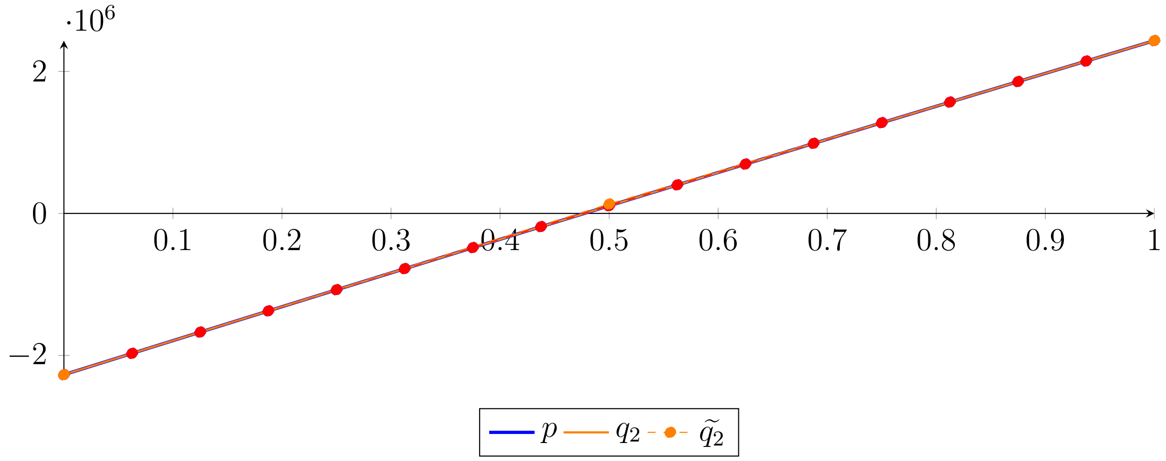
$$p = -1.54841\cdot10^{-10}X^{16} - 1.66965\cdot10^{-07}X^{15} - 2.92739\cdot10^{-07}X^{14} - 1.77943\cdot10^{-06}X^{13} - 1.17235\cdot10^{-06}X^{12} - 2.42234\cdot10^{-06}X^{11} - 6.86445\cdot10^{-07}X^{10} - 1.39162\cdot10^{-06}X^{9} + 1.07395\cdot10^{-06}X^{8} - 1.67072\cdot10^{-05}X^{7} - 0.00205879X^{6} + 0.132721X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874\cdot10^{6}X - 2.2715\cdot10^{6}$$
$$= -2.2715\cdot10^{6}B_{0,16}(X) - 1.97096\cdot10^{6}B_{1,16}(X) - 1.67127\cdot10^{6}B_{2,16}(X) - 1.37244\cdot10^{6}B_{3,16}(X) - 1.07448\cdot10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806\cdot10^{6}B_{12,16}(X) + 1.5682\cdot10^{6}B_{13,16}(X) + 1.85747\cdot10^{6}B_{14,16}(X) + 2.14586\cdot10^{6}B_{15,16}(X) + 2.43338\cdot10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265X^{2} + 4.80923\cdot10^{6}X - 2.27154\cdot10^{6}$$
$$= -2.27154\cdot10^{6}B_{0,2} + 133069B_{1,2} + 2.43342\cdot10^{6}B_{2,2}$$

$$\tilde{q}_2 = 22.7036X^{16} - 178.964X^{15} + 638.976X^{14} - 1366.64X^{13} + 1951.1X^{12} - 1960.92X^{11} + 1425.35X^{10} - 757.631X^{9} + 294.177X^{8} - 82.4368X^{7} + 16.2856X^{6} - 2.18949X^{5} + 0.191237X^{4} - 0.0101048X^{3} - 104265X^{2} + 4.80923\cdot10^{6}X - 2.27154\cdot10^{6}$$
$$= -2.27154\cdot10^{6}B_{0,16} - 1.97097\cdot10^{6}B_{1,16} - 1.67126\cdot10^{6}B_{2,16} - 1.37242\cdot10^{6}B_{3,16} - 1.07445\cdot10^{6}B_{4,16} - 777350B_{5,16} - 481118B_{6,16} - 185754B_{7,16} + 108740B_{8,16} + 402366B_{9,16} + 695123B_{10,16} + 987011B_{11,16} + 1.27803\cdot10^{6}B_{12,16} + 1.56818\cdot10^{6}B_{13,16} + 1.85746\cdot10^{6}B_{14,16} + 2.14587\cdot10^{6}B_{15,16} + 2.43342\cdot10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m:

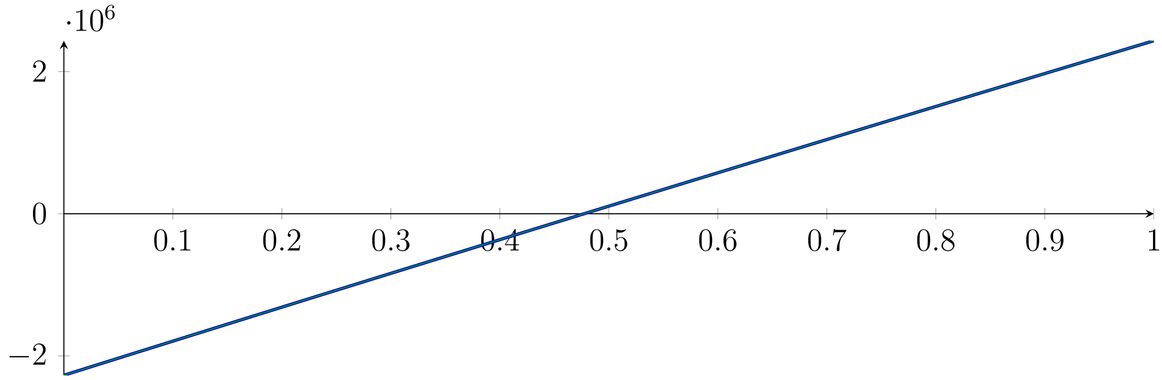
$$M = -104265X^{2} + 4.80923\cdot10^{6}X - 2.2715\cdot10^{6}$$

$$m = -104265X^2 + 4.80923{\cdot}10^6X - 2.27159{\cdot}10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



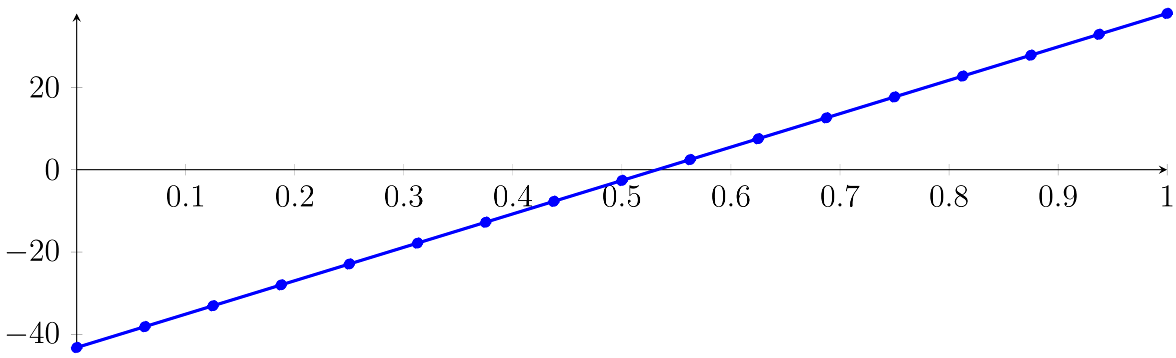
$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301{\cdot}10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

164.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

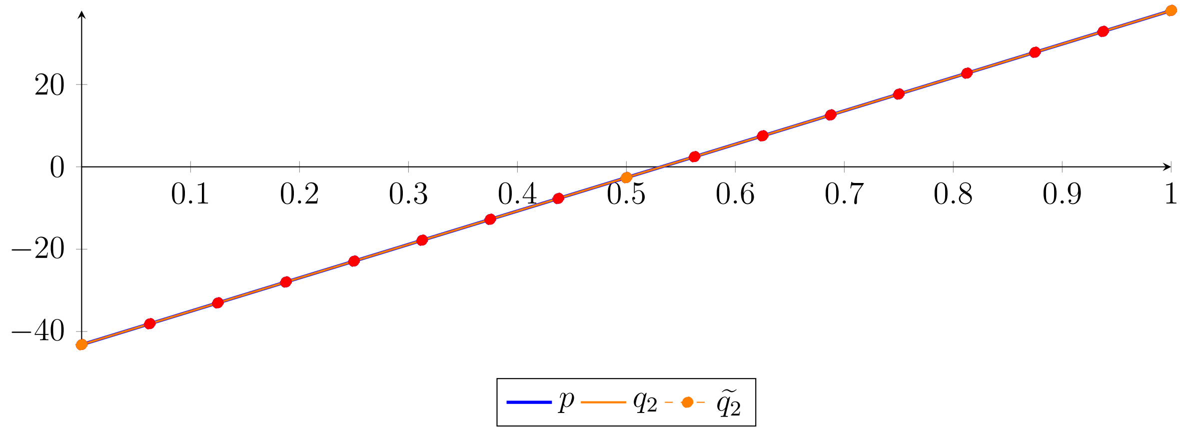
$$\begin{aligned} p = & -2.76723{\cdot}10^{-13}X^{16} - 2.40874{\cdot}10^{-12}X^{15} - 1.25233{\cdot}10^{-11}X^{14} - 3.02935{\cdot}10^{-11}X^{13} \\ & - 3.05617{\cdot}10^{-11}X^{12} - 3.83107{\cdot}10^{-11}X^{11} - 1.26692{\cdot}10^{-11}X^{10} - 2.6672{\cdot}10^{-11}X^{9} \\ & + 2.0004{\cdot}10^{-11}X^{8} + 4.12781{\cdot}10^{-12}X^{7} + 2.44493{\cdot}10^{-12}X^{6} - 1.21236{\cdot}10^{-13}X^{5} \\ & + 1.26288{\cdot}10^{-14}X^{4} - 4.1267{\cdot}10^{-12}X^{3} - 3.09388{\cdot}10^{-05}X^{2} + 81.1506X - 43.1911 \\ = & -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\ & - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X) \\ & + 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X) \\ & + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$\begin{aligned}\tilde{q}_2 &= 5.96265\cdot 10^{-05}X^{16} - 0.000436042X^{15} + 0.00141812X^{14} - 0.00269475X^{13} \\ &\quad + 0.00329809X^{12} - 0.00268757X^{11} + 0.00143268X^{10} - 0.000439599X^{9} \\ &\quad + 1.98418\cdot 10^{-05}X^{8} + 4.87608\cdot 10^{-05}X^{7} - 2.46333\cdot 10^{-05}X^{6} + 6.35808\cdot 10^{-06}X^{5} \\ &\quad - 9.62755\cdot 10^{-07}X^{4} + 8.21372\cdot 10^{-08}X^{3} - 3.09429\cdot 10^{-05}X^{2} + 81.1506X - 43.1911 \\ &= -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} \\ &\quad - 12.7597B_{6,16} - 7.68778B_{7,16} - 2.61587B_{8,16} + 2.45604B_{9,16} + 7.52795B_{10,16} + 12.5999B_{11,16} \\ &\quad + 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.5947\cdot 10^{-09}$.
Bounding polynomials M and m:

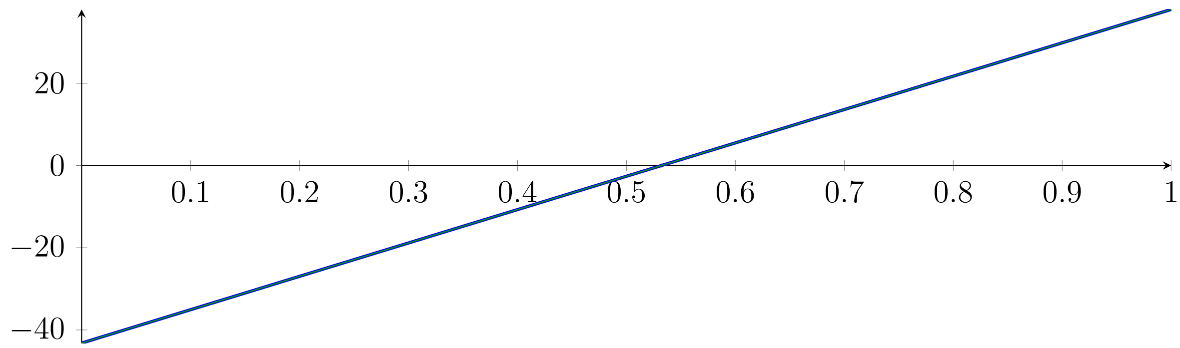
$$M = -3.09389\cdot 10^{-05}X^2 + 81.1506X - 43.1911$$

$$m = -3.09389\cdot 10^{-05}X^2 + 81.1506X - 43.1911$$

Root of M and m:

$$N(M) = \{0.532235, 2.62293\cdot 10^{6}\} \qquad N(m) = \{0.532235, 2.62293\cdot 10^{6}\}$$

Intersection intervals:



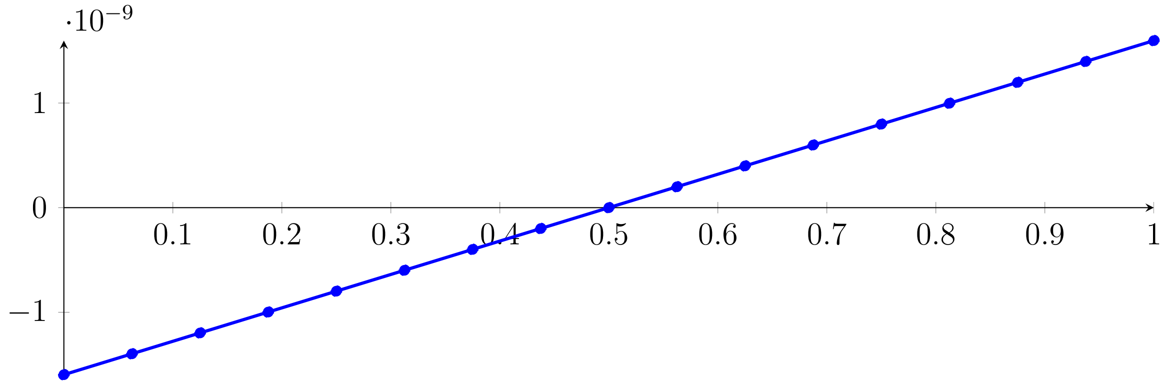
$$[0.532235, 0.532235]$$

Longest intersection interval: $3.93535\cdot 10^{-11}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

164.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

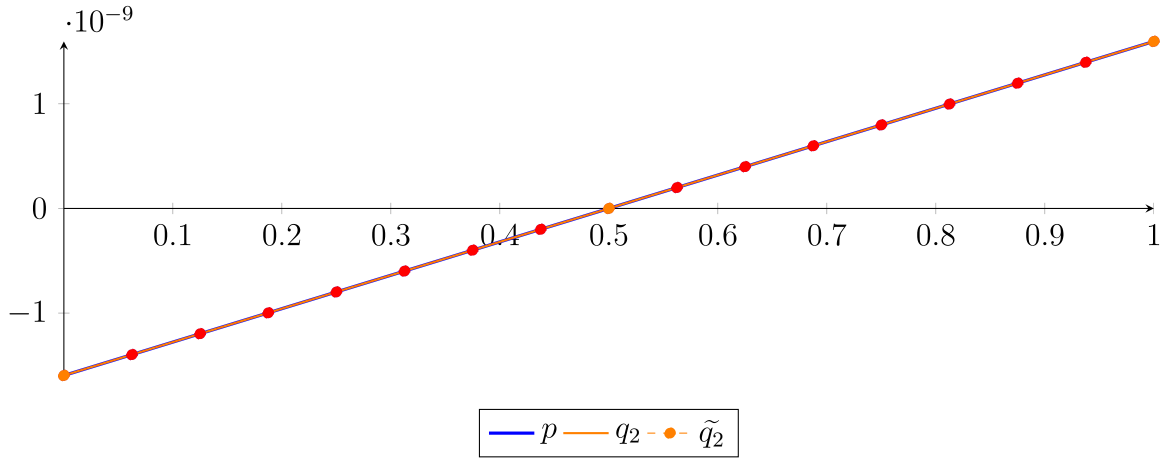
$$p = -4.89361\cdot10^{-24}X^{16} - 1.05466\cdot10^{-22}X^{15} - 3.09805\cdot10^{-22}X^{14} - 1.16981\cdot10^{-21}X^{13} - 9.76202\cdot10^{-22}X^{12} - 1.69365\cdot10^{-21}X^{11} - 5.95131\cdot10^{-22}X^{10} - 1.05349\cdot10^{-21}X^{9} + 7.5893\cdot10^{-22}X^{8} + 1.47859\cdot10^{-22}X^{7} + 9.70322\cdot10^{-23}X^{6} - 7.05688\cdot10^{-24}X^{5} + 1.47018\cdot10^{-24}X^{4} - 4.84676\cdot10^{-26}X^{2} + 3.19356\cdot10^{-09}X - 1.59674\cdot10^{-09}$$
$$= -1.59674\cdot10^{-09}B_{0,16}(X) - 1.39715\cdot10^{-09}B_{1,16}(X) - 1.19755\cdot10^{-09}B_{2,16}(X) - 9.97951\cdot10^{-10}B_{3,16}(X) - 7.98353\cdot10^{-10}B_{4,16}(X) - 5.98756\cdot10^{-10}B_{5,16}(X) - 3.99159\cdot10^{-10}B_{6,16}(X) - 1.99561\cdot10^{-10}B_{7,16}(X) + 3.6039\cdot10^{-14}B_{8,16}(X) + 1.99633\cdot10^{-10}B_{9,16}(X) + 3.99231\cdot10^{-10}B_{10,16}(X) + 5.98828\cdot10^{-10}B_{11,16}(X) + 7.98425\cdot10^{-10}B_{12,16}(X) + 9.98023\cdot10^{-10}B_{13,16}(X) + 1.19762\cdot10^{-09}B_{14,16}(X) + 1.39722\cdot10^{-09}B_{15,16}(X) + 1.59681\cdot10^{-09}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -4.83666\cdot10^{-26}X^{2} + 3.19356\cdot10^{-09}X - 1.59674\cdot10^{-09}$$
$$= -1.59674\cdot10^{-09}B_{0,2} + 3.6039\cdot10^{-14}B_{1,2} + 1.59681\cdot10^{-09}B_{2,2}$$

$$\widetilde{q}_2 = 9.45798\cdot10^{-15}X^{16} - 7.39324\cdot10^{-14}X^{15} + 2.61437\cdot10^{-13}X^{14} - 5.52989\cdot10^{-13}X^{13} + 7.79462\cdot10^{-13}X^{12} - 7.71893\cdot10^{-13}X^{11} + 5.51461\cdot10^{-13}X^{10} - 2.8712\cdot10^{-13}X^{9} + 1.08634\cdot10^{-13}X^{8} - 2.94042\cdot10^{-14}X^{7} + 5.52081\cdot10^{-15}X^{6} - 6.84058\cdot10^{-16}X^{5} + 5.20623\cdot10^{-17}X^{4} - 2.16513\cdot10^{-18}X^{3} + 1.74369\cdot10^{-20}X^{2} + 3.19356\cdot10^{-09}X - 1.59674\cdot10^{-09}$$
$$= -1.59674\cdot10^{-09}B_{0,16} - 1.39715\cdot10^{-09}B_{1,16} - 1.19755\cdot10^{-09}B_{2,16} - 9.97951\cdot10^{-10}B_{3,16} - 7.98353\cdot10^{-10}B_{4,16} - 5.98756\cdot10^{-10}B_{5,16} - 3.99159\cdot10^{-10}B_{6,16} - 1.99561\cdot10^{-10}B_{7,16} + 3.60393\cdot10^{-14}B_{8,16} + 1.99633\cdot10^{-10}B_{9,16} + 3.99231\cdot10^{-10}B_{10,16} + 5.98828\cdot10^{-10}B_{11,16} + 7.98425\cdot10^{-10}B_{12,16} + 9.98023\cdot10^{-10}B_{13,16} + 1.19762\cdot10^{-09}B_{14,16} + 1.39722\cdot10^{-09}B_{15,16} + 1.59681\cdot10^{-09}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.02367 \cdot 10^{-19}$.
Bounding polynomials M and m:

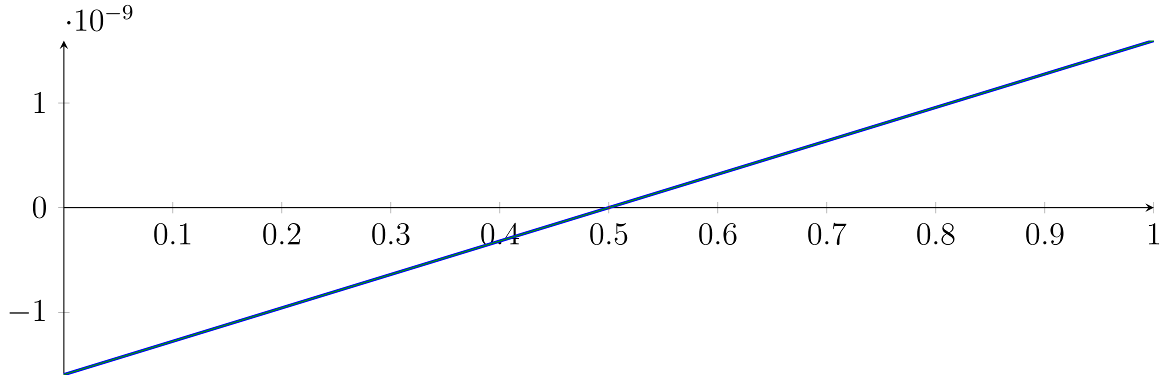
$$M = -4.82657 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$

$$m = -4.84676 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$

Root of M and m:

$$N(M) = \{0.5, 6.61662 \cdot 10^{16}\} \qquad N(m) = \{0.5, 6.58905 \cdot 10^{16}\}$$

Intersection intervals:



$[0.5, 0.5]$

Longest intersection interval: 0
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

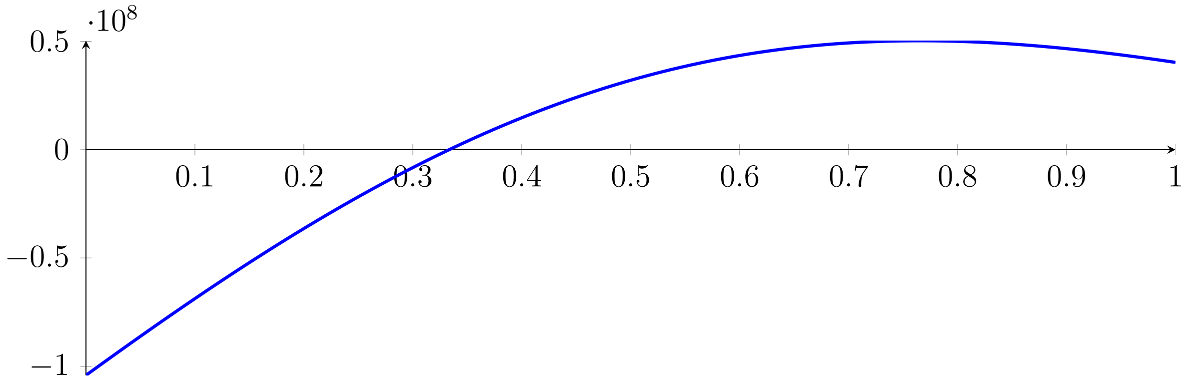
164.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

164.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} + 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

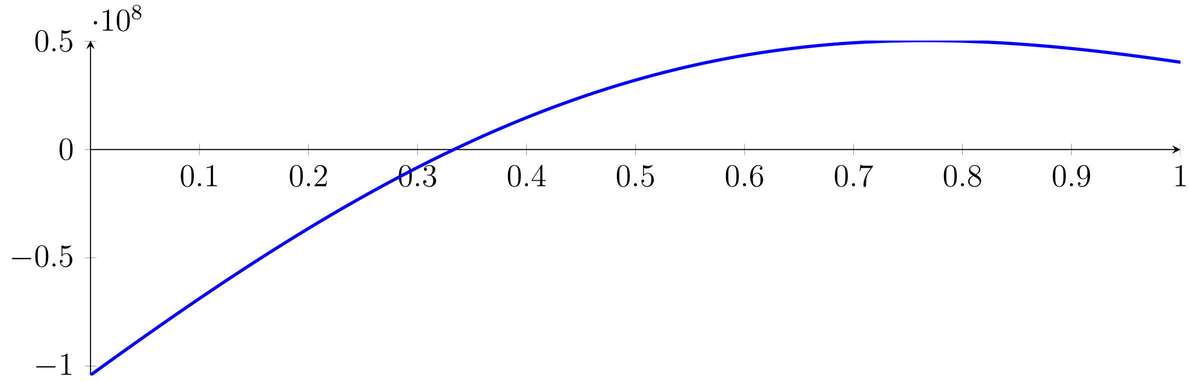
with precision $\varepsilon = 1{\cdot}10^{-64}$.

165 Running CubeClip on f_{16} with epsilon 64

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

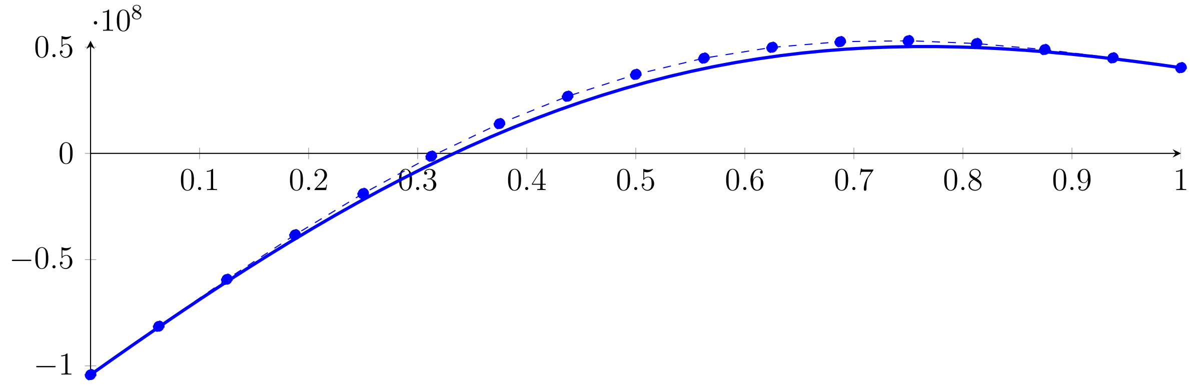
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



165.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

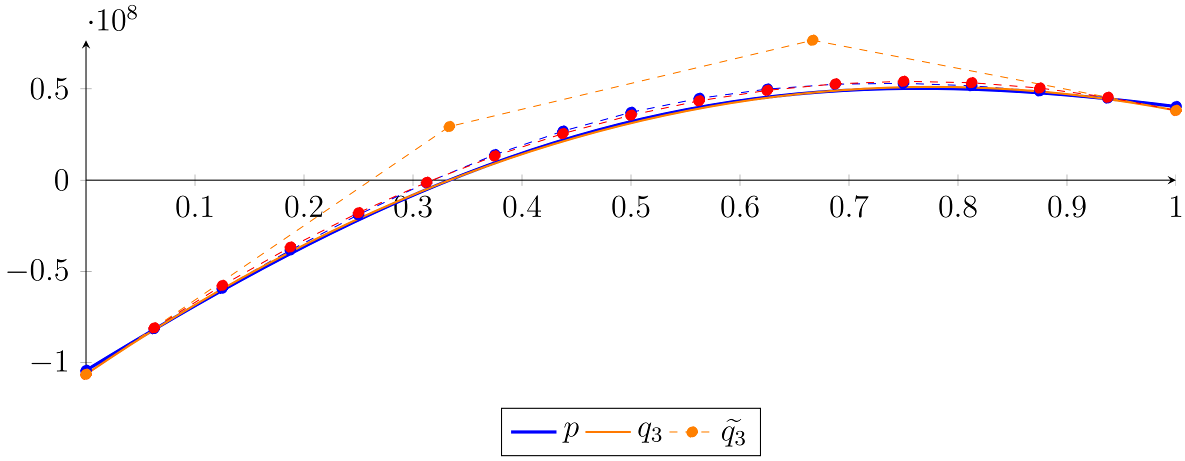
$$\begin{aligned} p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\quad \cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \\ &= -1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$
\begin{aligned}
q_3 &= 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.06335\cdot 10^8\\
&= -1.06335\cdot 10^8 B_{0,3} + 2.93558\cdot 10^7 B_{1,3} + 7.66068\cdot 10^7 B_{2,3} + 3.81764\cdot 10^7 B_{3,3}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{q_3} &= 2461.93X^{16} - 19614.9X^{15} + 70879.5X^{14} - 153661X^{13} + 222746X^{12} - 227755X^{11}\\
&\quad + 168826X^{10} - 91798.7X^9 + 36630.3X^8 - 10627.3X^7 + 2200.54X^6 - 316.059X^5\\
&\quad + 30.1958X^4 + 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.06335\cdot 10^8\\
&= -1.06335\cdot 10^8 B_{0,16} - 8.08928\cdot 10^7 B_{1,16} - 5.76618\cdot 10^7 B_{2,16} - 3.66369\cdot 10^7 B_{3,16} - 1.78131\\
&\quad \cdot 10^7 B_{4,16} - 1.18551\cdot 10^6 B_{5,16} + 1.32508\cdot 10^7 B_{6,16} + 2.55007\cdot 10^7 B_{7,16} + 3.55692\cdot 10^7 B_{8,16}\\
&\quad + 4.34611\cdot 10^7 B_{9,16} + 4.91815\cdot 10^7 B_{10,16} + 5.27353\cdot 10^7 B_{11,16} + 5.41273\cdot 10^7 B_{12,16}\\
&\quad + 5.33624\cdot 10^7 B_{13,16} + 5.04457\cdot 10^7 B_{14,16} + 4.53821\cdot 10^7 B_{15,16} + 3.81764\cdot 10^7 B_{16,16}
\end{aligned}
$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806\cdot 10^6$.

Bounding polynomials M and m:

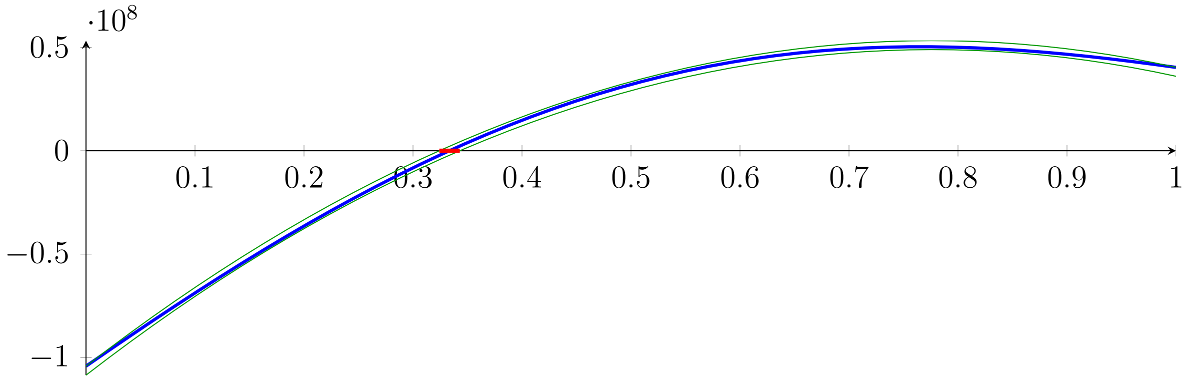
$$M = 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.04167\cdot 10^8$$

$$m = 2.75812\cdot 10^6 X^3 - 2.65319\cdot 10^8 X^2 + 4.07072\cdot 10^8 X - 1.08503\cdot 10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

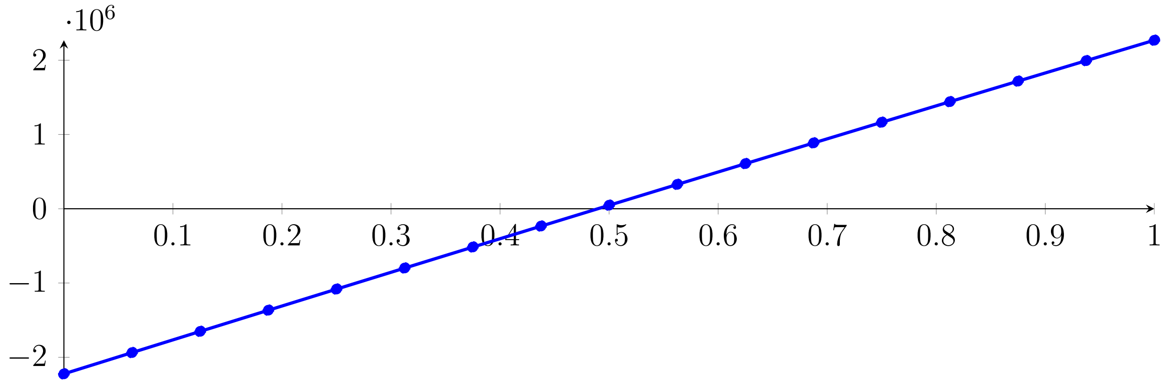
Longest intersection interval: 0.0187703

$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

165.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

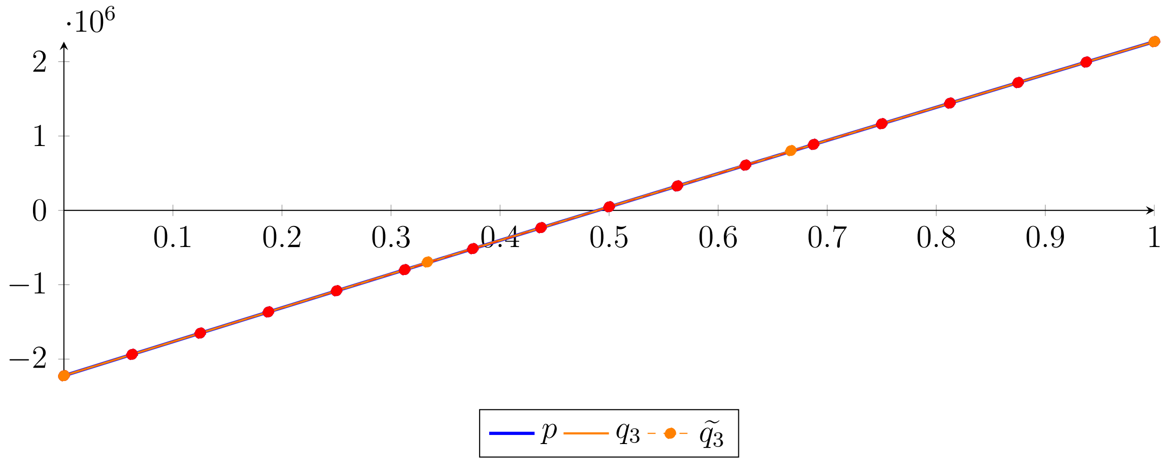
$$\begin{aligned} p &= -3.66617\cdot 10^{-09}X^{16} - 1.53217\cdot 10^{-07}X^{15} - 3.62234\cdot 10^{-07}X^{14} - 1.65579\cdot 10^{-06}X^{13} - 1.15373\cdot 10^{-06}X^{12} \\ &\quad - 2.3399\cdot 10^{-06}X^{11} - 5.02543\cdot 10^{-07}X^{10} - 1.38381\cdot 10^{-06}X^{9} + 1.1237\cdot 10^{-06}X^{8} - 1.19653\cdot 10^{-05}X^{7} \\ &\quad - 0.00155608X^{6} + 0.10496X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6} \\ &= -2.22335\cdot 10^{6}B_{0,16}(X) - 1.93666\cdot 10^{6}B_{1,16}(X) - 1.65074\cdot 10^{6}B_{2,16}(X) - 1.36561\cdot 10^{6}B_{3,16}(X) \\ &\quad - 1.08127\cdot 10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X) \\ &\quad + 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot 10^{6}B_{12,16}(X) + 1.44233 \\ &\quad \cdot 10^{6}B_{13,16}(X) + 1.71876\cdot 10^{6}B_{14,16}(X) + 1.99439\cdot 10^{6}B_{15,16}(X) + 2.26922\cdot 10^{6}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6} \\ &= -2.22335\cdot 10^{6}B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot 10^{6}B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 16.4956X^{16} - 129.161X^{15} + 457.83X^{14} - 971.671X^{13} + 1375.95X^{12} - 1370.96X^{11} \\ &\quad + 987.265X^{10} - 519.476X^{9} + 199.587X^{8} - 55.434X^{7} + 10.9237X^{6} - 1.48019X^{5} \\ &\quad + 0.129516X^{4} - 700.679X^{3} - 93879.9X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6} \\ &= -2.22335\cdot 10^{6}B_{0,16} - 1.93666\cdot 10^{6}B_{1,16} - 1.65074\cdot 10^{6}B_{2,16} - 1.36561\cdot 10^{6}B_{3,16} \\ &\quad - 1.08126\cdot 10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.4B_{8,16} \\ &\quad + 328650B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot 10^{6}B_{12,16} + 1.44233 \\ &\quad \cdot 10^{6}B_{13,16} + 1.71876\cdot 10^{6}B_{14,16} + 1.99439\cdot 10^{6}B_{15,16} + 2.26922\cdot 10^{6}B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m:

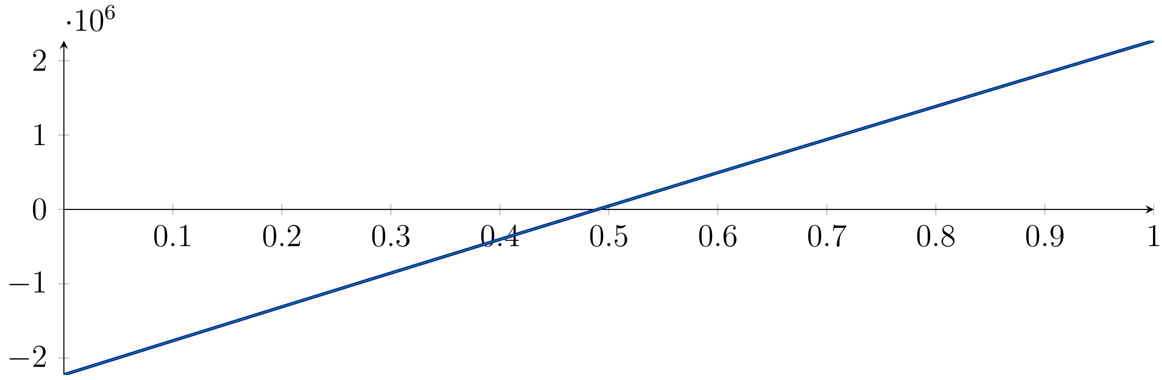
$$M = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6}$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715{\cdot}10^6X - 2.22335{\cdot}10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



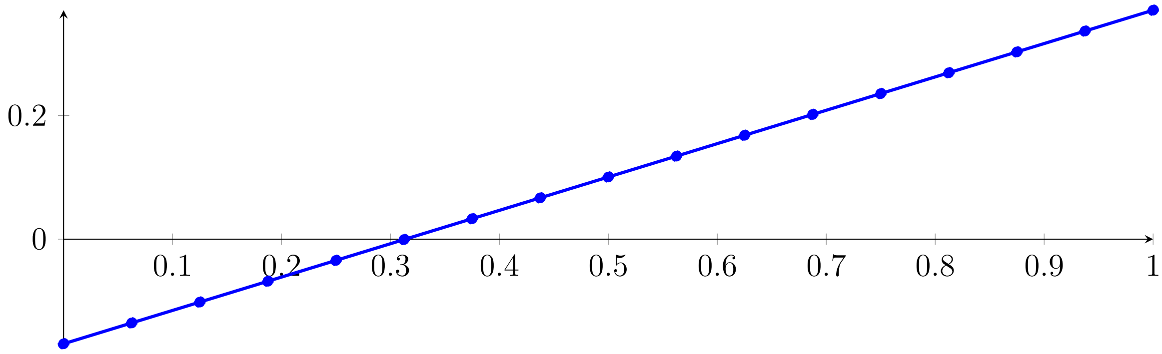
$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

165.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

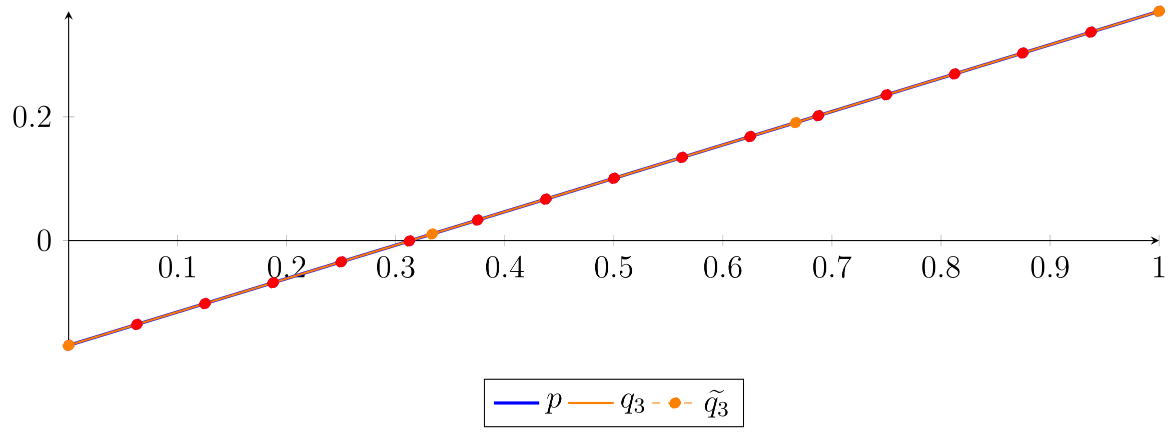
$$\begin{aligned}
p &= 5.55524{\cdot}10^{-15}X^{16} - 2.94313{\cdot}10^{-14}X^{15} + 1.19384{\cdot}10^{-13}X^{14} - 2.17482{\cdot}10^{-13}X^{13} + 7.26155{\cdot}10^{-14}X^{12} \\
&\quad - 3.44766{\cdot}10^{-13}X^{11} - 3.47292{\cdot}10^{-15}X^{10} - 1.1287{\cdot}10^{-13}X^9 + 2.93027{\cdot}10^{-14}X^8 + 8.06213{\cdot}10^{-15}X^7 \\
&\quad + 5.64349{\cdot}10^{-15}X^6 + 4.93312{\cdot}10^{-17}X^4 + 1.51788{\cdot}10^{-17}X^3 - 1.37072{\cdot}10^{-09}X^2 + 0.540148X - 0.169396 \\
&= -0.169396B_{0,16}(X) - 0.135637B_{1,16}(X) - 0.101877B_{2,16}(X) - 0.068118B_{3,16}(X) \\
&\quad - 0.0343587B_{4,16}(X) - 0.000599476B_{5,16}(X) + 0.0331598B_{6,16}(X) \\
&\quad + 0.0669191B_{7,16}(X) + 0.100678B_{8,16}(X) + 0.134438B_{9,16}(X) + 0.168197B_{10,16}(X) \\
&\quad + 0.201956B_{11,16}(X) + 0.235715B_{12,16}(X) + 0.269475B_{13,16}(X) \\
&\quad + 0.303234B_{14,16}(X) + 0.336993B_{15,16}(X) + 0.370752B_{16,16}(X)
\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
q_3 &= -1.07065{\cdot}10^{-17}X^3 - 1.37072{\cdot}10^{-09}X^2 + 0.540148X - 0.169396 \\
&= -0.169396B_{0,3} + 0.0106536B_{1,3} + 0.190703B_{2,3} + 0.370752B_{3,3}
\end{aligned}$$

$$
\begin{aligned}
\widetilde{q}_3 &= 8.59095{\cdot}10^{-06}X^{16} - 6.82648{\cdot}10^{-05}X^{15} + 0.000245968X^{14} - 0.000531568X^{13}\\
&\quad + 0.000767923X^{12} - 0.000782231X^{11} + 0.0005774X^{10} - 0.000312464X^{9}\\
&\quad + 0.000123994X^{8} - 3.57388{\cdot}10^{-05}X^{7} + 7.34249{\cdot}10^{-06}X^{6} - 1.04474{\cdot}10^{-06}X^{5}\\
&\quad + 9.86739{\cdot}10^{-08}X^{4} - 5.7553{\cdot}10^{-09}X^{3} - 1.19186{\cdot}10^{-09}X^{2} + 0.540148X - 0.169396\\
&= -0.169396B_{0,16} - 0.135637B_{1,16} - 0.101877B_{2,16} - 0.068118B_{3,16} - 0.0343587B_{4,16}\\
&\quad - 0.000599476B_{5,16} + 0.0331598B_{6,16} + 0.0669191B_{7,16} + 0.100678B_{8,16}\\
&\quad + 0.134438B_{9,16} + 0.168197B_{10,16} + 0.201956B_{11,16} + 0.235715B_{12,16}\\
&\quad + 0.269475B_{13,16} + 0.303234B_{14,16} + 0.336993B_{15,16} + 0.370752B_{16,16}
\end{aligned}
$$



The maximum difference of the Bézier coefficients is $\delta = 1.81206{\cdot}10^{-10}$.

Bounding polynomials M and m:

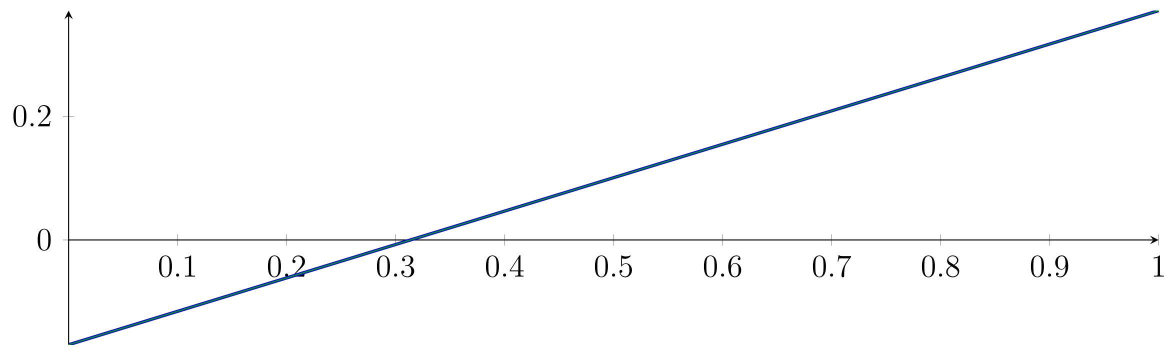
$$M = -1.07065{\cdot}10^{-17}X^3 - 1.37072{\cdot}10^{-09}X^2 + 0.540148X - 0.169396$$

$$m = -1.07065{\cdot}10^{-17}X^3 - 1.37072{\cdot}10^{-09}X^2 + 0.540148X - 0.169396$$

Root of M and m:

$$N(M) = \{-2.97569{\cdot}10^{8}, 0.31361, 1.69542{\cdot}10^{8}\} \quad N(m) = \{-2.97569{\cdot}10^{8}, 0.31361, 1.69542{\cdot}10^{8}\}$$

Intersection intervals:



$$[0.31361, 0.31361]$$

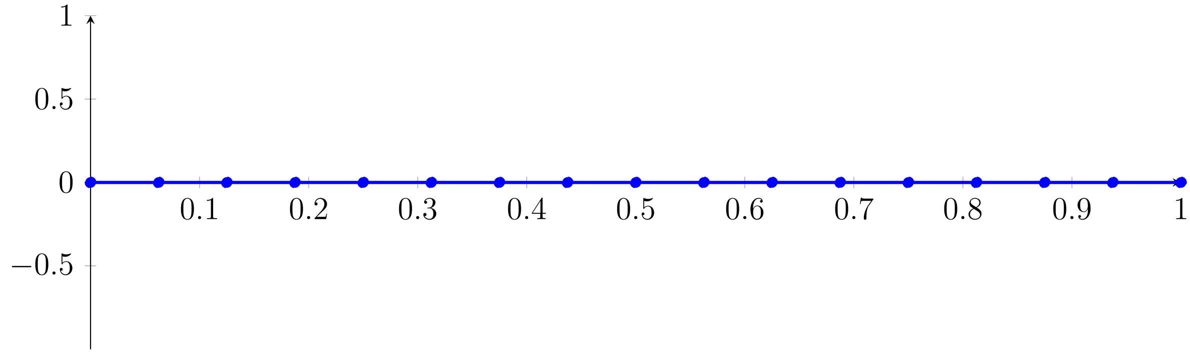
Longest intersection interval: $7.85803{\cdot}10^{-10}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

165.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

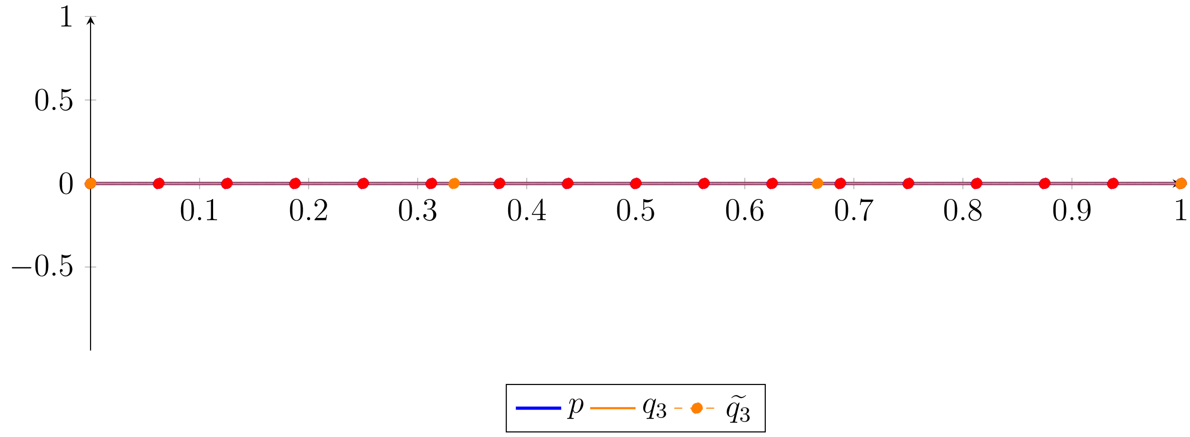
$$p = -1.51576\cdot 10^{-21}X^{16} + 2.62009\cdot 10^{-21}X^{15} - 3.98039\cdot 10^{-20}X^{14} + 3.2136\cdot 10^{-21}X^{13} - 5.16564\cdot 10^{-20}X^{12} + 1.52429\cdot 10^{-20}X^{11} - 1.44901\cdot 10^{-20}X^{10} - 1.40466\cdot 10^{-20}X^{9} + 2.34541\cdot 10^{-20}X^{8} + 3.25289\cdot 10^{-21}X^{7} + 2.38052\cdot 10^{-21}X^{6} - 2.2582\cdot 10^{-22}X^{5} + 3.52844\cdot 10^{-23}X^{4} + 4.2445\cdot 10^{-10}X - 2.39831\cdot 10^{-08}$$
$$= -2.39831\cdot 10^{-08}B_{0,16}(X) - 2.39566\cdot 10^{-08}B_{1,16}(X) - 2.39301\cdot 10^{-08}B_{2,16}(X) - 2.39036\cdot 10^{-08}B_{3,16}(X) - 2.3877\cdot 10^{-08}B_{4,16}(X) - 2.38505\cdot 10^{-08}B_{5,16}(X) - 2.3824\cdot 10^{-08}B_{6,16}(X) - 2.37974\cdot 10^{-08}B_{7,16}(X) - 2.37709\cdot 10^{-08}B_{8,16}(X) - 2.37444\cdot 10^{-08}B_{9,16}(X) - 2.37179\cdot 10^{-08}B_{10,16}(X) - 2.36913\cdot 10^{-08}B_{11,16}(X) - 2.36648\cdot 10^{-08}B_{12,16}(X) - 2.36383\cdot 10^{-08}B_{13,16}(X) - 2.36118\cdot 10^{-08}B_{14,16}(X) - 2.35852\cdot 10^{-08}B_{15,16}(X) - 2.35587\cdot 10^{-08}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -3.5866\cdot 10^{-25}X^{3} + 8.33643\cdot 10^{-25}X^{2} + 4.2445\cdot 10^{-10}X - 2.39831\cdot 10^{-08}$$
$$= -2.39831\cdot 10^{-08}B_{0,3} - 2.38417\cdot 10^{-08}B_{1,3} - 2.37002\cdot 10^{-08}B_{2,3} - 2.35587\cdot 10^{-08}B_{3,3}$$

$$\widetilde{q}_3 = -1.64958\cdot 10^{-12}X^{16} + 1.3166\cdot 10^{-11}X^{15} - 4.76688\cdot 10^{-11}X^{14} + 1.03558\cdot 10^{-10}X^{13} - 1.50448\cdot 10^{-10}X^{12} + 1.54183\cdot 10^{-10}X^{11} - 1.1456\cdot 10^{-10}X^{10} + 6.24452\cdot 10^{-11}X^{9} - 2.49793\cdot 10^{-11}X^{8} + 7.26358\cdot 10^{-12}X^{7} - 1.50649\cdot 10^{-12}X^{6} + 2.16616\cdot 10^{-13}X^{5} - 2.07725\cdot 10^{-14}X^{4} + 1.24748\cdot 10^{-15}X^{3} - 4.0727\cdot 10^{-17}X^{2} + 4.2445\cdot 10^{-10}X - 2.39831\cdot 10^{-08}$$
$$= -2.39831\cdot 10^{-08}B_{0,16} - 2.39566\cdot 10^{-08}B_{1,16} - 2.39301\cdot 10^{-08}B_{2,16} - 2.39036\cdot 10^{-08}B_{3,16} - 2.3877\cdot 10^{-08}B_{4,16} - 2.38505\cdot 10^{-08}B_{5,16} - 2.3824\cdot 10^{-08}B_{6,16} - 2.37974\cdot 10^{-08}B_{7,16} - 2.37709\cdot 10^{-08}B_{8,16} - 2.37444\cdot 10^{-08}B_{9,16} - 2.37179\cdot 10^{-08}B_{10,16} - 2.36913\cdot 10^{-08}B_{11,16} - 2.36648\cdot 10^{-08}B_{12,16} - 2.36383\cdot 10^{-08}B_{13,16} - 2.36118\cdot 10^{-08}B_{14,16} - 2.35852\cdot 10^{-08}B_{15,16} - 2.35587\cdot 10^{-08}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 3.51589{\cdot}10^{-17}$.

Bounding polynomials M and m:

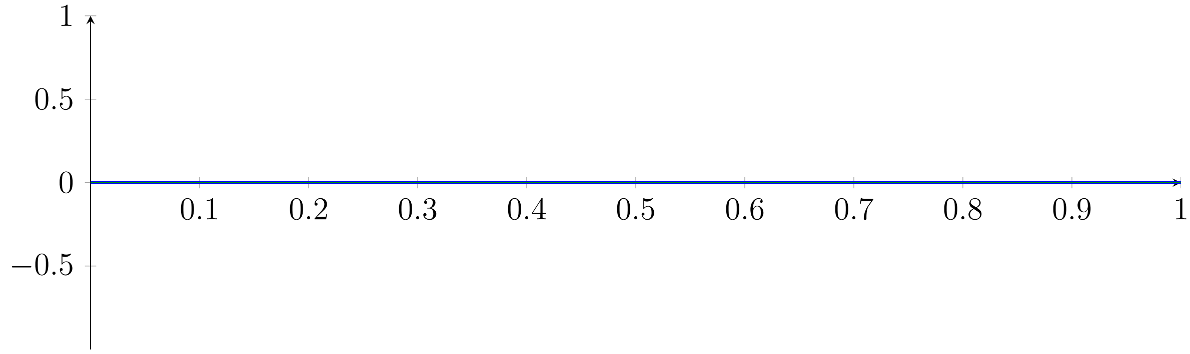
$$M = -3.5866{\cdot}10^{-25}X^3 + 8.33643{\cdot}10^{-25}X^2 + 4.2445{\cdot}10^{-10}X - 2.39831{\cdot}10^{-08}$$

$$m = -3.5866{\cdot}10^{-25}X^3 + 8.33643{\cdot}10^{-25}X^2 + 4.2445{\cdot}10^{-10}X - 2.39831{\cdot}10^{-08}$$

Root of M and m:

$$N(M) = \{-3.44011{\cdot}10^{7}, 56.504, 3.4401{\cdot}10^{7}\} \qquad N(m) = \{-3.44011{\cdot}10^{7}, 56.504, 3.4401{\cdot}10^{7}\}$$

Intersection intervals:

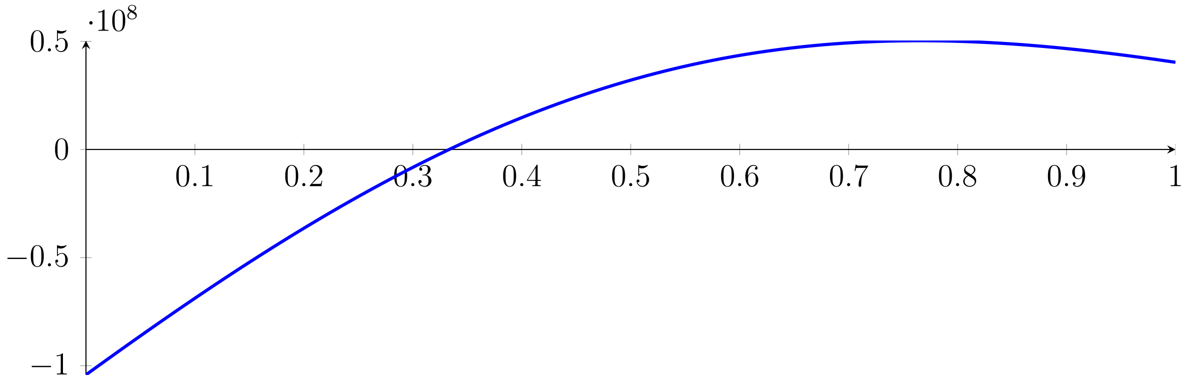


No intersection intervals with the x axis.

165.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

with precision $\varepsilon = 1\cdot10^{-64}$.

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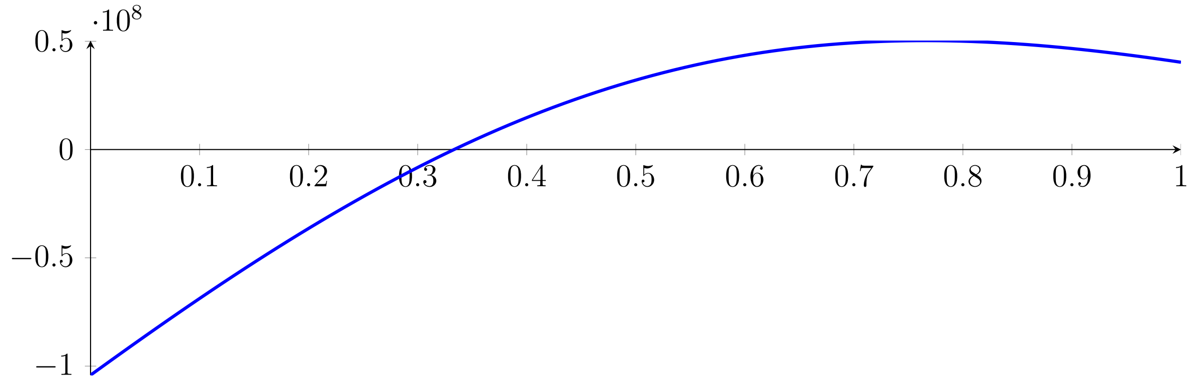
$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called BezClip with input polynomial on interval $[0,1]$:

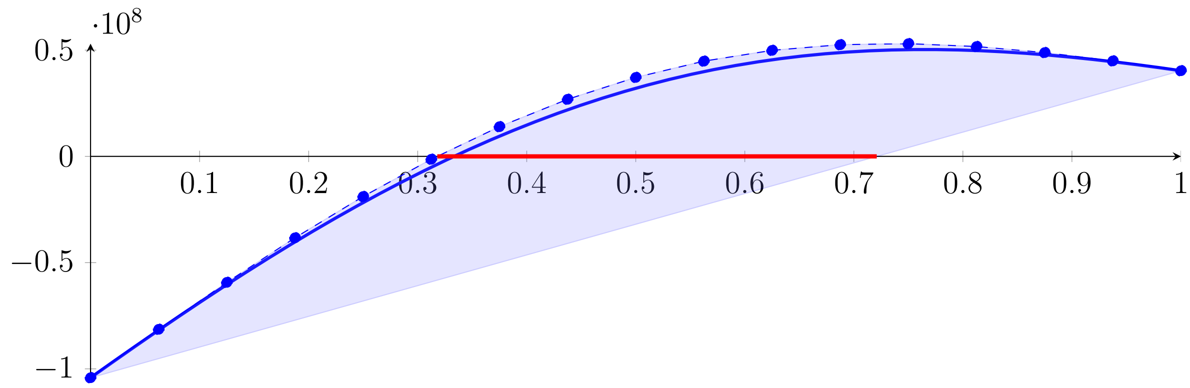
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} \\ &+ 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \end{aligned}$$



166.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &+ 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \\ = &-1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203 \\ &\cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X) \\ &+ 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901 \\ &\cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X) \\ &+ 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

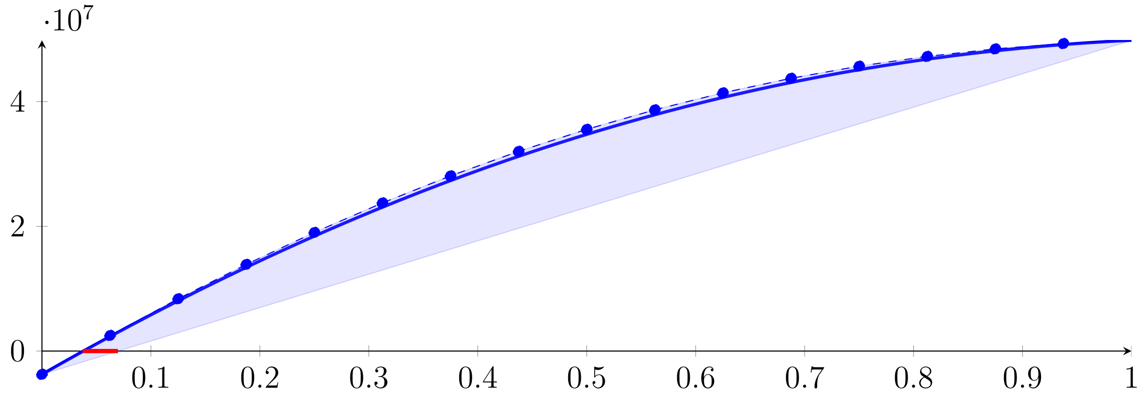
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

166.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.59825\cdot 10^{-06}X^{16} - 5.93153\cdot 10^{-05}X^{15} - 0.00248867X^{14} - 0.0650056X^{13} - 0.909142X^{12} \\ &\quad - 5.03931X^{11} + 36.4549X^{10} + 692.921X^{9} + 1886.97X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5} \\ &\quad + 3.91945\cdot 10^{6}X^{4} - 7.54561\cdot 10^{6}X^{3} - 4.29225\cdot 10^{7}X^{2} + 9.97982\cdot 10^{7}X - 3.73501\cdot 10^{6} \\ &= -3.73501\cdot 10^{6}B_{0,16}(X) + 2.50237\cdot 10^{6}B_{1,16}(X) + 8.38207\cdot 10^{6}B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^{7}B_{3,16}(X) + 1.90167\cdot 10^{7}B_{4,16}(X) + 2.37512\cdot 10^{7}B_{5,16}(X) + 2.80875\cdot 10^{7}B_{6,16}(X) \\ &\quad + 3.20213\cdot 10^{7}B_{7,16}(X) + 3.55508\cdot 10^{7}B_{8,16}(X) + 3.86768\cdot 10^{7}B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^{7}B_{10,16}(X) + 4.37336\cdot 10^{7}B_{11,16}(X) + 4.56783\cdot 10^{7}B_{12,16}(X) + 4.7247\cdot 10^{7}B_{13,16}(X) \\ &\quad + 4.84524\cdot 10^{7}B_{14,16}(X) + 4.93093\cdot 10^{7}B_{15,16}(X) + 4.98343\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

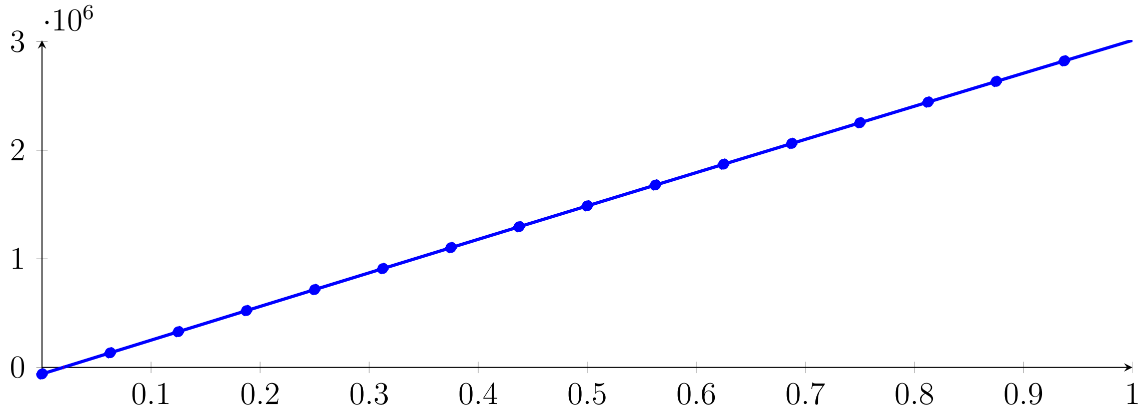
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

166.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 9.01396\cdot 10^{-08}X^{16} - 2.65848\cdot 10^{-07}X^{15} + 2.13948\cdot 10^{-06}X^{14} - 1.33627\cdot 10^{-06}X^{13} + 2.46973\cdot 10^{-06}X^{12} \\ &\quad - 2.45524\cdot 10^{-06}X^{11} + 5.50112\cdot 10^{-07}X^{10} - 1.64198\cdot 10^{-07}X^{9} - 7.35598\cdot 10^{-07}X^{8} - 1.00892\cdot 10^{-06}X^{7} \\ &\quad - 0.000177509X^{6} + 0.0161038X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot 10^{6}X - 60508.5 \\ &= -60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X) \\ &\quad + 910202B_{5,16}(X) + 1.1032\cdot 10^{6}B_{6,16}(X) + 1.29582\cdot 10^{6}B_{7,16}(X) + 1.48805\cdot 10^{6}B_{8,16}(X) \\ &\quad + 1.6799\cdot 10^{6}B_{9,16}(X) + 1.87137\cdot 10^{6}B_{10,16}(X) + 2.06245\cdot 10^{6}B_{11,16}(X) + 2.25315\cdot 10^{6}B_{12,16}(X) \\ &\quad + 2.44346\cdot 10^{6}B_{13,16}(X) + 2.63339\cdot 10^{6}B_{14,16}(X) + 2.82293\cdot 10^{6}B_{15,16}(X) + 3.01209\cdot 10^{6}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

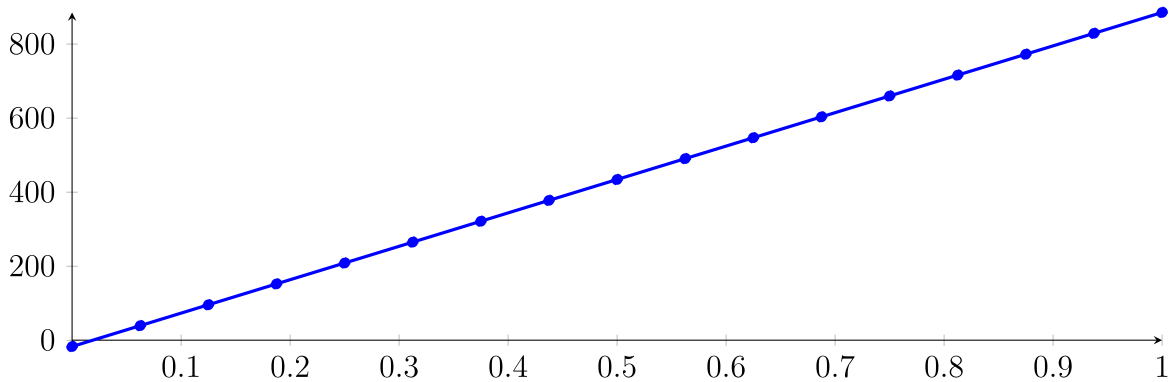
$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

166.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 2.55372\cdot 10^{-11}X^{16} - 7.21263\cdot 10^{-11}X^{15} + 6.24141\cdot 10^{-10}X^{14} - 4.11162\cdot 10^{-10}X^{13} \\ &\quad + 6.82359\cdot 10^{-10}X^{12} - 7.09475\cdot 10^{-10}X^{11} + 9.71305\cdot 10^{-11}X^{10} - 3.46101\cdot 10^{-11}X^{9} \\ &\quad - 2.13971\cdot 10^{-10}X^{8} - 1.46061\cdot 10^{-11}X^{7} - 1.63366\cdot 10^{-11}X^{6} + 1.87916\cdot 10^{-12}X^{5} \\ &\quad + 2.52576\cdot 10^{-14}X^{4} - 5.67777\cdot 10^{-09}X^{3} - 0.00382618X^{2} + 902.448X - 17.178 \\ &= -17.178B_{0,16}(X) + 39.225B_{1,16}(X) + 95.6279B_{2,16}(X) + 152.031B_{3,16}(X) + 208.434B_{4,16}(X) \\ &\quad + 264.837B_{5,16}(X) + 321.24B_{6,16}(X) + 377.642B_{7,16}(X) + 434.045B_{8,16}(X) \\ &\quad + 490.448B_{9,16}(X) + 546.851B_{10,16}(X) + 603.253B_{11,16}(X) + 659.656B_{12,16}(X) \\ &\quad + 716.059B_{13,16}(X) + 772.461B_{14,16}(X) + 828.864B_{15,16}(X) + 885.266B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

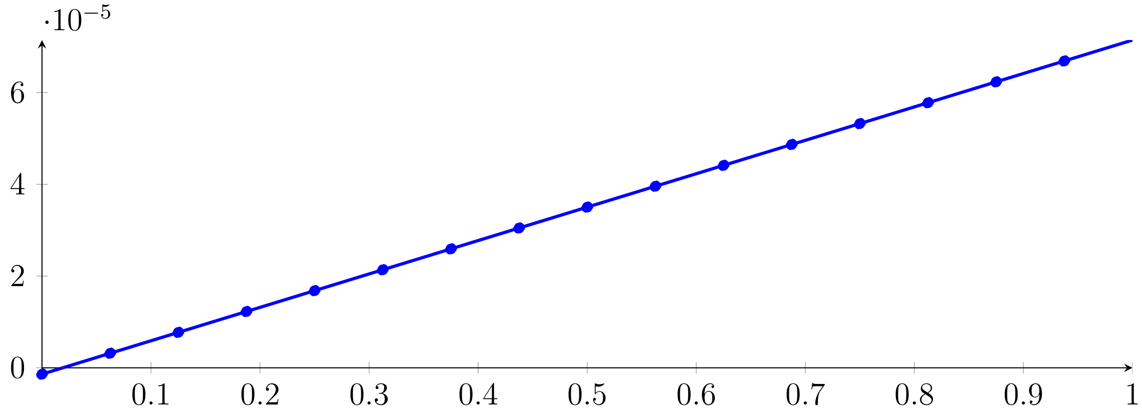
$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045\cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

166.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 2.14261{\cdot}10^{-18}X^{16} - 6.28573{\cdot}10^{-18}X^{15} + 5.28612{\cdot}10^{-17}X^{14} - 3.67279{\cdot}10^{-17}X^{13} + 6.10136{\cdot}10^{-17}X^{12} - 6.60335{\cdot}10^{-17}X^{11} + 1.66661{\cdot}10^{-17}X^{10} - 8.36524{\cdot}10^{-18}X^{9} - 1.56919{\cdot}10^{-17}X^{8} - 1.85474{\cdot}10^{-18}X^{7} - 1.4308{\cdot}10^{-18}X^{6} + 1.1562{\cdot}10^{-19}X^{5} - 1.20437{\cdot}10^{-20}X^{4} - 4.63221{\cdot}10^{-22}X^{3} - 2.49207{\cdot}10^{-17}X^{2} + 7.28316{\cdot}10^{-05}X - 1.38634{\cdot}10^{-06}$$
$$= -1.38634{\cdot}10^{-06}B_{0,16}(X) + 3.16564{\cdot}10^{-06}B_{1,16}(X) + 7.71761{\cdot}10^{-06}B_{2,16}(X) + 1.22696{\cdot}10^{-05}B_{3,16}(X) + 1.68216{\cdot}10^{-05}B_{4,16}(X) + 2.13735{\cdot}10^{-05}B_{5,16}(X) + 2.59255{\cdot}10^{-05}B_{6,16}(X) + 3.04775{\cdot}10^{-05}B_{7,16}(X) + 3.50295{\cdot}10^{-05}B_{8,16}(X) + 3.95814{\cdot}10^{-05}B_{9,16}(X) + 4.41334{\cdot}10^{-05}B_{10,16}(X) + 4.86854{\cdot}10^{-05}B_{11,16}(X) + 5.32374{\cdot}10^{-05}B_{12,16}(X) + 5.77893{\cdot}10^{-05}B_{13,16}(X) + 6.23413{\cdot}10^{-05}B_{14,16}(X) + 6.68933{\cdot}10^{-05}B_{15,16}(X) + 7.14453{\cdot}10^{-05}B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.51313{\cdot}10^{-15}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

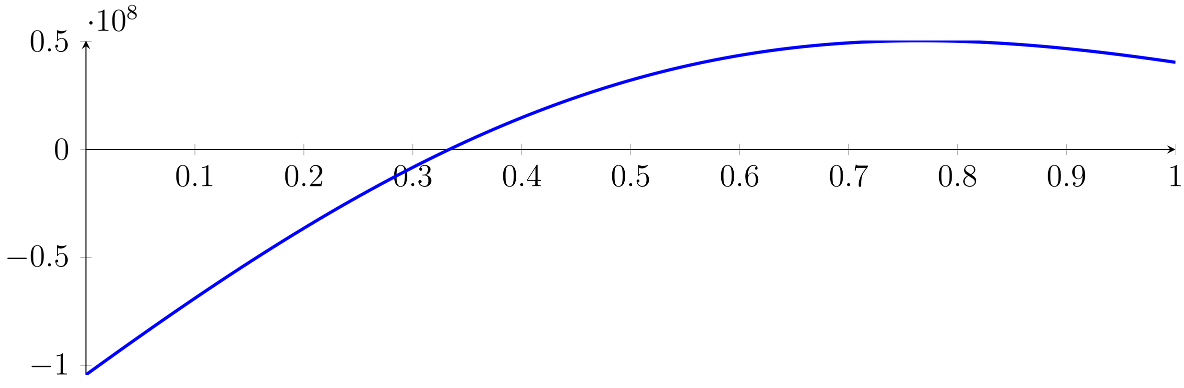
166.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

166.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1\cdot10^{-128}$.

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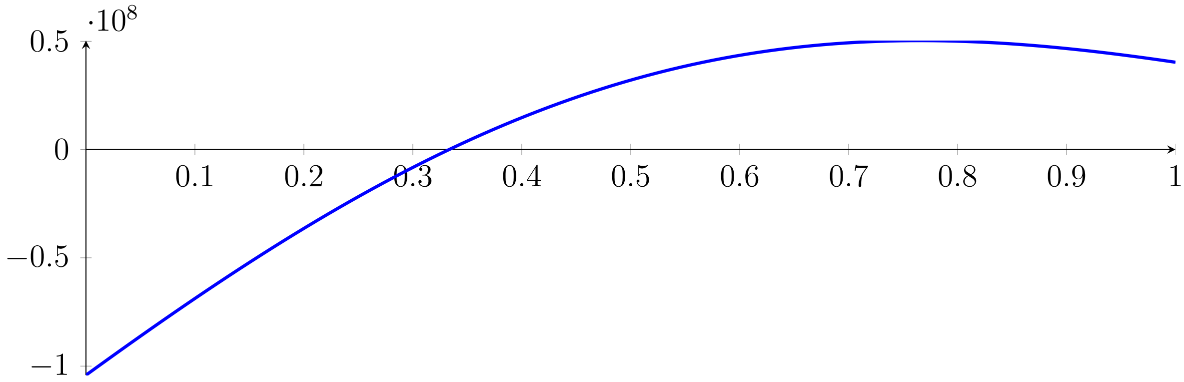
$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called QuadClip with input polynomial on interval $[0,1]$:

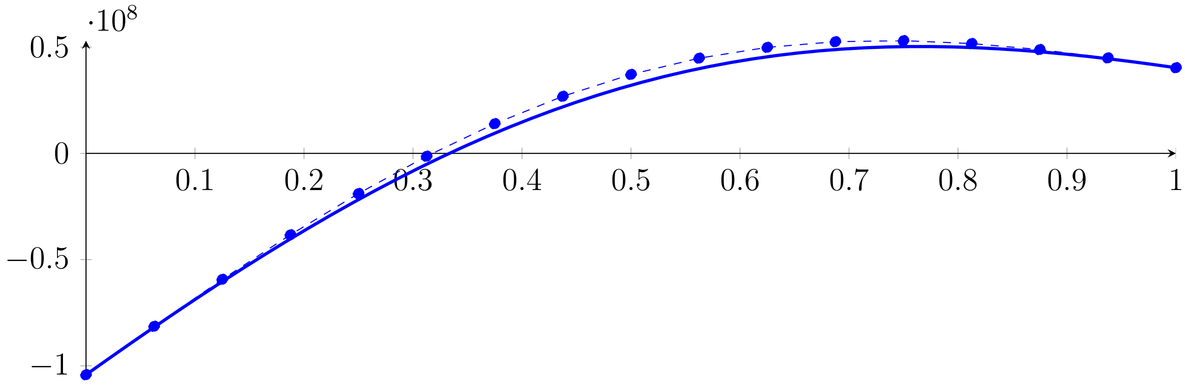
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



167.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

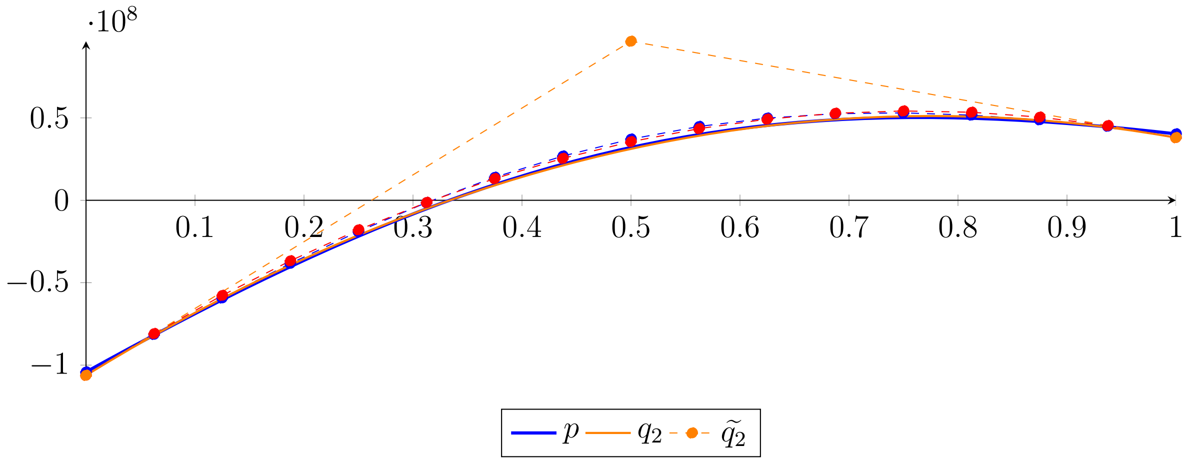
$$\begin{aligned} p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\quad \cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ &= -1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8 \\ &= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 6049.18X^{16} - 48305.2X^{15} + 174971X^{14} - 380294X^{13} + 552846X^{12} - 567203X^{11} \\ &\quad + 422303X^{10} - 231038X^9 + 93003.6X^8 - 27320.1X^7 + 5752.57X^6 - 843.63X^5 \\ &\quad + 82.5145X^4 - 5.01388X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8 \\ &= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017 \\ &\quad \cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16} \\ &\quad + 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16} \\ &\quad + 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m:

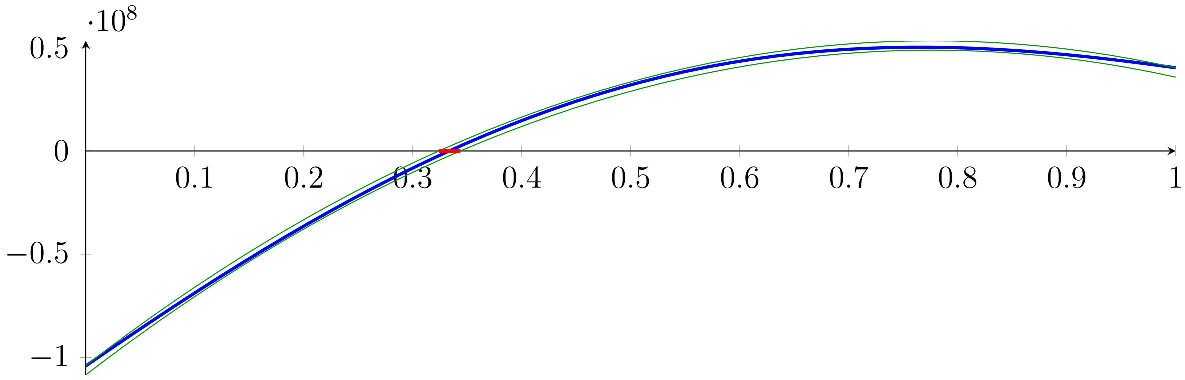
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

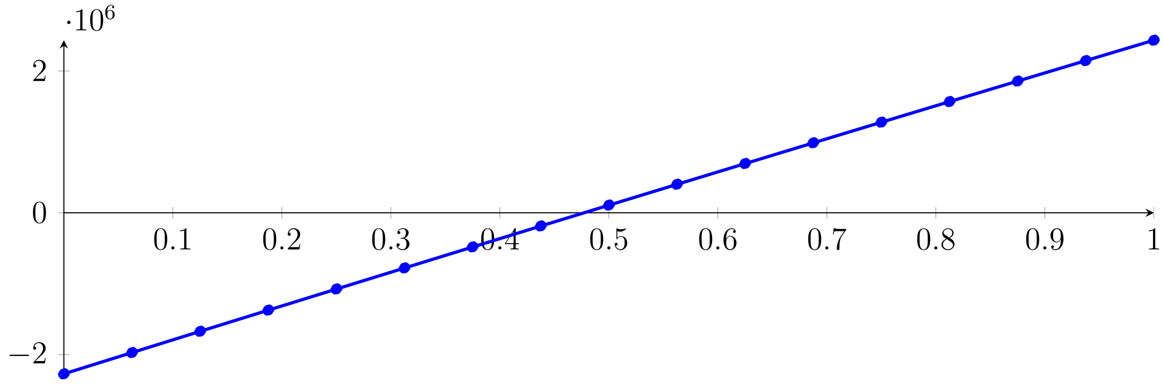
Longest intersection interval: 0.0196686

$\Longrightarrow$ Selective recursion: interval 1: [0.323946, 0.343615],

167.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

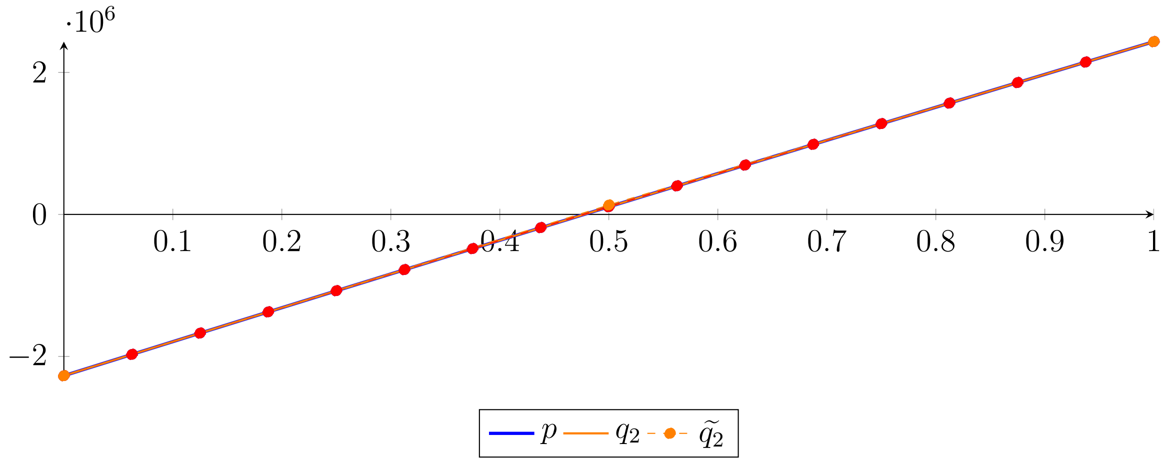
$$p = -1.54841{\cdot}10^{-10}X^{16} - 1.66965{\cdot}10^{-07}X^{15} - 2.92739{\cdot}10^{-07}X^{14} - 1.77943{\cdot}10^{-06}X^{13} - 1.17235{\cdot}10^{-06}X^{12} - 2.42234{\cdot}10^{-06}X^{11} - 6.86445{\cdot}10^{-07}X^{10} - 1.39162{\cdot}10^{-06}X^{9} + 1.07395{\cdot}10^{-06}X^{8} - 1.67072{\cdot}10^{-05}X^{7} - 0.00205879X^{6} + 0.132721X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874{\cdot}10^{6}X - 2.2715{\cdot}10^{6}$$
$$= -2.2715\cdot10^{6}B_{0,16}(X) - 1.97096\cdot10^{6}B_{1,16}(X) - 1.67127\cdot10^{6}B_{2,16}(X) - 1.37244\cdot10^{6}B_{3,16}(X) - 1.07448\cdot10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806\cdot10^{6}B_{12,16}(X) + 1.5682\cdot10^{6}B_{13,16}(X) + 1.85747\cdot10^{6}B_{14,16}(X) + 2.14586\cdot10^{6}B_{15,16}(X) + 2.43338\cdot10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265X^{2} + 4.80923\cdot10^{6}X - 2.27154\cdot10^{6}$$
$$= -2.27154\cdot10^{6}B_{0,2} + 133069B_{1,2} + 2.43342\cdot10^{6}B_{2,2}$$

$$\widetilde{q}_2 = 22.7036X^{16} - 178.964X^{15} + 638.976X^{14} - 1366.64X^{13} + 1951.1X^{12} - 1960.92X^{11} + 1425.35X^{10} - 757.631X^{9} + 294.177X^{8} - 82.4368X^{7} + 16.2856X^{6} - 2.18949X^{5} + 0.191237X^{4} - 0.0101048X^{3} - 104265X^{2} + 4.80923\cdot10^{6}X - 2.27154\cdot10^{6}$$
$$= -2.27154\cdot10^{6}B_{0,16} - 1.97097\cdot10^{6}B_{1,16} - 1.67126\cdot10^{6}B_{2,16} - 1.37242\cdot10^{6}B_{3,16} - 1.07445\cdot10^{6}B_{4,16} - 777350B_{5,16} - 481118B_{6,16} - 185754B_{7,16} + 108740B_{8,16} + 402366B_{9,16} + 695123B_{10,16} + 987011B_{11,16} + 1.27803\cdot10^{6}B_{12,16} + 1.56818\cdot10^{6}B_{13,16} + 1.85746\cdot10^{6}B_{14,16} + 2.14587\cdot10^{6}B_{15,16} + 2.43342\cdot10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m:

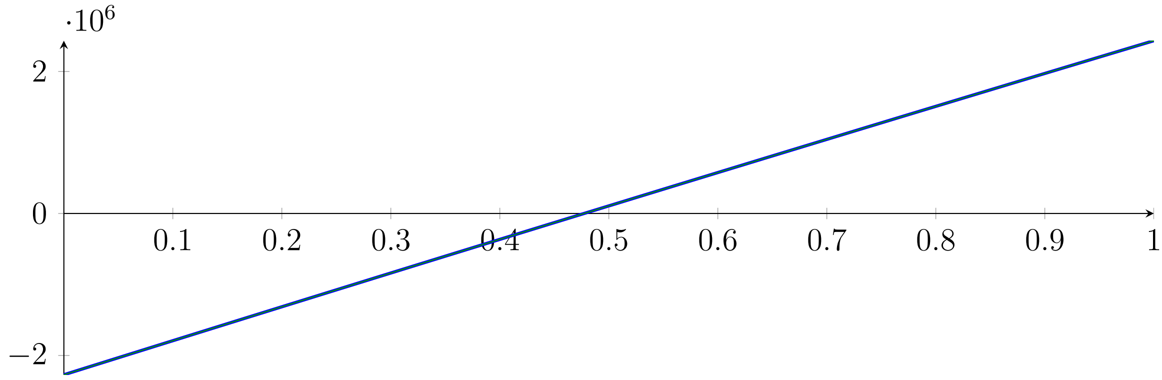
$$M = -104265X^{2} + 4.80923\cdot10^{6}X - 2.2715\cdot10^{6}$$

$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



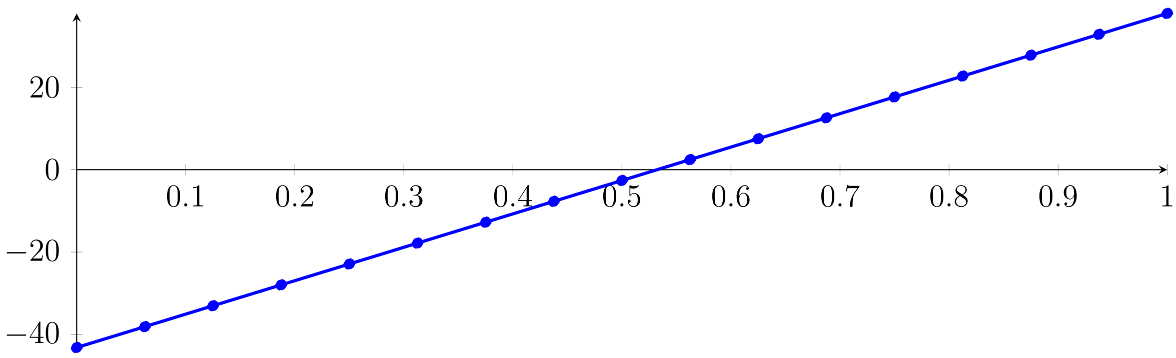
$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

167.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

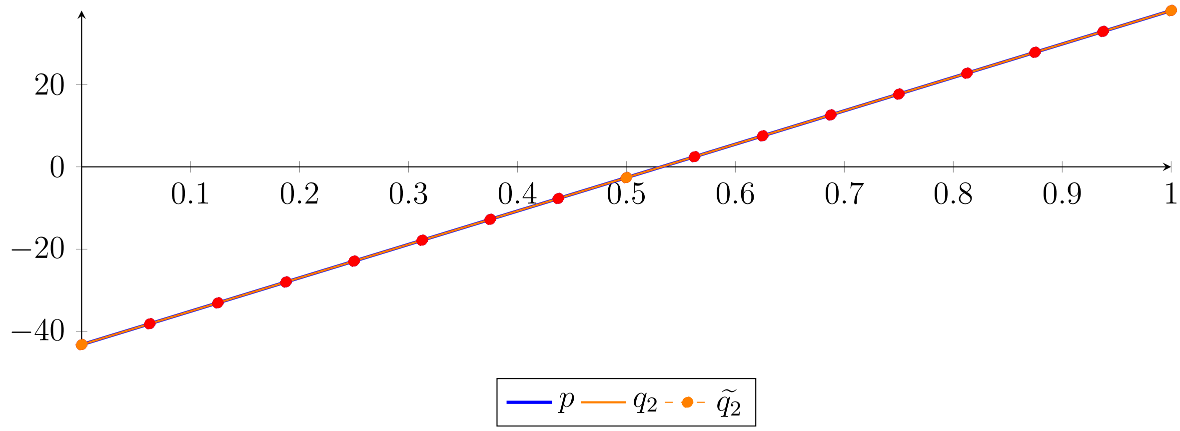
$$\begin{aligned} p = & -2.76723 \cdot 10^{-13} X^{16} - 2.40874 \cdot 10^{-12} X^{15} - 1.25233 \cdot 10^{-11} X^{14} - 3.02935 \cdot 10^{-11} X^{13} \\ & - 3.05617 \cdot 10^{-11} X^{12} - 3.83107 \cdot 10^{-11} X^{11} - 1.26692 \cdot 10^{-11} X^{10} - 2.6672 \cdot 10^{-11} X^{9} \\ & + 2.0004 \cdot 10^{-11} X^{8} + 4.12781 \cdot 10^{-12} X^{7} + 2.44493 \cdot 10^{-12} X^{6} - 1.21236 \cdot 10^{-13} X^{5} \\ & + 1.26288 \cdot 10^{-14} X^{4} - 4.1267 \cdot 10^{-12} X^{3} - 3.09388 \cdot 10^{-05} X^{2} + 81.1506X - 43.1911 \\ = & -43.1911 B_{0,16}(X) - 38.1192 B_{1,16}(X) - 33.0473 B_{2,16}(X) - 27.9754 B_{3,16}(X) - 22.9035 B_{4,16}(X) \\ & - 17.8316 B_{5,16}(X) - 12.7597 B_{6,16}(X) - 7.68778 B_{7,16}(X) - 2.61587 B_{8,16}(X) \\ & + 2.45604 B_{9,16}(X) + 7.52795 B_{10,16}(X) + 12.5999 B_{11,16}(X) + 17.6718 B_{12,16}(X) \\ & + 22.7437 B_{13,16}(X) + 27.8156 B_{14,16}(X) + 32.8875 B_{15,16}(X) + 37.9594 B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911 \\ &= -43.1911 B_{0,2} - 2.61586 B_{1,2} + 37.9594 B_{2,2} \end{aligned}$$

$$\begin{aligned}\widetilde{q}_2 &= 5.96265{\cdot}10^{-05}X^{16} - 0.000436042X^{15} + 0.00141812X^{14} - 0.00269475X^{13} \\ &\quad + 0.00329809X^{12} - 0.00268757X^{11} + 0.00143268X^{10} - 0.000439599X^{9} \\ &\quad + 1.98418{\cdot}10^{-05}X^{8} + 4.87608{\cdot}10^{-05}X^{7} - 2.46333{\cdot}10^{-05}X^{6} + 6.35808{\cdot}10^{-06}X^{5} \\ &\quad - 9.62755{\cdot}10^{-07}X^{4} + 8.21372{\cdot}10^{-08}X^{3} - 3.09429{\cdot}10^{-05}X^{2} + 81.1506X - 43.1911 \\ &= -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} \\ &\quad - 12.7597B_{6,16} - 7.68778B_{7,16} - 2.61587B_{8,16} + 2.45604B_{9,16} + 7.52795B_{10,16} + 12.5999B_{11,16} \\ &\quad + 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.5947{\cdot}10^{-09}$.

Bounding polynomials M and m:

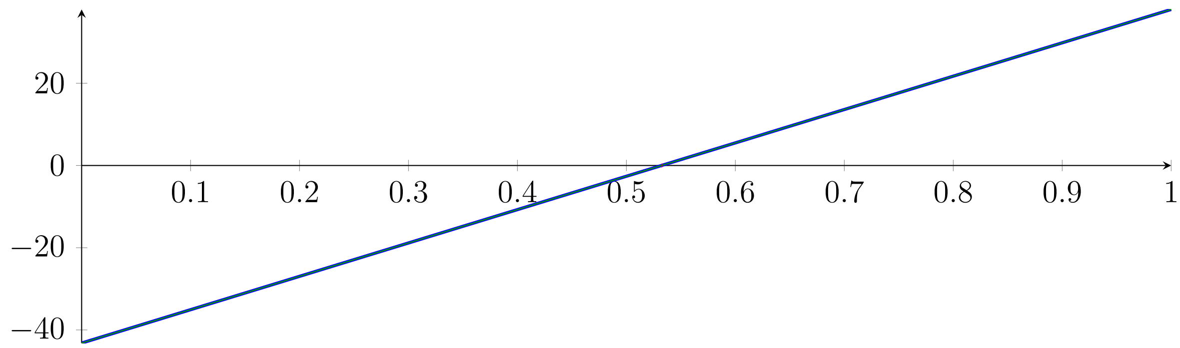
$$M = -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911$$

$$m = -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911$$

Root of M and m:

$$N(M) = \{0.532235, 2.62293{\cdot}10^{6}\} \qquad N(m) = \{0.532235, 2.62293{\cdot}10^{6}\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

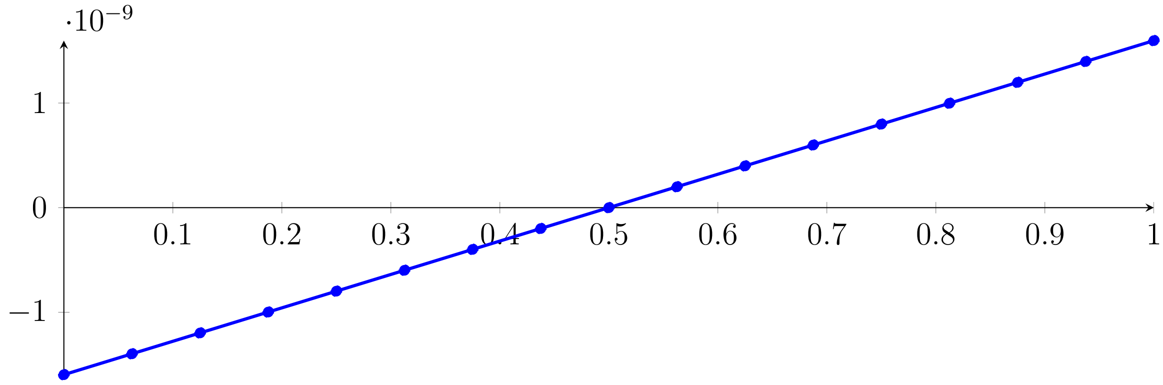
Longest intersection interval: $3.93535{\cdot}10^{-11}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

167.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

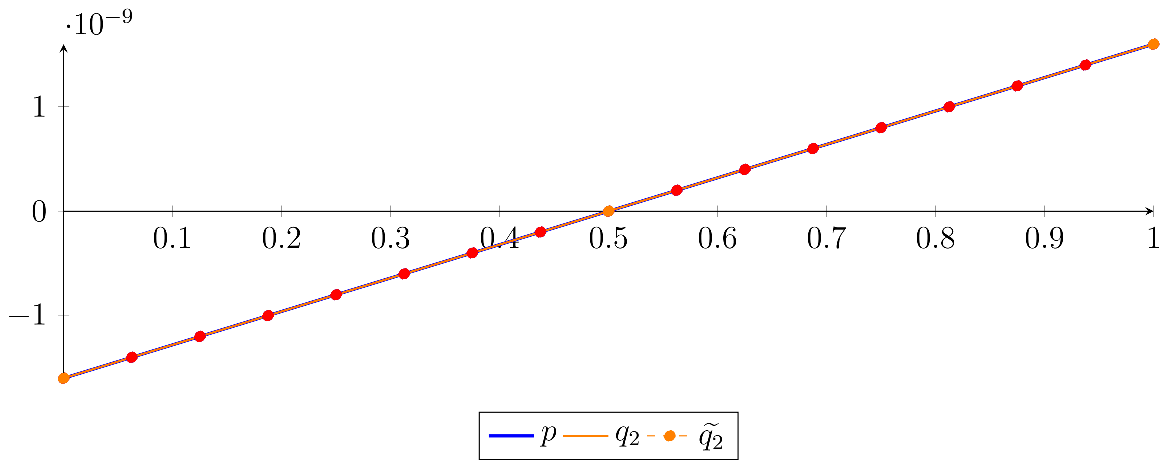
$$p = -4.89361\cdot10^{-24}X^{16} - 1.05466\cdot10^{-22}X^{15} - 3.09805\cdot10^{-22}X^{14} - 1.16981\cdot10^{-21}X^{13} - 9.76202\cdot10^{-22}X^{12} - 1.69365\cdot10^{-21}X^{11} - 5.95131\cdot10^{-22}X^{10} - 1.05349\cdot10^{-21}X^{9} + 7.5893\cdot10^{-22}X^{8} + 1.47859\cdot10^{-22}X^{7} + 9.70322\cdot10^{-23}X^{6} - 7.05688\cdot10^{-24}X^{5} + 1.47018\cdot10^{-24}X^{4} - 4.84676\cdot10^{-26}X^{2} + 3.19356\cdot10^{-09}X - 1.59674\cdot10^{-09}$$
$$= -1.59674\cdot10^{-09}B_{0,16}(X) - 1.39715\cdot10^{-09}B_{1,16}(X) - 1.19755\cdot10^{-09}B_{2,16}(X) - 9.97951\cdot10^{-10}B_{3,16}(X) - 7.98353\cdot10^{-10}B_{4,16}(X) - 5.98756\cdot10^{-10}B_{5,16}(X) - 3.99159\cdot10^{-10}B_{6,16}(X) - 1.99561\cdot10^{-10}B_{7,16}(X) + 3.6039\cdot10^{-14}B_{8,16}(X) + 1.99633\cdot10^{-10}B_{9,16}(X) + 3.99231\cdot10^{-10}B_{10,16}(X) + 5.98828\cdot10^{-10}B_{11,16}(X) + 7.98425\cdot10^{-10}B_{12,16}(X) + 9.98023\cdot10^{-10}B_{13,16}(X) + 1.19762\cdot10^{-09}B_{14,16}(X) + 1.39722\cdot10^{-09}B_{15,16}(X) + 1.59681\cdot10^{-09}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -4.83666\cdot10^{-26}X^{2} + 3.19356\cdot10^{-09}X - 1.59674\cdot10^{-09}$$
$$= -1.59674\cdot10^{-09}B_{0,2} + 3.6039\cdot10^{-14}B_{1,2} + 1.59681\cdot10^{-09}B_{2,2}$$

$$\widetilde{q}_2 = 9.45798\cdot10^{-15}X^{16} - 7.39324\cdot10^{-14}X^{15} + 2.61437\cdot10^{-13}X^{14} - 5.52989\cdot10^{-13}X^{13} + 7.79462\cdot10^{-13}X^{12} - 7.71893\cdot10^{-13}X^{11} + 5.51461\cdot10^{-13}X^{10} - 2.8712\cdot10^{-13}X^{9} + 1.08634\cdot10^{-13}X^{8} - 2.94042\cdot10^{-14}X^{7} + 5.52081\cdot10^{-15}X^{6} - 6.84058\cdot10^{-16}X^{5} + 5.20623\cdot10^{-17}X^{4} - 2.16513\cdot10^{-18}X^{3} + 1.74369\cdot10^{-20}X^{2} + 3.19356\cdot10^{-09}X - 1.59674\cdot10^{-09}$$
$$= -1.59674\cdot10^{-09}B_{0,16} - 1.39715\cdot10^{-09}B_{1,16} - 1.19755\cdot10^{-09}B_{2,16} - 9.97951\cdot10^{-10}B_{3,16} - 7.98353\cdot10^{-10}B_{4,16} - 5.98756\cdot10^{-10}B_{5,16} - 3.99159\cdot10^{-10}B_{6,16} - 1.99561\cdot10^{-10}B_{7,16} + 3.60393\cdot10^{-14}B_{8,16} + 1.99633\cdot10^{-10}B_{9,16} + 3.99231\cdot10^{-10}B_{10,16} + 5.98828\cdot10^{-10}B_{11,16} + 7.98425\cdot10^{-10}B_{12,16} + 9.98023\cdot10^{-10}B_{13,16} + 1.19762\cdot10^{-09}B_{14,16} + 1.39722\cdot10^{-09}B_{15,16} + 1.59681\cdot10^{-09}B_{16,16}$$



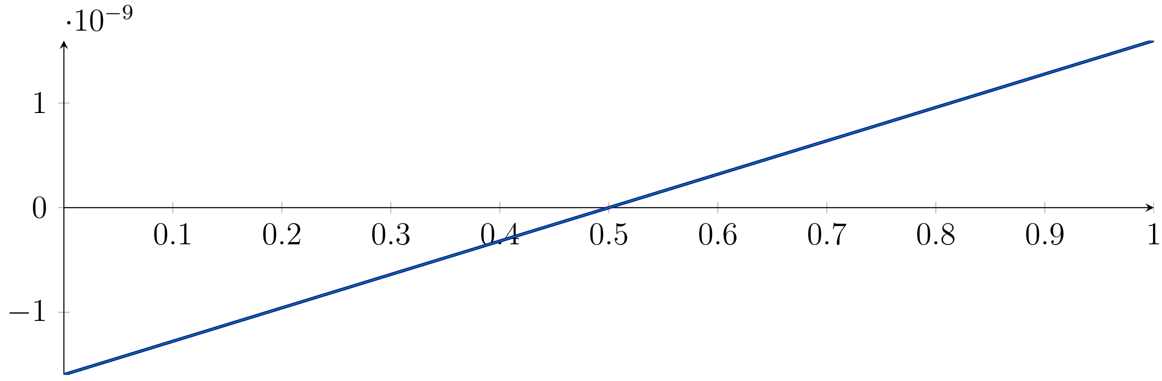
The maximum difference of the Bézier coefficients is $\delta = 2.02367 \cdot 10^{-19}$.
Bounding polynomials M and m:

$$M = -4.82657 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$
$$m = -4.84676 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$

Root of M and m:

$$N(M) = \{0.5, 6.61662 \cdot 10^{16}\} \qquad N(m) = \{0.5, 6.58905 \cdot 10^{16}\}$$

Intersection intervals:



$[0.5, 0.5]$

Longest intersection interval: 0
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

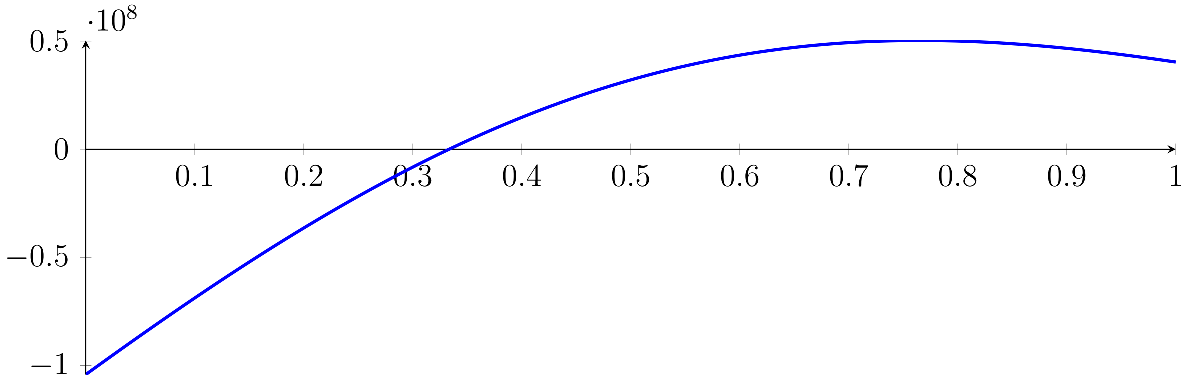
167.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

167.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

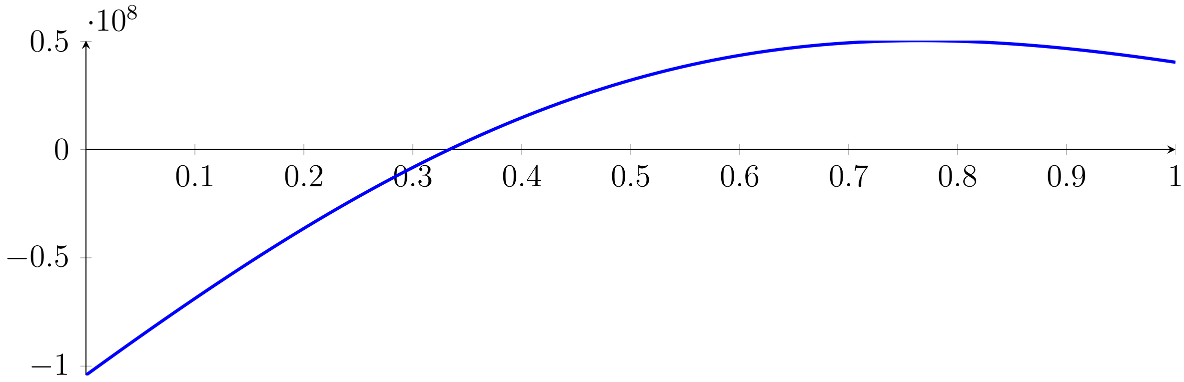
with precision $\varepsilon = 1\cdot10^{-128}$.

168 Running CubeClip on f_{16} with epsilon 128

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

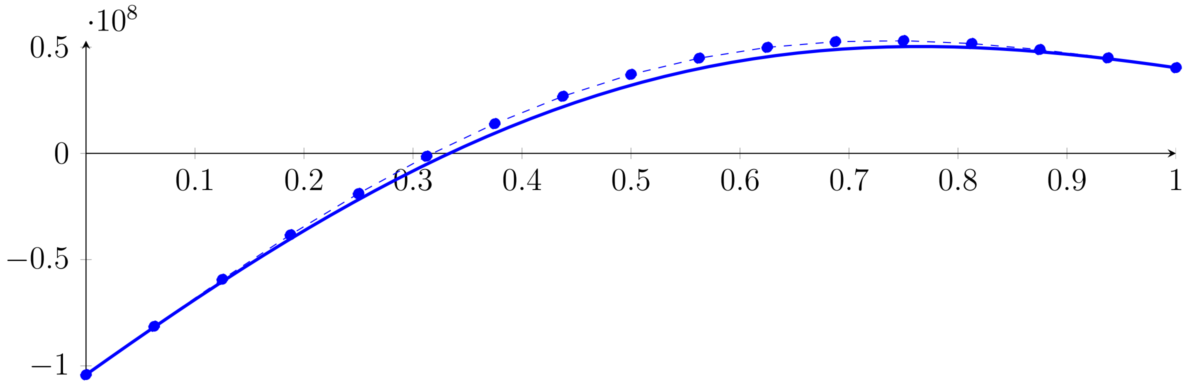
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



168.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

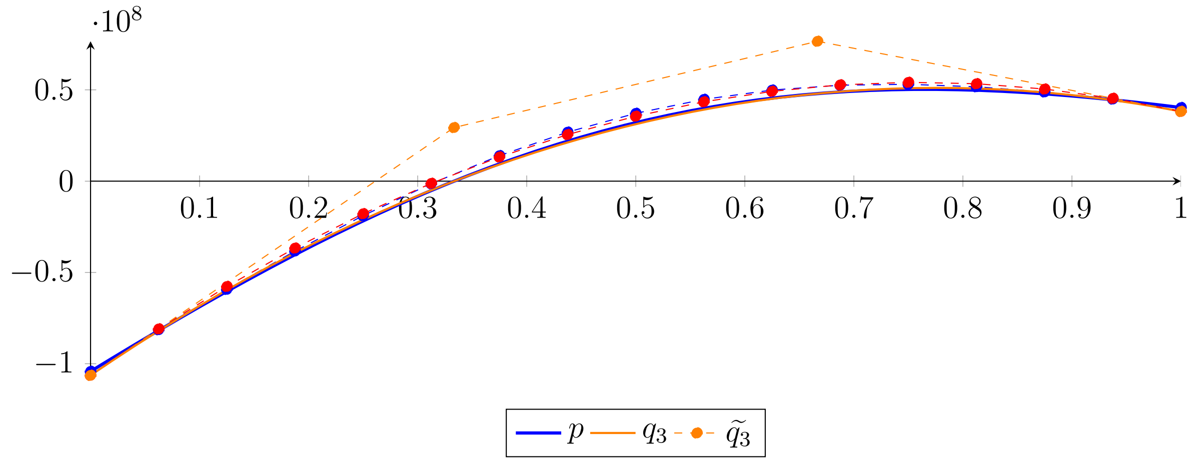
$$\begin{aligned} p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\quad \cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ &= -1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 2.75812\cdot10^6X^3 - 2.65319\cdot10^8X^2 + 4.07072\cdot10^8X - 1.06335\cdot10^8 \\ &= -1.06335\cdot10^8B_{0,3} + 2.93558\cdot10^7B_{1,3} + 7.66068\cdot10^7B_{2,3} + 3.81764\cdot10^7B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 2461.93X^{16} - 19614.9X^{15} + 70879.5X^{14} - 153661X^{13} + 222746X^{12} - 227755X^{11} \\ &\quad + 168826X^{10} - 91798.7X^9 + 36630.3X^8 - 10627.3X^7 + 2200.54X^6 - 316.059X^5 \\ &\quad + 30.1958X^4 + 2.75812\cdot10^6X^3 - 2.65319\cdot10^8X^2 + 4.07072\cdot10^8X - 1.06335\cdot10^8 \\ &= -1.06335\cdot10^8B_{0,16} - 8.08928\cdot10^7B_{1,16} - 5.76618\cdot10^7B_{2,16} - 3.66369\cdot10^7B_{3,16} - 1.78131 \\ &\quad \cdot10^7B_{4,16} - 1.18551\cdot10^6B_{5,16} + 1.32508\cdot10^7B_{6,16} + 2.55007\cdot10^7B_{7,16} + 3.55692\cdot10^7B_{8,16} \\ &\quad + 4.34611\cdot10^7B_{9,16} + 4.91815\cdot10^7B_{10,16} + 5.27353\cdot10^7B_{11,16} + 5.41273\cdot10^7B_{12,16} \\ &\quad + 5.33624\cdot10^7B_{13,16} + 5.04457\cdot10^7B_{14,16} + 4.53821\cdot10^7B_{15,16} + 3.81764\cdot10^7B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806\cdot10^6$.

Bounding polynomials M and m:

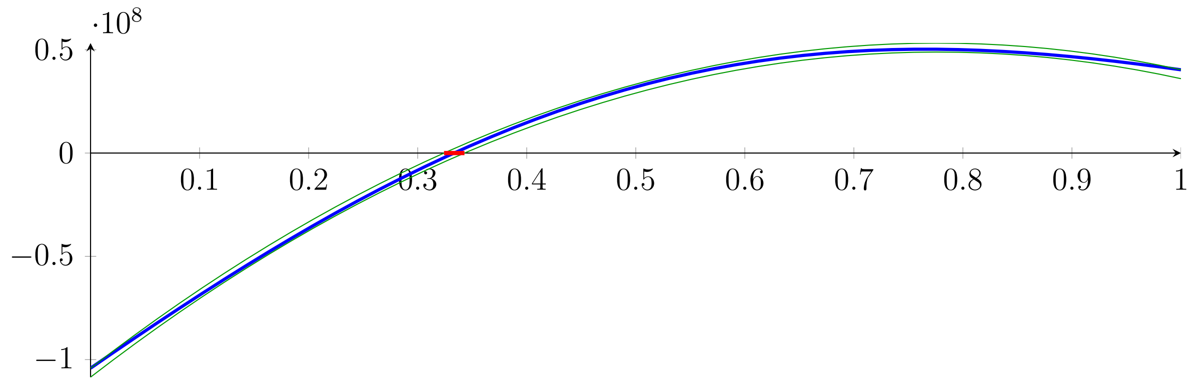
$$M = 2.75812\cdot10^6X^3 - 2.65319\cdot10^8X^2 + 4.07072\cdot10^8X - 1.04167\cdot10^8$$

$$m = 2.75812\cdot10^6X^3 - 2.65319\cdot10^8X^2 + 4.07072\cdot10^8X - 1.08503\cdot10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

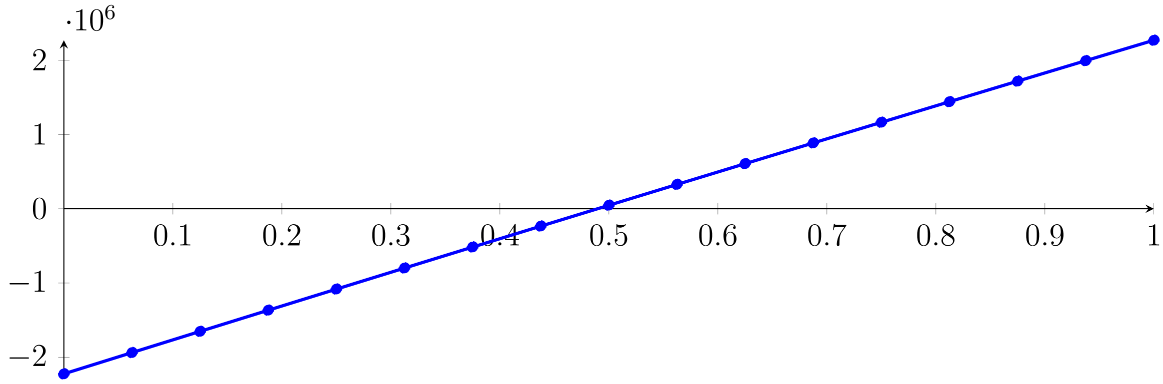
Longest intersection interval: 0.0187703

$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

168.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

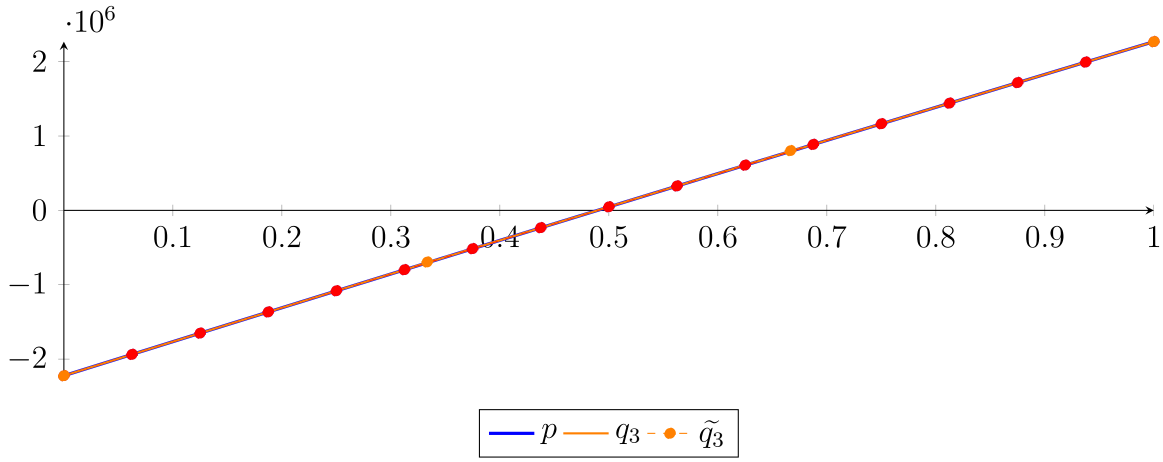
$$\begin{aligned}p = &-3.66617\cdot10^{-09}X^{16} - 1.53217\cdot10^{-07}X^{15} - 3.62234\cdot10^{-07}X^{14} - 1.65579\cdot10^{-06}X^{13} - 1.15373\cdot10^{-06}X^{12}\\ &- 2.3399\cdot10^{-06}X^{11} - 5.02543\cdot10^{-07}X^{10} - 1.38381\cdot10^{-06}X^{9} + 1.1237\cdot10^{-06}X^{8} - 1.19653\cdot10^{-05}X^{7}\\ &- 0.00155608X^{6} + 0.10496X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}\\ = &-2.22335\cdot10^{6}B_{0,16}(X) - 1.93666\cdot10^{6}B_{1,16}(X) - 1.65074\cdot10^{6}B_{2,16}(X) - 1.36561\cdot10^{6}B_{3,16}(X)\\ &- 1.08127\cdot10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X)\\ &+ 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot10^{6}B_{12,16}(X) + 1.44233\\ &\cdot10^{6}B_{13,16}(X) + 1.71876\cdot10^{6}B_{14,16}(X) + 1.99439\cdot10^{6}B_{15,16}(X) + 2.26922\cdot10^{6}B_{16,16}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= -700.673X^3 - 93879.9X^2 + 4.58715\cdot10^6X - 2.22335\cdot10^6\\ &= -2.22335\cdot10^6B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot10^6B_{3,3}\end{aligned}$$

$$\begin{aligned}\tilde{q}_3 &= 16.4956X^{16} - 129.161X^{15} + 457.83X^{14} - 971.671X^{13} + 1375.95X^{12} - 1370.96X^{11}\\ &+ 987.265X^{10} - 519.476X^{9} + 199.587X^{8} - 55.434X^{7} + 10.9237X^{6} - 1.48019X^{5}\\ &+ 0.129516X^{4} - 700.679X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}\\ &= -2.22335\cdot10^{6}B_{0,16} - 1.93666\cdot10^{6}B_{1,16} - 1.65074\cdot10^{6}B_{2,16} - 1.36561\cdot10^{6}B_{3,16}\\ &- 1.08126\cdot10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.4B_{8,16}\\ &+ 328650B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot10^{6}B_{12,16} + 1.44233\\ &\cdot10^{6}B_{13,16} + 1.71876\cdot10^{6}B_{14,16} + 1.99439\cdot10^{6}B_{15,16} + 2.26922\cdot10^{6}B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m:

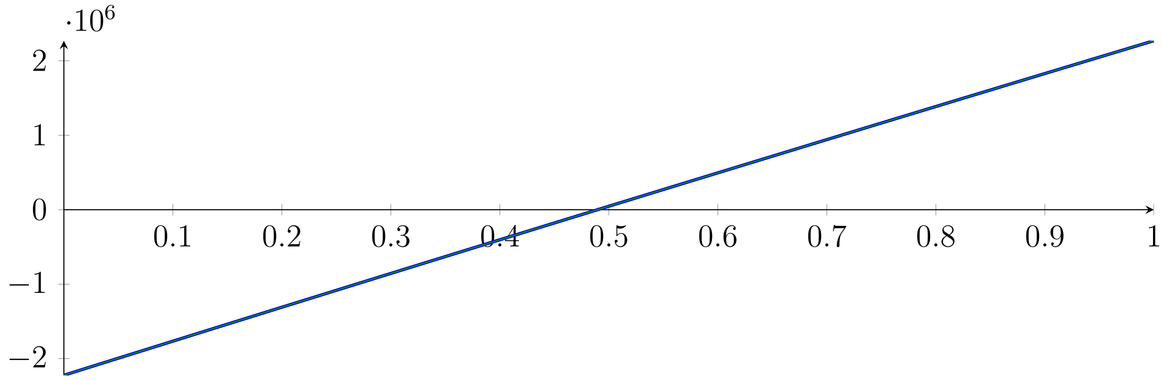
$$M = -700.673X^3 - 93879.9X^2 + 4.58715\cdot10^6X - 2.22335\cdot10^6$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715{\cdot}10^6X - 2.22335{\cdot}10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



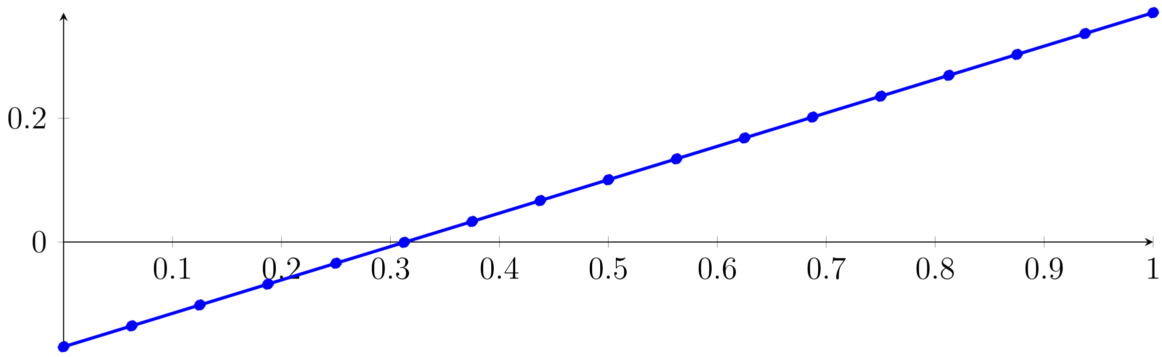
$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

168.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

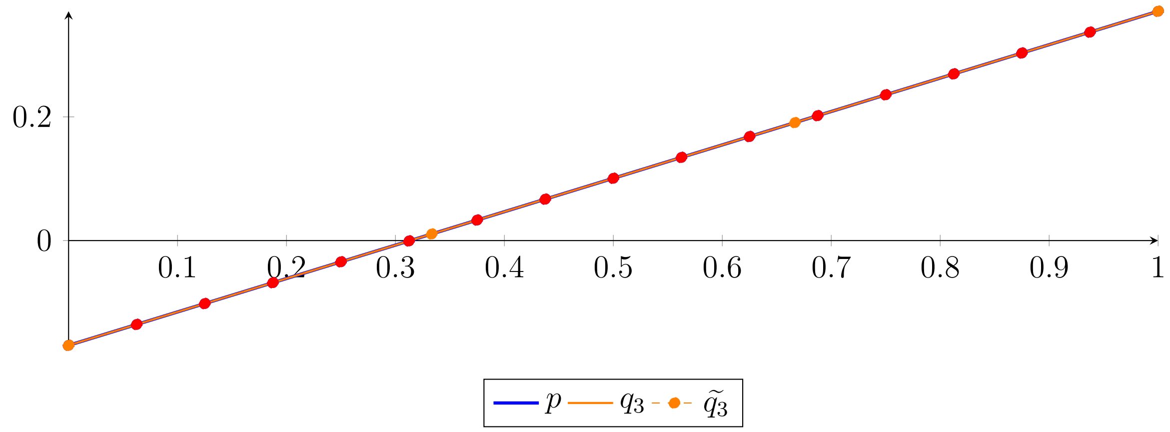
$$\begin{aligned} p = {} & 5.55524{\cdot}10^{-15}X^{16} - 2.94313{\cdot}10^{-14}X^{15} + 1.19384{\cdot}10^{-13}X^{14} - 2.17482{\cdot}10^{-13}X^{13} + 7.26155{\cdot}10^{-14}X^{12} \\ & - 3.44766{\cdot}10^{-13}X^{11} - 3.47292{\cdot}10^{-15}X^{10} - 1.1287{\cdot}10^{-13}X^9 + 2.93027{\cdot}10^{-14}X^8 + 8.06213{\cdot}10^{-15}X^7 \\ & + 5.64349{\cdot}10^{-15}X^6 + 4.93312{\cdot}10^{-17}X^4 + 1.51788{\cdot}10^{-17}X^3 - 1.37072{\cdot}10^{-09}X^2 + 0.540148X - 0.169396 \\ = {} & -0.169396B_{0,16}(X) - 0.135637B_{1,16}(X) - 0.101877B_{2,16}(X) - 0.068118B_{3,16}(X) \\ & - 0.0343587B_{4,16}(X) - 0.000599476B_{5,16}(X) + 0.0331598B_{6,16}(X) \\ & + 0.0669191B_{7,16}(X) + 0.100678B_{8,16}(X) + 0.134438B_{9,16}(X) + 0.168197B_{10,16}(X) \\ & + 0.201956B_{11,16}(X) + 0.235715B_{12,16}(X) + 0.269475B_{13,16}(X) \\ & + 0.303234B_{14,16}(X) + 0.336993B_{15,16}(X) + 0.370752B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07065{\cdot}10^{-17}X^3 - 1.37072{\cdot}10^{-09}X^2 + 0.540148X - 0.169396 \\ &= -0.169396B_{0,3} + 0.0106536B_{1,3} + 0.190703B_{2,3} + 0.370752B_{3,3} \end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= 8.59095\cdot 10^{-06}X^{16} - 6.82648\cdot 10^{-05}X^{15} + 0.000245968X^{14} - 0.000531568X^{13} \\ &\quad + 0.000767923X^{12} - 0.000782231X^{11} + 0.0005774X^{10} - 0.000312464X^{9} \\ &\quad + 0.000123994X^{8} - 3.57388\cdot 10^{-05}X^{7} + 7.34249\cdot 10^{-06}X^{6} - 1.04474\cdot 10^{-06}X^{5} \\ &\quad + 9.86739\cdot 10^{-08}X^{4} - 5.7553\cdot 10^{-09}X^{3} - 1.19186\cdot 10^{-09}X^{2} + 0.540148X - 0.169396 \\ &= -0.169396B_{0,16} - 0.135637B_{1,16} - 0.101877B_{2,16} - 0.068118B_{3,16} - 0.0343587B_{4,16} \\ &\quad - 0.000599476B_{5,16} + 0.0331598B_{6,16} + 0.0669191B_{7,16} + 0.100678B_{8,16} \\ &\quad + 0.134438B_{9,16} + 0.168197B_{10,16} + 0.201956B_{11,16} + 0.235715B_{12,16} \\ &\quad + 0.269475B_{13,16} + 0.303234B_{14,16} + 0.336993B_{15,16} + 0.370752B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.81206\cdot 10^{-10}$.
Bounding polynomials M and m:

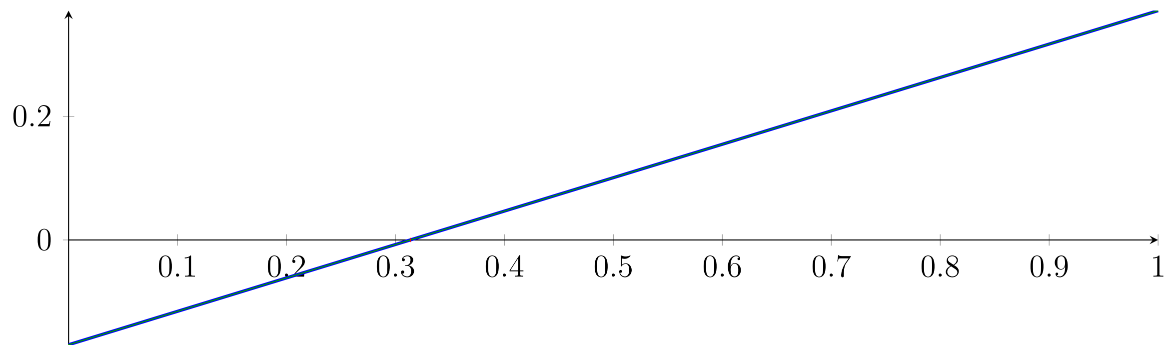
$$M = -1.07065\cdot 10^{-17}X^3 - 1.37072\cdot 10^{-09}X^2 + 0.540148X - 0.169396$$

$$m = -1.07065\cdot 10^{-17}X^3 - 1.37072\cdot 10^{-09}X^2 + 0.540148X - 0.169396$$

Root of M and m:

$$N(M) = \{-2.97569\cdot 10^{8}, 0.31361, 1.69542\cdot 10^{8}\} \quad N(m) = \{-2.97569\cdot 10^{8}, 0.31361, 1.69542\cdot 10^{8}\}$$

Intersection intervals:



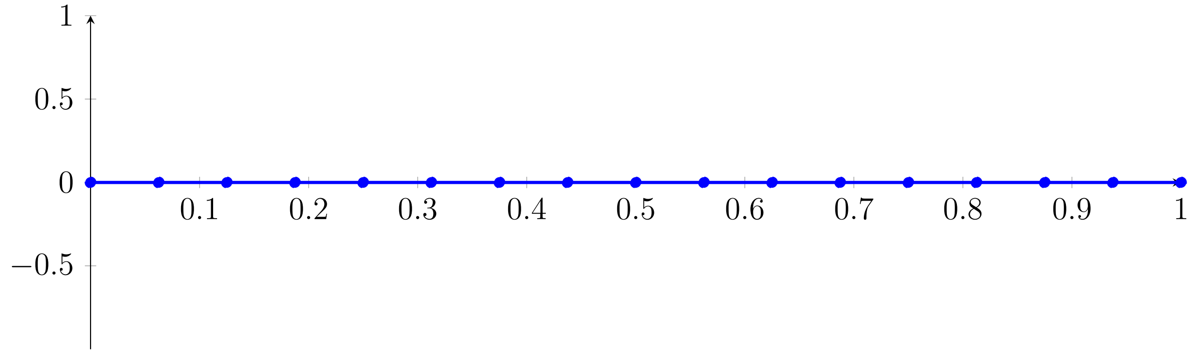
$$[0.31361, 0.31361]$$

Longest intersection interval: $7.85803\cdot 10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

168.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

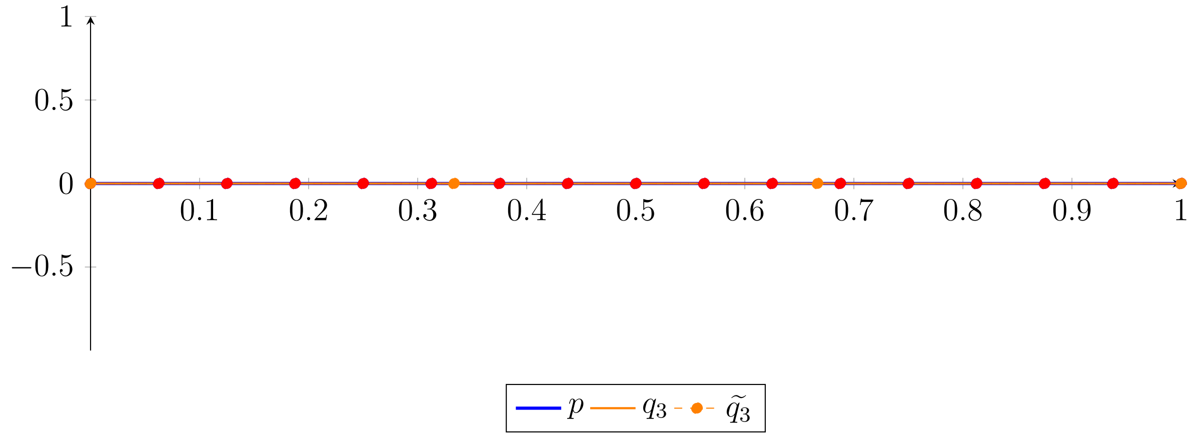
$$p = -1.51576\cdot 10^{-21}X^{16} + 2.62009\cdot 10^{-21}X^{15} - 3.98039\cdot 10^{-20}X^{14} + 3.2136\cdot 10^{-21}X^{13} - 5.16564\cdot 10^{-20}X^{12} + 1.52429\cdot 10^{-20}X^{11} - 1.44901\cdot 10^{-20}X^{10} - 1.40466\cdot 10^{-20}X^{9} + 2.34541\cdot 10^{-20}X^{8} + 3.25289\cdot 10^{-21}X^{7} + 2.38052\cdot 10^{-21}X^{6} - 2.2582\cdot 10^{-22}X^{5} + 3.52844\cdot 10^{-23}X^{4} + 4.2445\cdot 10^{-10}X - 2.39831\cdot 10^{-08}$$
$$= -2.39831\cdot 10^{-08}B_{0,16}(X) - 2.39566\cdot 10^{-08}B_{1,16}(X) - 2.39301\cdot 10^{-08}B_{2,16}(X) - 2.39036\cdot 10^{-08}B_{3,16}(X) - 2.3877\cdot 10^{-08}B_{4,16}(X) - 2.38505\cdot 10^{-08}B_{5,16}(X) - 2.3824\cdot 10^{-08}B_{6,16}(X) - 2.37974\cdot 10^{-08}B_{7,16}(X) - 2.37709\cdot 10^{-08}B_{8,16}(X) - 2.37444\cdot 10^{-08}B_{9,16}(X) - 2.37179\cdot 10^{-08}B_{10,16}(X) - 2.36913\cdot 10^{-08}B_{11,16}(X) - 2.36648\cdot 10^{-08}B_{12,16}(X) - 2.36383\cdot 10^{-08}B_{13,16}(X) - 2.36118\cdot 10^{-08}B_{14,16}(X) - 2.35852\cdot 10^{-08}B_{15,16}(X) - 2.35587\cdot 10^{-08}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -3.5866\cdot 10^{-25}X^{3} + 8.33643\cdot 10^{-25}X^{2} + 4.2445\cdot 10^{-10}X - 2.39831\cdot 10^{-08}$$
$$= -2.39831\cdot 10^{-08}B_{0,3} - 2.38417\cdot 10^{-08}B_{1,3} - 2.37002\cdot 10^{-08}B_{2,3} - 2.35587\cdot 10^{-08}B_{3,3}$$

$$\widetilde{q}_3 = -1.64958\cdot 10^{-12}X^{16} + 1.3166\cdot 10^{-11}X^{15} - 4.76688\cdot 10^{-11}X^{14} + 1.03558\cdot 10^{-10}X^{13} - 1.50448\cdot 10^{-10}X^{12} + 1.54183\cdot 10^{-10}X^{11} - 1.1456\cdot 10^{-10}X^{10} + 6.24452\cdot 10^{-11}X^{9} - 2.49793\cdot 10^{-11}X^{8} + 7.26358\cdot 10^{-12}X^{7} - 1.50649\cdot 10^{-12}X^{6} + 2.16616\cdot 10^{-13}X^{5} - 2.07725\cdot 10^{-14}X^{4} + 1.24748\cdot 10^{-15}X^{3} - 4.0727\cdot 10^{-17}X^{2} + 4.2445\cdot 10^{-10}X - 2.39831\cdot 10^{-08}$$
$$= -2.39831\cdot 10^{-08}B_{0,16} - 2.39566\cdot 10^{-08}B_{1,16} - 2.39301\cdot 10^{-08}B_{2,16} - 2.39036\cdot 10^{-08}B_{3,16} - 2.3877\cdot 10^{-08}B_{4,16} - 2.38505\cdot 10^{-08}B_{5,16} - 2.3824\cdot 10^{-08}B_{6,16} - 2.37974\cdot 10^{-08}B_{7,16} - 2.37709\cdot 10^{-08}B_{8,16} - 2.37444\cdot 10^{-08}B_{9,16} - 2.37179\cdot 10^{-08}B_{10,16} - 2.36913\cdot 10^{-08}B_{11,16} - 2.36648\cdot 10^{-08}B_{12,16} - 2.36383\cdot 10^{-08}B_{13,16} - 2.36118\cdot 10^{-08}B_{14,16} - 2.35852\cdot 10^{-08}B_{15,16} - 2.35587\cdot 10^{-08}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 3.51589{\cdot}10^{-17}$.
Bounding polynomials M and m:

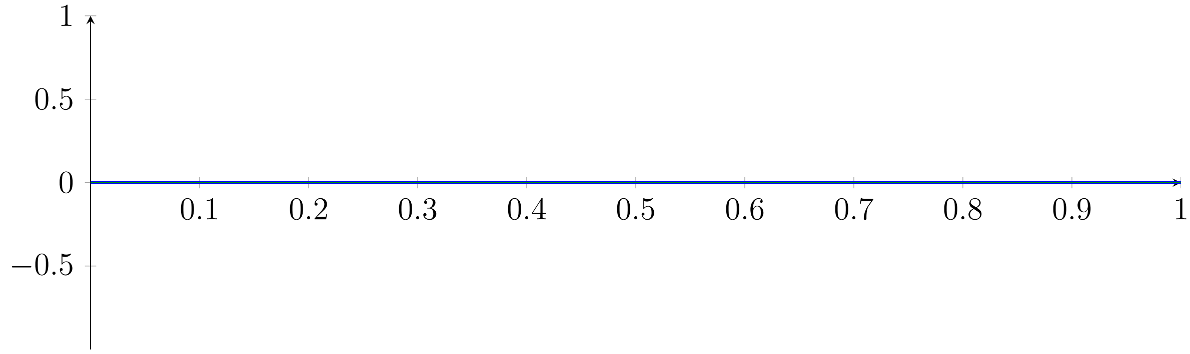
$$M = -3.5866{\cdot}10^{-25}X^3 + 8.33643{\cdot}10^{-25}X^2 + 4.2445{\cdot}10^{-10}X - 2.39831{\cdot}10^{-08}$$

$$m = -3.5866{\cdot}10^{-25}X^3 + 8.33643{\cdot}10^{-25}X^2 + 4.2445{\cdot}10^{-10}X - 2.39831{\cdot}10^{-08}$$

Root of M and m:

$$N(M) = \{-3.44011{\cdot}10^7, 56.504, 3.4401{\cdot}10^7\} \qquad N(m) = \{-3.44011{\cdot}10^7, 56.504, 3.4401{\cdot}10^7\}$$

Intersection intervals:

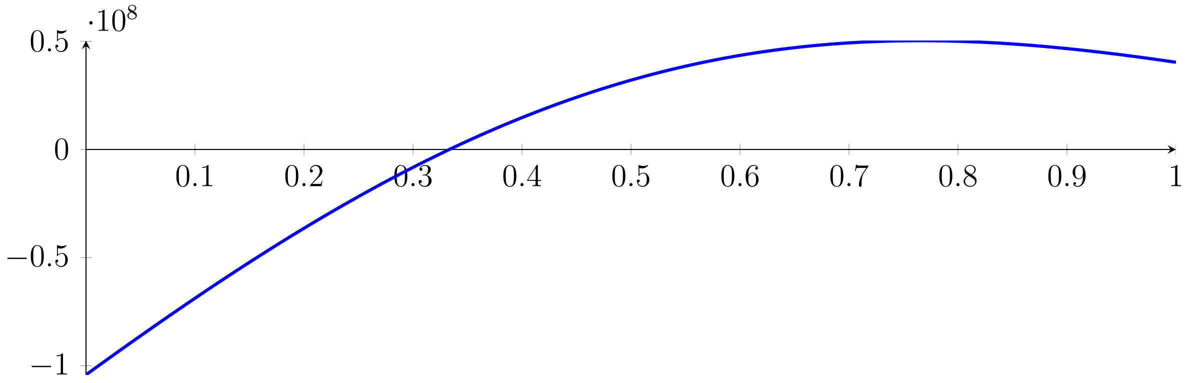


No intersection intervals with the x axis.

168.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} + 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8}$$



Result: Root Intervals

with precision $\varepsilon = 1{\cdot}10^{-128}$.

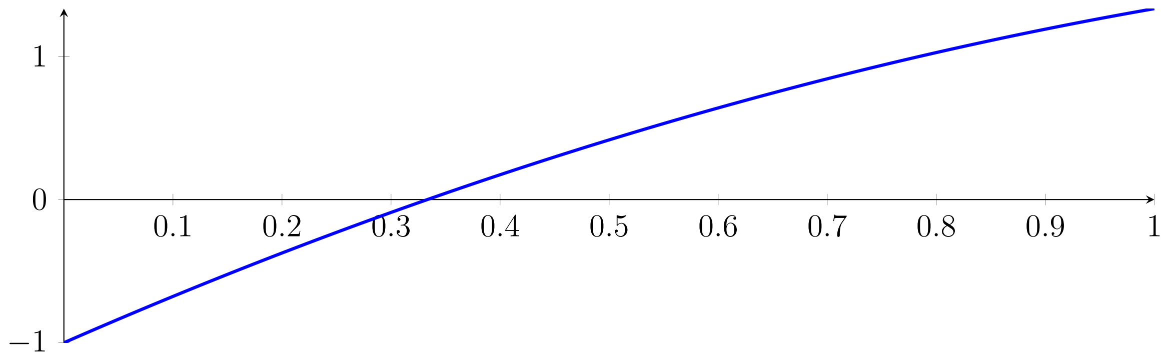
Part III
Numeric = MpfrFloat with precision 1024

169 Running BezClip on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

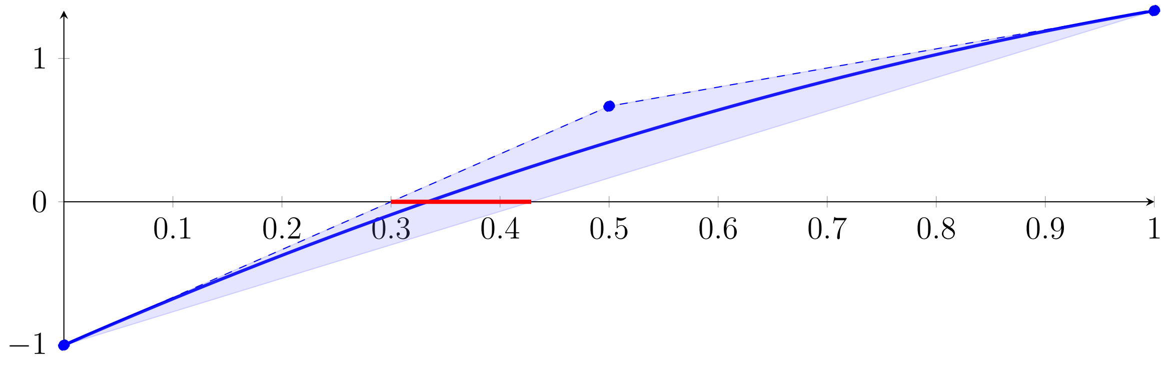
$$p = -1X^2 + 3.33333X - 1$$



169.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

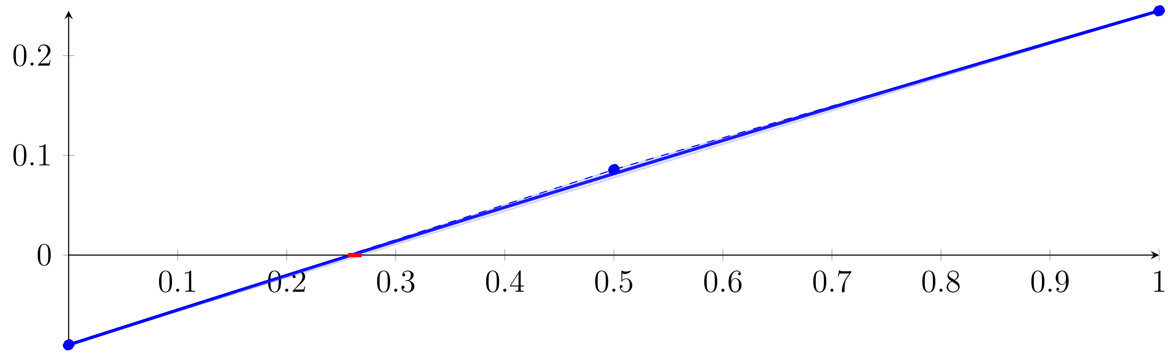
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

169.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -0.0165306X^2 + 0.351429X - 0.09$$
$$= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

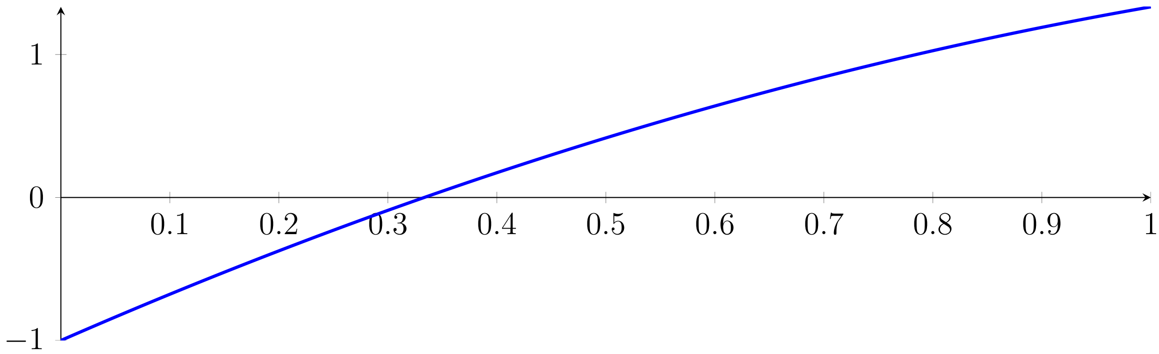
169.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Found root in interval $[0.332927, 0.334552]$ at recursion depth 3!

169.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.332927, 0.334552]$$

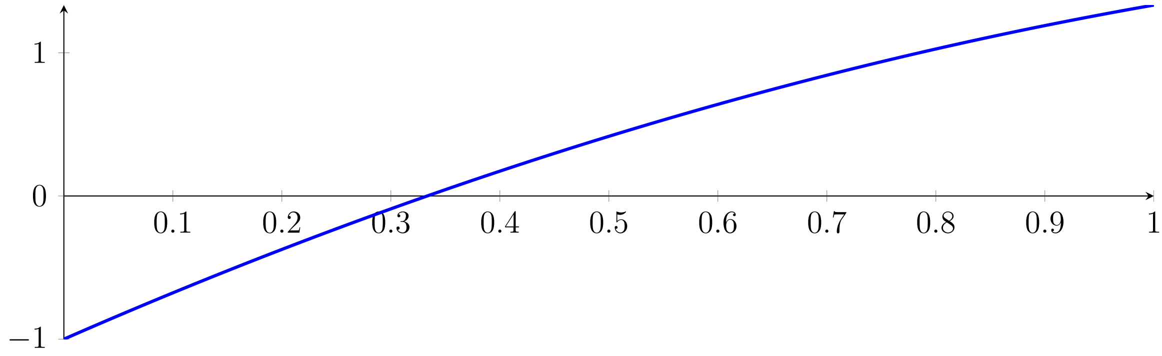
with precision $\varepsilon = 0.01$.

170 Running QuadClip on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

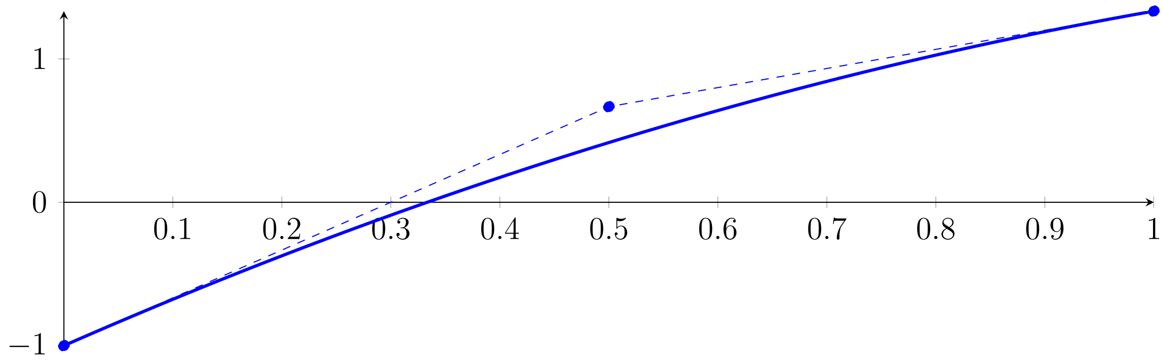
$$p = -1X^2 + 3.33333X - 1$$



170.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

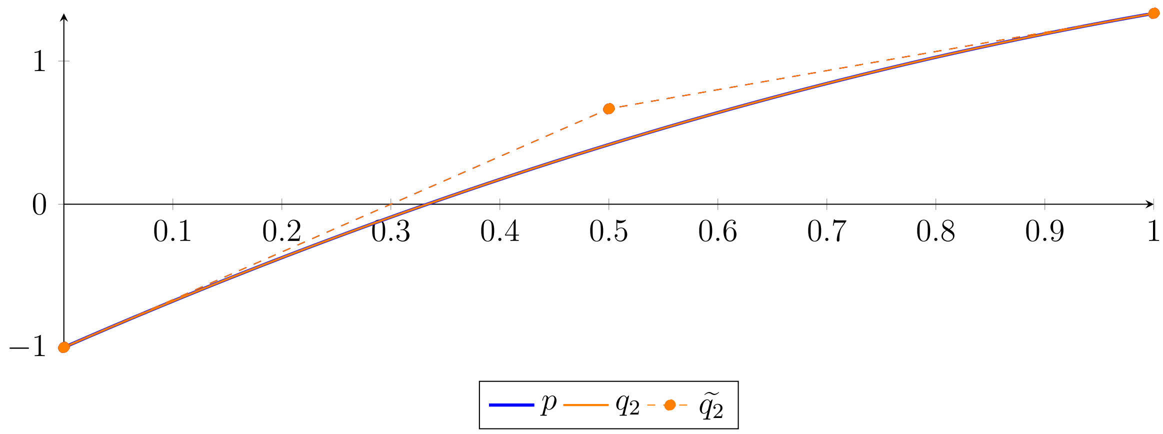
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.22507 \cdot 10^{-308}$.
Bounding polynomials M and m:

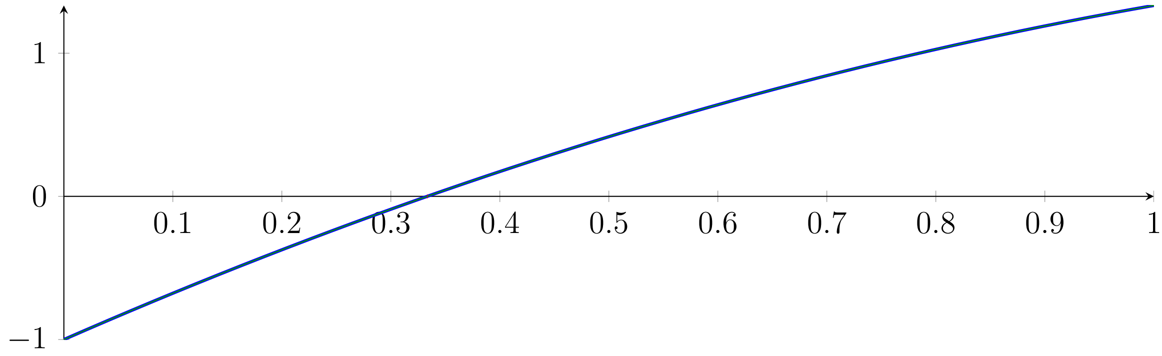
$$M = -1X^2 + 3.33333X - 1$$

$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $1.11254 \cdot 10^{-308}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

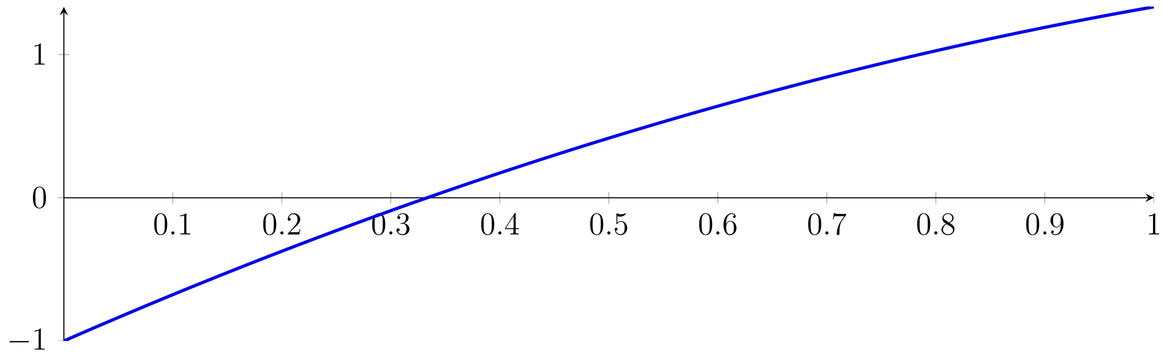
170.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

170.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

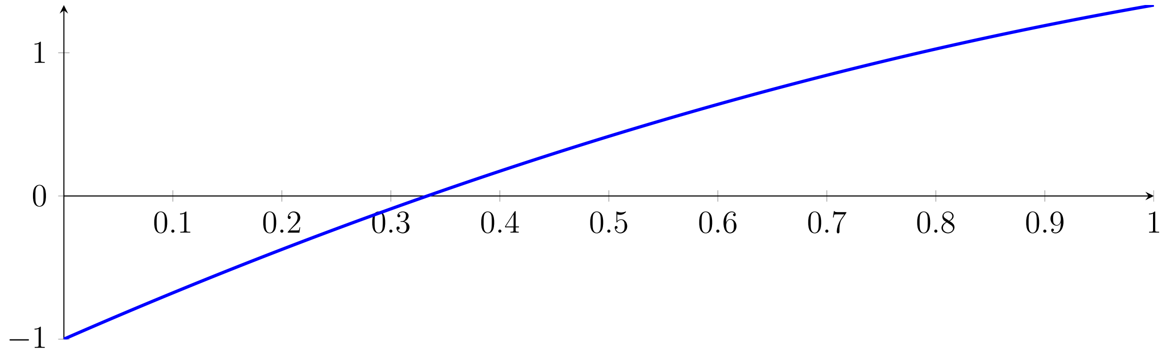
with precision $\varepsilon = 0.01$.

171 Running `CubeClip` on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

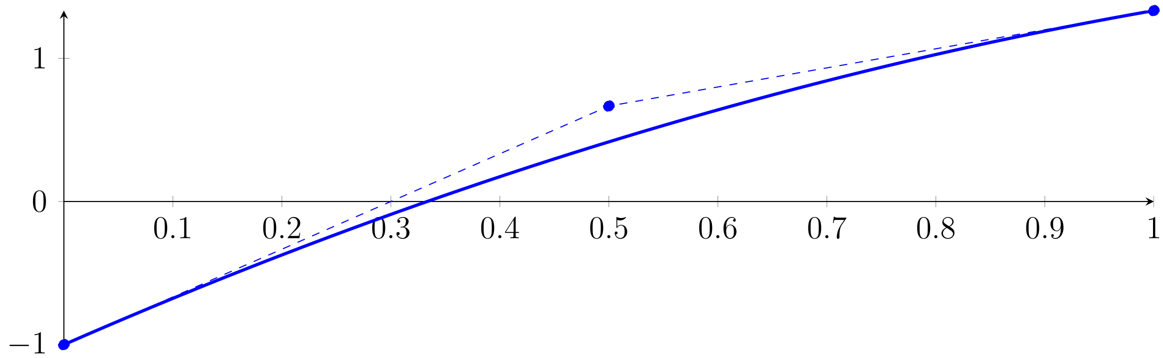
$$p = -1X^2 + 3.33333X - 1$$



171.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

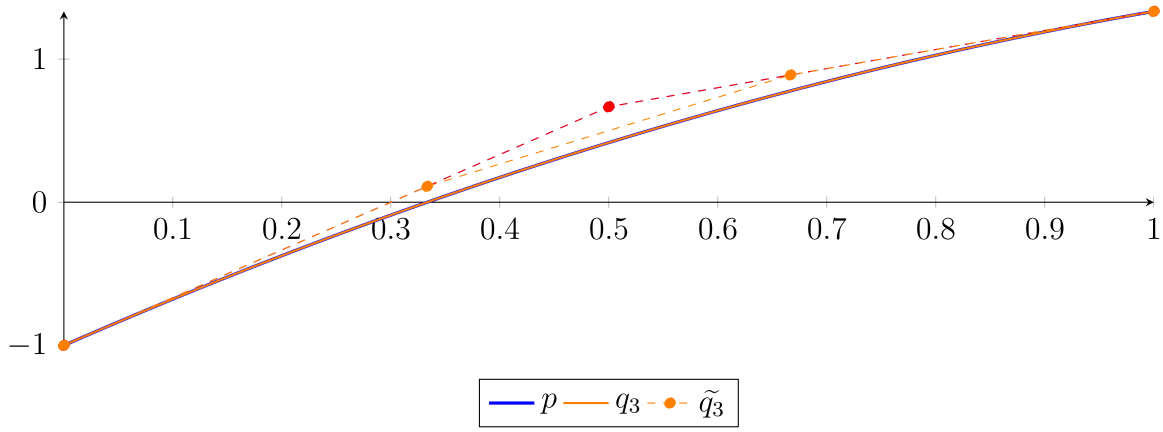
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.66881 \cdot 10^{-308}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.00128{\cdot}10^{-307}$.
Bounding polynomials M and m:

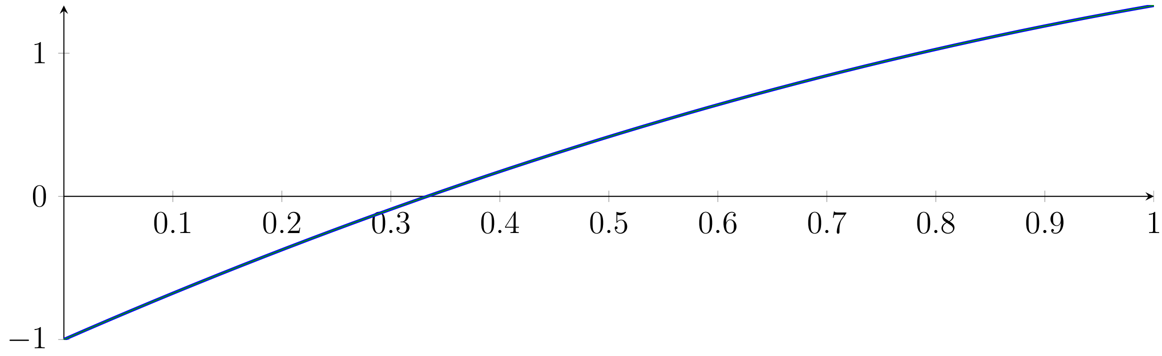
$$M = -2.78134{\cdot}10^{-308}X^3 - 1X^2 + 3.33333X - 1$$

$$m = -2.78134{\cdot}10^{-308}X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$N(M) = \{-3.59539{\cdot}10^{307}, -3.38949{\cdot}10^{291}, 1.03213{\cdot}10^{153}\}$ $N(m) = \{-3.59539{\cdot}10^{307}, -3.38949{\cdot}10^{291}, 1.75\}$

Intersection intervals:

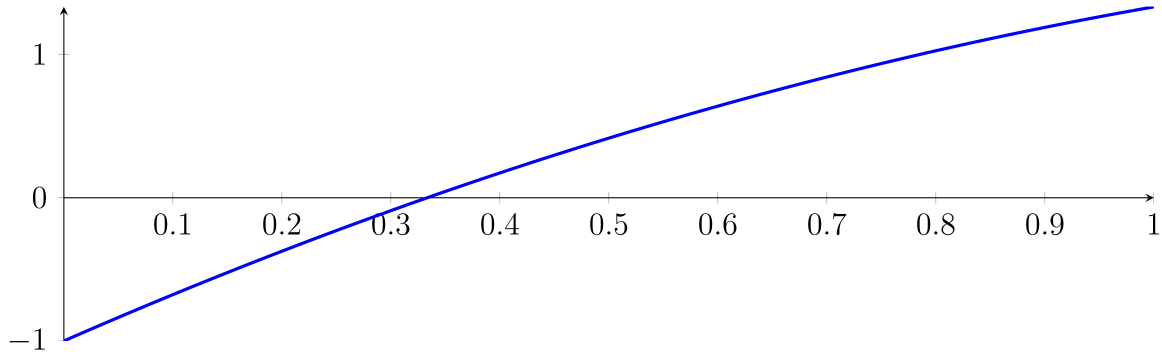


No intersection intervals with the x axis.

171.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

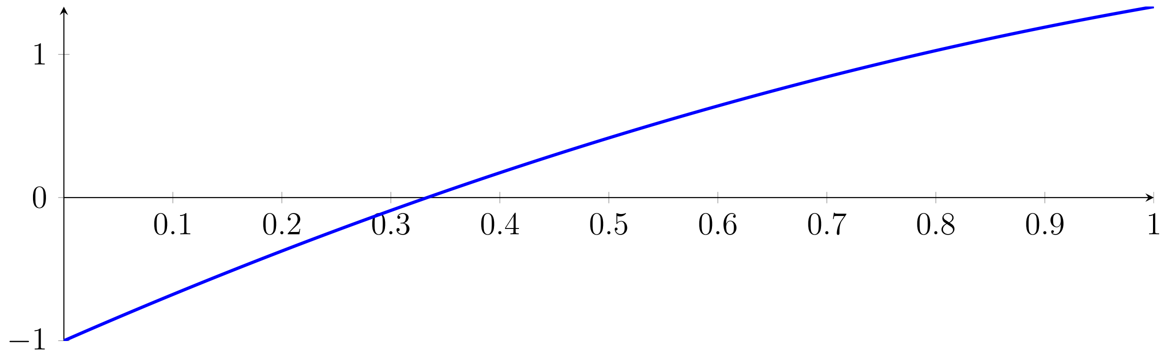
with precision $\varepsilon = 0.01$.

172 Running `BezClip` on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

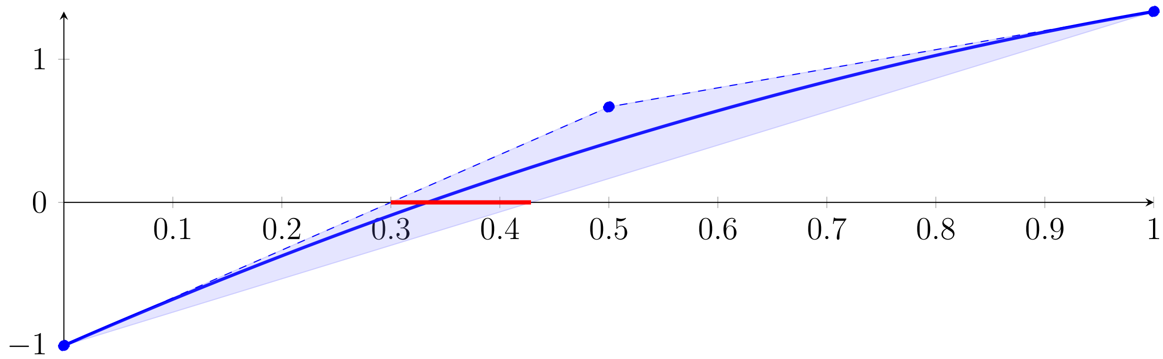
$$p = -1X^2 + 3.33333X - 1$$



172.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

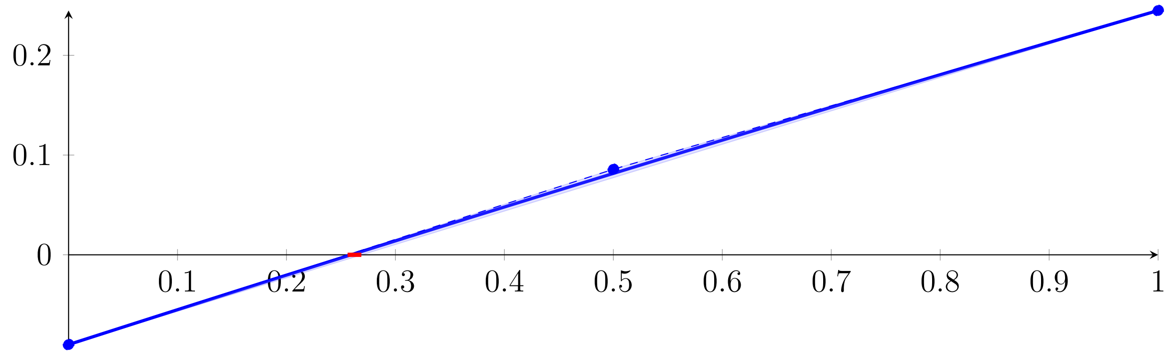
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\implies$ Selective recursion: interval 1: $[0.3, 0.428571]$,

172.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

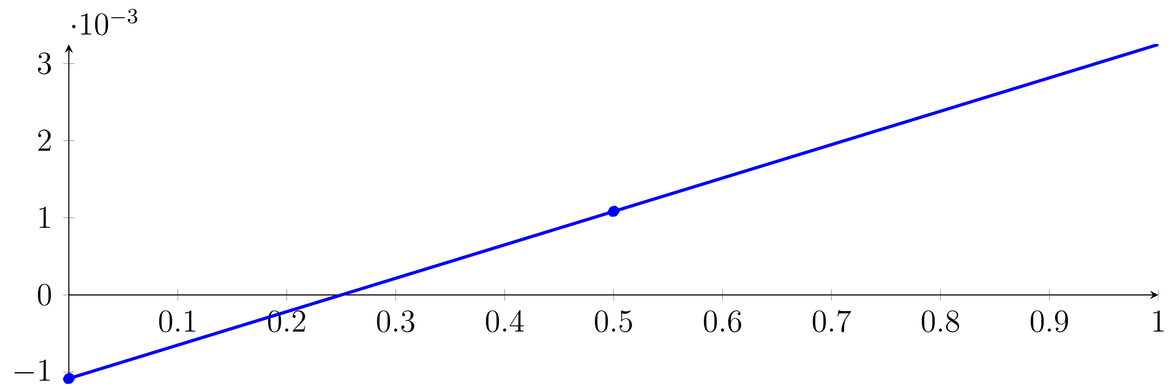
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

172.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538X - 0.00108418 \\ &= -0.00108418B_{0,2}(X) + 0.00108352B_{1,2}(X) + 0.00324857B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

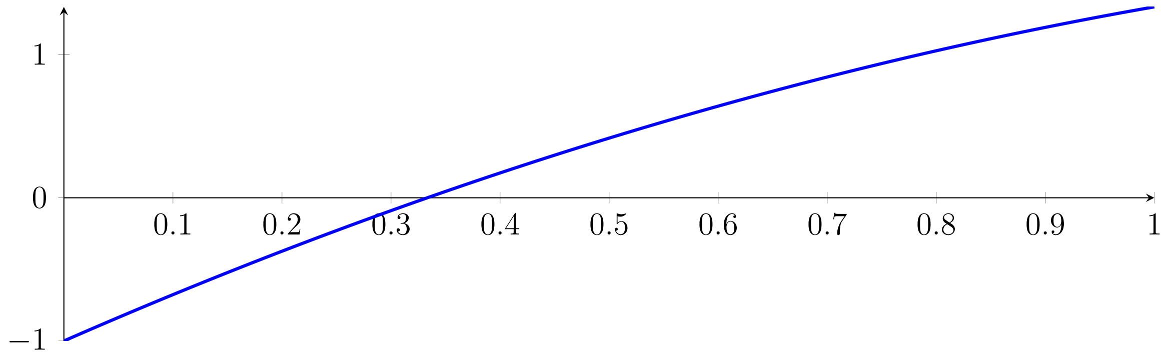
172.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Found root in interval $[0.333333, 0.333334]$ at recursion depth 4!

172.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333334]$$

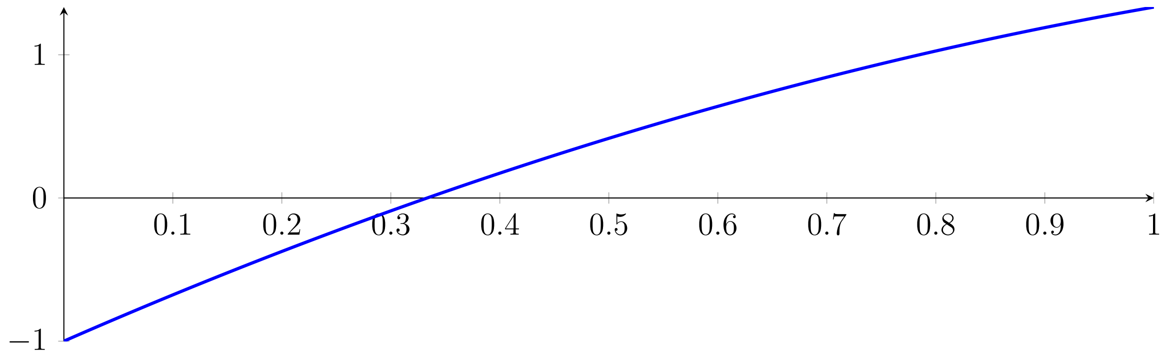
with precision $\varepsilon = 0.0001$.

173 Running QuadClip on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

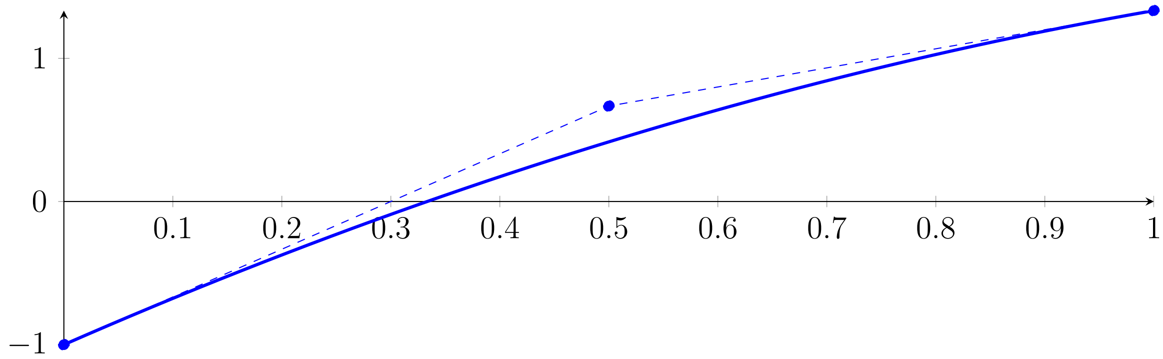
$$p = -1X^2 + 3.33333X - 1$$



173.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

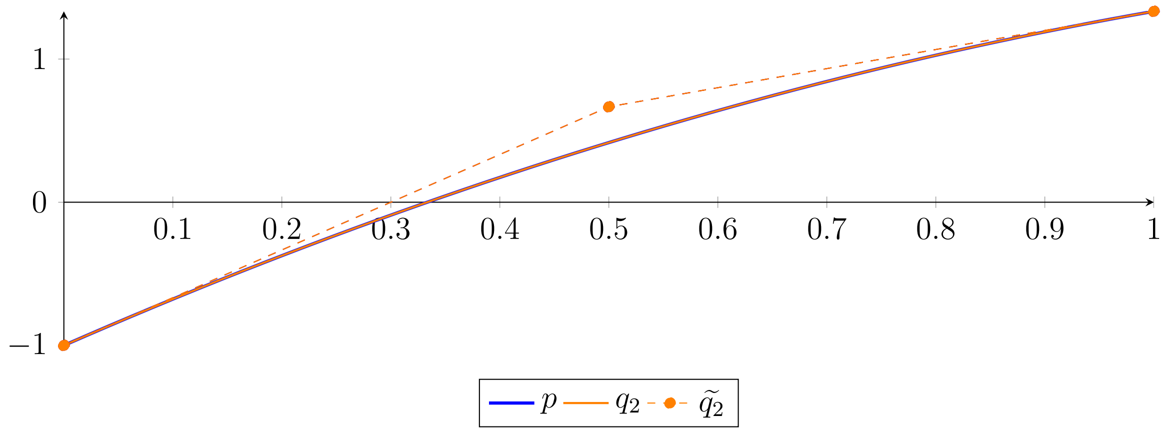
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.22507 \cdot 10^{-308}$.
Bounding polynomials M and m:

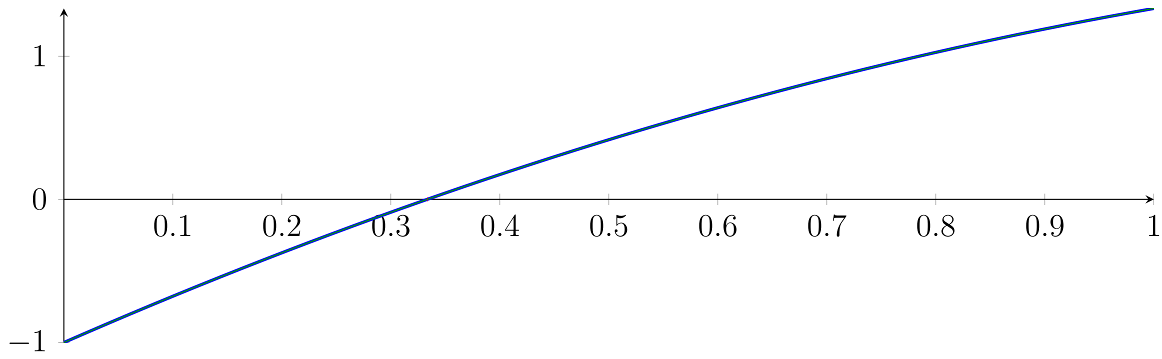
$$M = -1X^2 + 3.33333X - 1$$

$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $1.11254 \cdot 10^{-308}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

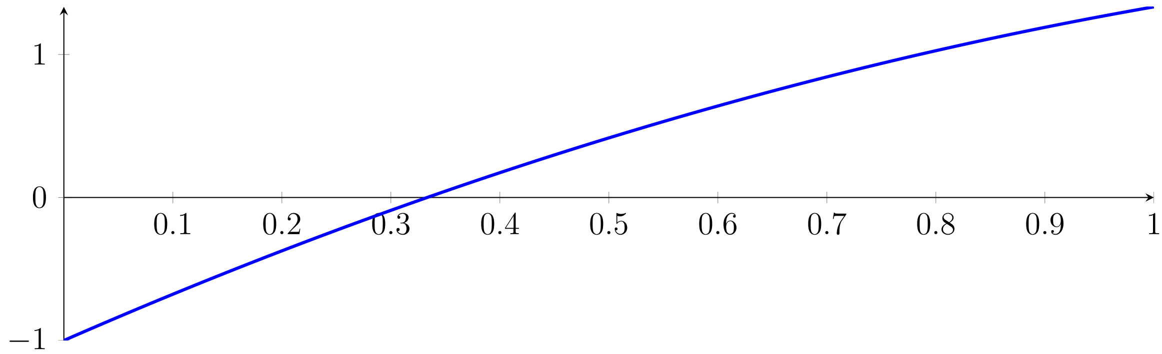
173.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

173.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

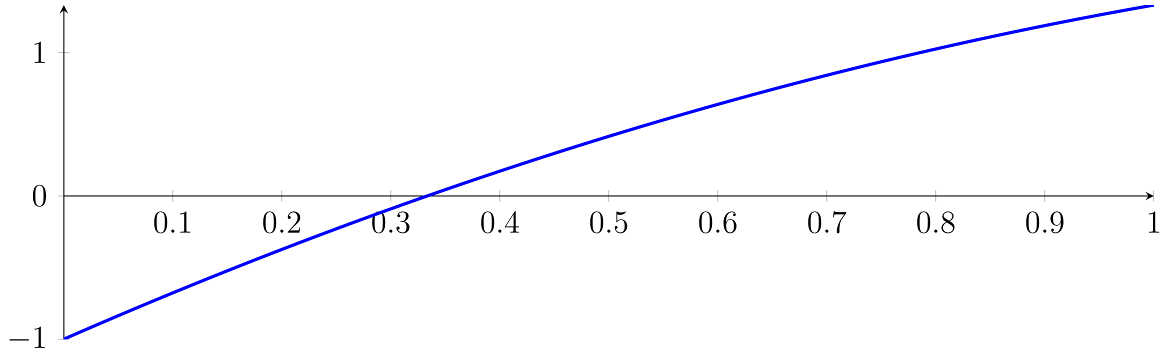
with precision $\varepsilon = 0.0001$.

174 Running CubeClip on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval [0, 1]:

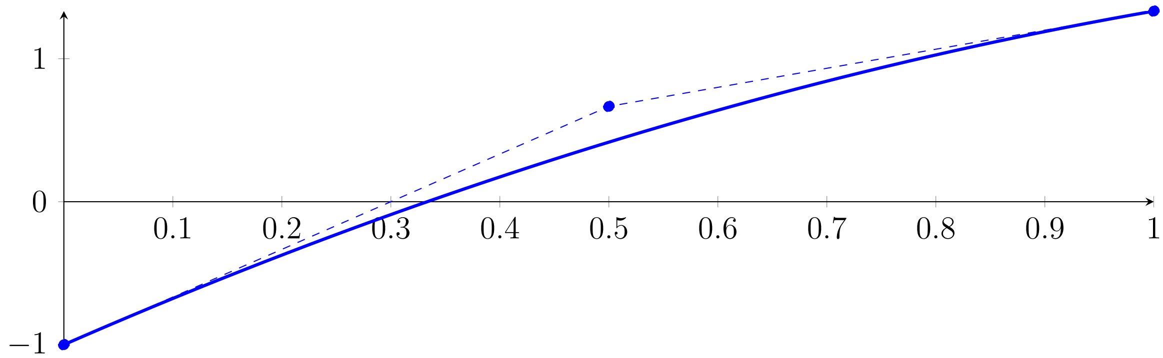
$$p = -1X^2 + 3.33333X - 1$$



174.1 Recursion Branch 1 for Input Interval [0, 1]

Normalized monomial und Bézier representations and the Bézier polygon:

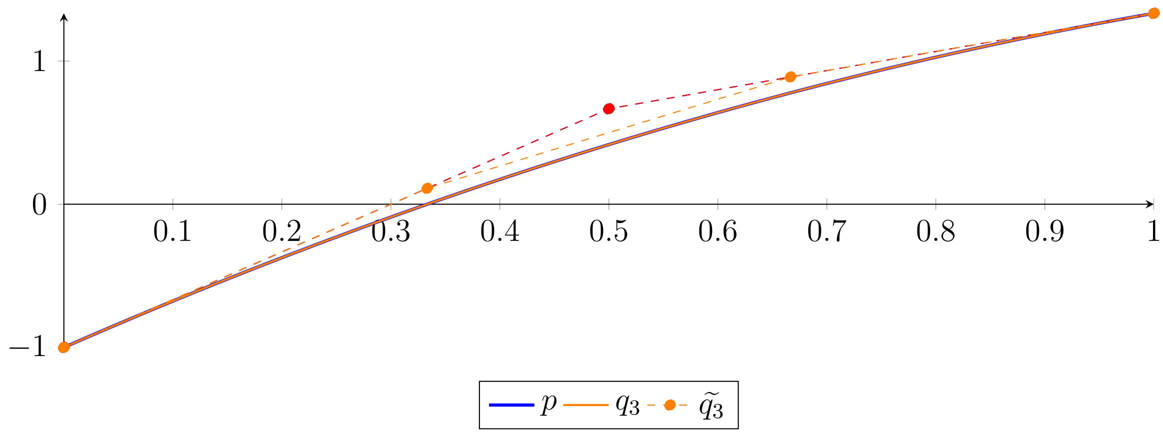
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.66881 \cdot 10^{-308}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.00128{\cdot}10^{-307}$.
Bounding polynomials M and m:

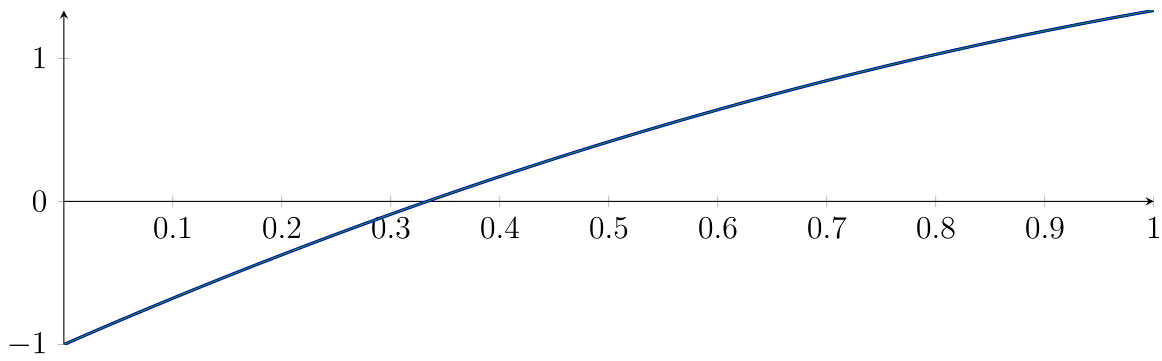
$$M = -2.78134{\cdot}10^{-308}X^3 - 1X^2 + 3.33333X - 1$$

$$m = -2.78134{\cdot}10^{-308}X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$N(M) = \{-3.59539{\cdot}10^{307}, -3.38949{\cdot}10^{291}, 1.03213{\cdot}10^{153}\}$ $N(m) = \{-3.59539{\cdot}10^{307}, -3.38949{\cdot}10^{291}, 1.75\}$

Intersection intervals:

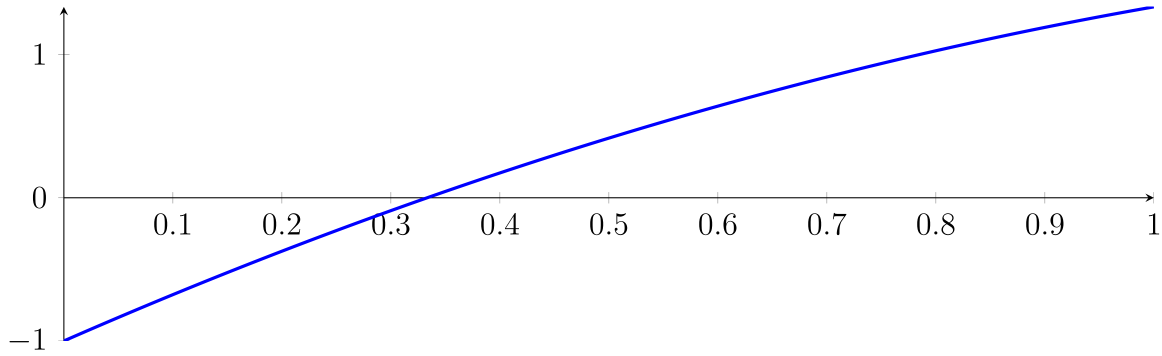


No intersection intervals with the x axis.

174.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

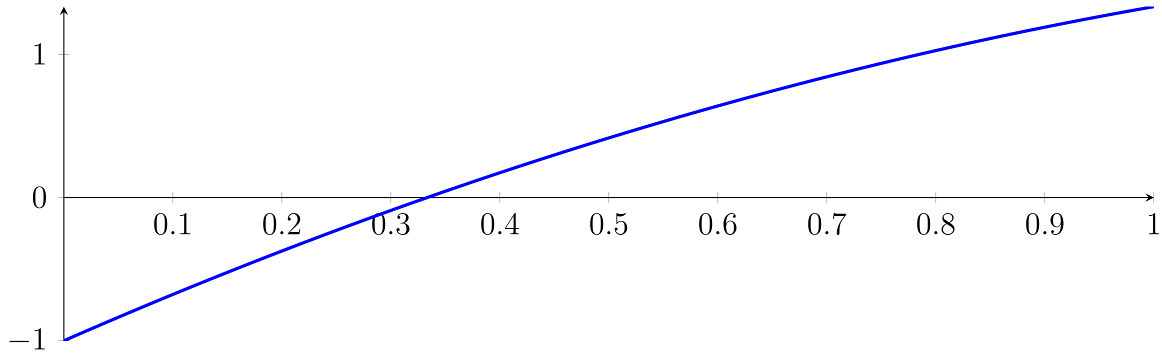
with precision $\varepsilon = 0.0001$.

175 Running BezClip on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

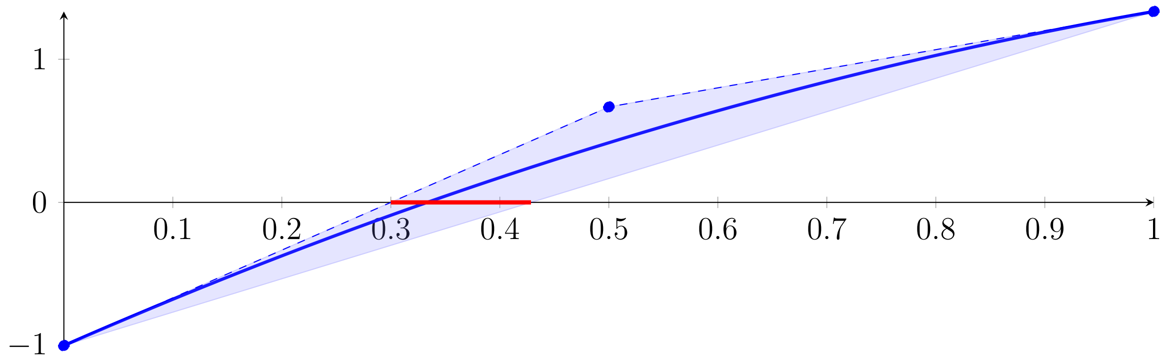
$$p = -1X^2 + 3.33333X - 1$$



175.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

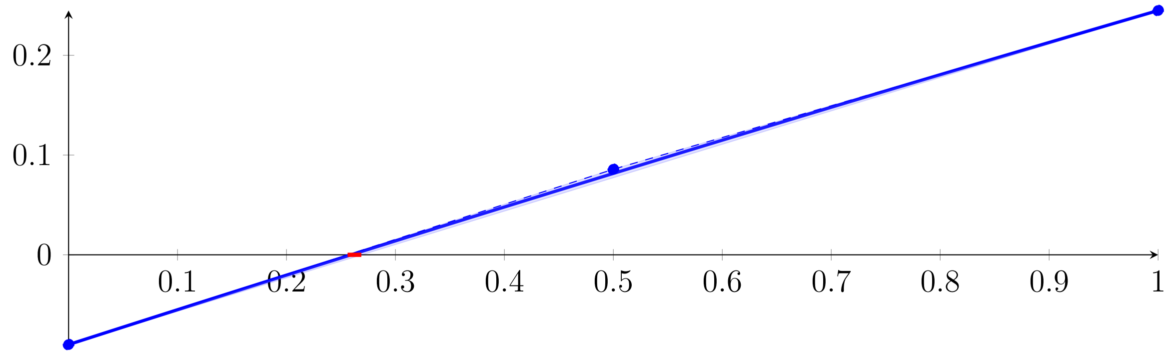
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

175.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

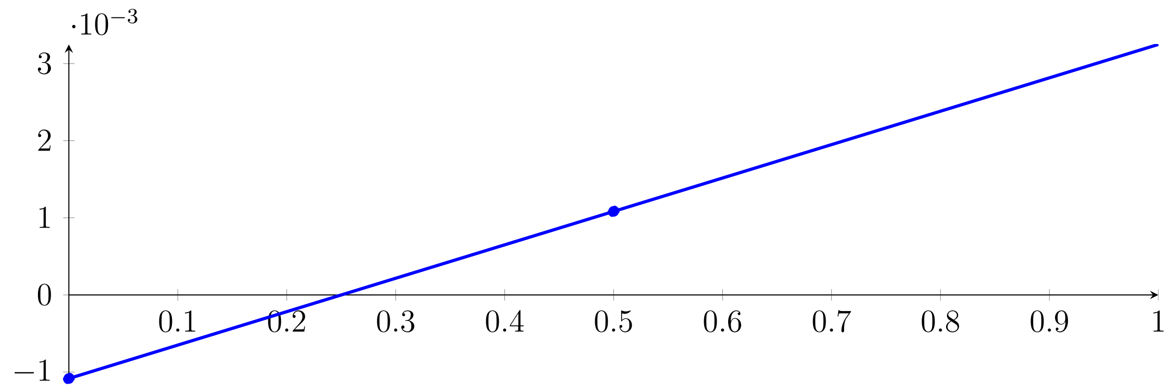
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

175.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538X - 0.00108418 \\ &= -0.00108418B_{0,2}(X) + 0.00108352B_{1,2}(X) + 0.00324857B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

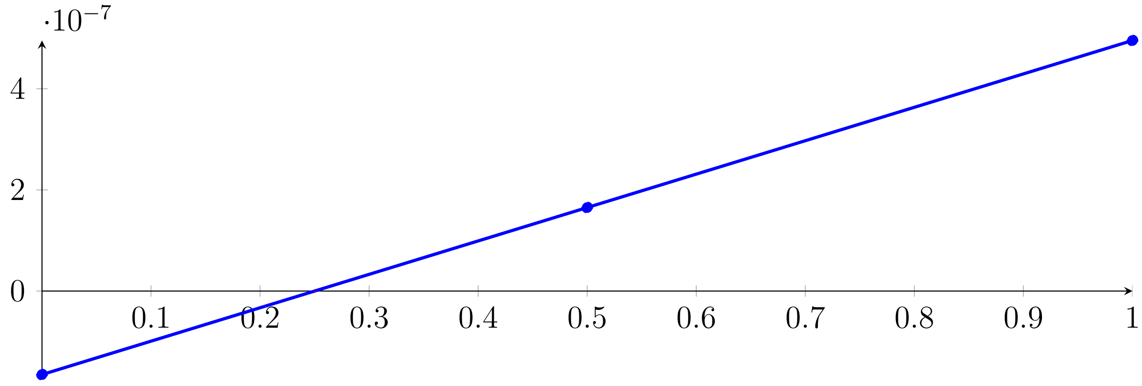
$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

175.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07}$$
$$= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $2.32306 \cdot 10^{-08}$
$\implies$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

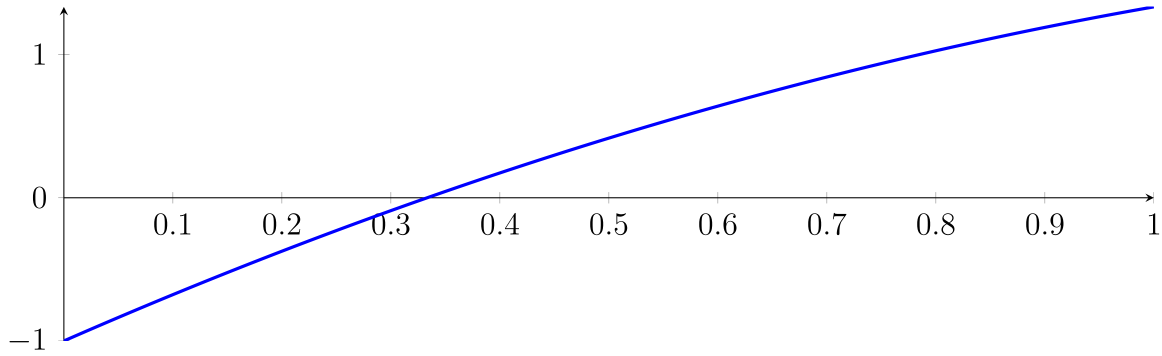
175.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

175.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

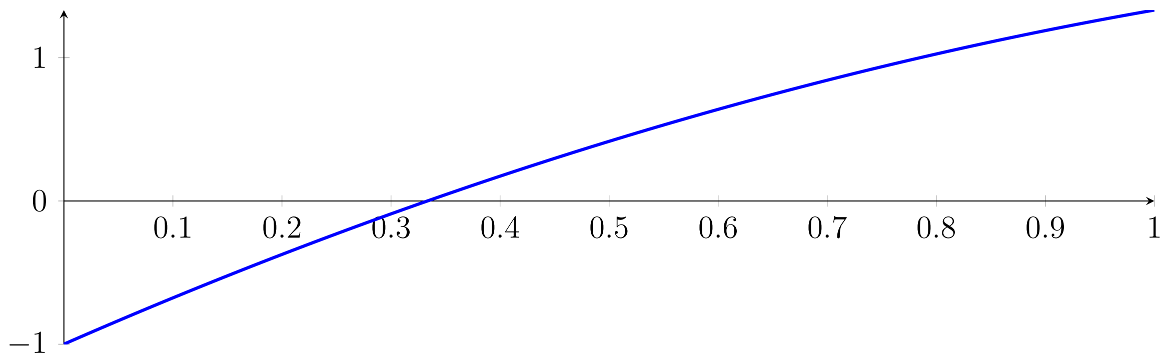
with precision $\varepsilon = 1 \cdot 10^{-08}$.

176 Running QuadClip on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval [0, 1]:

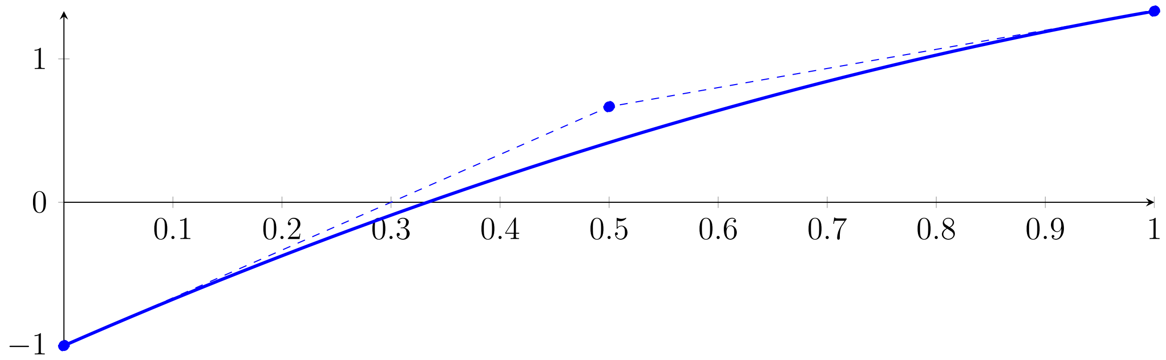
$$p = -1X^2 + 3.33333X - 1$$



176.1 Recursion Branch 1 for Input Interval [0, 1]

Normalized monomial und Bézier representations and the Bézier polygon:

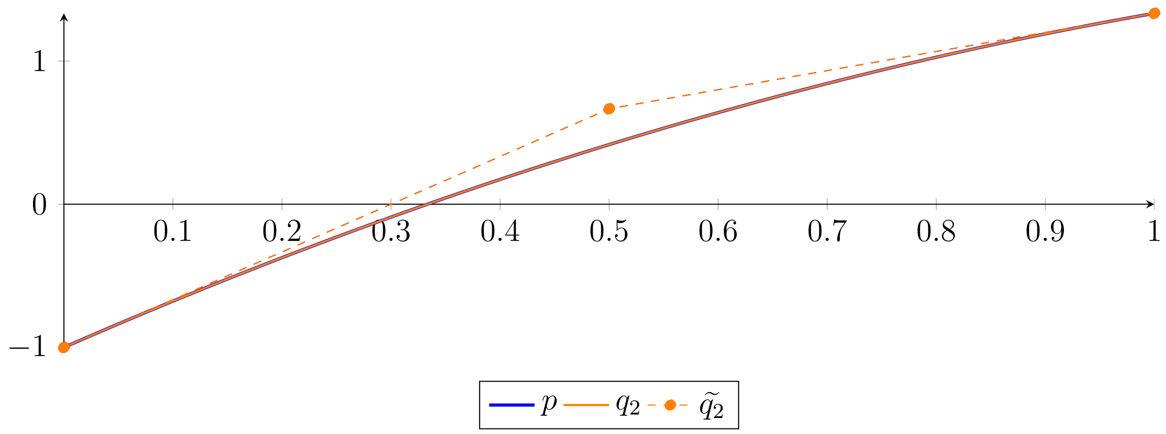
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.22507 \cdot 10^{-308}$.
Bounding polynomials M and m:

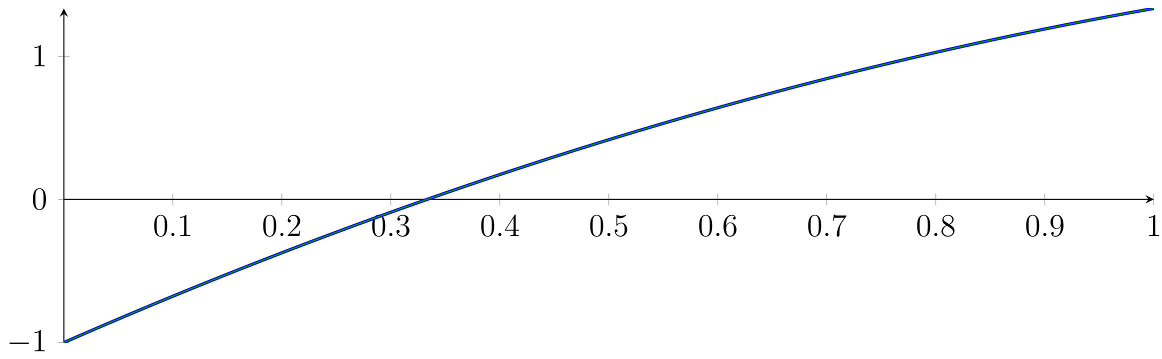
$$M = -1X^2 + 3.33333X - 1$$

$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $1.11254 \cdot 10^{-308}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

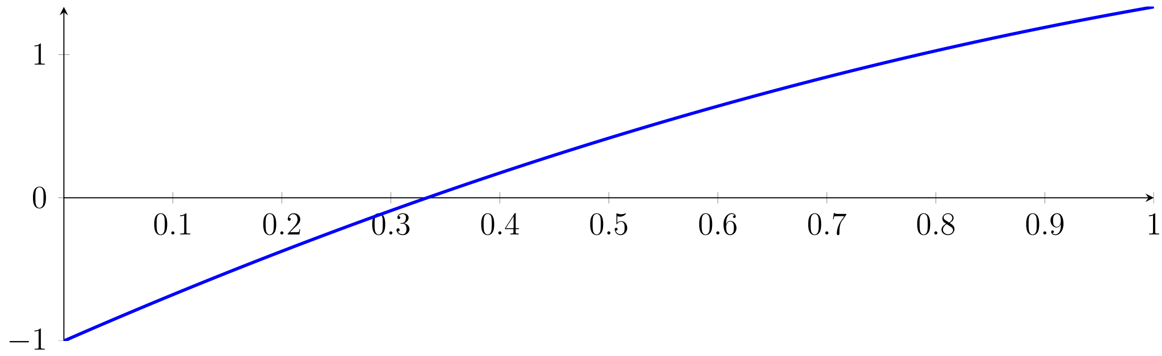
176.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

176.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

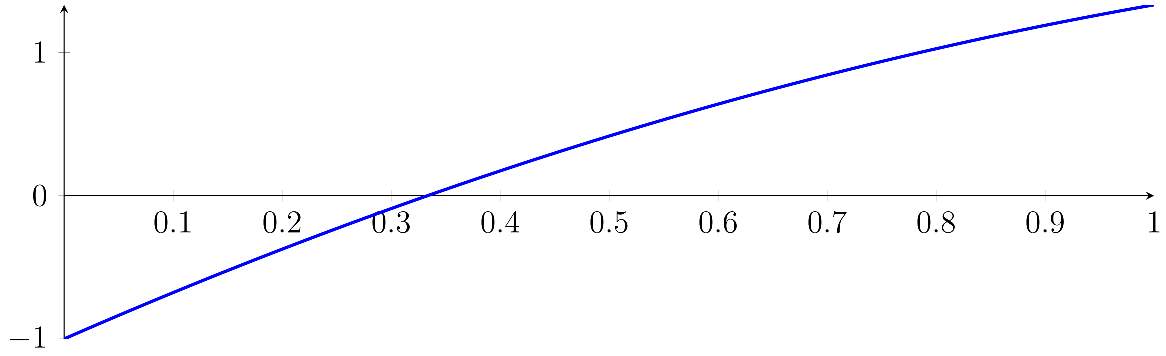
with precision $\varepsilon = 1 \cdot 10^{-08}$.

177 Running CubeClip on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0,1]$:

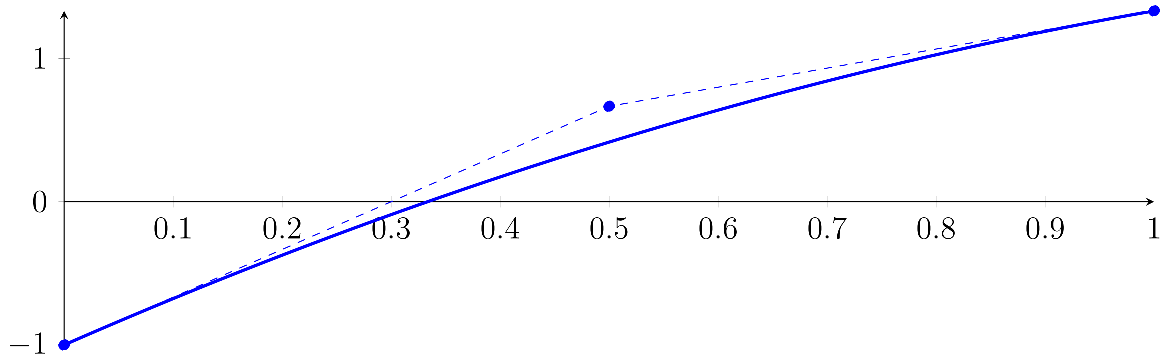
$$p = -1X^2 + 3.33333X - 1$$



177.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

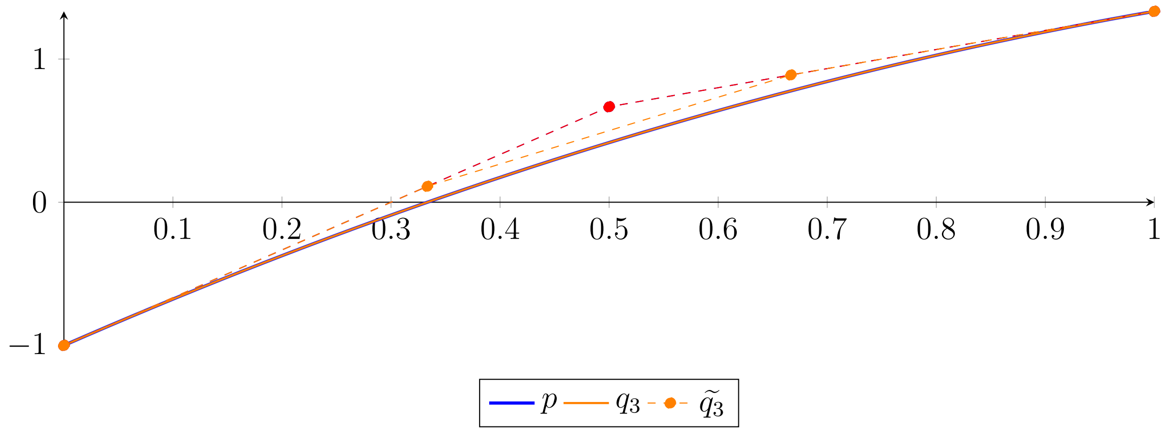
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.66881 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.00128{\cdot}10^{-307}$.
Bounding polynomials M and m:

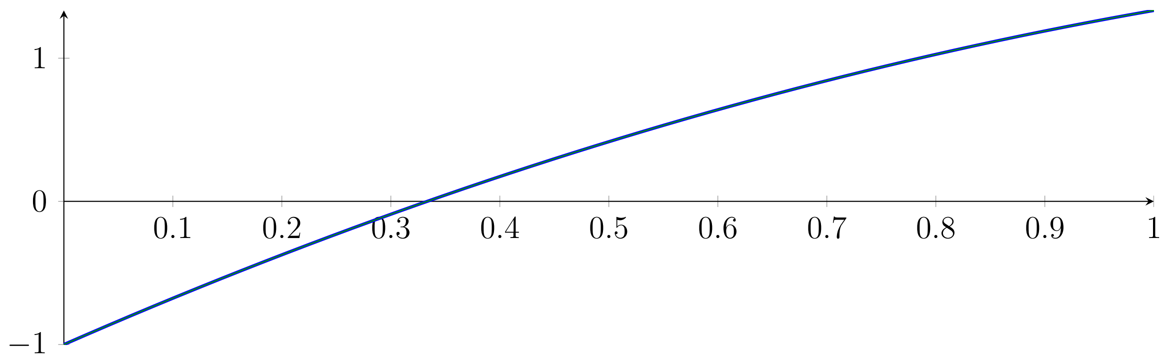
$$M = -2.78134{\cdot}10^{-308}X^3 - 1X^2 + 3.33333X - 1$$

$$m = -2.78134{\cdot}10^{-308}X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$N(M) = \{-3.59539{\cdot}10^{307}, -3.38949{\cdot}10^{291}, 1.03213{\cdot}10^{153}\}$ $N(m) = \{-3.59539{\cdot}10^{307}, -3.38949{\cdot}10^{291}, 1.75\}$

Intersection intervals:

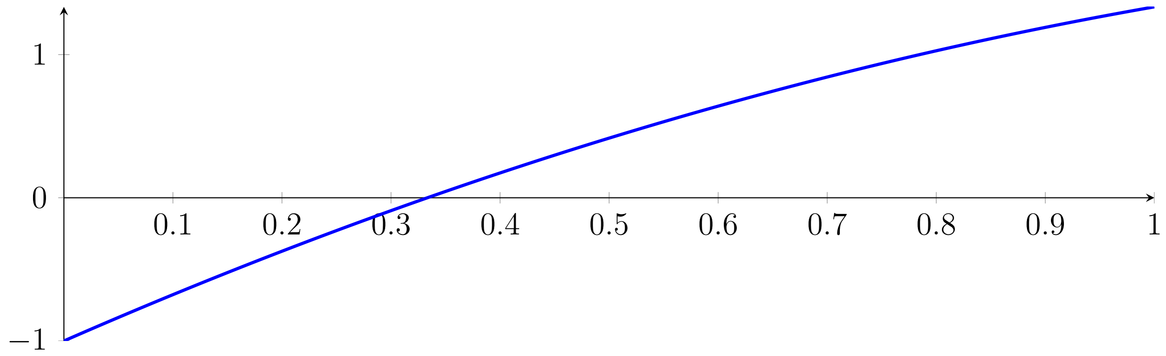


No intersection intervals with the x axis.

177.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

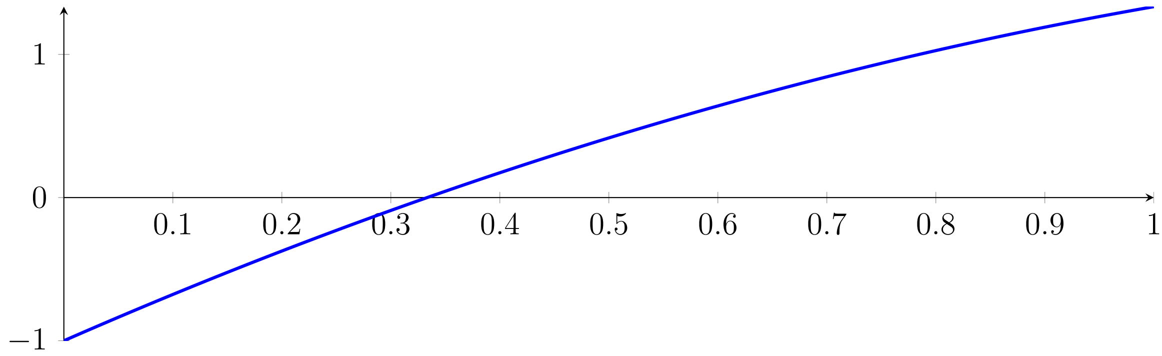
with precision $\varepsilon = 1 \cdot 10^{-08}$.

178 Running BezClip on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

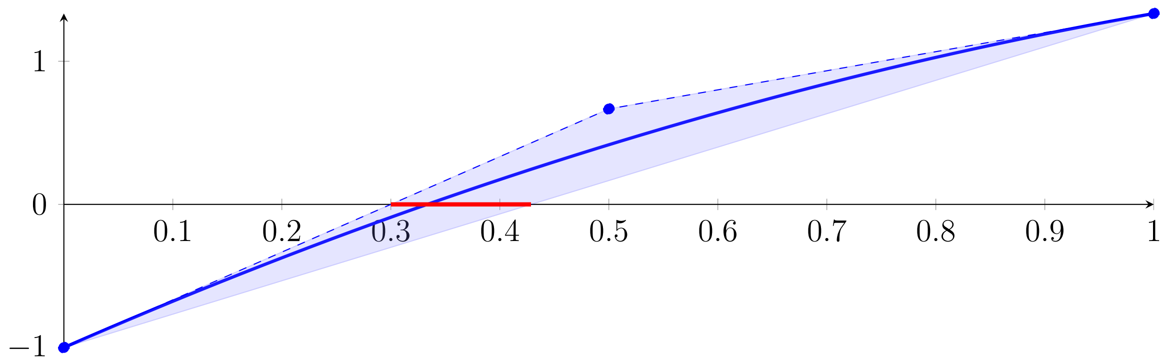
$$p = -1X^2 + 3.33333X - 1$$



178.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

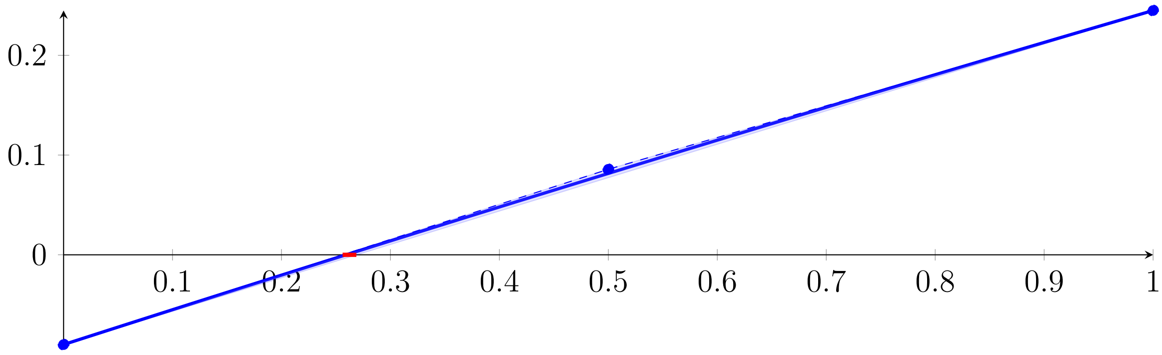
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

178.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

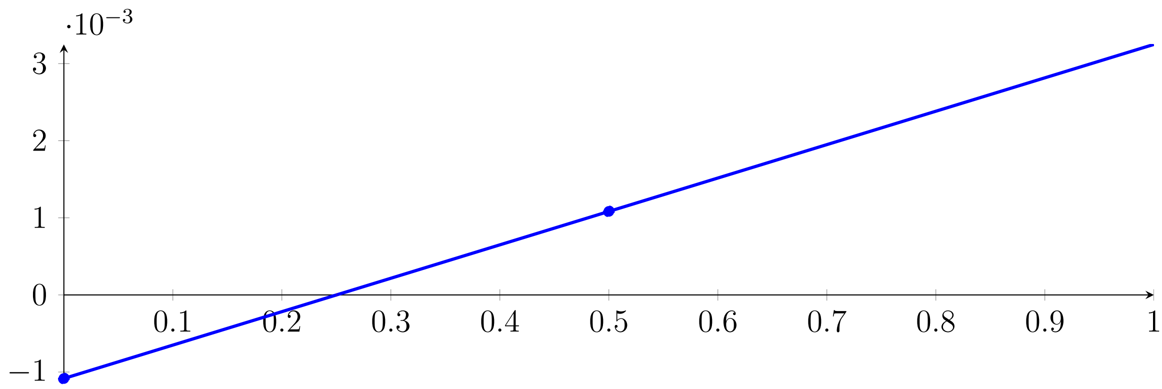
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

178.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

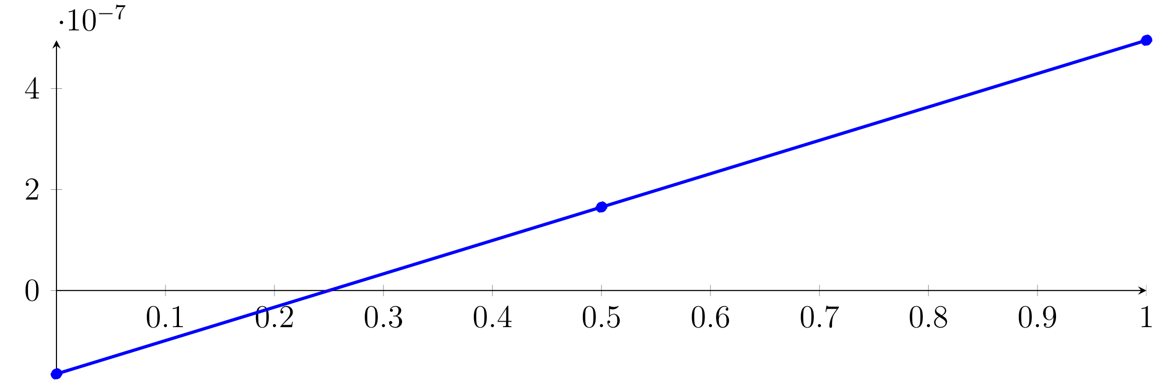
$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

178.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07}$$
$$= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

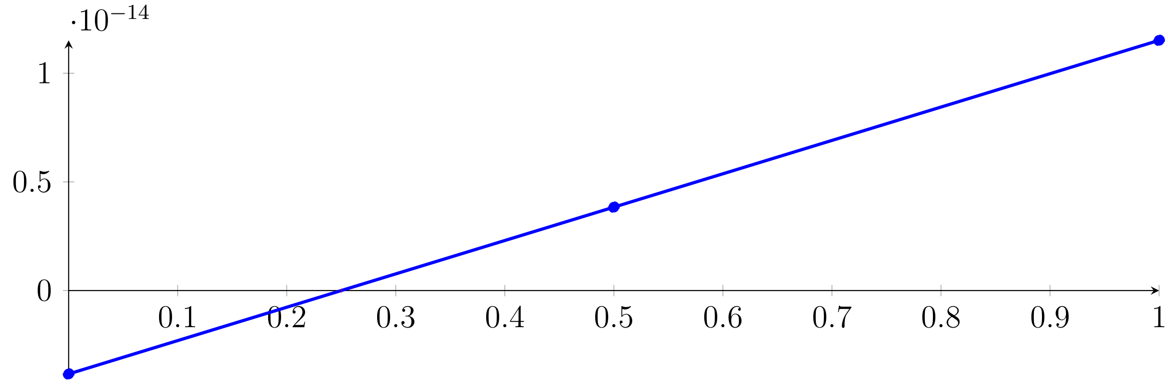
Longest intersection interval: $2.32306 \cdot 10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

178.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.31358 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15}$$
$$= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.3966 \cdot 10^{-16}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

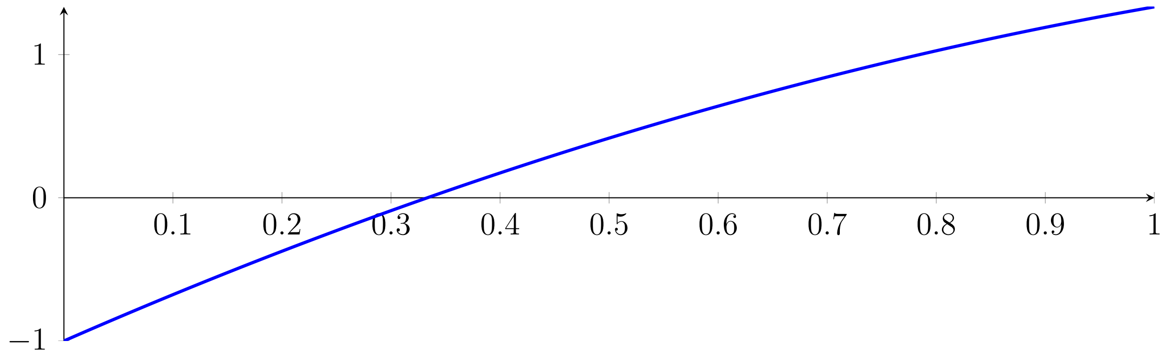
178.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

178.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

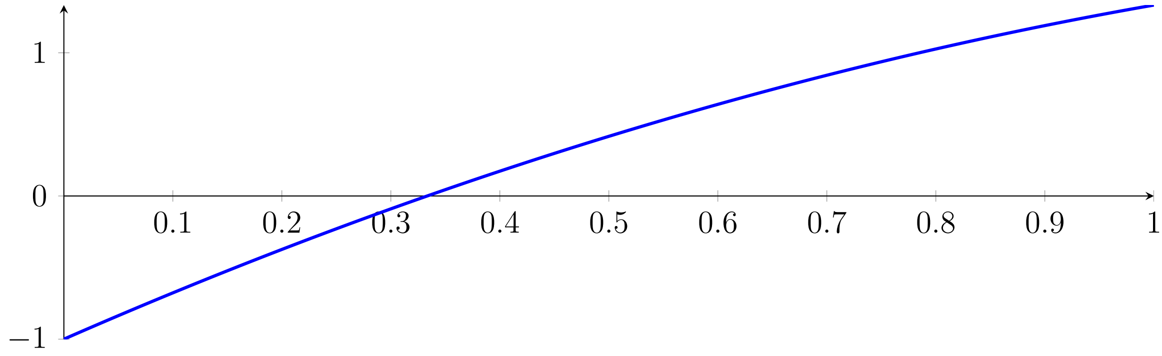
with precision $\varepsilon = 1 \cdot 10^{-16}$.

179 Running QuadClip on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

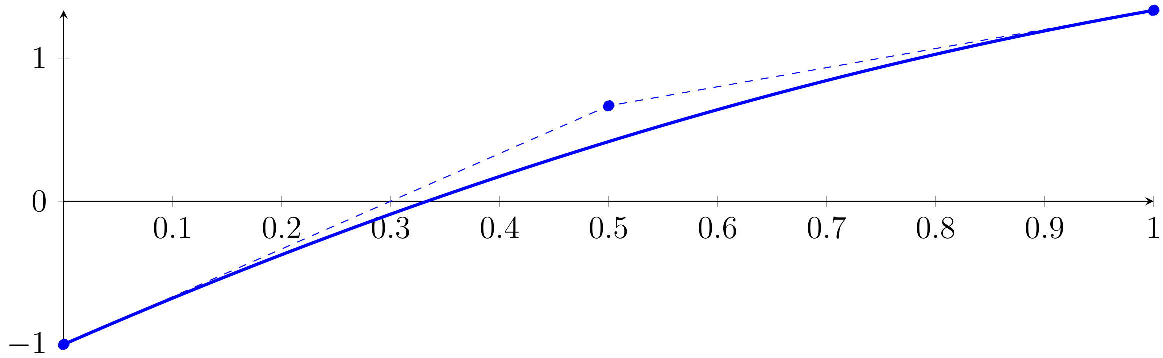
$$p = -1X^2 + 3.33333X - 1$$



179.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

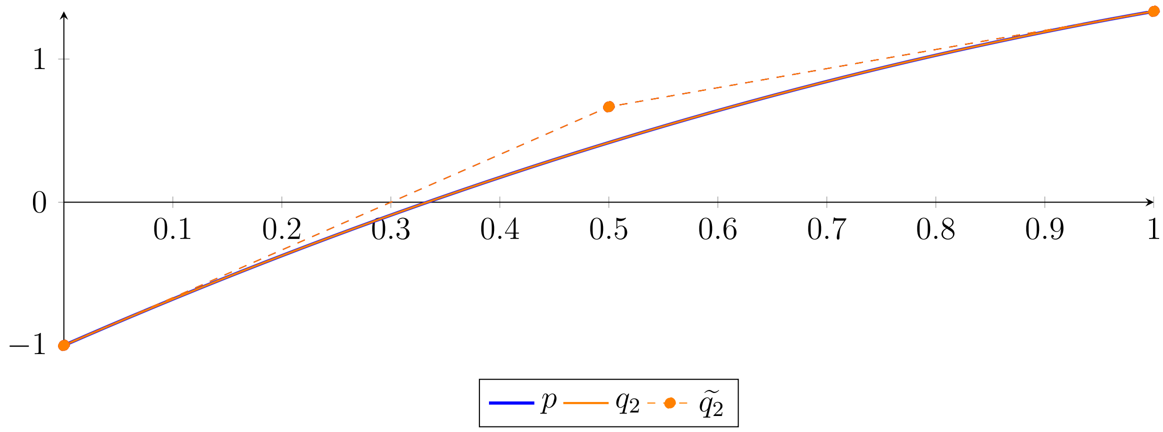
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.22507 \cdot 10^{-308}$.
Bounding polynomials M and m:

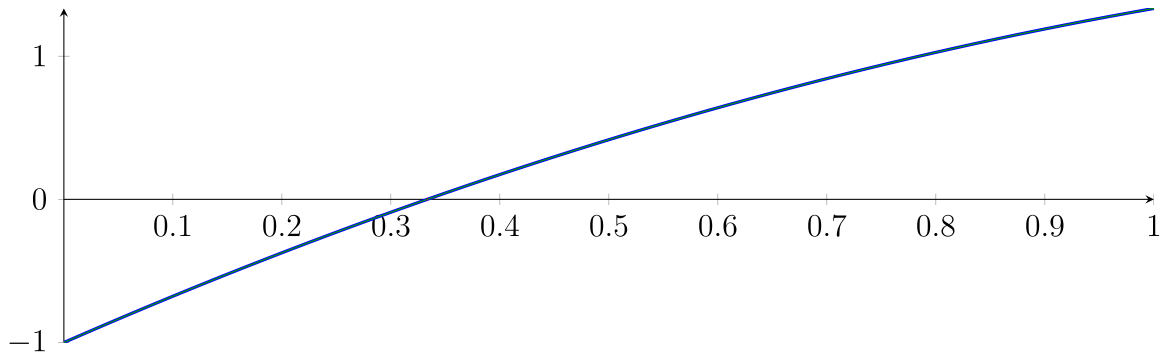
$$M = -1X^2 + 3.33333X - 1$$

$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $1.11254 \cdot 10^{-308}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

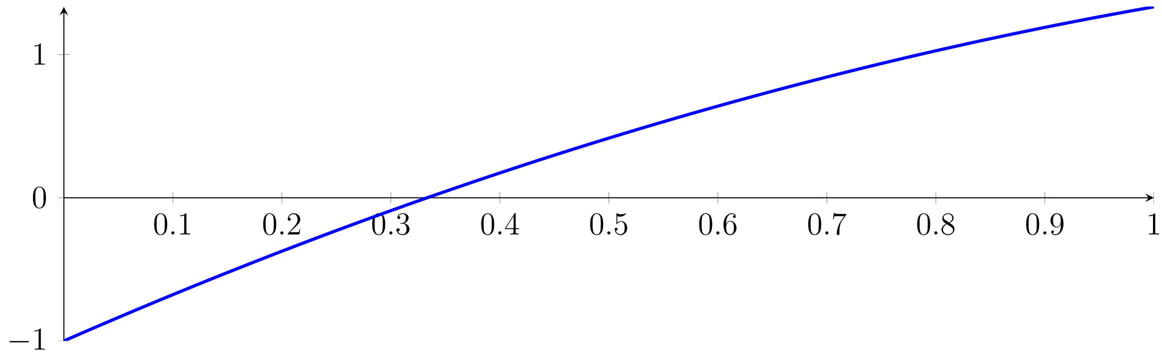
179.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

179.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

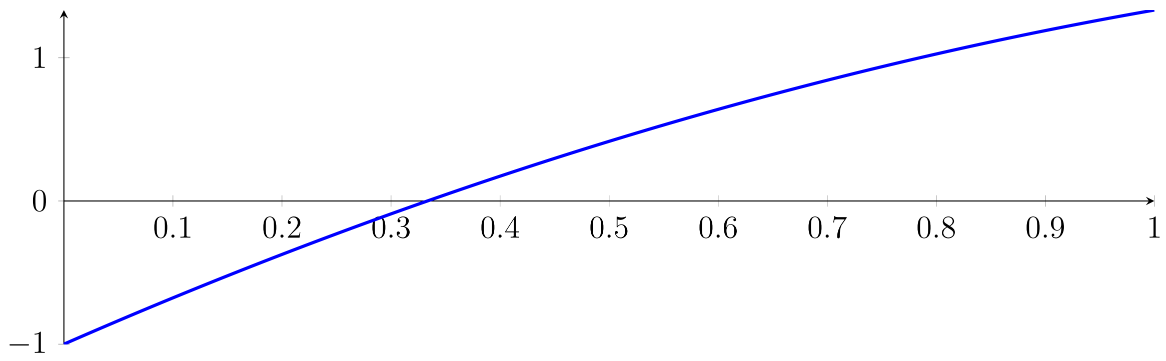
with precision $\varepsilon = 1 \cdot 10^{-16}$.

180 Running CubeClip on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

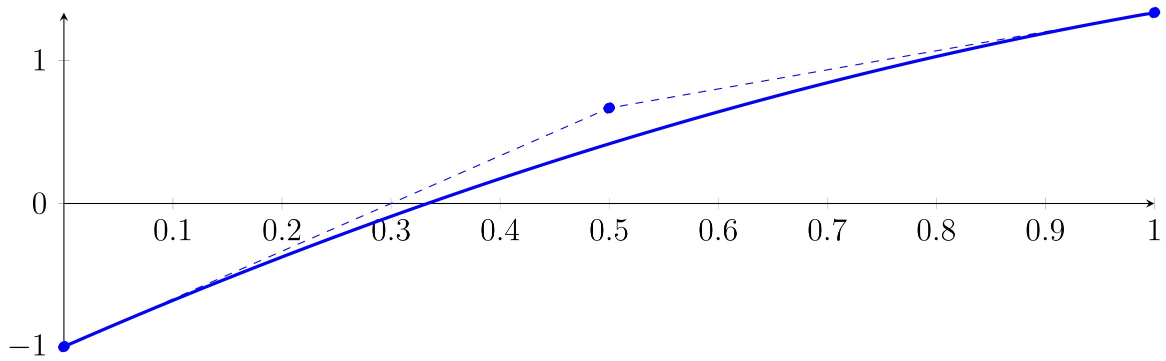
$$p = -1X^2 + 3.33333X - 1$$



180.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

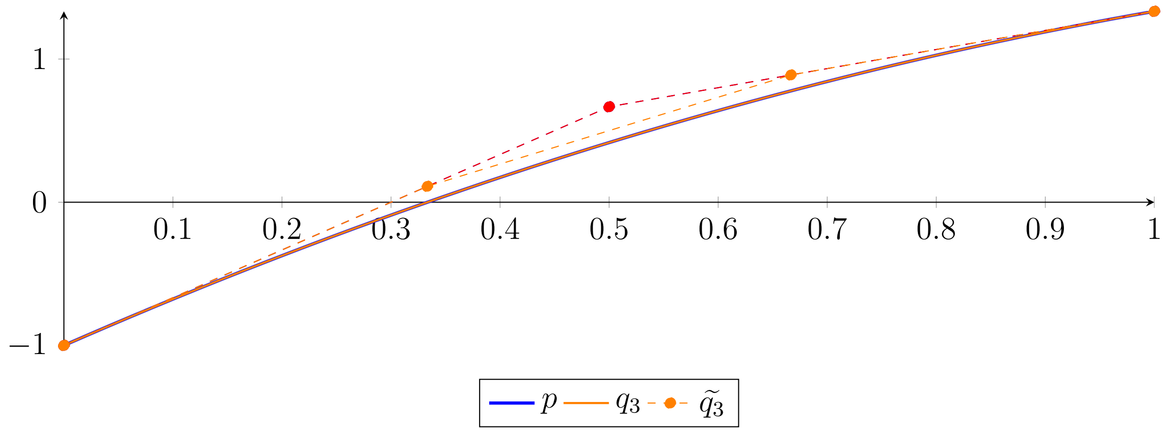
$$p = -1X^2 + 3.33333X - 1$$
$$= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X)$$



Degree reduction and raising:

$$q_3 = -1.66881 \cdot 10^{-308}X^3 - 1X^2 + 3.33333X - 1$$
$$= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3}$$

$$\widetilde{q}_3 = -1X^2 + 3.33333X - 1$$
$$= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2}$$



The maximum difference of the Bézier coefficients is $\delta = 1.00128{\cdot}10^{-307}$.
Bounding polynomials M and m:

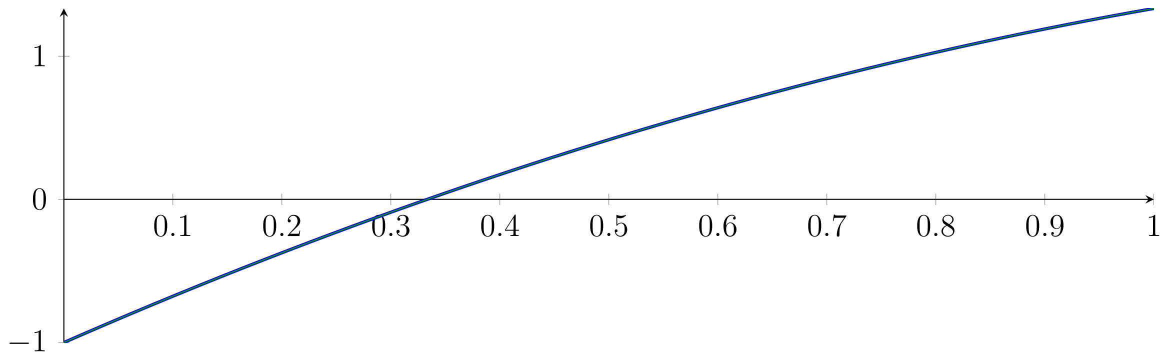
$$M = -2.78134{\cdot}10^{-308}X^3 - 1X^2 + 3.33333X - 1$$

$$m = -2.78134{\cdot}10^{-308}X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$N(M) = \{-3.59539{\cdot}10^{307}, -3.38949{\cdot}10^{291}, 1.03213{\cdot}10^{153}\}$ $N(m) = \{-3.59539{\cdot}10^{307}, -3.38949{\cdot}10^{291}, 1.75\}$

Intersection intervals:

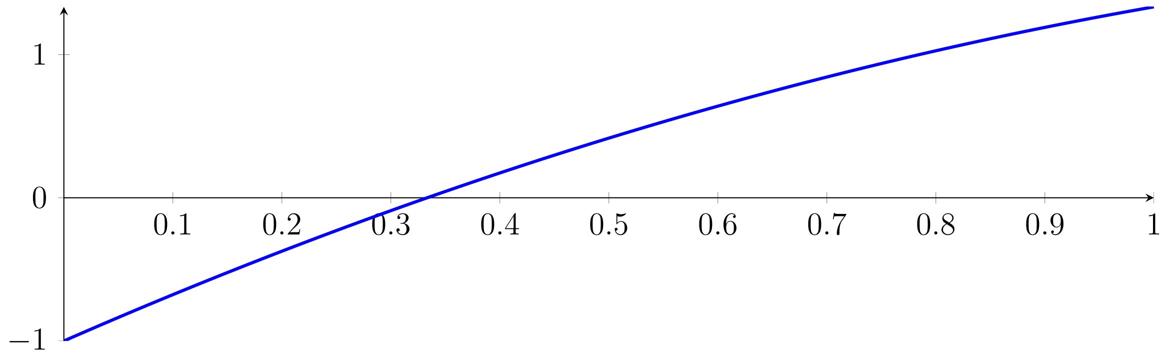


No intersection intervals with the x axis.

180.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

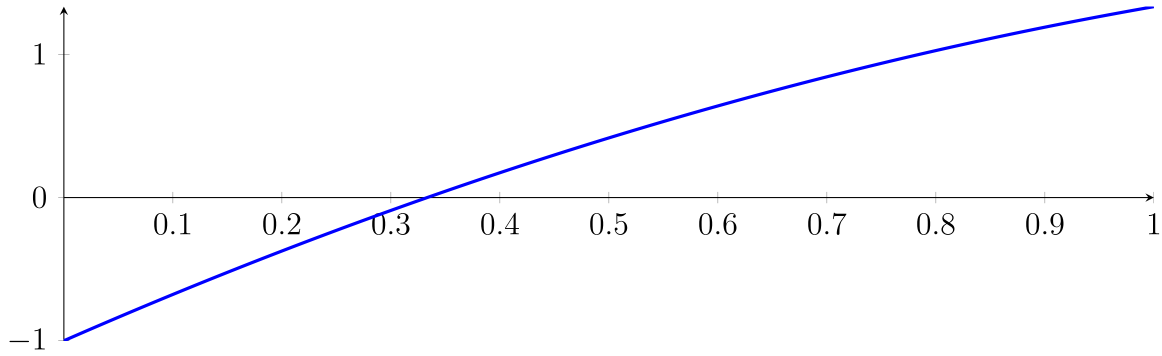
with precision $\varepsilon = 1{\cdot}10^{-16}$.

181 Running BezClip on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

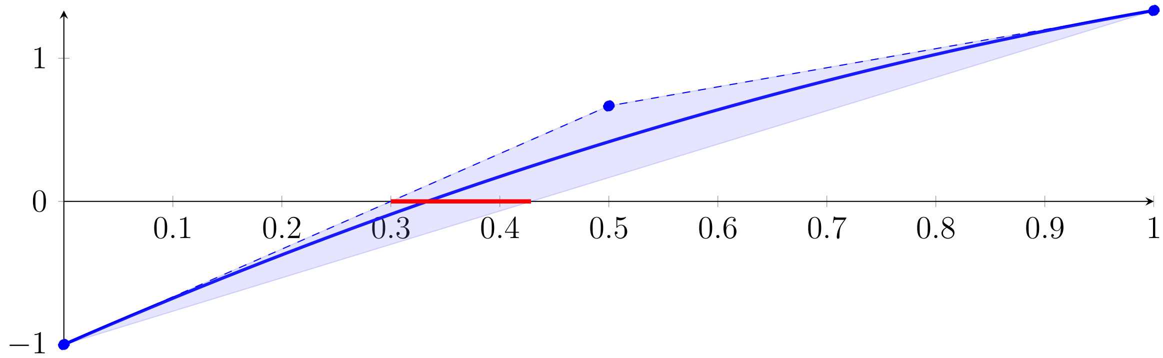
$$p = -1X^2 + 3.33333X - 1$$



181.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

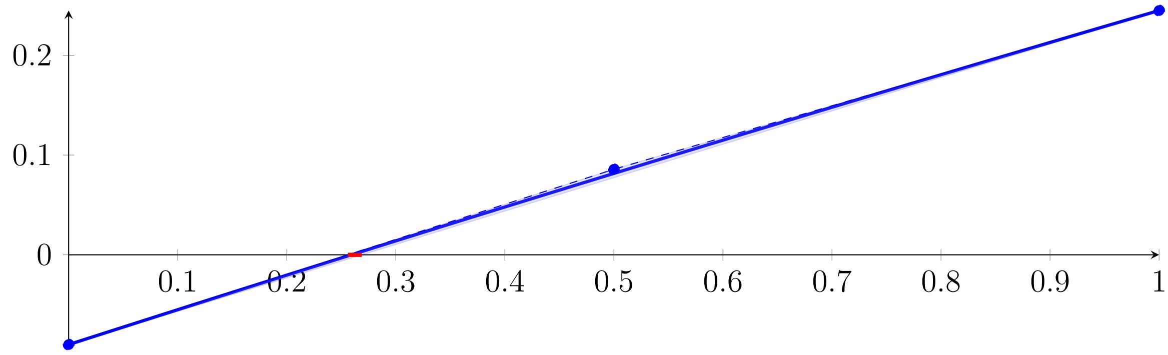
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

181.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

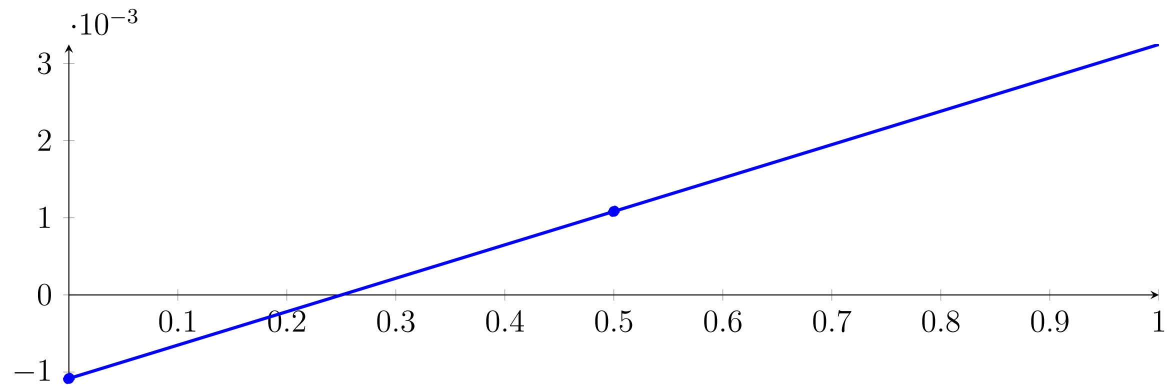
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

181.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -2.64151 \cdot 10^{-06} X^2 + 0.00433538X - 0.00108418$$
$$= -0.00108418B_{0,2}(X) + 0.00108352B_{1,2}(X) + 0.00324857B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

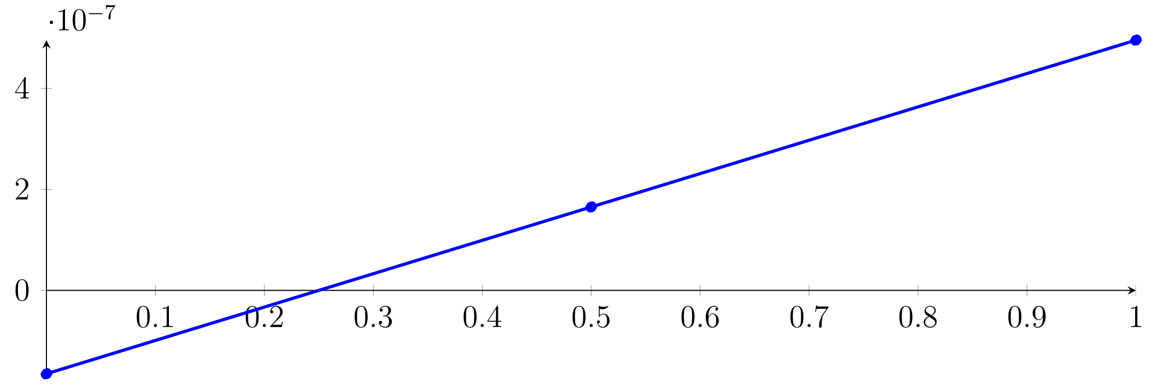
$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

181.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07}$$
$$= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

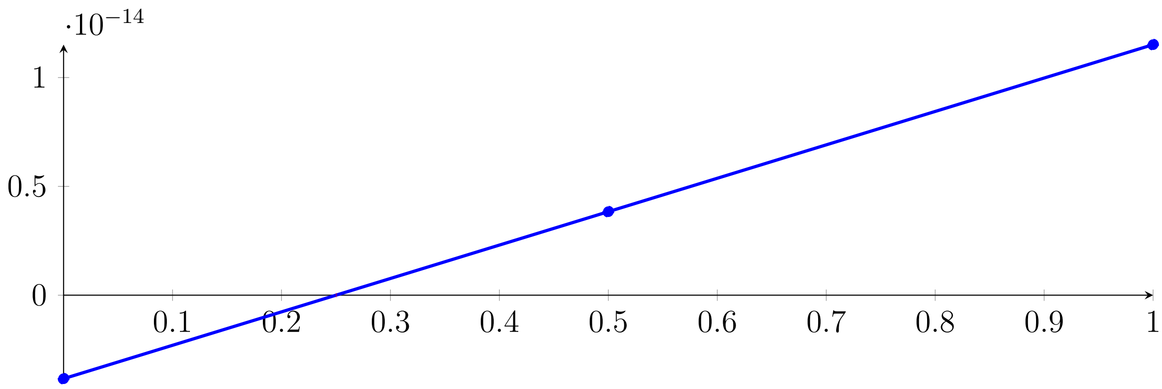
$$[0.25, 0.25]$$

Longest intersection interval: $2.32306 \cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

181.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.31358 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15}$$
$$= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

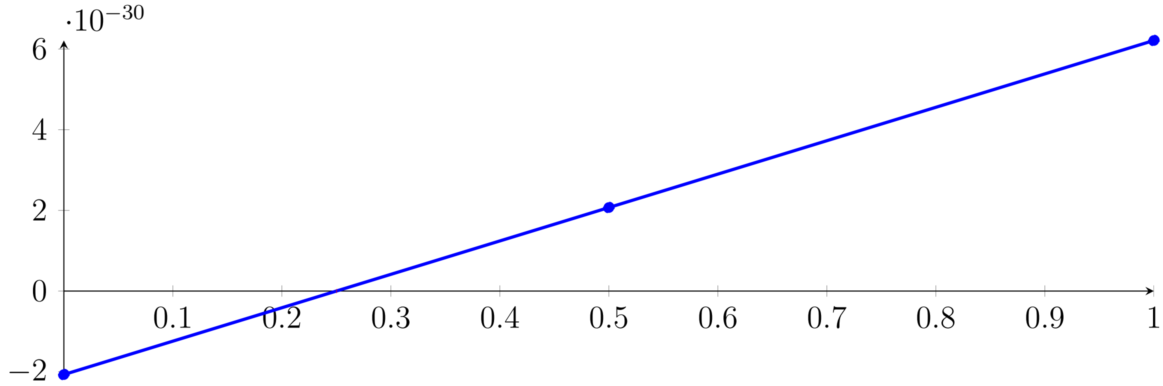
$$[0.25, 0.25]$$

Longest intersection interval: $5.3966 \cdot 10^{-16}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

181.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -9.65021 \cdot 10^{-60} X^2 + 8.28394 \cdot 10^{-30} X - 2.07099 \cdot 10^{-30}$$
$$= -2.07099 \cdot 10^{-30} B_{0,2}(X) + 2.07099 \cdot 10^{-30} B_{1,2}(X) + 6.21296 \cdot 10^{-30} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $2.91232 \cdot 10^{-31}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

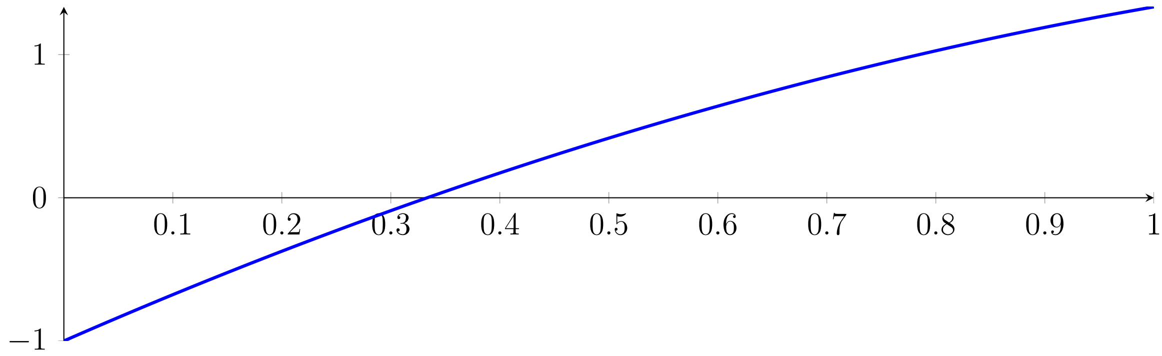
181.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 7!

181.8 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

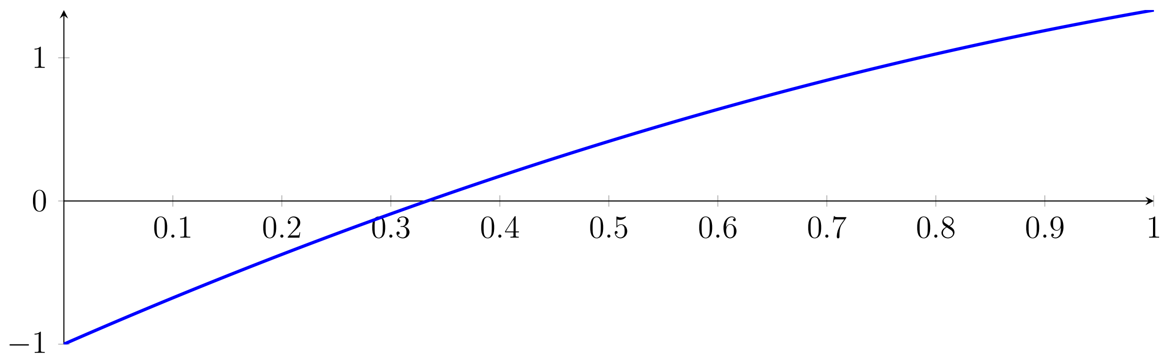
with precision $\varepsilon = 1 \cdot 10^{-32}$.

182 Running `QuadClip` on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called `QuadClip` with input polynomial on interval $[0, 1]$:

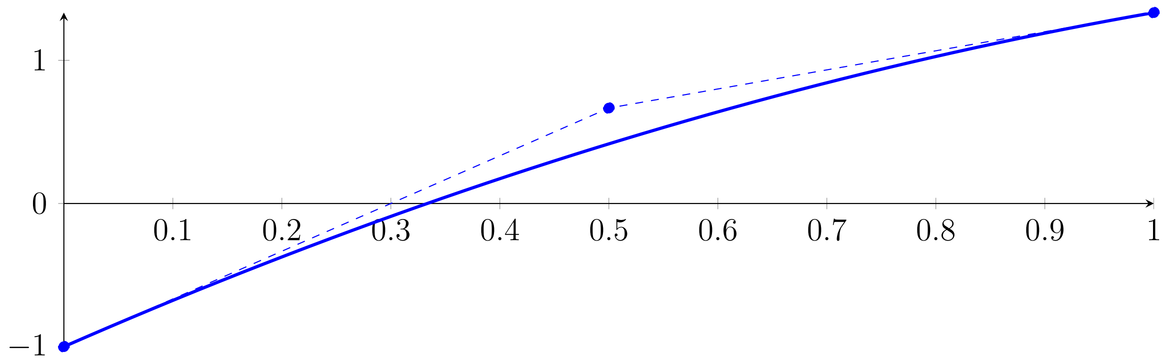
$$p = -1X^2 + 3.33333X - 1$$



182.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

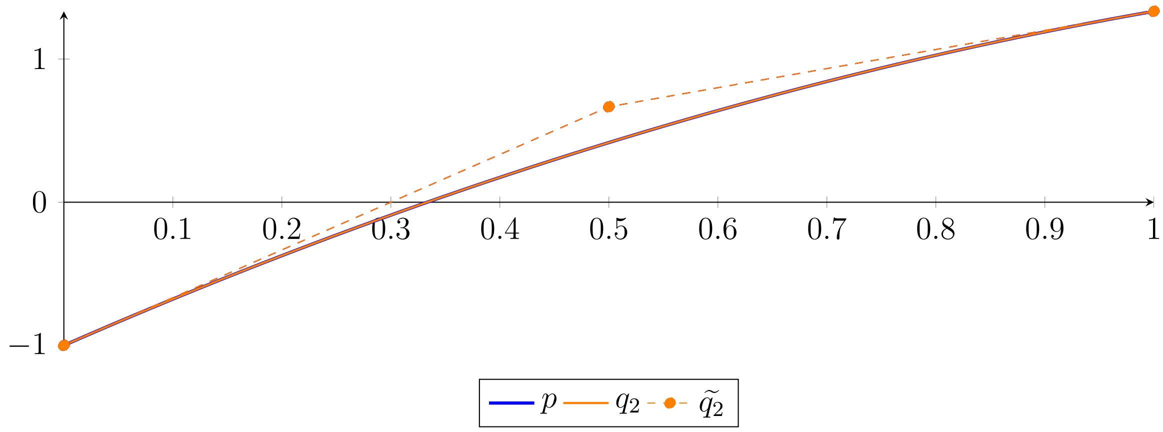
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.22507 \cdot 10^{-308}$.
Bounding polynomials M and m:

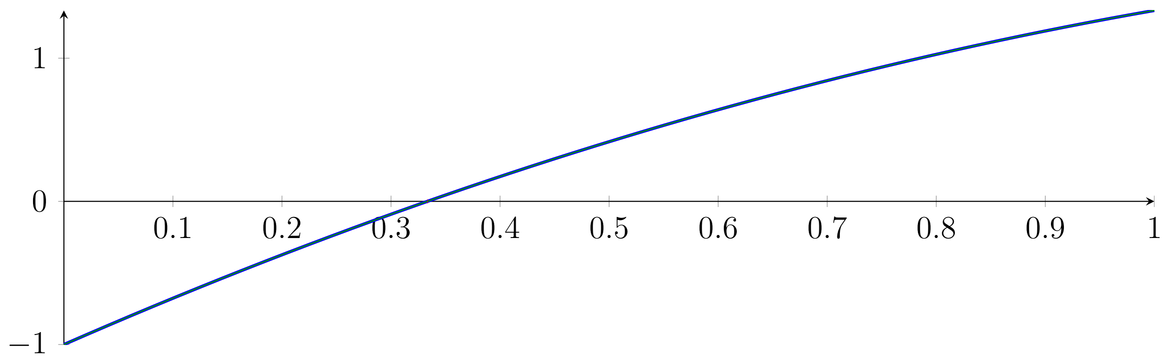
$$M = -1X^2 + 3.33333X - 1$$

$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $1.11254 \cdot 10^{-308}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

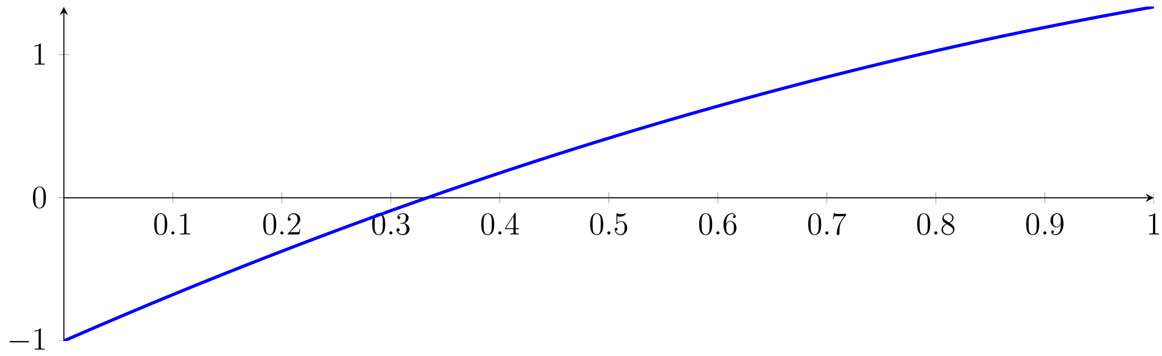
182.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

182.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

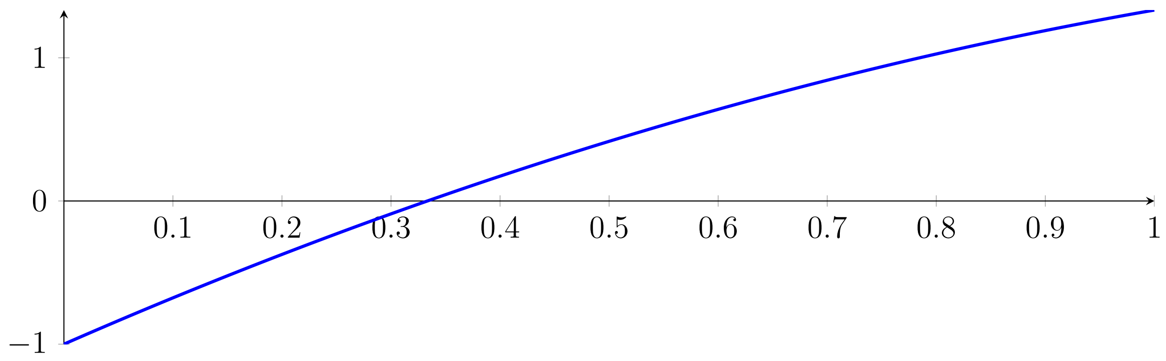
with precision $\varepsilon = 1 \cdot 10^{-32}$.

183 Running CubeClip on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

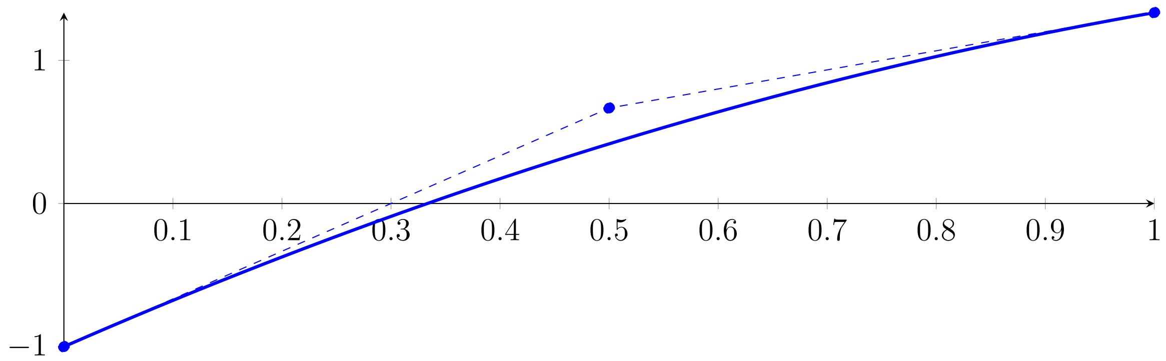
$$p = -1X^2 + 3.33333X - 1$$



183.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

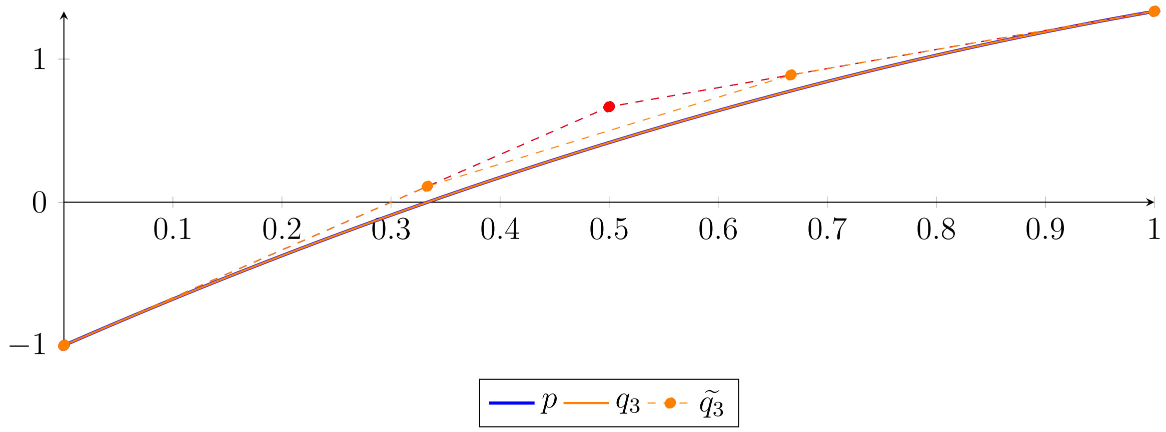
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.66881 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.00128{\cdot}10^{-307}$.
Bounding polynomials M and m:

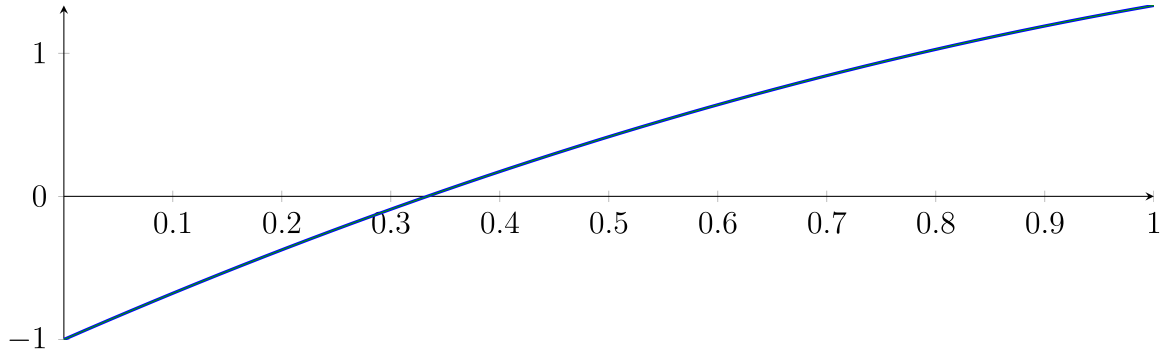
$$M = -2.78134{\cdot}10^{-308}X^3 - 1X^2 + 3.33333X - 1$$

$$m = -2.78134{\cdot}10^{-308}X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$N(M) = \{-3.59539{\cdot}10^{307}, -3.38949{\cdot}10^{291}, 1.03213{\cdot}10^{153}\}$ $N(m) = \{-3.59539{\cdot}10^{307}, -3.38949{\cdot}10^{291}, 1.75\}$

Intersection intervals:

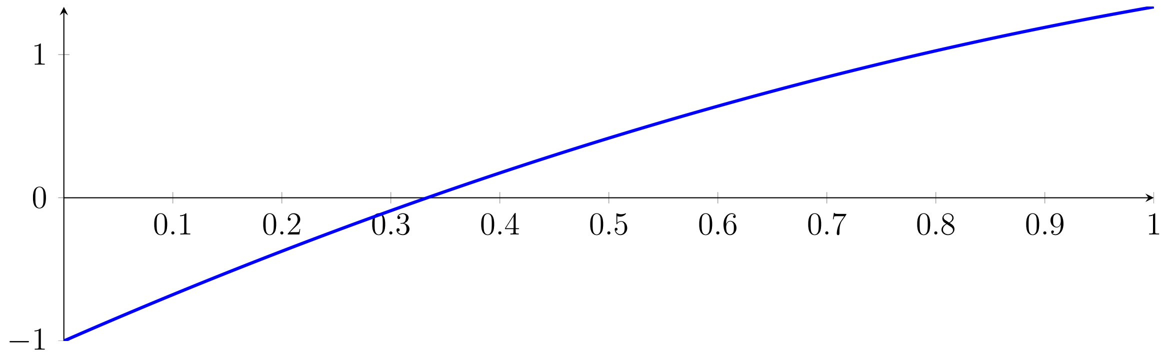


No intersection intervals with the x axis.

183.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

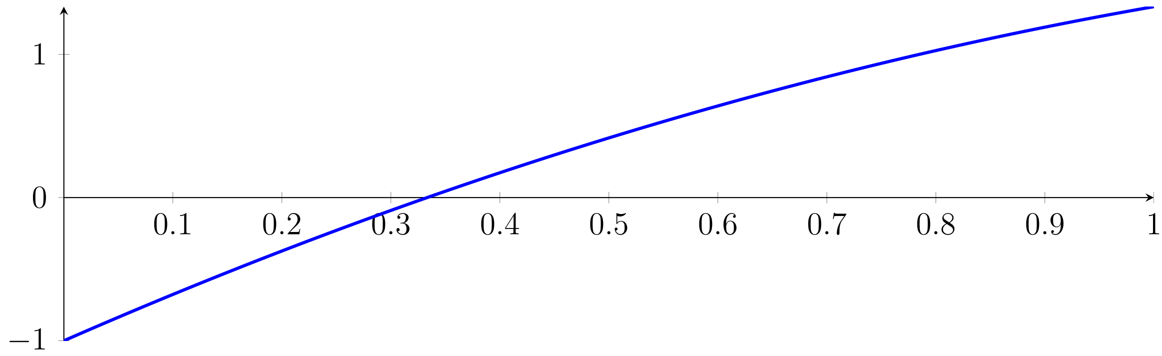
with precision $\varepsilon = 1 \cdot 10^{-32}$.

184 Running `BezClip` on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

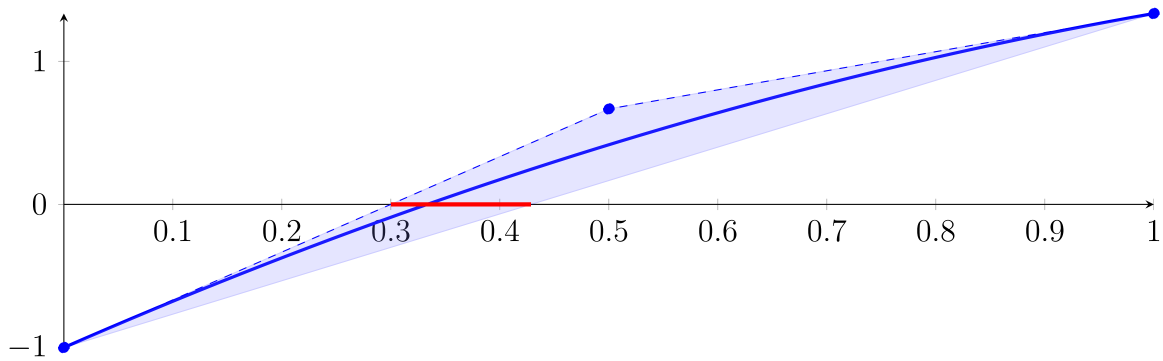
$$p = -1X^2 + 3.33333X - 1$$



184.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

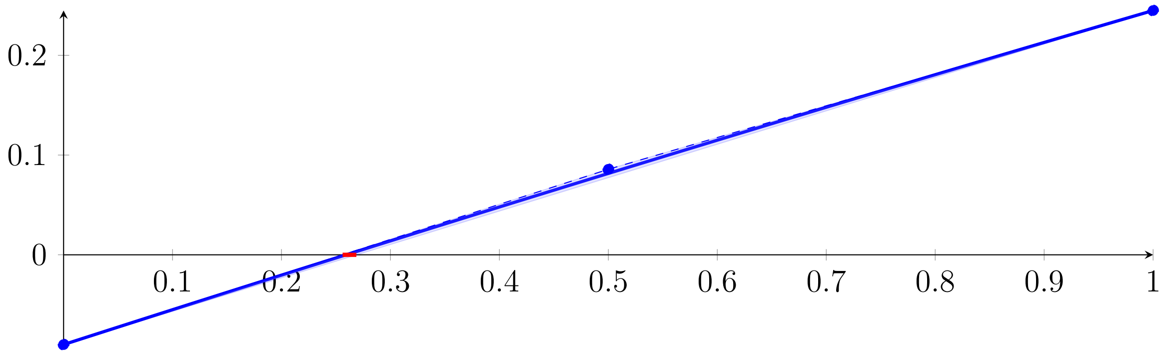
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

184.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

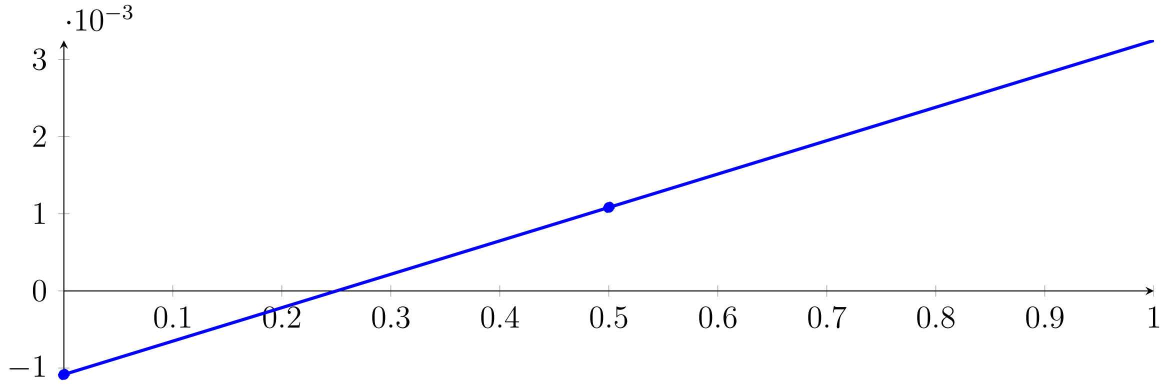
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

184.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

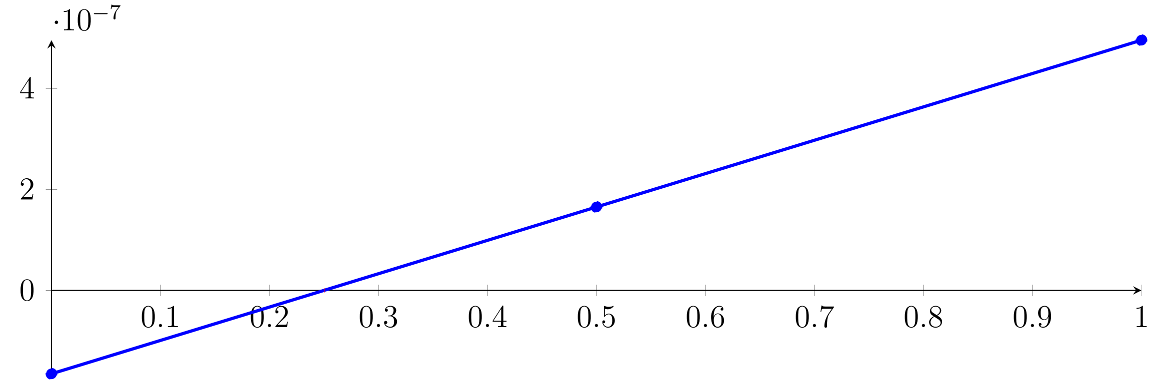
$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

184.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -6.14013\cdot10^{-14}X^2 + 6.60781\cdot10^{-07}X - 1.65195\cdot10^{-07}$$
$$= -1.65195\cdot10^{-07}B_{0,2}(X) + 1.65195\cdot10^{-07}B_{1,2}(X) + 4.95585\cdot10^{-07}B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

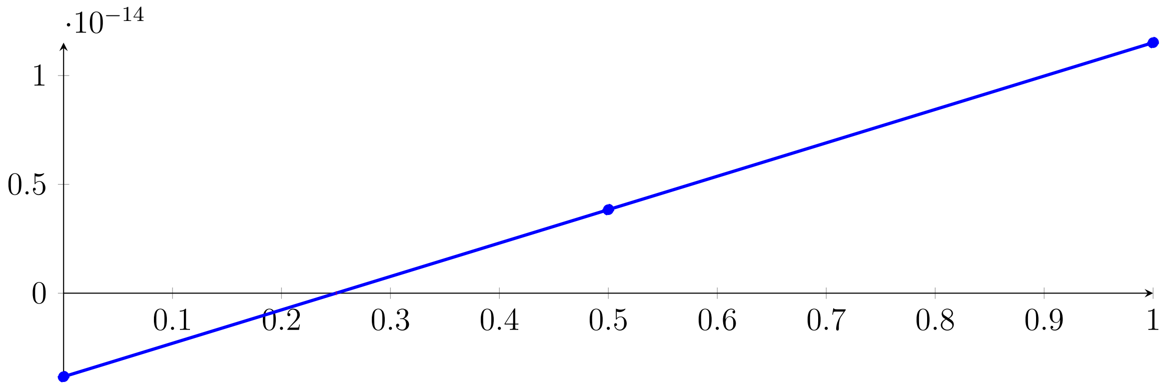
$$[0.25, 0.25]$$

Longest intersection interval: $2.32306\cdot10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

184.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.31358\cdot10^{-29}X^2 + 1.53503\cdot10^{-14}X - 3.83758\cdot10^{-15}$$
$$= -3.83758\cdot10^{-15}B_{0,2}(X) + 3.83758\cdot10^{-15}B_{1,2}(X) + 1.15127\cdot10^{-14}B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

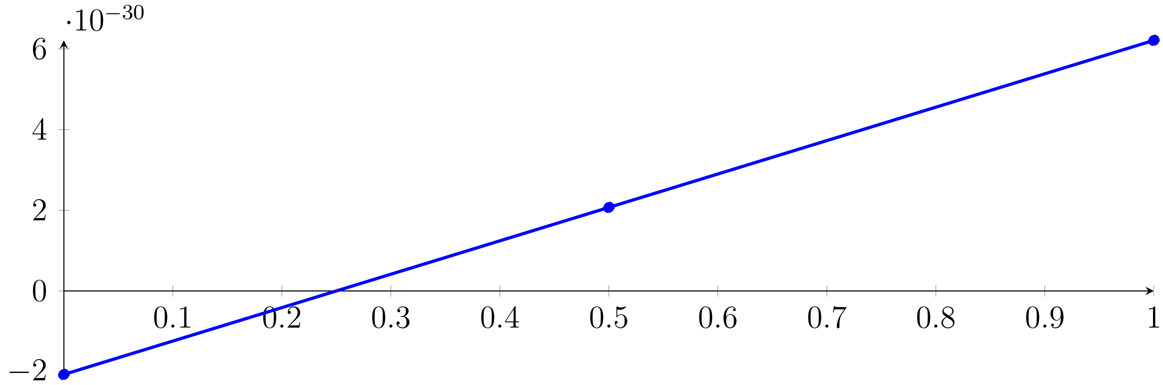
$$[0.25, 0.25]$$

Longest intersection interval: $5.3966\cdot10^{-16}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

184.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -9.65021 \cdot 10^{-60} X^2 + 8.28394 \cdot 10^{-30} X - 2.07099 \cdot 10^{-30}$$
$$= -2.07099 \cdot 10^{-30} B_{0,2}(X) + 2.07099 \cdot 10^{-30} B_{1,2}(X) + 6.21296 \cdot 10^{-30} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

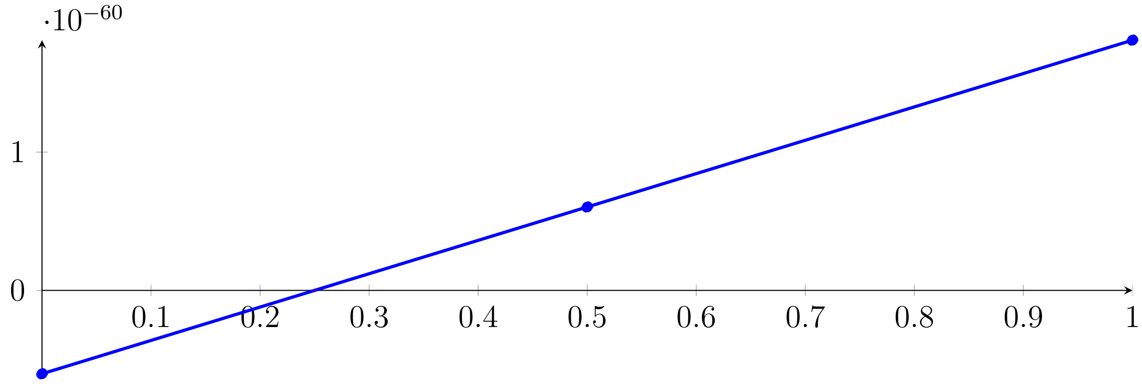
$$[0.25, 0.25]$$

Longest intersection interval: $2.91232 \cdot 10^{-31}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

184.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -8.18495 \cdot 10^{-121} X^2 + 2.41255 \cdot 10^{-60} X - 6.03138 \cdot 10^{-61}$$
$$= -6.03138 \cdot 10^{-61} B_{0,2}(X) + 6.03138 \cdot 10^{-61} B_{1,2}(X) + 1.80941 \cdot 10^{-60} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $8.48163 \cdot 10^{-62}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

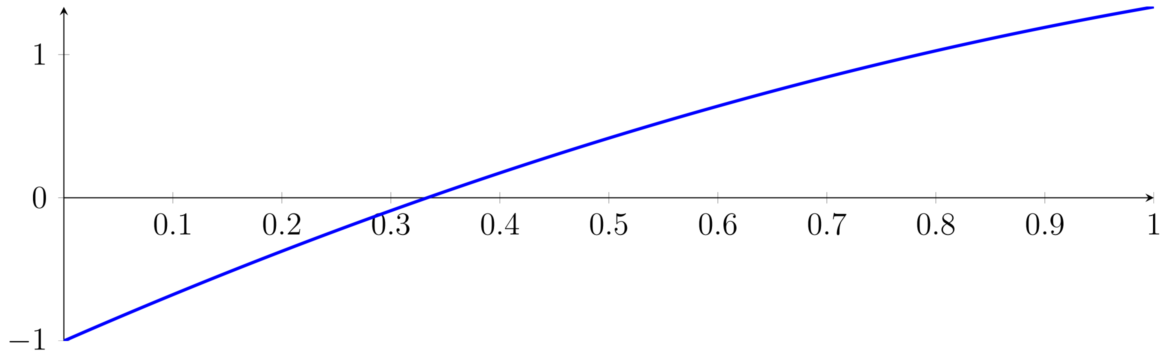
184.8 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 8!

184.9 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

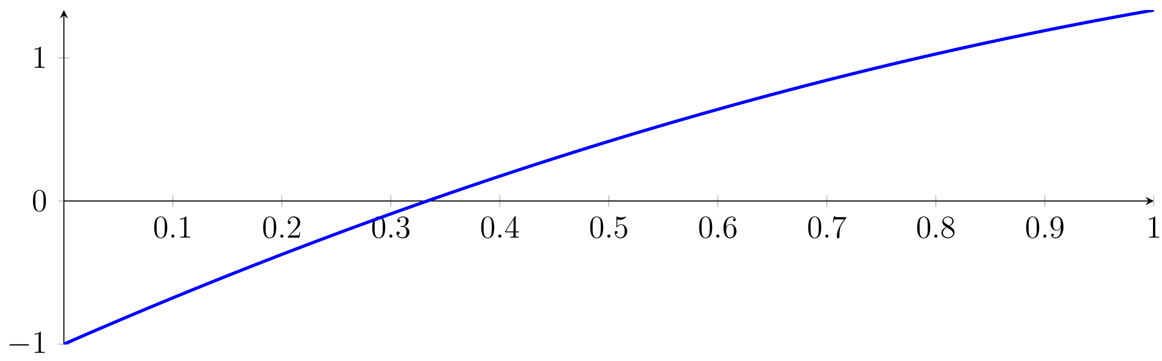
with precision $\varepsilon = 1 \cdot 10^{-64}$.

185 Running `QuadClip` on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called `QuadClip` with input polynomial on interval $[0, 1]$:

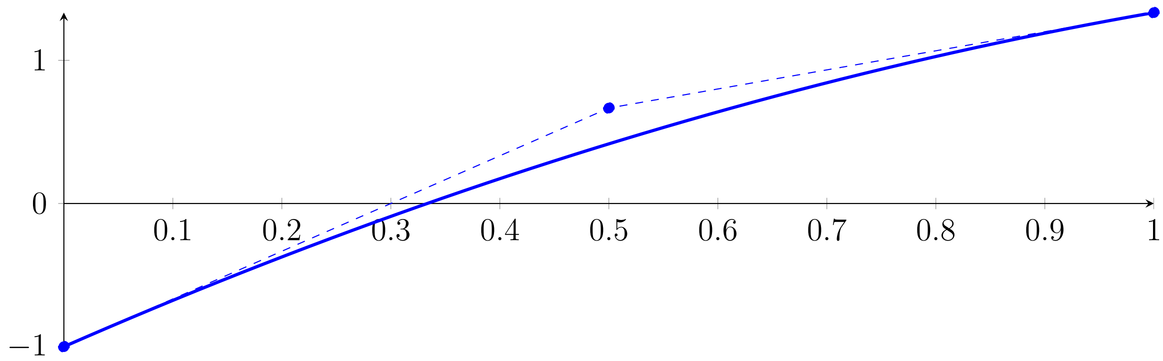
$$p = -1X^2 + 3.33333X - 1$$



185.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

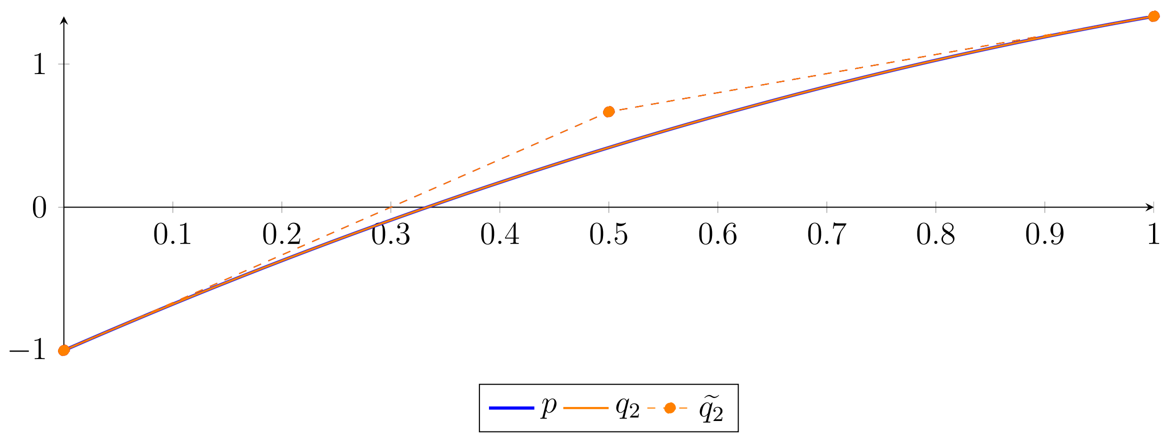
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



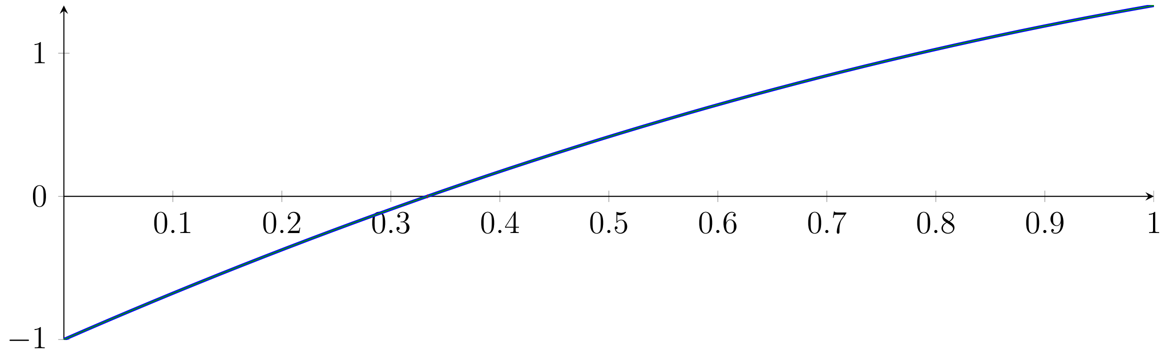
The maximum difference of the Bézier coefficients is $\delta = 2.22507 \cdot 10^{-308}$.
Bounding polynomials M and m:

$$M = -1X^2 + 3.33333X - 1$$
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $1.11254 \cdot 10^{-308}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

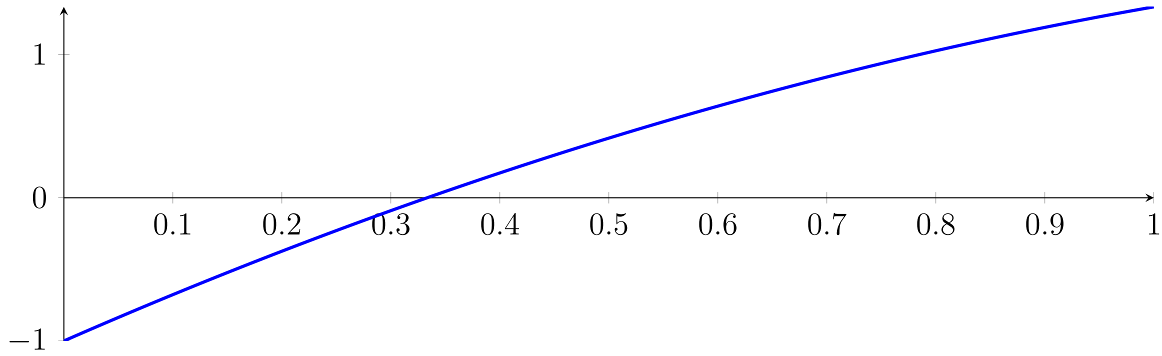
185.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

185.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

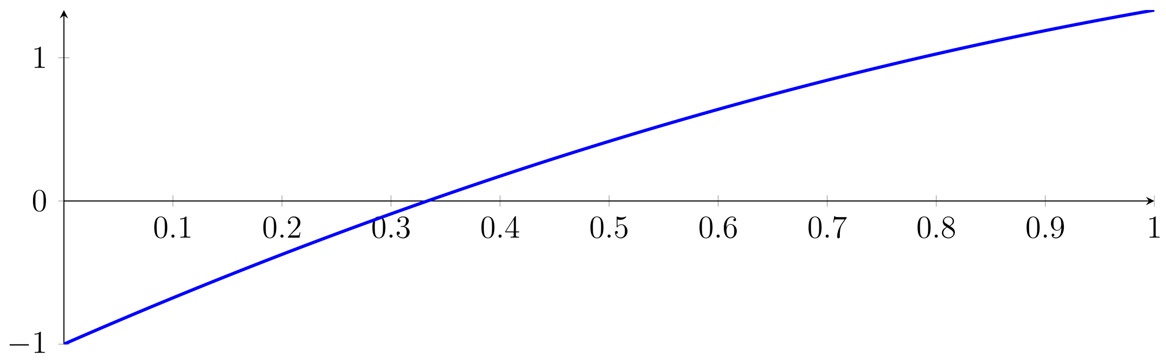
with precision $\varepsilon = 1 \cdot 10^{-64}$.

186 Running `CubeClip` on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called `CubeClip` with input polynomial on interval $[0, 1]$:

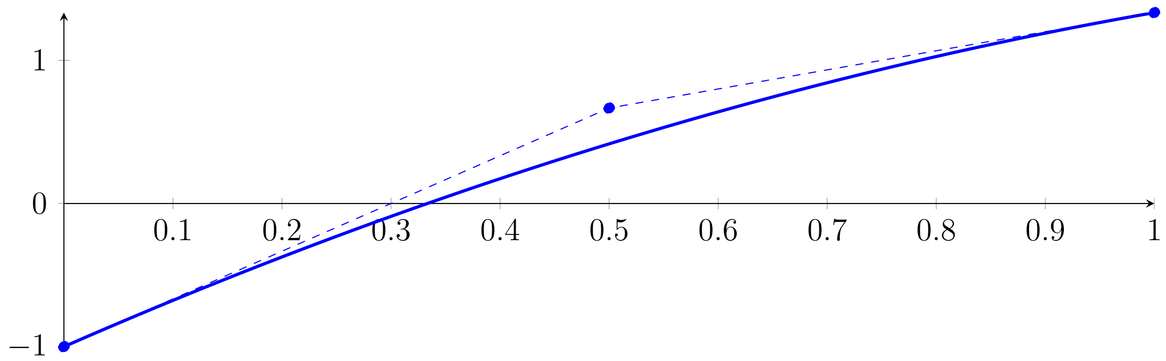
$$p = -1X^2 + 3.33333X - 1$$



186.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

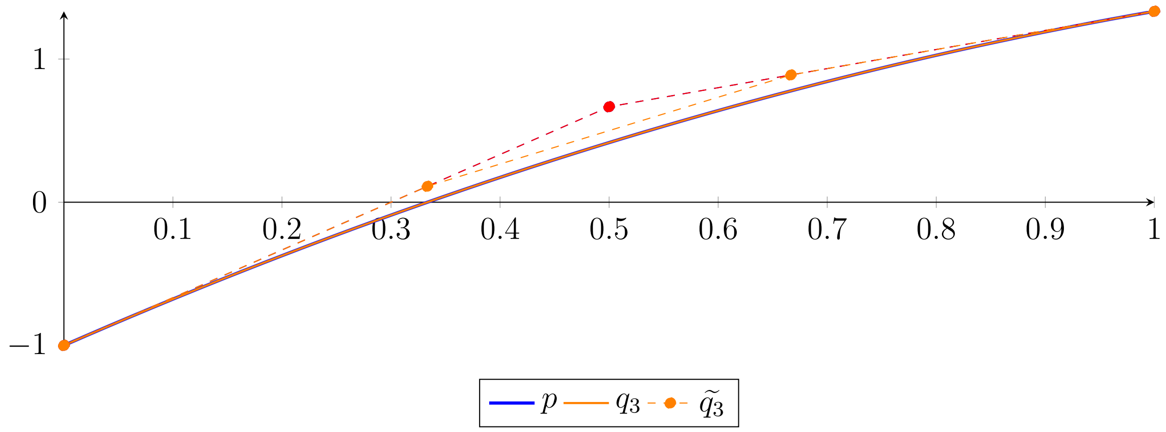
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.66881 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.00128 \cdot 10^{-307}$.

Bounding polynomials M and m:

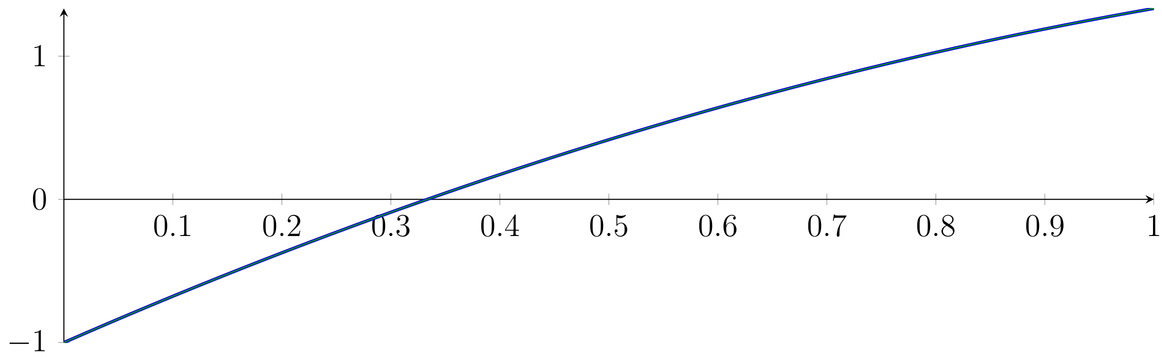
$$M = -2.78134 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1$$

$$m = -2.78134 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$N(M) = \{-3.59539 \cdot 10^{307}, -3.38949 \cdot 10^{291}, 1.03213 \cdot 10^{153}\}$ $N(m) = \{-3.59539 \cdot 10^{307}, -3.38949 \cdot 10^{291}, 1.75\}$

Intersection intervals:

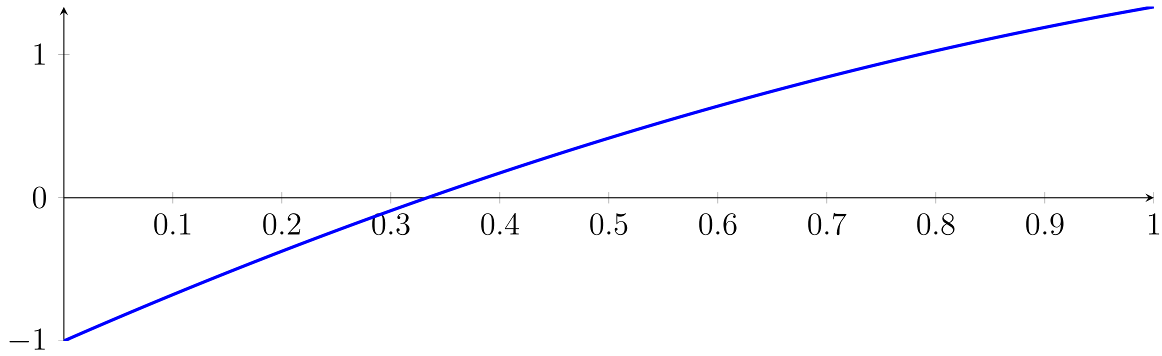


No intersection intervals with the x axis.

186.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

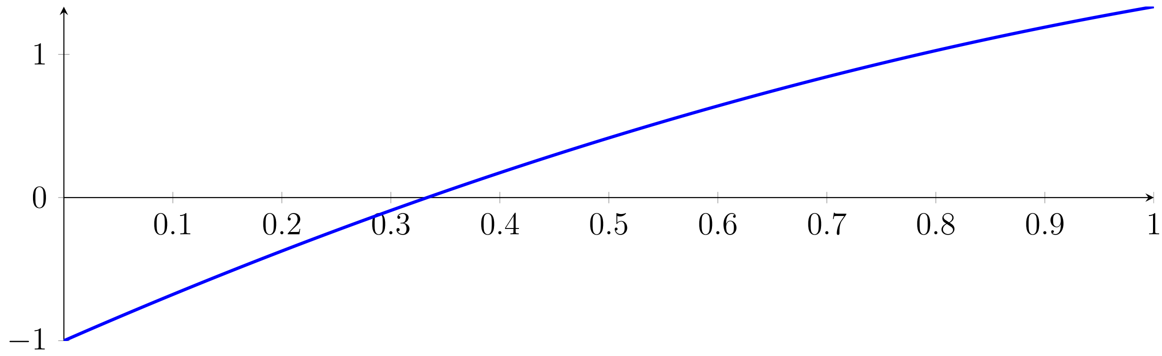
with precision $\varepsilon = 1 \cdot 10^{-64}$.

187 Running BezClip on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

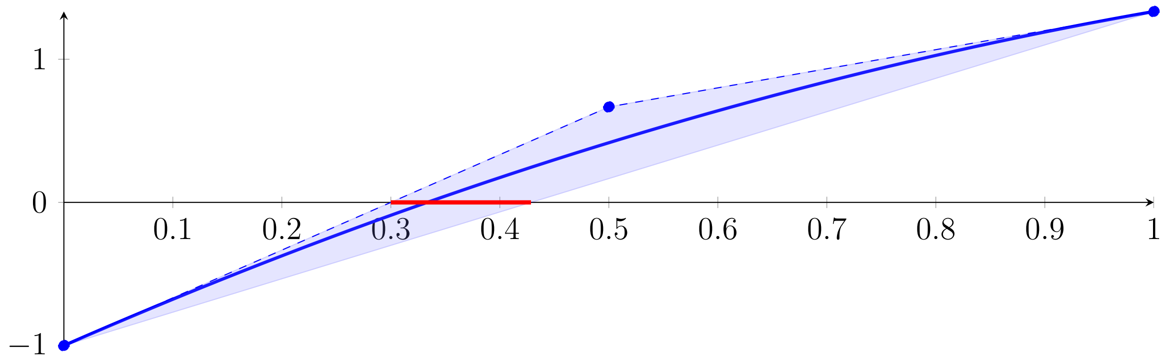
$$p = -1X^2 + 3.33333X - 1$$



187.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

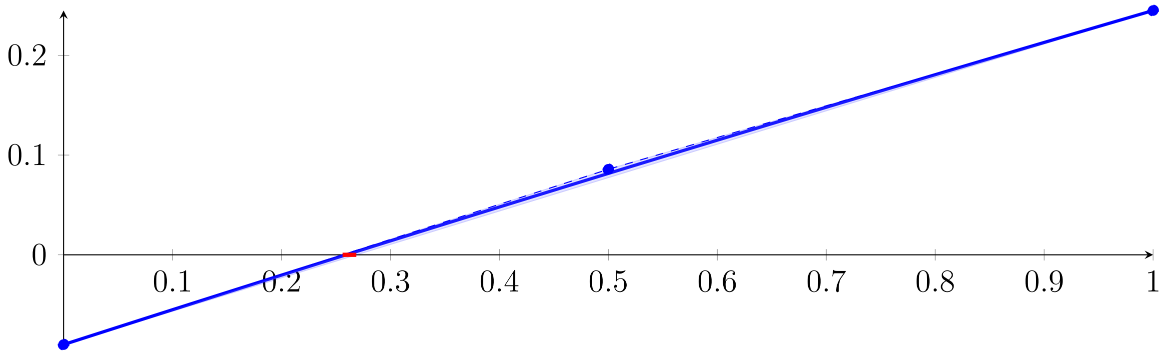
$$[0.3, 0.428571]$$

Longest intersection interval: 0.128571
$\Longrightarrow$ Selective recursion: interval 1: $[0.3, 0.428571]$,

187.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

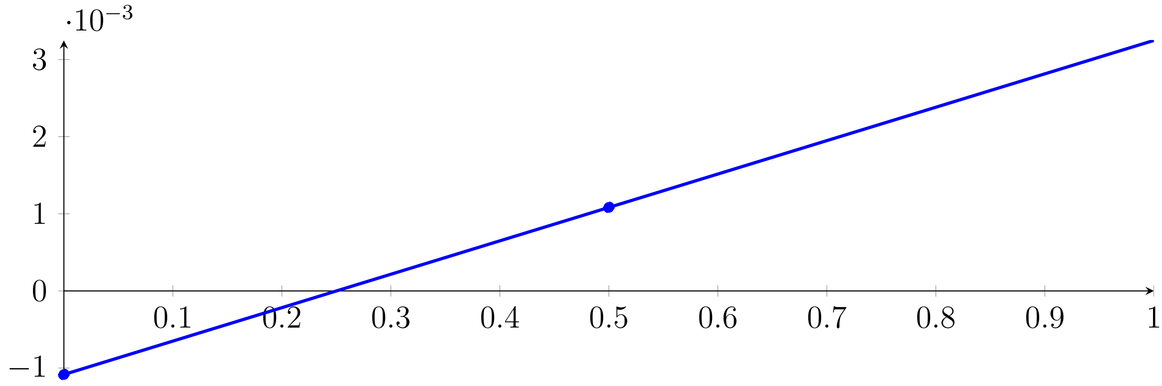
$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641
$\Longrightarrow$ Selective recursion: interval 1: $[0.332927, 0.334552]$,

187.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

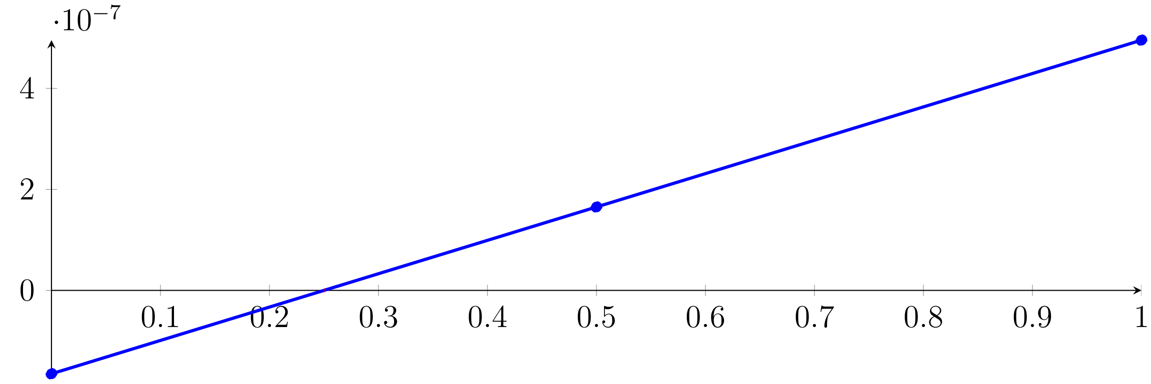
$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333334]$,

187.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07}$$
$$= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

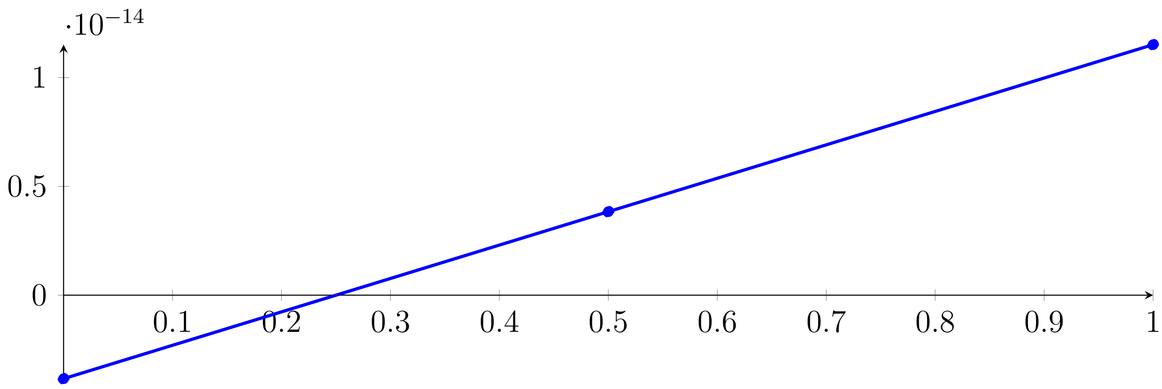
Longest intersection interval: $2.32306 \cdot 10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

187.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.31358 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15}$$
$$= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

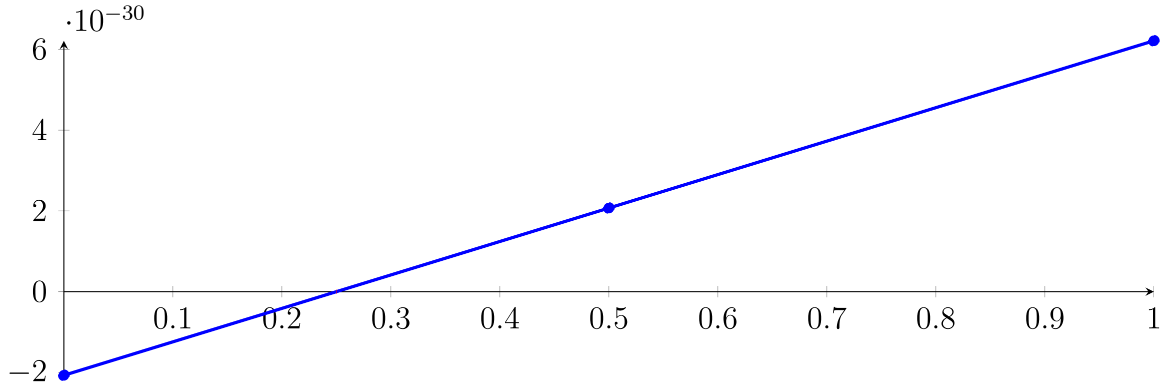
Longest intersection interval: $5.3966 \cdot 10^{-16}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

187.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -9.65021 \cdot 10^{-60} X^2 + 8.28394 \cdot 10^{-30} X - 2.07099 \cdot 10^{-30}$$
$$= -2.07099 \cdot 10^{-30} B_{0,2}(X) + 2.07099 \cdot 10^{-30} B_{1,2}(X) + 6.21296 \cdot 10^{-30} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

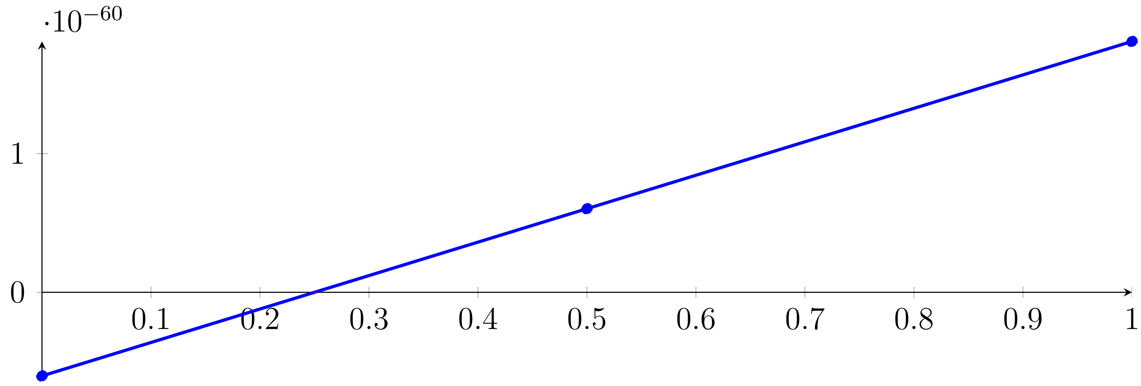
$$[0.25, 0.25]$$

Longest intersection interval: $2.91232 \cdot 10^{-31}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

187.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -8.18495 \cdot 10^{-121} X^2 + 2.41255 \cdot 10^{-60} X - 6.03138 \cdot 10^{-61}$$
$$= -6.03138 \cdot 10^{-61} B_{0,2}(X) + 6.03138 \cdot 10^{-61} B_{1,2}(X) + 1.80941 \cdot 10^{-60} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

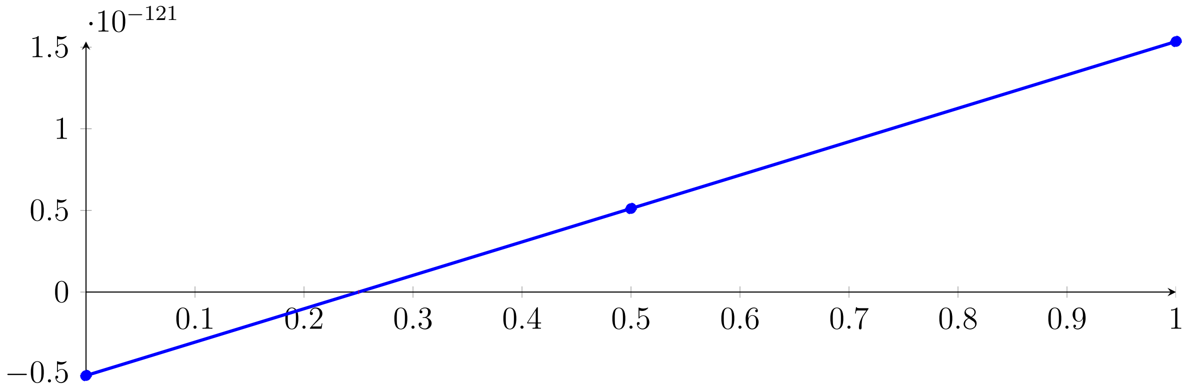
$$[0.25, 0.25]$$

Longest intersection interval: $8.48163 \cdot 10^{-62}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

187.8 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -5.8881 \cdot 10^{-243} X^2 + 2.04624 \cdot 10^{-121} X - 5.1156 \cdot 10^{-122}$$
$$= -5.1156 \cdot 10^{-122} B_{0,2}(X) + 5.1156 \cdot 10^{-122} B_{1,2}(X) + 1.53468 \cdot 10^{-121} B_{2,2}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $7.19381 \cdot 10^{-123}$
$\implies$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

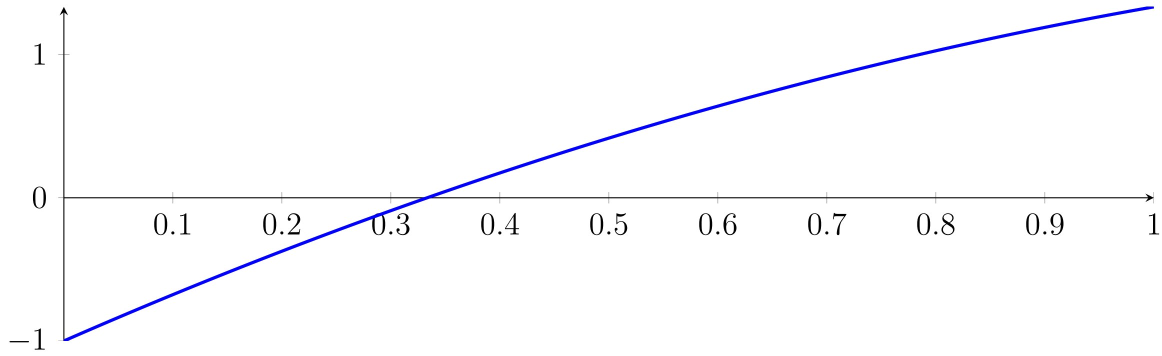
187.9 Recursion Branch 1 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 9!

187.10 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

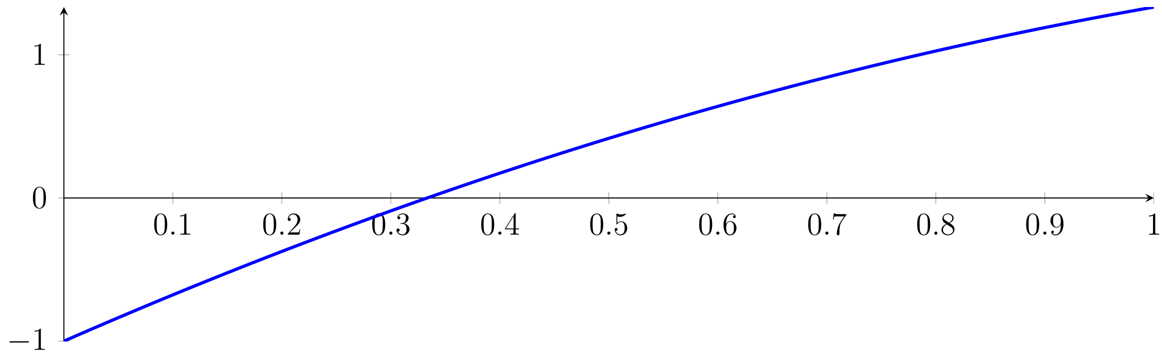
with precision $\varepsilon = 1 \cdot 10^{-128}$.

188 Running QuadClip on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

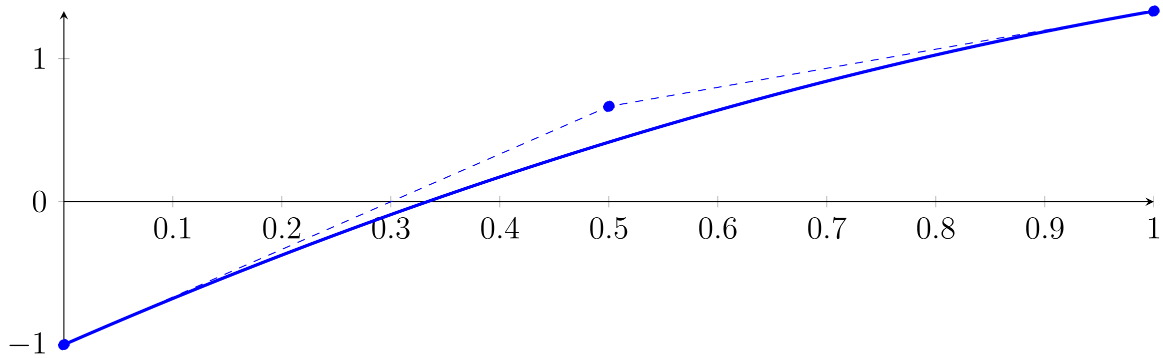
$$p = -1X^2 + 3.33333X - 1$$



188.1 Recursion Branch 1 for Input Interval $[0, 1]$

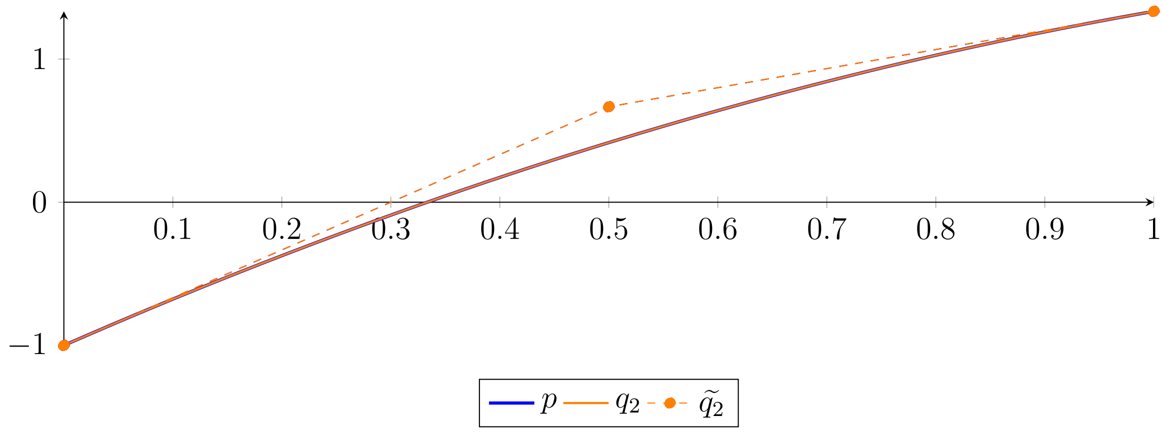
Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1X^2 + 3.33333X - 1$$
$$= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X)$$



Degree reduction and raising:

$$q_2 = -1X^2 + 3.33333X - 1$$
$$= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2}$$
$$\tilde{q}_2 = -1X^2 + 3.33333X - 1$$
$$= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2}$$



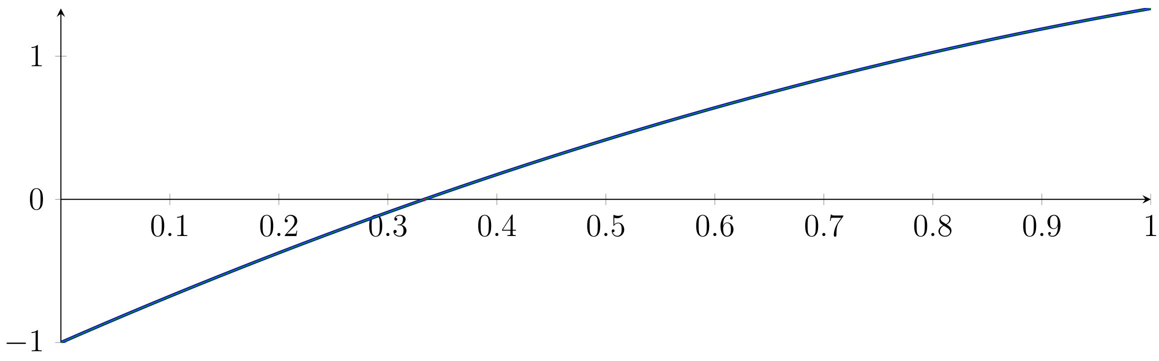
The maximum difference of the Bézier coefficients is $\delta = 2.22507 \cdot 10^{-308}$.
Bounding polynomials M and m:

$$M = -1X^2 + 3.33333X - 1$$
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m:

$$N(M) = \{0.333333, 3\} \qquad N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $1.11254 \cdot 10^{-308}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

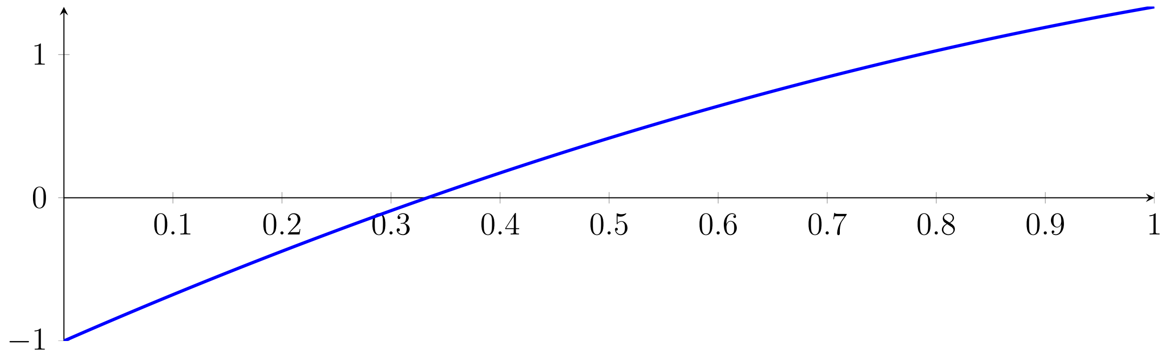
188.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

188.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

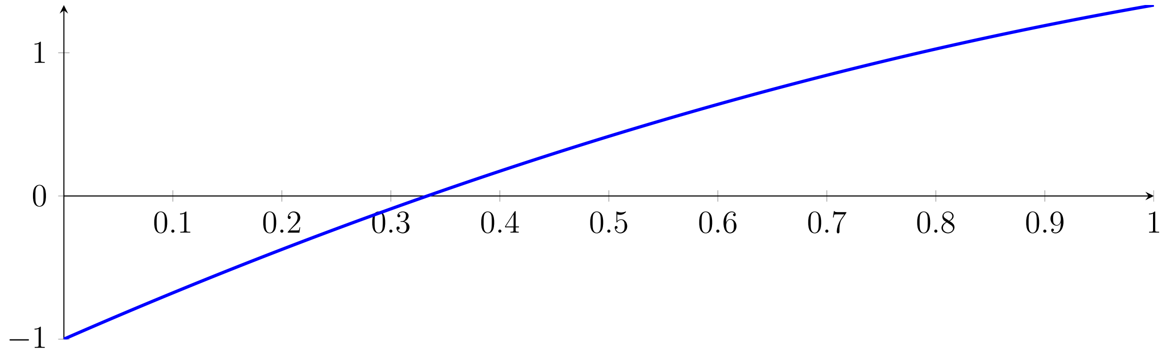
with precision $\varepsilon = 1 \cdot 10^{-128}$.

189 Running CubeClip on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

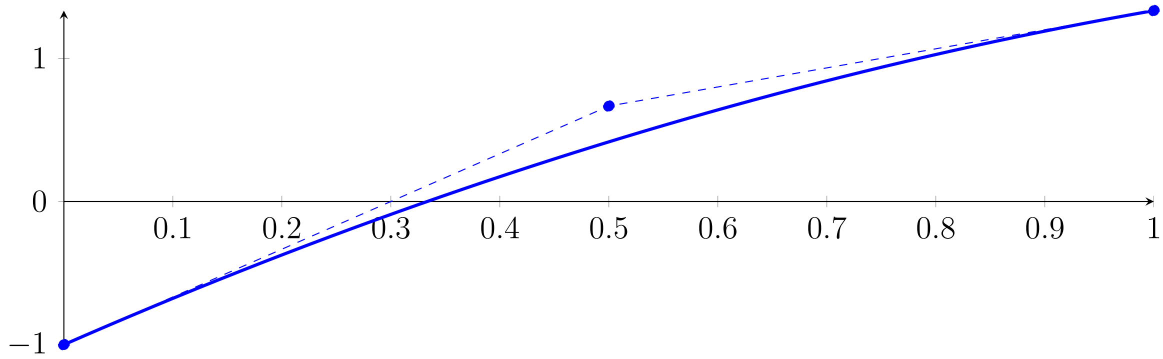
$$p = -1X^2 + 3.33333X - 1$$



189.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

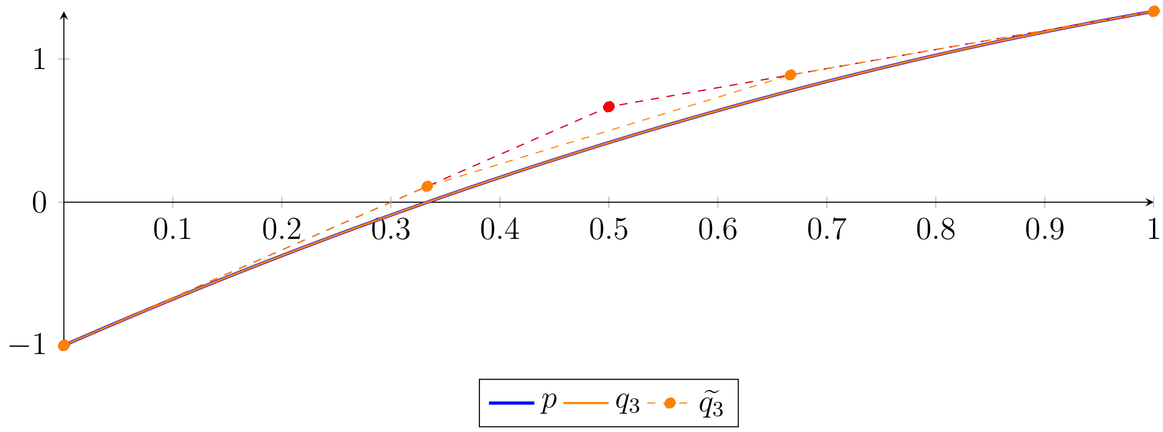
$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.66881 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.00128{\cdot}10^{-307}$.
Bounding polynomials M and m:

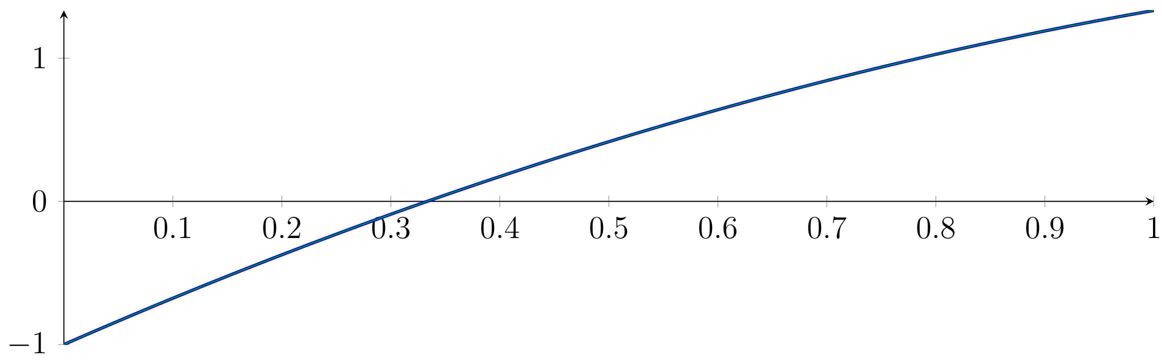
$$M = -2.78134{\cdot}10^{-308}X^3 - 1X^2 + 3.33333X - 1$$

$$m = -2.78134{\cdot}10^{-308}X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m:

$N(M) = \{-3.59539{\cdot}10^{307}, -3.38949{\cdot}10^{291}, 1.03213{\cdot}10^{153}\}$ $N(m) = \{-3.59539{\cdot}10^{307}, -3.38949{\cdot}10^{291}, 1.75\}$

Intersection intervals:

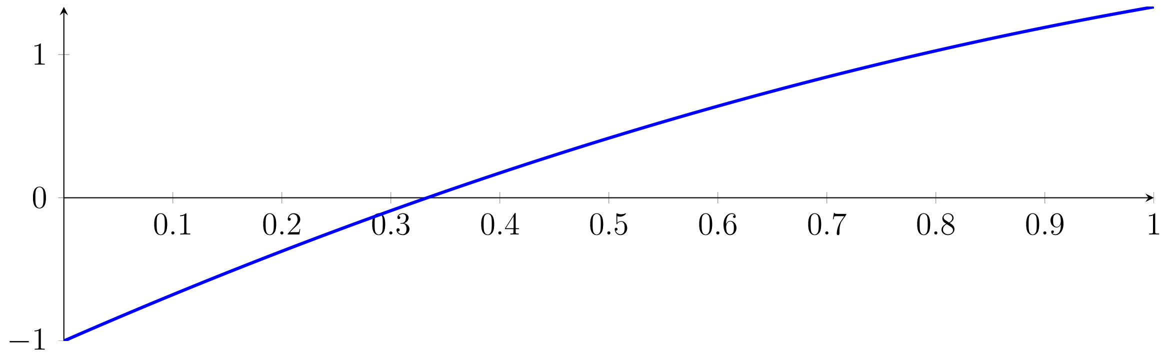


No intersection intervals with the x axis.

189.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

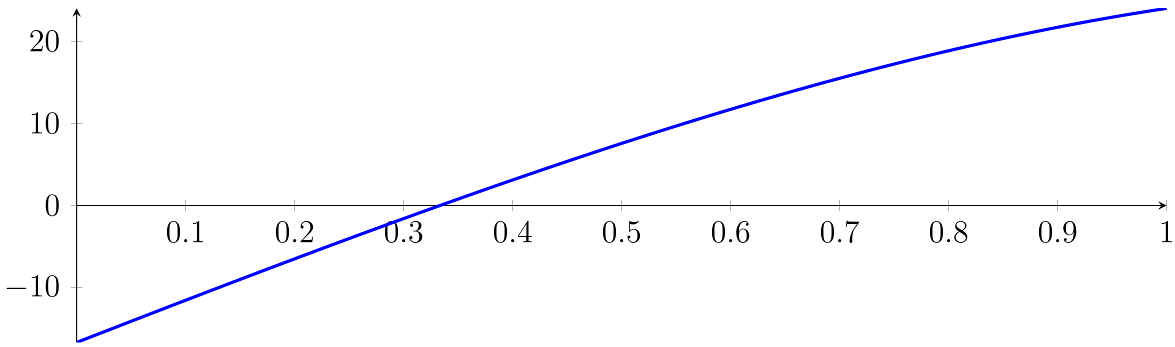
with precision $\varepsilon = 1 \cdot 10^{-128}$.

190 Running BezClip on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

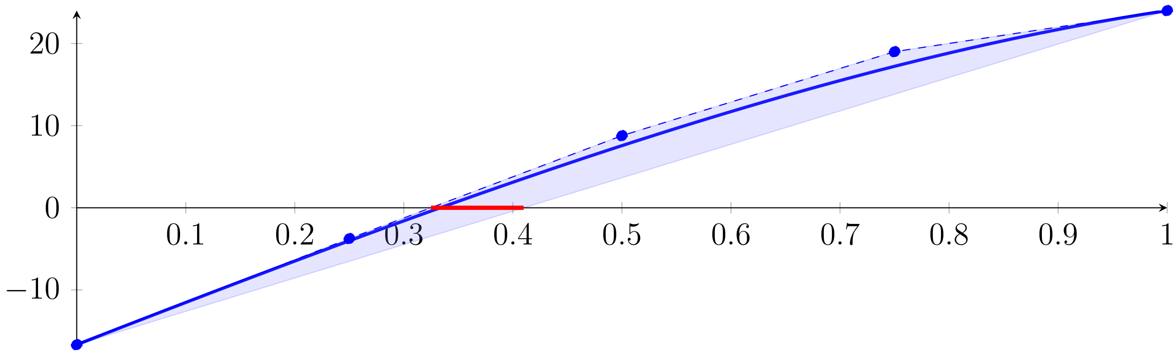
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



190.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

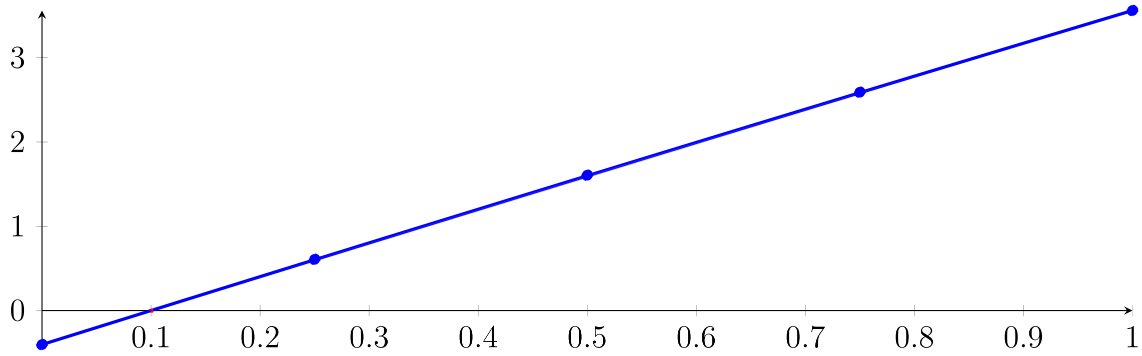
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

190.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

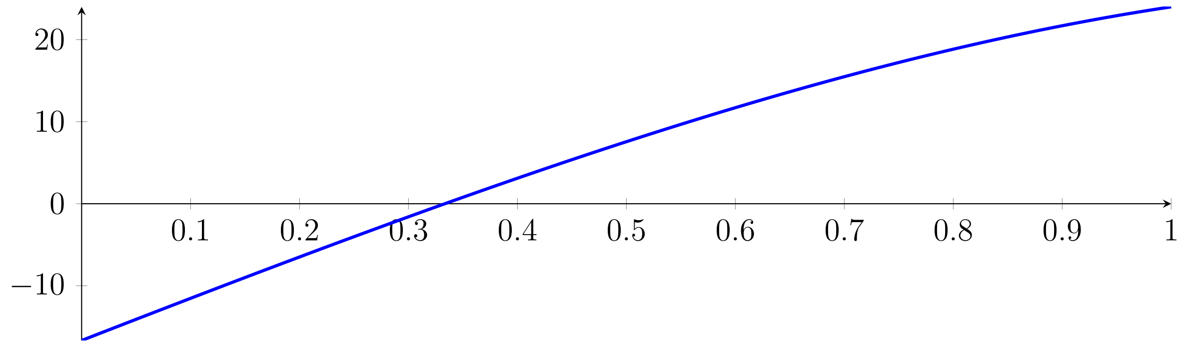
190.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Found root in interval $[0.333317, 0.333491]$ at recursion depth 3!

190.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333317, 0.333491]$$

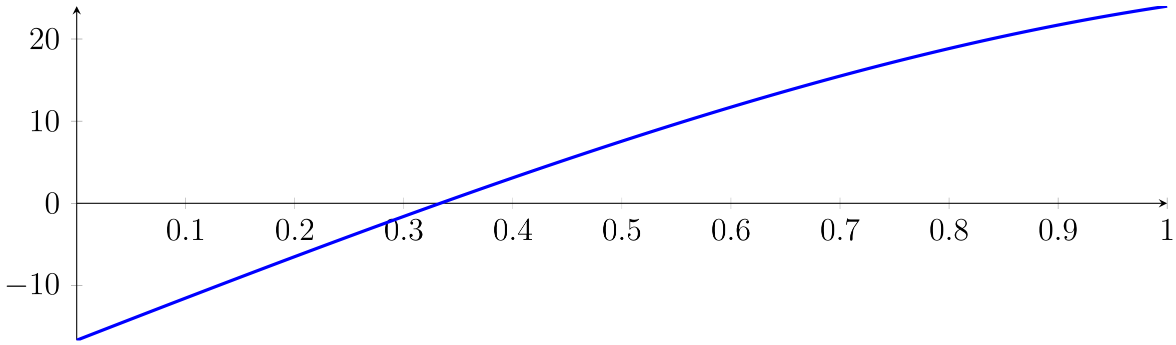
with precision $\varepsilon = 0.01$.

191 Running QuadClip on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

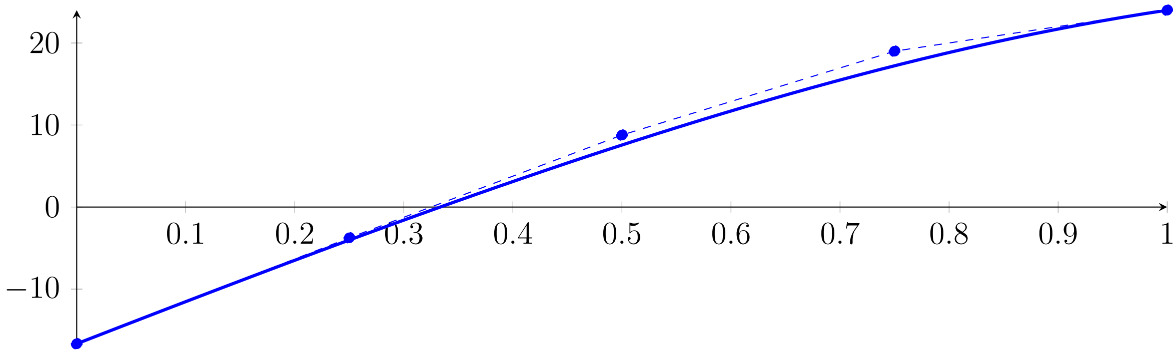
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



191.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

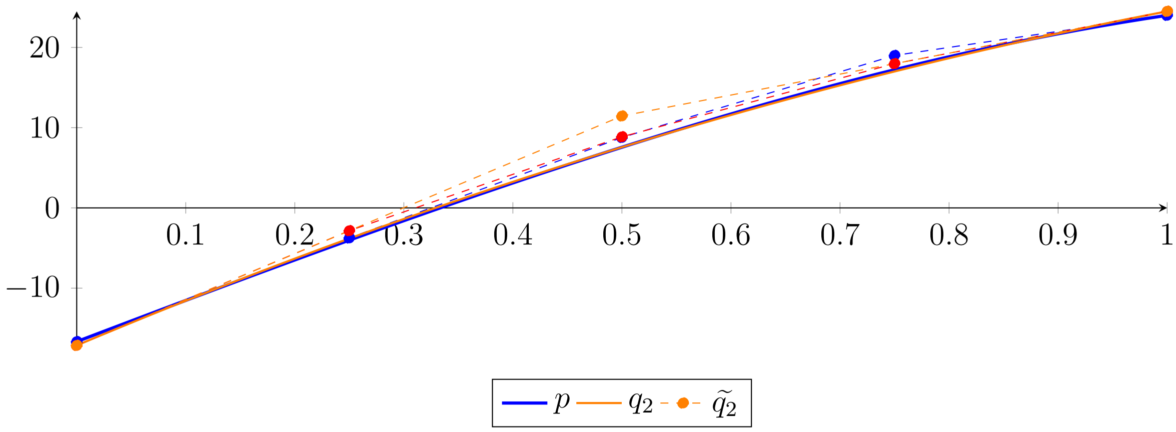
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -2.13607 \cdot 10^{-306}X^4 + 7.12024 \cdot 10^{-306}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

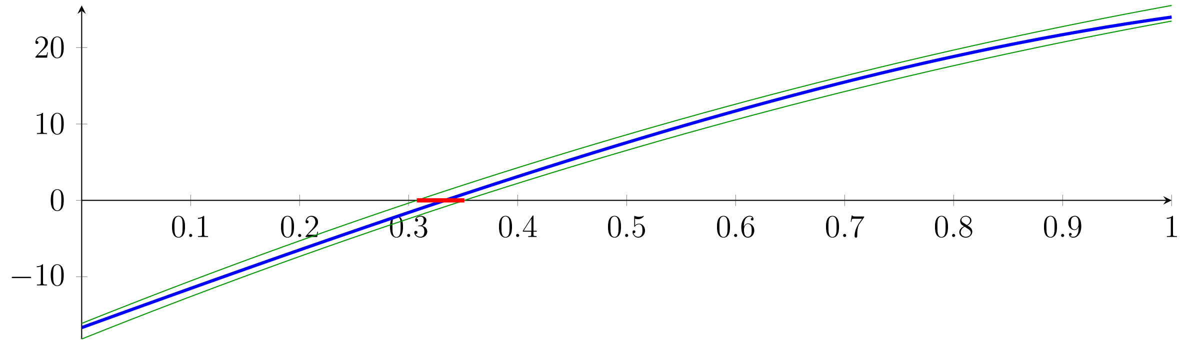
$$M = -15.5476X^2 + 57.181X - 16.1119$$

$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



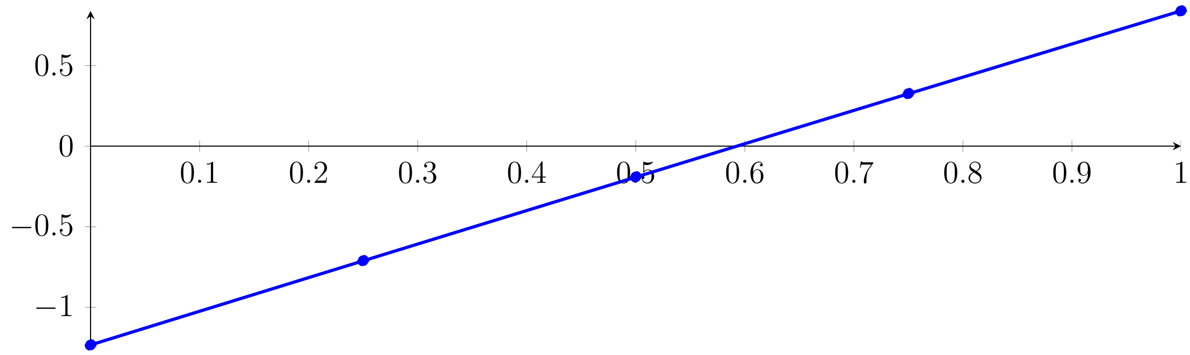
$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205
$\implies$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

191.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

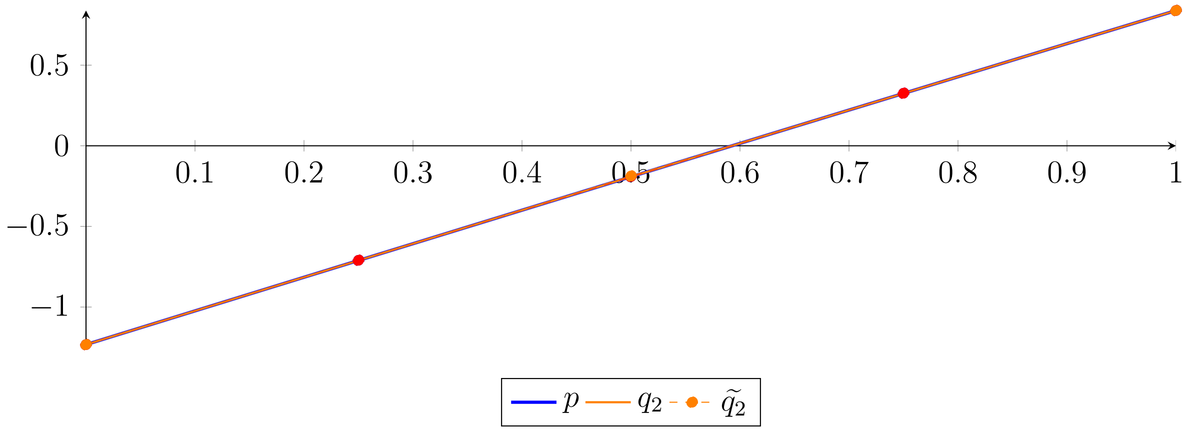
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -4.5614 \cdot 10^{-307} X^4 + 1.02353 \cdot 10^{-306} X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



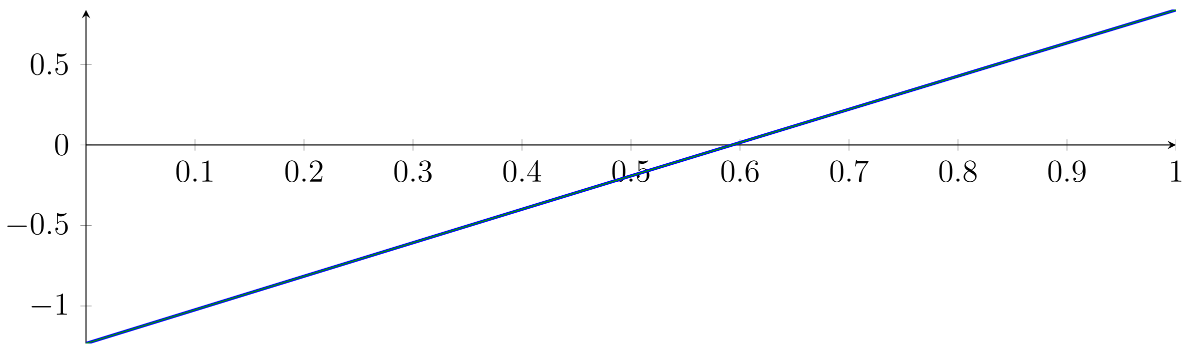
The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.
Bounding polynomials M and m:

$$M = -0.020089X^2 + 2.09166X - 1.23274$$
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$
$\implies$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

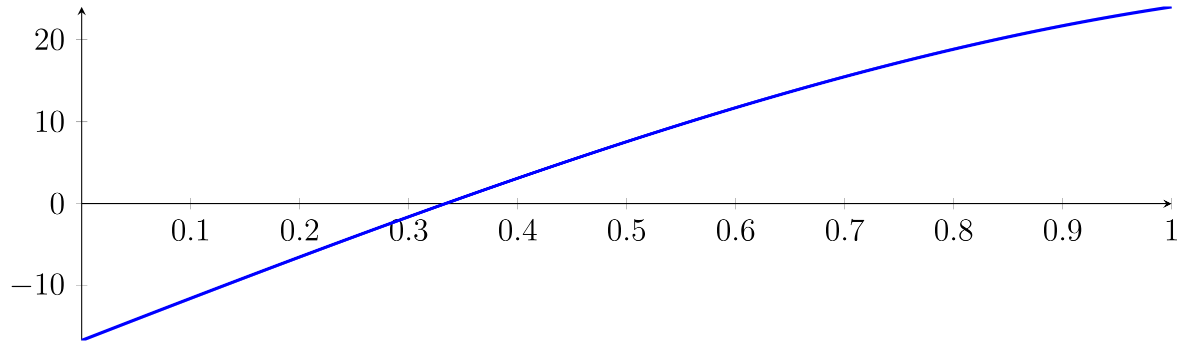
191.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Found root in interval $[0.333332, 0.333335]$ at recursion depth 3!

191.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333332, 0.333335]$$

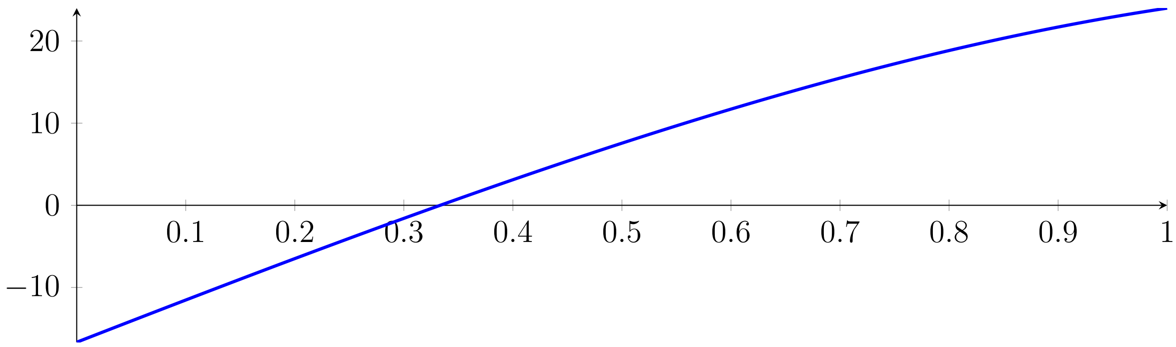
with precision $\varepsilon = 0.01$.

192 Running CubeClip on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

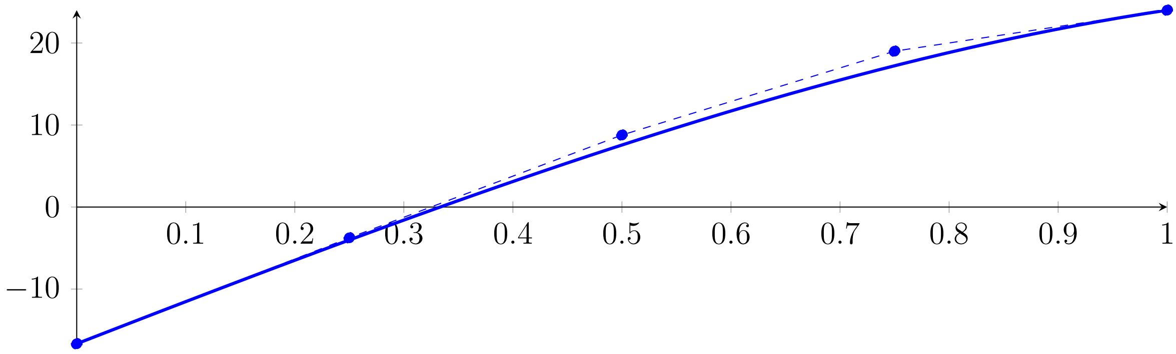
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



192.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

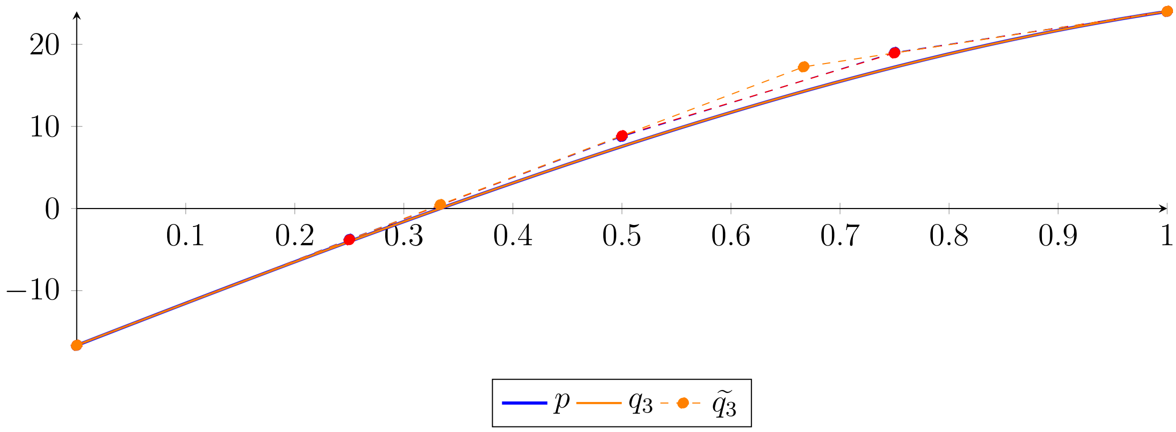
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$
$$= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X)$$



Degree reduction and raising:

$$q_3 = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524$$
$$= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3}$$

$$\tilde{q}_3 = -3.56012 \cdot 10^{-307}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524$$
$$= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

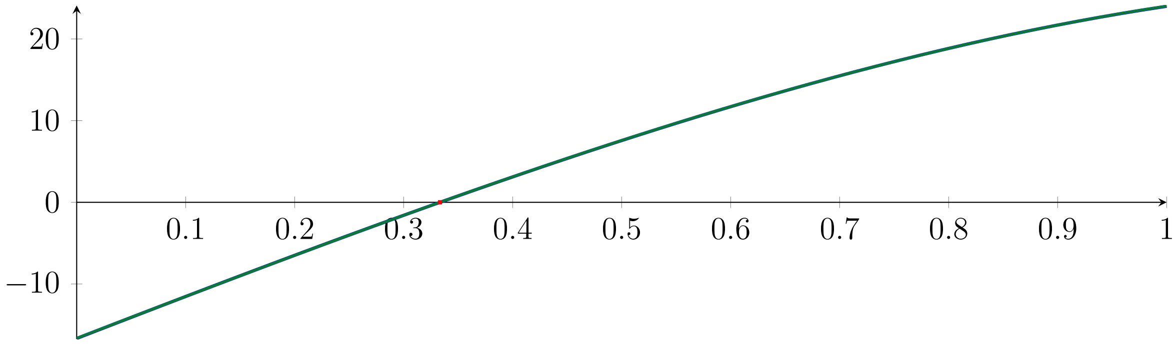
$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

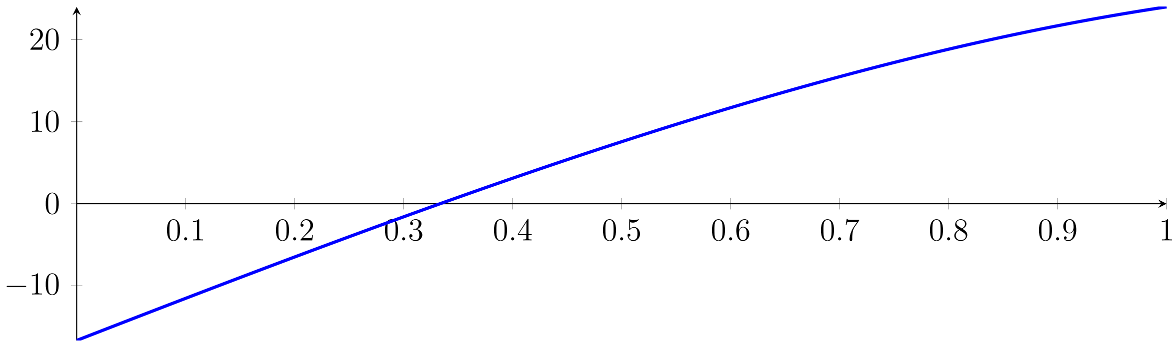
192.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Found root in interval $[0.331524, 0.335136]$ at recursion depth 2!

192.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.331524, 0.335136]$$

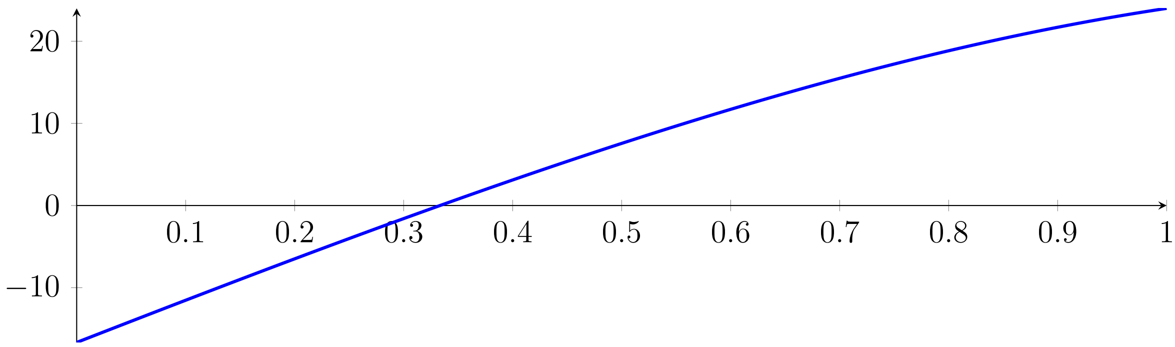
with precision $\varepsilon = 0.01$.

193 Running BezClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

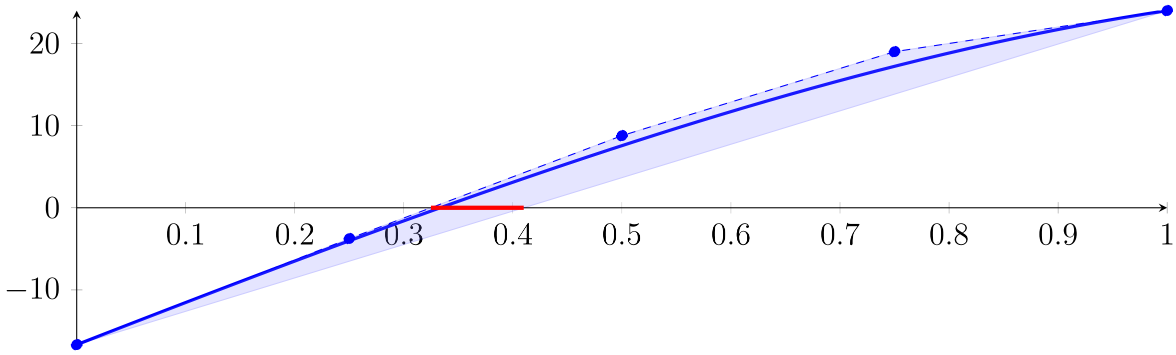
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



193.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$
$$= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

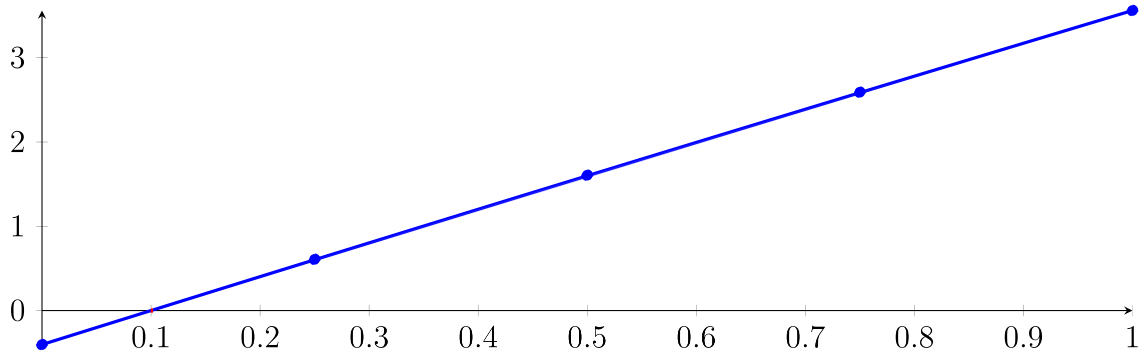
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

193.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711$$
$$= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

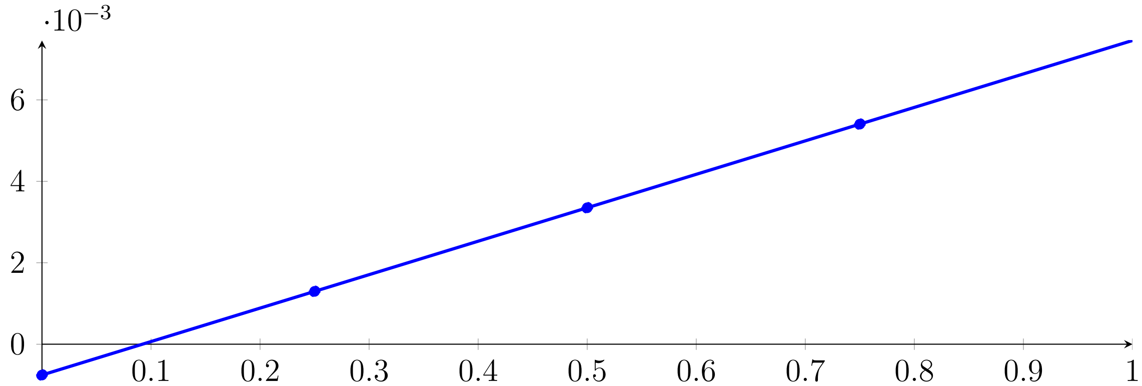
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

193.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -9.01975 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

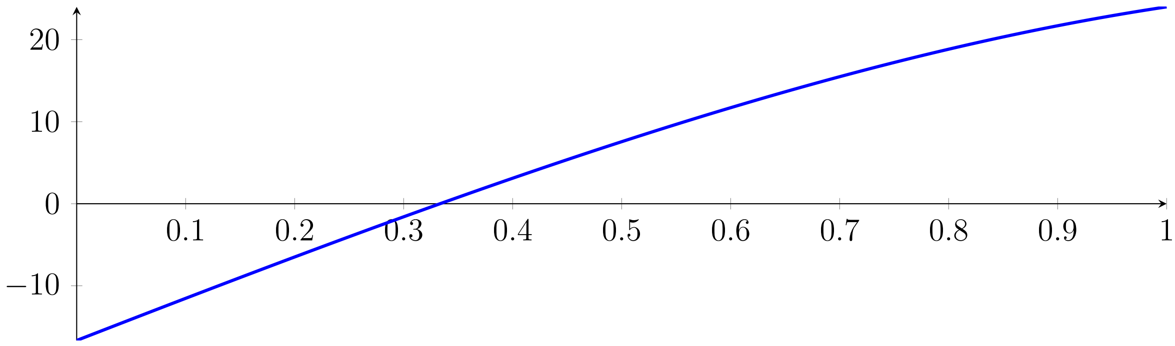
193.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

193.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

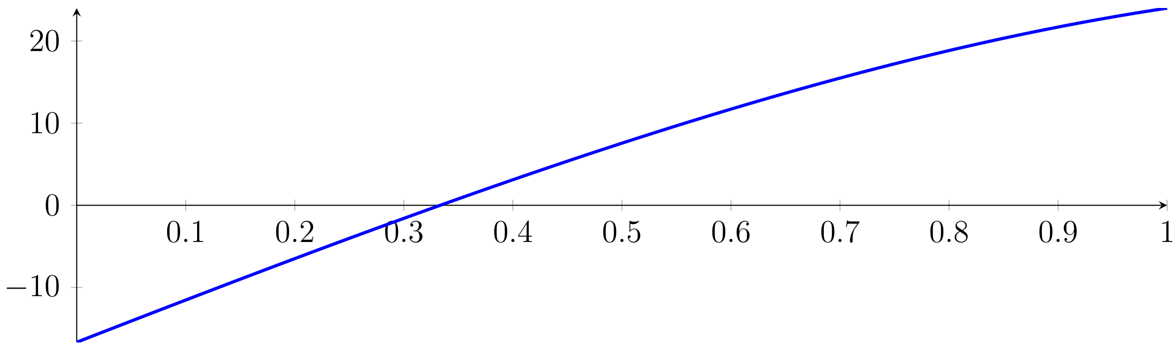
with precision $\varepsilon = 0.0001$.

194 Running QuadClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

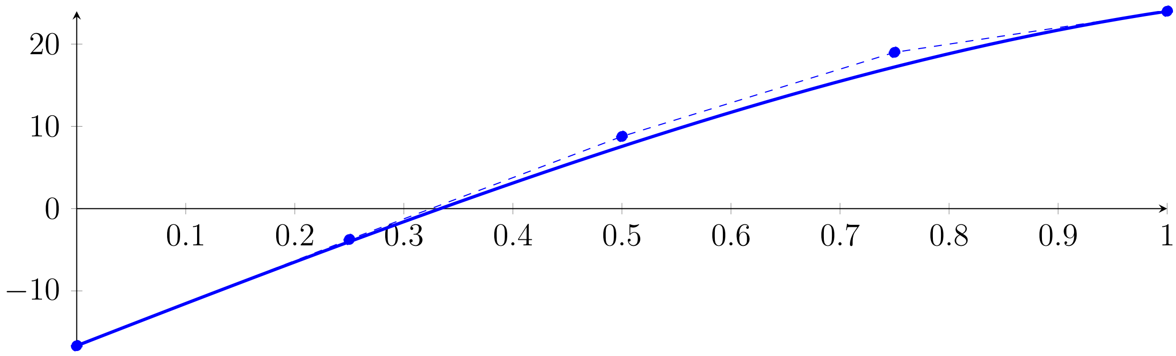
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



194.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

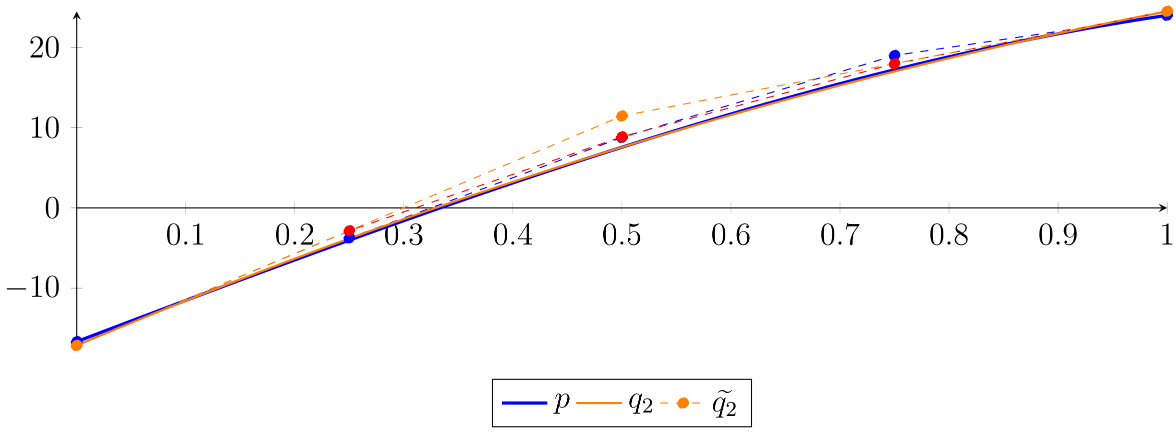
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -2.13607 \cdot 10^{-306}X^4 + 7.12024 \cdot 10^{-306}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

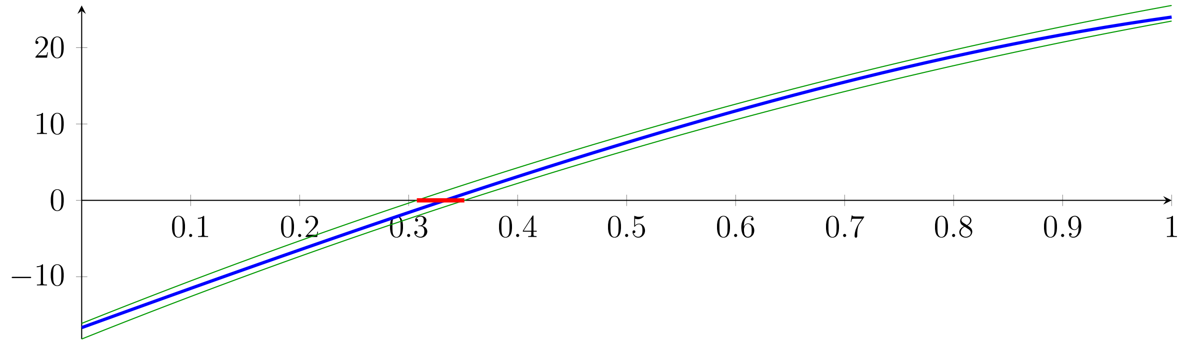
$$M = -15.5476X^2 + 57.181X - 16.1119$$

$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



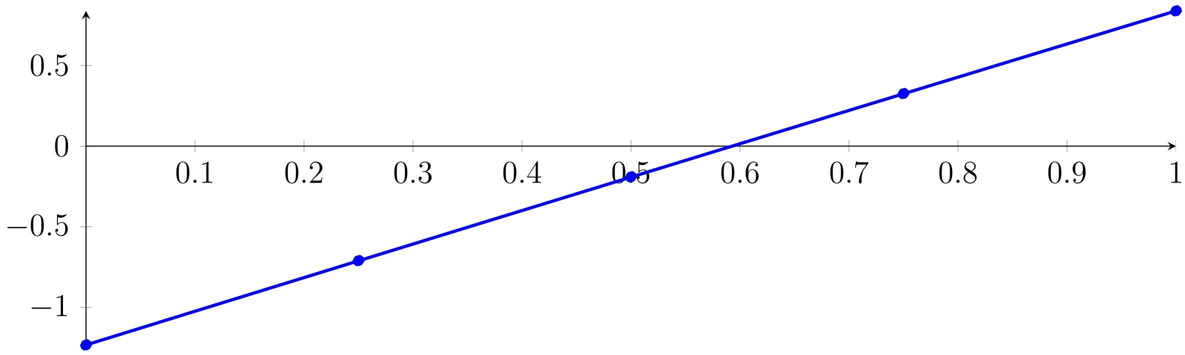
$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

194.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

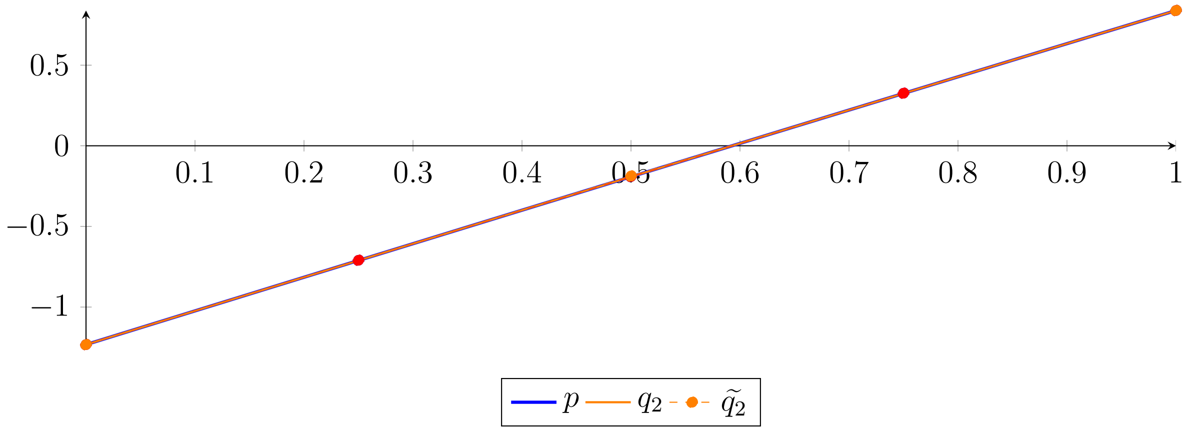
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -4.5614 \cdot 10^{-307} X^4 + 1.02353 \cdot 10^{-306} X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.
Bounding polynomials M and m:

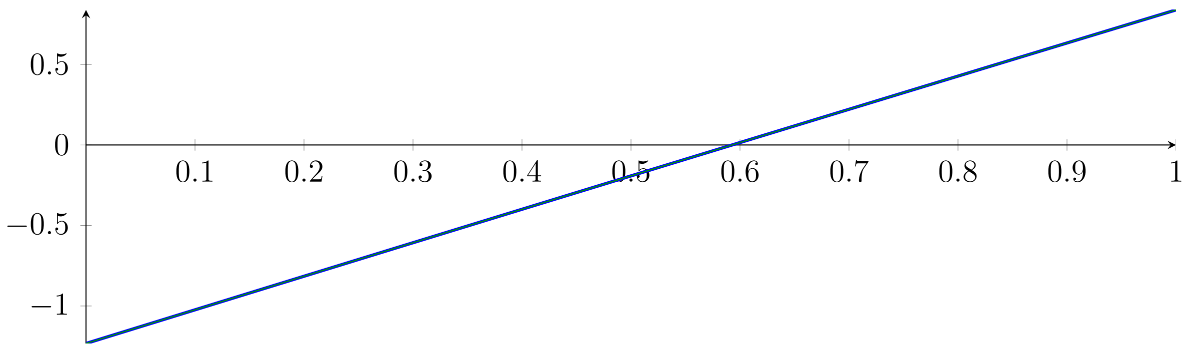
$$M = -0.020089X^2 + 2.09166X - 1.23274$$

$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

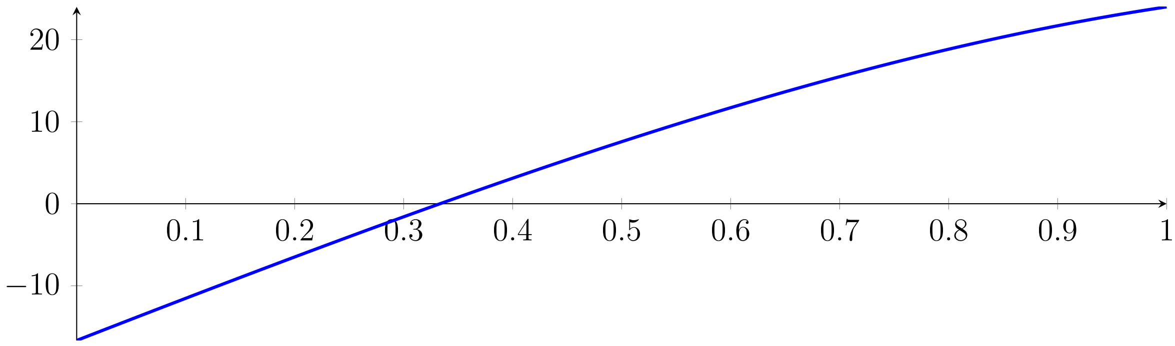
194.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Found root in interval $[0.333332, 0.333335]$ at recursion depth 3!

194.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333332, 0.333335]$$

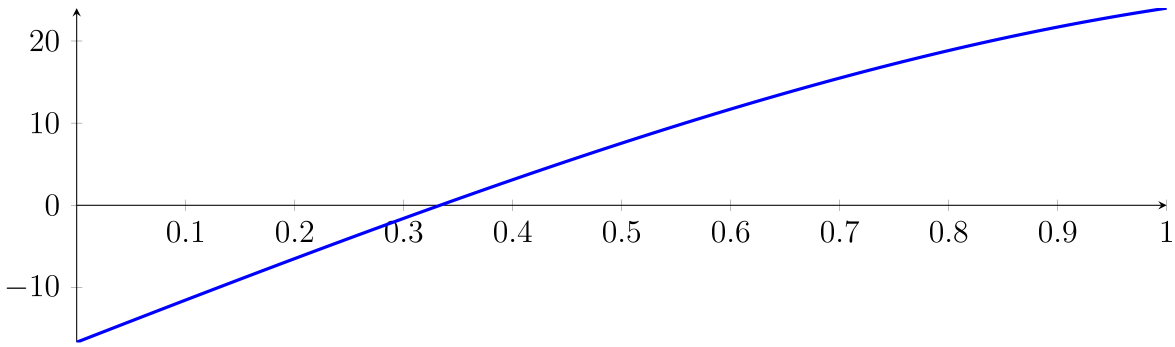
with precision $\varepsilon = 0.0001$.

195 Running CubeClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0,1]$:

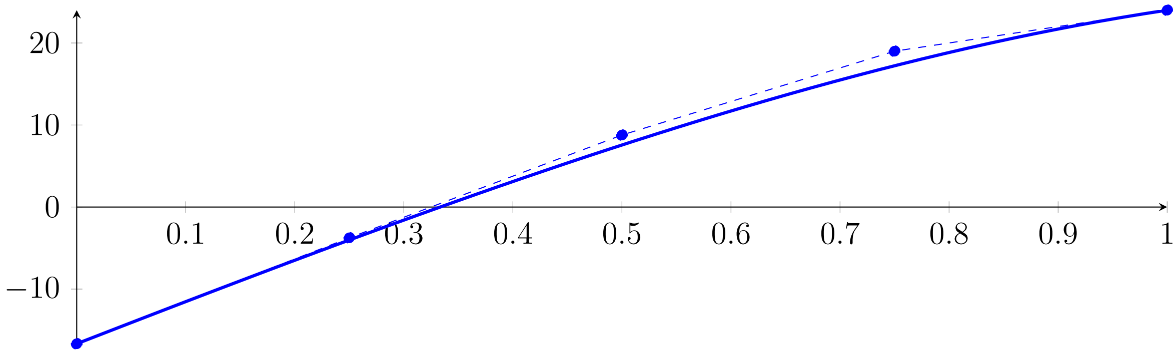
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



195.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

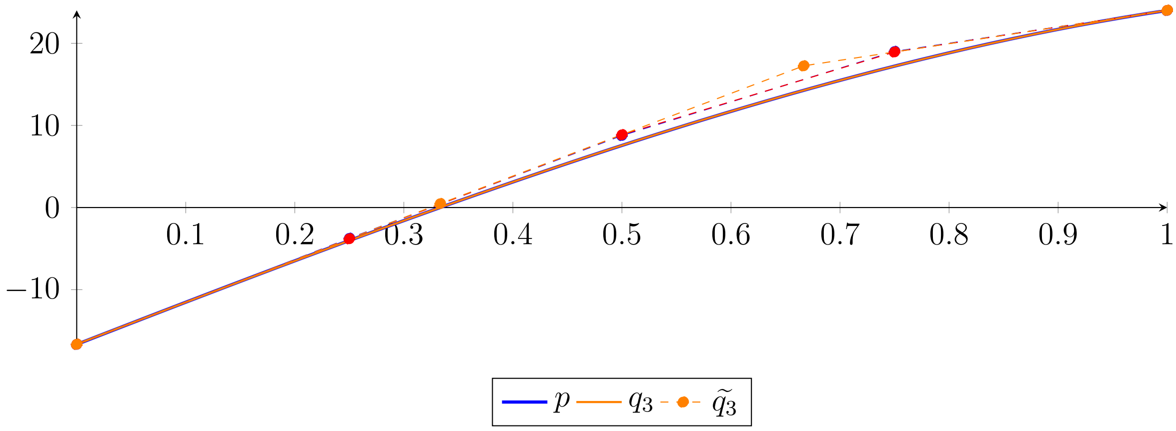
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -3.56012 \cdot 10^{-307}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



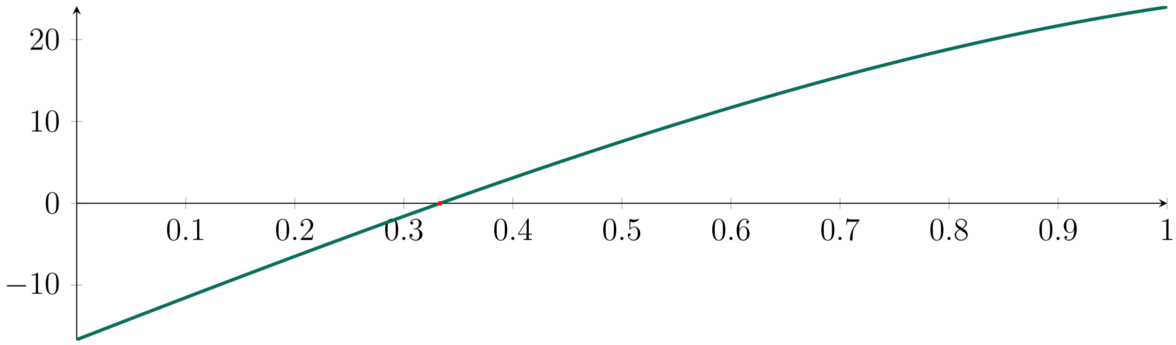
The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



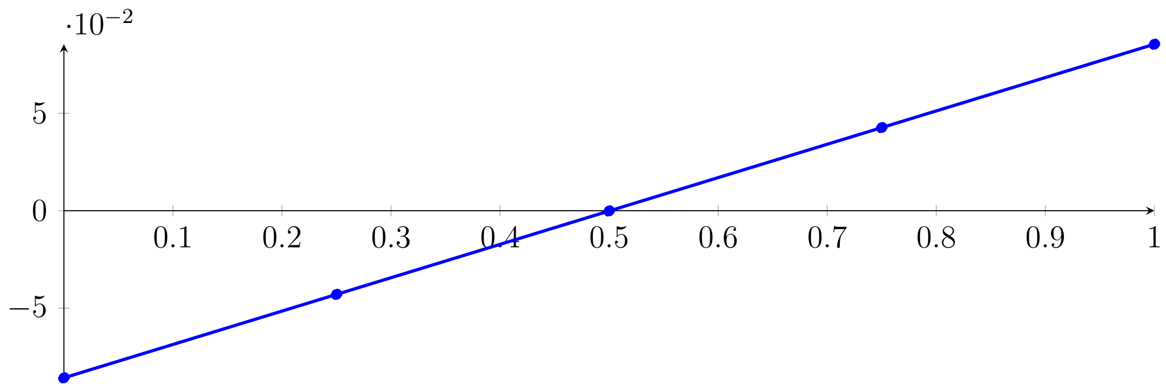
$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

195.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

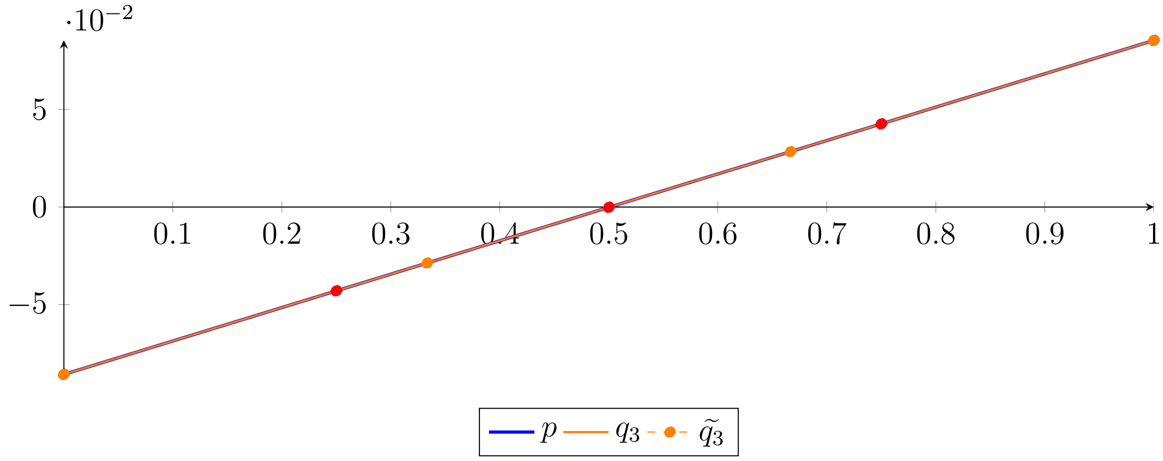
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1.39067 \cdot 10^{-309} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.
Bounding polynomials M and m:

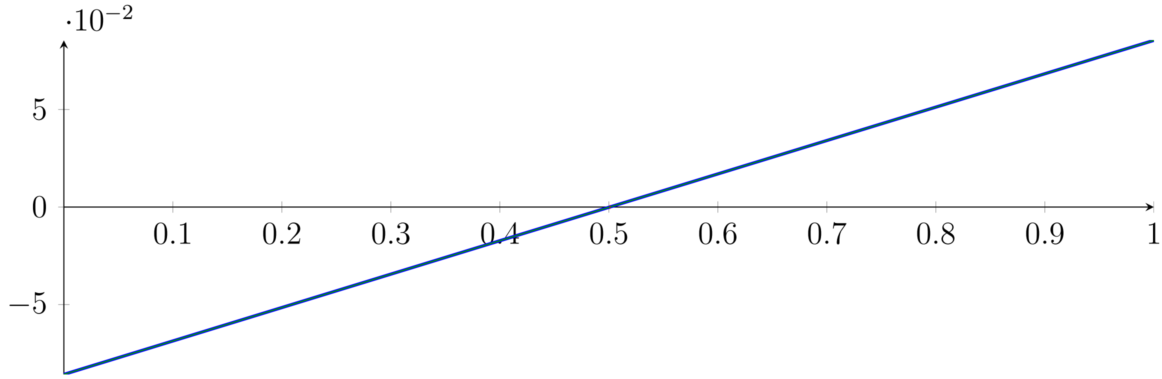
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

Longest intersection interval: $1.7041 \cdot 10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

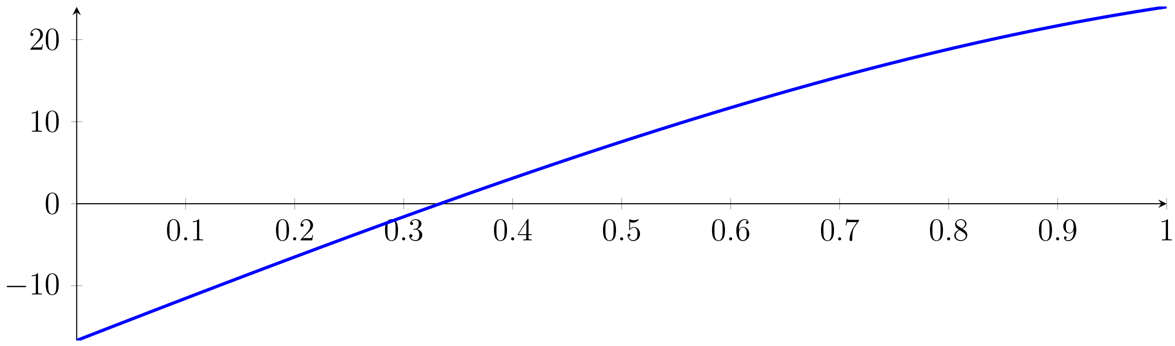
195.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

195.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

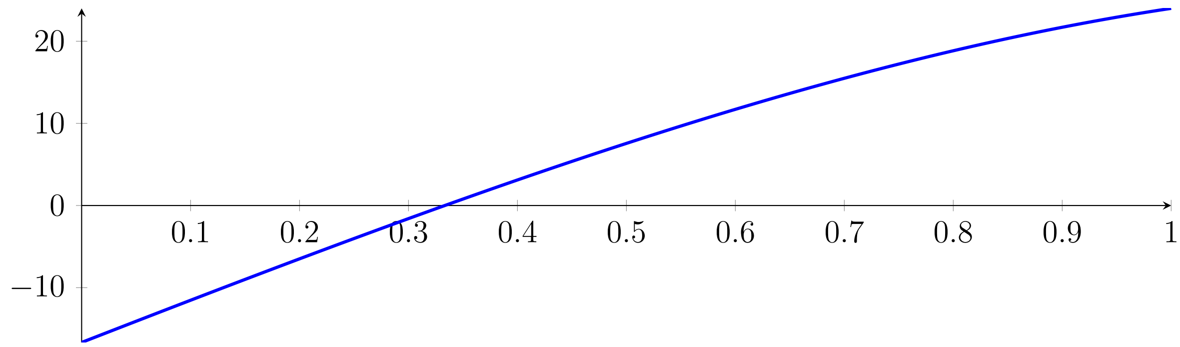
with precision $\varepsilon = 0.0001$.

196 Running BezClip on f_4 with epsilon 8

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

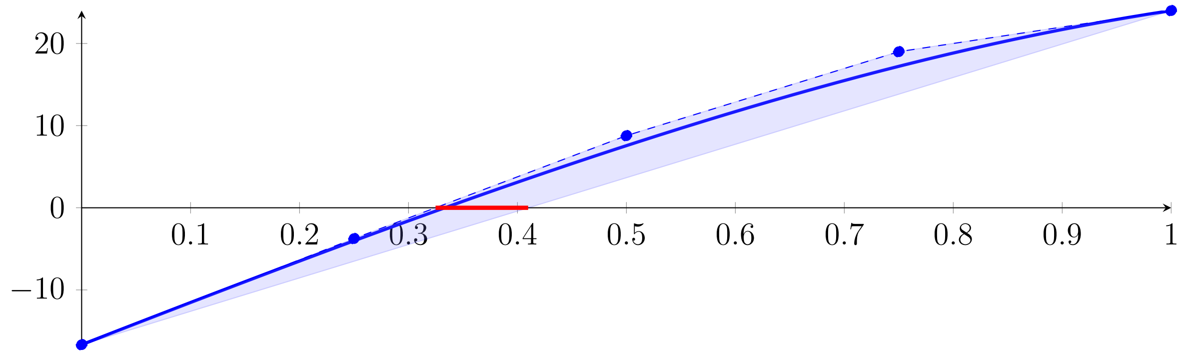
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



196.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

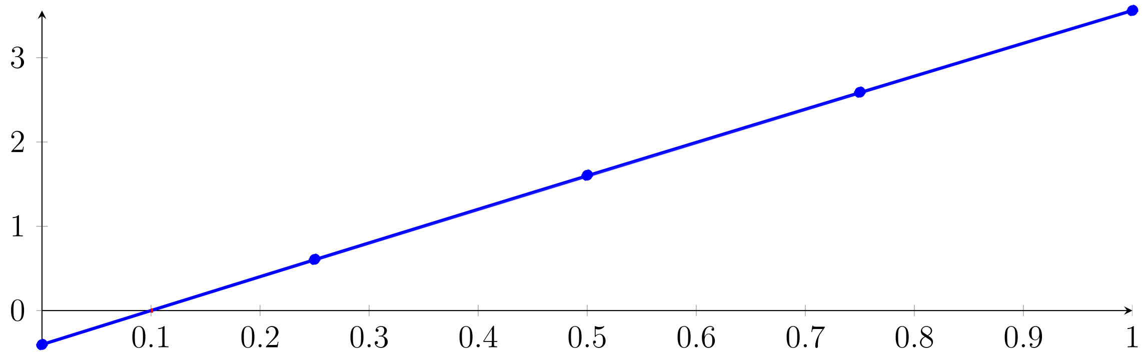
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

196.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

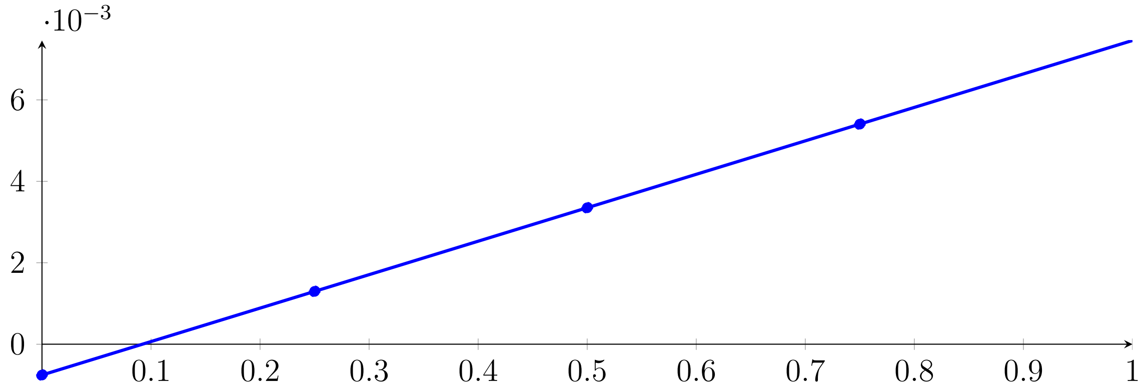
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

196.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01975 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

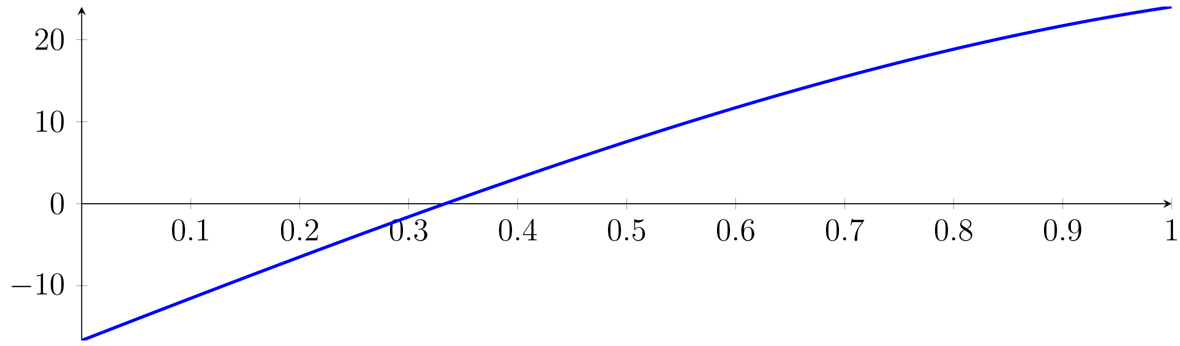
196.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

196.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

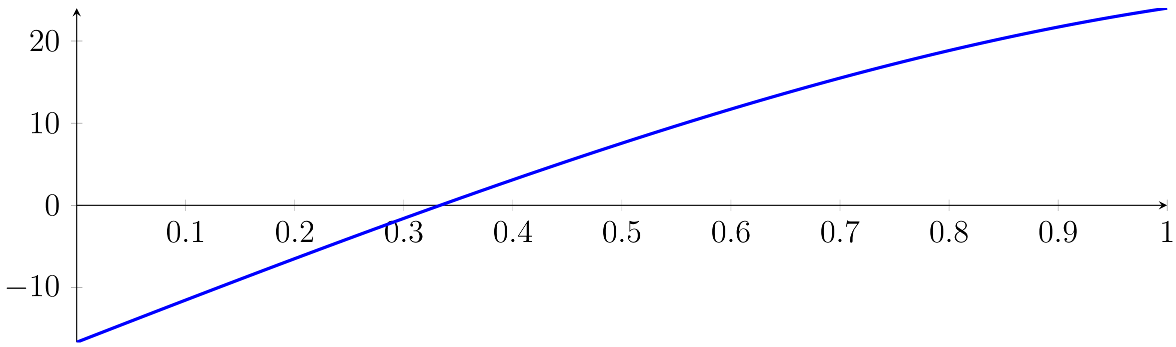
with precision $\varepsilon = 1 \cdot 10^{-08}$.

197 Running QuadClip on f_4 with epsilon 8

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

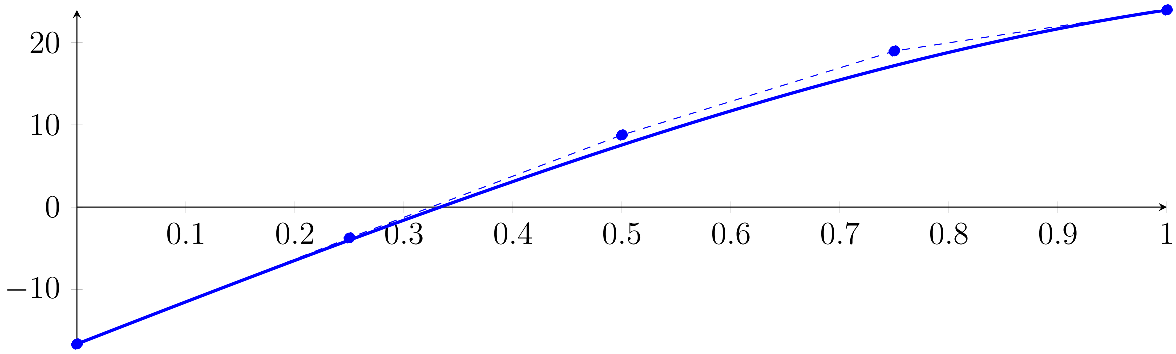
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



197.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

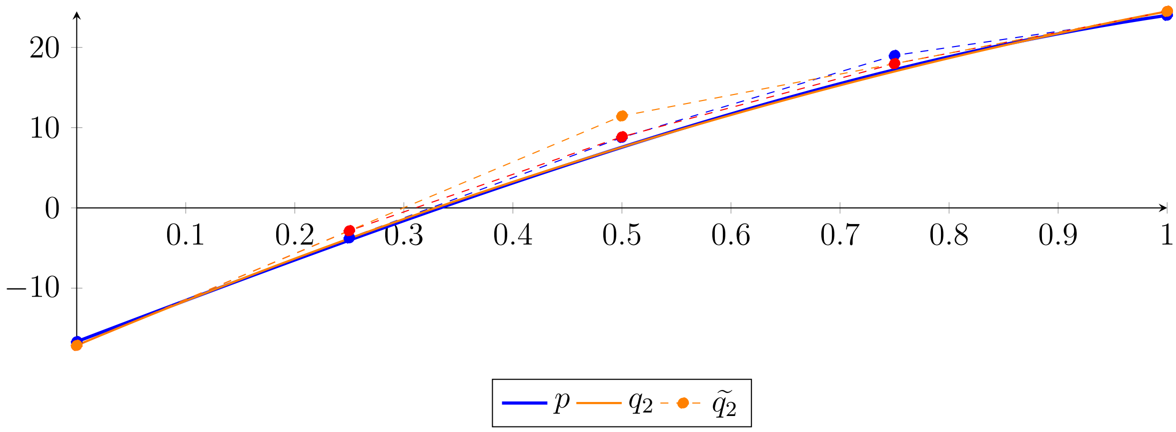
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -2.13607 \cdot 10^{-306}X^4 + 7.12024 \cdot 10^{-306}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



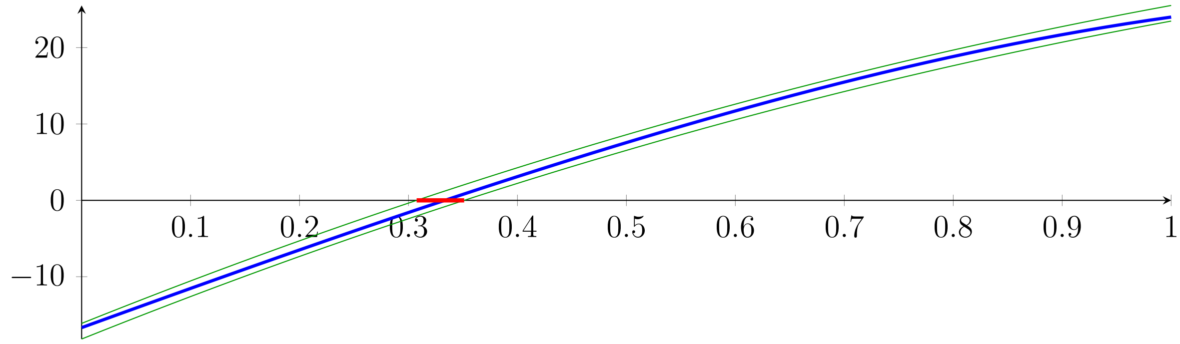
The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

$$M = -15.5476X^2 + 57.181X - 16.1119$$
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



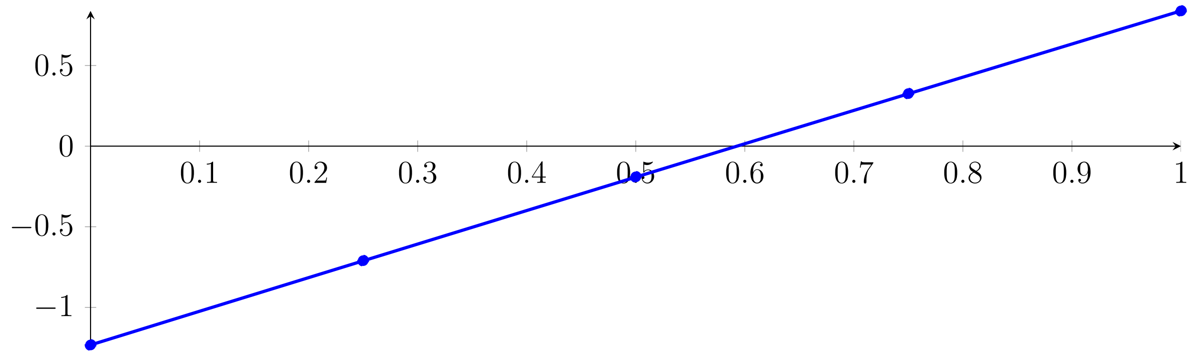
$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

197.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

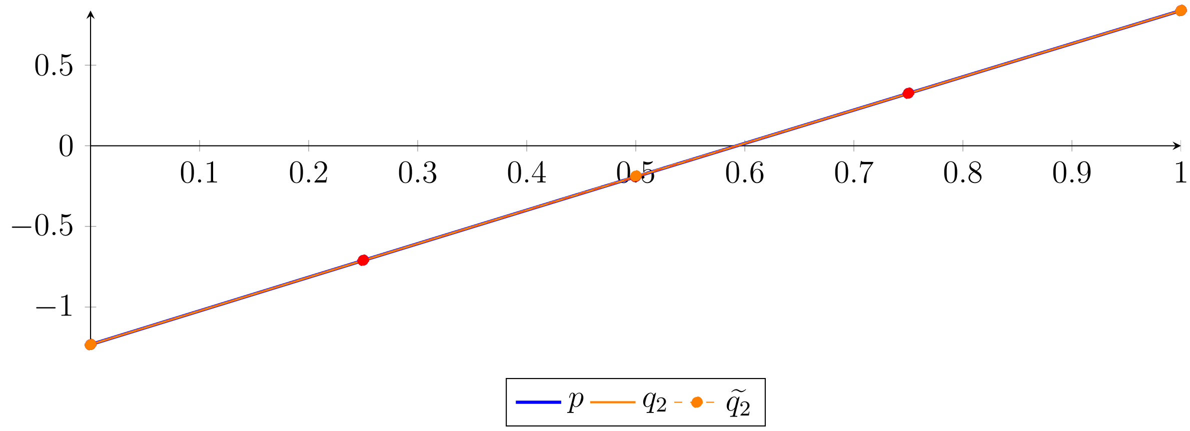
$$p = -3.62044 \cdot 10^{-06}X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278$$
$$= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -0.020089X^2 + 2.09166X - 1.23281$$
$$= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2}$$

$$\widetilde{q}_2 = -4.5614 \cdot 10^{-307}X^4 + 1.02353 \cdot 10^{-306}X^3 - 0.020089X^2 + 2.09166X - 1.23281$$
$$= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

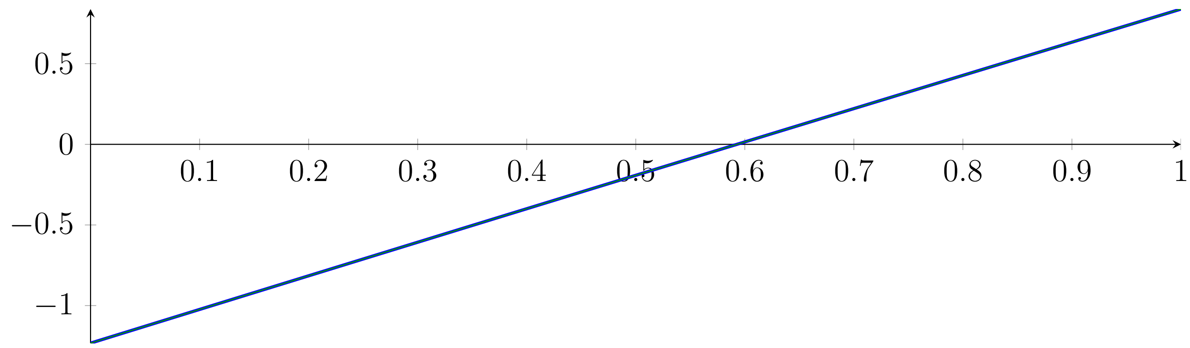
Bounding polynomials M and m:

$$M = -0.020089X^2 + 2.09166X - 1.23274$$
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

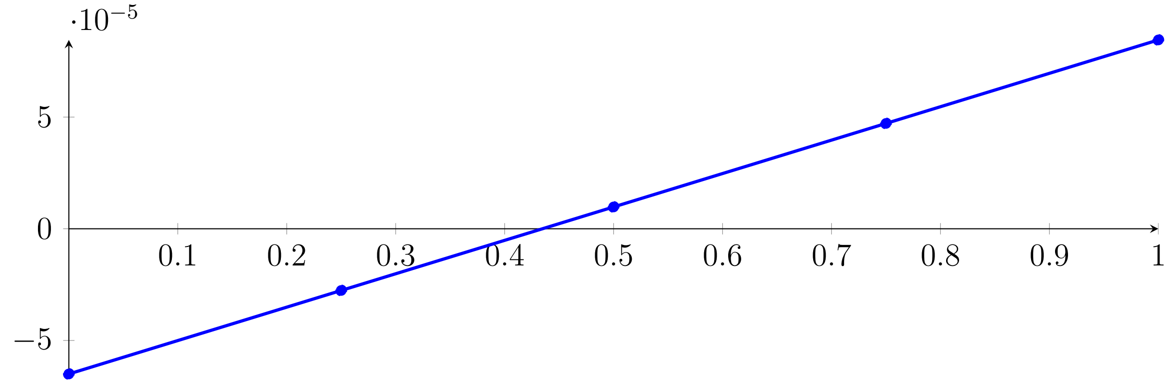
Longest intersection interval: $7.23183 \cdot 10^{-05}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

197.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

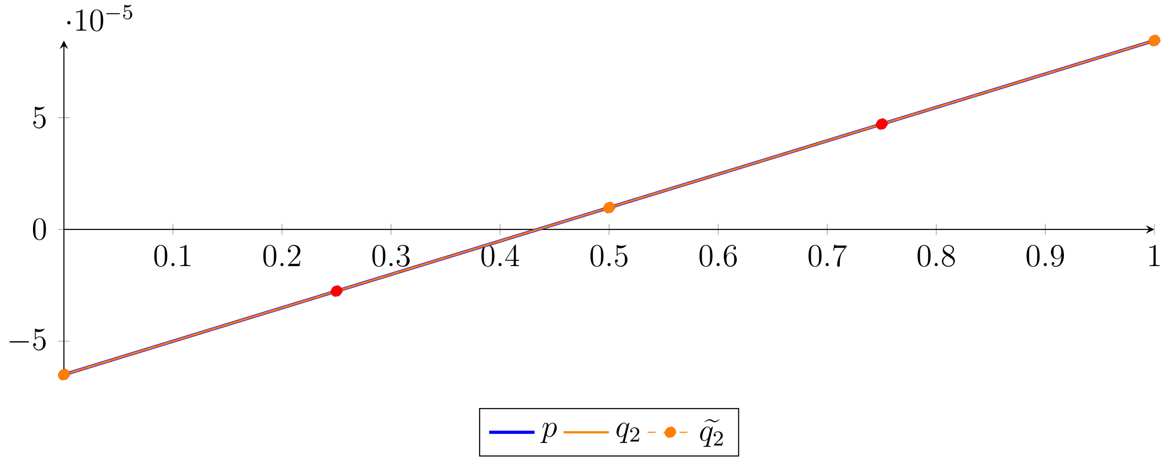
$$p = -9.9027 \cdot 10^{-23}X^4 - 2.82525 \cdot 10^{-16}X^3 - 1.06146 \cdot 10^{-10}X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05}B_{0,4}(X) - 2.76196 \cdot 10^{-05}B_{1,4}(X) + 9.76777$$
$$\cdot 10^{-06}B_{2,4}(X) + 4.71551 \cdot 10^{-05}B_{3,4}(X) + 8.45424 \cdot 10^{-05}B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\widetilde{q}_2 = 1.22227 \cdot 10^{-311} X^4 - 1.62969 \cdot 10^{-311} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



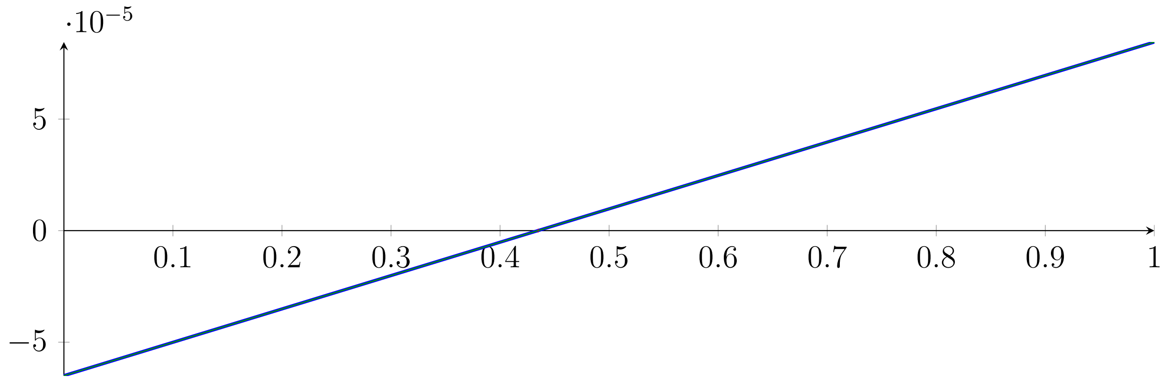
The maximum difference of the Bézier coefficients is $\delta = 2.82526 \cdot 10^{-17}$.
Bounding polynomials M and m:

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.434685, 1.4089 \cdot 10^{6}\} \qquad N(m) = \{0.434685, 1.4089 \cdot 10^{6}\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

Longest intersection interval: $3.77836 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

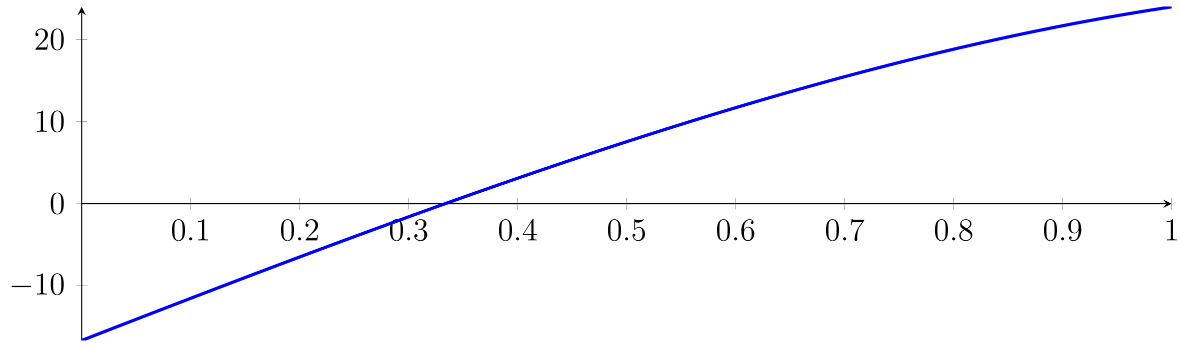
197.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

197.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-08}$.

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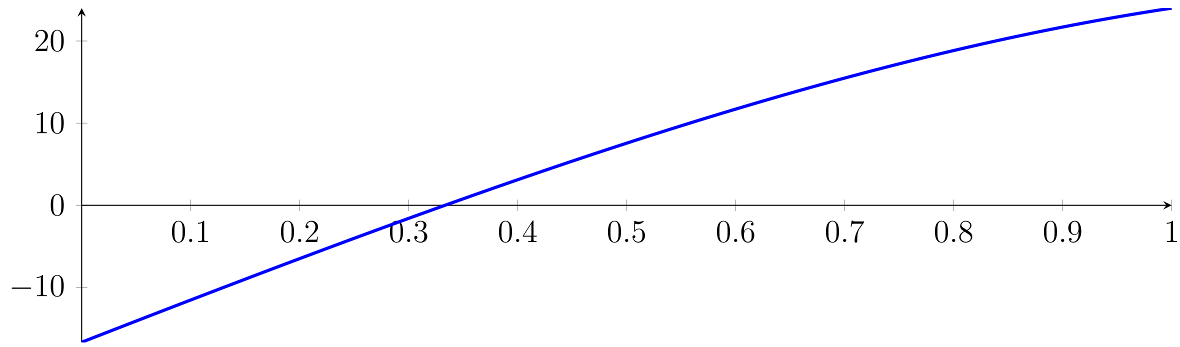
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

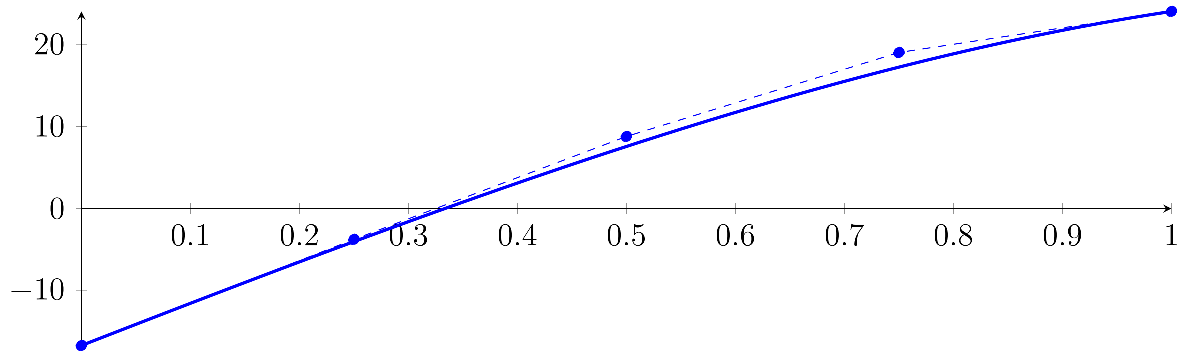
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



198.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

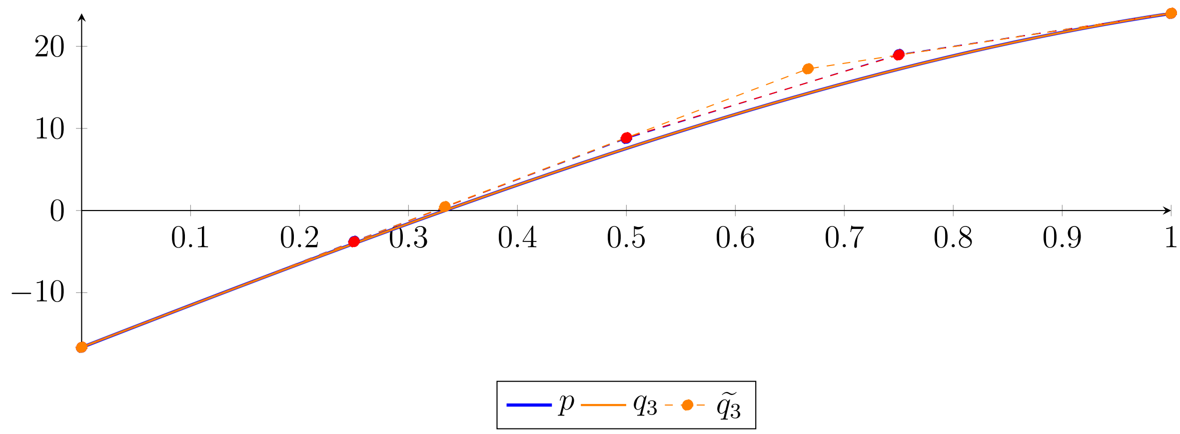
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -3.56012 \cdot 10^{-307}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

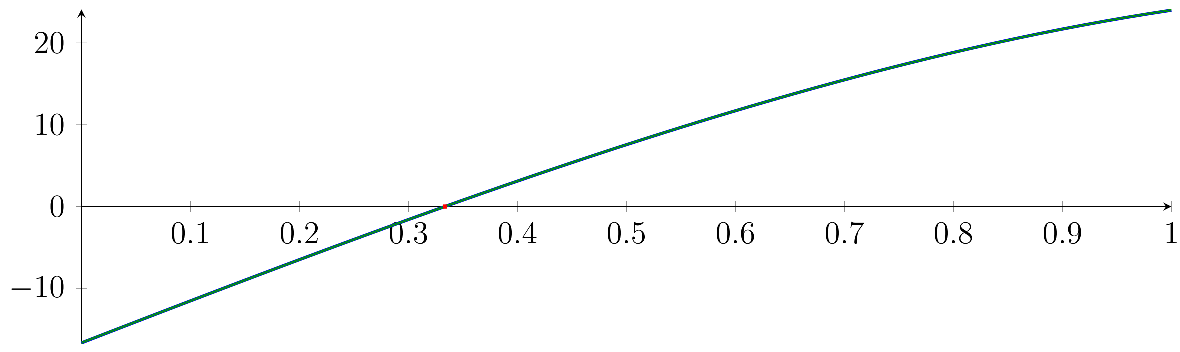
$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



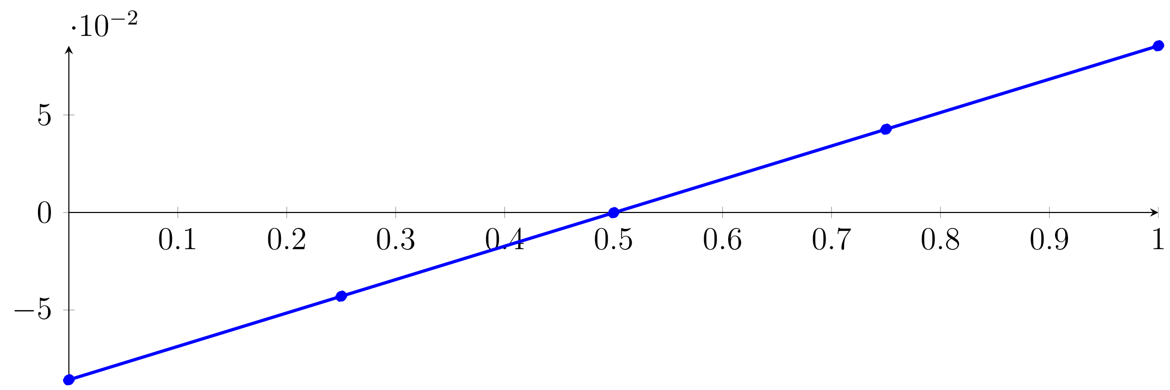
$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

198.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

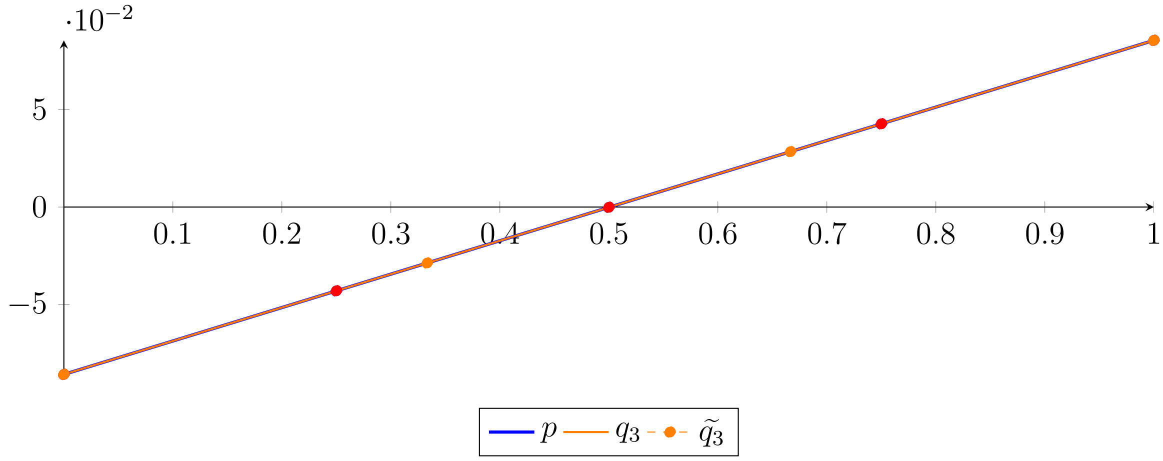
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1.39067 \cdot 10^{-309} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.
Bounding polynomials M and m:

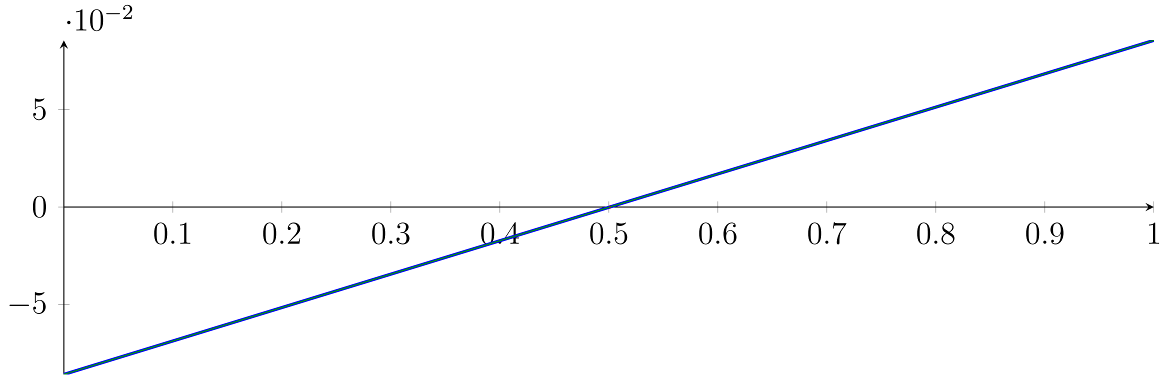
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

Longest intersection interval: $1.7041 \cdot 10^{-10}$
$\implies$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

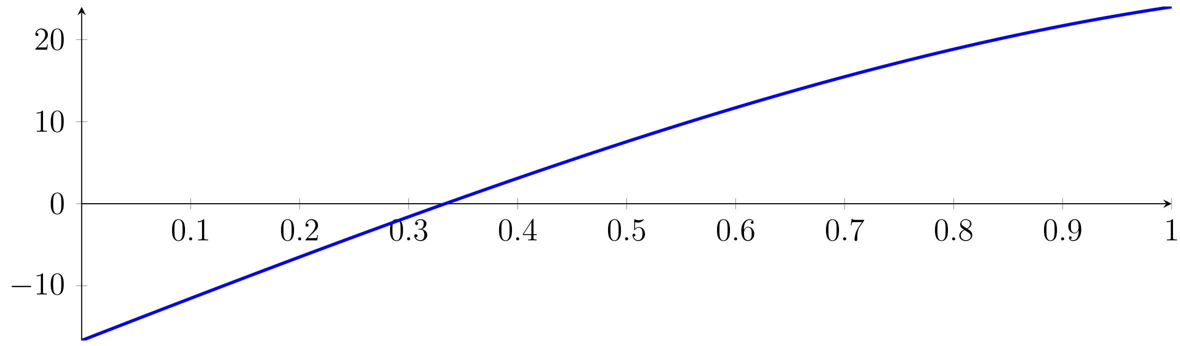
198.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

198.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-08}$.

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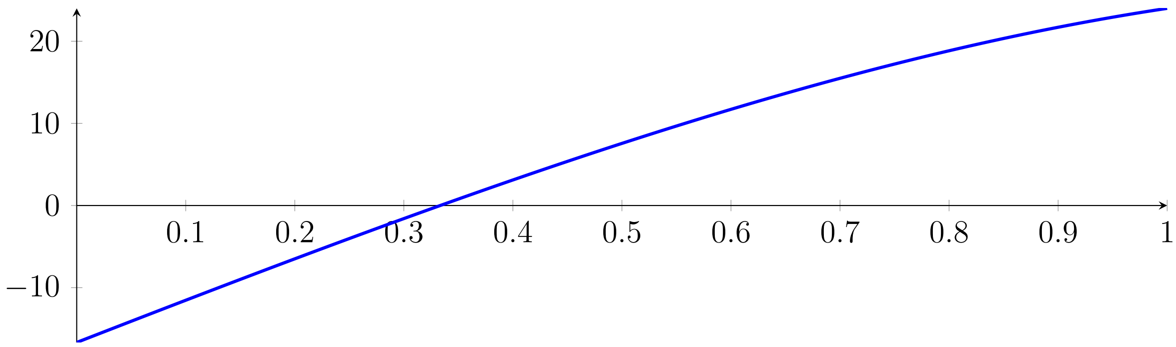
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

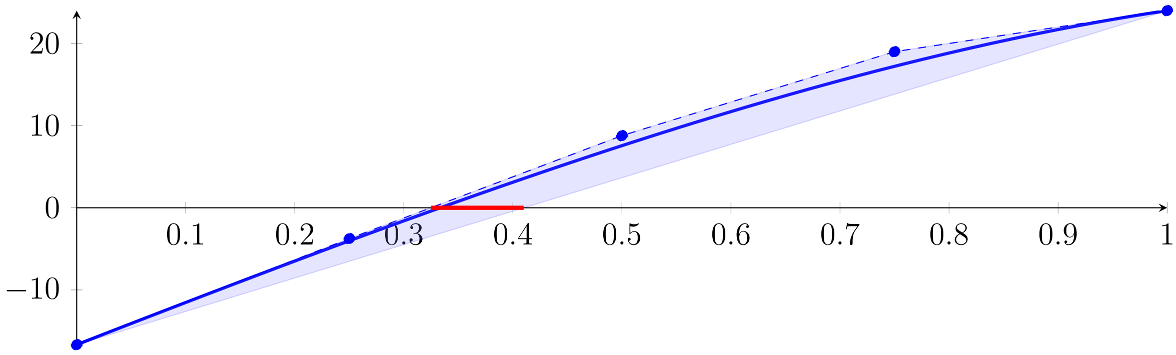
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



199.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

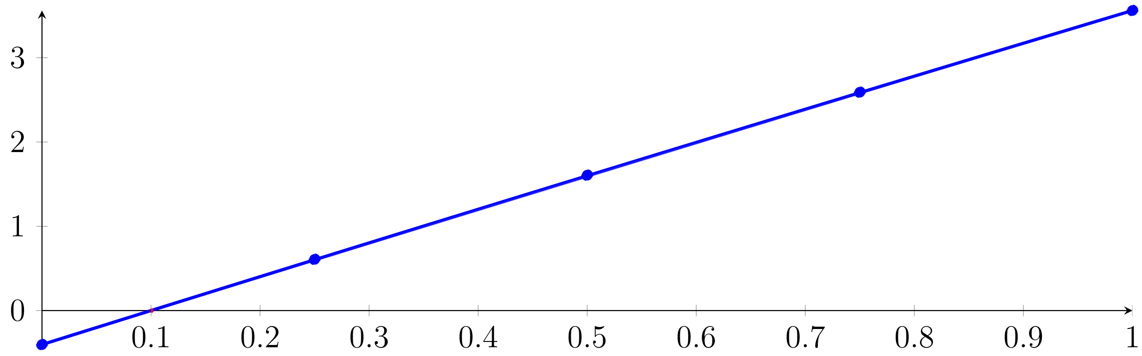
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

199.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

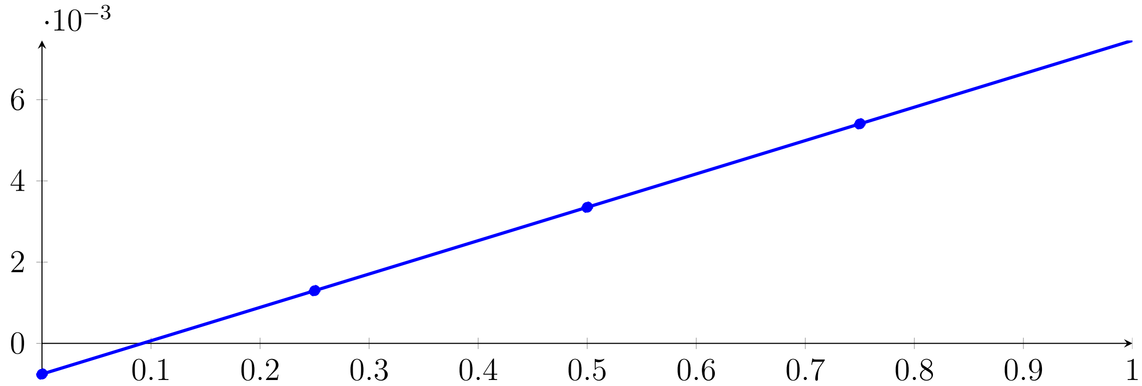
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

199.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01975 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

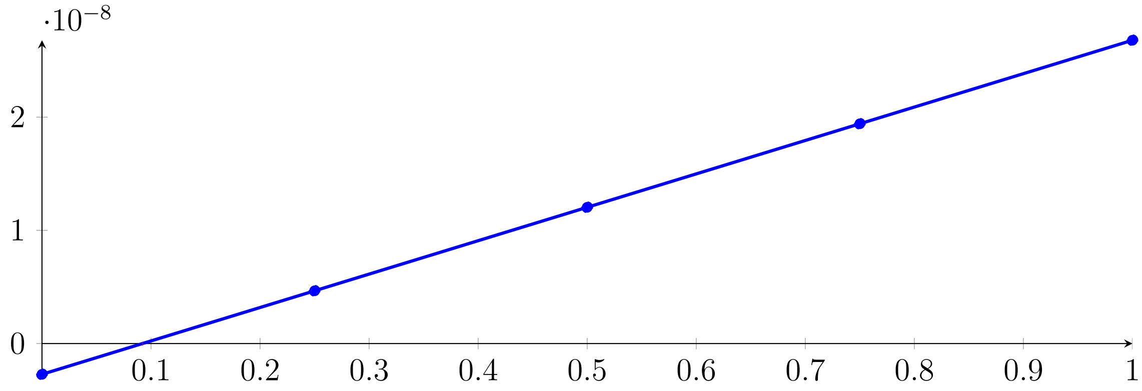
$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

199.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -1.50129\cdot10^{-37}X^4 - 2.17066\cdot10^{-27}X^3 - 4.13296\cdot10^{-18}X^2 + 2.95096\cdot10^{-08}X - 2.71749\cdot10^{-09}\\ &= -2.71749\cdot10^{-09}B_{0,4}(X) + 4.6599\cdot10^{-09}B_{1,4}(X) + 1.20373\\ &\quad\cdot10^{-08}B_{2,4}(X) + 1.94147\cdot10^{-08}B_{3,4}(X) + 2.67921\cdot10^{-08}B_{4,4}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28975\cdot10^{-11}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

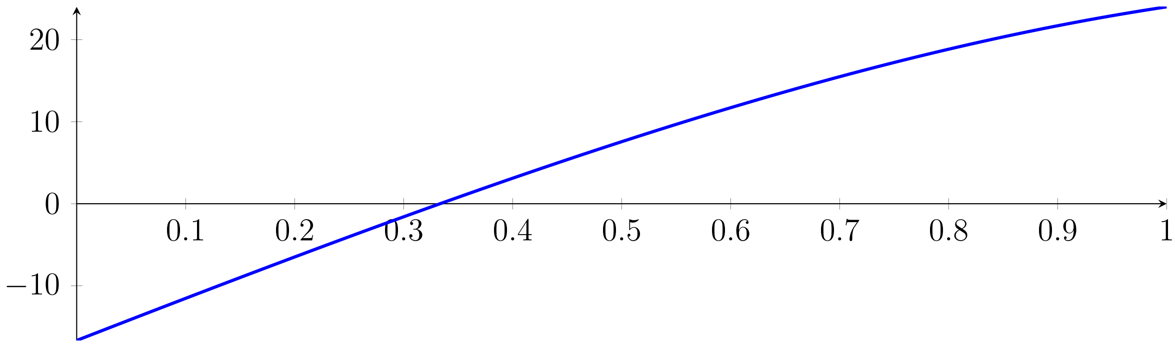
199.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

199.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

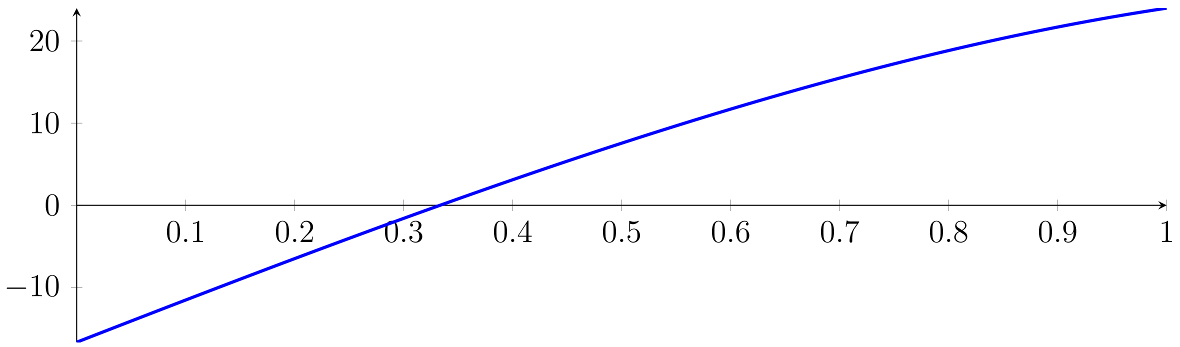
with precision $\varepsilon = 1 \cdot 10^{-16}$.

200 Running QuadClip on f_4 with epsilon 16

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

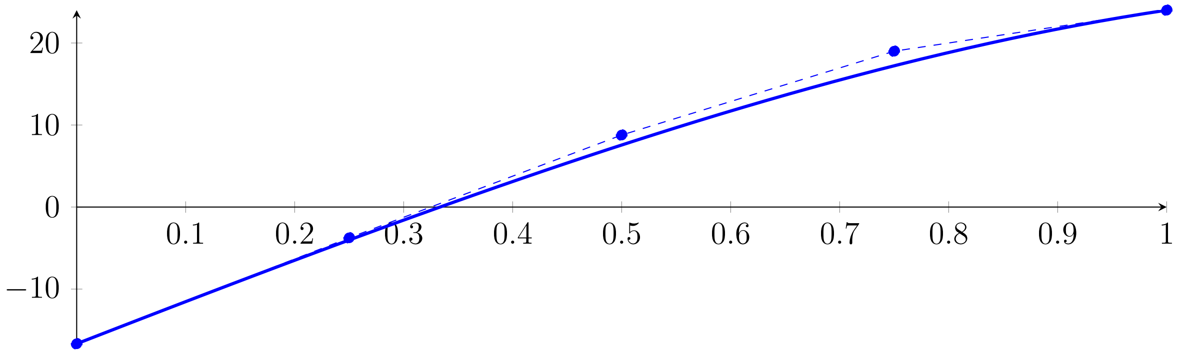
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



200.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

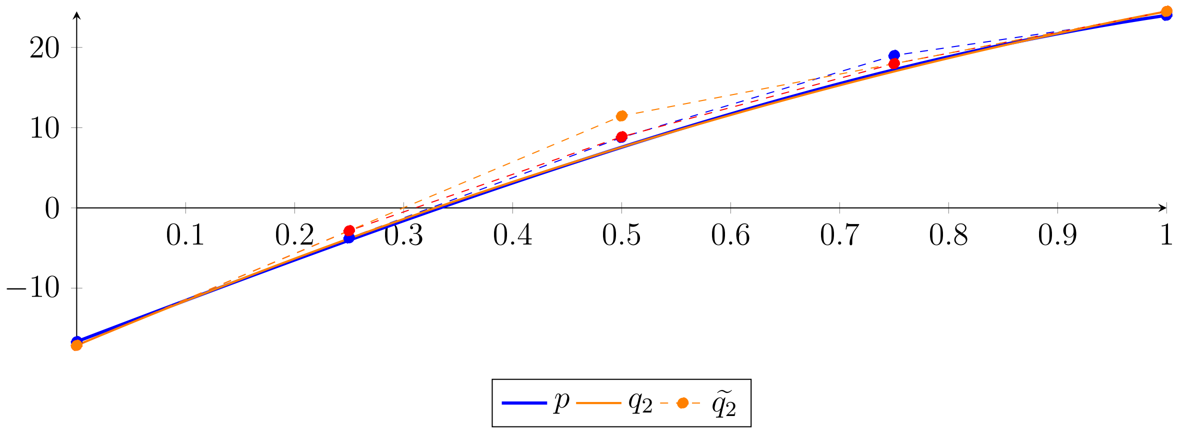
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -2.13607 \cdot 10^{-306}X^4 + 7.12024 \cdot 10^{-306}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

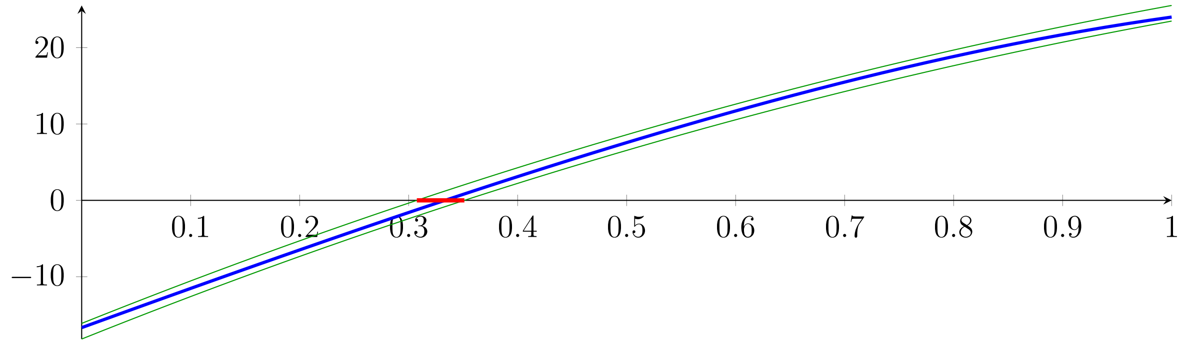
$$M = -15.5476X^2 + 57.181X - 16.1119$$

$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



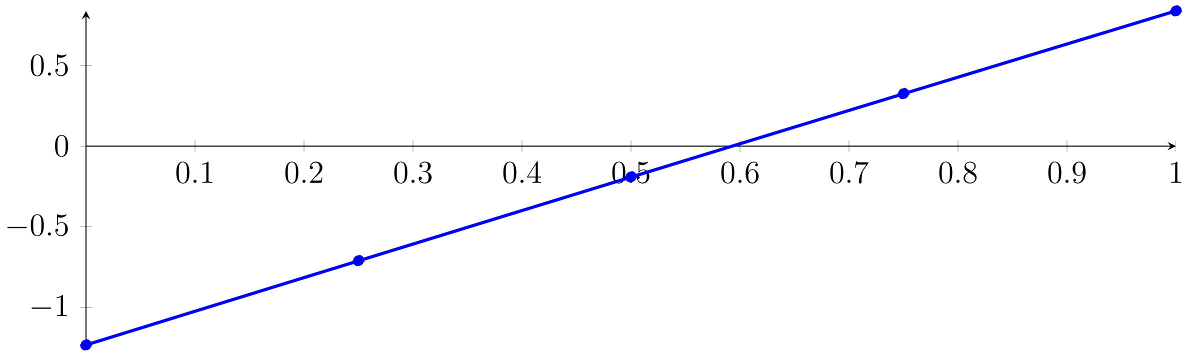
$[0.307477, 0.351097]$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

200.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

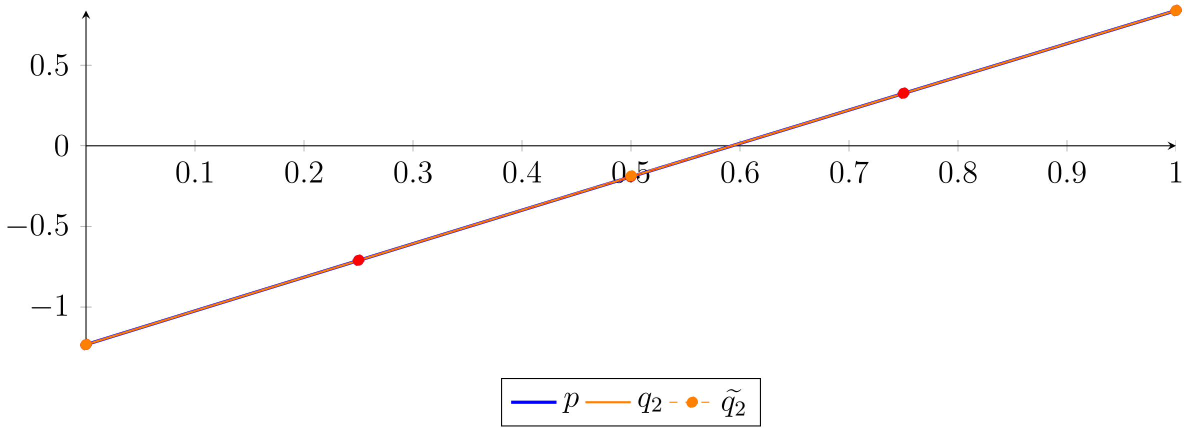
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06}X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -4.5614 \cdot 10^{-307}X^4 + 1.02353 \cdot 10^{-306}X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

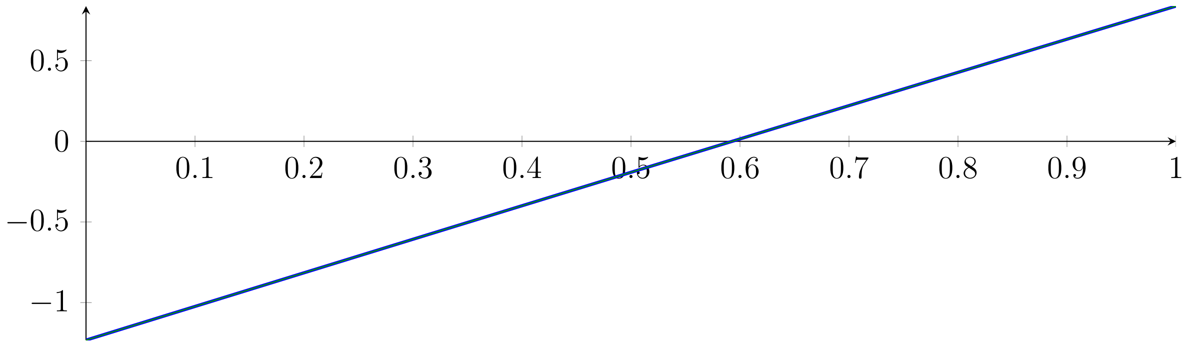
Bounding polynomials M and m:

$$M = -0.020089X^2 + 2.09166X - 1.23274$$
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

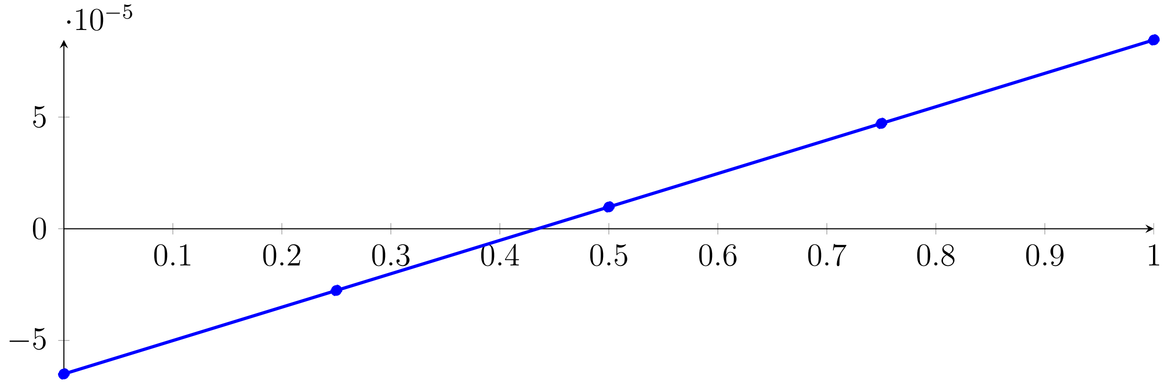
Longest intersection interval: $7.23183 \cdot 10^{-05}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

200.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

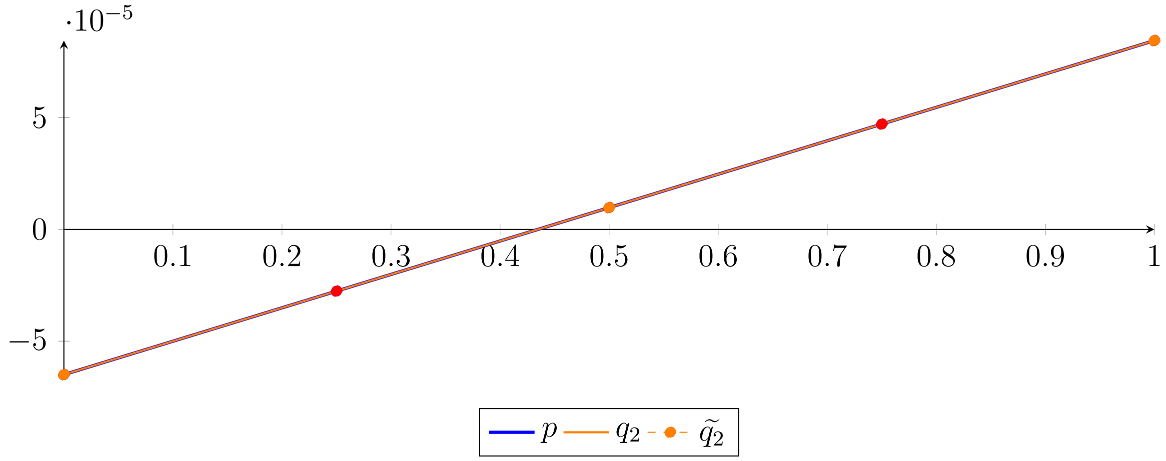
$$p = -9.9027 \cdot 10^{-23}X^4 - 2.82525 \cdot 10^{-16}X^3 - 1.06146 \cdot 10^{-10}X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05}B_{0,4}(X) - 2.76196 \cdot 10^{-05}B_{1,4}(X) + 9.76777 \cdot 10^{-06}B_{2,4}(X) + 4.71551 \cdot 10^{-05}B_{3,4}(X) + 8.45424 \cdot 10^{-05}B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\widetilde{q}_2 = 1.22227 \cdot 10^{-311} X^4 - 1.62969 \cdot 10^{-311} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



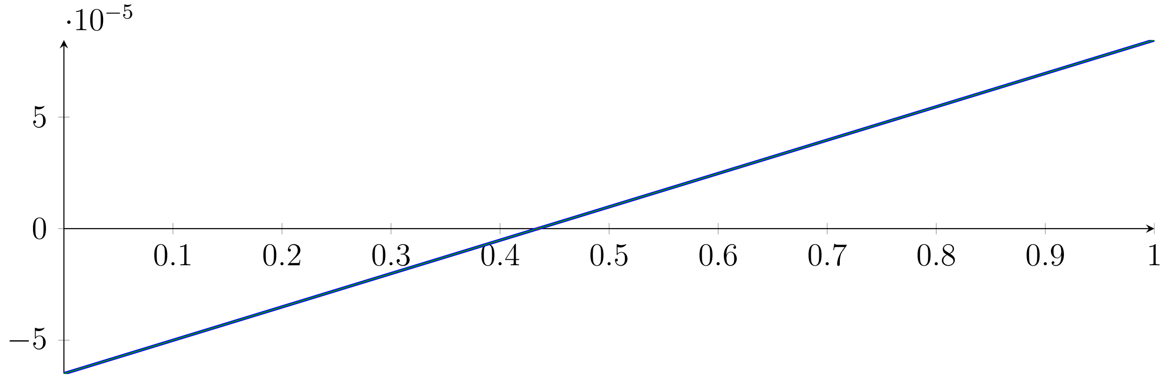
The maximum difference of the Bézier coefficients is $\delta = 2.82526 \cdot 10^{-17}$.
Bounding polynomials M and m:

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.434685, 1.4089 \cdot 10^{6}\} \qquad N(m) = \{0.434685, 1.4089 \cdot 10^{6}\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

Longest intersection interval: $3.77836 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

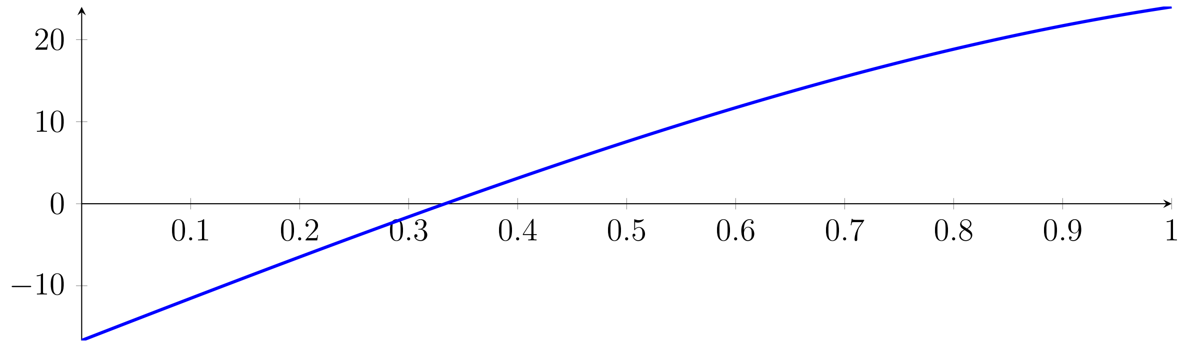
200.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

200.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

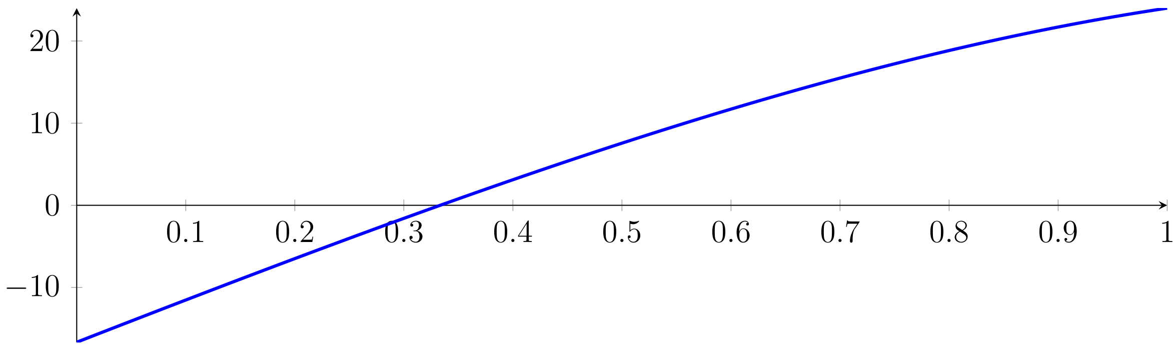
with precision $\varepsilon = 1 \cdot 10^{-16}$.

201 Running CubeClip on f_4 with epsilon 16

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

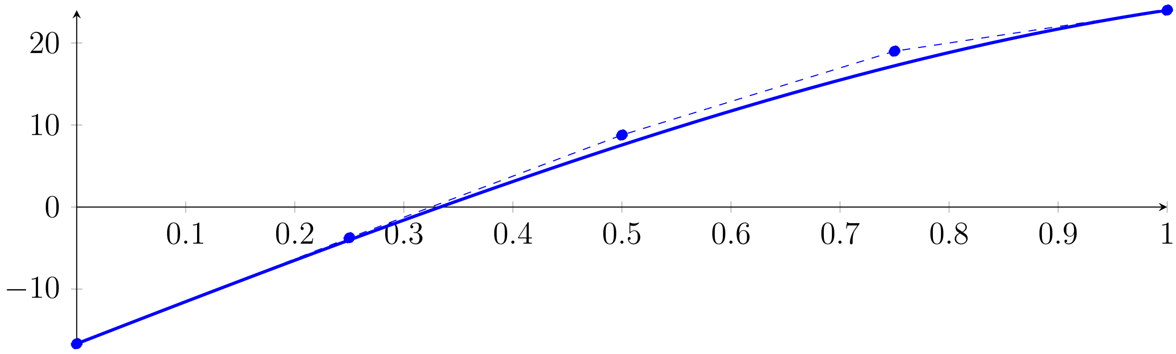
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



201.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

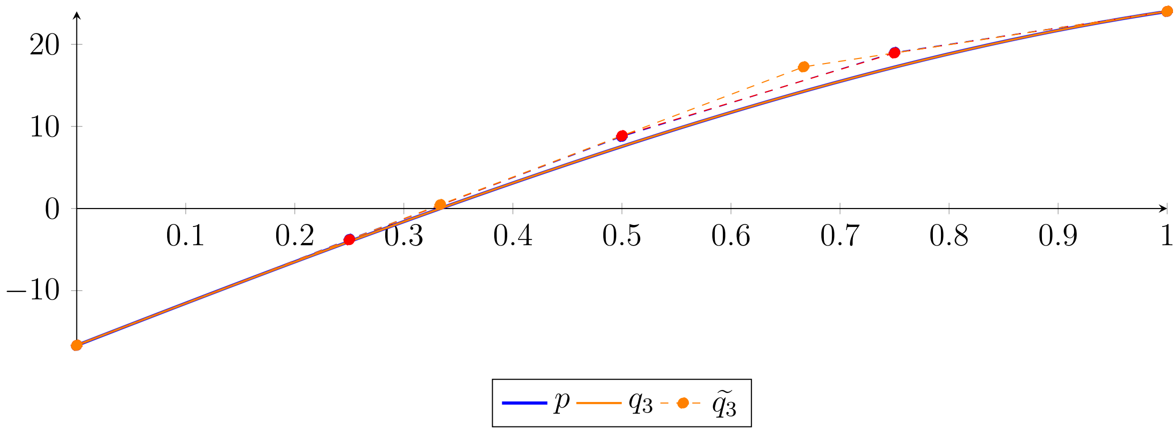
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -3.56012\cdot 10^{-307}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

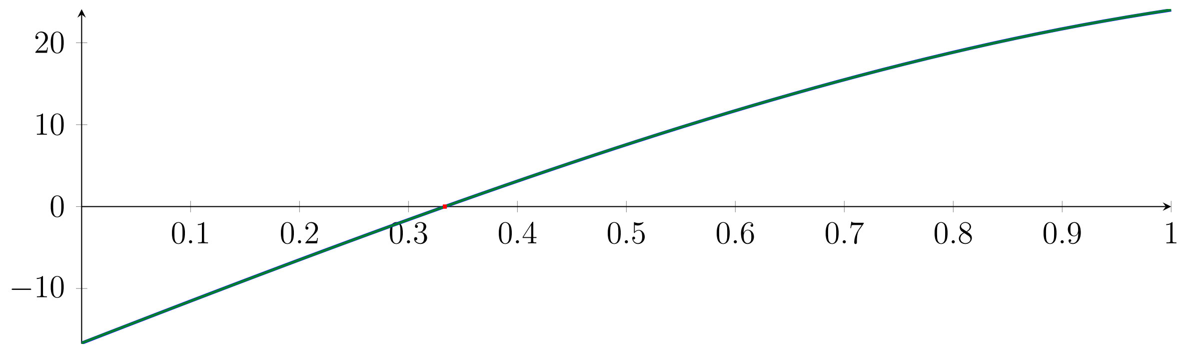
$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



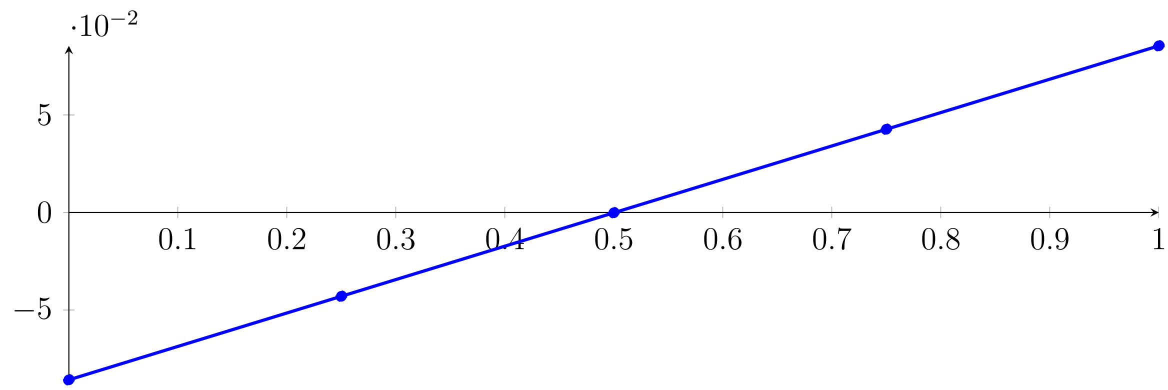
$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

201.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

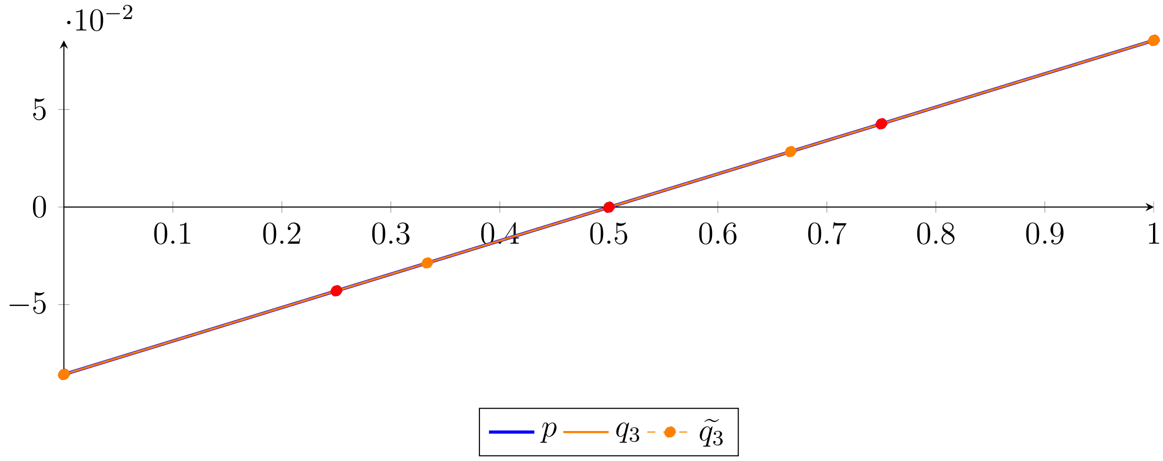
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1.39067 \cdot 10^{-309} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m:

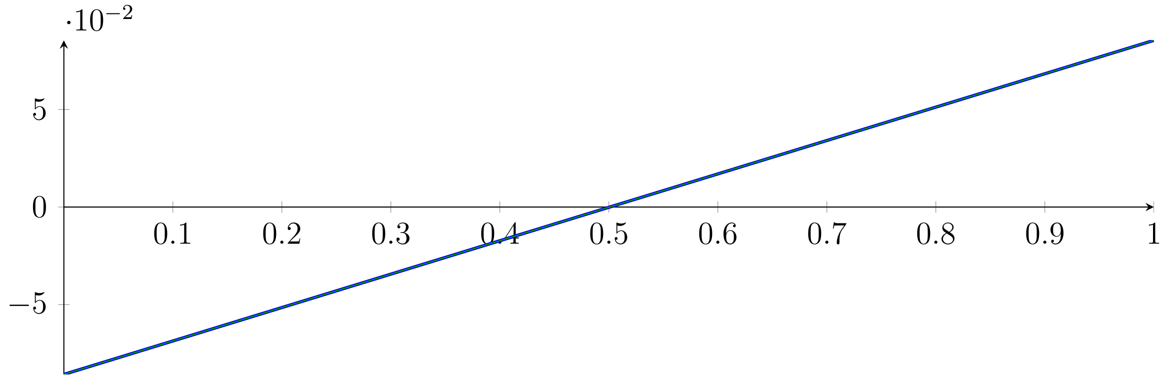
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

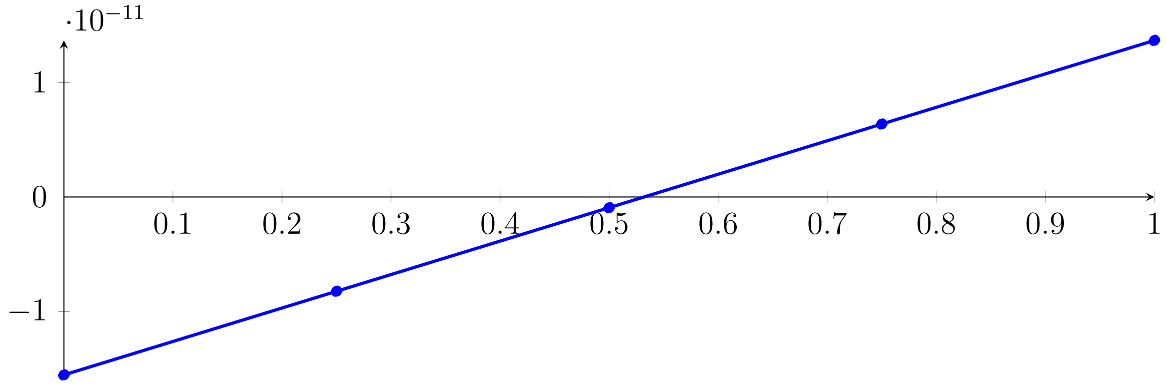
Longest intersection interval: $1.7041 \cdot 10^{-10}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

201.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

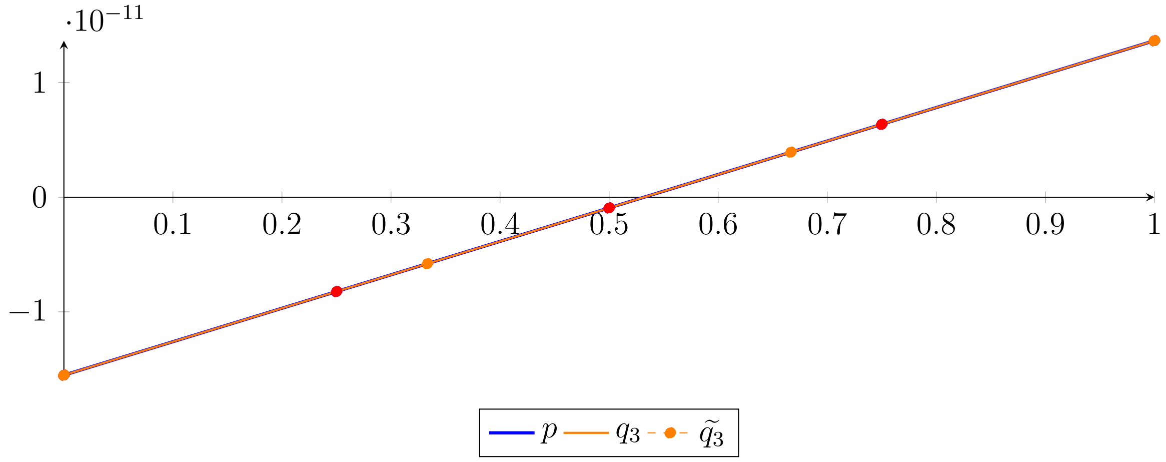
Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.43544 \cdot 10^{-49} X^4 - 2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}$$
$$= -1.55233 \cdot 10^{-11} B_{0,4}(X) - 8.22817 \cdot 10^{-12} B_{1,4}(X) - 9.33052 \cdot 10^{-13} B_{2,4}(X) + 6.36207 \cdot 10^{-12} B_{3,4}(X) + 1.36572 \cdot 10^{-11} B_{4,4}(X)$$



Degree reduction and raising:

$$q_3 = -2.09885\cdot10^{-36}X^3 - 4.0413\cdot10^{-24}X^2 + 2.91805\cdot10^{-11}X - 1.55233\cdot10^{-11}$$
$$= -1.55233\cdot10^{-11}B_{0,3} - 5.79646\cdot10^{-12}B_{1,3} + 3.93036\cdot10^{-12}B_{2,3} + 1.36572\cdot10^{-11}B_{3,3}$$
$$\widetilde{q}_3 = -3.23791\cdot10^{-319}X^4 - 2.09885\cdot10^{-36}X^3 - 4.0413\cdot10^{-24}X^2 + 2.91805\cdot10^{-11}X - 1.55233\cdot10^{-11}$$
$$= -1.55233\cdot10^{-11}B_{0,4} - 8.22817\cdot10^{-12}B_{1,4} - 9.33052\cdot10^{-13}B_{2,4} + 6.36207\cdot10^{-12}B_{3,4} + 1.36572\cdot10^{-11}B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 1.23038\cdot10^{-50}$.

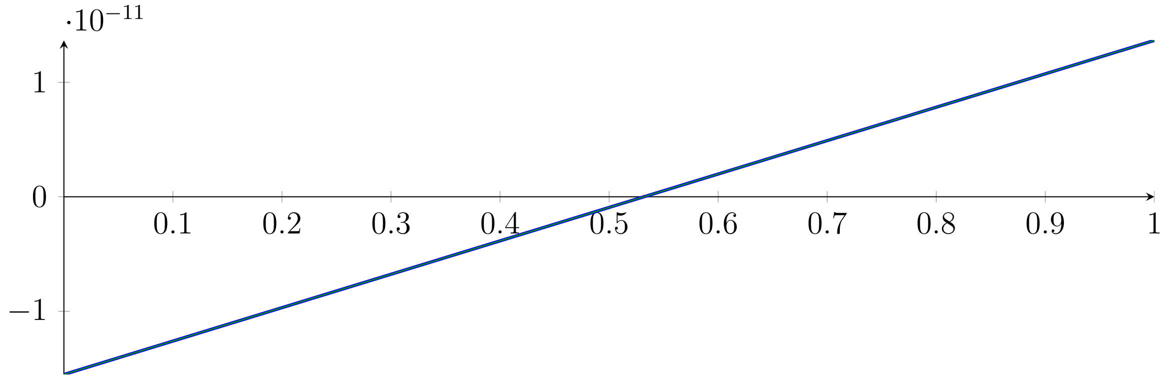
Bounding polynomials M and m:

$$M = -2.09885\cdot10^{-36}X^3 - 4.0413\cdot10^{-24}X^2 + 2.91805\cdot10^{-11}X - 1.55233\cdot10^{-11}$$
$$m = -2.09885\cdot10^{-36}X^3 - 4.0413\cdot10^{-24}X^2 + 2.91805\cdot10^{-11}X - 1.55233\cdot10^{-11}$$

Root of M and m:

$N(M) = \{-4.81371\cdot10^{12}, 0.531249, 2.88823\cdot10^{12}\}$ $\quad N(m) = \{-4.81371\cdot10^{12}, 0.531249, 2.88823\cdot10^{12}\}$

Intersection intervals:



$[0.531249, 0.531249]$

Longest intersection interval: $8.43287\cdot10^{-40}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

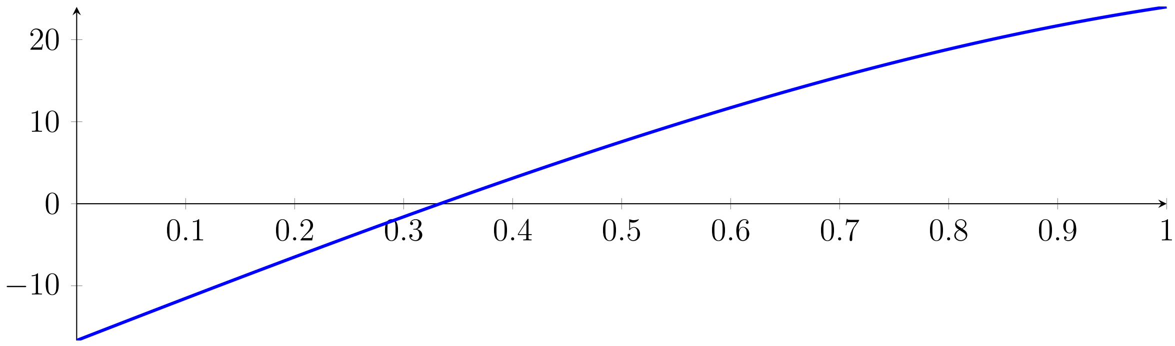
201.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 4!
p(0) = -2.11876e-14 - p(1) -2.11876e-14

201.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

with precision $\varepsilon = 1 \cdot 10^{-16}$.

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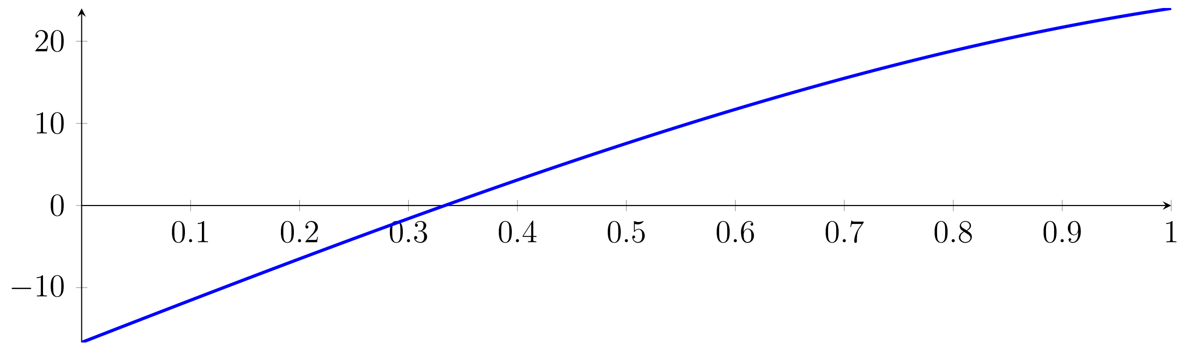
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

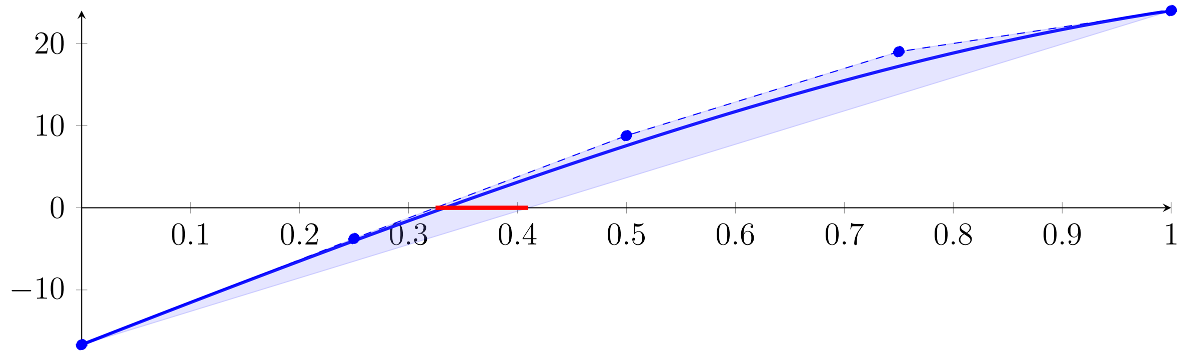
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



202.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$
$$= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

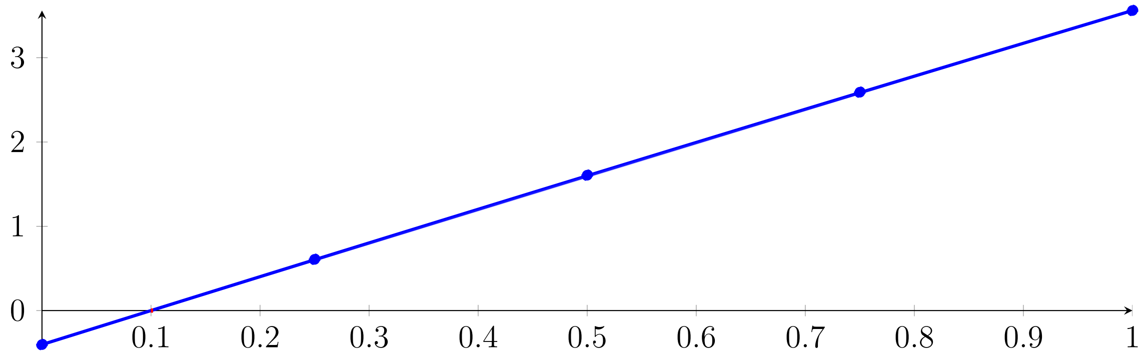
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

202.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711$$
$$= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

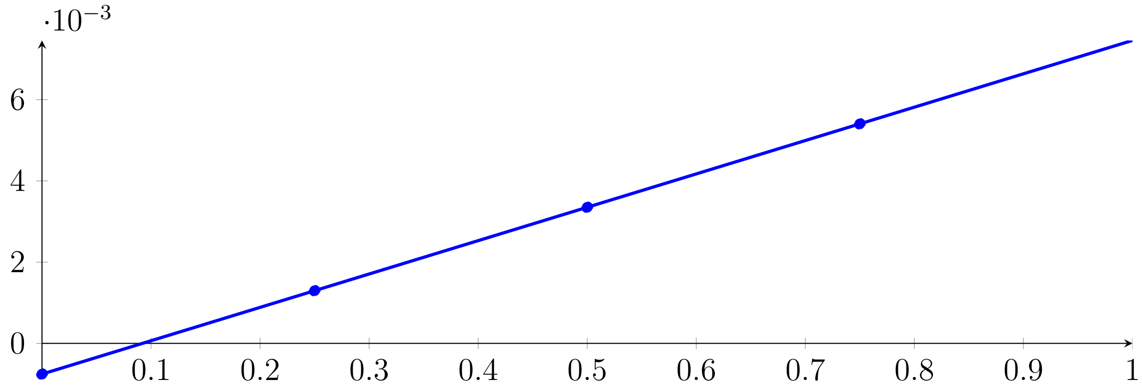
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

202.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01975 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

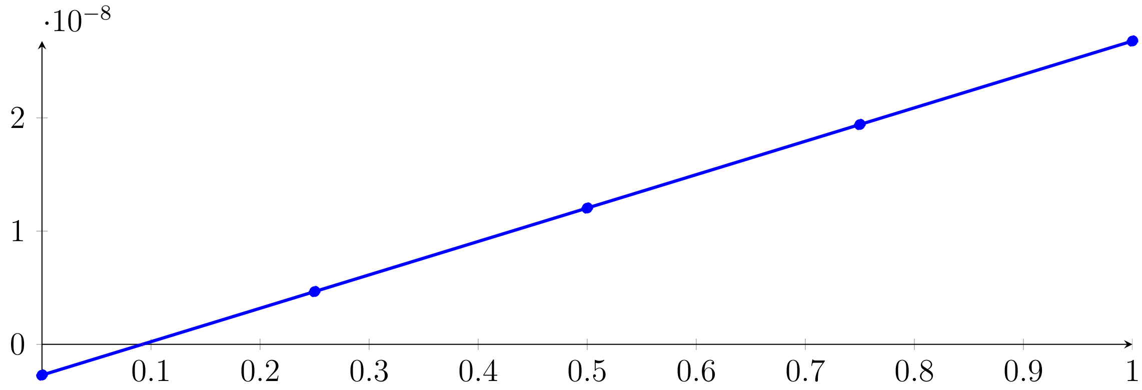
$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

202.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.50129 \cdot 10^{-37} X^4 - 2.17066 \cdot 10^{-27} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09}$$
$$= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

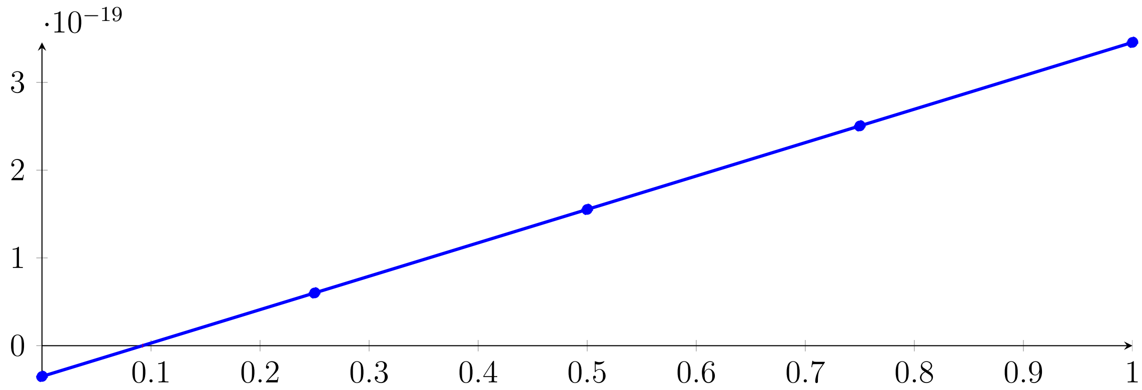
Longest intersection interval: $1.28975 \cdot 10^{-11}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

202.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -4.15417 \cdot 10^{-81} X^4 - 4.65699 \cdot 10^{-60} X^3 - 6.87497 \cdot 10^{-40} X^2 + 3.80599 \cdot 10^{-19} X - 3.50488 \cdot 10^{-20}$$
$$= -3.50488 \cdot 10^{-20} B_{0,4}(X) + 6.01009 \cdot 10^{-20} B_{1,4}(X) + 1.55251 \cdot 10^{-19} B_{2,4}(X) + 2.504 \cdot 10^{-19} B_{3,4}(X) + 3.4555 \cdot 10^{-19} B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.66345 \cdot 10^{-22}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

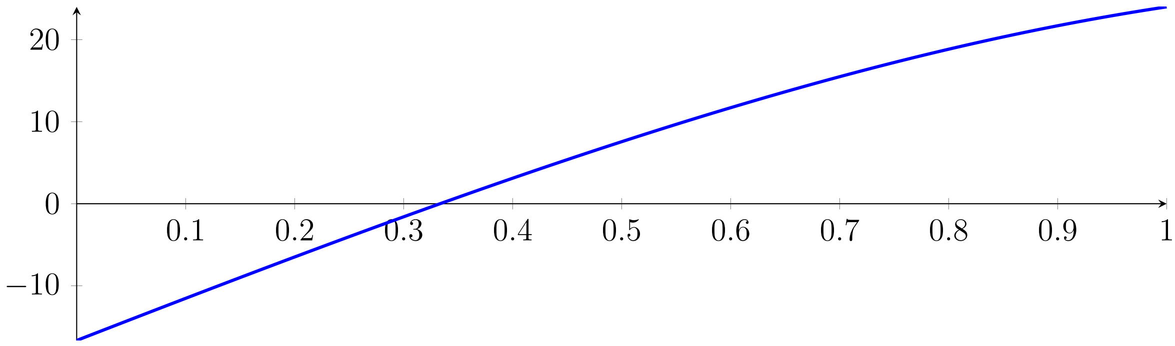
202.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

202.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

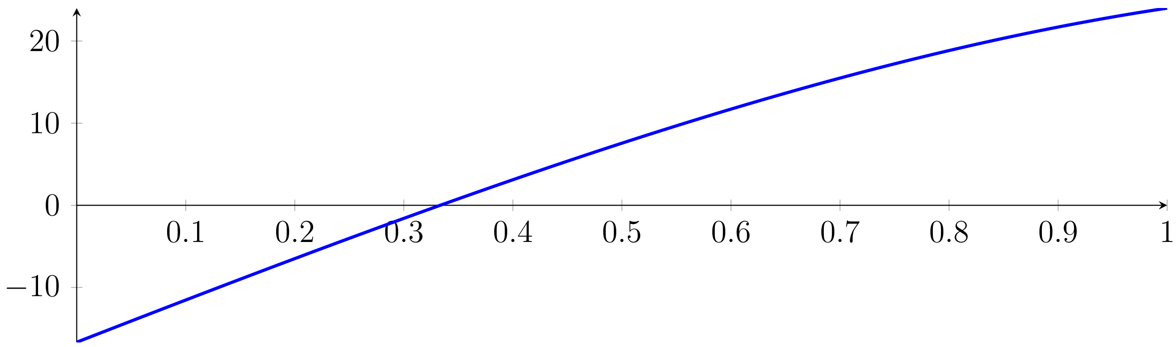
with precision $\varepsilon = 1 \cdot 10^{-32}$.

203 Running `QuadClip` on f_4 with epsilon 32

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called `QuadClip` with input polynomial on interval $[0, 1]$:

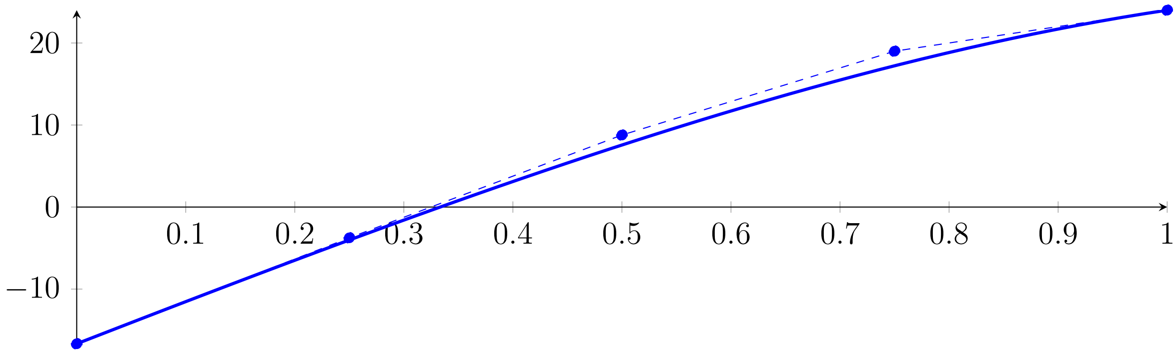
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



203.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

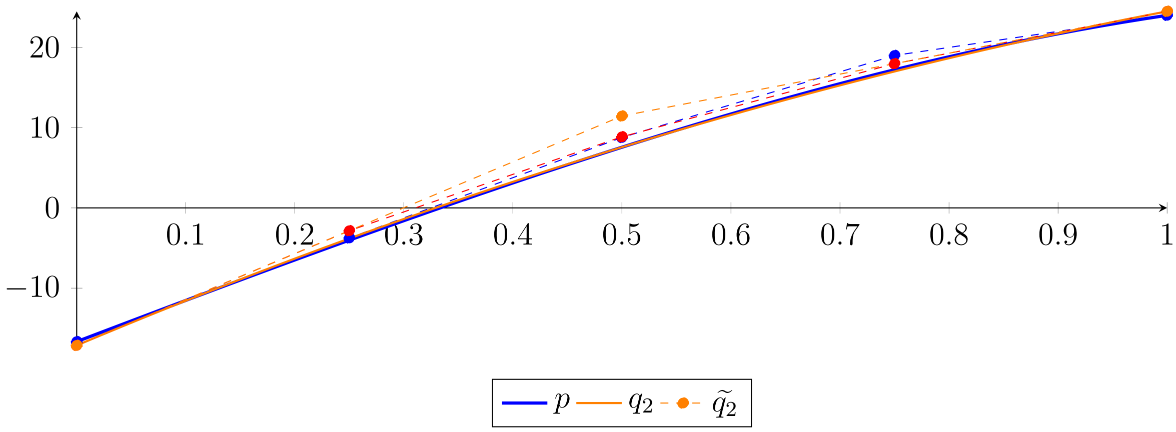
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -2.13607 \cdot 10^{-306}X^4 + 7.12024 \cdot 10^{-306}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

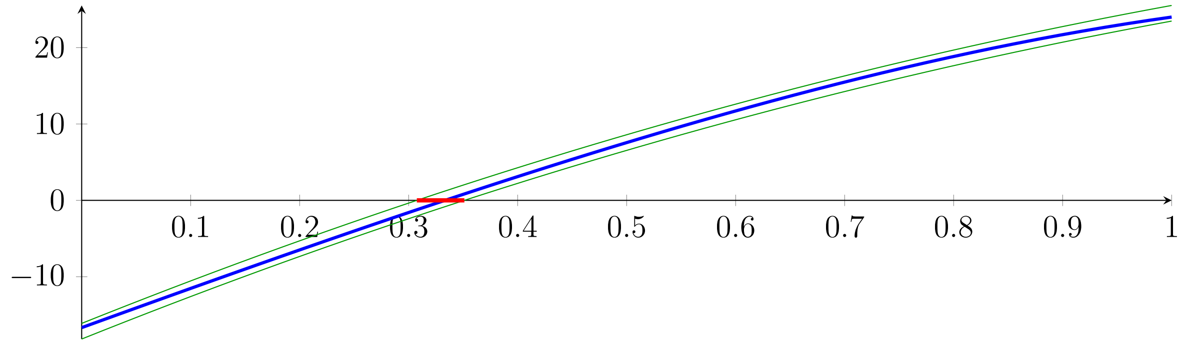
$$M = -15.5476X^2 + 57.181X - 16.1119$$

$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



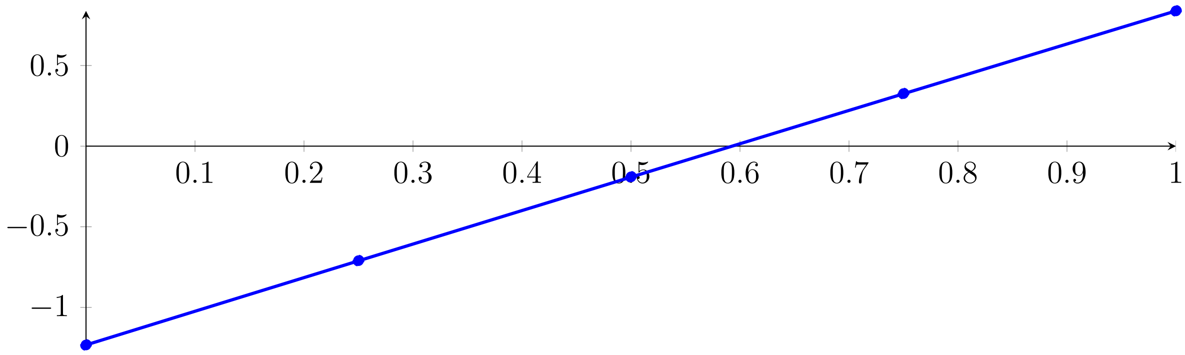
$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

203.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

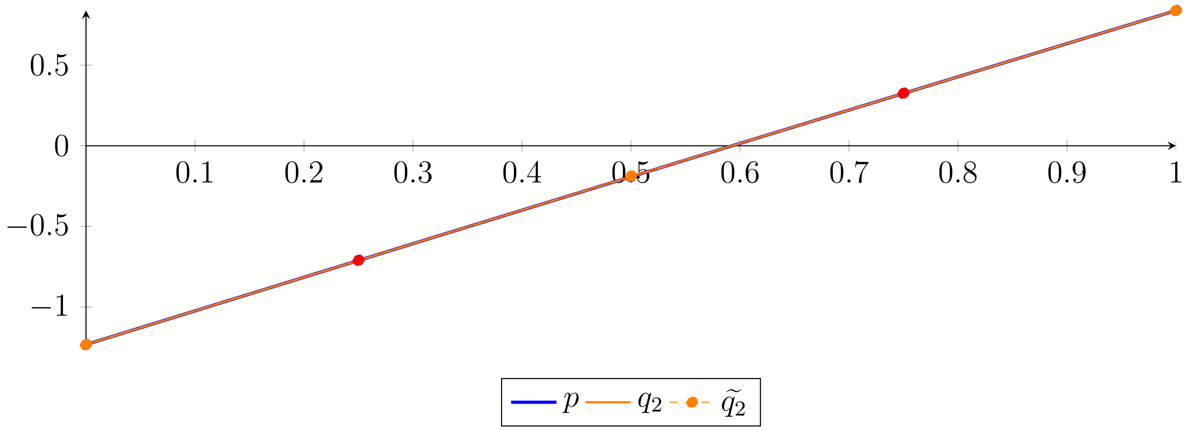
$$p = -3.62044 \cdot 10^{-06} X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278$$
$$= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -0.020089X^2 + 2.09166X - 1.23281$$
$$= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2}$$

$$\widetilde{q}_2 = -4.5614 \cdot 10^{-307} X^4 + 1.02353 \cdot 10^{-306} X^3 - 0.020089X^2 + 2.09166X - 1.23281$$
$$= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4}$$



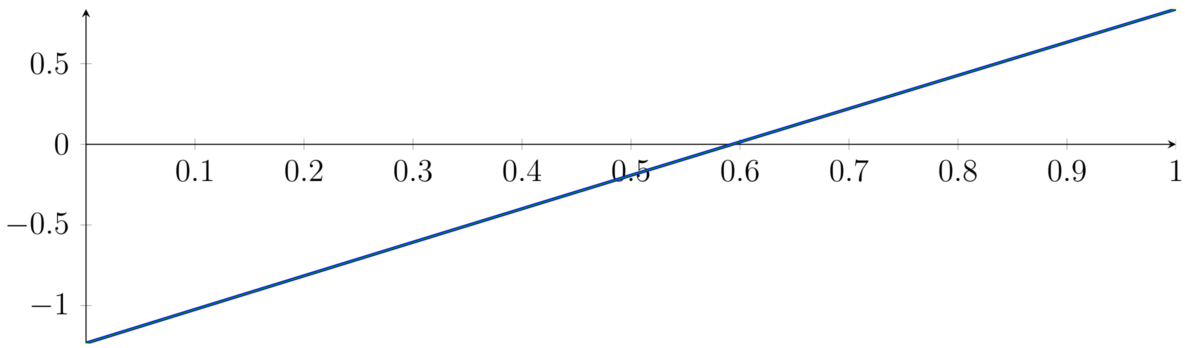
The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.
Bounding polynomials M and m:

$$M = -0.020089X^2 + 2.09166X - 1.23274$$
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



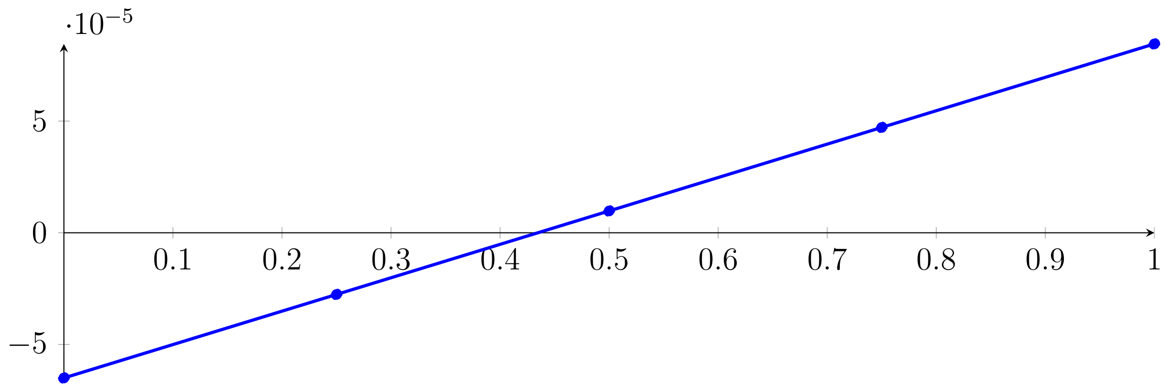
$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

203.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

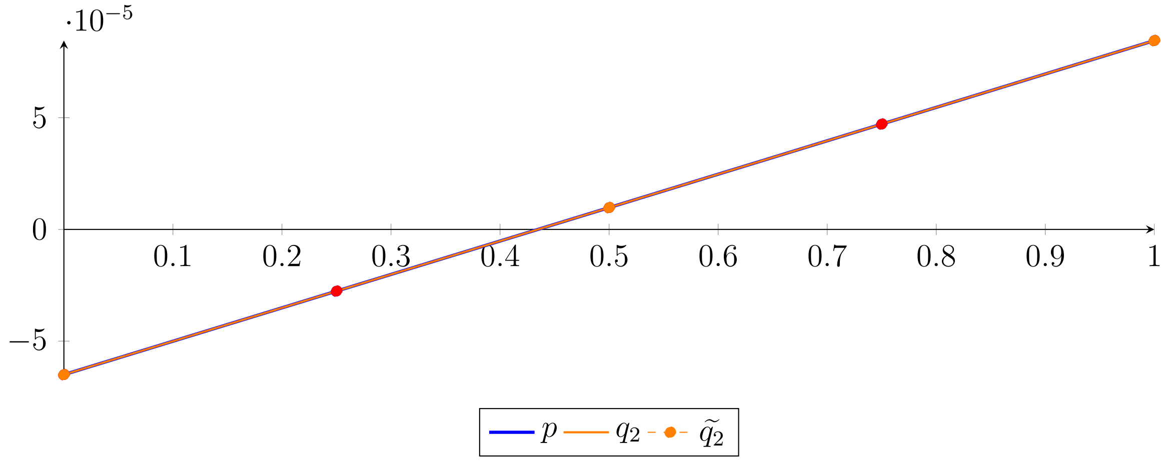
$$p = -9.9027 \cdot 10^{-23}X^4 - 2.82525 \cdot 10^{-16}X^3 - 1.06146 \cdot 10^{-10}X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05}B_{0,4}(X) - 2.76196 \cdot 10^{-05}B_{1,4}(X) + 9.76777$$
$$\cdot 10^{-06}B_{2,4}(X) + 4.71551 \cdot 10^{-05}B_{3,4}(X) + 8.45424 \cdot 10^{-05}B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.06147\cdot10^{-10}X^2 + 0.000149549X - 6.50069\cdot10^{-05}$$
$$= -6.50069\cdot10^{-05}B_{0,2} + 9.76779\cdot10^{-06}B_{1,2} + 8.45424\cdot10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = 1.22227\cdot10^{-311}X^4 - 1.62969\cdot10^{-311}X^3 - 1.06147\cdot10^{-10}X^2 + 0.000149549X - 6.50069\cdot10^{-05}$$
$$= -6.50069\cdot10^{-05}B_{0,4} - 2.76196\cdot10^{-05}B_{1,4} + 9.76777\cdot10^{-06}B_{2,4} + 4.71551\cdot10^{-05}B_{3,4} + 8.45424\cdot10^{-05}B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.82526\cdot10^{-17}$.

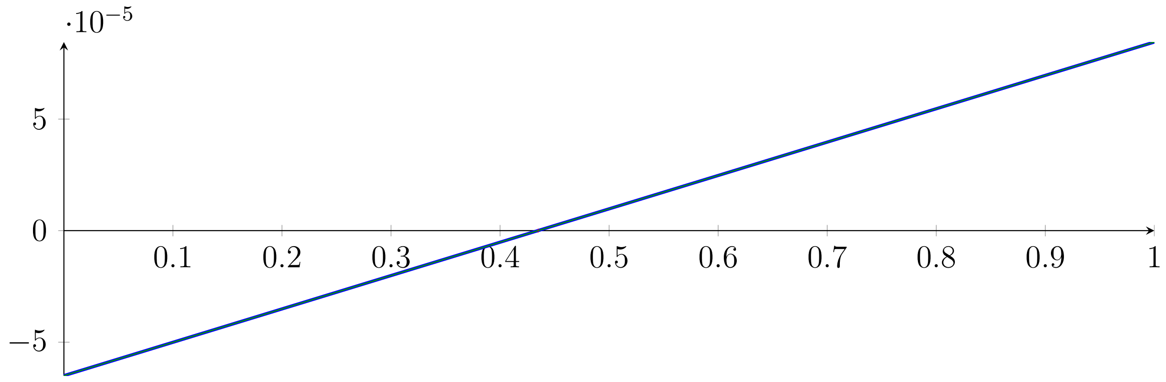
Bounding polynomials M and m:

$$M = -1.06147\cdot10^{-10}X^2 + 0.000149549X - 6.50069\cdot10^{-05}$$
$$m = -1.06147\cdot10^{-10}X^2 + 0.000149549X - 6.50069\cdot10^{-05}$$

Root of M and m:

$$N(M) = \{0.434685, 1.4089\cdot10^{6}\} \qquad N(m) = \{0.434685, 1.4089\cdot10^{6}\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

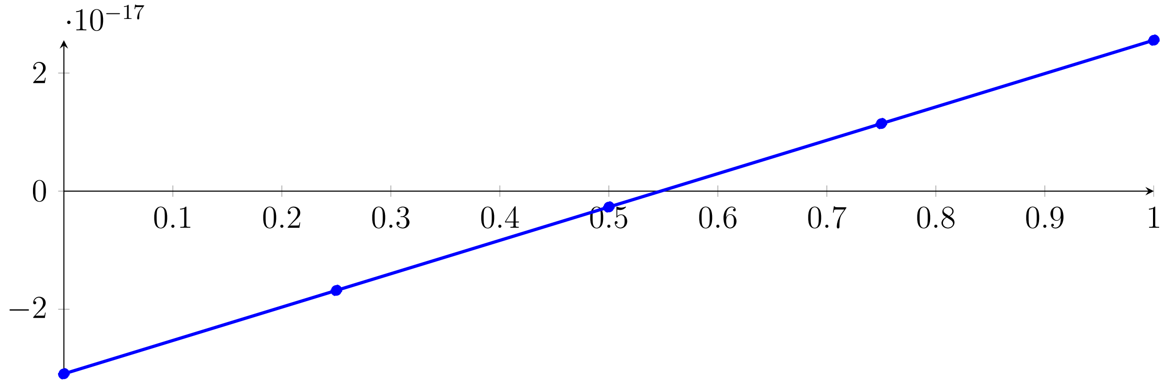
Longest intersection interval: $3.77836\cdot10^{-13}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

203.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

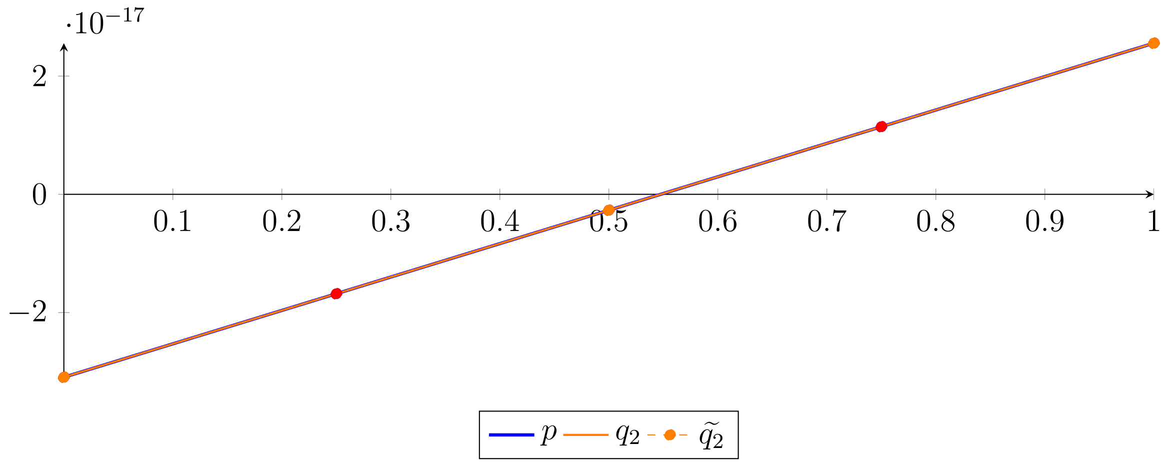
$$p = -2.01821\cdot 10^{-72}X^4 - 1.52394\cdot 10^{-53}X^3 - 1.51535\cdot 10^{-35}X^2 + 5.65051\cdot 10^{-17}X - 3.09418\cdot 10^{-17}$$
$$= -3.09418\cdot 10^{-17}B_{0,4}(X) - 1.68155\cdot 10^{-17}B_{1,4}(X) - 2.68924 \cdot 10^{-18}B_{2,4}(X) + 1.1437\cdot 10^{-17}B_{3,4}(X) + 2.55633\cdot 10^{-17}B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.51535\cdot 10^{-35}X^2 + 5.65051\cdot 10^{-17}X - 3.09418\cdot 10^{-17}$$
$$= -3.09418\cdot 10^{-17}B_{0,2} - 2.68924\cdot 10^{-18}B_{1,2} + 2.55633\cdot 10^{-17}B_{2,2}$$

$$\widetilde{q}_2 = -7.41098\cdot 10^{-324}X^4 + 1.72923\cdot 10^{-323}X^3 - 1.51535\cdot 10^{-35}X^2 + 5.65051\cdot 10^{-17}X - 3.09418\cdot 10^{-17}$$
$$= -3.09418\cdot 10^{-17}B_{0,4} - 1.68155\cdot 10^{-17}B_{1,4} - 2.68924\cdot 10^{-18}B_{2,4} + 1.1437\cdot 10^{-17}B_{3,4} + 2.55633\cdot 10^{-17}B_{4,4}$$



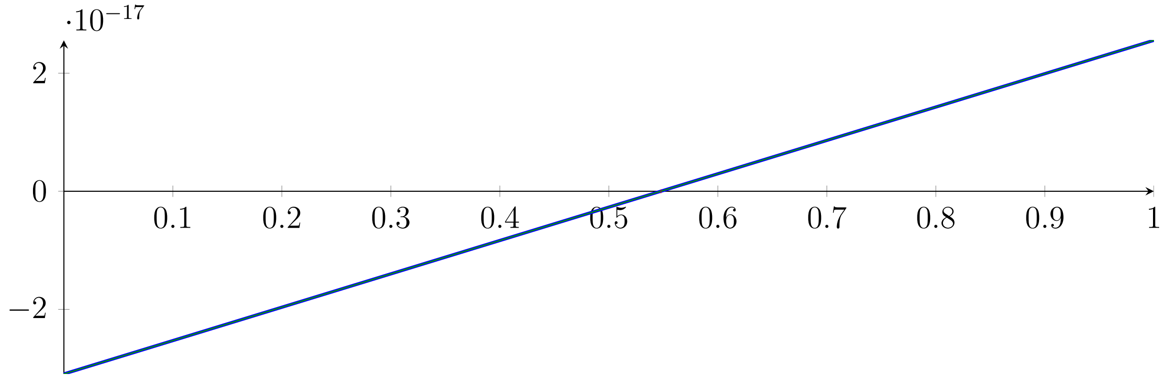
The maximum difference of the Bézier coefficients is $\delta = 1.52394\cdot 10^{-54}$.
Bounding polynomials M and m:

$$M = -1.51535\cdot 10^{-35}X^2 + 5.65051\cdot 10^{-17}X - 3.09418\cdot 10^{-17}$$
$$m = -1.51535\cdot 10^{-35}X^2 + 5.65051\cdot 10^{-17}X - 3.09418\cdot 10^{-17}$$

Root of M and m:

$$N(M) = \{0.547593, 3.72886\cdot 10^{18}\} \qquad N(m) = \{0.547593, 3.72886\cdot 10^{18}\}$$

Intersection intervals:



$[0.547593, 0.547593]$

Longest intersection interval: $5.39398 \cdot 10^{-38}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

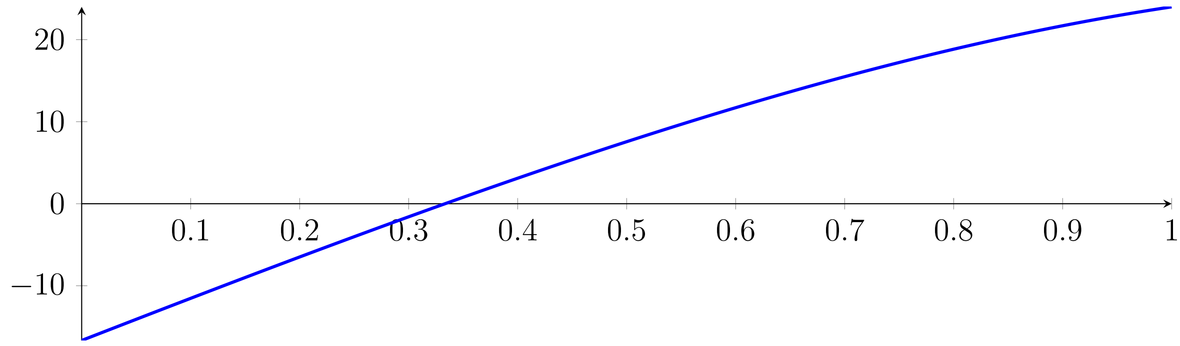
203.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

203.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

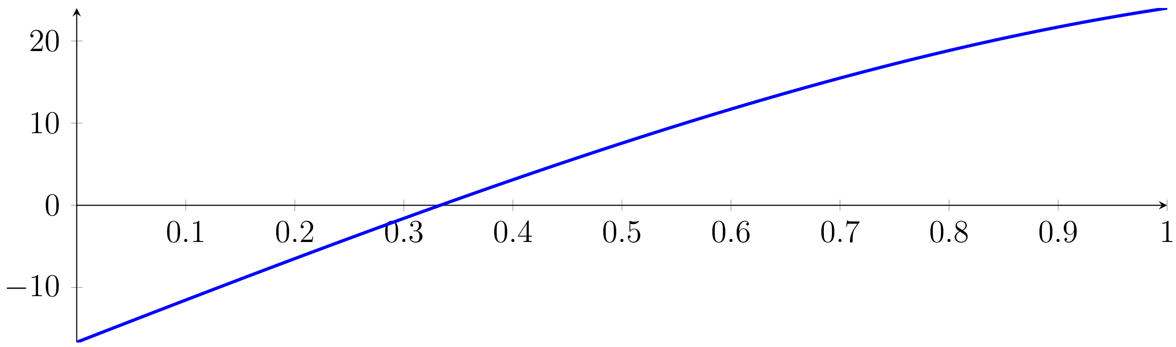
with precision $\varepsilon = 1 \cdot 10^{-32}$.

204 Running CubeClip on f_4 with epsilon 32

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0,1]$:

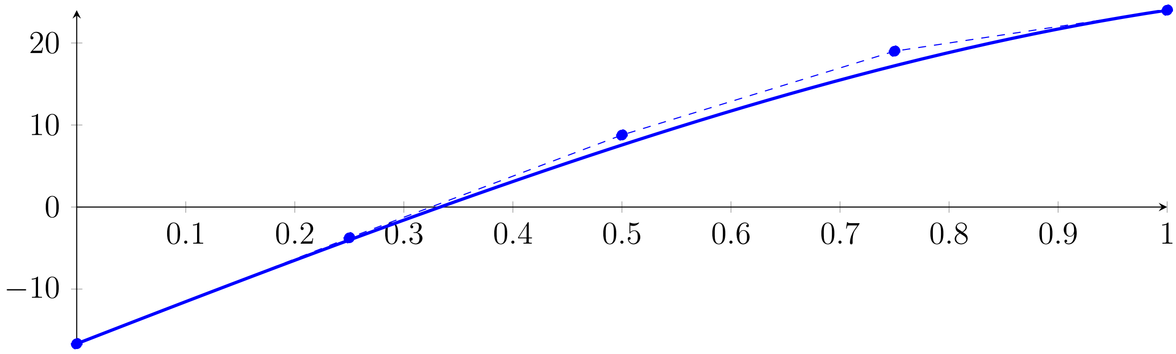
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



204.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

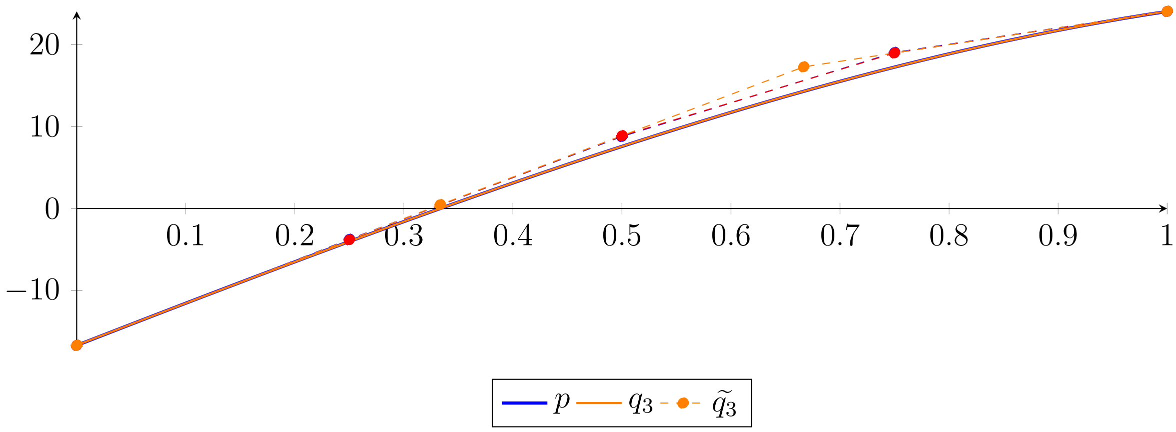
$$\begin{aligned}p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667\\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524\\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3}\end{aligned}$$

$$\begin{aligned}\widetilde{q_3} &= -3.56012{\cdot}10^{-307}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524\\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4}\end{aligned}$$



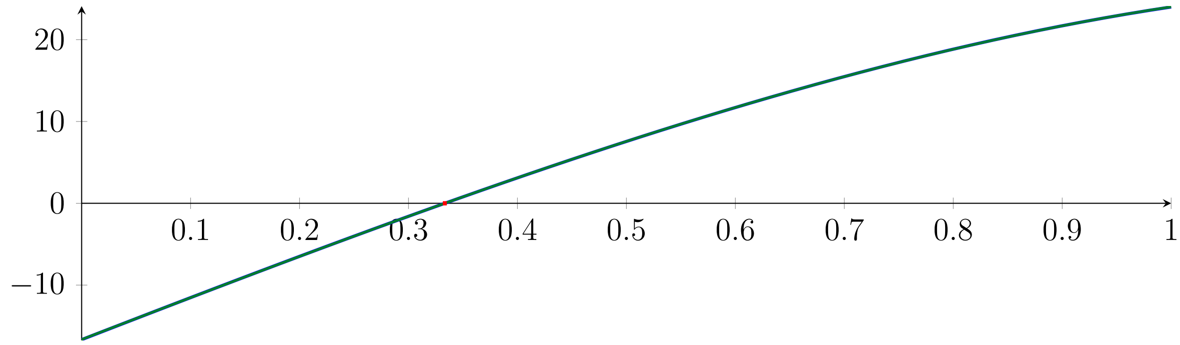
The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



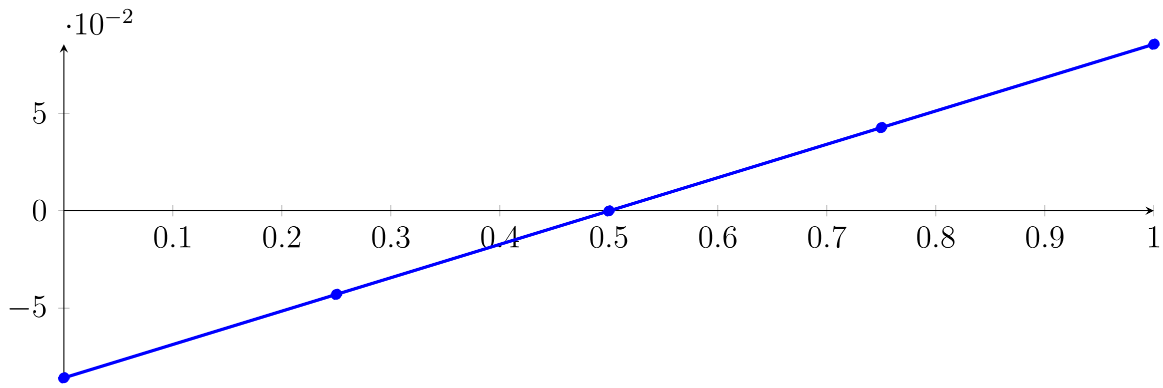
$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

204.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

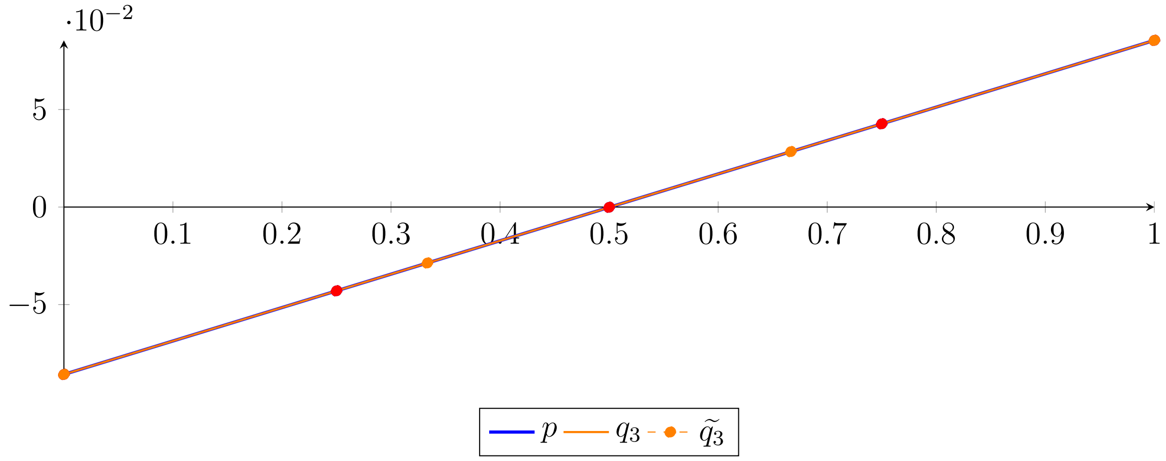
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1.39067 \cdot 10^{-309} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.
Bounding polynomials M and m:

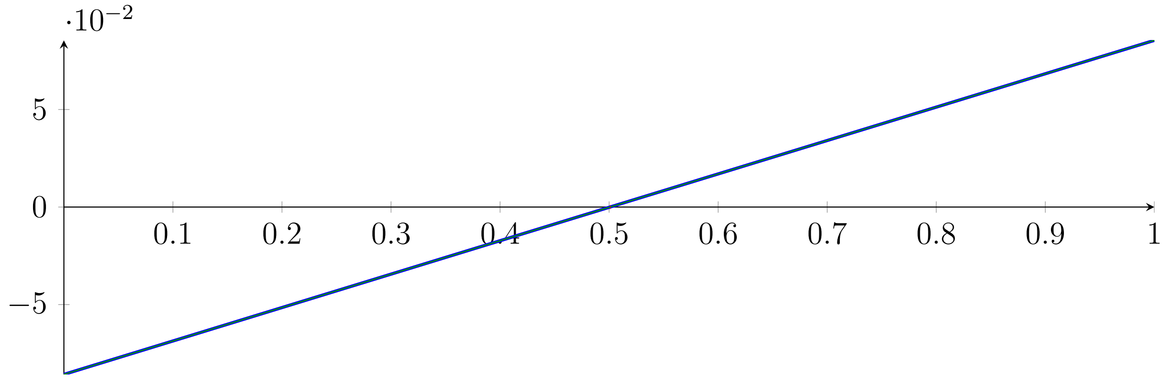
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



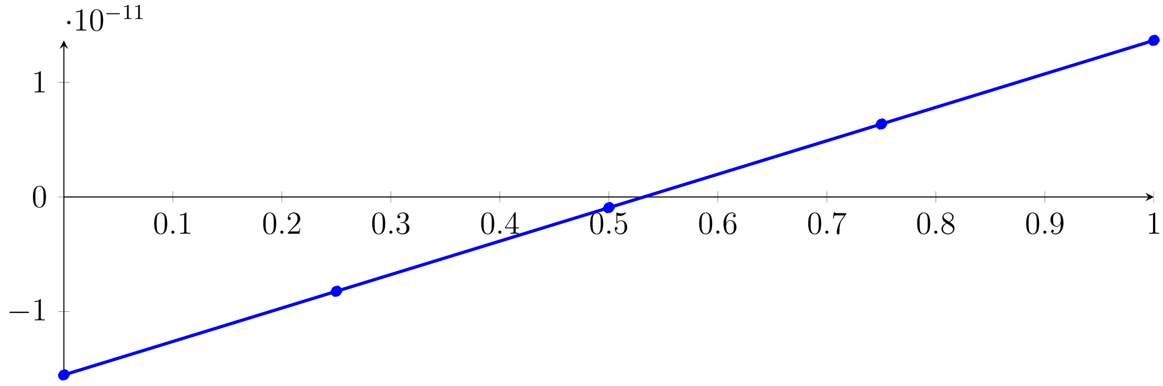
$$[0.500825, 0.500825]$$

Longest intersection interval: $1.7041 \cdot 10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

204.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

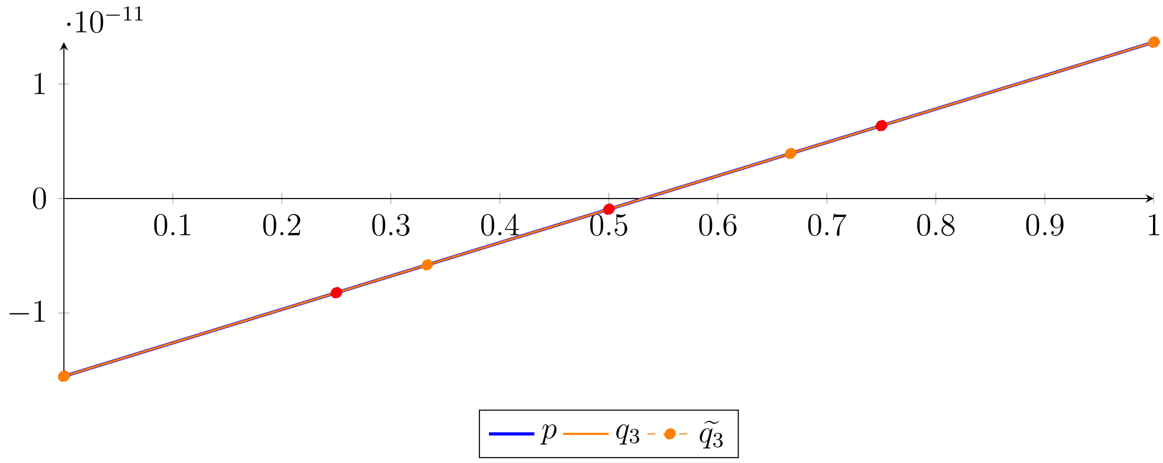
Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.43544 \cdot 10^{-49} X^4 - 2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}$$
$$= -1.55233 \cdot 10^{-11} B_{0,4}(X) - 8.22817 \cdot 10^{-12} B_{1,4}(X) - 9.33052 \cdot 10^{-13} B_{2,4}(X) + 6.36207 \cdot 10^{-12} B_{3,4}(X) + 1.36572 \cdot 10^{-11} B_{4,4}(X)$$



Degree reduction and raising:

$$q_3 = -2.09885\cdot 10^{-36} X^3 - 4.0413\cdot 10^{-24} X^2 + 2.91805\cdot 10^{-11} X - 1.55233\cdot 10^{-11}$$
$$= -1.55233\cdot 10^{-11} B_{0,3} - 5.79646\cdot 10^{-12} B_{1,3} + 3.93036\cdot 10^{-12} B_{2,3} + 1.36572\cdot 10^{-11} B_{3,3}$$
$$\widetilde{q}_3 = -3.23791\cdot 10^{-319} X^4 - 2.09885\cdot 10^{-36} X^3 - 4.0413\cdot 10^{-24} X^2 + 2.91805\cdot 10^{-11} X - 1.55233\cdot 10^{-11}$$
$$= -1.55233\cdot 10^{-11} B_{0,4} - 8.22817\cdot 10^{-12} B_{1,4} - 9.33052\cdot 10^{-13} B_{2,4} + 6.36207\cdot 10^{-12} B_{3,4} + 1.36572\cdot 10^{-11} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 1.23038\cdot 10^{-50}$.

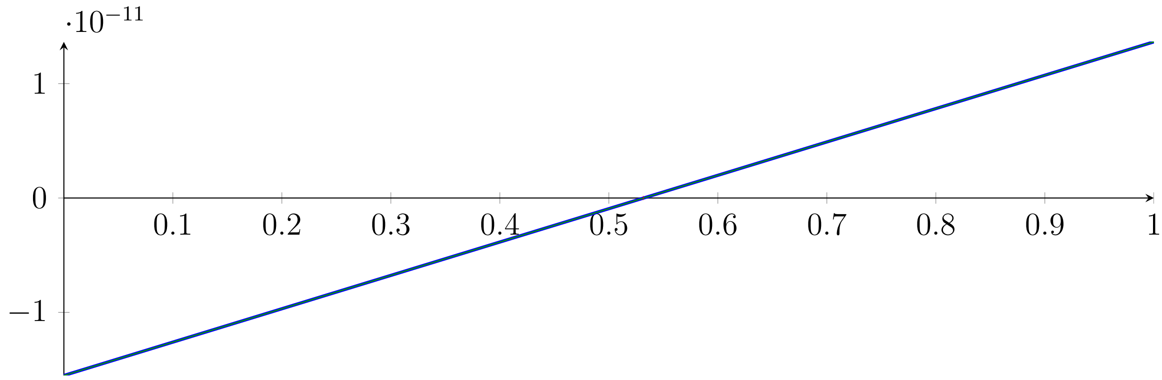
Bounding polynomials M and m:

$$M = -2.09885\cdot 10^{-36} X^3 - 4.0413\cdot 10^{-24} X^2 + 2.91805\cdot 10^{-11} X - 1.55233\cdot 10^{-11}$$
$$m = -2.09885\cdot 10^{-36} X^3 - 4.0413\cdot 10^{-24} X^2 + 2.91805\cdot 10^{-11} X - 1.55233\cdot 10^{-11}$$

Root of M and m:

$$N(M) = \{-4.81371\cdot 10^{12}, 0.531249, 2.88823\cdot 10^{12}\} \quad N(m) = \{-4.81371\cdot 10^{12}, 0.531249, 2.88823\cdot 10^{12}\}$$

Intersection intervals:



$$[0.531249, 0.531249]$$

Longest intersection interval: $8.43287\cdot 10^{-40}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

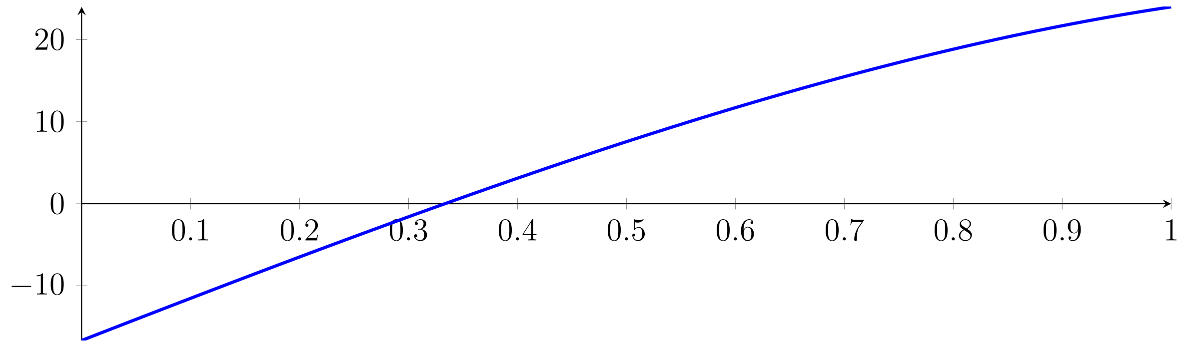
204.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 4!
p(0) = -2.11876e-14 - p(1) -2.11876e-14

204.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

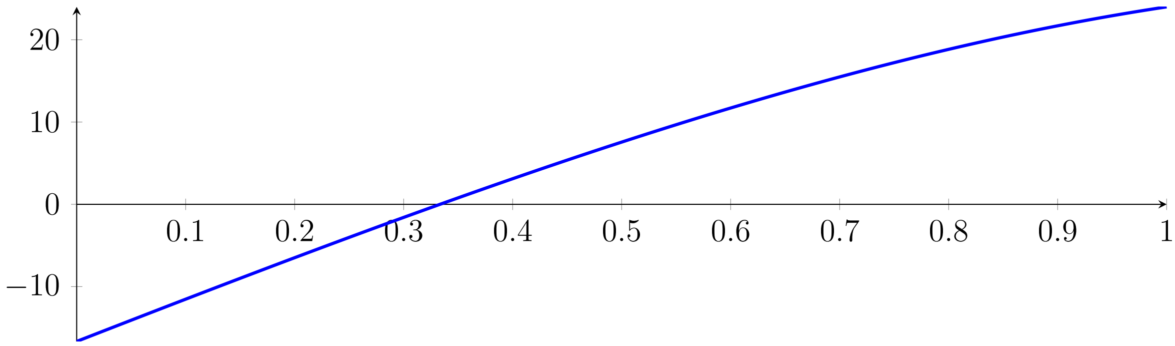
with precision $\varepsilon = 1 \cdot 10^{-32}$.

205 Running BezClip on f_4 with epsilon 64

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

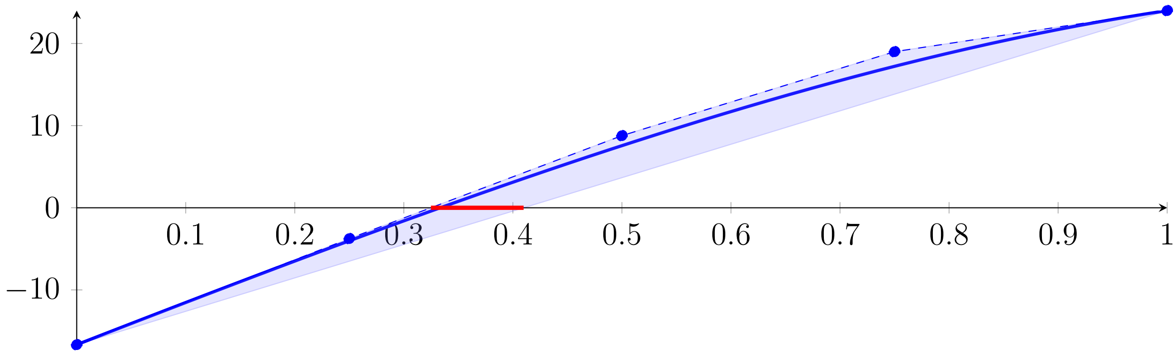
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



205.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$
$$= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

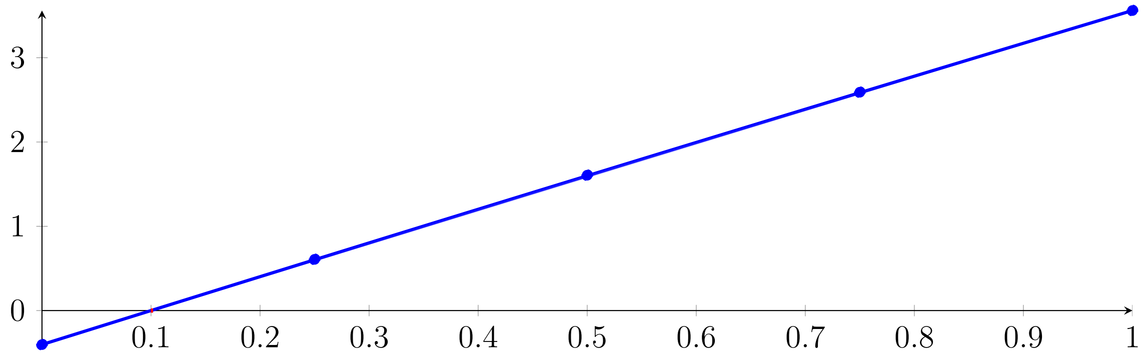
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

205.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711$$
$$= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

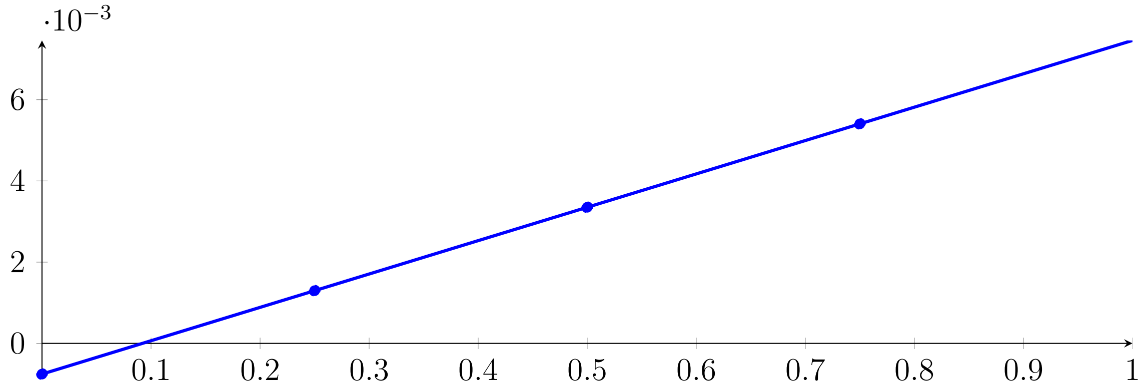
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: [0.333317, 0.333491],

205.3 Recursion Branch 1 1 1 in Interval 1: [0.333317, 0.333491]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01975 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

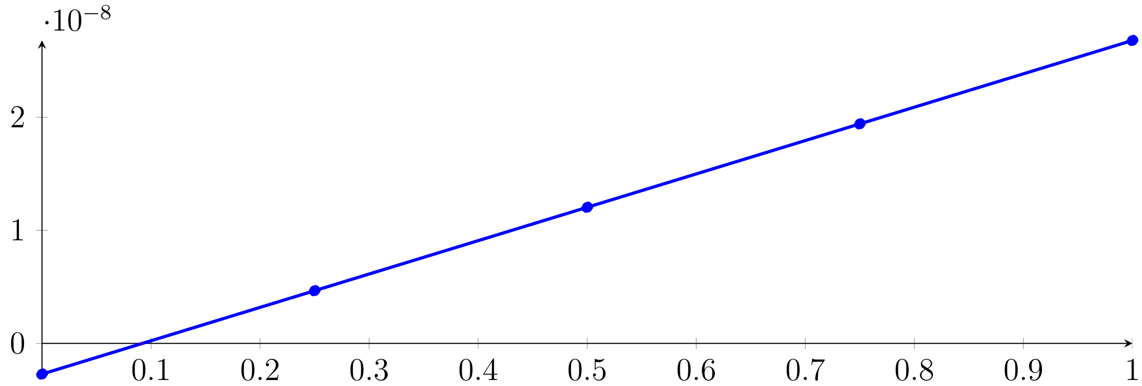
$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: [0.333333, 0.333333],

205.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.50129\cdot 10^{-37}X^4 - 2.17066\cdot 10^{-27}X^3 - 4.13296\cdot 10^{-18}X^2 + 2.95096\cdot 10^{-08}X - 2.71749\cdot 10^{-09}$$
$$= -2.71749\cdot 10^{-09}B_{0,4}(X) + 4.6599\cdot 10^{-09}B_{1,4}(X) + 1.20373 \cdot 10^{-08}B_{2,4}(X) + 1.94147\cdot 10^{-08}B_{3,4}(X) + 2.67921\cdot 10^{-08}B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

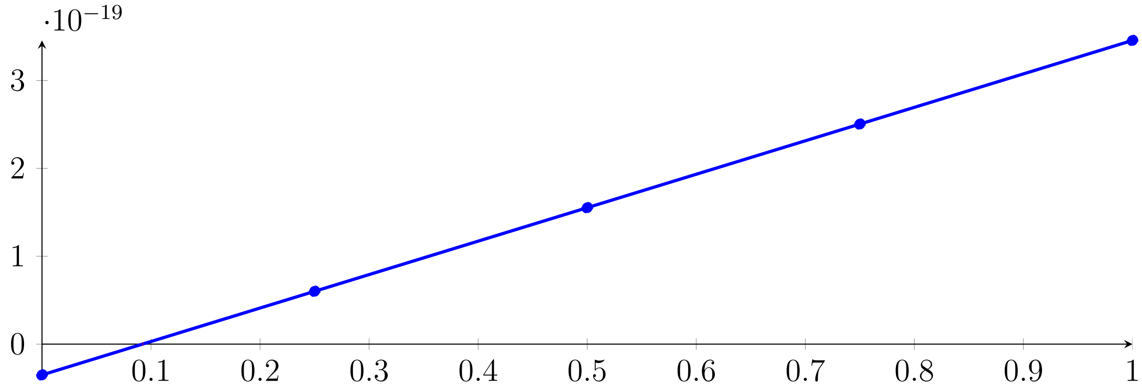
$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28975\cdot 10^{-11}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

205.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -4.15417\cdot 10^{-81}X^4 - 4.65699\cdot 10^{-60}X^3 - 6.87497\cdot 10^{-40}X^2 + 3.80599\cdot 10^{-19}X - 3.50488\cdot 10^{-20}$$
$$= -3.50488\cdot 10^{-20}B_{0,4}(X) + 6.01009\cdot 10^{-20}B_{1,4}(X) + 1.55251 \cdot 10^{-19}B_{2,4}(X) + 2.504\cdot 10^{-19}B_{3,4}(X) + 3.4555\cdot 10^{-19}B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

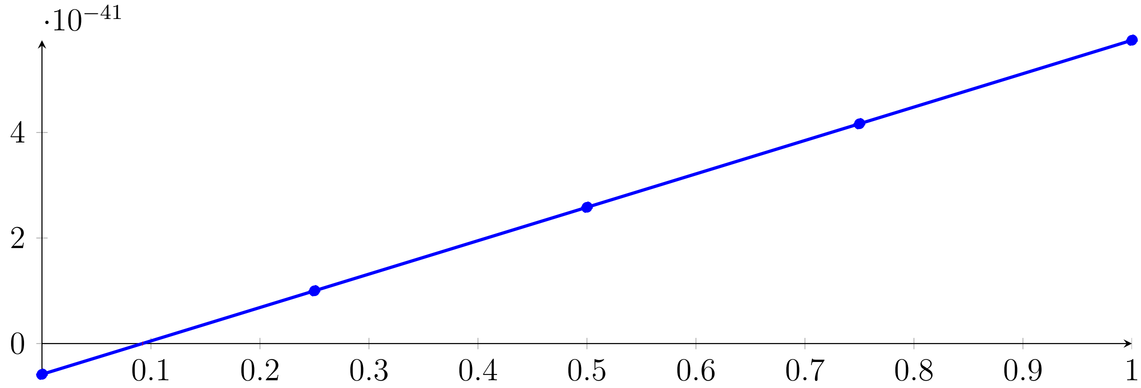
$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.66345\cdot 10^{-22}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

205.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.18068\cdot10^{-168}X^4 - 2.14355\cdot10^{-125}X^3 - 1.90234\cdot10^{-83}X^2 + 6.33106\cdot10^{-41}X - 5.83018\cdot10^{-42}$$
$$= -5.83018\cdot10^{-42}B_{0,4}(X) + 9.99747\cdot10^{-42}B_{1,4}(X) + 2.58251$$
$$\cdot10^{-41}B_{2,4}(X) + 4.16528\cdot10^{-41}B_{3,4}(X) + 5.74804\cdot10^{-41}B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $2.76706\cdot10^{-44}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

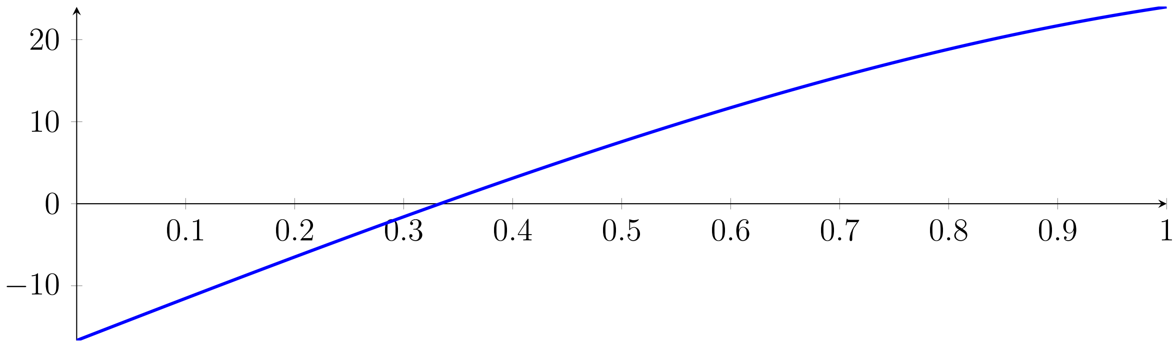
205.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 7!

205.8 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

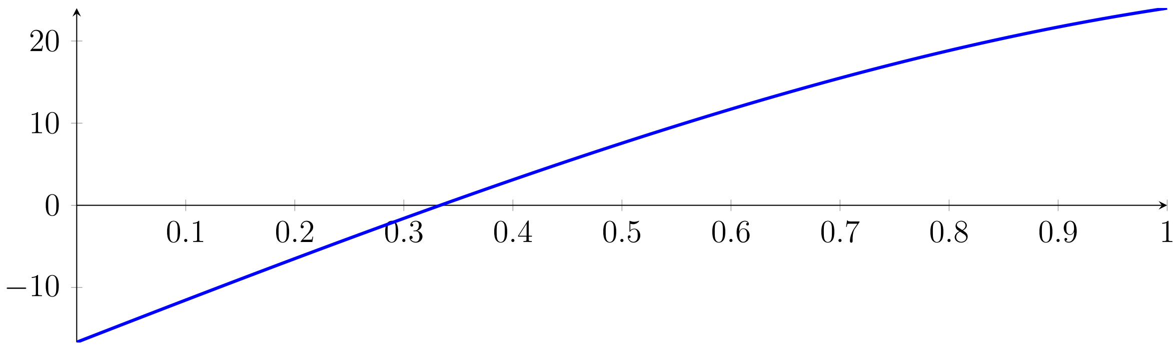
with precision $\varepsilon = 1 \cdot 10^{-64}$.

206 Running QuadClip on f_4 with epsilon 64

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

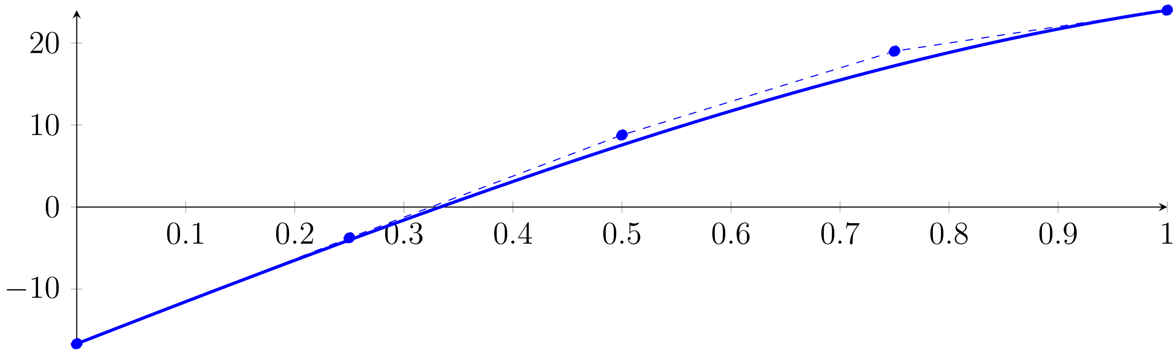
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



206.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

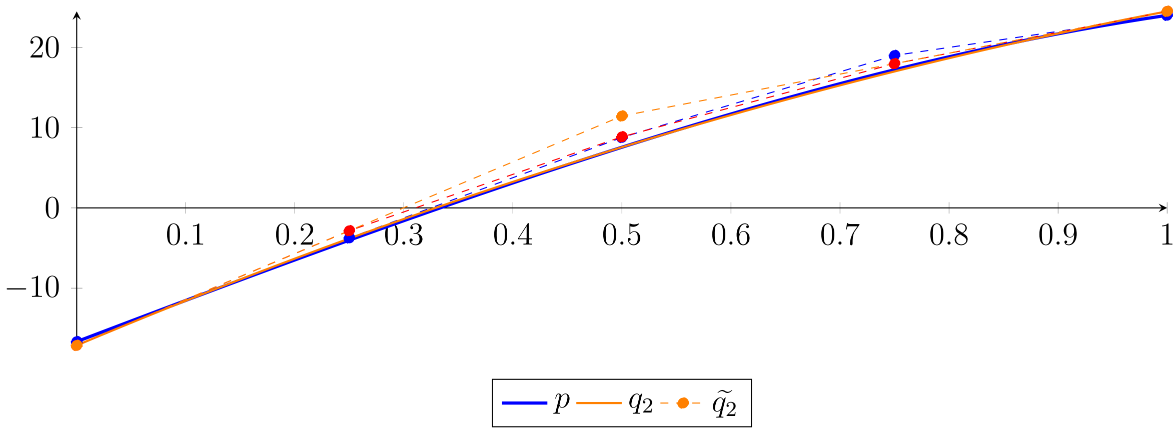
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -2.13607 \cdot 10^{-306}X^4 + 7.12024 \cdot 10^{-306}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

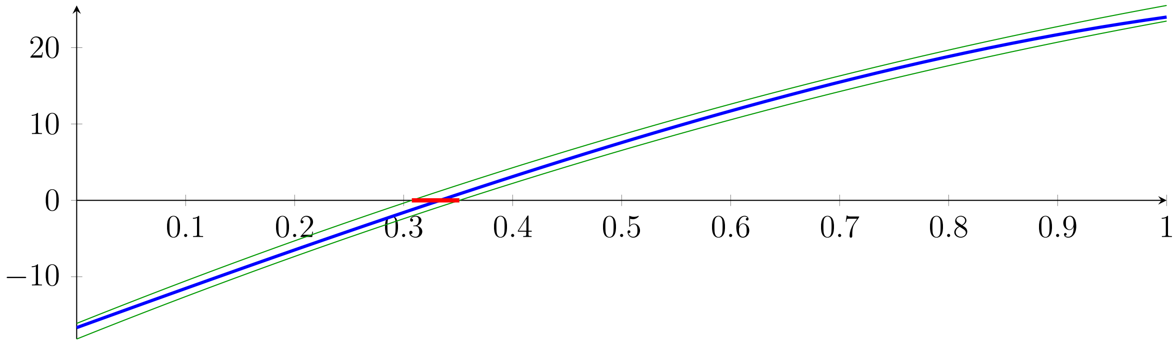
$$M = -15.5476X^2 + 57.181X - 16.1119$$

$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



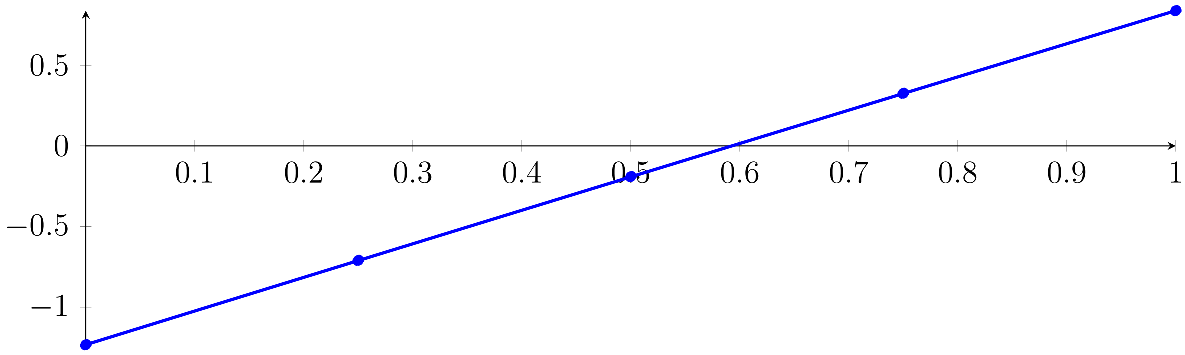
$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

206.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

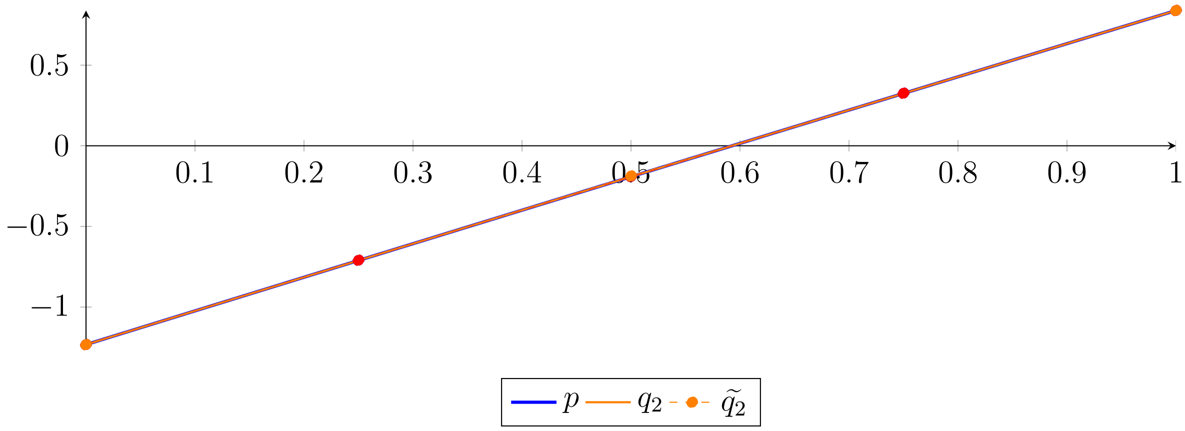
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -4.5614 \cdot 10^{-307} X^4 + 1.02353 \cdot 10^{-306} X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

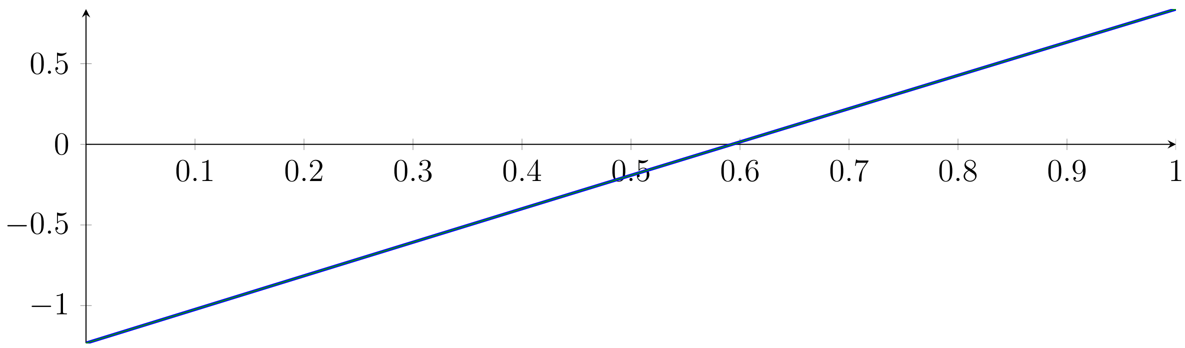
Bounding polynomials M and m:

$$M = -0.020089X^2 + 2.09166X - 1.23274$$
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

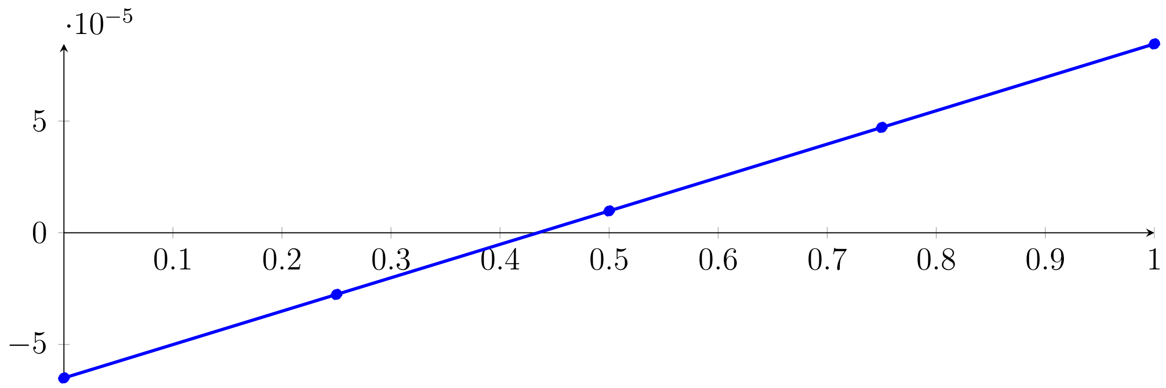
Longest intersection interval: $7.23183 \cdot 10^{-05}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

206.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

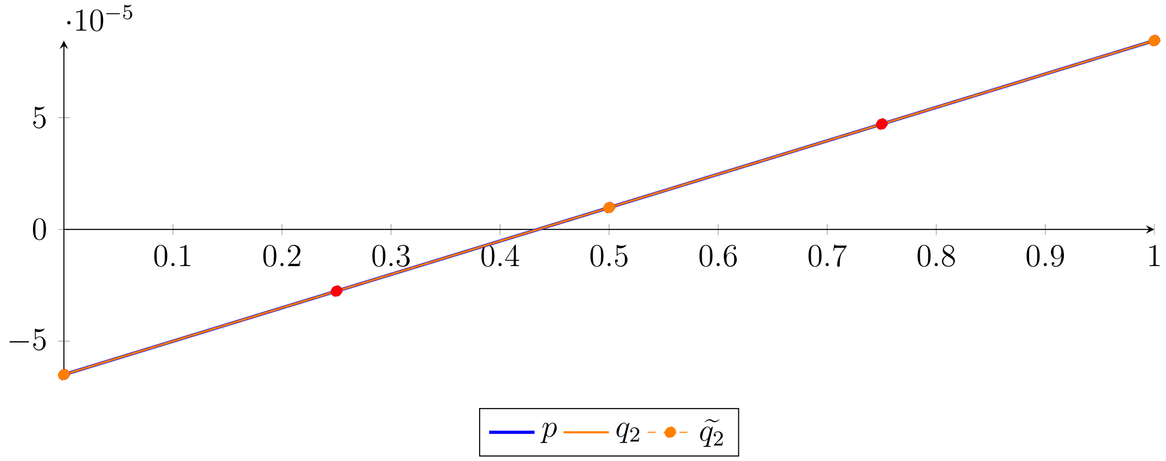
$$p = -9.9027 \cdot 10^{-23}X^4 - 2.82525 \cdot 10^{-16}X^3 - 1.06146 \cdot 10^{-10}X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05}B_{0,4}(X) - 2.76196 \cdot 10^{-05}B_{1,4}(X) + 9.76777 \cdot 10^{-06}B_{2,4}(X) + 4.71551 \cdot 10^{-05}B_{3,4}(X) + 8.45424 \cdot 10^{-05}B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.06147{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$
$$= -6.50069{\cdot}10^{-05}B_{0,2} + 9.76779{\cdot}10^{-06}B_{1,2} + 8.45424{\cdot}10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = 1.22227{\cdot}10^{-311}X^4 - 1.62969{\cdot}10^{-311}X^3 - 1.06147{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$
$$= -6.50069{\cdot}10^{-05}B_{0,4} - 2.76196{\cdot}10^{-05}B_{1,4} + 9.76777{\cdot}10^{-06}B_{2,4} + 4.71551{\cdot}10^{-05}B_{3,4} + 8.45424{\cdot}10^{-05}B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.82526{\cdot}10^{-17}$.

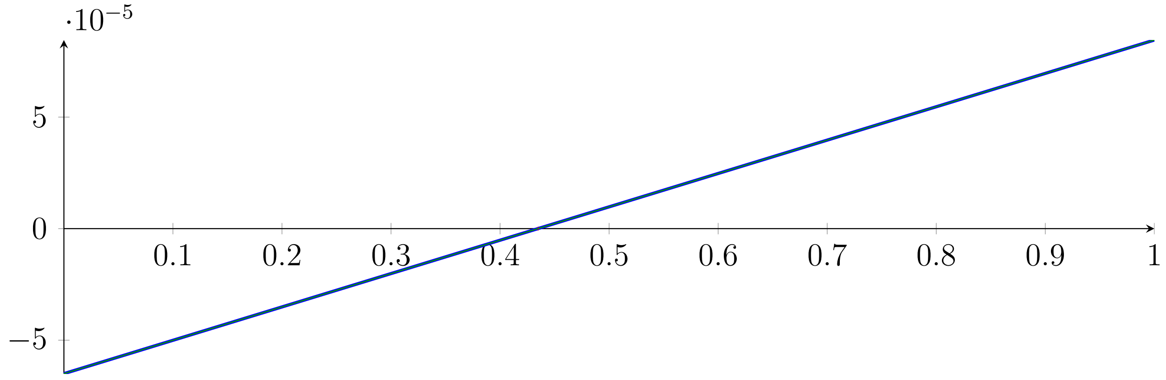
Bounding polynomials M and m:

$$M = -1.06147{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$
$$m = -1.06147{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{0.434685, 1.4089{\cdot}10^{6}\} \qquad N(m) = \{0.434685, 1.4089{\cdot}10^{6}\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

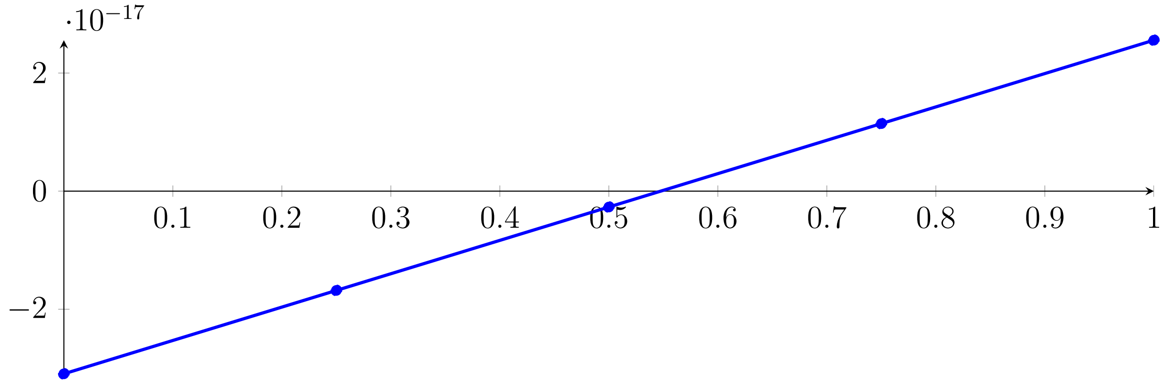
Longest intersection interval: $3.77836{\cdot}10^{-13}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

206.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

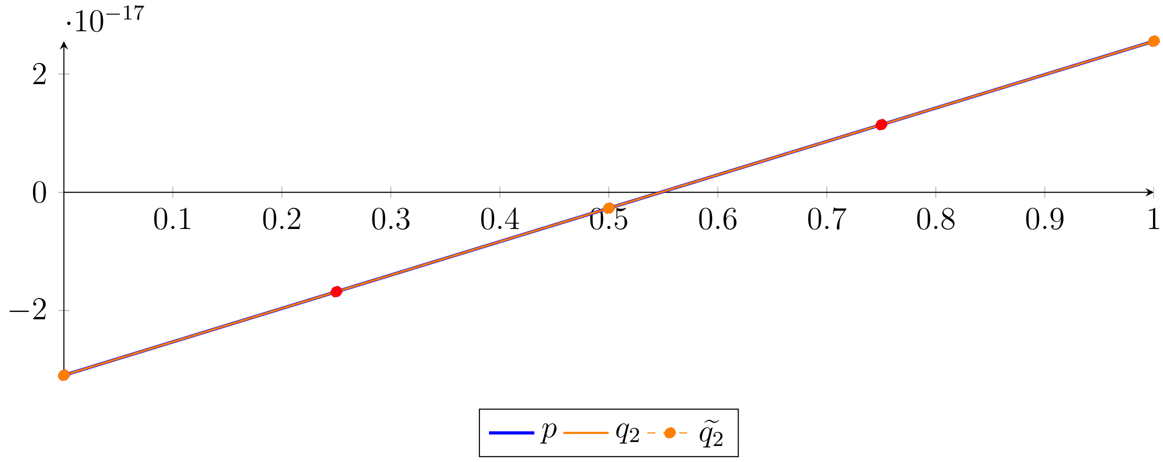
$$p = -2.01821 \cdot 10^{-72} X^4 - 1.52394 \cdot 10^{-53} X^3 - 1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17}$$
$$= -3.09418 \cdot 10^{-17} B_{0,4}(X) - 1.68155 \cdot 10^{-17} B_{1,4}(X) - 2.68924 \cdot 10^{-18} B_{2,4}(X) + 1.1437 \cdot 10^{-17} B_{3,4}(X) + 2.55633 \cdot 10^{-17} B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17}$$
$$= -3.09418 \cdot 10^{-17} B_{0,2} - 2.68924 \cdot 10^{-18} B_{1,2} + 2.55633 \cdot 10^{-17} B_{2,2}$$

$$\widetilde{q}_2 = -7.41098 \cdot 10^{-324} X^4 + 1.72923 \cdot 10^{-323} X^3 - 1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17}$$
$$= -3.09418 \cdot 10^{-17} B_{0,4} - 1.68155 \cdot 10^{-17} B_{1,4} - 2.68924 \cdot 10^{-18} B_{2,4} + 1.1437 \cdot 10^{-17} B_{3,4} + 2.55633 \cdot 10^{-17} B_{4,4}$$



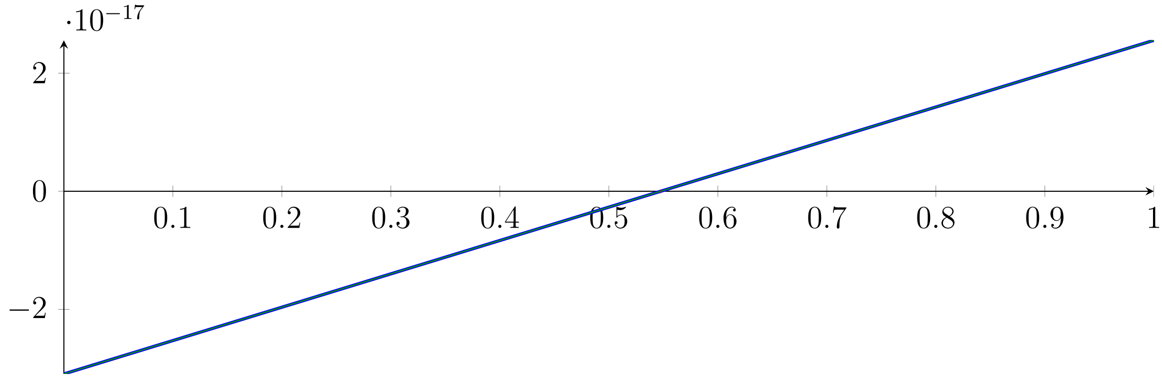
The maximum difference of the Bézier coefficients is $\delta = 1.52394 \cdot 10^{-54}$.
Bounding polynomials M and m:

$$M = -1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17}$$
$$m = -1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17}$$

Root of M and m:

$$N(M) = \{0.547593, 3.72886 \cdot 10^{18}\} \qquad N(m) = \{0.547593, 3.72886 \cdot 10^{18}\}$$

Intersection intervals:



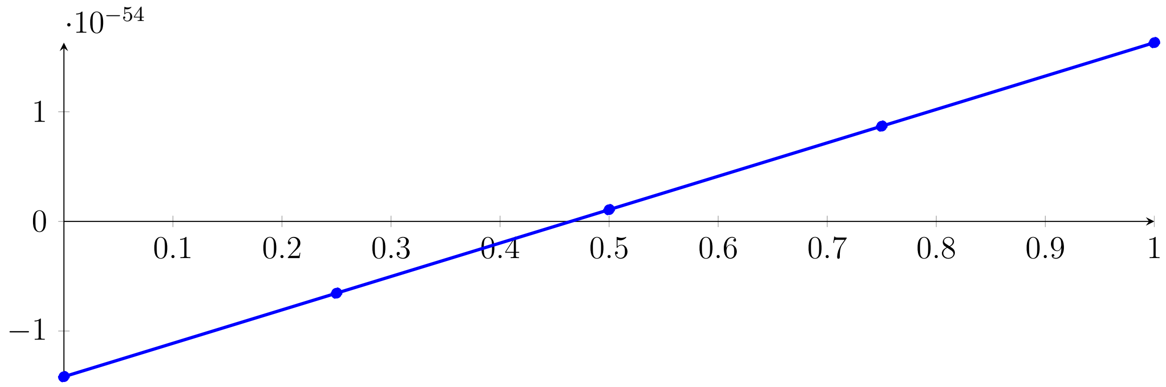
$[0.547593, 0.547593]$

Longest intersection interval: $5.39398 \cdot 10^{-38}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

206.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

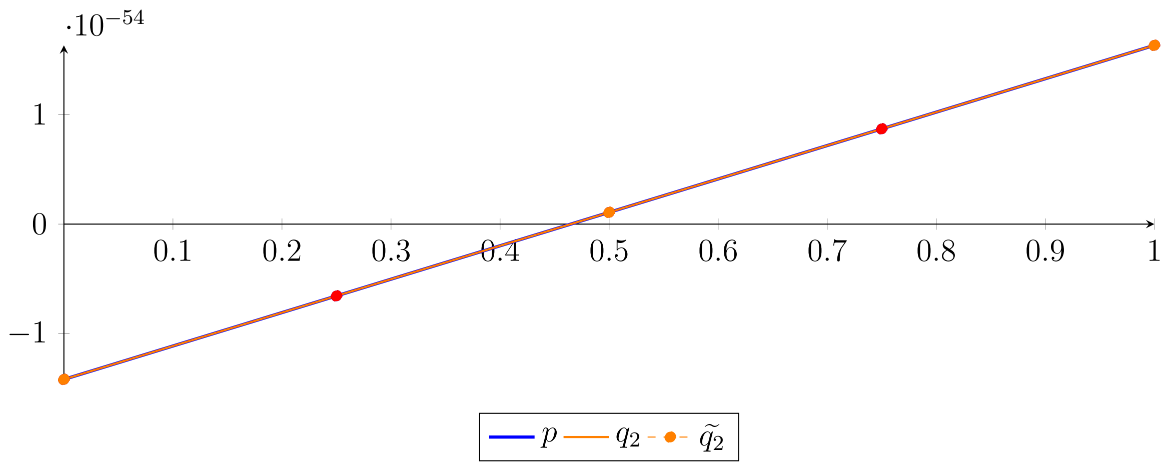
$$p = -1.70846 \cdot 10^{-221} X^4 - 2.39164 \cdot 10^{-165} X^3 - 4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54}$$
$$= -1.41679 \cdot 10^{-54} B_{0,4}(X) - 6.54819 \cdot 10^{-55} B_{1,4}(X) + 1.0715 \cdot 10^{-55} B_{2,4}(X) + 8.6912 \cdot 10^{-55} B_{3,4}(X) + 1.63109 \cdot 10^{-54} B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54}$$
$$= -1.41679 \cdot 10^{-54} B_{0,2} + 1.0715 \cdot 10^{-55} B_{1,2} + 1.63109 \cdot 10^{-54} B_{2,2}$$

$$\widetilde{q}_2 = 7.25964 \cdot 10^{-362} X^4 - 5.80771 \cdot 10^{-362} X^3 - 4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54}$$
$$= -1.41679 \cdot 10^{-54} B_{0,4} - 6.54819 \cdot 10^{-55} B_{1,4} + 1.0715 \cdot 10^{-55} B_{2,4} + 8.6912 \cdot 10^{-55} B_{3,4} + 1.63109 \cdot 10^{-54} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.39164 \cdot 10^{-166}$.
Bounding polynomials M and m:

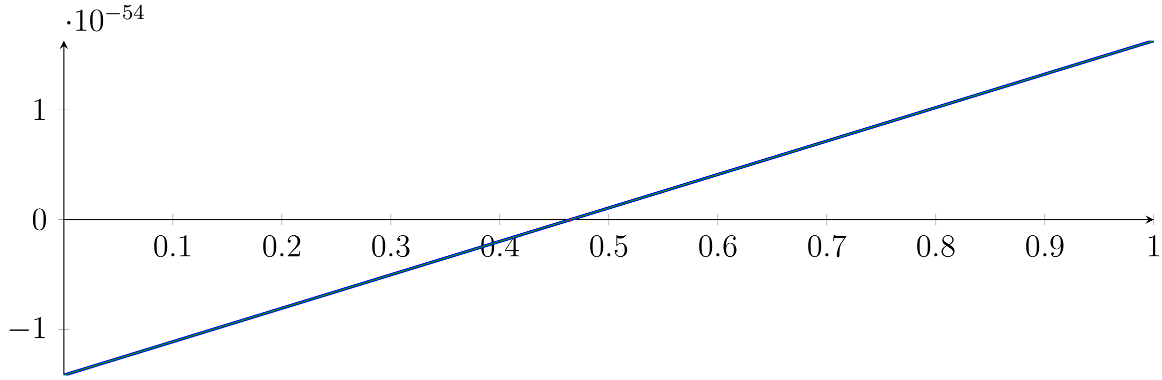
$$M = -4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54}$$

$$m = -4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54}$$

Root of M and m:

$$N(M) = \{0.464844, 6.91299 \cdot 10^{55}\} \qquad N(m) = \{0.464844, 6.91299 \cdot 10^{55}\}$$

Intersection intervals:



$$[0.464844, 0.464844]$$

Longest intersection interval: $1.56938 \cdot 10^{-112}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

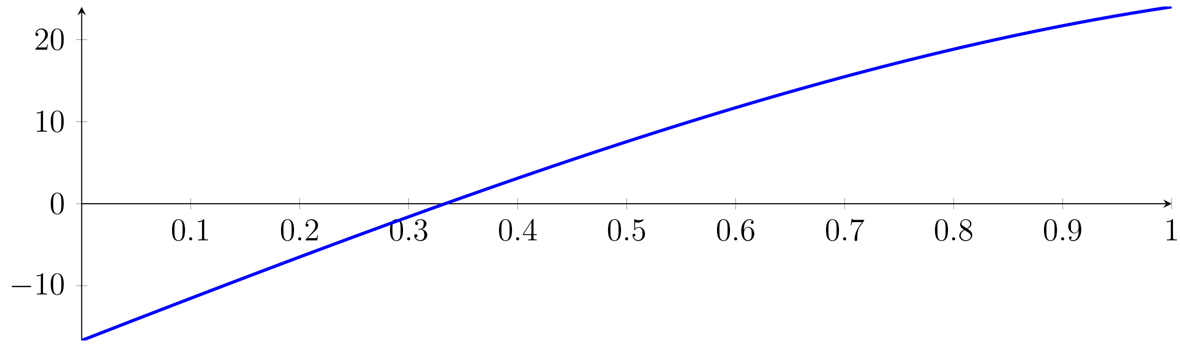
206.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

206.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-64}$.

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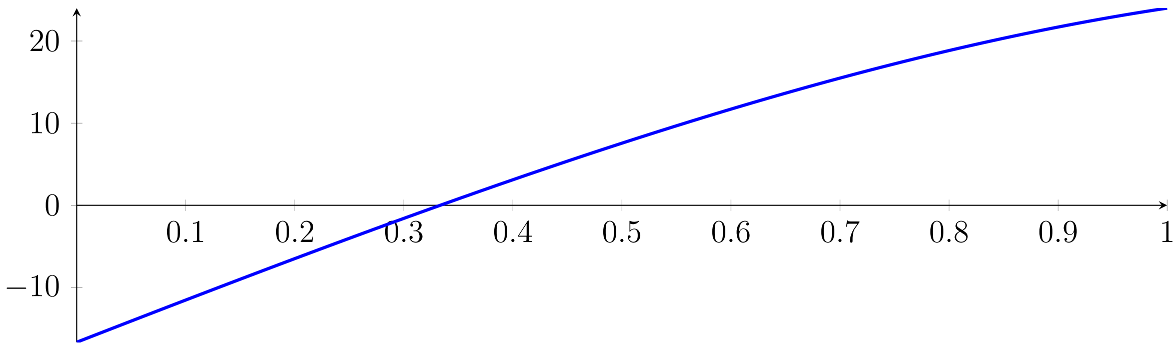
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0,1]$:

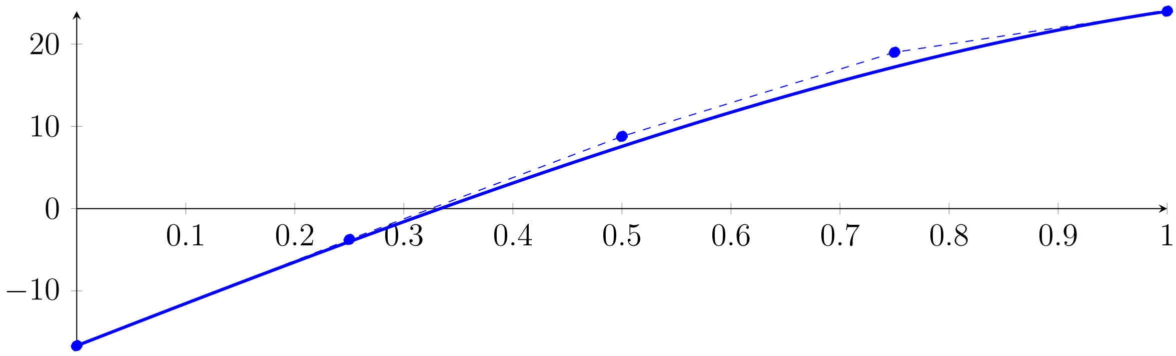
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



207.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

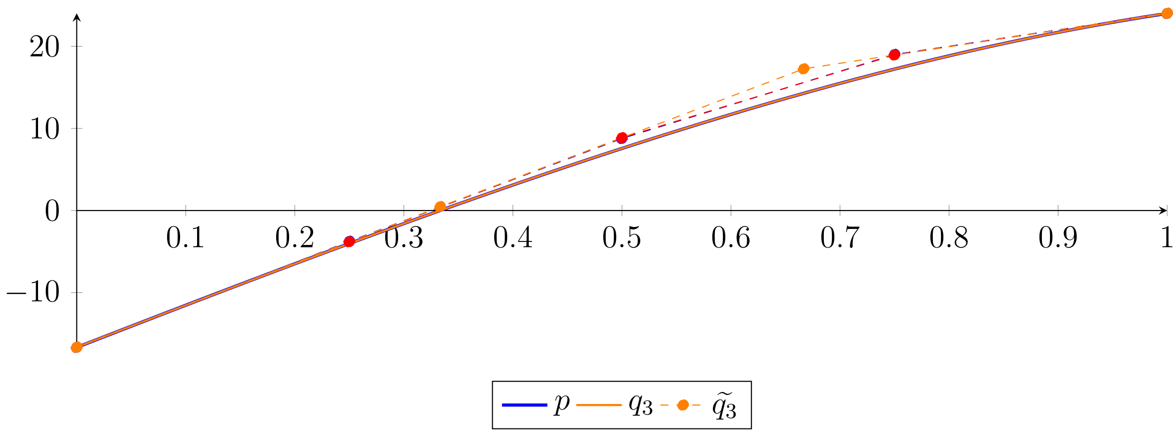
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -3.56012 \cdot 10^{-307}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

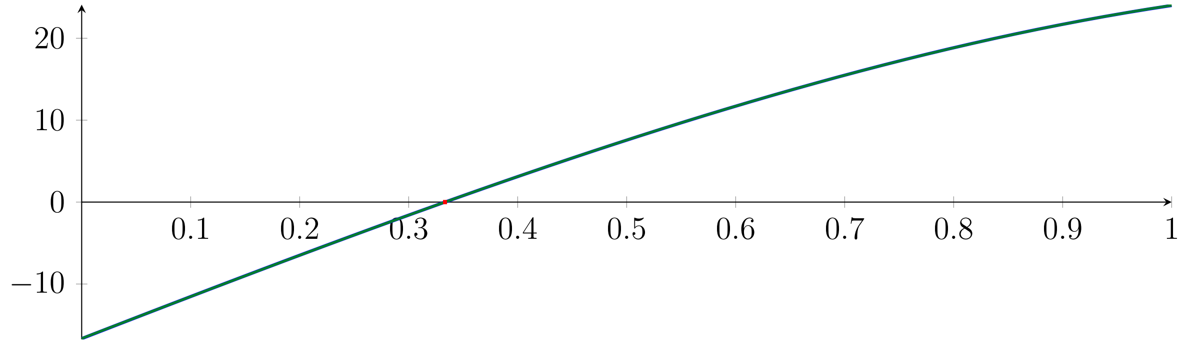
Bounding polynomials M and m:

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



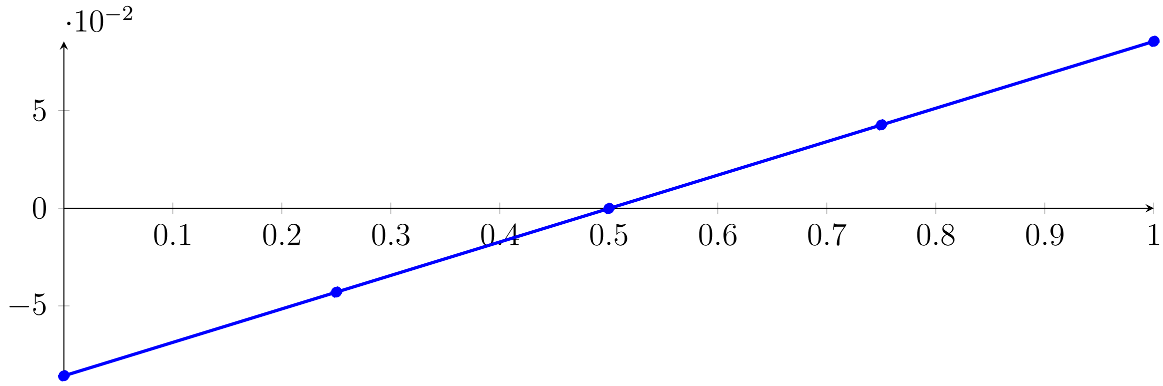
$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

207.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

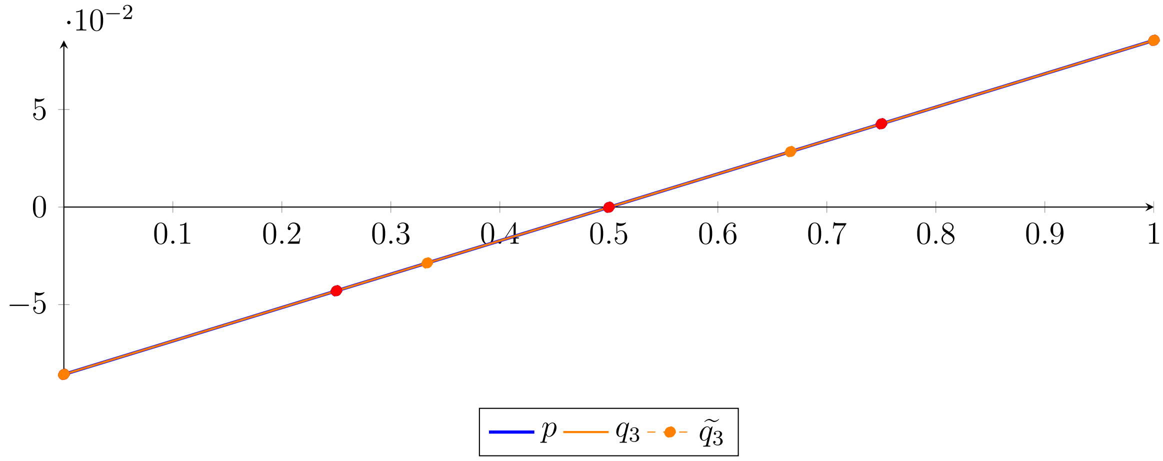
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4}(X) - 0.0429507B_{1,4}(X) - 0.000129666B_{2,4}(X) \\ &\quad + 0.0426682B_{3,4}(X) + 0.0854427B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,3} - 0.0286693B_{1,3} + 0.02841B_{2,3} + 0.0854427B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1.39067 \cdot 10^{-309} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529X^2 + 0.171376X - 0.0857948 \\ &= -0.0857948B_{0,4} - 0.0429507B_{1,4} - 0.000129666B_{2,4} + 0.0426682B_{3,4} + 0.0854427B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.
Bounding polynomials M and m:

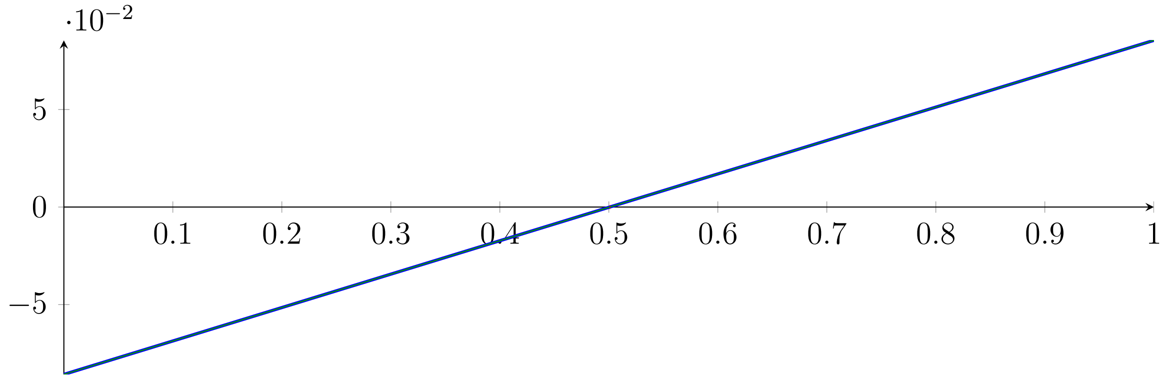
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



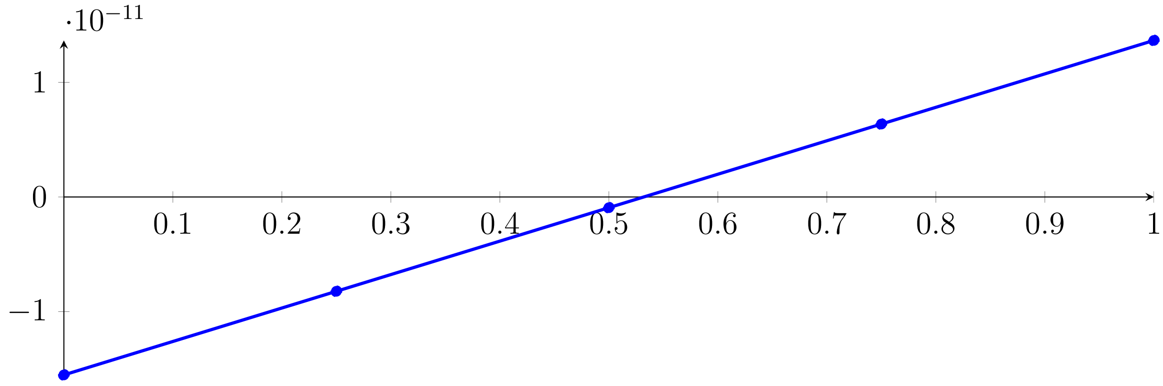
$$[0.500825, 0.500825]$$

Longest intersection interval: $1.7041 \cdot 10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

207.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

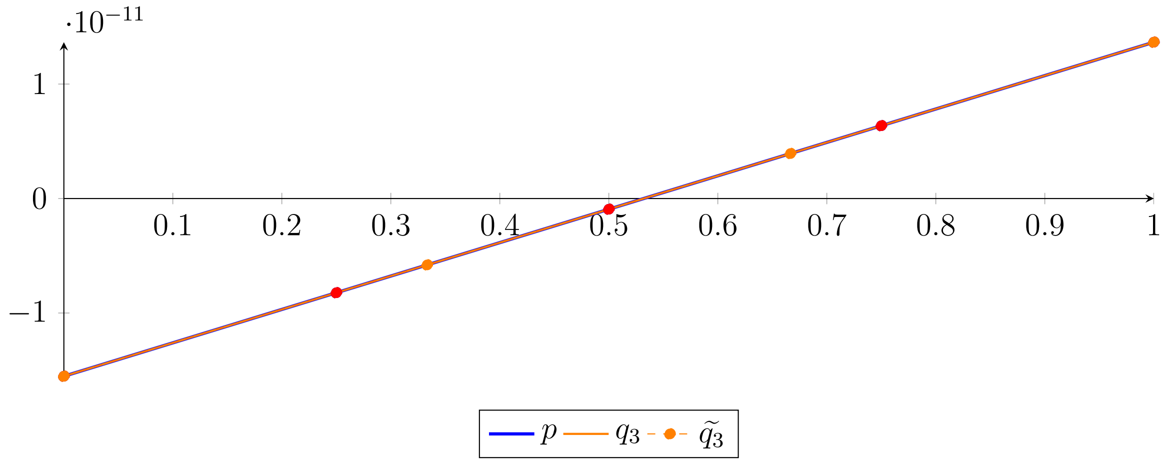
$$\begin{aligned} p &= -1.43544 \cdot 10^{-49} X^4 - 2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\ &= -1.55233 \cdot 10^{-11} B_{0,4}(X) - 8.22817 \cdot 10^{-12} B_{1,4}(X) - 9.33052 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36207 \cdot 10^{-12} B_{3,4}(X) + 1.36572 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$q_3 = -2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}$
$\quad = -1.55233 \cdot 10^{-11} B_{0,3} - 5.79646 \cdot 10^{-12} B_{1,3} + 3.93036 \cdot 10^{-12} B_{2,3} + 1.36572 \cdot 10^{-11} B_{3,3}$

$\widetilde{q_3} = -3.23791 \cdot 10^{-319} X^4 - 2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}$
$\quad = -1.55233 \cdot 10^{-11} B_{0,4} - 8.22817 \cdot 10^{-12} B_{1,4} - 9.33052 \cdot 10^{-13} B_{2,4} + 6.36207 \cdot 10^{-12} B_{3,4} + 1.36572 \cdot 10^{-11} B_{4,4}$



The maximum difference of the Bézier coefficients is $\delta = 1.23038 \cdot 10^{-50}$.

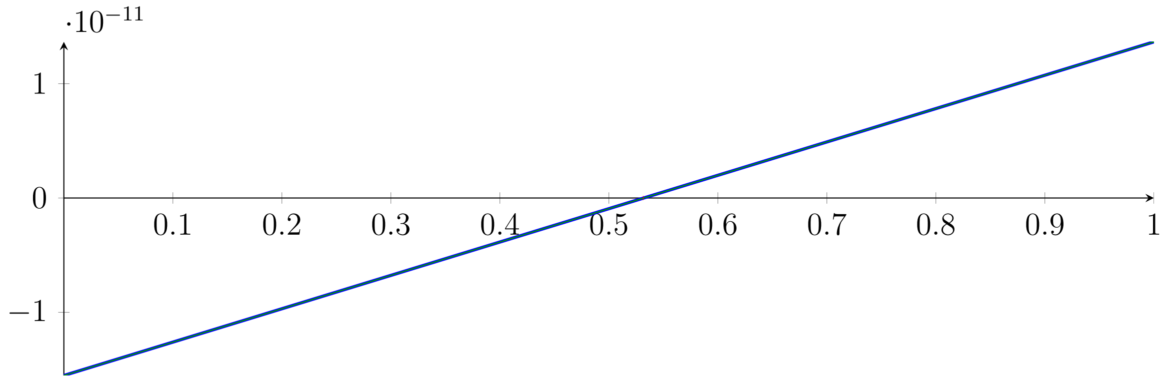
Bounding polynomials M and m:

$$M = -2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}$$
$$m = -2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}$$

Root of M and m:

$N(M) = \{-4.81371 \cdot 10^{12}, 0.531249, 2.88823 \cdot 10^{12}\}$ $\quad N(m) = \{-4.81371 \cdot 10^{12}, 0.531249, 2.88823 \cdot 10^{12}\}$

Intersection intervals:



$[0.531249, 0.531249]$

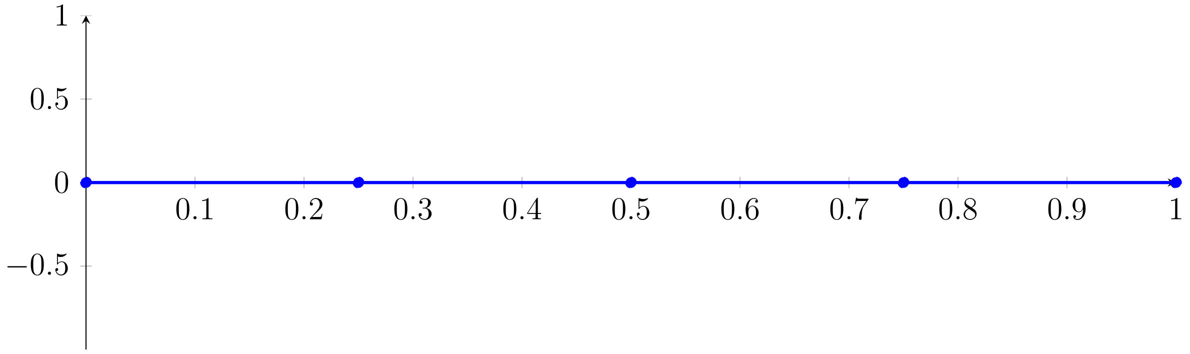
Longest intersection interval: $8.43287 \cdot 10^{-40}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

207.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

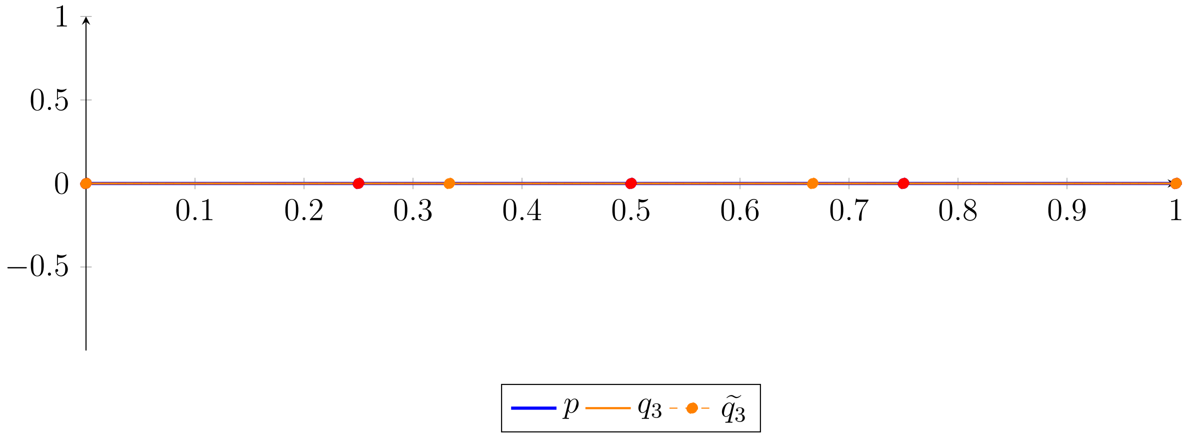
$$p = -7.25914{\cdot}10^{-206}X^4 - 1.25865{\cdot}10^{-153}X^3 - 2.8739{\cdot}10^{-102}X^2 + 2.46075{\cdot}10^{-50}X - 2.11876{\cdot}10^{-14}$$
$$= -2.11876{\cdot}10^{-14}B_{0,4}(X) - 2.11876{\cdot}10^{-14}B_{1,4}(X) - 2.11876{\cdot}10^{-14}B_{2,4}(X) - 2.11876{\cdot}10^{-14}B_{3,4}(X) - 2.11876{\cdot}10^{-14}B_{4,4}(X)$$



Degree reduction and raising:

$$q_3 = -1.25865{\cdot}10^{-153}X^3 - 2.8739{\cdot}10^{-102}X^2 + 2.46075{\cdot}10^{-50}X - 2.11876{\cdot}10^{-14}$$
$$= -2.11876{\cdot}10^{-14}B_{0,3} - 2.11876{\cdot}10^{-14}B_{1,3} - 2.11876{\cdot}10^{-14}B_{2,3} - 2.11876{\cdot}10^{-14}B_{3,3}$$

$$\widetilde{q_3} = 6.32404{\cdot}10^{-322}X^4 - 1.25865{\cdot}10^{-153}X^3 - 2.8739{\cdot}10^{-102}X^2 + 2.46075{\cdot}10^{-50}X - 2.11876{\cdot}10^{-14}$$
$$= -2.11876{\cdot}10^{-14}B_{0,4} - 2.11876{\cdot}10^{-14}B_{1,4} - 2.11876{\cdot}10^{-14}B_{2,4} - 2.11876{\cdot}10^{-14}B_{3,4} - 2.11876{\cdot}10^{-14}B_{4,4}$$



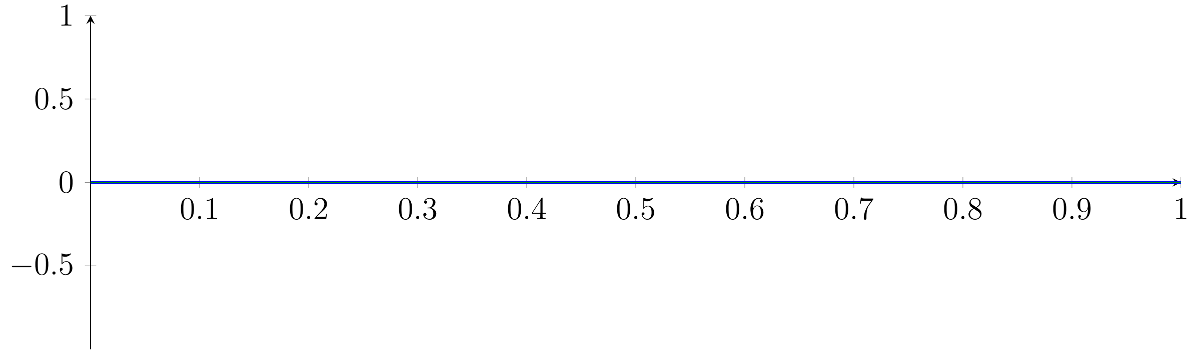
The maximum difference of the Bézier coefficients is $\delta = 6.22212{\cdot}10^{-207}$.
Bounding polynomials M and m:

$$M = -1.25865{\cdot}10^{-153}X^3 - 2.8739{\cdot}10^{-102}X^2 + 2.46075{\cdot}10^{-50}X - 2.11876{\cdot}10^{-14}$$
$$m = -1.25865{\cdot}10^{-153}X^3 - 2.8739{\cdot}10^{-102}X^2 + 2.46075{\cdot}10^{-50}X - 2.11876{\cdot}10^{-14}$$

Root of M and m:

$N(M) = \{-5.70827{\cdot}10^{51}, 3.91034{\cdot}10^{21}, 3.42496{\cdot}10^{51}\}$ $N(m) = \{-5.70827{\cdot}10^{51}, 3.91034{\cdot}10^{21}, 3.42496{\cdot}10^{51}\}$

Intersection intervals:

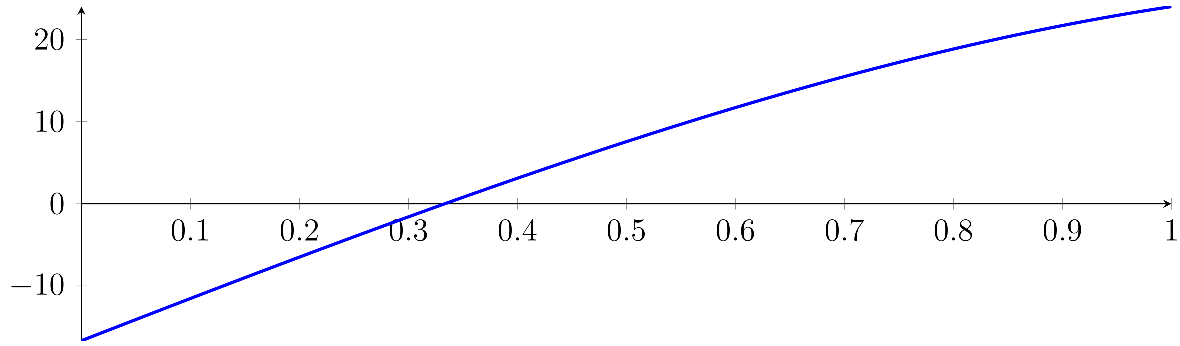


No intersection intervals with the x axis.

207.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

with precision $\varepsilon = 1{\cdot}10^{-64}$.

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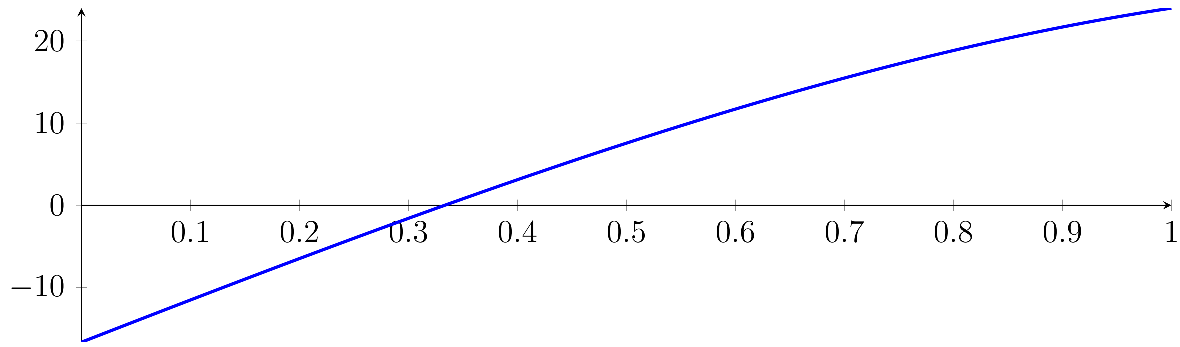
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

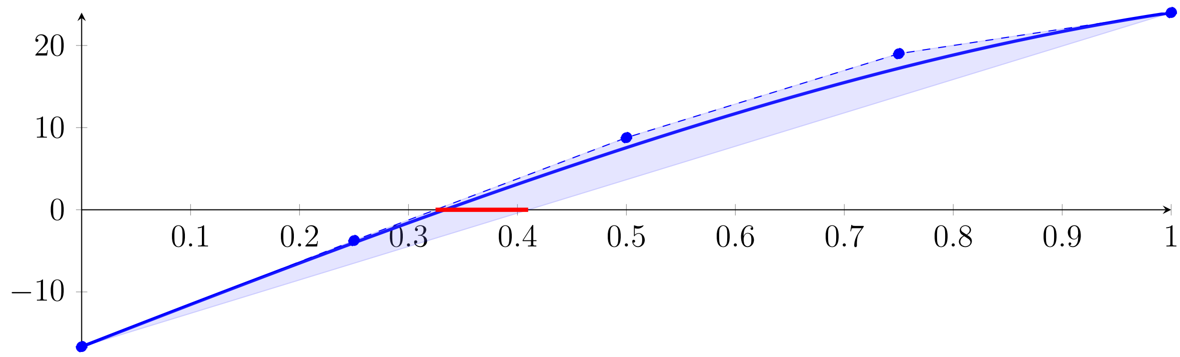
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



208.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$
$$= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

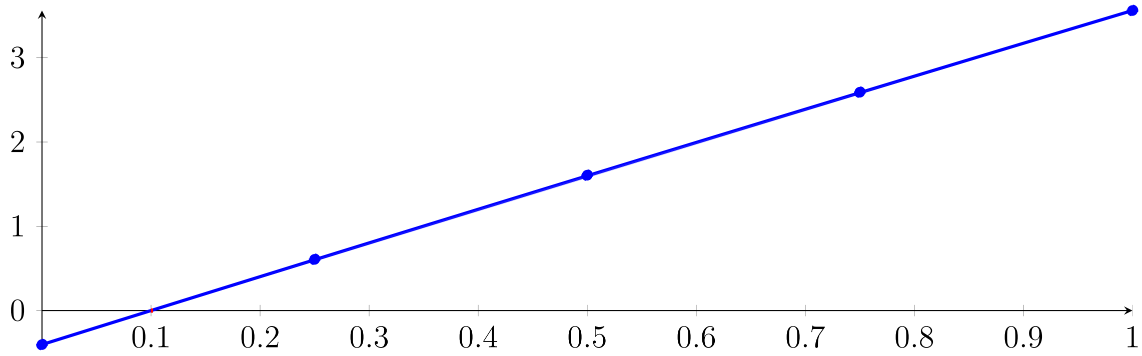
$$[0.324834, 0.409836]$$

Longest intersection interval: 0.0850024
$\Longrightarrow$ Selective recursion: interval 1: $[0.324834, 0.409836]$,

208.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -5.22064 \cdot 10^{-05}X^4 - 0.0055067X^3 - 0.0754159X^2 + 4.04499X - 0.403711$$
$$= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

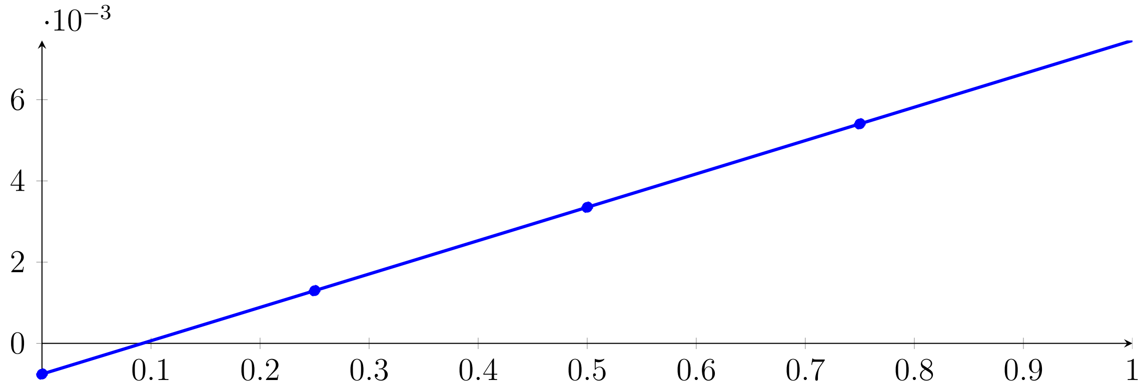
$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877
$\Longrightarrow$ Selective recursion: interval 1: $[0.333317, 0.333491]$,

208.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01975 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

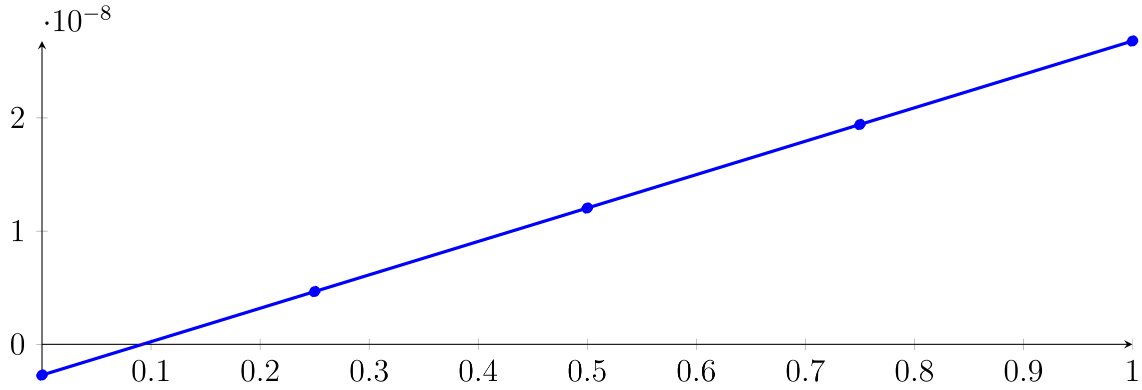
$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

208.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.50129 \cdot 10^{-37} X^4 - 2.17066 \cdot 10^{-27} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09}$$
$$= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

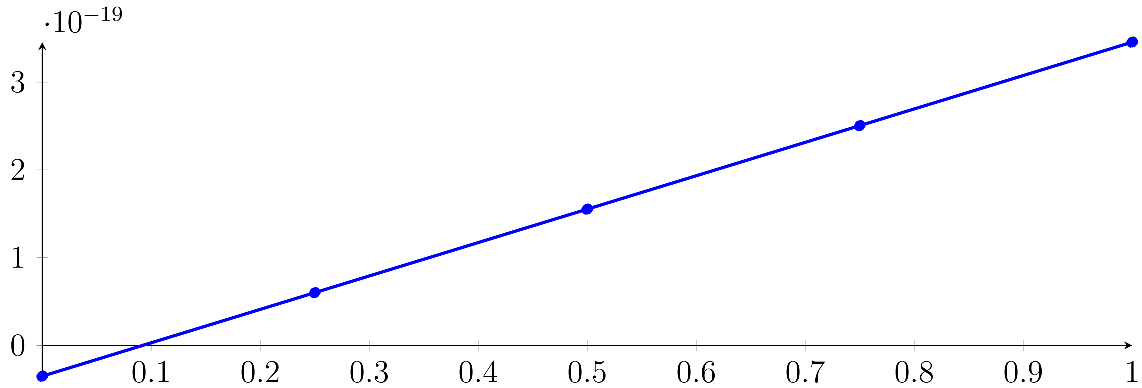
$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28975 \cdot 10^{-11}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

208.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -4.15417 \cdot 10^{-81} X^4 - 4.65699 \cdot 10^{-60} X^3 - 6.87497 \cdot 10^{-40} X^2 + 3.80599 \cdot 10^{-19} X - 3.50488 \cdot 10^{-20}$$
$$= -3.50488 \cdot 10^{-20} B_{0,4}(X) + 6.01009 \cdot 10^{-20} B_{1,4}(X) + 1.55251 \cdot 10^{-19} B_{2,4}(X) + 2.504 \cdot 10^{-19} B_{3,4}(X) + 3.4555 \cdot 10^{-19} B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

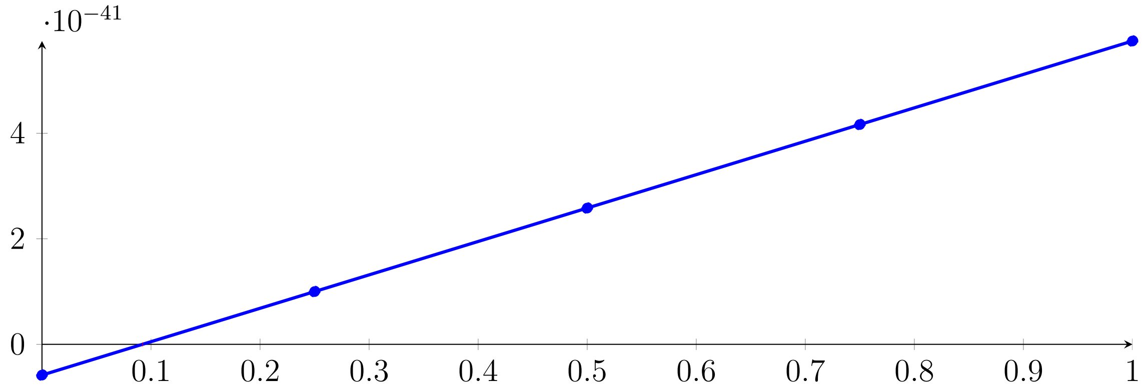
$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.66345 \cdot 10^{-22}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

208.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.18068 \cdot 10^{-168} X^4 - 2.14355 \cdot 10^{-125} X^3 - 1.90234 \cdot 10^{-83} X^2 + 6.33106 \cdot 10^{-41} X - 5.83018 \cdot 10^{-42}$$
$$= -5.83018 \cdot 10^{-42} B_{0,4}(X) + 9.99747 \cdot 10^{-42} B_{1,4}(X) + 2.58251 \cdot 10^{-41} B_{2,4}(X) + 4.16528 \cdot 10^{-41} B_{3,4}(X) + 5.74804 \cdot 10^{-41} B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

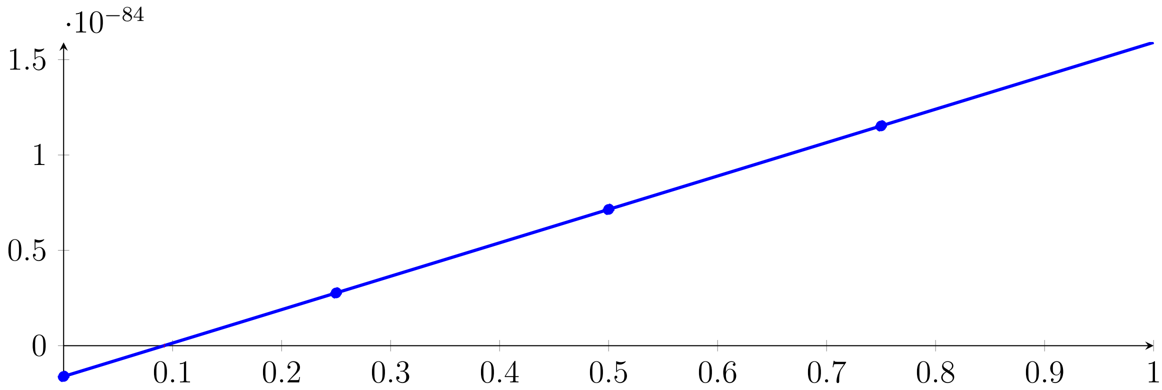
Longest intersection interval: $2.76706 \cdot 10^{-44}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

208.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.86463 \cdot 10^{-342} X^4 - 4.54137 \cdot 10^{-256} X^3 - 1.45655 \cdot 10^{-170} X^2 + 1.75184 \cdot 10^{-84} X - 1.61324 \cdot 10^{-85}$$
$$= -1.61324 \cdot 10^{-85} B_{0,4}(X) + 2.76636 \cdot 10^{-85} B_{1,4}(X) + 7.14596 \cdot 10^{-85} B_{2,4}(X) + 1.15256 \cdot 10^{-84} B_{3,4}(X) + 1.59052 \cdot 10^{-84} B_{4,4}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $7.65661 \cdot 10^{-88}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

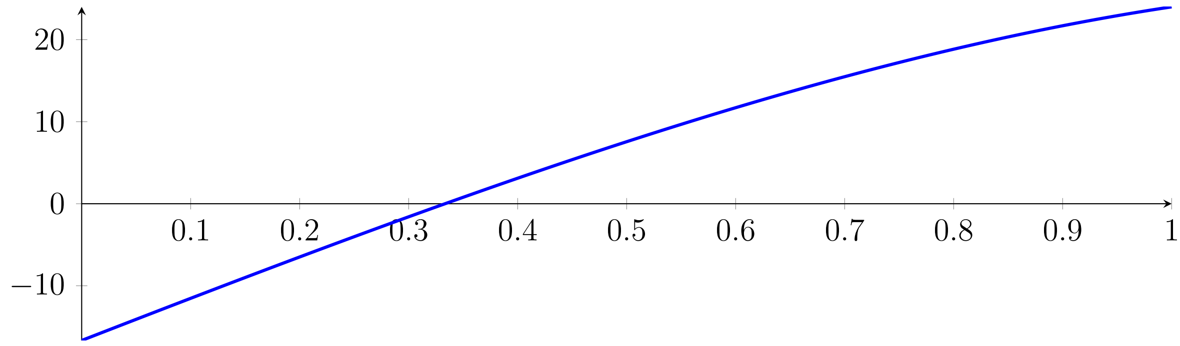
208.8 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 8!

208.9 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-128}$.

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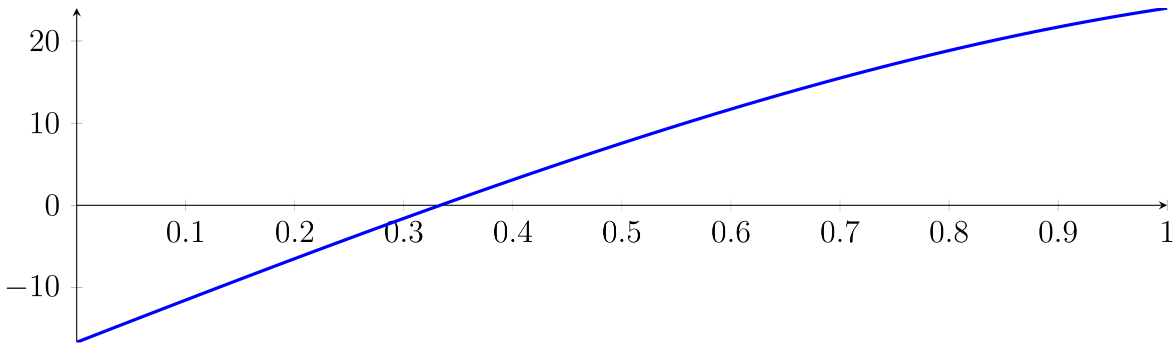
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0,1]$:

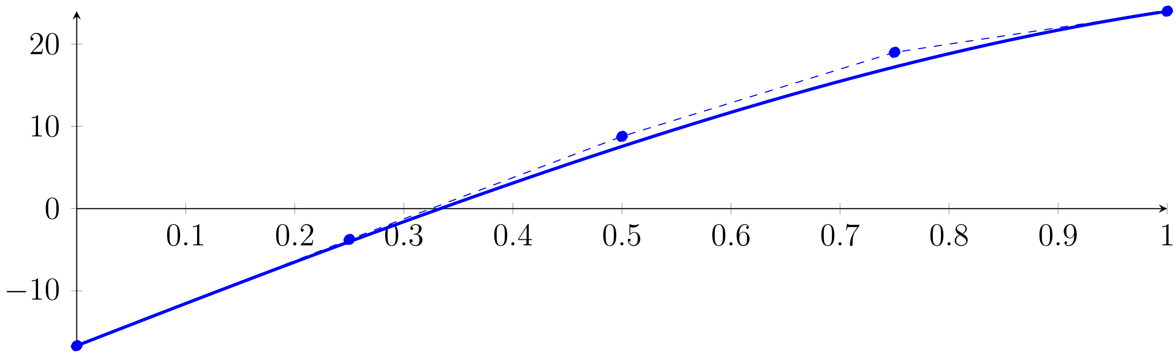
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



209.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

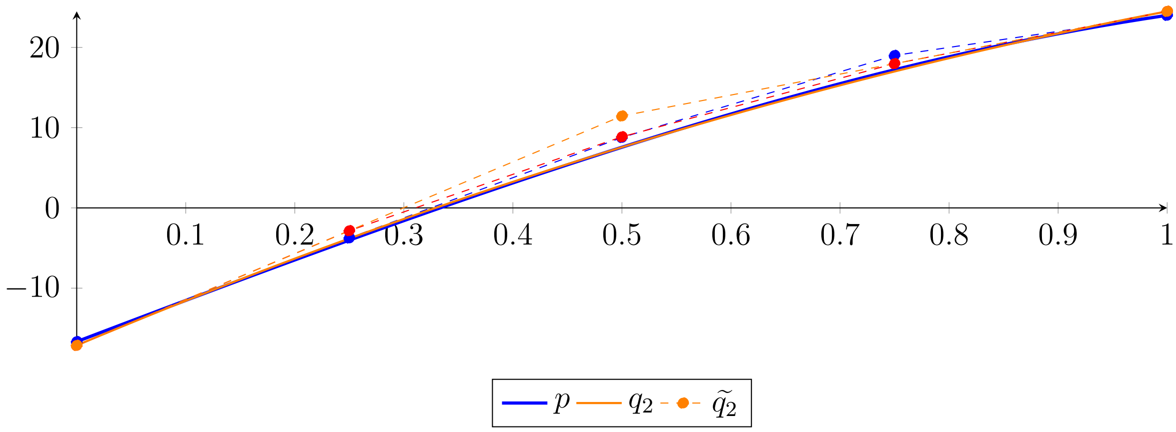
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= -2.13607\cdot10^{-306}X^4 + 7.12024\cdot10^{-306}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.
Bounding polynomials M and m:

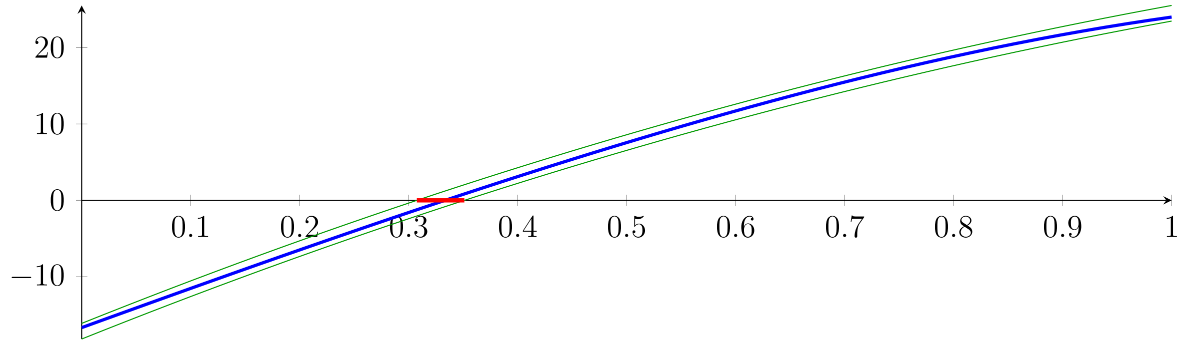
$$M = -15.5476X^2 + 57.181X - 16.1119$$

$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m:

$$N(M) = \{0.307477, 3.37032\} \qquad N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



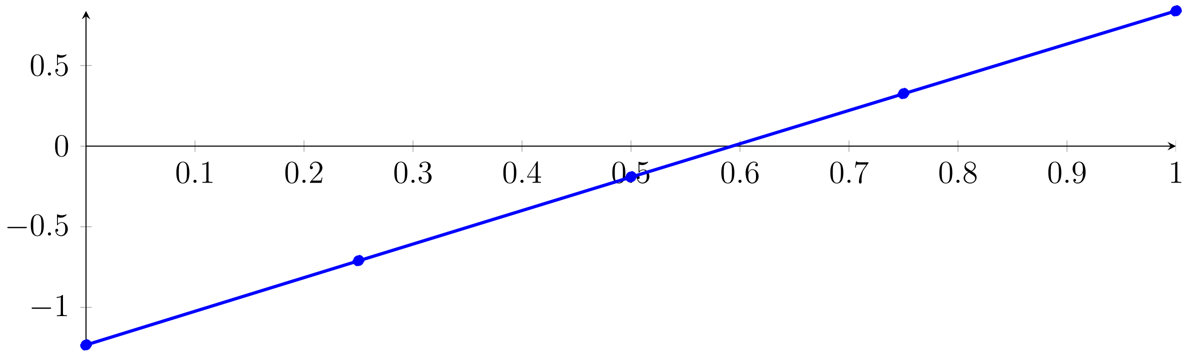
$[0.307477, 0.351097]$

Longest intersection interval: 0.0436205
$\Longrightarrow$ Selective recursion: interval 1: $[0.307477, 0.351097]$,

209.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

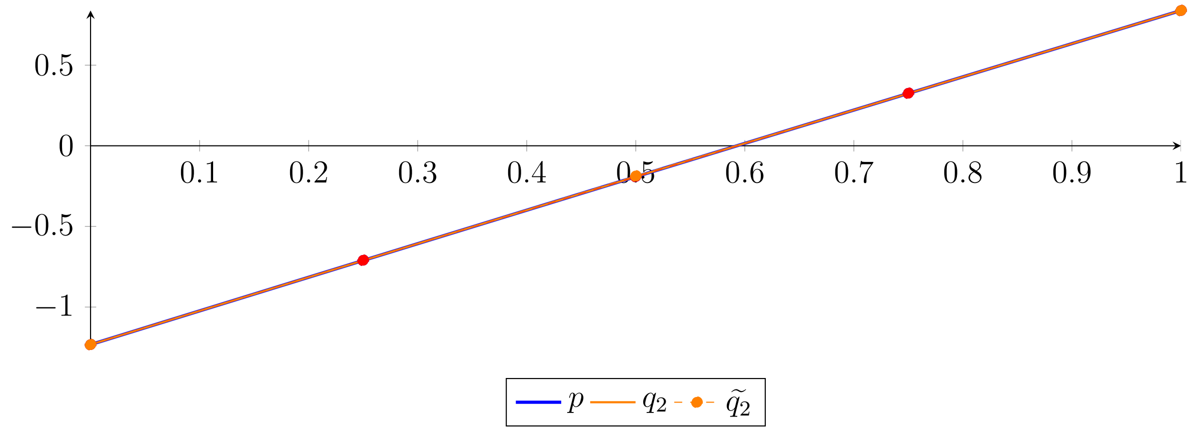
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404X^3 - 0.0189752X^2 + 2.09121X - 1.23278 \\ &= -1.23278B_{0,4}(X) - 0.709974B_{1,4}(X) - 0.190333B_{2,4}(X) + 0.325959B_{3,4}(X) + 0.838717B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,2} - 0.186985B_{1,2} + 0.838754B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -4.5614 \cdot 10^{-307} X^4 + 1.02353 \cdot 10^{-306} X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281B_{0,4} - 0.709899B_{1,4} - 0.190333B_{2,4} + 0.325885B_{3,4} + 0.838754B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

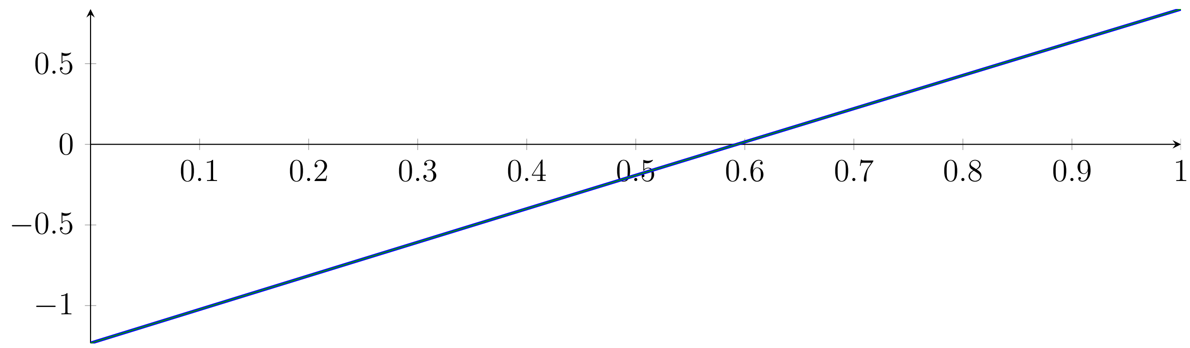
Bounding polynomials M and m:

$$M = -0.020089X^2 + 2.09166X - 1.23274$$
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m:

$$N(M) = \{0.592734, 103.527\} \qquad N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

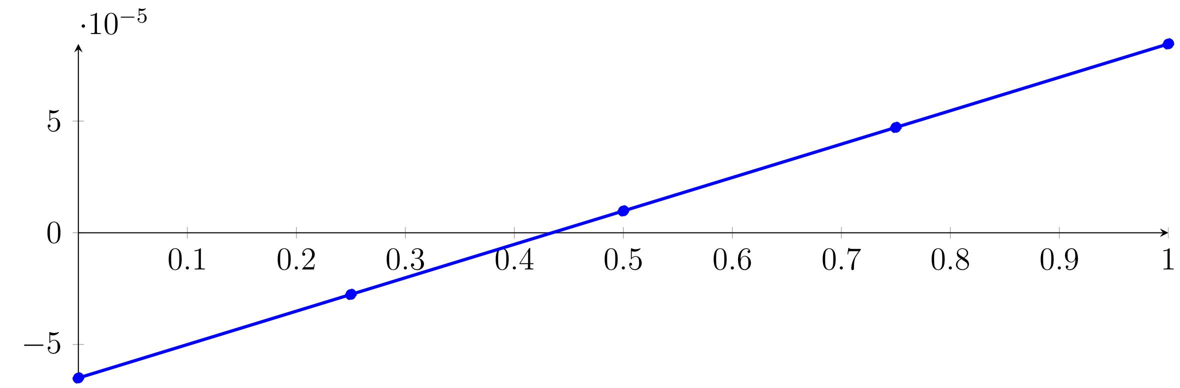
Longest intersection interval: $7.23183 \cdot 10^{-05}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333332, 0.333335]$,

209.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

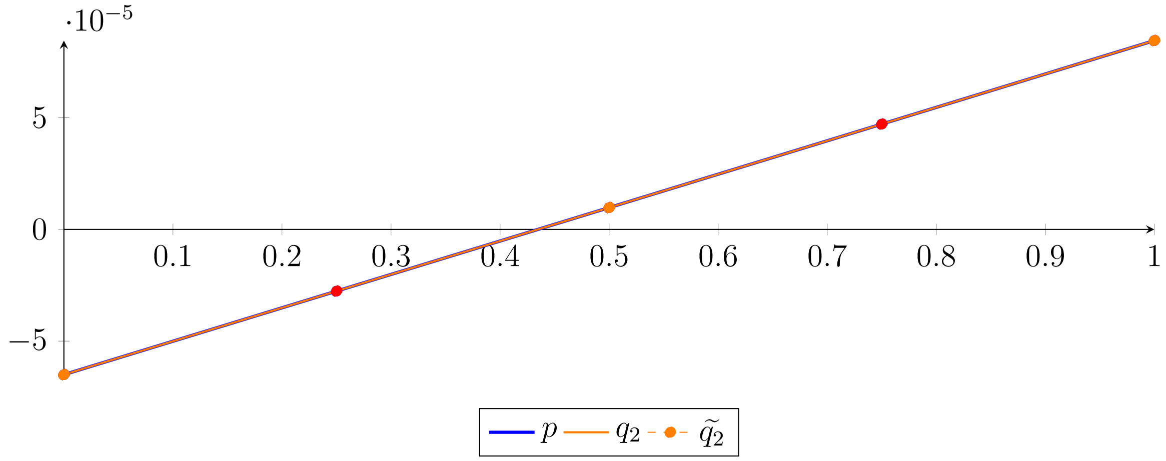
$$p = -9.9027 \cdot 10^{-23} X^4 - 2.82525 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549X - 6.50069 \cdot 10^{-05}$$
$$= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777$$
$$\cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.06147{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$
$$= -6.50069{\cdot}10^{-05}B_{0,2} + 9.76779{\cdot}10^{-06}B_{1,2} + 8.45424{\cdot}10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = 1.22227{\cdot}10^{-311}X^4 - 1.62969{\cdot}10^{-311}X^3 - 1.06147{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$
$$= -6.50069{\cdot}10^{-05}B_{0,4} - 2.76196{\cdot}10^{-05}B_{1,4} + 9.76777{\cdot}10^{-06}B_{2,4} + 4.71551{\cdot}10^{-05}B_{3,4} + 8.45424{\cdot}10^{-05}B_{4,4}$$



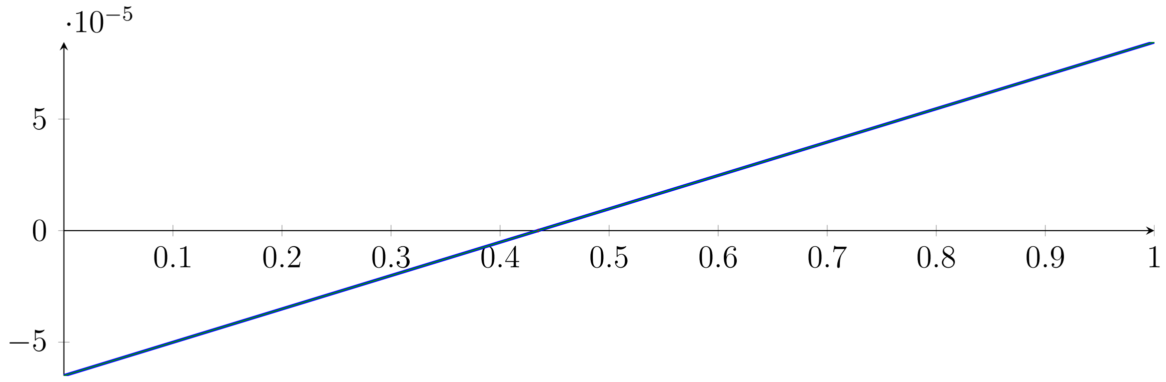
The maximum difference of the Bézier coefficients is $\delta = 2.82526{\cdot}10^{-17}$.
Bounding polynomials M and m:

$$M = -1.06147{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$
$$m = -1.06147{\cdot}10^{-10}X^2 + 0.000149549X - 6.50069{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{0.434685, 1.4089{\cdot}10^{6}\} \qquad N(m) = \{0.434685, 1.4089{\cdot}10^{6}\}$$

Intersection intervals:



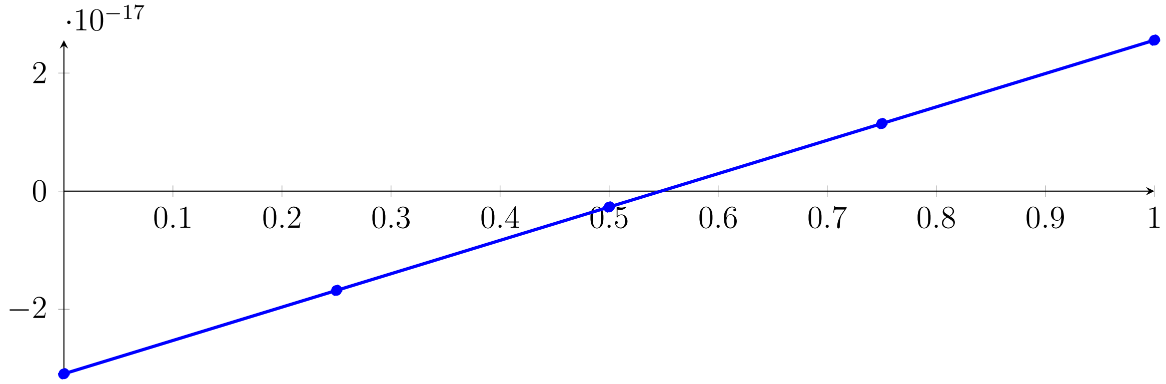
$$[0.434685, 0.434685]$$

Longest intersection interval: $3.77836{\cdot}10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

209.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

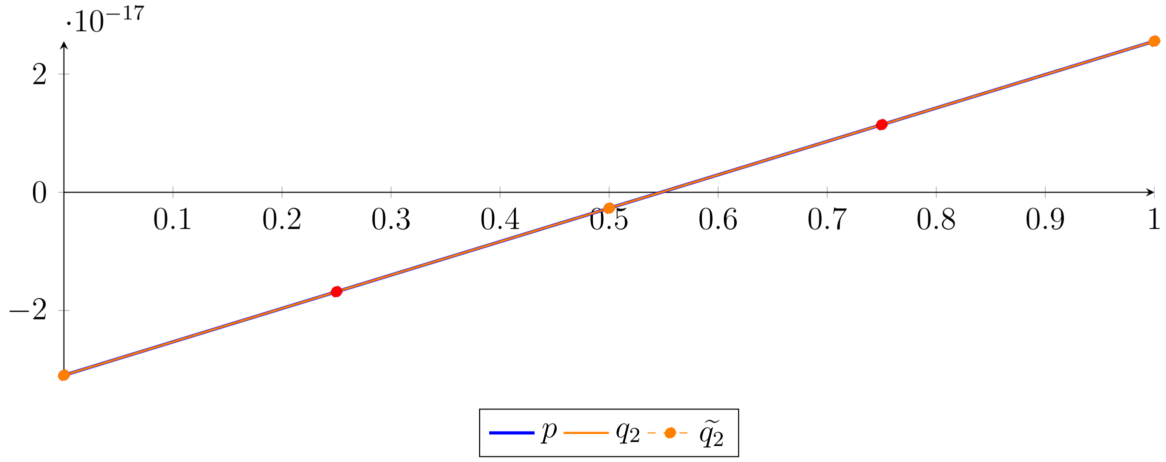
$$p = -2.01821\cdot10^{-72}X^4 - 1.52394\cdot10^{-53}X^3 - 1.51535\cdot10^{-35}X^2 + 5.65051\cdot10^{-17}X - 3.09418\cdot10^{-17}$$
$$= -3.09418\cdot10^{-17}B_{0,4}(X) - 1.68155\cdot10^{-17}B_{1,4}(X) - 2.68924\cdot10^{-18}B_{2,4}(X) + 1.1437\cdot10^{-17}B_{3,4}(X) + 2.55633\cdot10^{-17}B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -1.51535\cdot10^{-35}X^2 + 5.65051\cdot10^{-17}X - 3.09418\cdot10^{-17}$$
$$= -3.09418\cdot10^{-17}B_{0,2} - 2.68924\cdot10^{-18}B_{1,2} + 2.55633\cdot10^{-17}B_{2,2}$$

$$\widetilde{q}_2 = -7.41098\cdot10^{-324}X^4 + 1.72923\cdot10^{-323}X^3 - 1.51535\cdot10^{-35}X^2 + 5.65051\cdot10^{-17}X - 3.09418\cdot10^{-17}$$
$$= -3.09418\cdot10^{-17}B_{0,4} - 1.68155\cdot10^{-17}B_{1,4} - 2.68924\cdot10^{-18}B_{2,4} + 1.1437\cdot10^{-17}B_{3,4} + 2.55633\cdot10^{-17}B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 1.52394\cdot10^{-54}$.

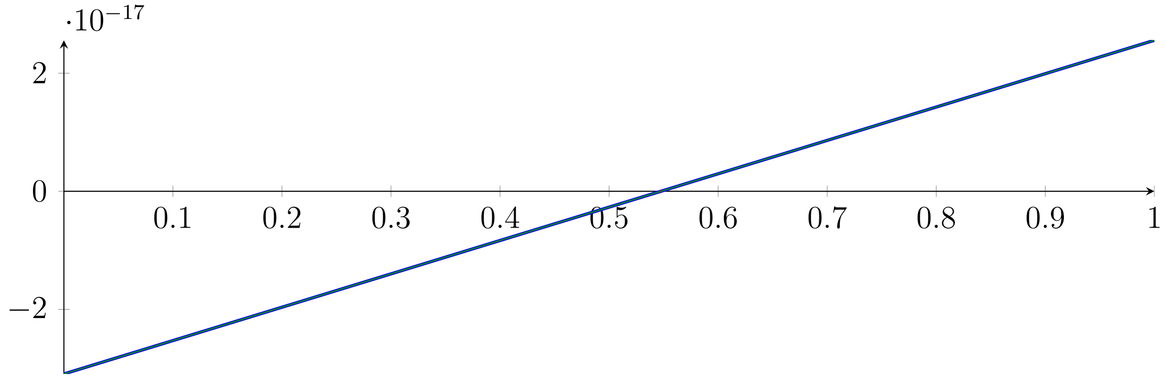
Bounding polynomials M and m:

$$M = -1.51535\cdot10^{-35}X^2 + 5.65051\cdot10^{-17}X - 3.09418\cdot10^{-17}$$
$$m = -1.51535\cdot10^{-35}X^2 + 5.65051\cdot10^{-17}X - 3.09418\cdot10^{-17}$$

Root of M and m:

$$N(M) = \{0.547593, 3.72886\cdot10^{18}\} \qquad N(m) = \{0.547593, 3.72886\cdot10^{18}\}$$

Intersection intervals:



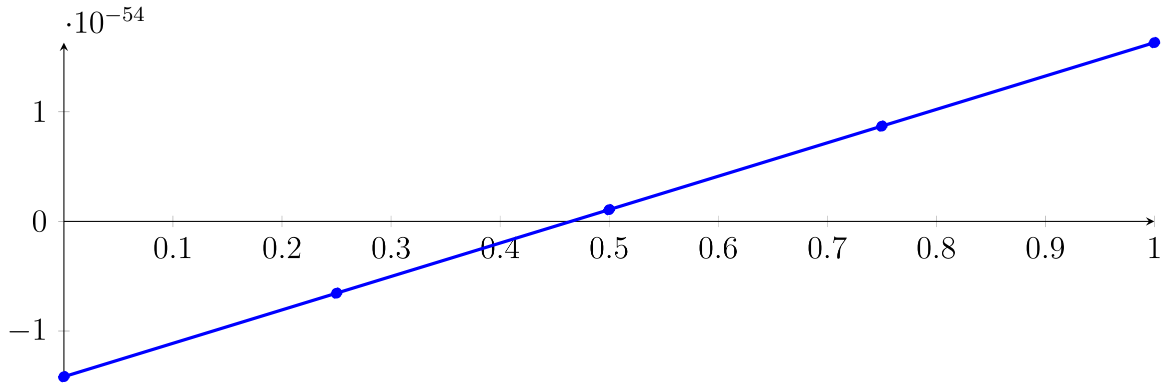
$[0.547593, 0.547593]$

Longest intersection interval: $5.39398 \cdot 10^{-38}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

209.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

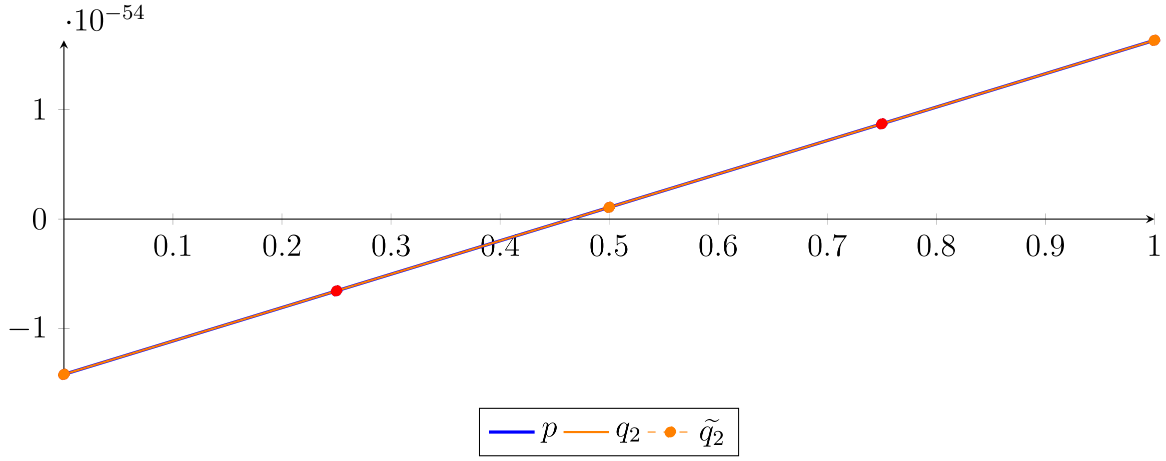
$$p = -1.70846 \cdot 10^{-221} X^4 - 2.39164 \cdot 10^{-165} X^3 - 4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54}$$
$$= -1.41679 \cdot 10^{-54} B_{0,4}(X) - 6.54819 \cdot 10^{-55} B_{1,4}(X) + 1.0715 \cdot 10^{-55} B_{2,4}(X) + 8.6912 \cdot 10^{-55} B_{3,4}(X) + 1.63109 \cdot 10^{-54} B_{4,4}(X)$$



Degree reduction and raising:

$$q_2 = -4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54}$$
$$= -1.41679 \cdot 10^{-54} B_{0,2} + 1.0715 \cdot 10^{-55} B_{1,2} + 1.63109 \cdot 10^{-54} B_{2,2}$$

$$\widetilde{q}_2 = 7.25964 \cdot 10^{-362} X^4 - 5.80771 \cdot 10^{-362} X^3 - 4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54}$$
$$= -1.41679 \cdot 10^{-54} B_{0,4} - 6.54819 \cdot 10^{-55} B_{1,4} + 1.0715 \cdot 10^{-55} B_{2,4} + 8.6912 \cdot 10^{-55} B_{3,4} + 1.63109 \cdot 10^{-54} B_{4,4}$$



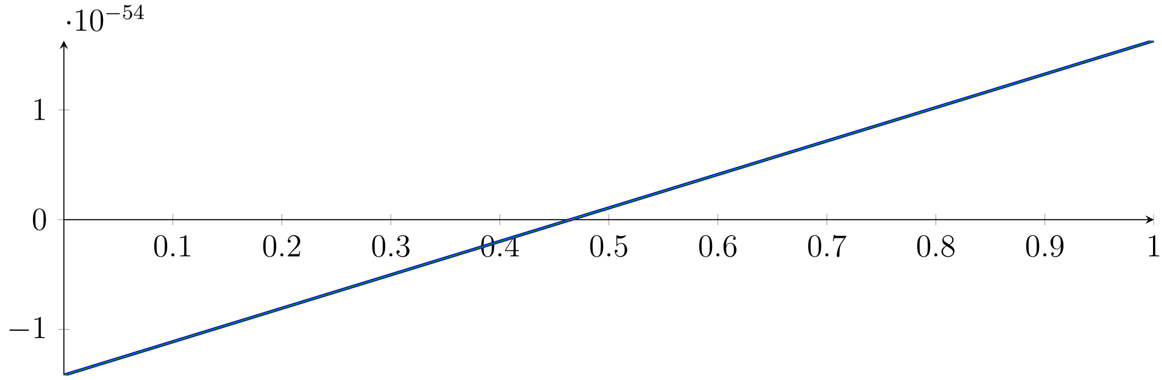
The maximum difference of the Bézier coefficients is $\delta = 2.39164 \cdot 10^{-166}$.
Bounding polynomials M and m:

$$M = -4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54}$$
$$m = -4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54}$$

Root of M and m:

$$N(M) = \{0.464844, 6.91299 \cdot 10^{55}\} \qquad N(m) = \{0.464844, 6.91299 \cdot 10^{55}\}$$

Intersection intervals:



$$[0.464844, 0.464844]$$

Longest intersection interval: $1.56938 \cdot 10^{-112}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

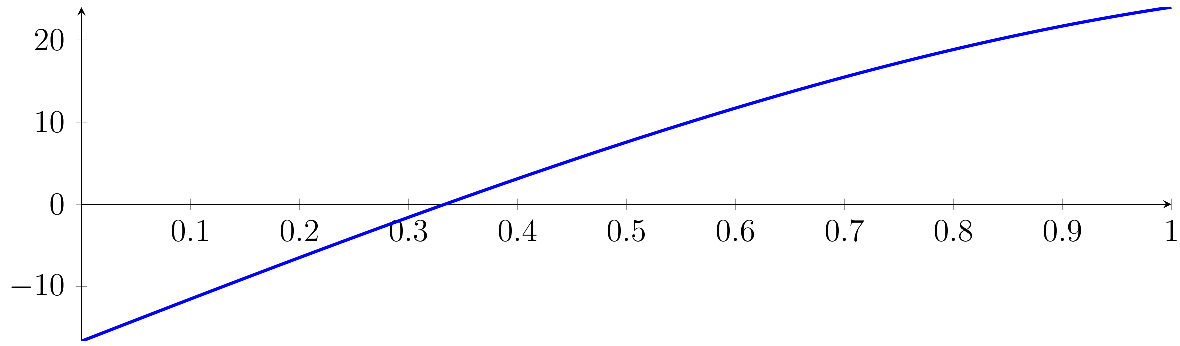
209.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

209.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-128}$.

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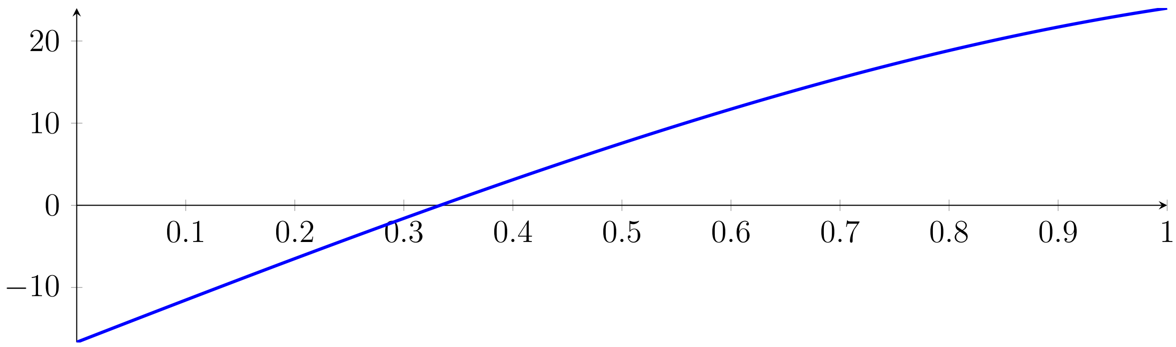
$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

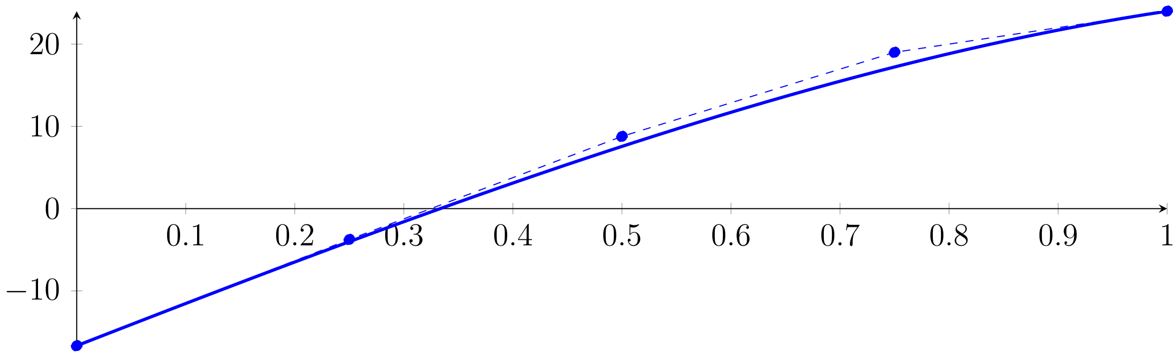
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



210.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

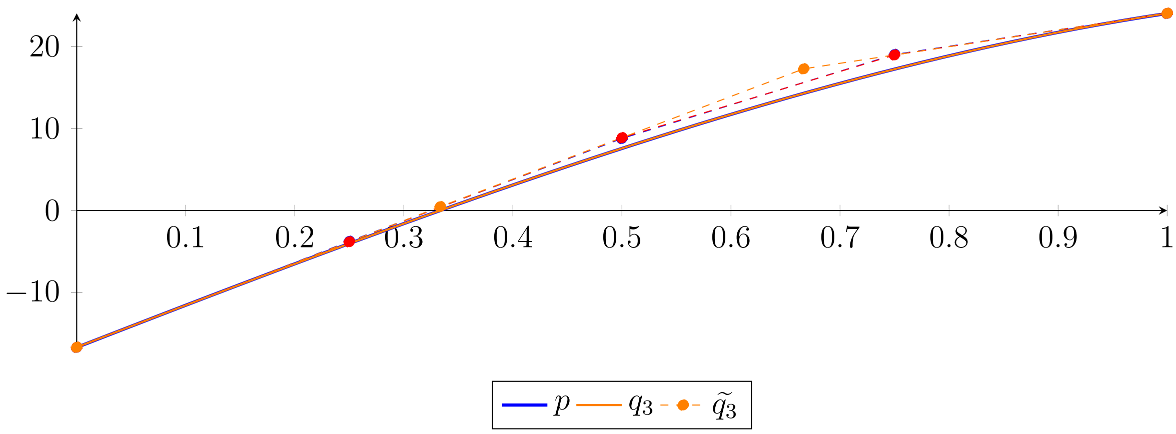
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -3.56012 \cdot 10^{-307}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



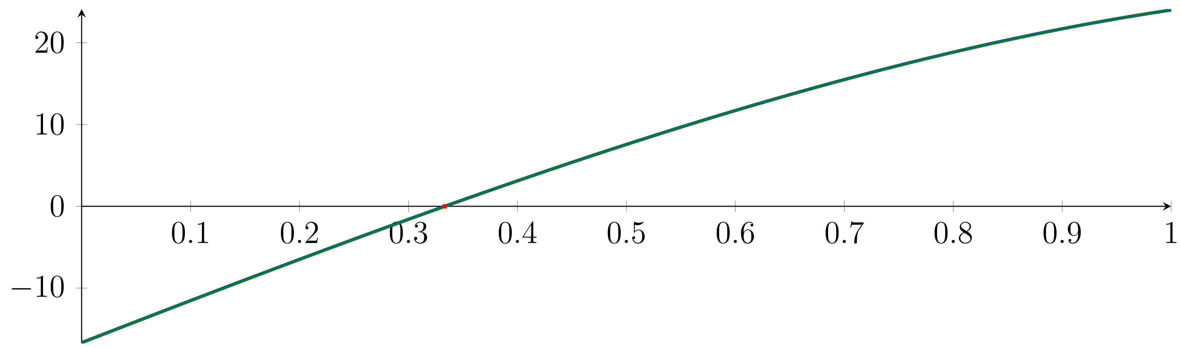
The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.
Bounding polynomials M and m:

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m:

$$N(M) = \{-2.5042, 0.331524, 2.0643\} \qquad N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



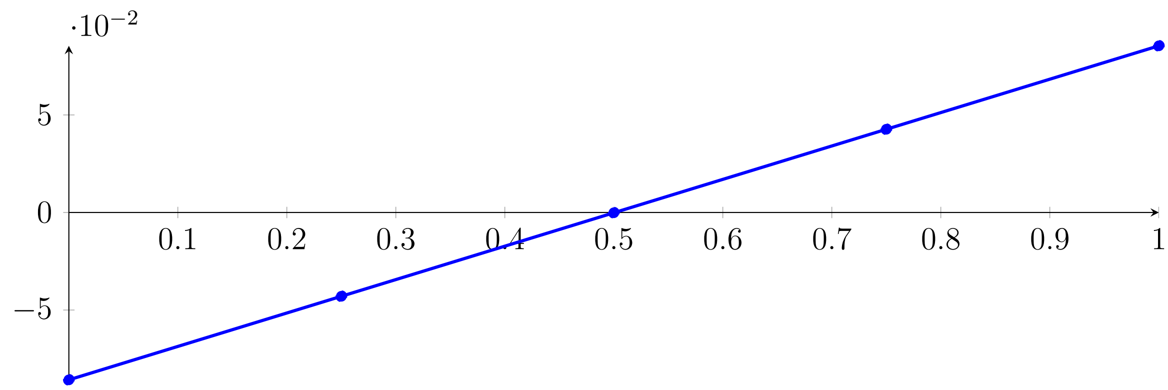
$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204
$\Longrightarrow$ Selective recursion: interval 1: $[0.331524, 0.335136]$,

210.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

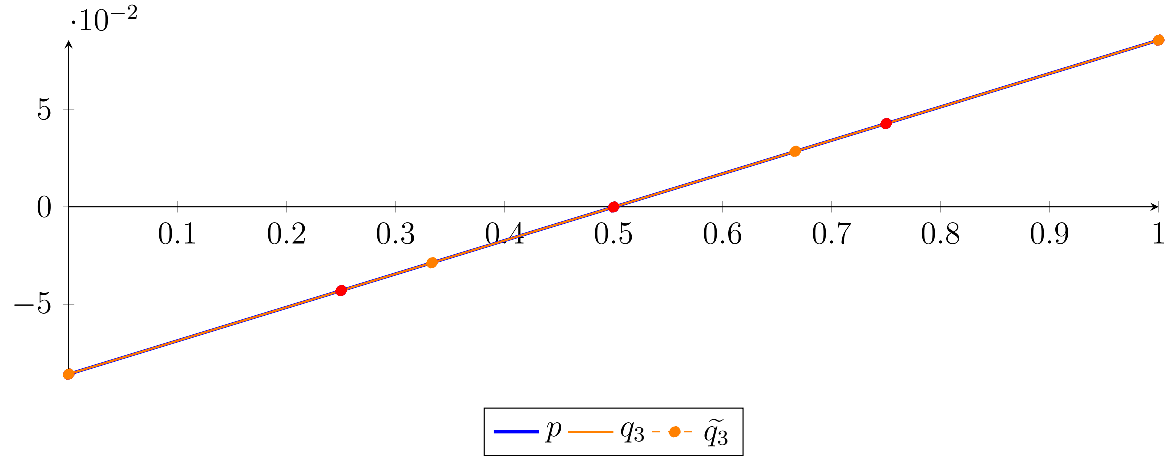
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1.39067 \cdot 10^{-309} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



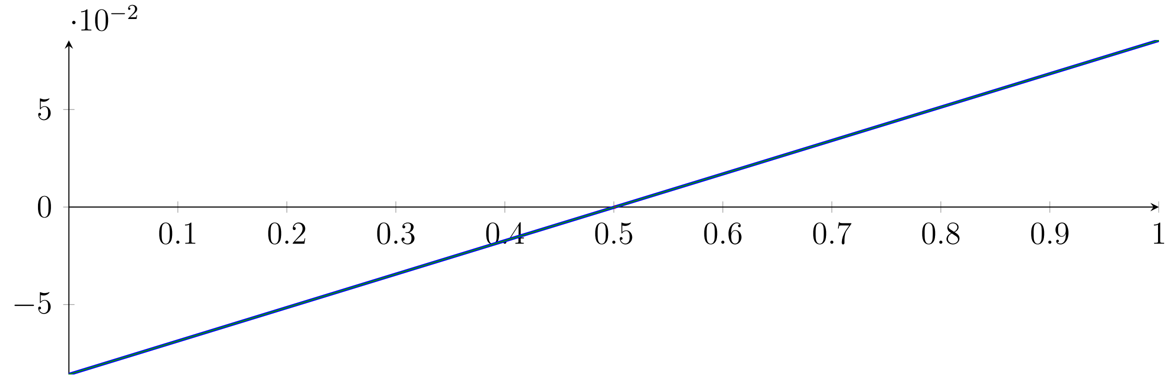
The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.
Bounding polynomials M and m:

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m:

$$N(M) = \{-819.802, 0.500825, 492.682\} \qquad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



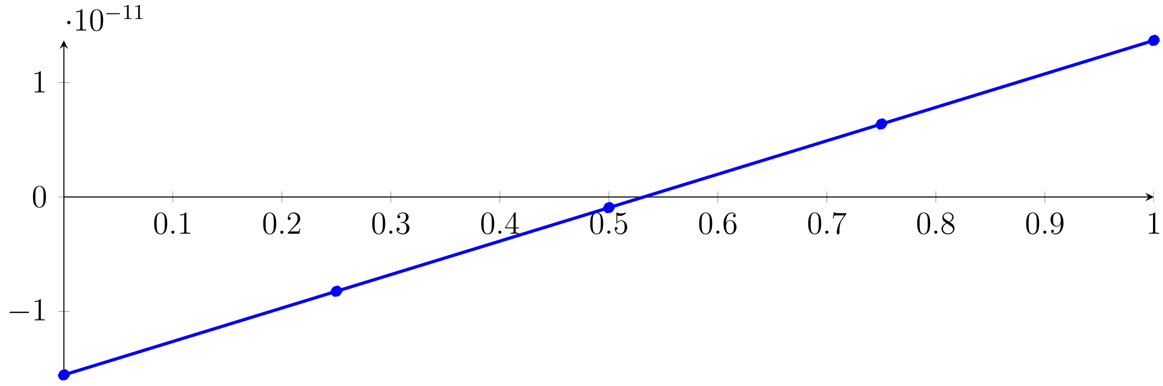
$$[0.500825, 0.500825]$$

Longest intersection interval: $1.7041 \cdot 10^{-10}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

210.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

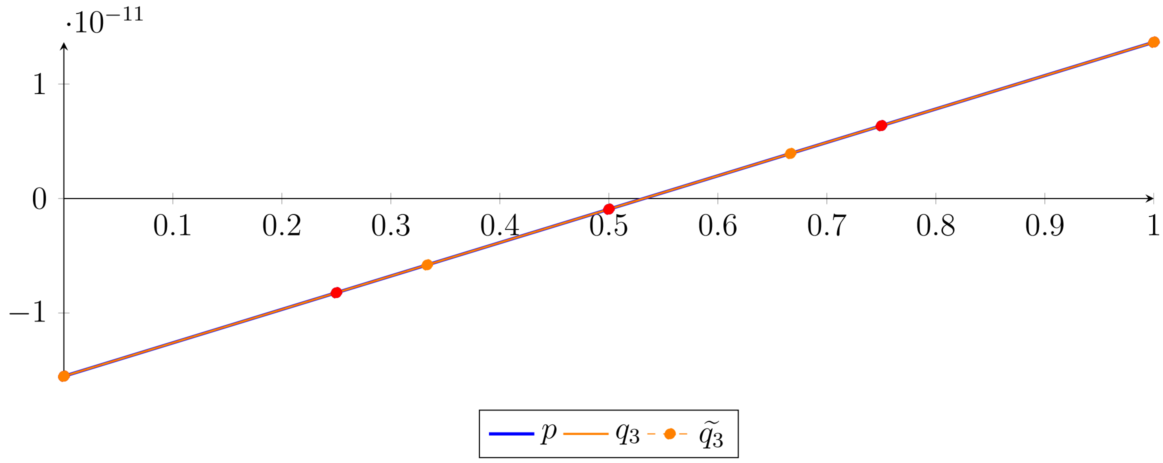
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.43544 \cdot 10^{-49} X^4 - 2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\ &= -1.55233 \cdot 10^{-11} B_{0,4}(X) - 8.22817 \cdot 10^{-12} B_{1,4}(X) - 9.33052 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36207 \cdot 10^{-12} B_{3,4}(X) + 1.36572 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -2.09885\cdot 10^{-36}X^3 - 4.0413\cdot 10^{-24}X^2 + 2.91805\cdot 10^{-11}X - 1.55233\cdot 10^{-11}$$
$$= -1.55233\cdot 10^{-11}B_{0,3} - 5.79646\cdot 10^{-12}B_{1,3} + 3.93036\cdot 10^{-12}B_{2,3} + 1.36572\cdot 10^{-11}B_{3,3}$$
$$\widetilde{q}_3 = -3.23791\cdot 10^{-319}X^4 - 2.09885\cdot 10^{-36}X^3 - 4.0413\cdot 10^{-24}X^2 + 2.91805\cdot 10^{-11}X - 1.55233\cdot 10^{-11}$$
$$= -1.55233\cdot 10^{-11}B_{0,4} - 8.22817\cdot 10^{-12}B_{1,4} - 9.33052\cdot 10^{-13}B_{2,4} + 6.36207\cdot 10^{-12}B_{3,4} + 1.36572\cdot 10^{-11}B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 1.23038\cdot 10^{-50}$.

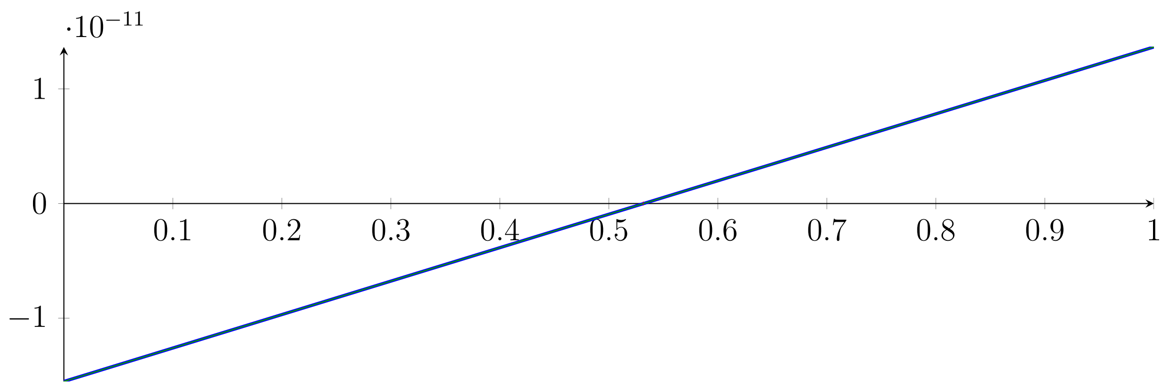
Bounding polynomials M and m:

$$M = -2.09885\cdot 10^{-36}X^3 - 4.0413\cdot 10^{-24}X^2 + 2.91805\cdot 10^{-11}X - 1.55233\cdot 10^{-11}$$
$$m = -2.09885\cdot 10^{-36}X^3 - 4.0413\cdot 10^{-24}X^2 + 2.91805\cdot 10^{-11}X - 1.55233\cdot 10^{-11}$$

Root of M and m:

$N(M) = \{-4.81371\cdot 10^{12}, 0.531249, 2.88823\cdot 10^{12}\}$ $N(m) = \{-4.81371\cdot 10^{12}, 0.531249, 2.88823\cdot 10^{12}\}$

Intersection intervals:



$[0.531249, 0.531249]$

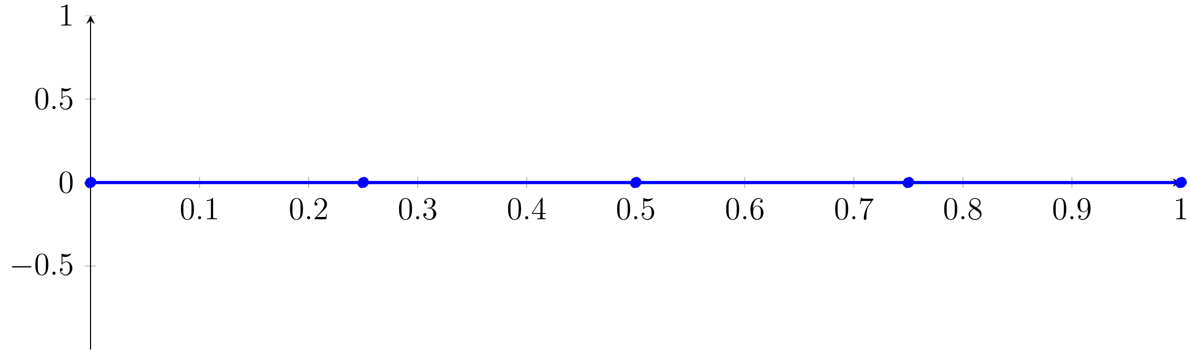
Longest intersection interval: $8.43287\cdot 10^{-40}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

210.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

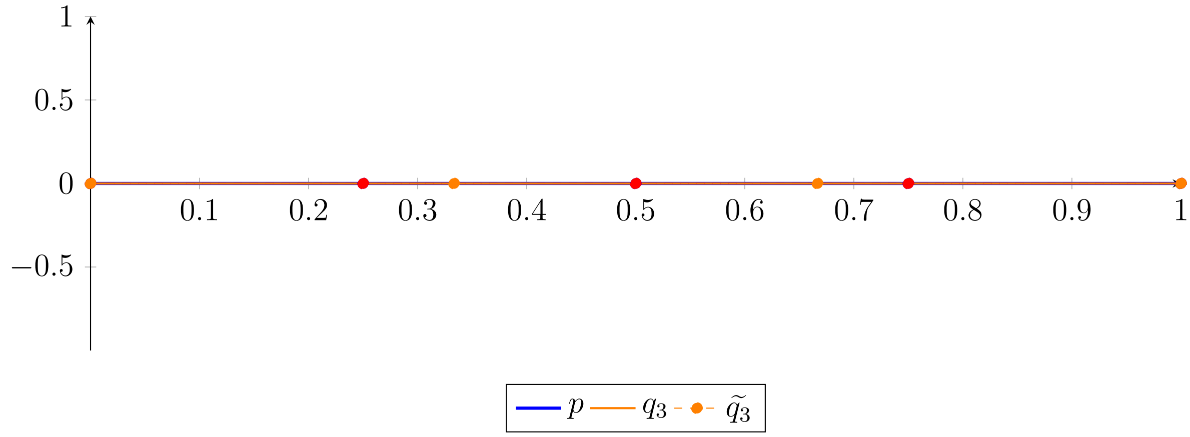
$$p = -7.25914\cdot 10^{-206}X^4 - 1.25865\cdot 10^{-153}X^3 - 2.8739\cdot 10^{-102}X^2 + 2.46075\cdot 10^{-50}X - 2.11876\cdot 10^{-14}$$
$$= -2.11876\cdot 10^{-14}B_{0,4}(X) - 2.11876\cdot 10^{-14}B_{1,4}(X) - 2.11876 \cdot 10^{-14}B_{2,4}(X) - 2.11876\cdot 10^{-14}B_{3,4}(X) - 2.11876\cdot 10^{-14}B_{4,4}(X)$$



Degree reduction and raising:

$$q_3 = -1.25865\cdot 10^{-153}X^3 - 2.8739\cdot 10^{-102}X^2 + 2.46075\cdot 10^{-50}X - 2.11876\cdot 10^{-14}$$
$$= -2.11876\cdot 10^{-14}B_{0,3} - 2.11876\cdot 10^{-14}B_{1,3} - 2.11876\cdot 10^{-14}B_{2,3} - 2.11876\cdot 10^{-14}B_{3,3}$$

$$\widetilde{q}_3 = 6.32404\cdot 10^{-322}X^4 - 1.25865\cdot 10^{-153}X^3 - 2.8739\cdot 10^{-102}X^2 + 2.46075\cdot 10^{-50}X - 2.11876\cdot 10^{-14}$$
$$= -2.11876 10^{-14}B_{0,4} - 2.11876 10^{-14}B_{1,4} - 2.11876 10^{-14}B_{2,4} - 2.11876 10^{-14}B_{3,4} - 2.11876 10^{-14}B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 6.22212\cdot 10^{-207}$.

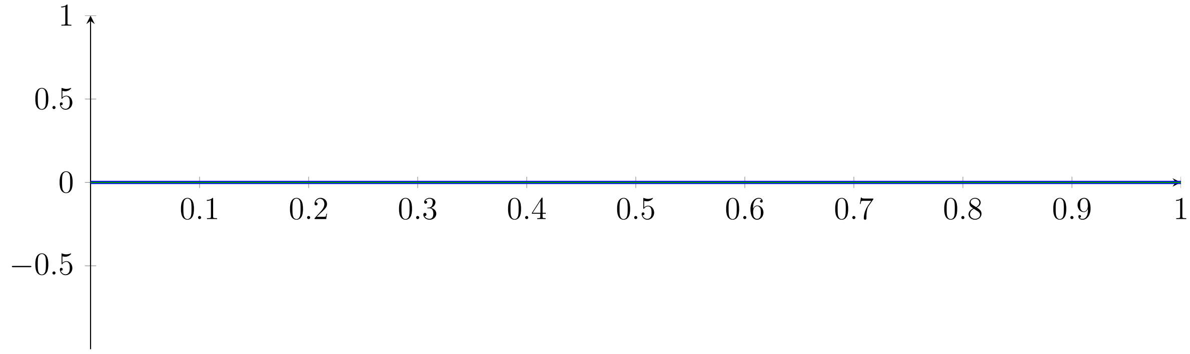
Bounding polynomials M and m:

$$M = -1.25865\cdot 10^{-153}X^3 - 2.8739\cdot 10^{-102}X^2 + 2.46075\cdot 10^{-50}X - 2.11876\cdot 10^{-14}$$
$$m = -1.25865\cdot 10^{-153}X^3 - 2.8739\cdot 10^{-102}X^2 + 2.46075\cdot 10^{-50}X - 2.11876\cdot 10^{-14}$$

Root of M and m:

$N(M) = \{-5.70827\cdot 10^{51}, 3.91034\cdot 10^{21}, 3.42496\cdot 10^{51}\}$ $N(m) = \{-5.70827\cdot 10^{51}, 3.91034\cdot 10^{21}, 3.42496\cdot 10^{51}\}$

Intersection intervals:

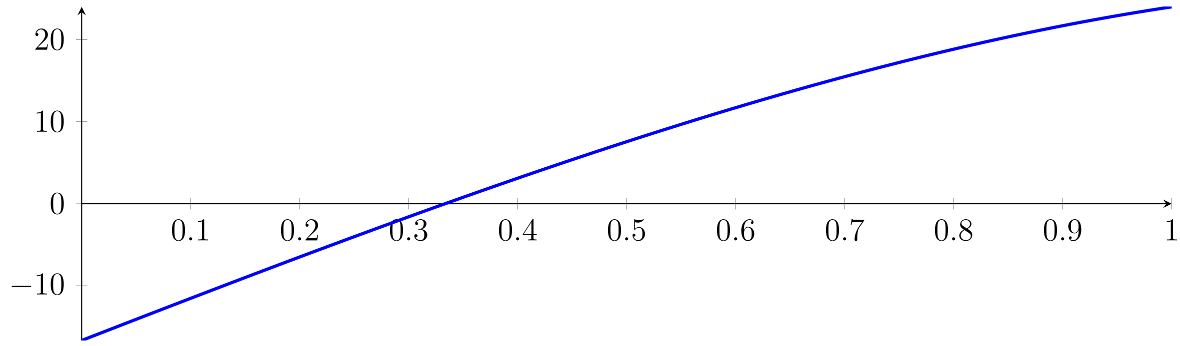


No intersection intervals with the x axis.

210.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

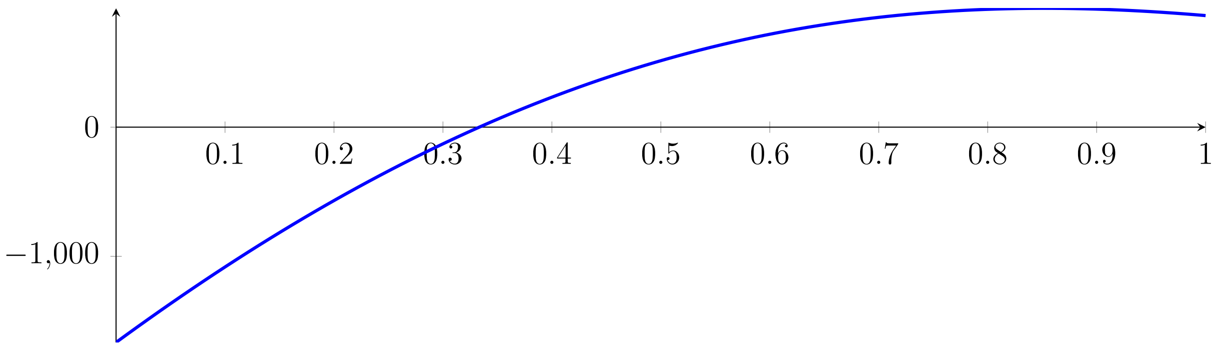
with precision $\varepsilon = 1 \cdot 10^{-128}$.

211 Running BezClip on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

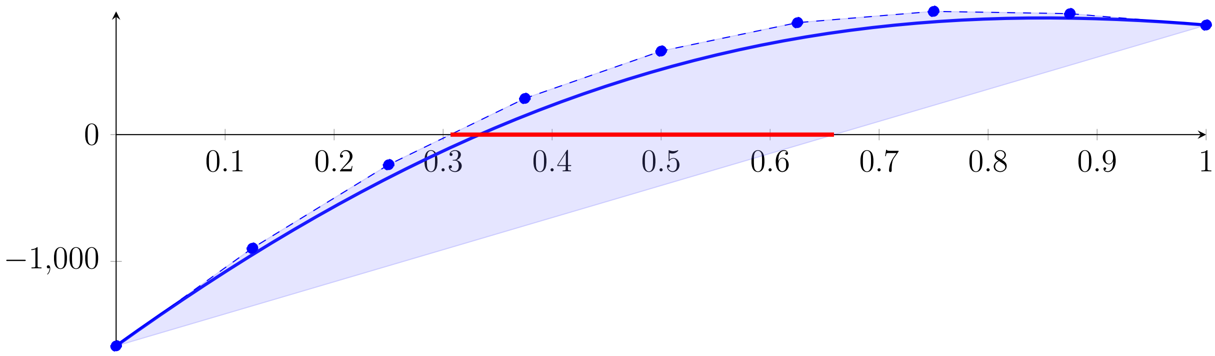
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



211.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$
$$= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X)$$
$$+ 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

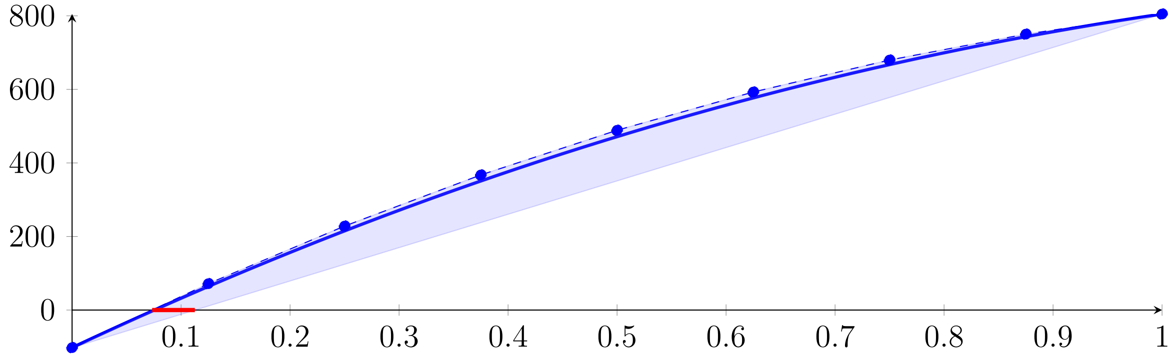
$$[0.306796, 0.658588]$$

Longest intersection interval: 0.351792
$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

211.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\ &+ 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\ = &-102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\ &+ 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

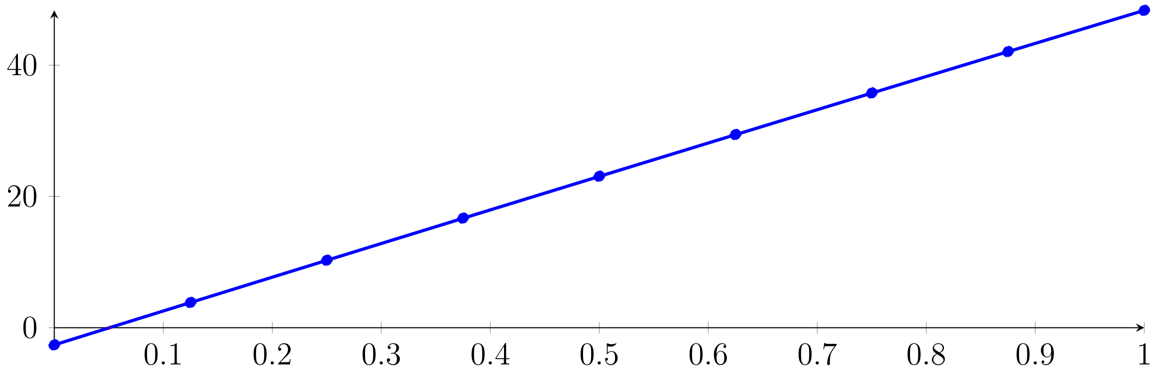
Longest intersection interval: 0.0391855

$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

211.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-1.30406 \cdot 10^{-15}X^8 - 1.54459 \cdot 10^{-12}X^7 - 4.9583 \cdot 10^{-10}X^6 + 3.66751 \cdot 10^{-08}X^5 \\ &+ 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\ = &-2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\ &+ 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

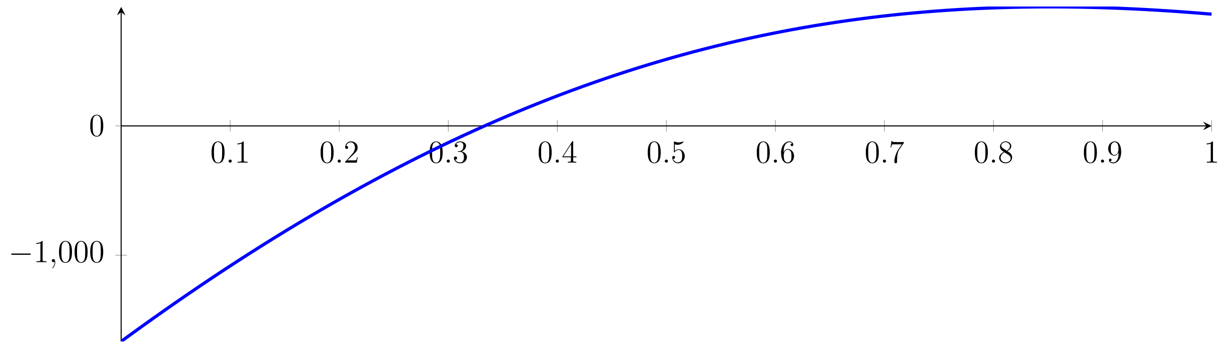
211.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Found root in interval $[0.333333, 0.333343]$ at recursion depth 4!

211.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333343]$$

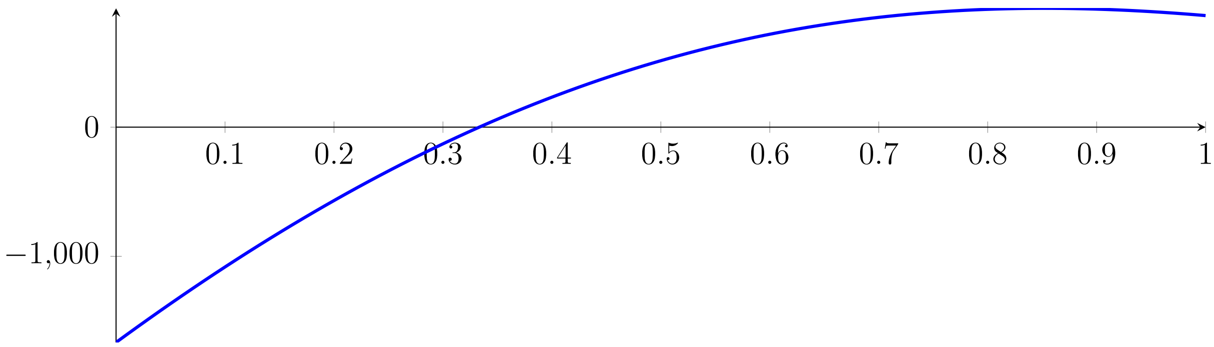
with precision $\varepsilon = 0.01$.

212 Running QuadClip on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

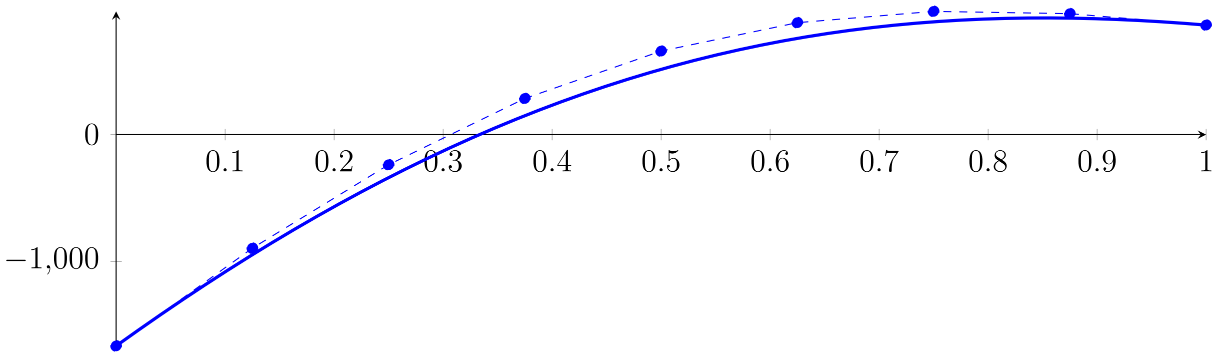
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



212.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

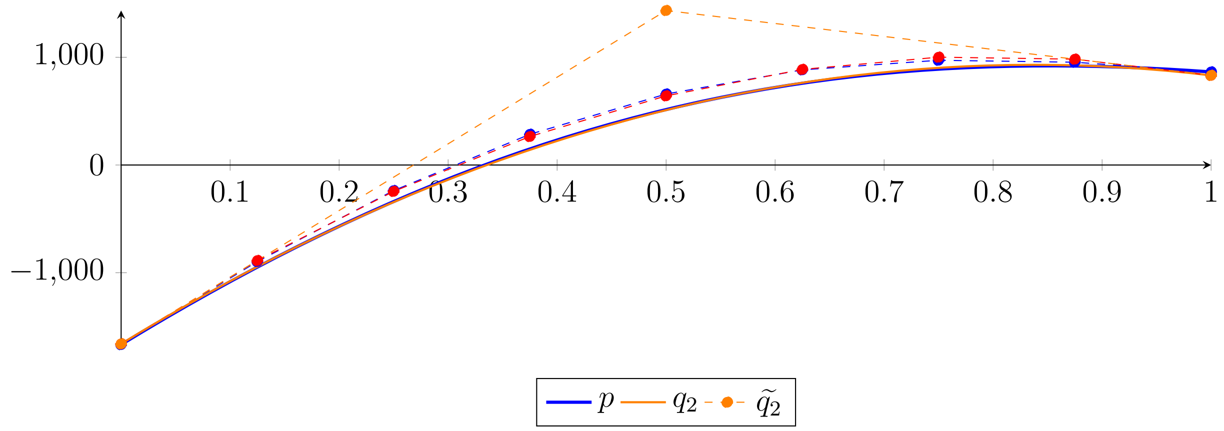
$$\begin{aligned}p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67\\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X)\\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -3695.78X^2 + 6187.64X - 1660\\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2}\end{aligned}$$

$$\begin{aligned}\tilde{q}_2 &= 6.38292 \cdot 10^{-300}X^8 - 2.86993 \cdot 10^{-299}X^7 + 5.15342 \cdot 10^{-299}X^6 - 4.6928 \cdot 10^{-299}X^5\\ &\quad + 2.29296 \cdot 10^{-299}X^4 - 6.02693 \cdot 10^{-300}X^3 - 3695.78X^2 + 6187.64X - 1660\\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8}\\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

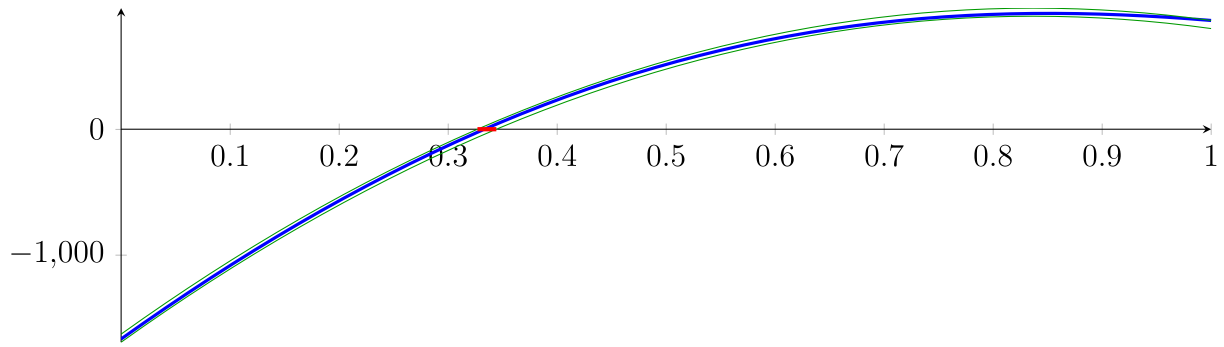
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

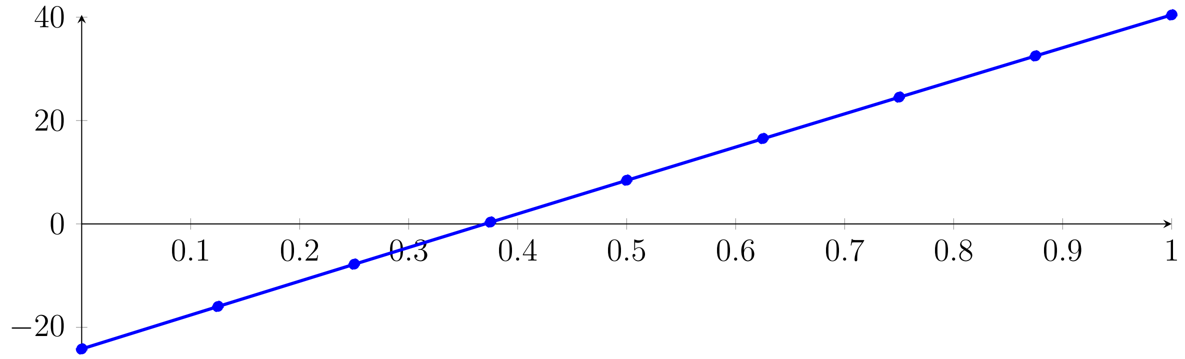
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

212.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

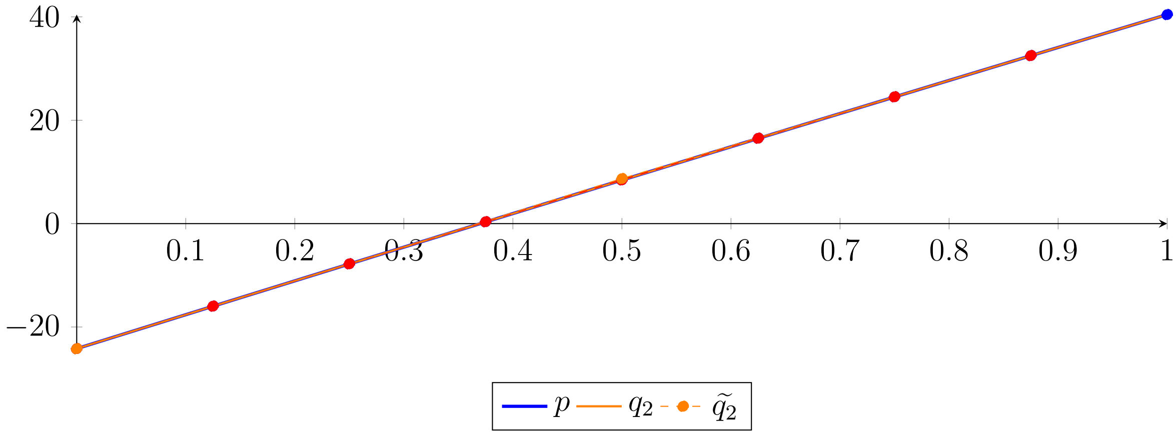
$$\begin{aligned} p = &-8.16249\cdot 10^{-15}X^8 - 7.66571\cdot 10^{-12}X^7 - 1.9444\cdot 10^{-09}X^6 + 1.19263\cdot 10^{-07}X^5 \\ &+ 8.12335\cdot 10^{-05}X^4 - 0.0012733X^3 - 1.18084X^2 + 65.8155X - 24.1945 \\ = &-24.1945B_{0,8}(X) - 15.9676B_{1,8}(X) - 7.78282B_{2,8}(X) + 0.35975B_{3,8}(X) \\ &+ 8.4601B_{4,8}(X) + 16.5182B_{5,8}(X) + 24.5341B_{6,8}(X) + 32.5077B_{7,8}(X) + 40.4389B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 4.27533{\cdot}10^{-302}X^8 - 2.42468{\cdot}10^{-301}X^7 + 4.90537{\cdot}10^{-301}X^6 - 4.6286{\cdot}10^{-301}X^5 \\ &\quad + 2.17546{\cdot}10^{-301}X^4 - 5.36695{\cdot}10^{-302}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894{\cdot}10^{-05}$.

Bounding polynomials M and m:

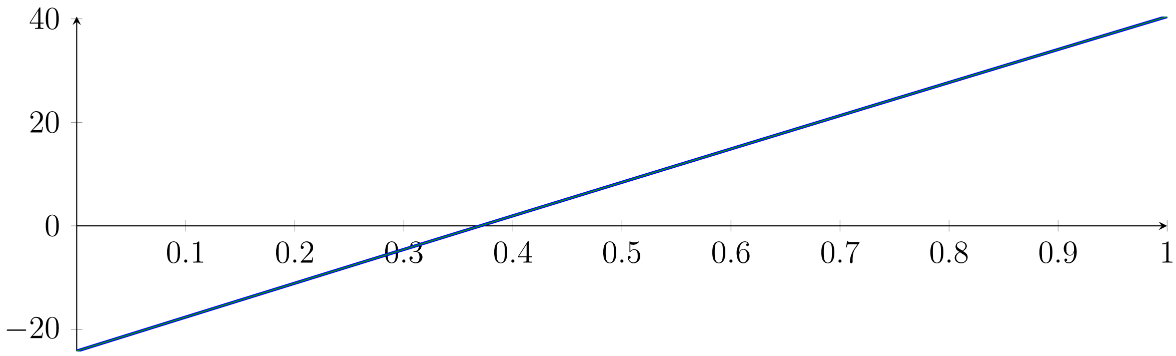
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588{\cdot}10^{-06}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

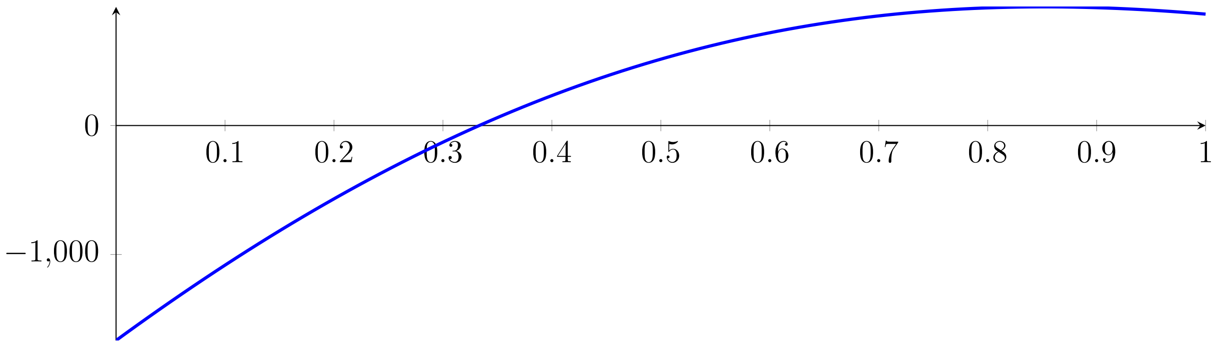
212.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

212.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

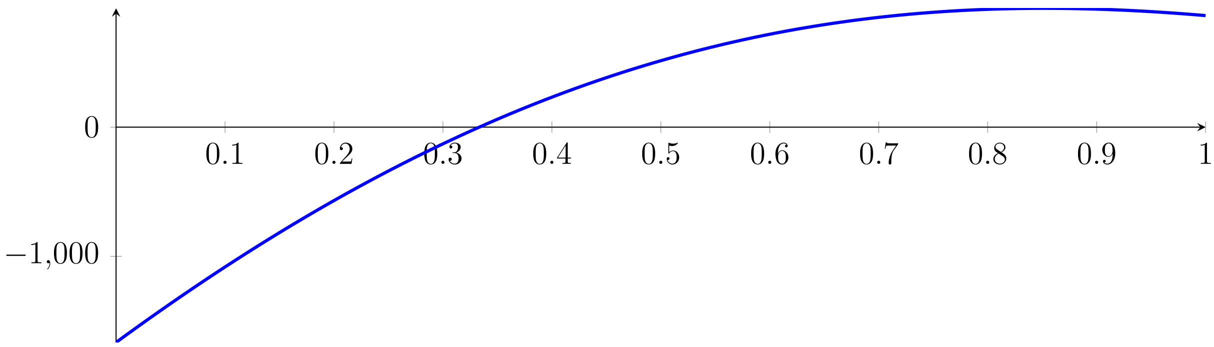
with precision $\varepsilon = 0.01$.

213 Running CubeClip on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

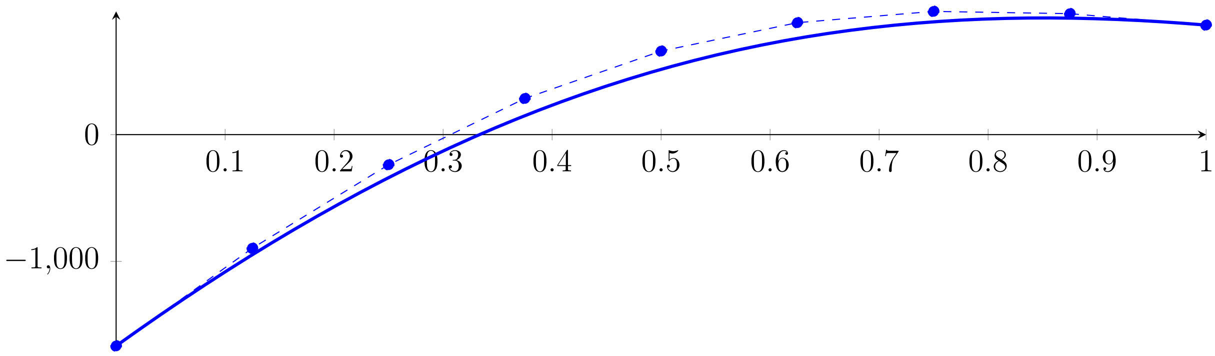
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



213.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

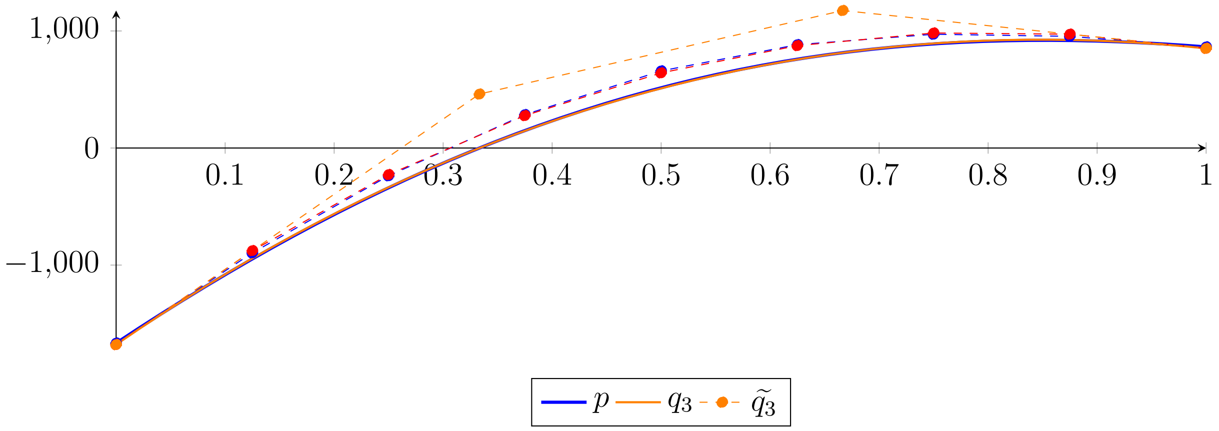
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.80826 \cdot 10^{-300}X^8 + 1.16606 \cdot 10^{-299}X^7 - 2.1197 \cdot 10^{-299}X^6 + 2.15003 \cdot 10^{-299}X^5 \\ &\quad - 1.20234 \cdot 10^{-299}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

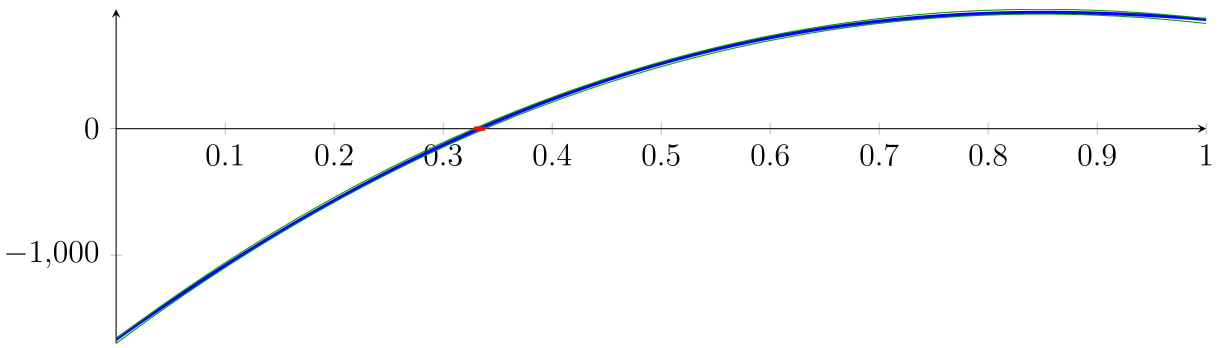
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$[0.328258, 0.338551]$

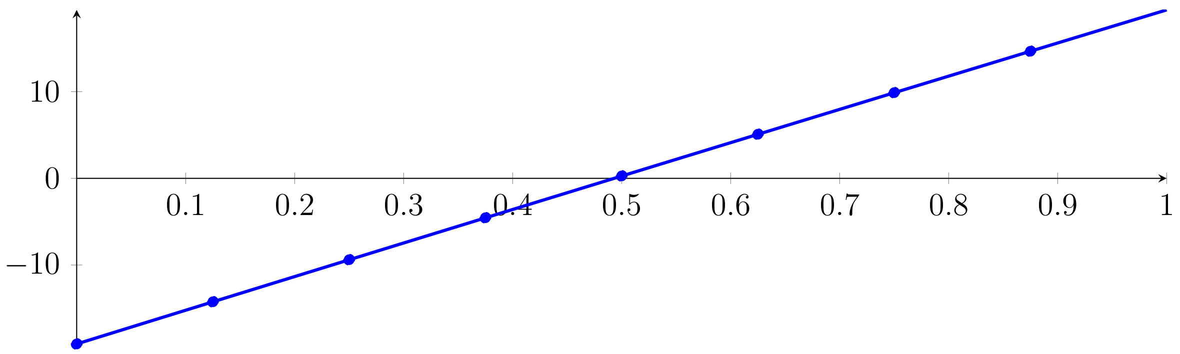
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

213.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

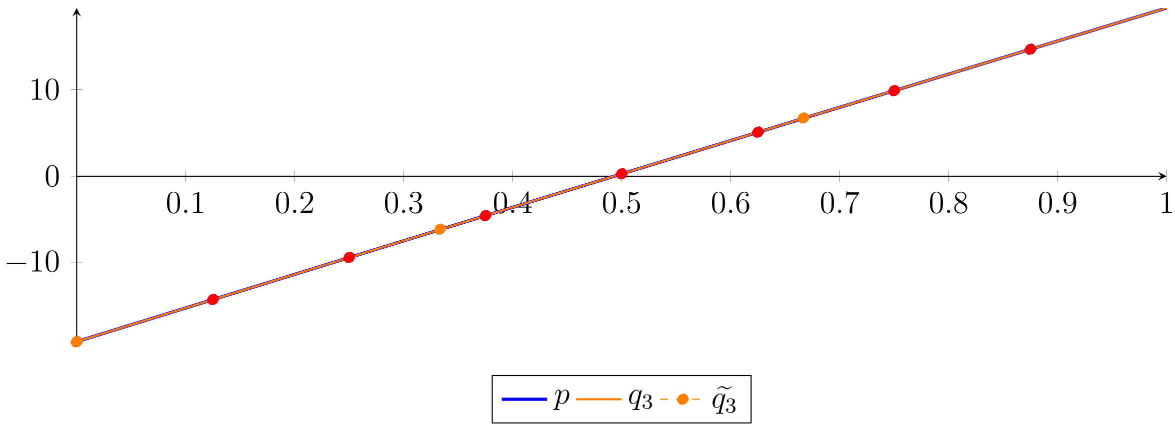
$$p = -1.25947 \cdot 10^{-16} X^8 - 1.9937 \cdot 10^{-13} X^7 - 8.53073 \cdot 10^{-11} X^6 + 8.72839 \cdot 10^{-09} X^5 + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124$$
$$= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X)$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1.96643 \cdot 10^{-303}X^8 + 1.82947 \cdot 10^{-302}X^7 - 4.89395 \cdot 10^{-302}X^6 + 5.49554 \cdot 10^{-302}X^5 \\ &\quad - 2.47838 \cdot 10^{-302}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m:

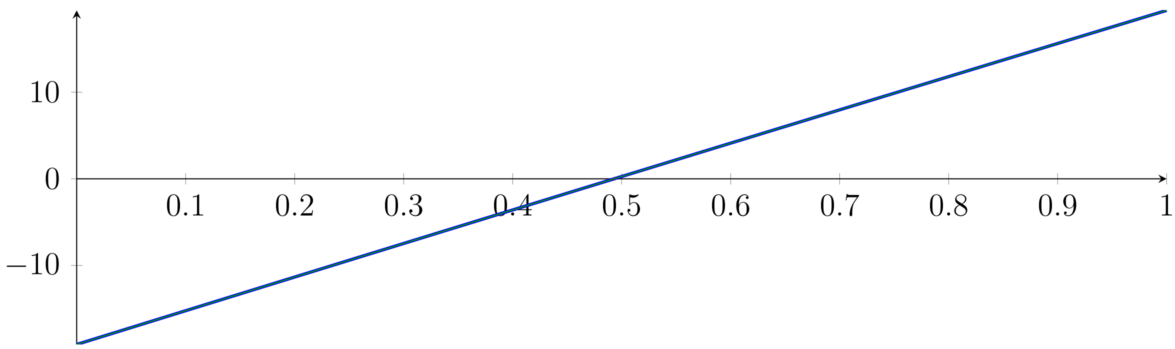
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.1252 \cdot 10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

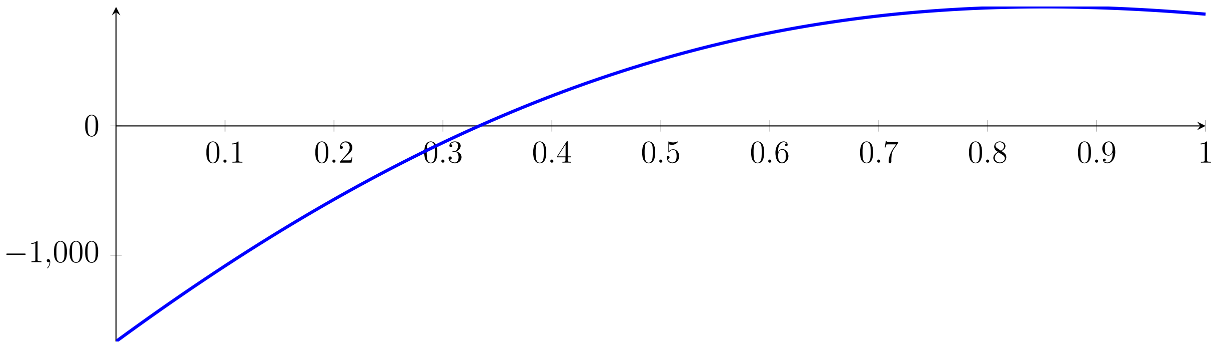
213.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

213.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

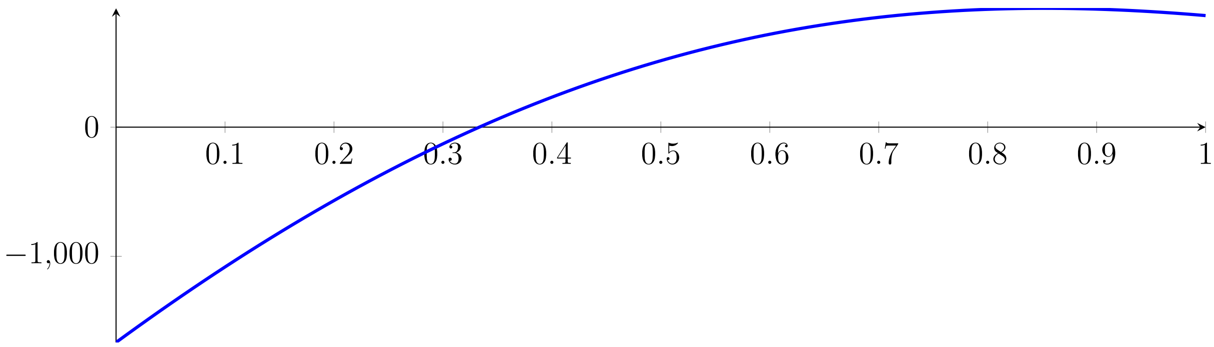
with precision $\varepsilon = 0.01$.

214 Running `BezClip` on f_8 with epsilon 4

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

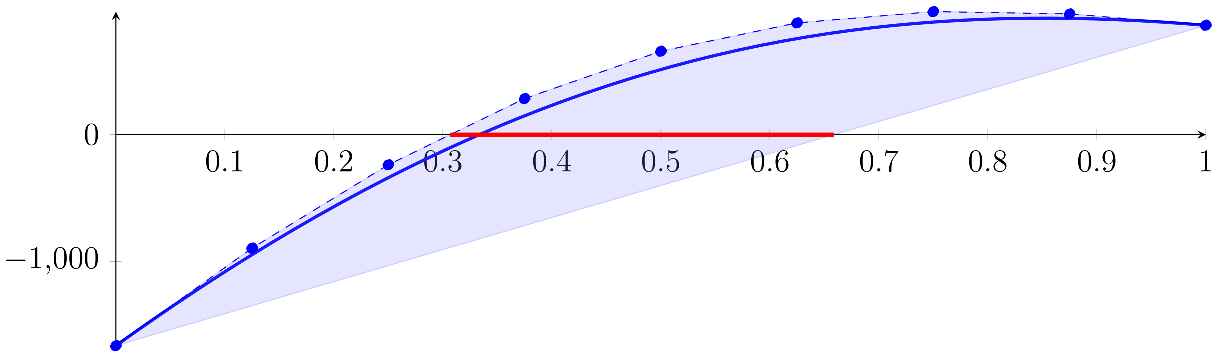
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



214.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

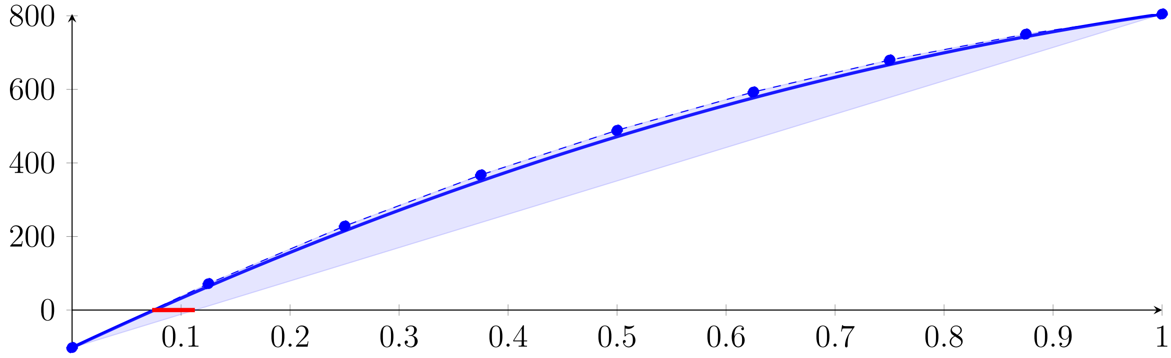
$$[0.306796, 0.658588]$$

Longest intersection interval: 0.351792
$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

214.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\ &+ 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\ = &-102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\ &+ 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

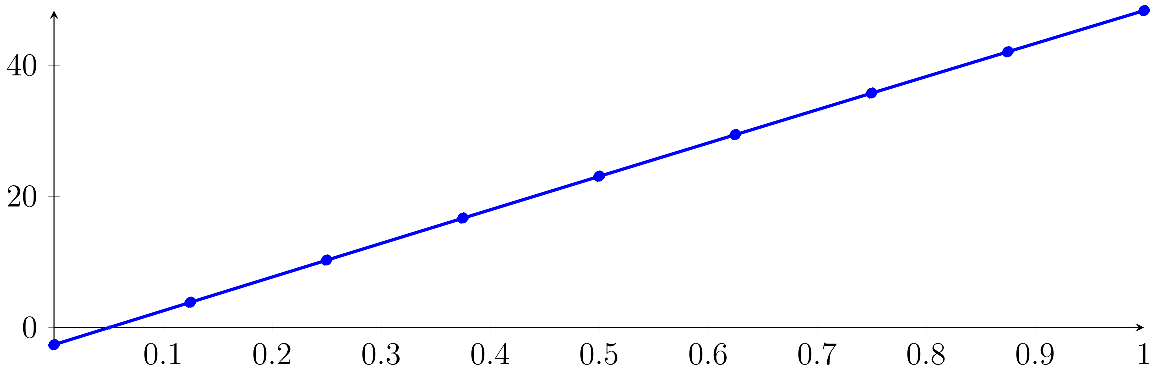
Longest intersection interval: 0.0391855

$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

214.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-1.30406 \cdot 10^{-15}X^8 - 1.54459 \cdot 10^{-12}X^7 - 4.9583 \cdot 10^{-10}X^6 + 3.66751 \cdot 10^{-08}X^5 \\ &+ 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\ = &-2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\ &+ 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

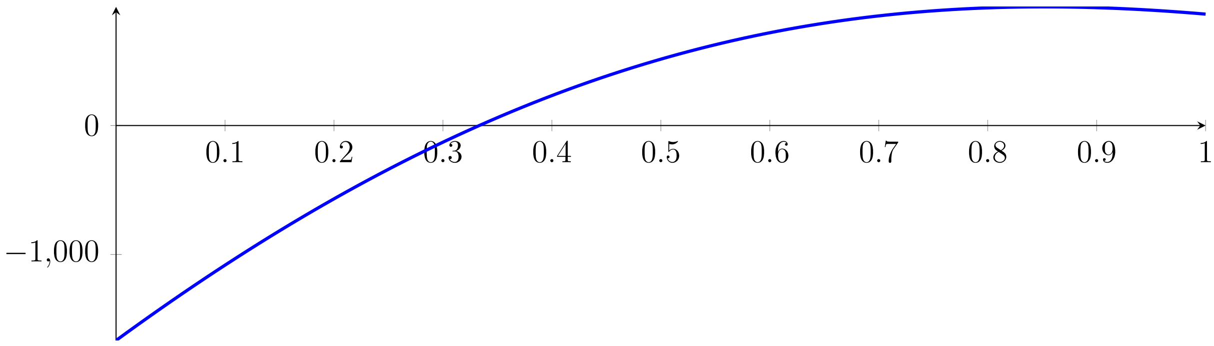
214.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Found root in interval $[0.333333, 0.333343]$ at recursion depth 4!

214.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333343]$$

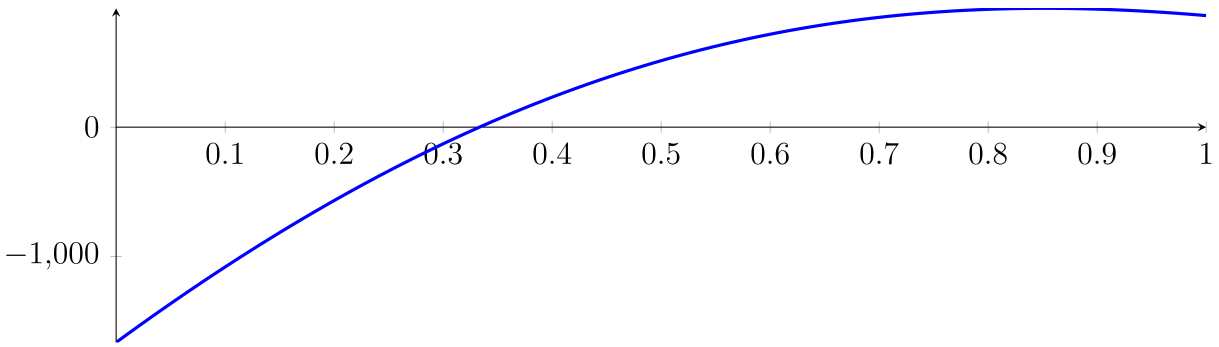
with precision $\varepsilon = 0.0001$.

215 Running QuadClip on f_8 with epsilon 4

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

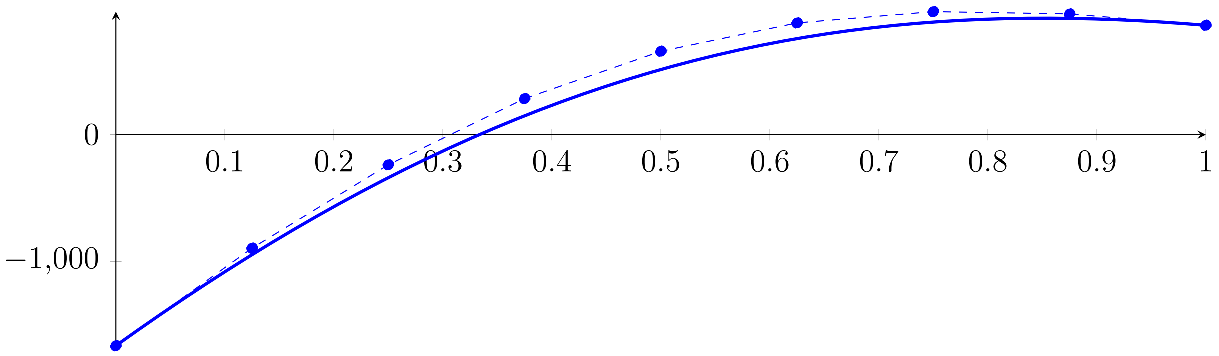
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



215.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

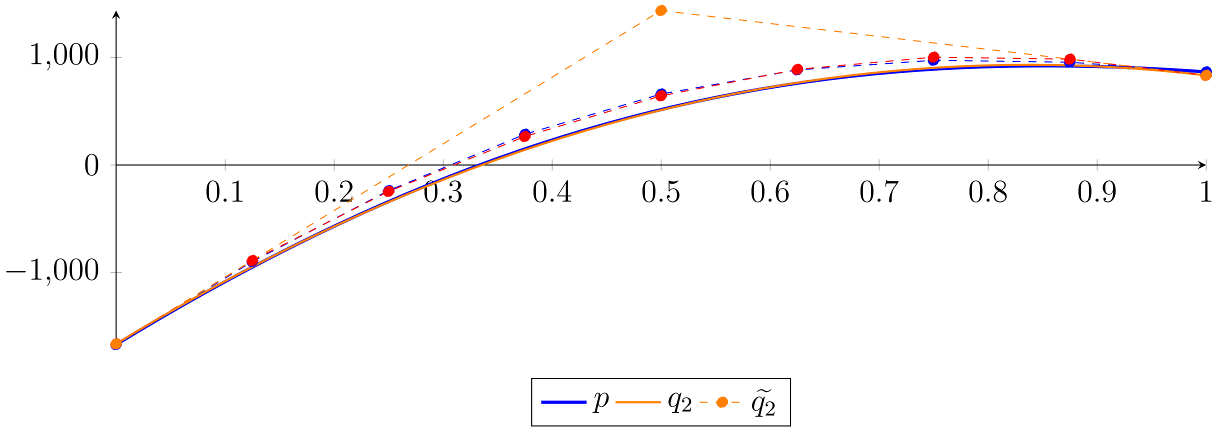
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 6.38292 \cdot 10^{-300}X^8 - 2.86993 \cdot 10^{-299}X^7 + 5.15342 \cdot 10^{-299}X^6 - 4.6928 \cdot 10^{-299}X^5 \\ &\quad + 2.29296 \cdot 10^{-299}X^4 - 6.02693 \cdot 10^{-300}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

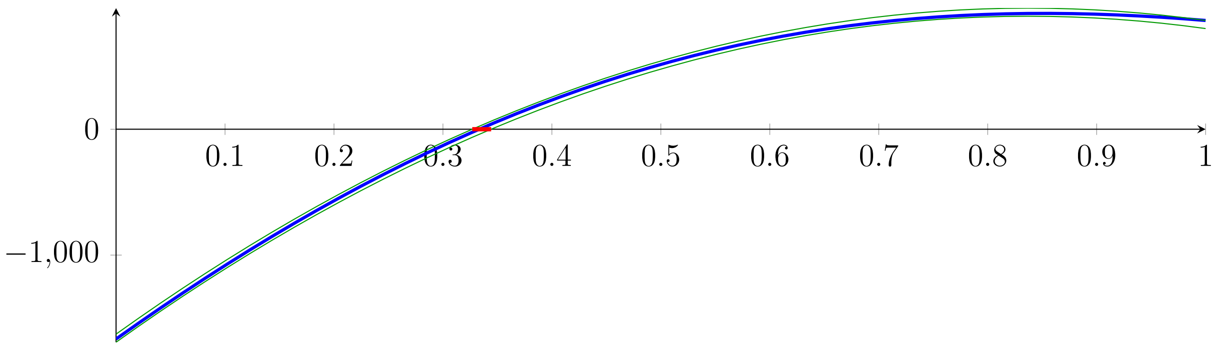
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

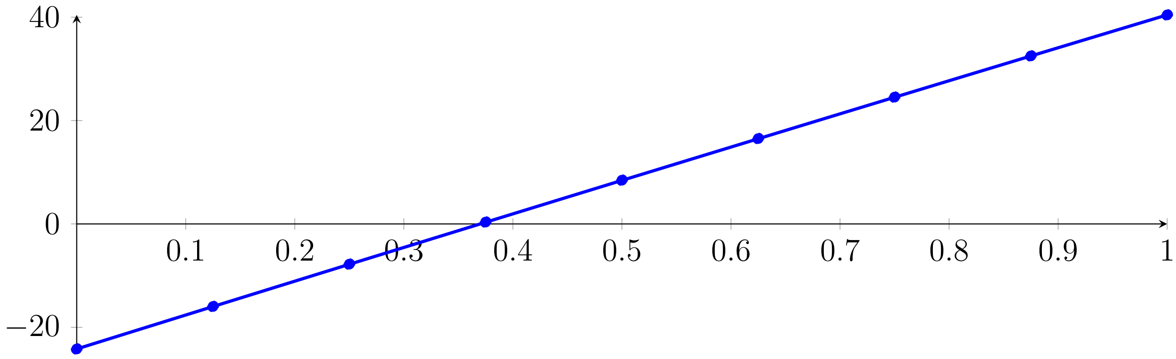
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

215.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

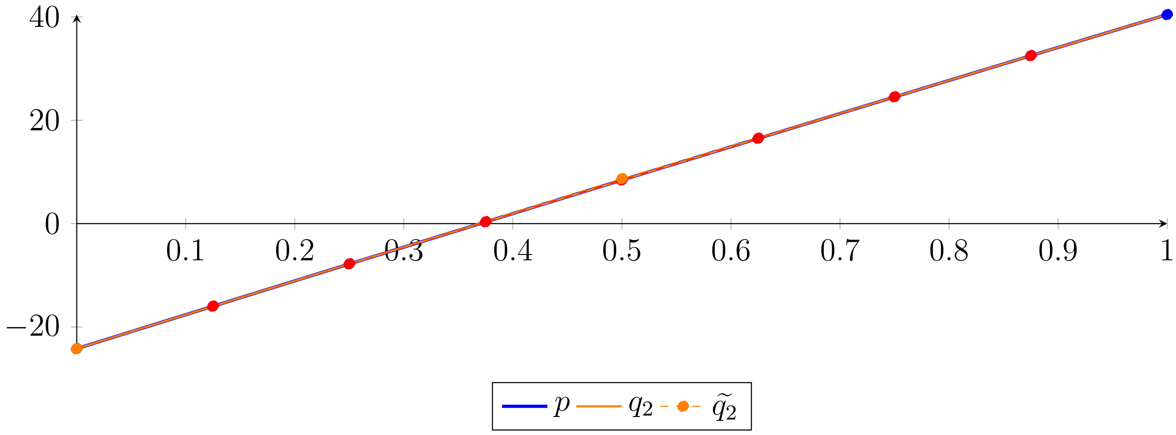
$$\begin{aligned} p = & -8.16249\cdot 10^{-15}X^8 - 7.66571\cdot 10^{-12}X^7 - 1.9444\cdot 10^{-09}X^6 + 1.19263\cdot 10^{-07}X^5 \\ & + 8.12335\cdot 10^{-05}X^4 - 0.0012733X^3 - 1.18084X^2 + 65.8155X - 24.1945 \\ = & -24.1945B_{0,8}(X) - 15.9676B_{1,8}(X) - 7.78282B_{2,8}(X) + 0.35975B_{3,8}(X) \\ & + 8.4601B_{4,8}(X) + 16.5182B_{5,8}(X) + 24.5341B_{6,8}(X) + 32.5077B_{7,8}(X) + 40.4389B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 4.27533{\cdot}10^{-302}X^8 - 2.42468{\cdot}10^{-301}X^7 + 4.90537{\cdot}10^{-301}X^6 - 4.6286{\cdot}10^{-301}X^5 \\ &\quad + 2.17546{\cdot}10^{-301}X^4 - 5.36695{\cdot}10^{-302}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894{\cdot}10^{-05}$.

Bounding polynomials M and m:

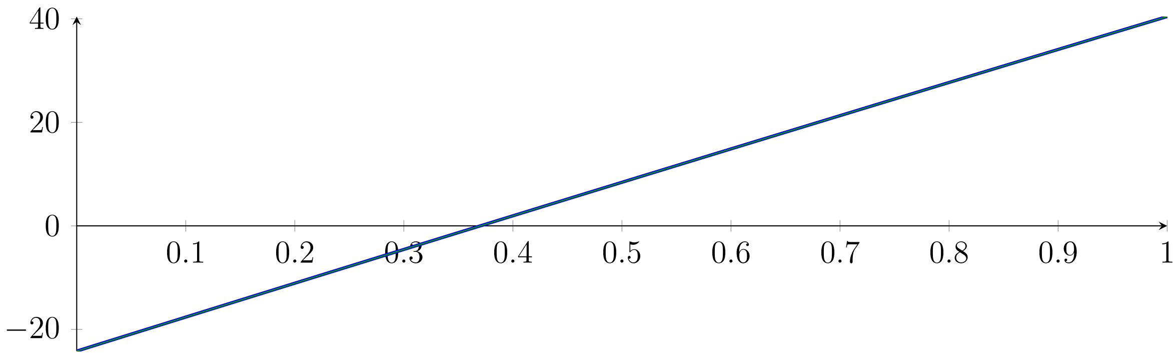
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588{\cdot}10^{-06}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

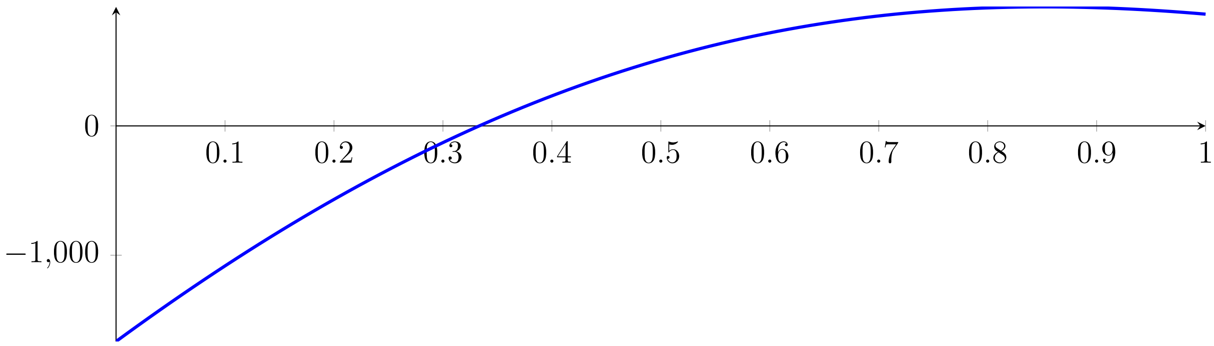
215.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

215.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

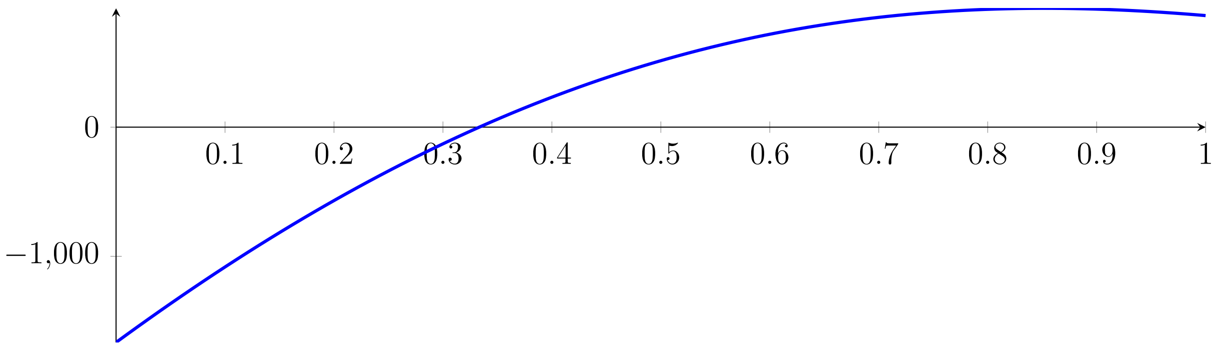
with precision $\varepsilon = 0.0001$.

216 Running CubeClip on f_8 with epsilon 4

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0,1]$:

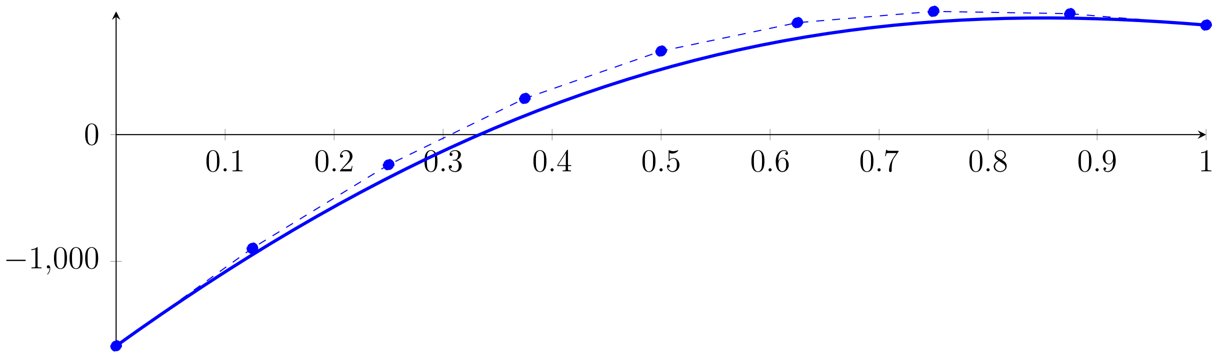
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



216.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

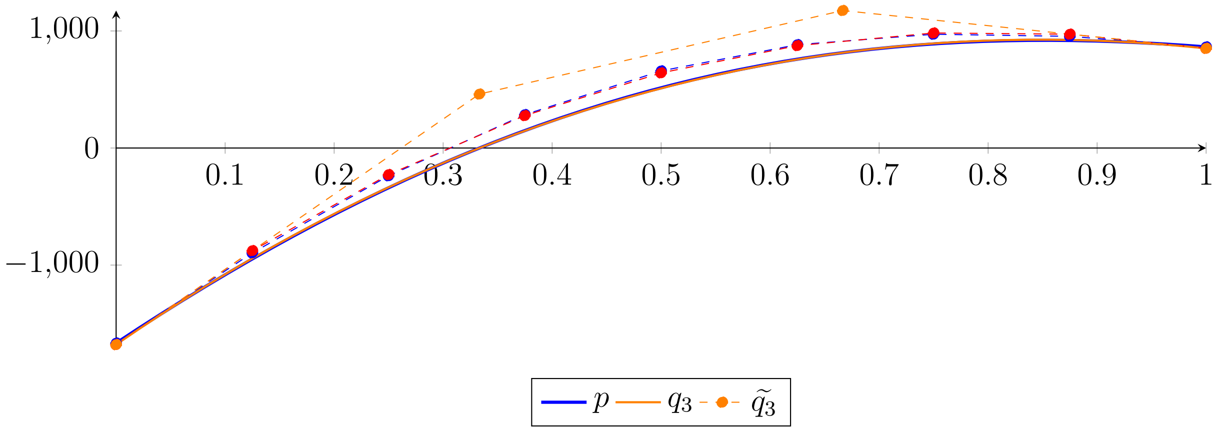
$$\begin{aligned}p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67\\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X)\\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47\\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3}\end{aligned}$$

$$\begin{aligned}\tilde{q}_3 &= -2.80826\cdot 10^{-300}X^8 + 1.16606\cdot 10^{-299}X^7 - 2.1197\cdot 10^{-299}X^6 + 2.15003\cdot 10^{-299}X^5\\ &\quad - 1.20234\cdot 10^{-299}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47\\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8}\\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

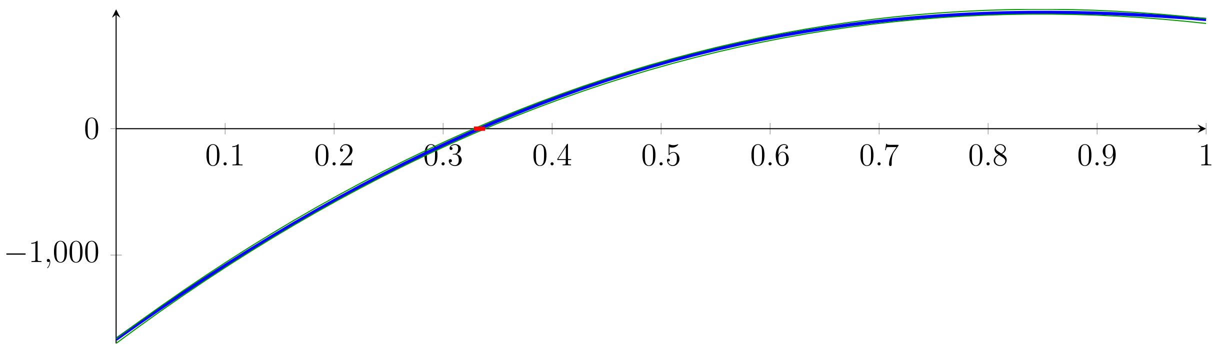
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$[0.328258, 0.338551]$

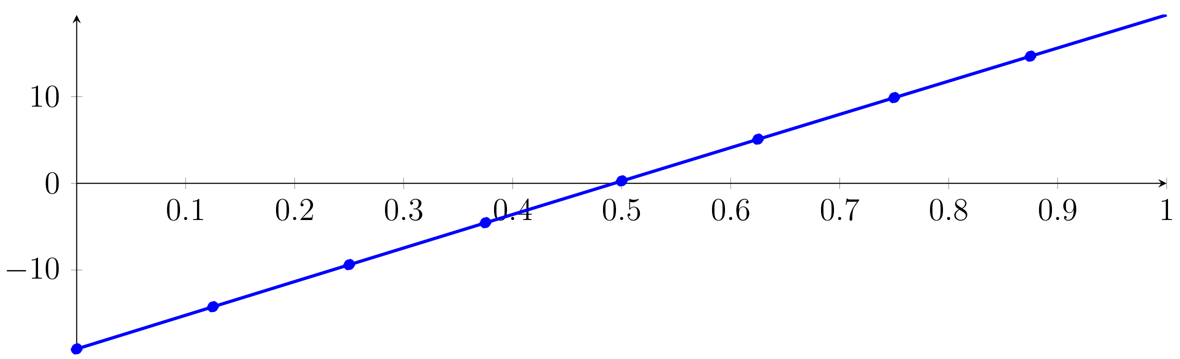
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

216.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

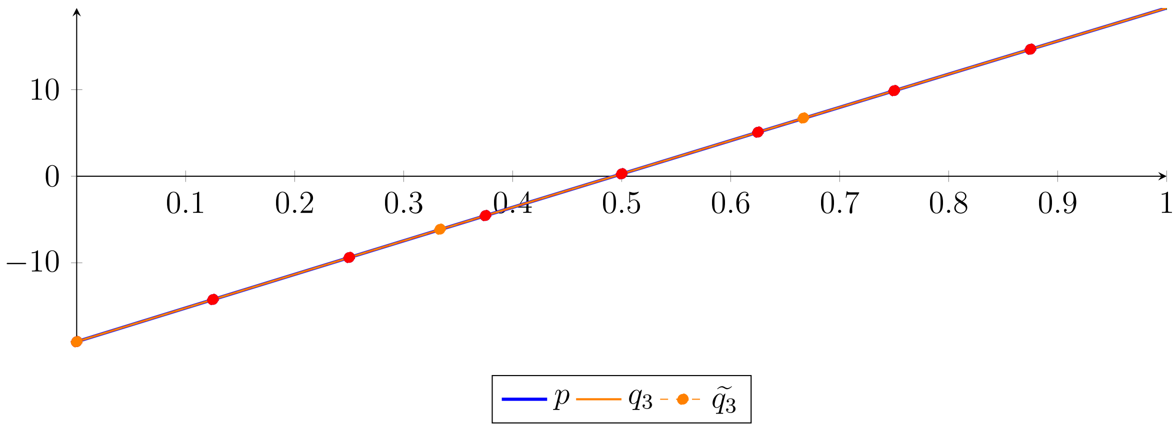
$$\begin{aligned} p &= -1.25947\cdot 10^{-16}X^8 - 1.9937\cdot 10^{-13}X^7 - 8.53073\cdot 10^{-11}X^6 + 8.72839\cdot 10^{-09}X^5 \\ &\quad + 1.00963\cdot 10^{-05}X^4 - 0.000261161X^3 - 0.416284X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8}(X) - 14.2419B_{1,8}(X) - 9.38619B_{2,8}(X) - 4.54539B_{3,8}(X) \\ &\quad + 0.280537B_{4,8}(X) + 5.09158B_{5,8}(X) + 9.88774B_{6,8}(X) + 14.669B_{7,8}(X) + 19.4354B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1.96643 \cdot 10^{-303}X^8 + 1.82947 \cdot 10^{-302}X^7 - 4.89395 \cdot 10^{-302}X^6 + 5.49554 \cdot 10^{-302}X^5 \\ &\quad - 2.47838 \cdot 10^{-302}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.
Bounding polynomials M and m:

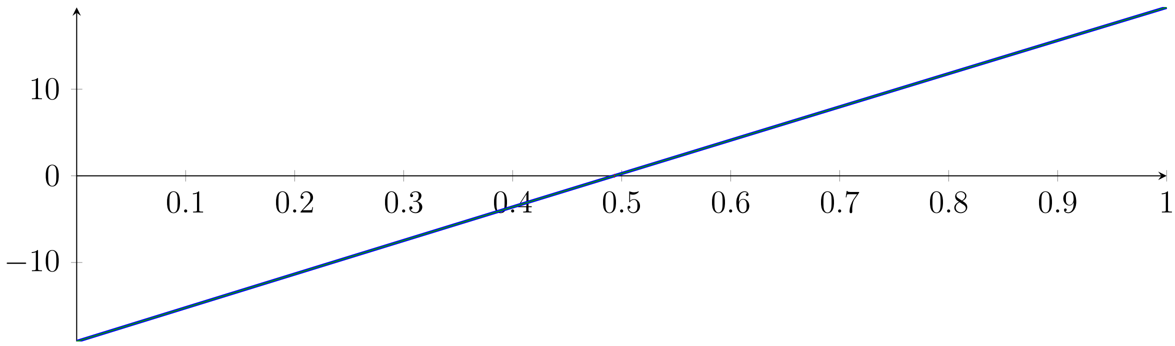
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.1252 \cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

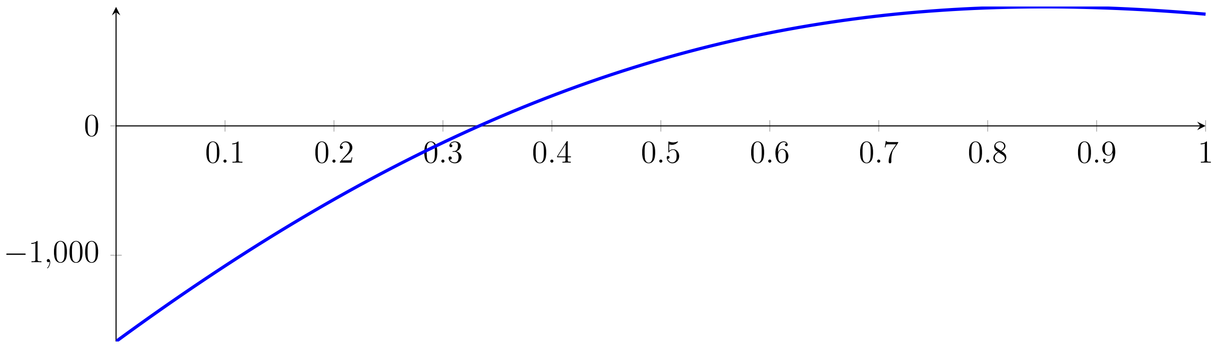
216.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

216.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 0.0001$.

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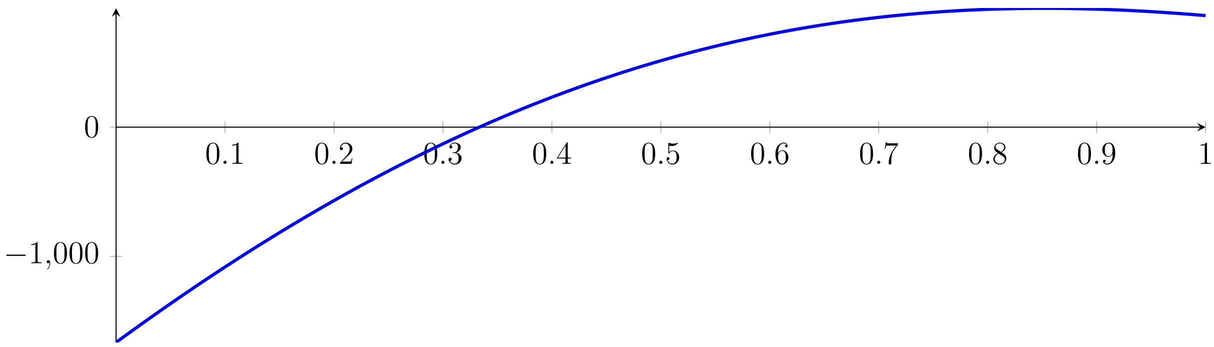
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0,1]$:

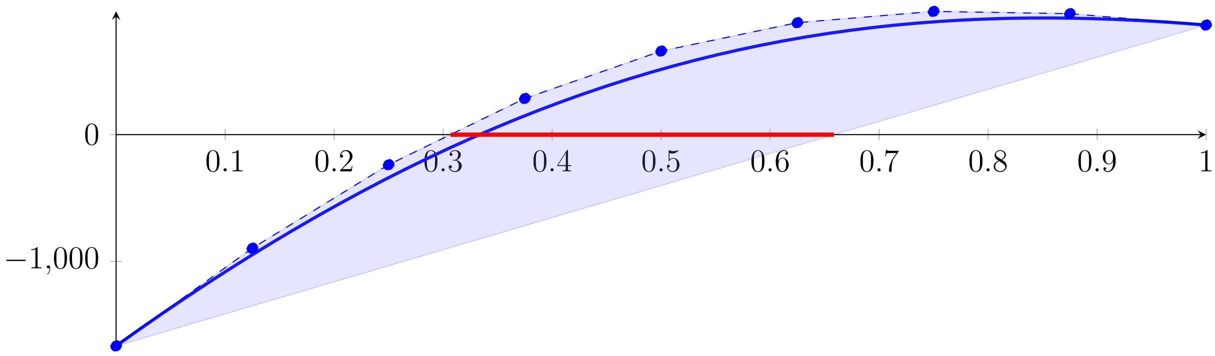
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



217.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

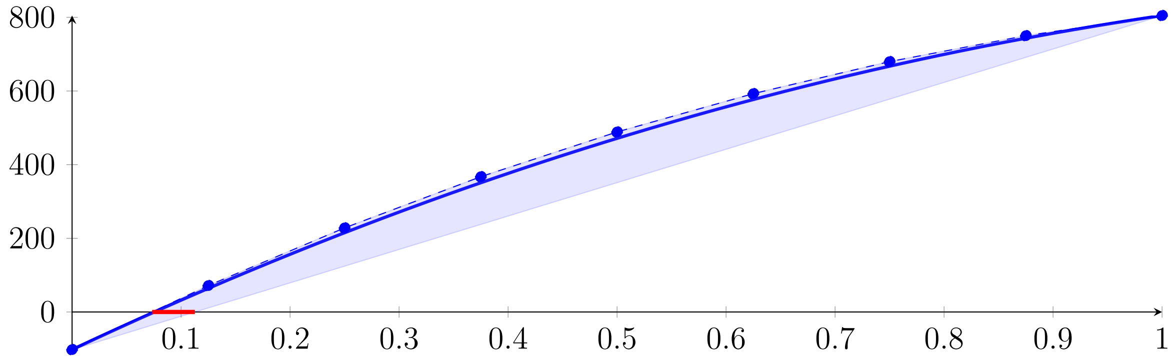
Longest intersection interval: 0.351792

$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

217.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\ &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\ &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\ &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

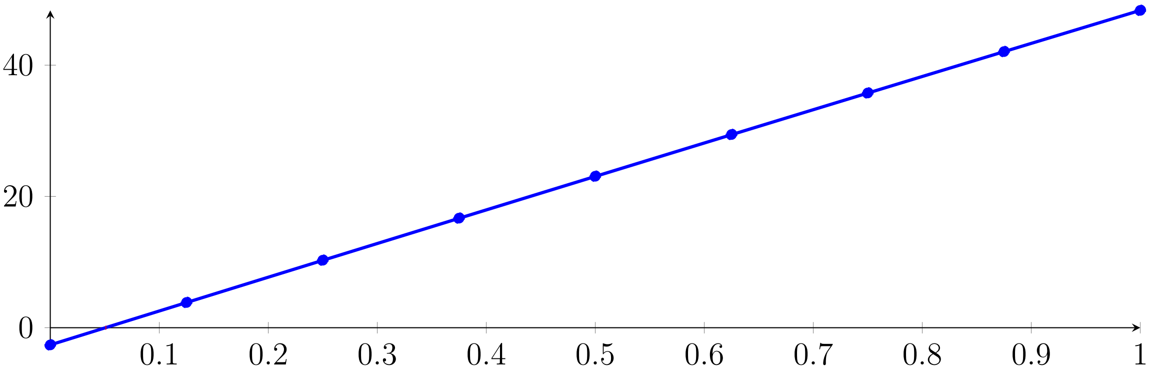
$$[0.0734515, 0.112637]$$

Longest intersection interval: 0.0391855
$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

217.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.30406 \cdot 10^{-15}X^8 - 1.54459 \cdot 10^{-12}X^7 - 4.9583 \cdot 10^{-10}X^6 + 3.66751 \cdot 10^{-08}X^5 \\ &\quad + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\ &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\ &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

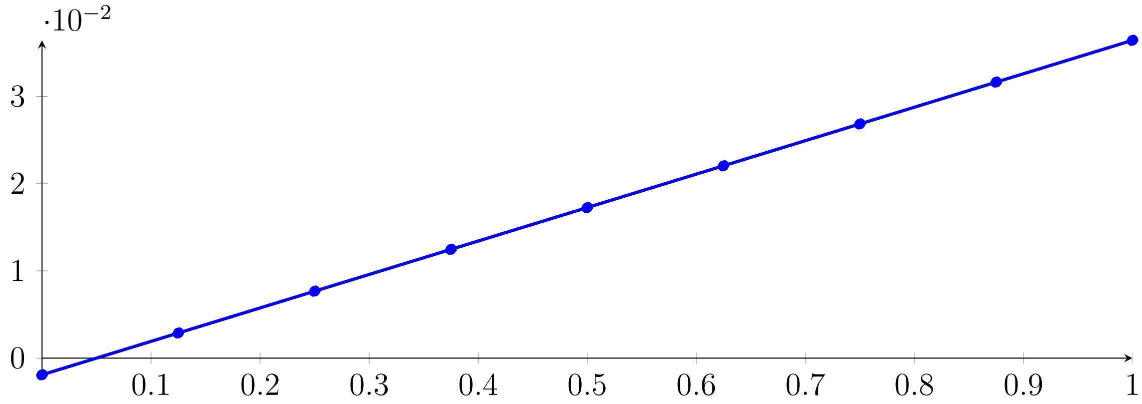
$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

217.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -1.20583\cdot10^{-40}X^8 - 1.92397\cdot10^{-34}X^7 - 8.32342\cdot10^{-29}X^6 + 8.24755\cdot10^{-24}X^5\\ &\quad + 9.89972\cdot10^{-18}X^4 - 2.37322\cdot10^{-13}X^3 - 4.12146\cdot10^{-07}X^2 + 0.0383444X - 0.00191378\\ &= -0.00191378B_{0,8}(X) + 0.00287927B_{1,8}(X) + 0.00767231B_{2,8}(X)\\ &\quad + 0.0124653B_{3,8}(X) + 0.0172583B_{4,8}(X) + 0.0220513B_{5,8}(X)\\ &\quad + 0.0268443B_{6,8}(X) + 0.0316373B_{7,8}(X) + 0.0364302B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

Longest intersection interval: $5.36469\cdot10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

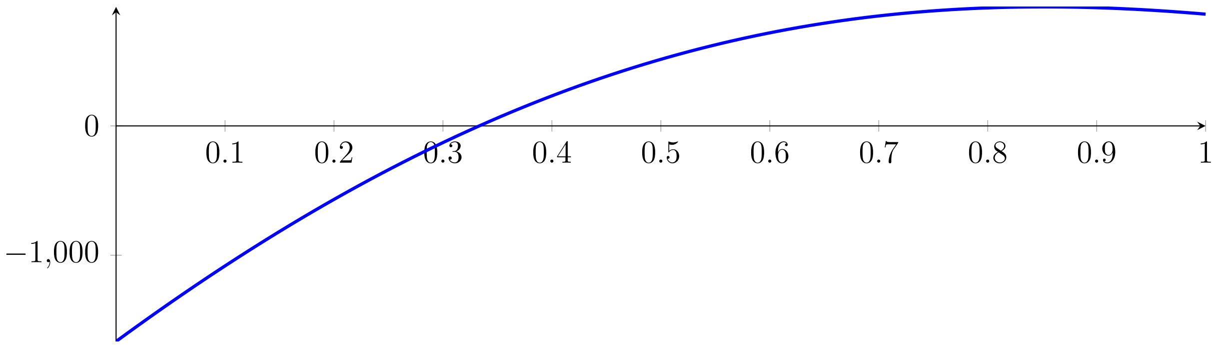
217.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

217.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-08}$.

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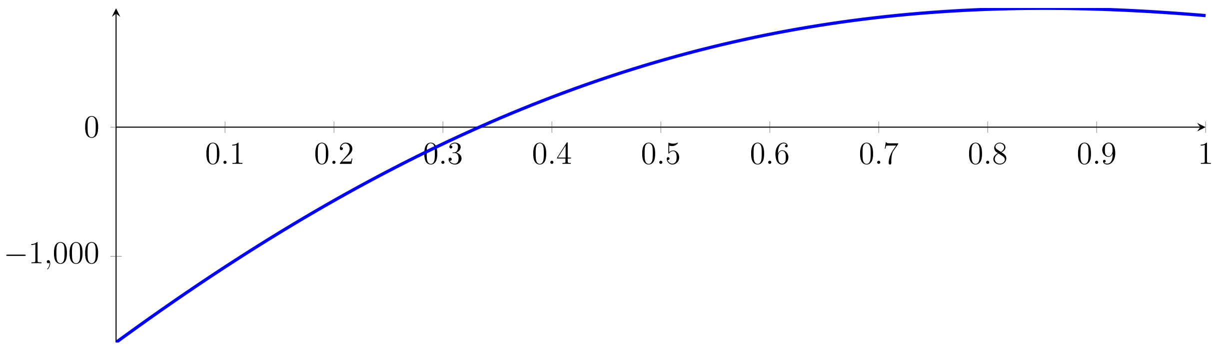
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

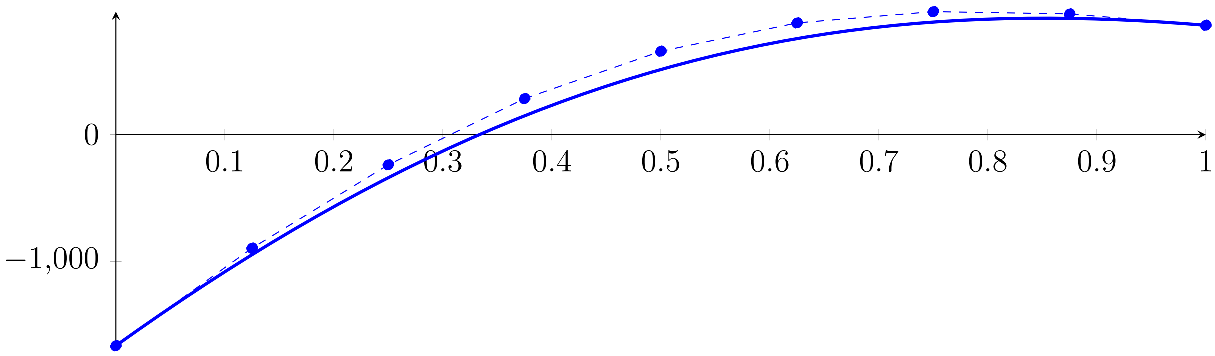
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



218.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

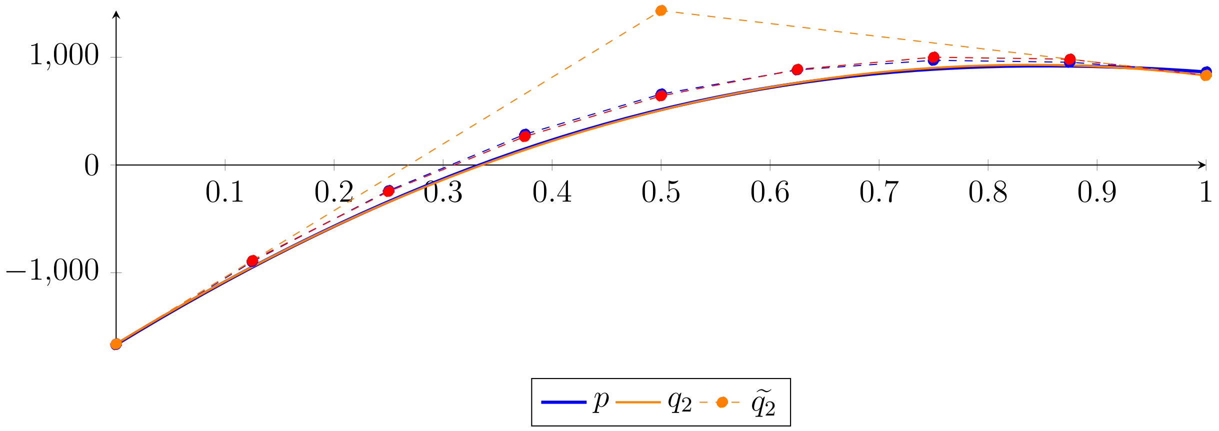
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 6.38292 \cdot 10^{-300}X^8 - 2.86993 \cdot 10^{-299}X^7 + 5.15342 \cdot 10^{-299}X^6 - 4.6928 \cdot 10^{-299}X^5 \\ &\quad + 2.29296 \cdot 10^{-299}X^4 - 6.02693 \cdot 10^{-300}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

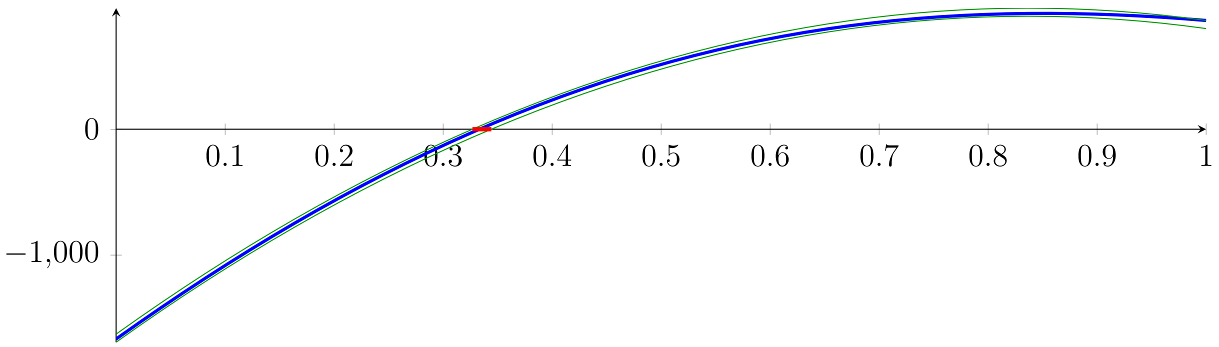
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

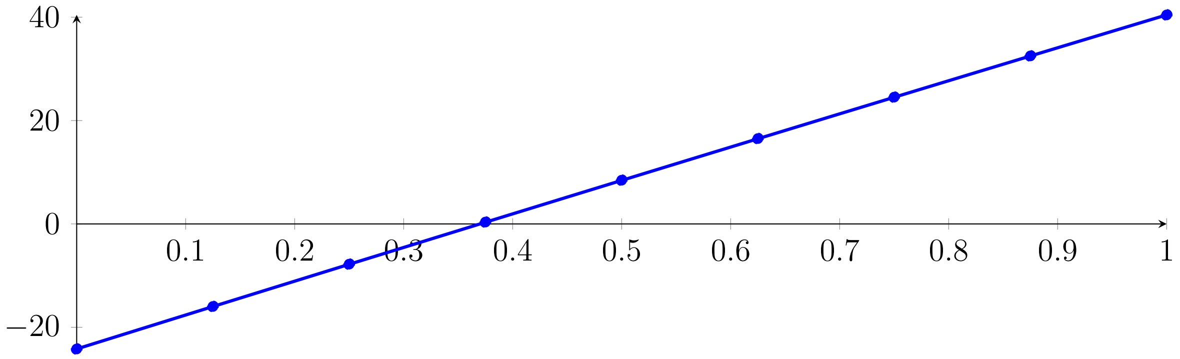
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

218.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

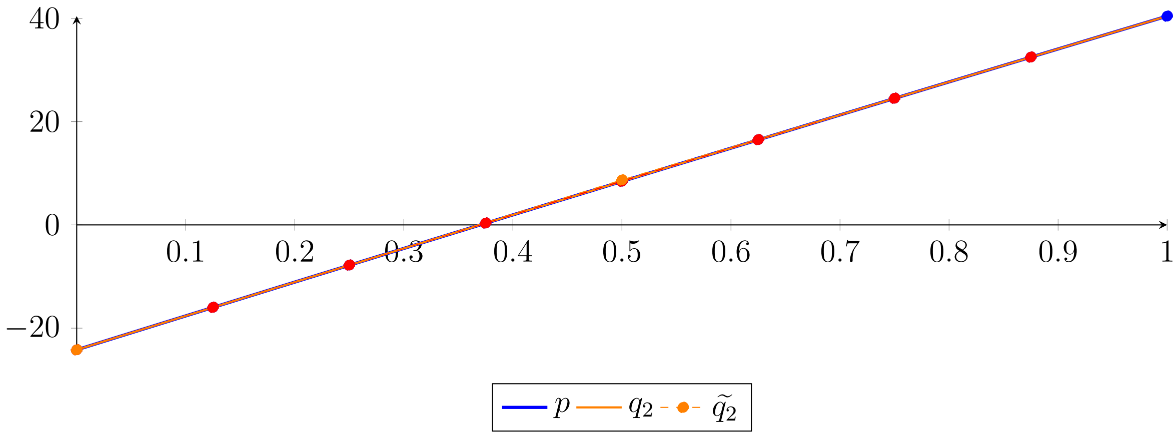
$$\begin{aligned} p = &-8.16249 \cdot 10^{-15} X^8 - 7.66571 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ &+ 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ = &-24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &+ 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 4.27533{\cdot}10^{-302}X^8 - 2.42468{\cdot}10^{-301}X^7 + 4.90537{\cdot}10^{-301}X^6 - 4.6286{\cdot}10^{-301}X^5 \\ &\quad + 2.17546{\cdot}10^{-301}X^4 - 5.36695{\cdot}10^{-302}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894{\cdot}10^{-05}$.

Bounding polynomials M and m:

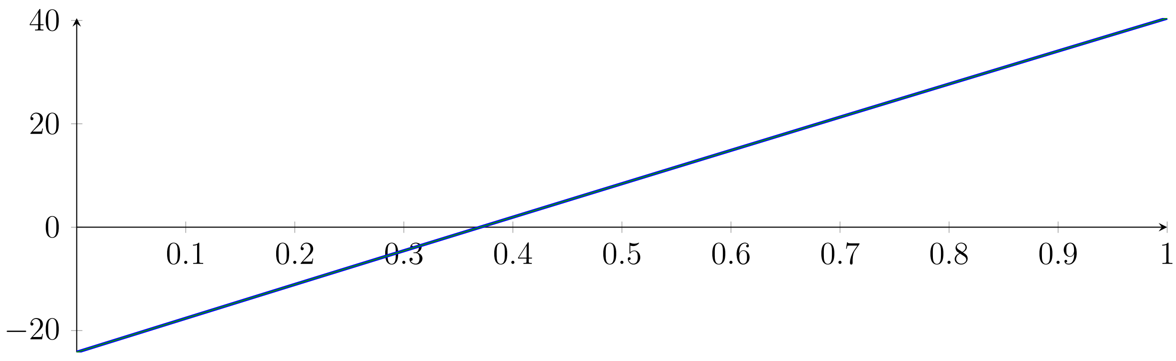
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

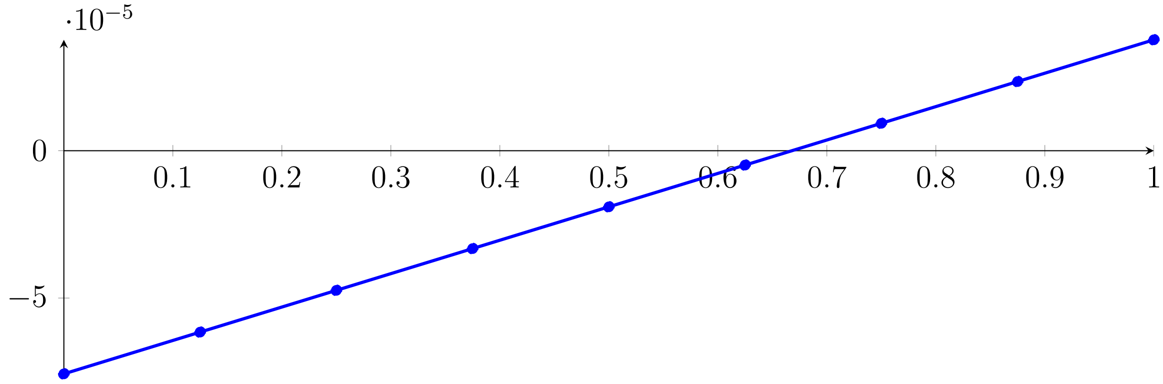
Longest intersection interval: $1.74588{\cdot}10^{-06}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

218.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

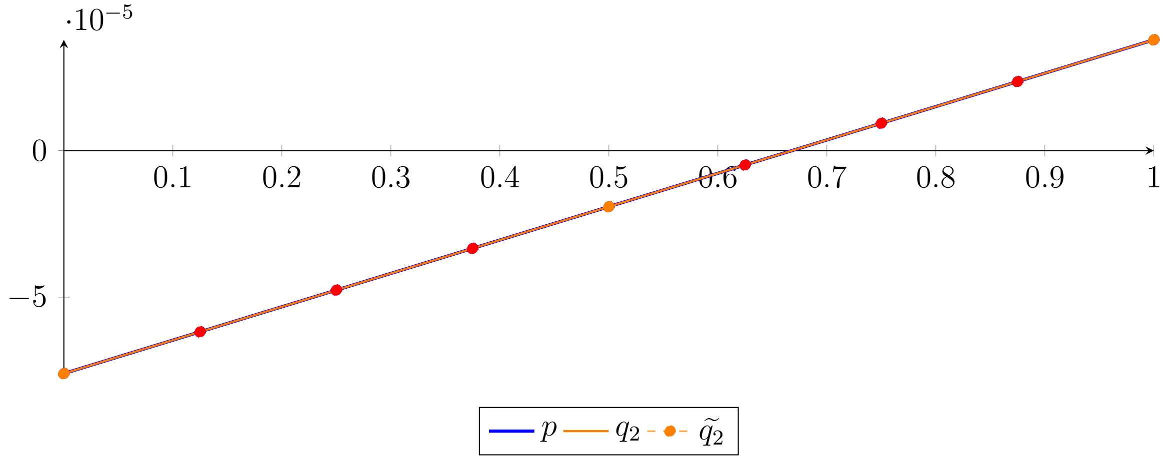
$$p = -7.04578\cdot10^{-61}X^8 - 3.80201\cdot10^{-52}X^7 - 5.5627\cdot10^{-44}X^6 + 1.86413\cdot10^{-36}X^5 + 7.56737\cdot10^{-28}X^4 - 6.13517\cdot10^{-21}X^3 - 3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$= -7.5732\cdot10^{-05}B_{0,8}(X) - 6.15596\cdot10^{-05}B_{1,8}(X) - 4.73873\cdot10^{-05}B_{2,8}(X) - 3.32149\cdot10^{-05}B_{3,8}(X) - 1.90425\cdot10^{-05}B_{4,8}(X) - 4.87016\cdot10^{-06}B_{5,8}(X) + 9.3022\cdot10^{-06}B_{6,8}(X) + 2.34746\cdot10^{-05}B_{7,8}(X) + 3.76469\cdot10^{-05}B_{8,8}(X)$$



Degree reduction and raising:

$$q_2 = -3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$= -7.5732\cdot10^{-05}B_{0,2} - 1.90425\cdot10^{-05}B_{1,2} + 3.76469\cdot10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = -3.62396\cdot10^{-308}X^8 + 2.32992\cdot10^{-308}X^7 + 2.61753\cdot10^{-307}X^6 - 5.97049\cdot10^{-307}X^5 + 5.13401\cdot10^{-307}X^4 - 1.82868\cdot10^{-307}X^3 - 3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$= -7.5732\cdot10^{-05}B_{0,8} - 6.15596\cdot10^{-05}B_{1,8} - 4.73873\cdot10^{-05}B_{2,8} - 3.32149\cdot10^{-05}B_{3,8} - 1.90425\cdot10^{-05}B_{4,8} - 4.87016\cdot10^{-06}B_{5,8} + 9.3022\cdot10^{-06}B_{6,8} + 2.34746\cdot10^{-05}B_{7,8} + 3.76469\cdot10^{-05}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 3.06758\cdot10^{-22}$.

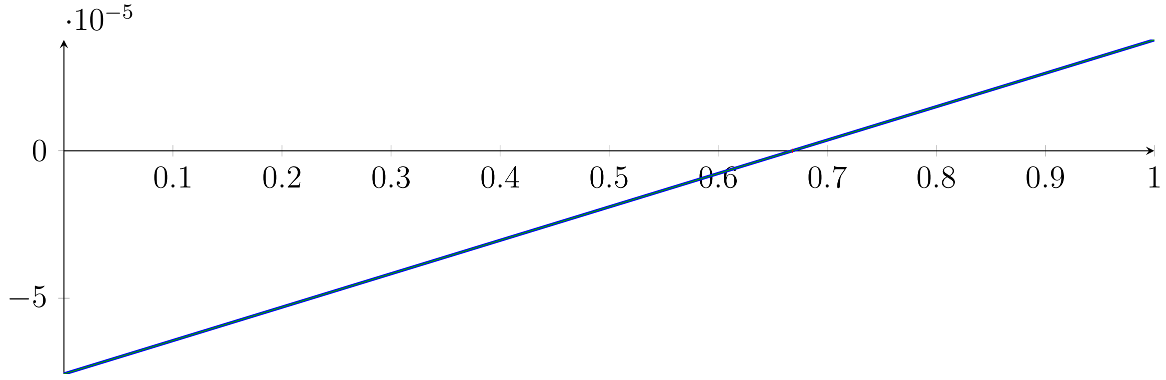
Bounding polynomials M and m:

$$M = -3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$
$$m = -3.6034\cdot10^{-12}X^2 + 0.000113379X - 7.5732\cdot10^{-05}$$

Root of M and m:

$$N(M) = \{0.667955, 3.14644\cdot10^{7}\} \qquad N(m) = \{0.667955, 3.14644\cdot10^{7}\}$$

Intersection intervals:



$[0.667955, 0.667955]$

Longest intersection interval: $5.41121 \cdot 10^{-18}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

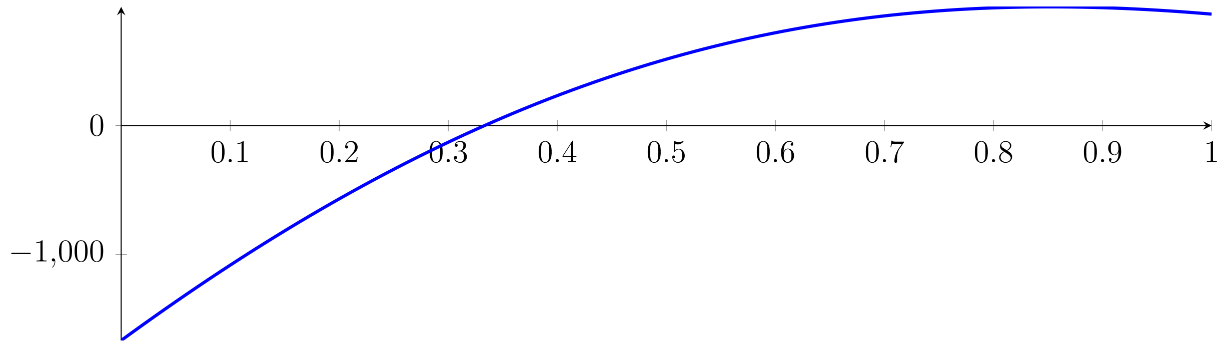
218.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

218.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-08}$.

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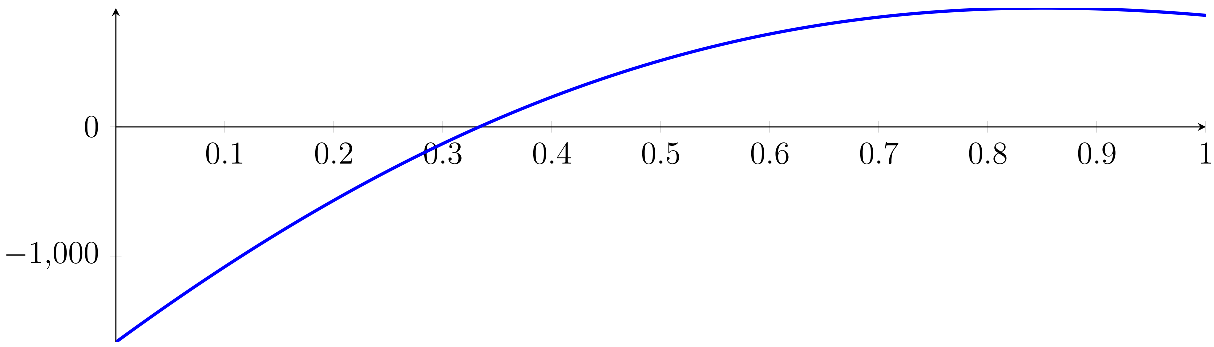
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

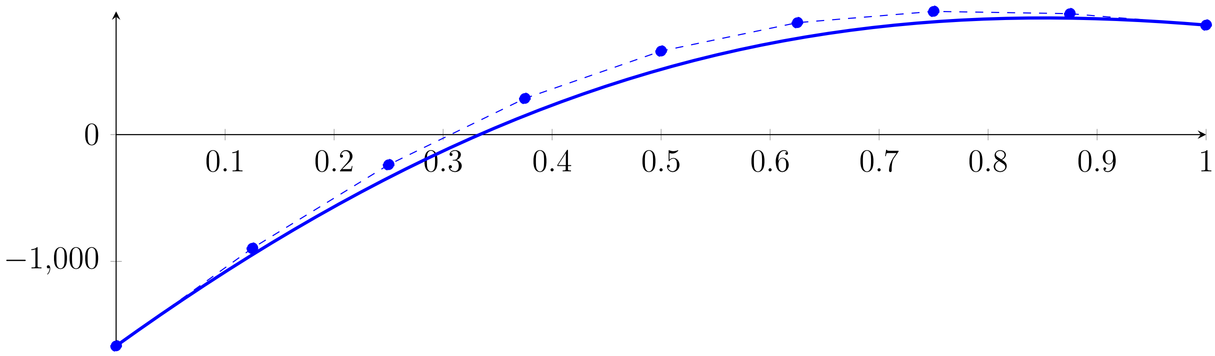
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



219.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

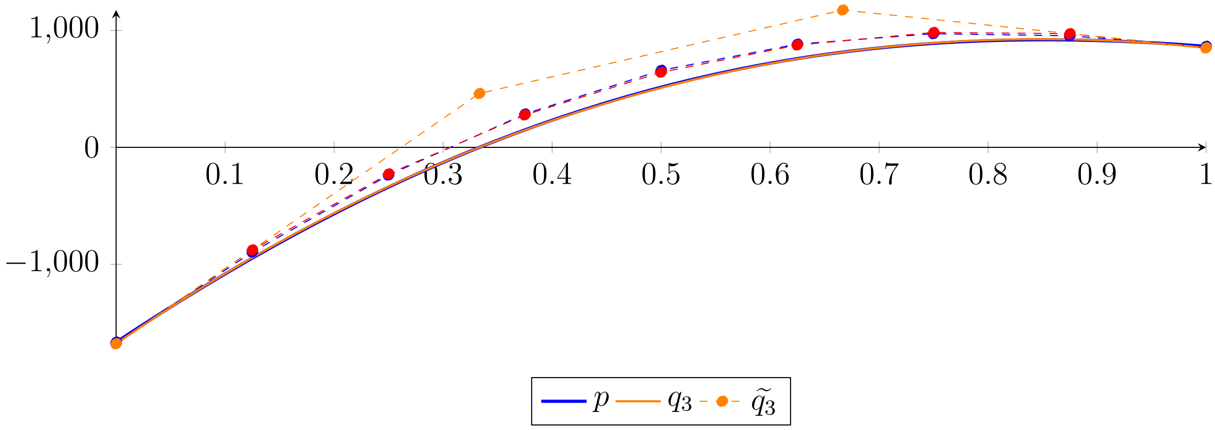
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -2.80826 \cdot 10^{-300}X^8 + 1.16606 \cdot 10^{-299}X^7 - 2.1197 \cdot 10^{-299}X^6 + 2.15003 \cdot 10^{-299}X^5 \\ &\quad - 1.20234 \cdot 10^{-299}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



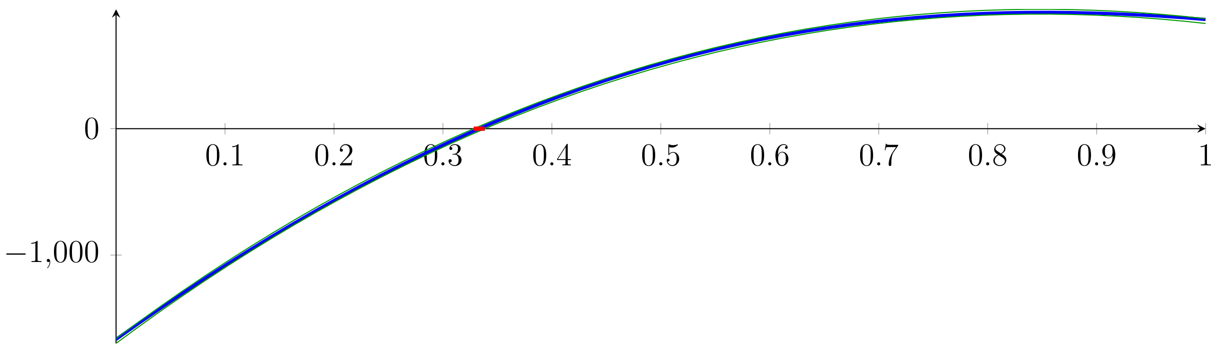
The maximum difference of the Bézier coefficients is $\delta = 19.0273$.
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



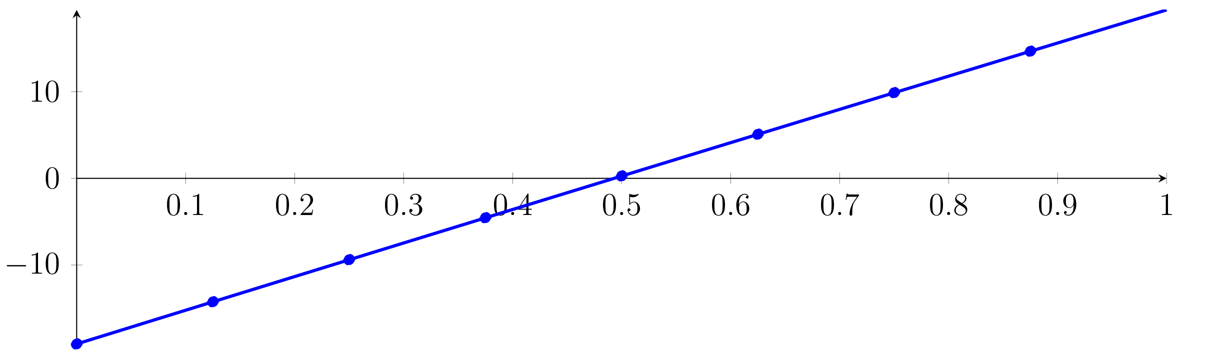
$[0.328258, 0.338551]$

Longest intersection interval: 0.0102926
$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

219.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

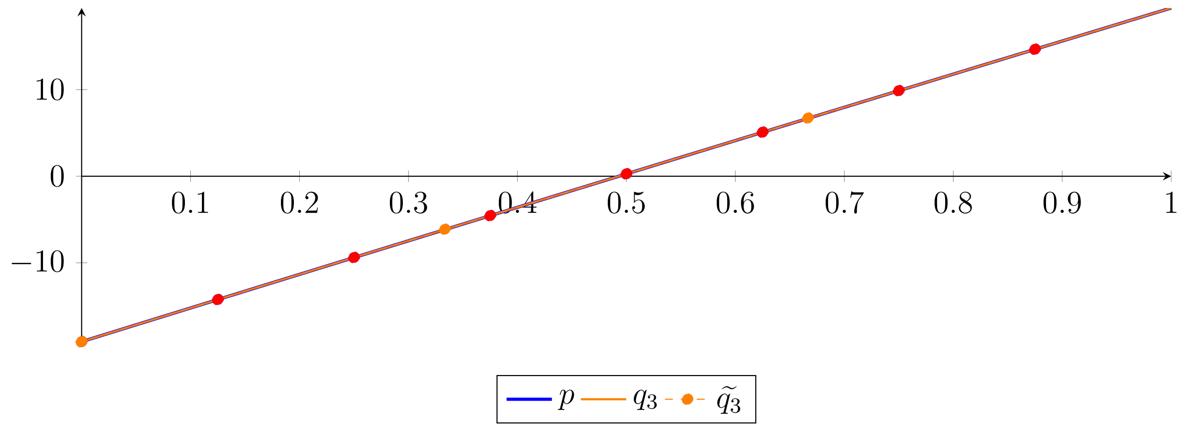
$$\begin{aligned} p = &-1.25947 \cdot 10^{-16} X^8 - 1.9937 \cdot 10^{-13} X^7 - 8.53073 \cdot 10^{-11} X^6 + 8.72839 \cdot 10^{-09} X^5 \\ &+ 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ = &-19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ &+ 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\widetilde{q}_3 = -1.96643\cdot10^{-303}X^8 + 1.82947\cdot10^{-302}X^7 - 4.89395\cdot10^{-302}X^6 + 5.49554\cdot10^{-302}X^5$$
$$- 2.47838\cdot10^{-302}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$
$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$
$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



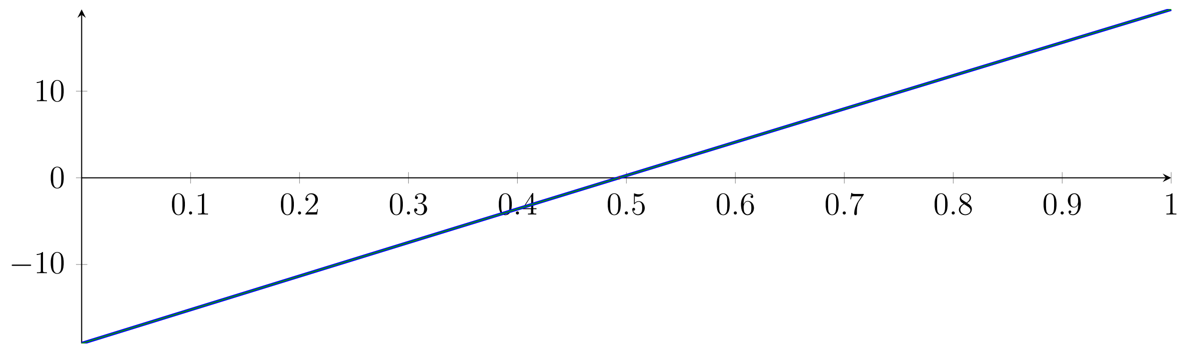
The maximum difference of the Bézier coefficients is $\delta = 2.16902\cdot10^{-07}$.
Bounding polynomials M and m:

$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$
$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.1252\cdot10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

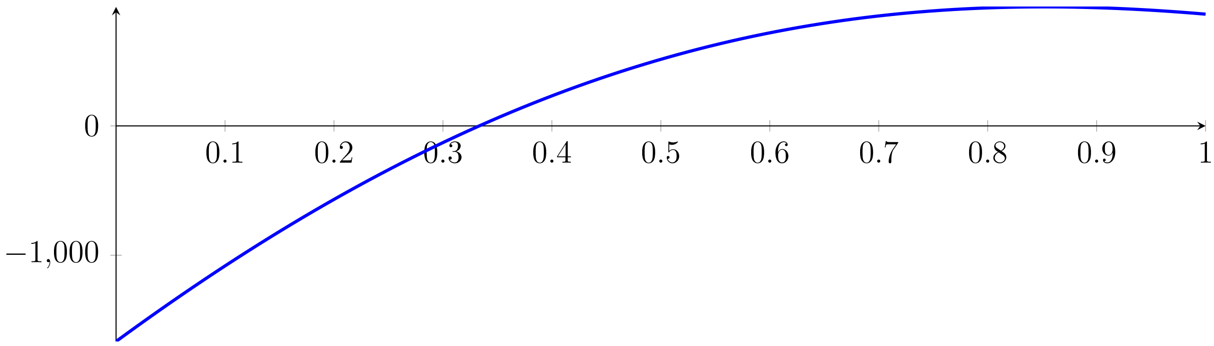
219.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

219.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-08}$.

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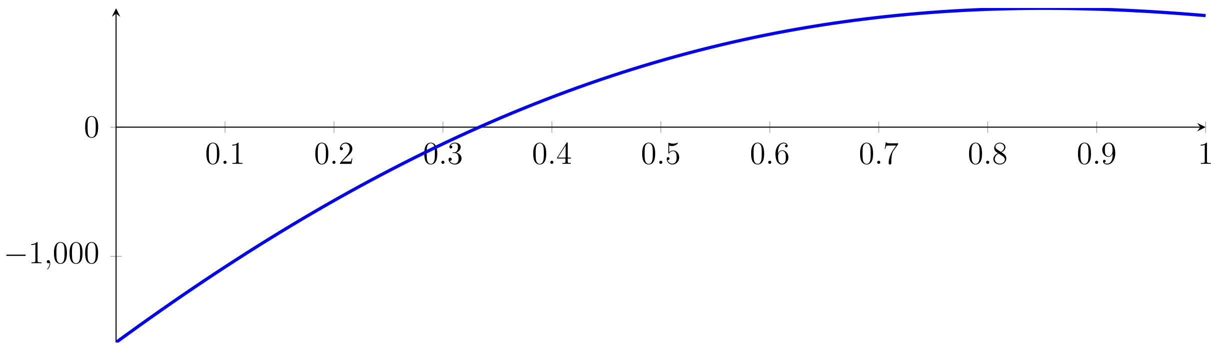
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0,1]$:

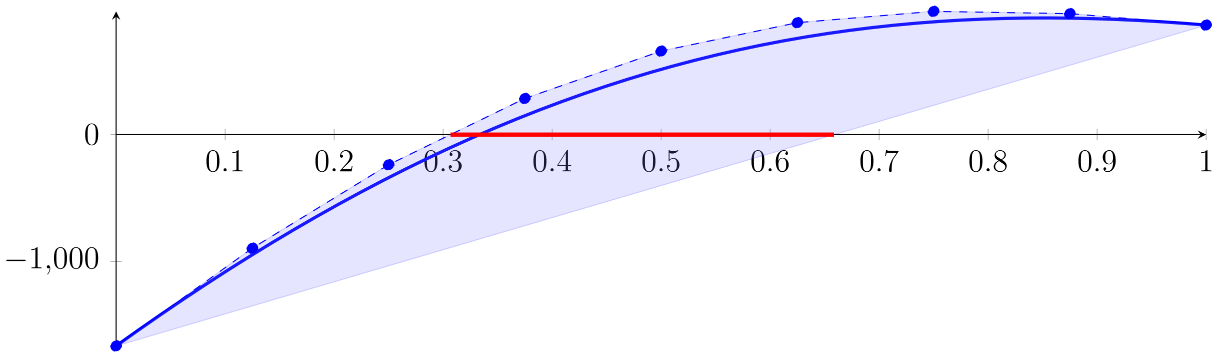
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



220.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

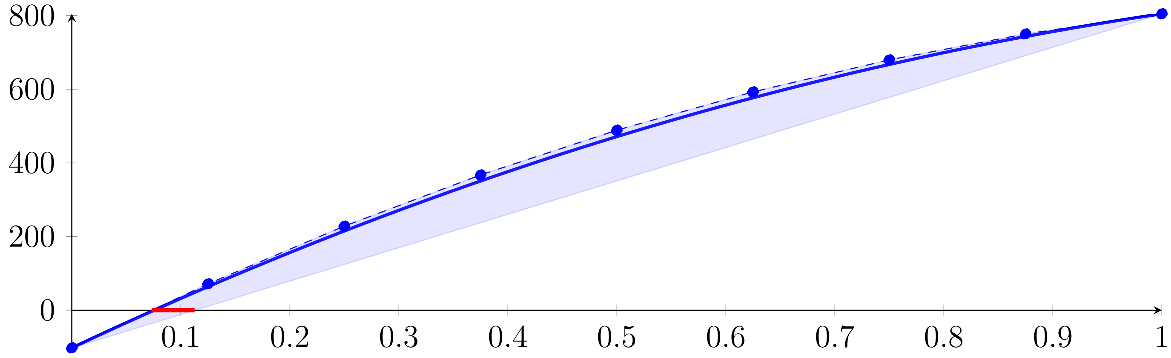
Longest intersection interval: 0.351792

$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

220.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17$$
$$= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

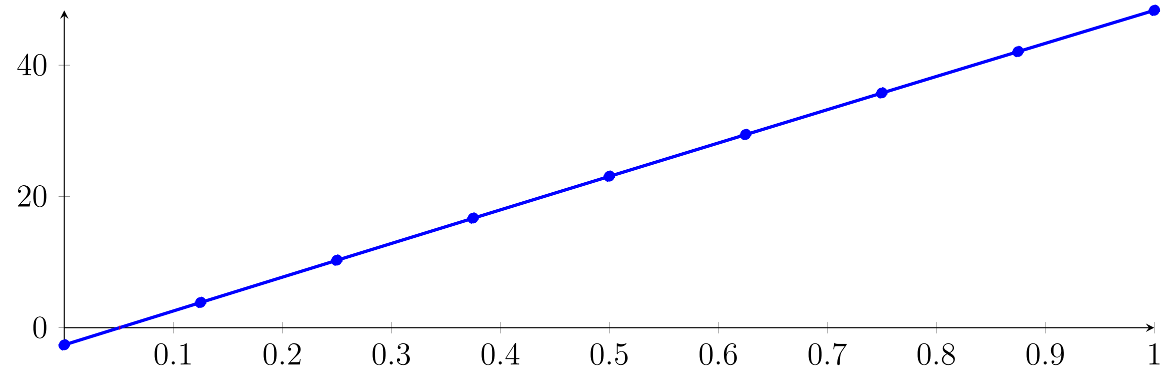
$$[0.0734515, 0.112637]$$

Longest intersection interval: 0.0391855
$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

220.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.30406 \cdot 10^{-15}X^8 - 1.54459 \cdot 10^{-12}X^7 - 4.9583 \cdot 10^{-10}X^6 + 3.66751 \cdot 10^{-08}X^5 + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683$$
$$= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

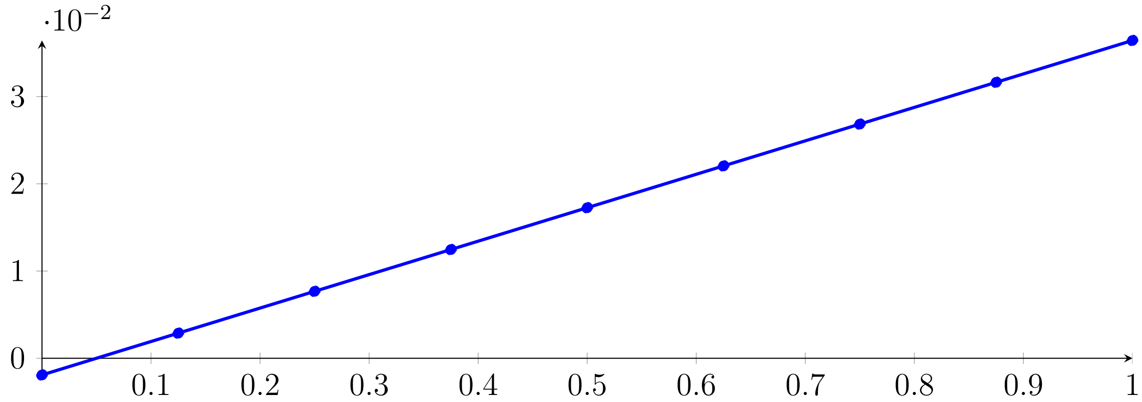
$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

220.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.20583 \cdot 10^{-40} X^8 - 1.92397 \cdot 10^{-34} X^7 - 8.32342 \cdot 10^{-29} X^6 + 8.24755 \cdot 10^{-24} X^5 + 9.89972 \cdot 10^{-18} X^4 - 2.37322 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378$$
$$= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

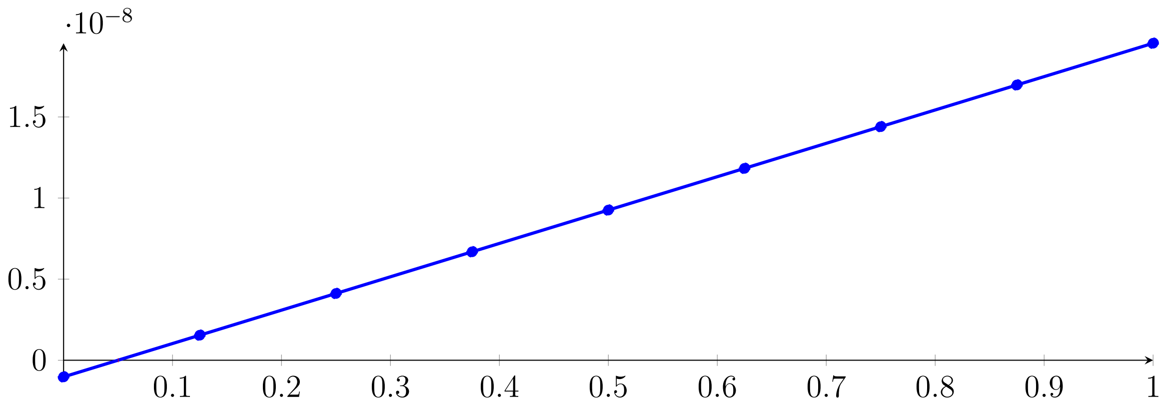
$$[0.0499103, 0.0499109]$$

Longest intersection interval: $5.36469 \cdot 10^{-07}$
$\implies$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

220.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -8.27263 \cdot 10^{-91} X^8 - 2.46044 \cdot 10^{-78} X^7 - 1.98413 \cdot 10^{-66} X^6 + 3.66478 \cdot 10^{-55} X^5 + 8.19978 \cdot 10^{-43} X^4 - 3.66412 \cdot 10^{-32} X^3 - 1.18615 \cdot 10^{-19} X^2 + 2.05706 \cdot 10^{-08} X - 1.02667 \cdot 10^{-09}$$
$$= -1.02667 \cdot 10^{-09} B_{0,8}(X) + 1.54465 \cdot 10^{-09} B_{1,8}(X) + 4.11597 \cdot 10^{-09} B_{2,8}(X) + 6.68729 \cdot 10^{-09} B_{3,8}(X) + 9.25862 \cdot 10^{-09} B_{4,8}(X) + 1.18299 \cdot 10^{-08} B_{5,8}(X) + 1.44013 \cdot 10^{-08} B_{6,8}(X) + 1.69726 \cdot 10^{-08} B_{7,8}(X) + 1.95439 \cdot 10^{-08} B_{8,8}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87793 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

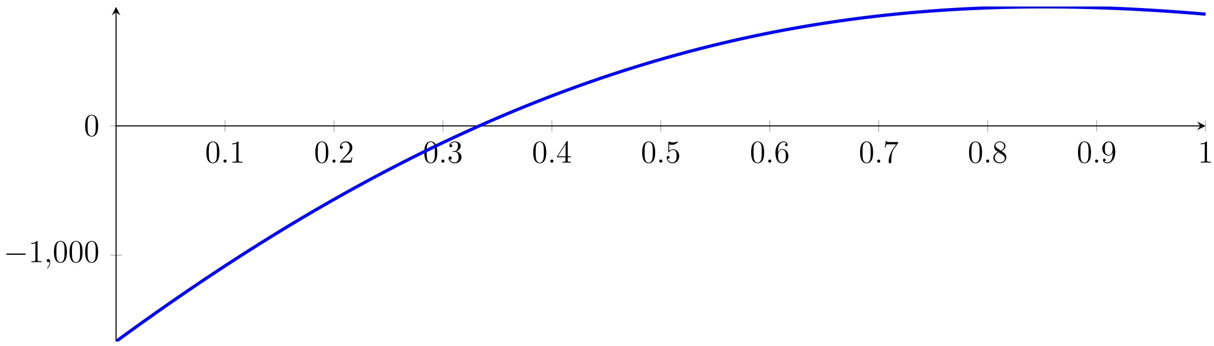
220.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

220.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-16}$.

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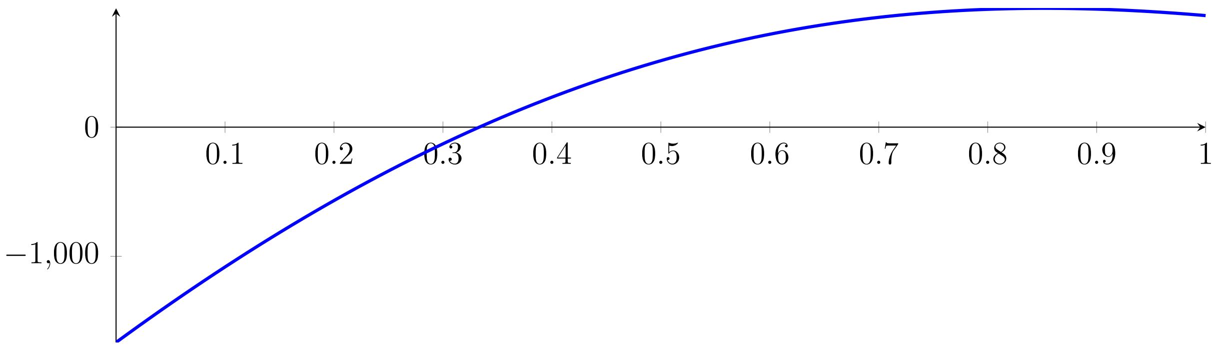
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

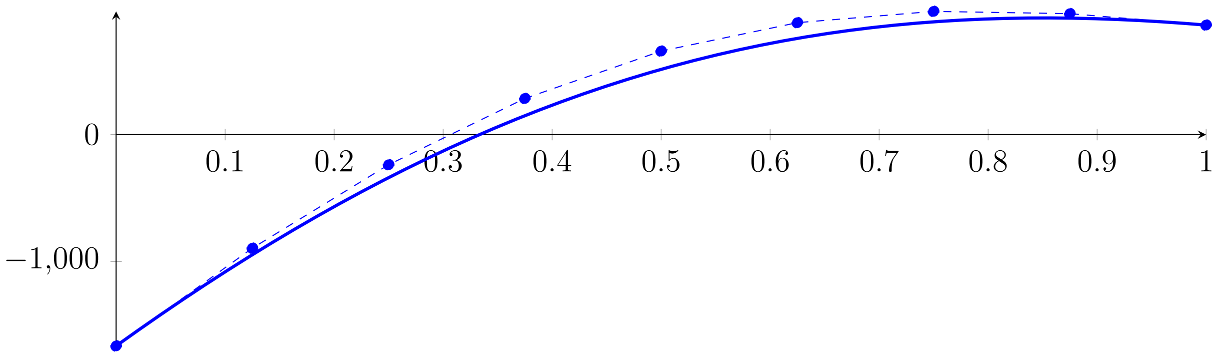
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



221.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

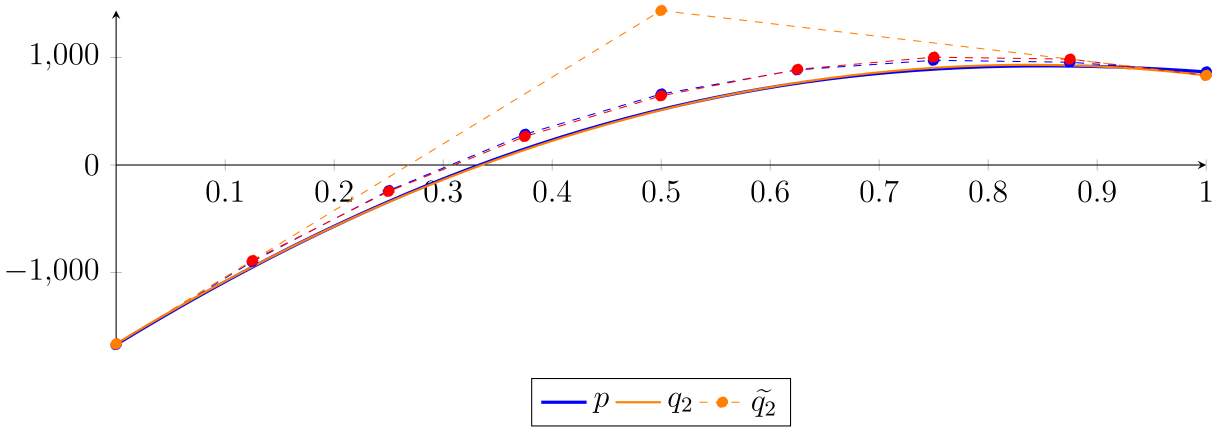
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 6.38292 \cdot 10^{-300}X^8 - 2.86993 \cdot 10^{-299}X^7 + 5.15342 \cdot 10^{-299}X^6 - 4.6928 \cdot 10^{-299}X^5 \\ &\quad + 2.29296 \cdot 10^{-299}X^4 - 6.02693 \cdot 10^{-300}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

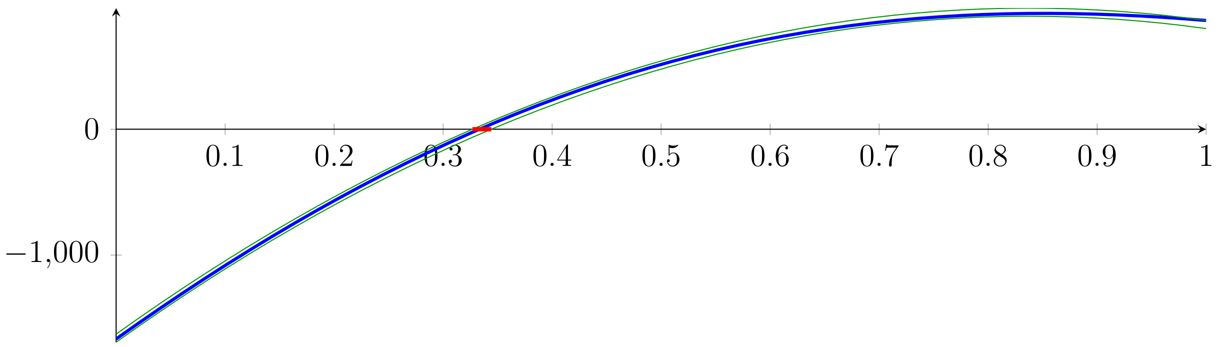
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

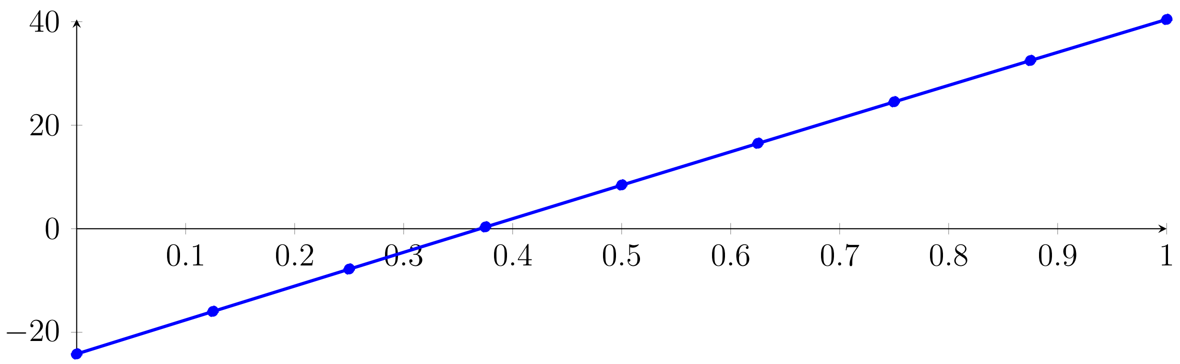
Longest intersection interval: 0.0173372

$\implies$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

221.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

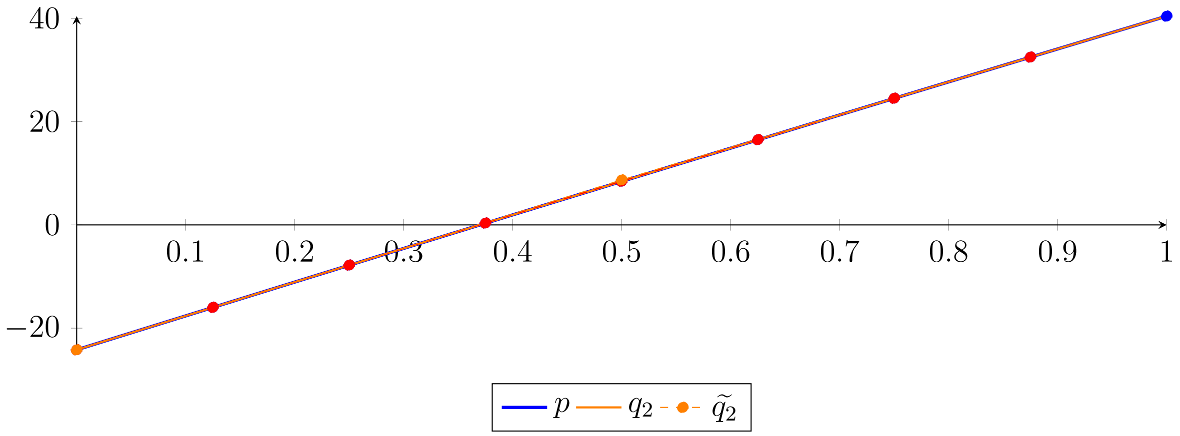
$$\begin{aligned} p = & -8.16249\cdot 10^{-15}X^8 - 7.66571\cdot 10^{-12}X^7 - 1.9444\cdot 10^{-09}X^6 + 1.19263\cdot 10^{-07}X^5 \\ & + 8.12335\cdot 10^{-05}X^4 - 0.0012733X^3 - 1.18084X^2 + 65.8155X - 24.1945 \\ = & -24.1945B_{0,8}(X) - 15.9676B_{1,8}(X) - 7.78282B_{2,8}(X) + 0.35975B_{3,8}(X) \\ & + 8.4601B_{4,8}(X) + 16.5182B_{5,8}(X) + 24.5341B_{6,8}(X) + 32.5077B_{7,8}(X) + 40.4389B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\widetilde{q}_2 = 4.27533\cdot 10^{-302}X^8 - 2.42468\cdot 10^{-301}X^7 + 4.90537\cdot 10^{-301}X^6 - 4.6286\cdot 10^{-301}X^5$$
$$+ 2.17546\cdot 10^{-301}X^4 - 5.36695\cdot 10^{-302}X^3 - 1.18261X^2 + 65.8162X - 24.1946$$
$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$
$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894\cdot 10^{-05}$.
Bounding polynomials M and m:

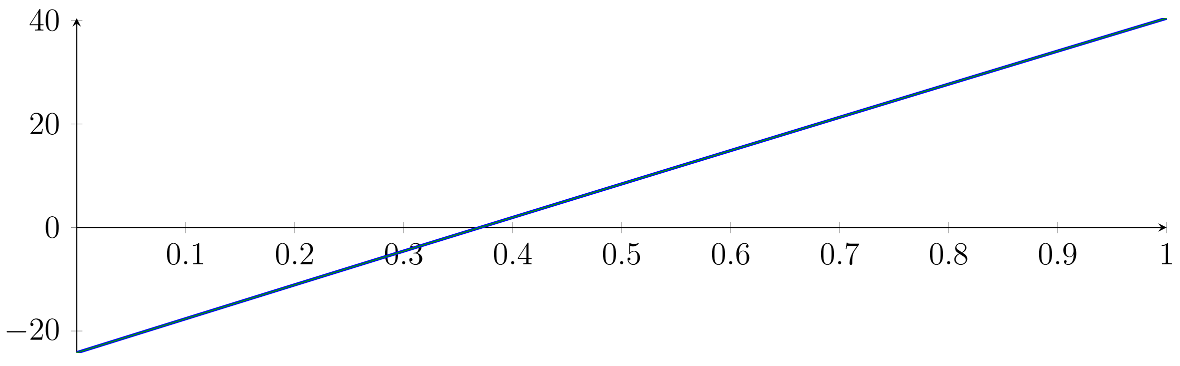
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



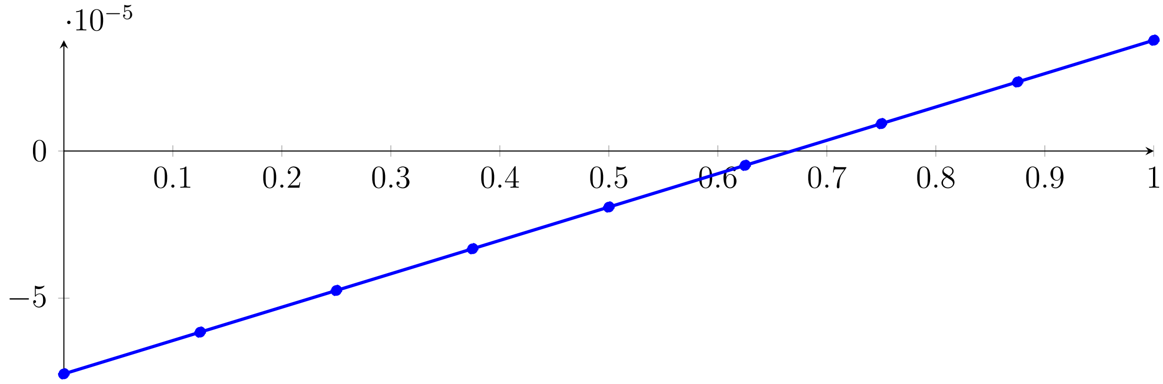
$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588\cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

221.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

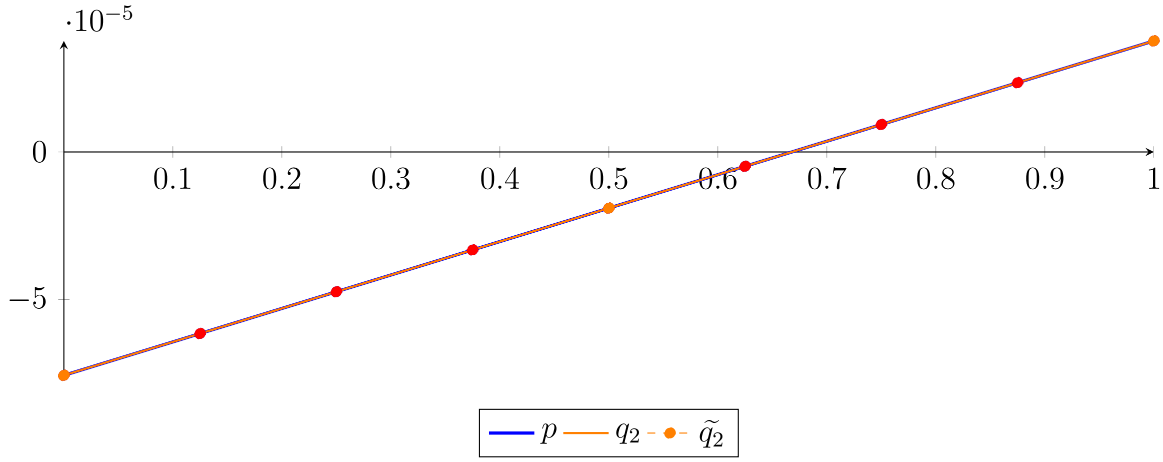
$$p = -7.04578 \cdot 10^{-61} X^8 - 3.80201 \cdot 10^{-52} X^7 - 5.5627 \cdot 10^{-44} X^6 + 1.86413 \cdot 10^{-36} X^5 + 7.56737 \cdot 10^{-28} X^4 - 6.13517 \cdot 10^{-21} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379X - 7.5732 \cdot 10^{-05}$$
$$= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)$$



Degree reduction and raising:

$$q_2 = -3.6034 \cdot 10^{-12} X^2 + 0.000113379X - 7.5732 \cdot 10^{-05}$$
$$= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = -3.62396 \cdot 10^{-308} X^8 + 2.32992 \cdot 10^{-308} X^7 + 2.61753 \cdot 10^{-307} X^6 - 5.97049 \cdot 10^{-307} X^5 + 5.13401 \cdot 10^{-307} X^4 - 1.82868 \cdot 10^{-307} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379X - 7.5732 \cdot 10^{-05}$$
$$= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 3.06758 \cdot 10^{-22}$.

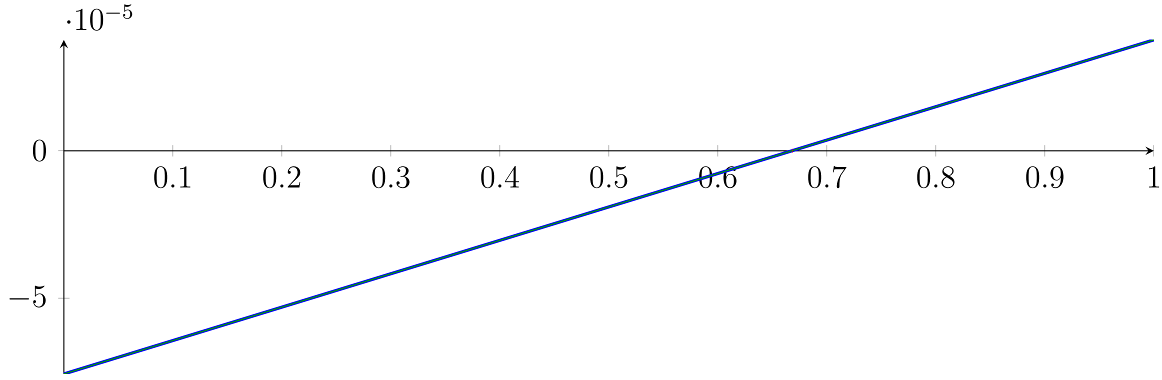
Bounding polynomials M and m:

$$M = -3.6034 \cdot 10^{-12} X^2 + 0.000113379X - 7.5732 \cdot 10^{-05}$$
$$m = -3.6034 \cdot 10^{-12} X^2 + 0.000113379X - 7.5732 \cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.667955, 3.14644 \cdot 10^{7}\} \qquad N(m) = \{0.667955, 3.14644 \cdot 10^{7}\}$$

Intersection intervals:



$[0.667955, 0.667955]$

Longest intersection interval: $5.41121 \cdot 10^{-18}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

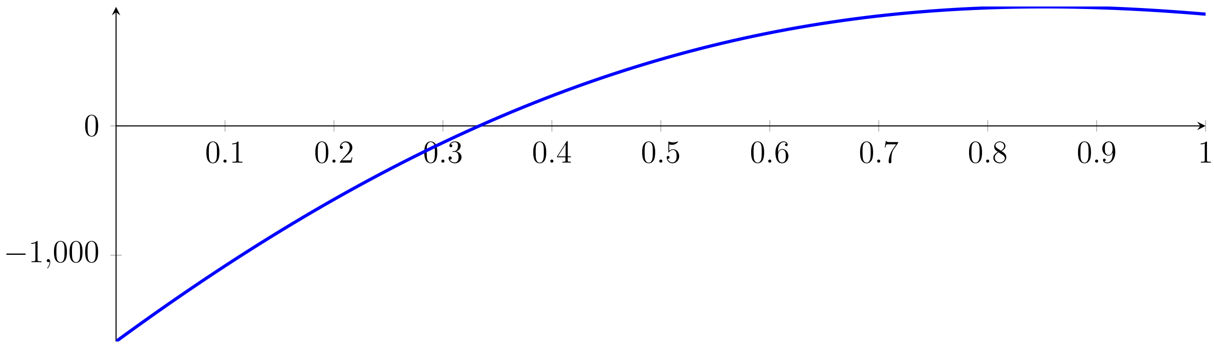
221.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

221.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-16}$.

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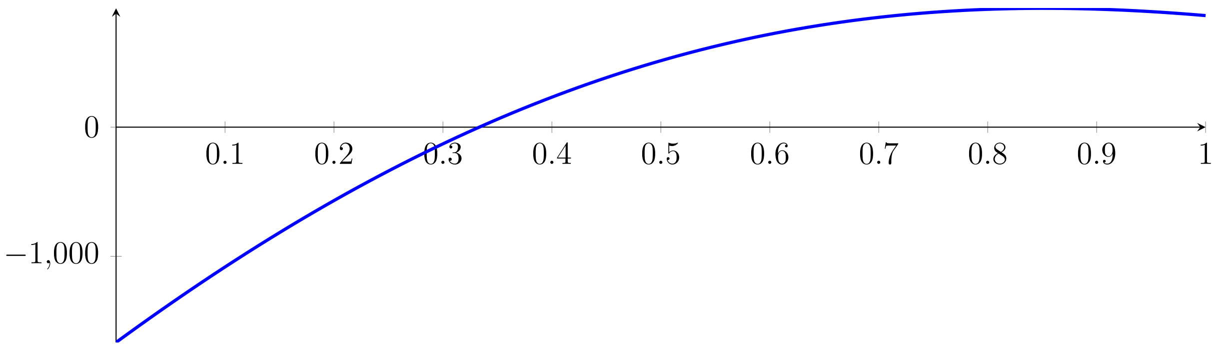
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

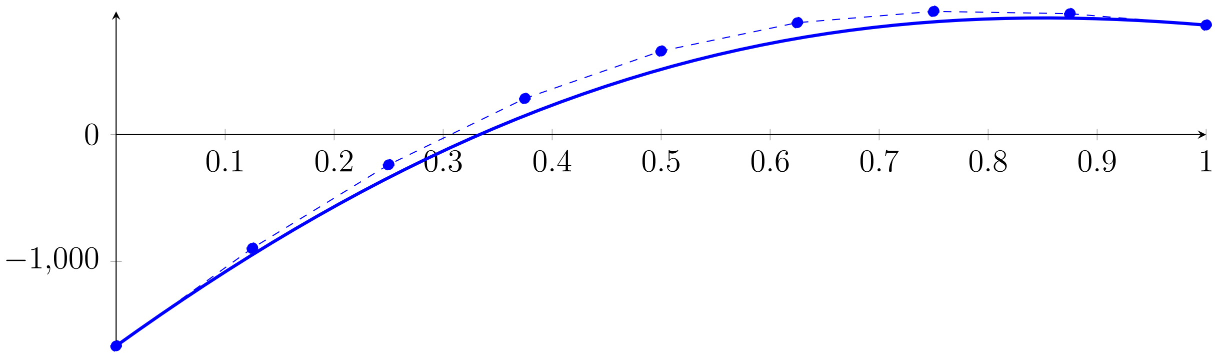
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



222.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

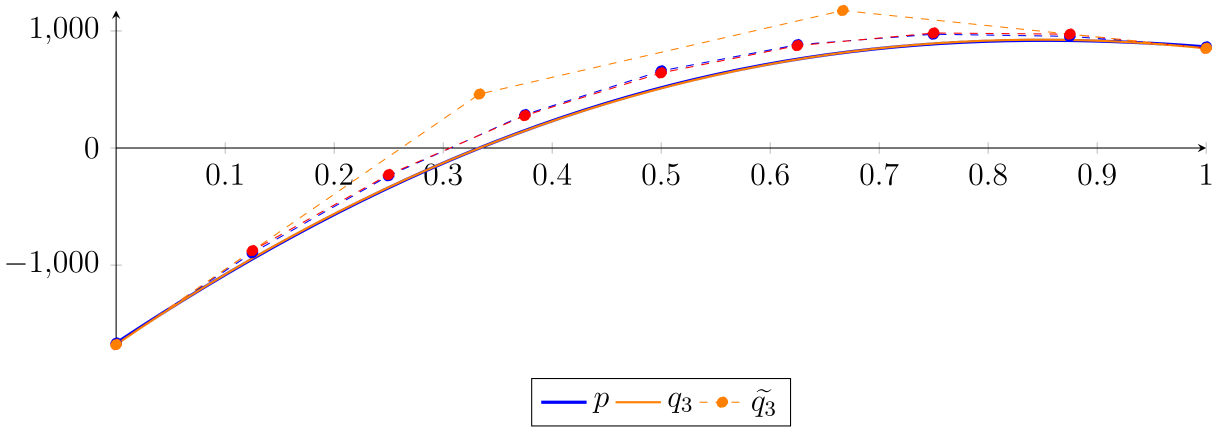
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -2.80826 \cdot 10^{-300}X^8 + 1.16606 \cdot 10^{-299}X^7 - 2.1197 \cdot 10^{-299}X^6 + 2.15003 \cdot 10^{-299}X^5 \\ &\quad - 1.20234 \cdot 10^{-299}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

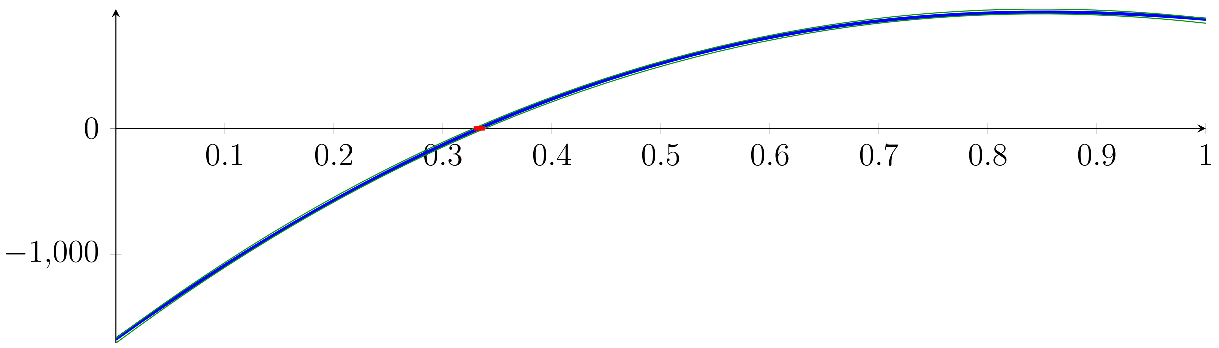
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$[0.328258, 0.338551]$

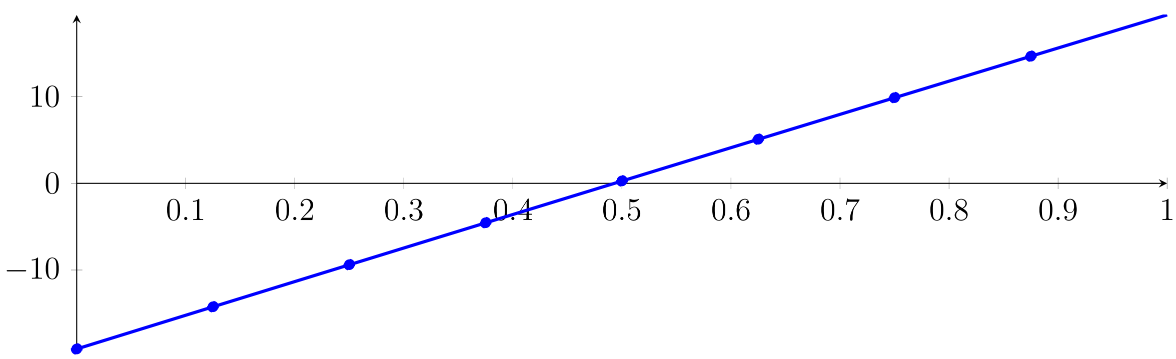
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

222.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

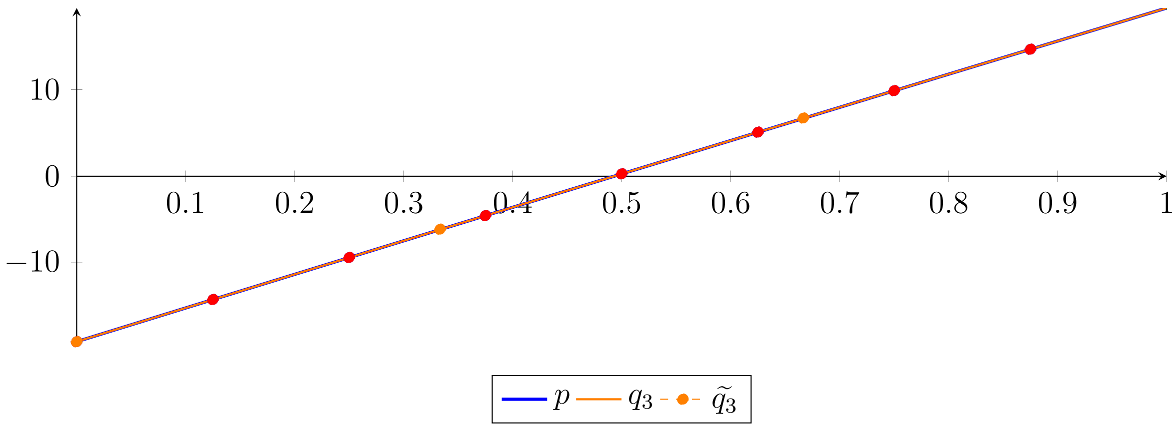
$$\begin{aligned} p &= -1.25947 \cdot 10^{-16} X^8 - 1.9937 \cdot 10^{-13} X^7 - 8.53073 \cdot 10^{-11} X^6 + 8.72839 \cdot 10^{-09} X^5 \\ &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1.96643\cdot 10^{-303}X^8 + 1.82947\cdot 10^{-302}X^7 - 4.89395\cdot 10^{-302}X^6 + 5.49554\cdot 10^{-302}X^5 \\ &\quad - 2.47838\cdot 10^{-302}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902\cdot 10^{-07}$.
Bounding polynomials M and m:

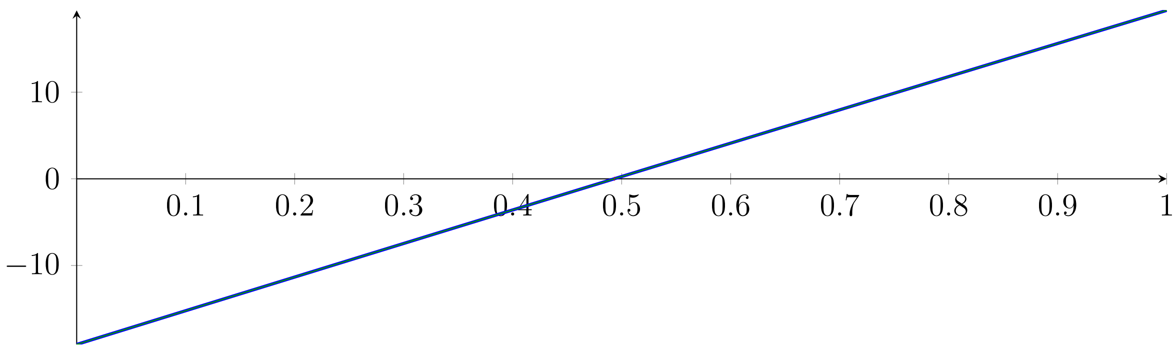
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



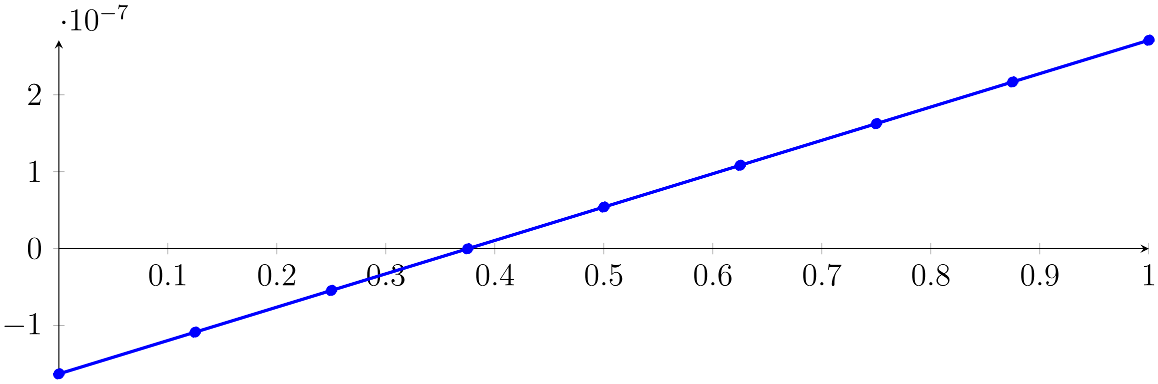
$$[0.493109, 0.493109]$$

Longest intersection interval: $1.1252\cdot 10^{-08}$
$\implies$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

222.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

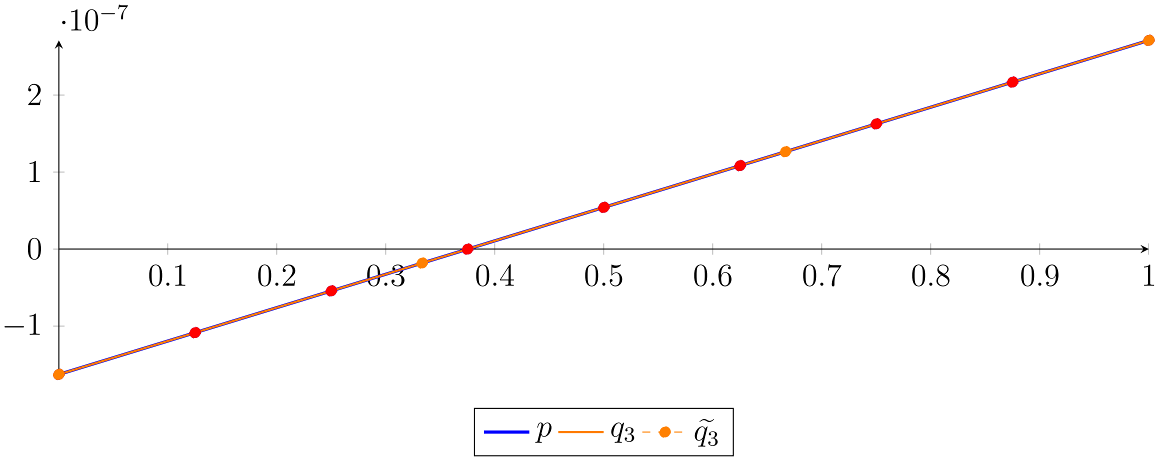
$$p = -3.2361\cdot10^{-80}X^8 - 4.56398\cdot10^{-69}X^7 - 1.74524\cdot10^{-58}X^6 + 1.52857\cdot10^{-48}X^5 + 1.62178\cdot10^{-37}X^4 - 3.43646\cdot10^{-28}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,8}(X) - 1.08585\cdot10^{-07}B_{1,8}(X) - 5.43592\cdot10^{-08}B_{2,8}(X) - 1.33711\cdot10^{-10}B_{3,8}(X) + 5.40918\cdot10^{-08}B_{4,8}(X) + 1.08317\cdot10^{-07}B_{5,8}(X) + 1.62543\cdot10^{-07}B_{6,8}(X) + 2.16768\cdot10^{-07}B_{7,8}(X) + 2.70994\cdot10^{-07}B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = -3.43646\cdot10^{-28}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,3} - 1.82089\cdot10^{-08}B_{1,3} + 1.26393\cdot10^{-07}B_{2,3} + 2.70994\cdot10^{-07}B_{3,3}$$

$$\tilde{q}_3 = 8.86066\cdot10^{-312}X^8 + 7.41744\cdot10^{-311}X^7 - 5.24902\cdot10^{-310}X^6 + 1.01267\cdot10^{-309}X^5 - 7.8921\cdot10^{-310}X^4 - 3.43646\cdot10^{-28}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,8} - 1.08585\cdot10^{-07}B_{1,8} - 5.43592\cdot10^{-08}B_{2,8} - 1.33711\cdot10^{-10}B_{3,8} + 5.40918\cdot10^{-08}B_{4,8} + 1.08317\cdot10^{-07}B_{5,8} + 1.62543\cdot10^{-07}B_{6,8} + 2.16768\cdot10^{-07}B_{7,8} + 2.70994\cdot10^{-07}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 3.47524\cdot10^{-39}$.

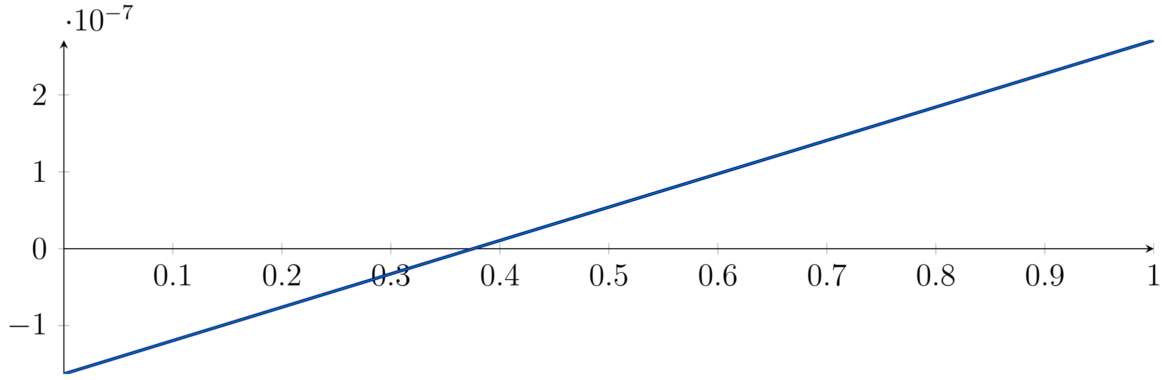
Bounding polynomials M and m:

$$M = -3.43646\cdot10^{-28}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$m = -3.43646\cdot10^{-28}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$

Root of M and m:

$$N(M) = \{-1.6133\cdot10^{11}, 0.375292, 7.82468\cdot10^{9}\} \quad N(m) = \{-1.6133\cdot10^{11}, 0.375292, 7.82468\cdot10^{9}\}$$

Intersection intervals:



$[0.375292, 0.375292]$

Longest intersection interval: $1.60221 \cdot 10^{-32}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

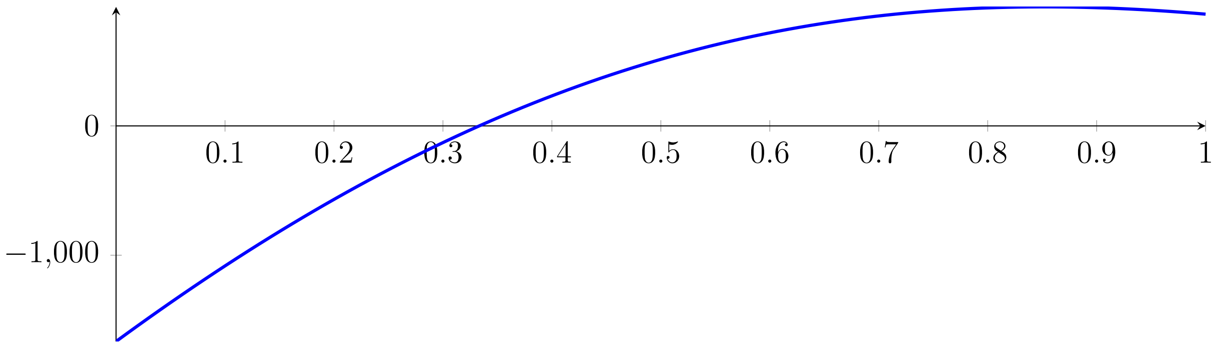
222.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 4!
p(0) = -6.9178e-12 - p(1) -6.9178e-12

222.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

with precision $\varepsilon = 1 \cdot 10^{-16}$.

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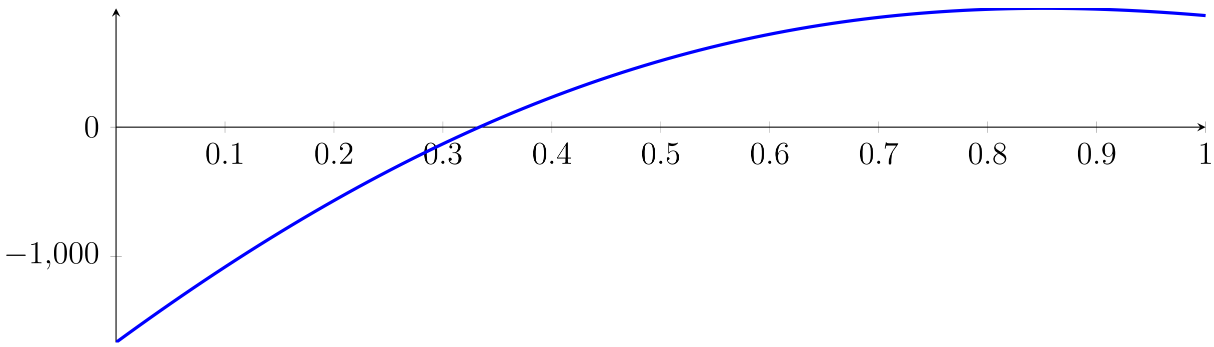
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

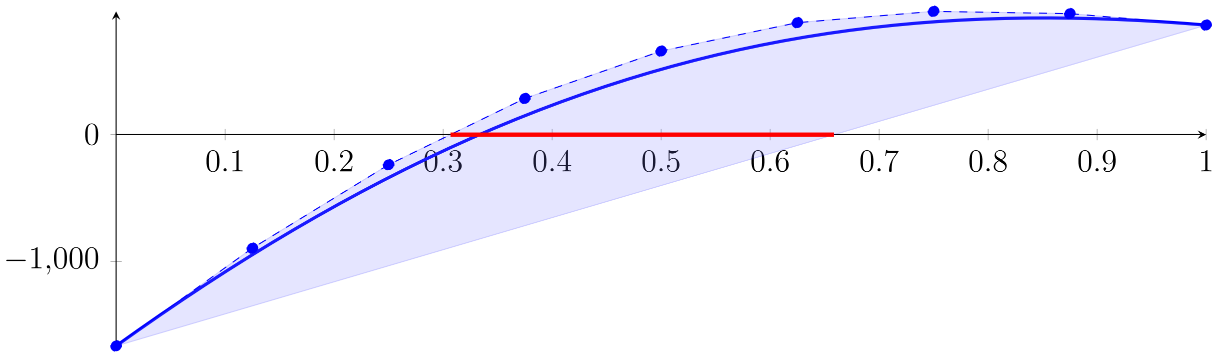
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



223.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

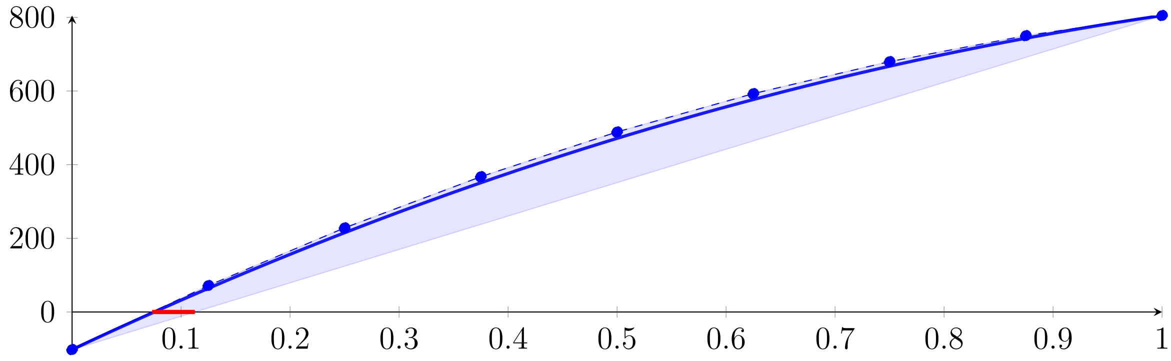
$$[0.306796, 0.658588]$$

Longest intersection interval: 0.351792
$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

223.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\ &+ 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\ = &-102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\ &+ 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

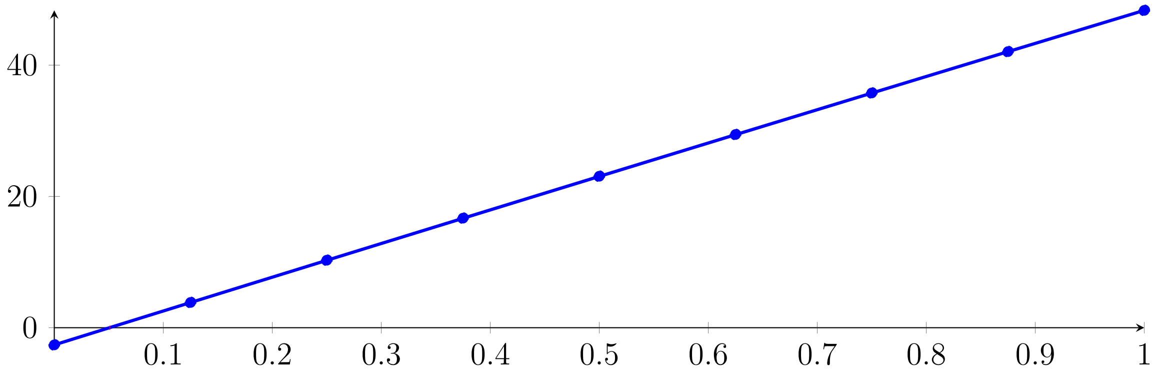
Longest intersection interval: 0.0391855

$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

223.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-1.30406\cdot 10^{-15}X^8 - 1.54459\cdot 10^{-12}X^7 - 4.9583\cdot 10^{-10}X^6 + 3.66751\cdot 10^{-08}X^5 \\ &+ 3.25466\cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\ = &-2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\ &+ 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

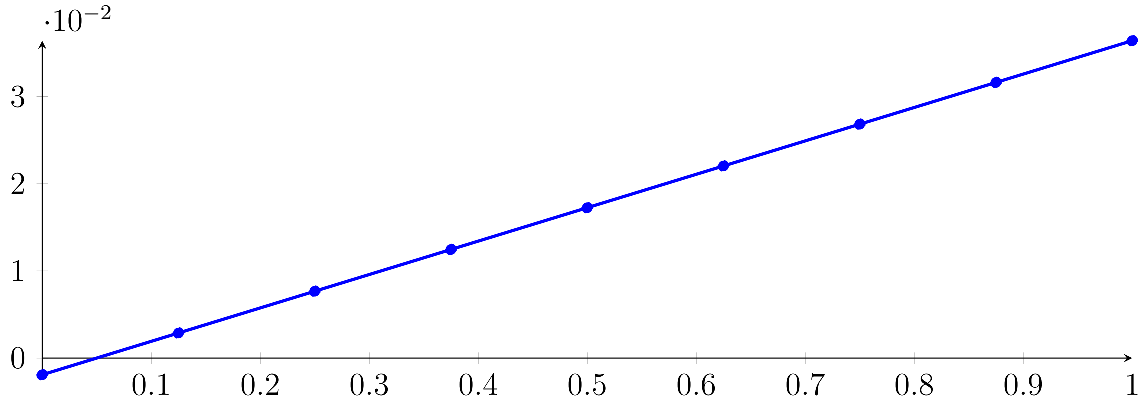
Longest intersection interval: 0.000742589

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

223.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$
\begin{aligned}
p &= -1.20583\cdot 10^{-40}X^8 - 1.92397\cdot 10^{-34}X^7 - 8.32342\cdot 10^{-29}X^6 + 8.24755\cdot 10^{-24}X^5 \\
&\quad + 9.89972\cdot 10^{-18}X^4 - 2.37322\cdot 10^{-13}X^3 - 4.12146\cdot 10^{-07}X^2 + 0.0383444X - 0.00191378 \\
&= -0.00191378B_{0,8}(X) + 0.00287927B_{1,8}(X) + 0.00767231B_{2,8}(X) \\
&\quad + 0.0124653B_{3,8}(X) + 0.0172583B_{4,8}(X) + 0.0220513B_{5,8}(X) \\
&\quad + 0.0268443B_{6,8}(X) + 0.0316373B_{7,8}(X) + 0.0364302B_{8,8}(X)
\end{aligned}
$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

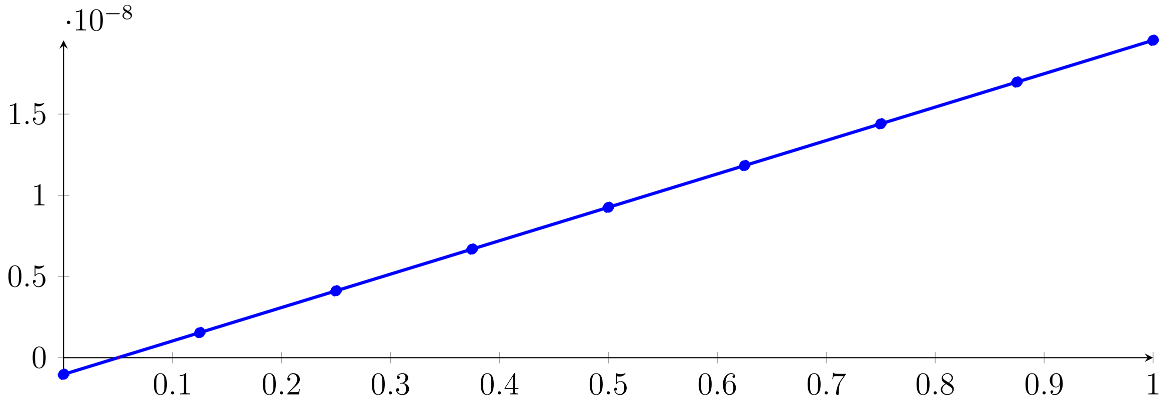
Longest intersection interval: $5.36469\cdot 10^{-07}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

223.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$
\begin{aligned}
p &= -8.27263\cdot 10^{-91}X^8 - 2.46044\cdot 10^{-78}X^7 - 1.98413\cdot 10^{-66}X^6 + 3.66478\cdot 10^{-55}X^5 + 8.19978 \\
&\quad \cdot 10^{-43}X^4 - 3.66412\cdot 10^{-32}X^3 - 1.18615\cdot 10^{-19}X^2 + 2.05706\cdot 10^{-08}X - 1.02667\cdot 10^{-09} \\
&= -1.02667\cdot 10^{-09}B_{0,8}(X) + 1.54465\cdot 10^{-09}B_{1,8}(X) + 4.11597\cdot 10^{-09}B_{2,8}(X) \\
&\quad + 6.68729\cdot 10^{-09}B_{3,8}(X) + 9.25862\cdot 10^{-09}B_{4,8}(X) + 1.18299\cdot 10^{-08}B_{5,8}(X) \\
&\quad + 1.44013\cdot 10^{-08}B_{6,8}(X) + 1.69726\cdot 10^{-08}B_{7,8}(X) + 1.95439\cdot 10^{-08}B_{8,8}(X)
\end{aligned}
$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

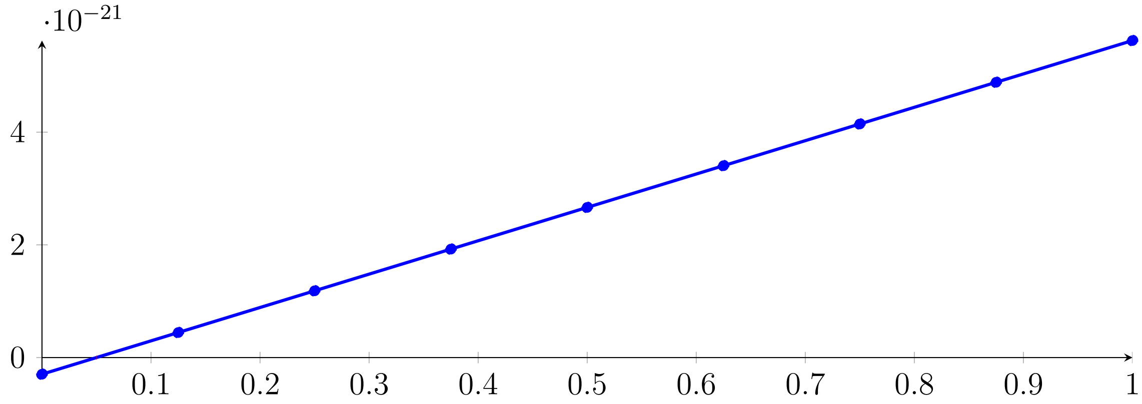
$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87793 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

223.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -3.89305 \cdot 10^{-191} X^8 - 4.02327 \cdot 10^{-166} X^7 - 1.12734 \cdot 10^{-141} X^6 + 7.23523 \cdot 10^{-118} X^5 + 5.62504 \\ &\quad \cdot 10^{-93} X^4 - 8.73397 \cdot 10^{-70} X^3 - 9.82433 \cdot 10^{-45} X^2 + 5.92008 \cdot 10^{-21} X - 2.9547 \cdot 10^{-22} \\ &= -2.9547 \cdot 10^{-22} B_{0,8}(X) + 4.4454 \cdot 10^{-22} B_{1,8}(X) + 1.18455 \cdot 10^{-21} B_{2,8}(X) \\ &\quad + 1.92456 \cdot 10^{-21} B_{3,8}(X) + 2.66457 \cdot 10^{-21} B_{4,8}(X) + 3.40458 \cdot 10^{-21} B_{5,8}(X) \\ &\quad + 4.14459 \cdot 10^{-21} B_{6,8}(X) + 4.8846 \cdot 10^{-21} B_{7,8}(X) + 5.62461 \cdot 10^{-21} B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $8.28251 \cdot 10^{-26}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

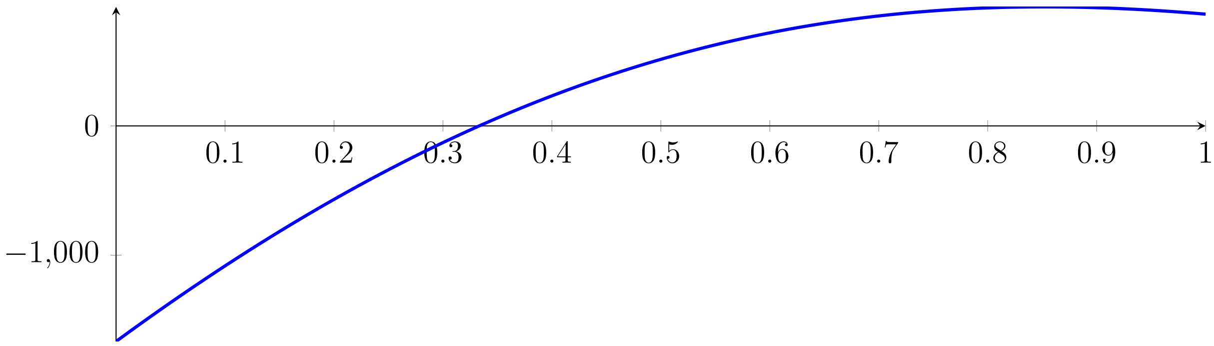
223.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 7!

223.8 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-32}$.

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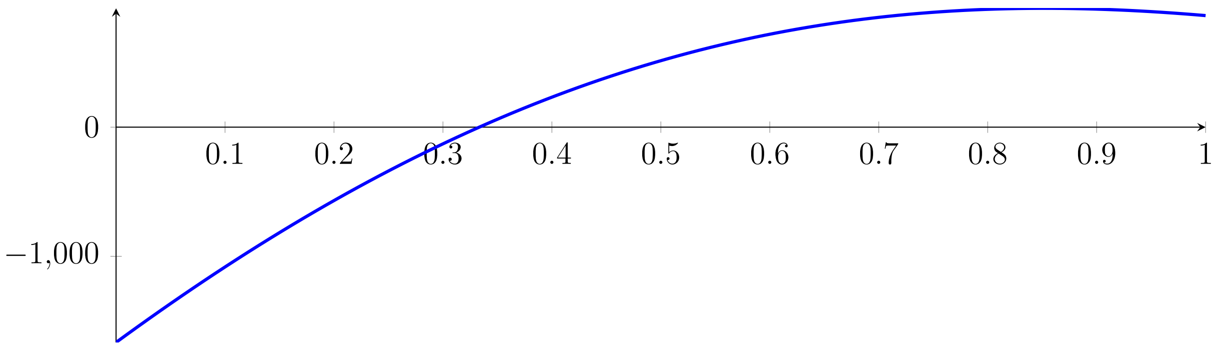
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

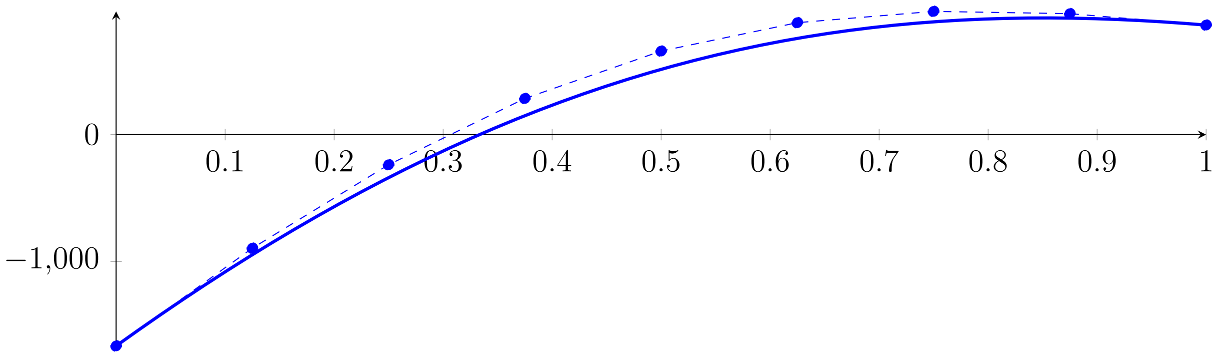
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



224.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

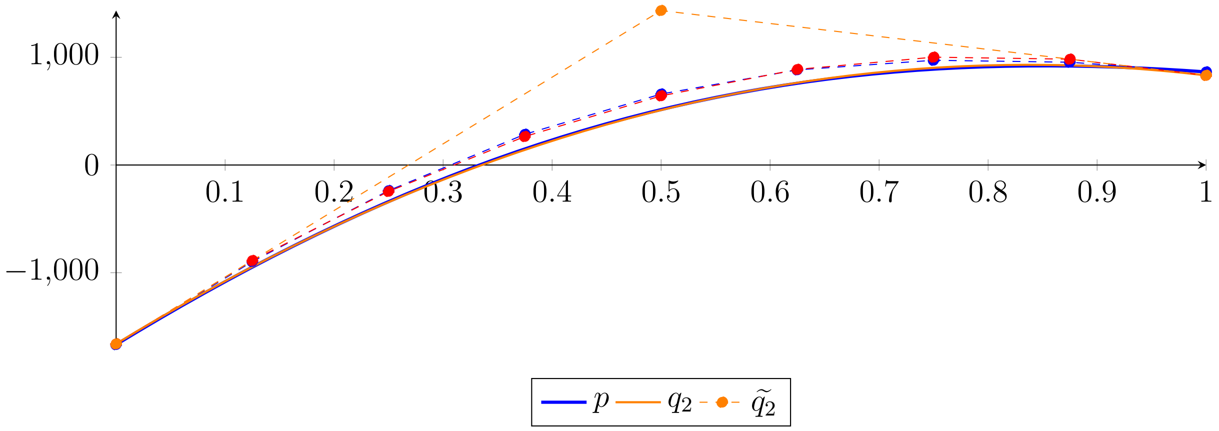
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 6.38292 \cdot 10^{-300}X^8 - 2.86993 \cdot 10^{-299}X^7 + 5.15342 \cdot 10^{-299}X^6 - 4.6928 \cdot 10^{-299}X^5 \\ &\quad + 2.29296 \cdot 10^{-299}X^4 - 6.02693 \cdot 10^{-300}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

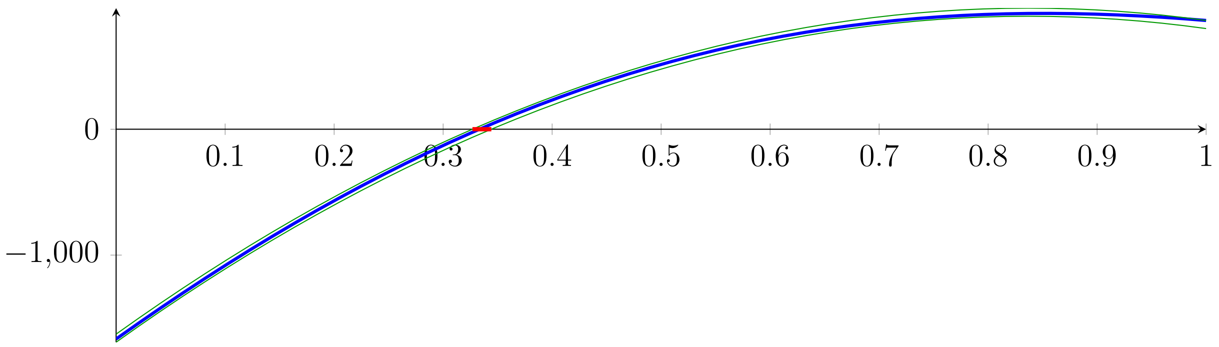
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

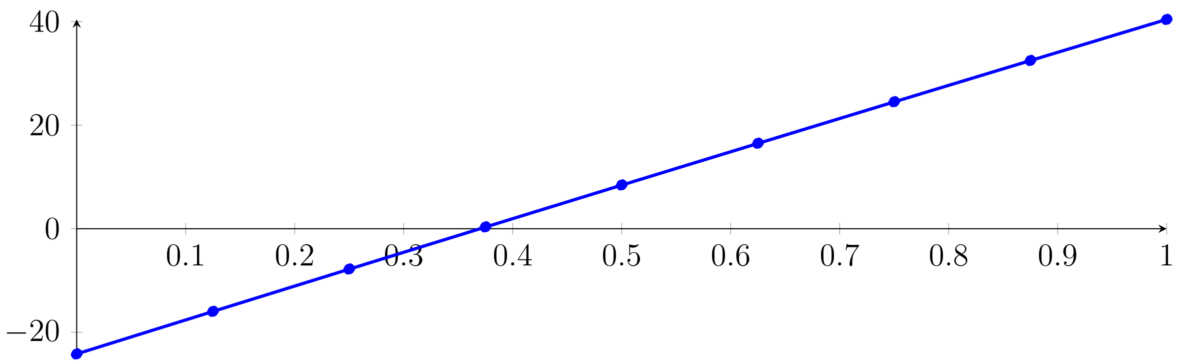
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

224.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

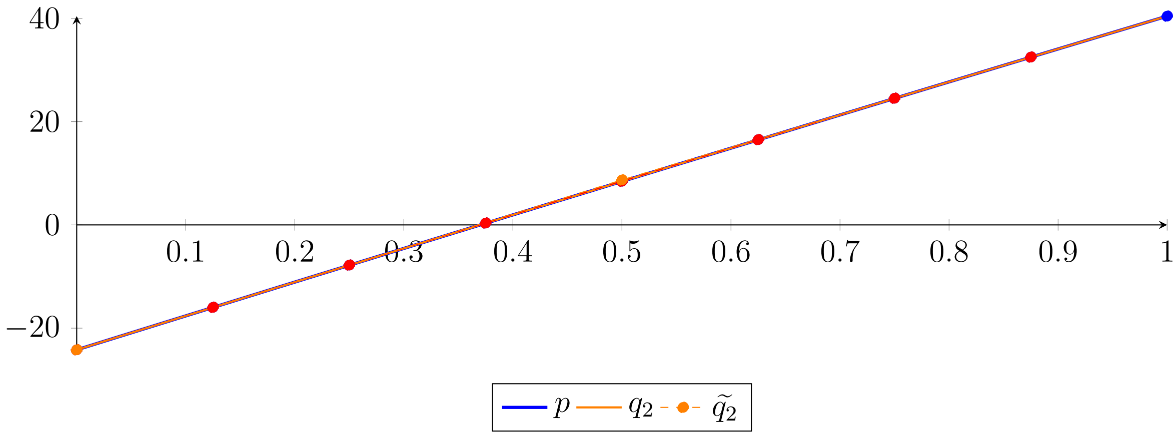
$$\begin{aligned} p = & -8.16249 \cdot 10^{-15} X^8 - 7.66571 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ & + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ = & -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ & + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 = &\ 4.27533\cdot 10^{-302}X^8 - 2.42468\cdot 10^{-301}X^7 + 4.90537\cdot 10^{-301}X^6 - 4.6286\cdot 10^{-301}X^5 \\ &+ 2.17546\cdot 10^{-301}X^4 - 5.36695\cdot 10^{-302}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ = &-24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894\cdot 10^{-05}$.

Bounding polynomials M and m:

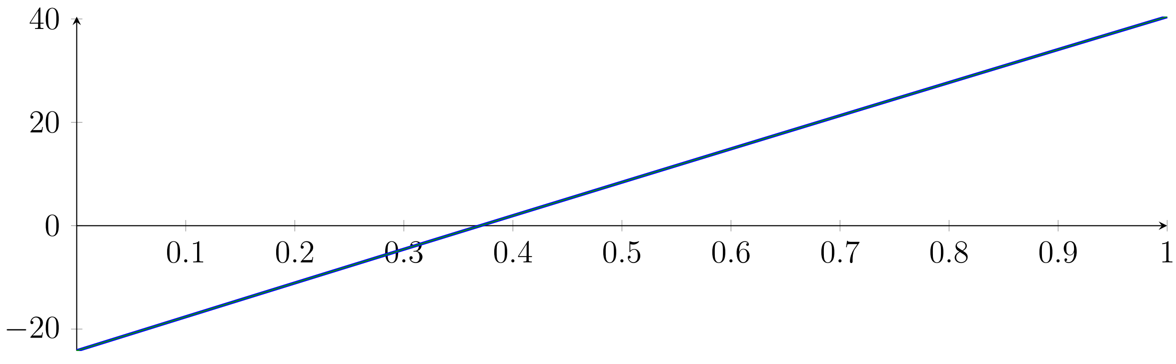
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

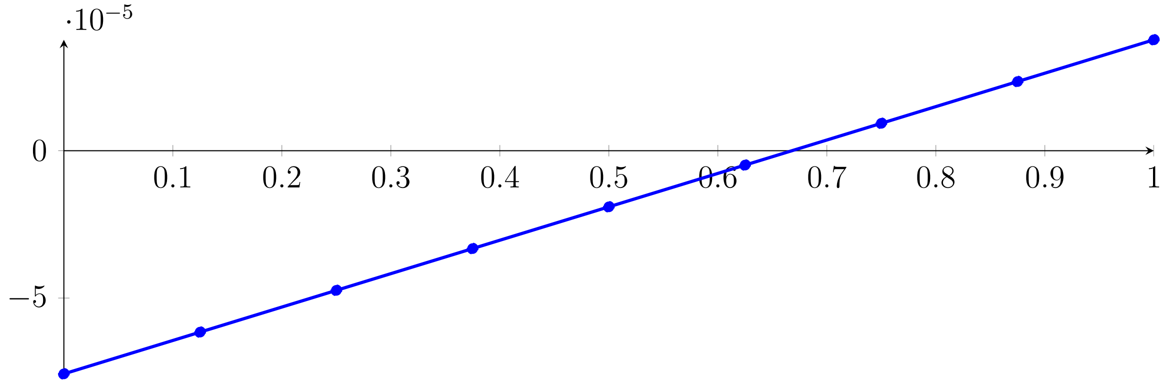
Longest intersection interval: $1.74588\cdot 10^{-06}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

224.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

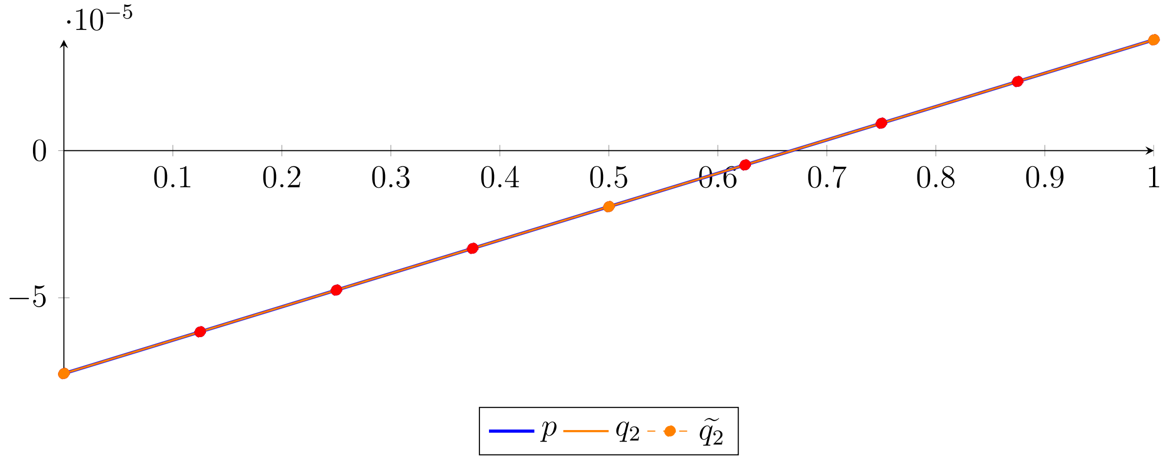
$$p = -7.04578 \cdot 10^{-61} X^8 - 3.80201 \cdot 10^{-52} X^7 - 5.5627 \cdot 10^{-44} X^6 + 1.86413 \cdot 10^{-36} X^5 + 7.56737 \cdot 10^{-28} X^4 - 6.13517 \cdot 10^{-21} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$
$$= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)$$



Degree reduction and raising:

$$q_2 = -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$
$$= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}$$

$$\widetilde{q}_2 = -3.62396 \cdot 10^{-308} X^8 + 2.32992 \cdot 10^{-308} X^7 + 2.61753 \cdot 10^{-307} X^6 - 5.97049 \cdot 10^{-307} X^5 + 5.13401 \cdot 10^{-307} X^4 - 1.82868 \cdot 10^{-307} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$
$$= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 3.06758 \cdot 10^{-22}$.

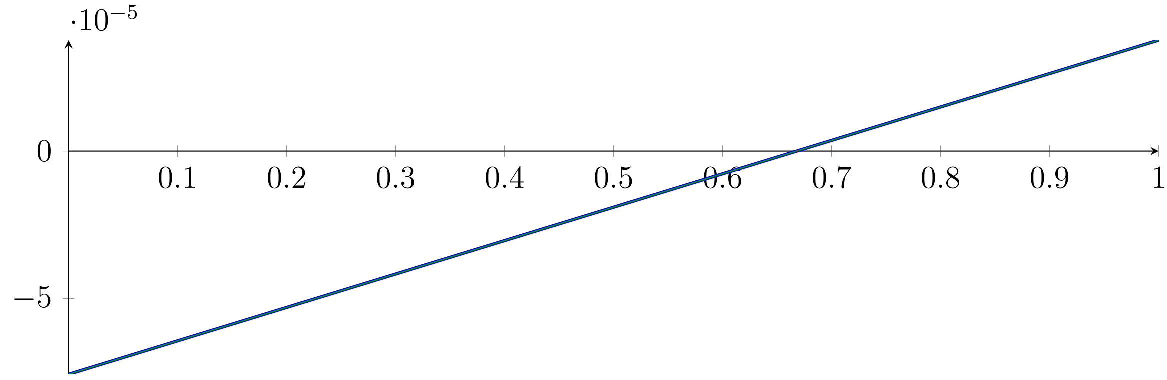
Bounding polynomials M and m:

$$M = -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$
$$m = -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.667955, 3.14644 \cdot 10^{7}\} \qquad N(m) = \{0.667955, 3.14644 \cdot 10^{7}\}$$

Intersection intervals:



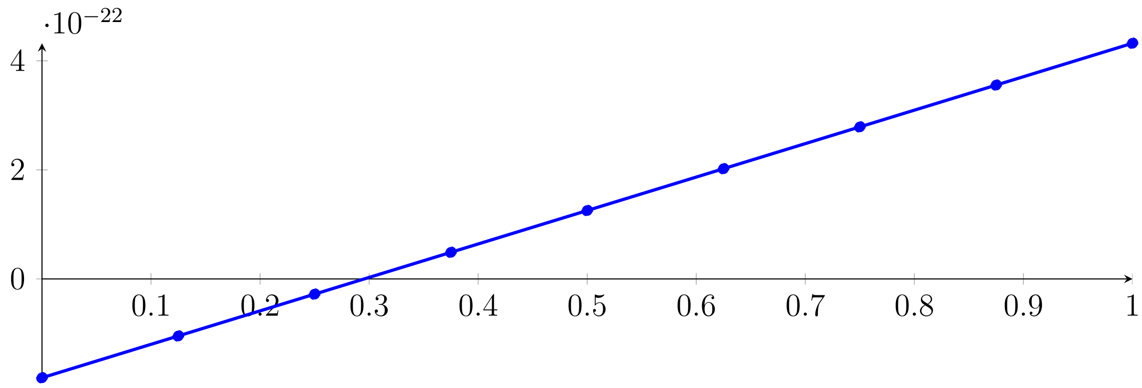
$[0.667955, 0.667955]$

Longest intersection interval: $5.41121 \cdot 10^{-18}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

224.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

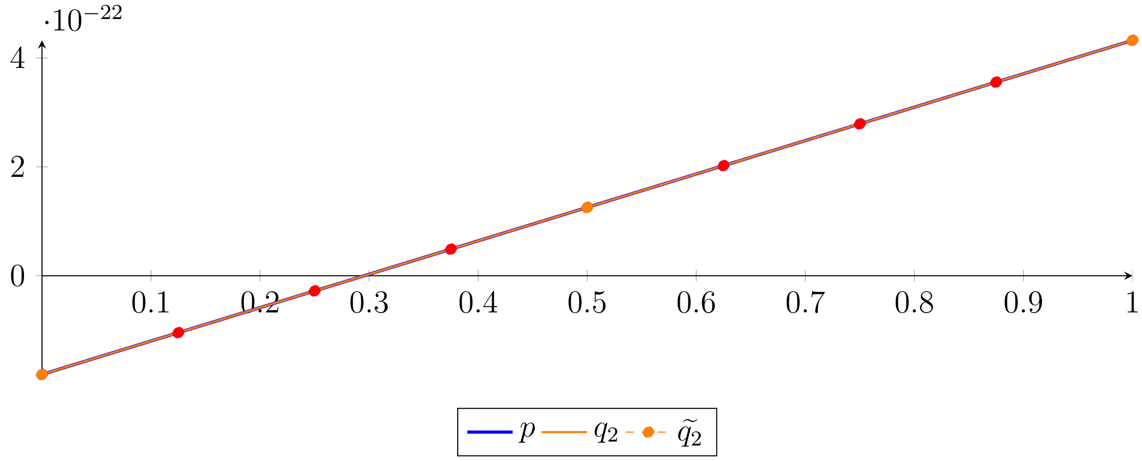
$$\begin{aligned} p = & -5.17944 \cdot 10^{-199} X^8 - 5.16502 \cdot 10^{-173} X^7 - 1.39653 \cdot 10^{-147} X^6 + 8.64863 \cdot 10^{-123} X^5 + 6.48817 \\ & \cdot 10^{-97} X^4 - 9.72096 \cdot 10^{-73} X^3 - 1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22} \\ = & -1.81261 \cdot 10^{-22} B_{0,8}(X) - 1.04571 \cdot 10^{-22} B_{1,8}(X) - 2.78818 \cdot 10^{-23} B_{2,8}(X) \\ & + 4.88078 \cdot 10^{-23} B_{3,8}(X) + 1.25497 \cdot 10^{-22} B_{4,8}(X) + 2.02187 \cdot 10^{-22} B_{5,8}(X) \\ & + 2.78877 \cdot 10^{-22} B_{6,8}(X) + 3.55566 \cdot 10^{-22} B_{7,8}(X) + 4.32256 \cdot 10^{-22} B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 = & -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22} \\ = & -1.81261 \cdot 10^{-22} B_{0,2} + 1.25497 \cdot 10^{-22} B_{1,2} + 4.32256 \cdot 10^{-22} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 = & \; 5.39888 \cdot 10^{-325} X^8 - 2.84119 \cdot 10^{-324} X^7 + 5.35011 \cdot 10^{-324} X^6 - 4.57499 \cdot 10^{-324} X^5 + 1.82797 \\ & \cdot 10^{-324} X^4 - 3.72306 \cdot 10^{-325} X^3 - 1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22} \\ = & -1.81261 \cdot 10^{-22} B_{0,8} - 1.04571 \cdot 10^{-22} B_{1,8} - 2.78818 \cdot 10^{-23} B_{2,8} + 4.88078 \cdot 10^{-23} B_{3,8} + 1.25497 \\ & \cdot 10^{-22} B_{4,8} + 2.02187 \cdot 10^{-22} B_{5,8} + 2.78877 \cdot 10^{-22} B_{6,8} + 3.55566 \cdot 10^{-22} B_{7,8} + 4.32256 \cdot 10^{-22} B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.86048 \cdot 10^{-74}$.
Bounding polynomials M and m:

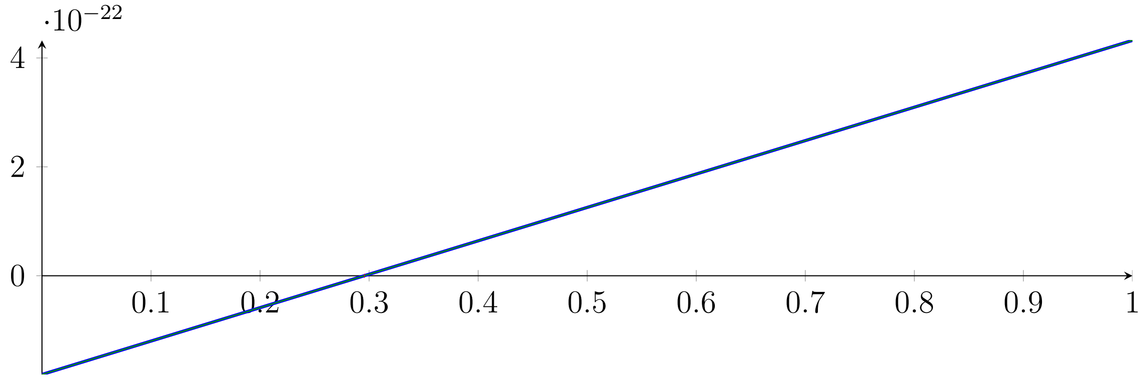
$$M = -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22}$$

$$m = -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22}$$

Root of M and m:

$N(M) = \{0.295446, 5.81467 \cdot 10^{24}\}$ $\qquad$ $N(m) = \{0.295446, 5.81467 \cdot 10^{24}\}$

Intersection intervals:



$$[0.295446, 0.295446]$$

Longest intersection interval: $1.58446 \cdot 10^{-52}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

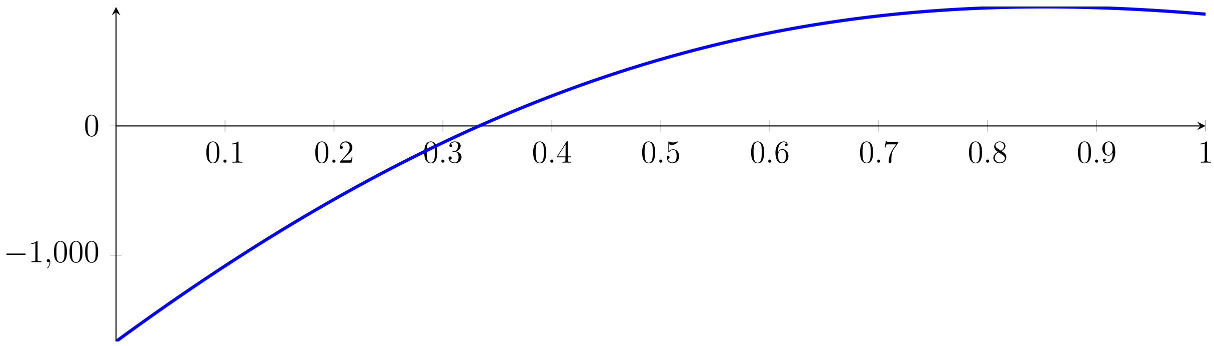
224.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

224.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-32}$.

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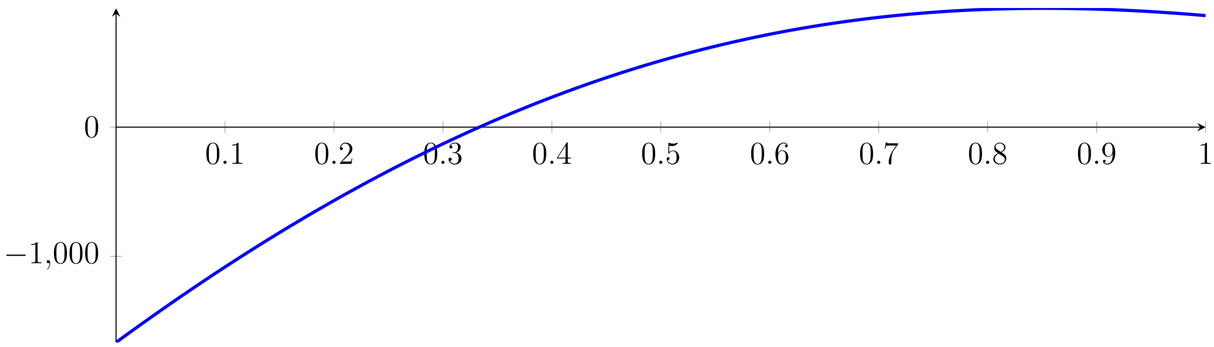
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

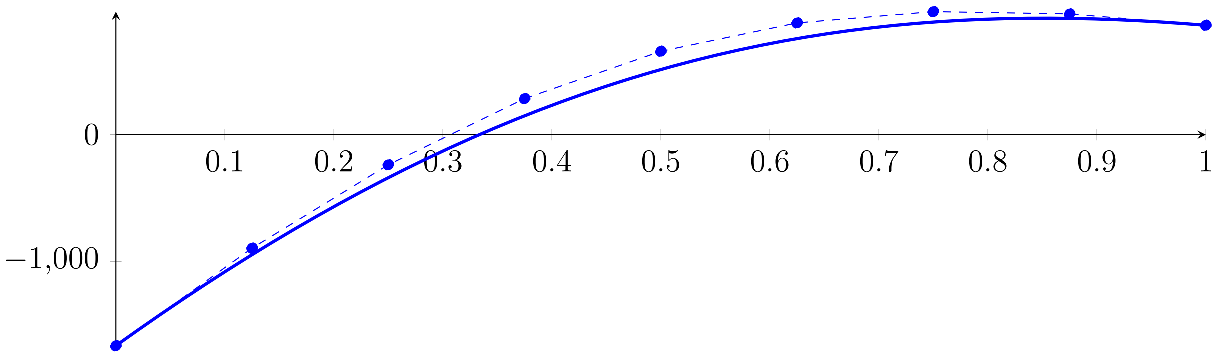
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



225.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

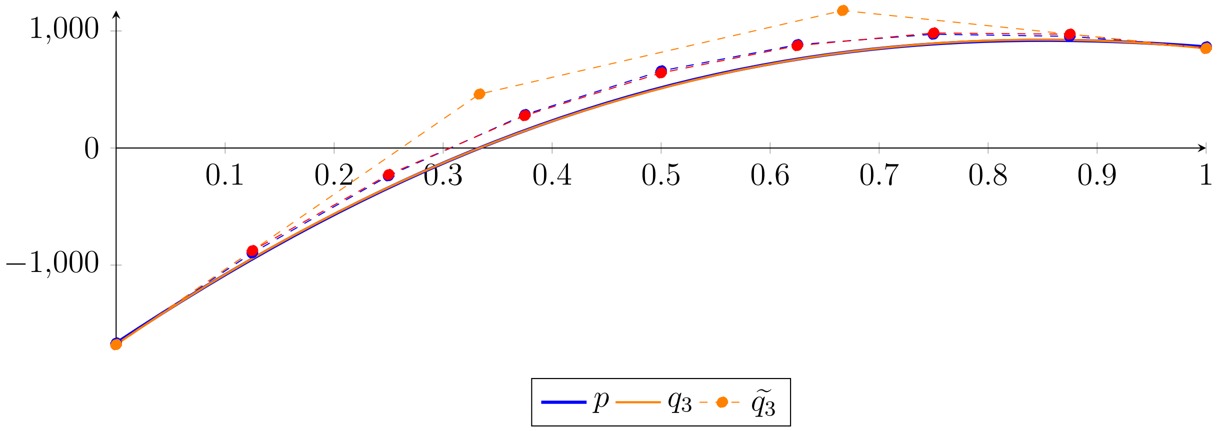
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.80826\cdot10^{-300}X^8 + 1.16606\cdot10^{-299}X^7 - 2.1197\cdot10^{-299}X^6 + 2.15003\cdot10^{-299}X^5 \\ &\quad - 1.20234\cdot10^{-299}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

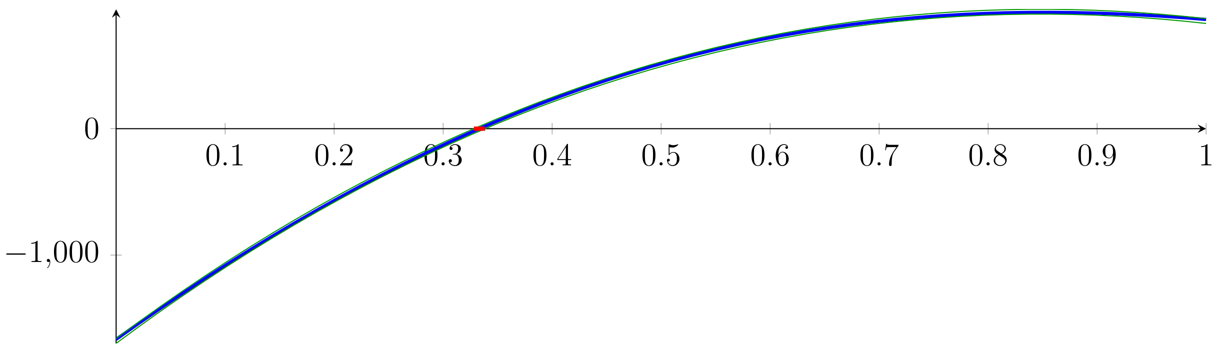
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$[0.328258, 0.338551]$

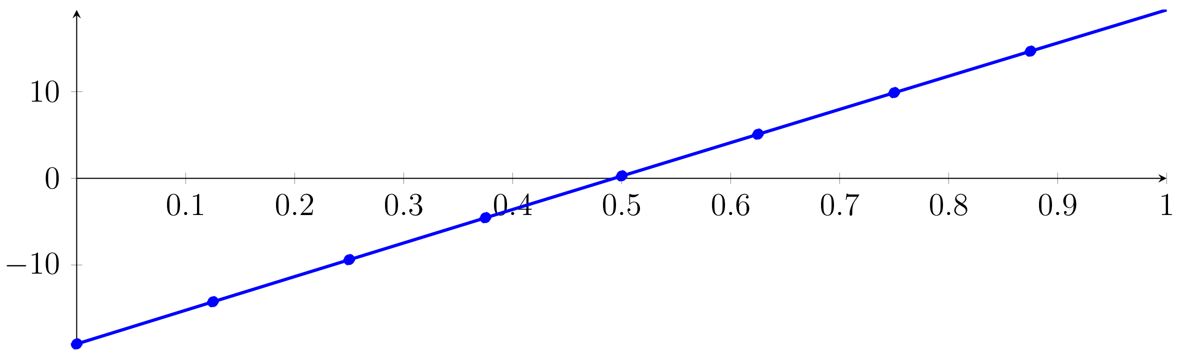
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

225.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

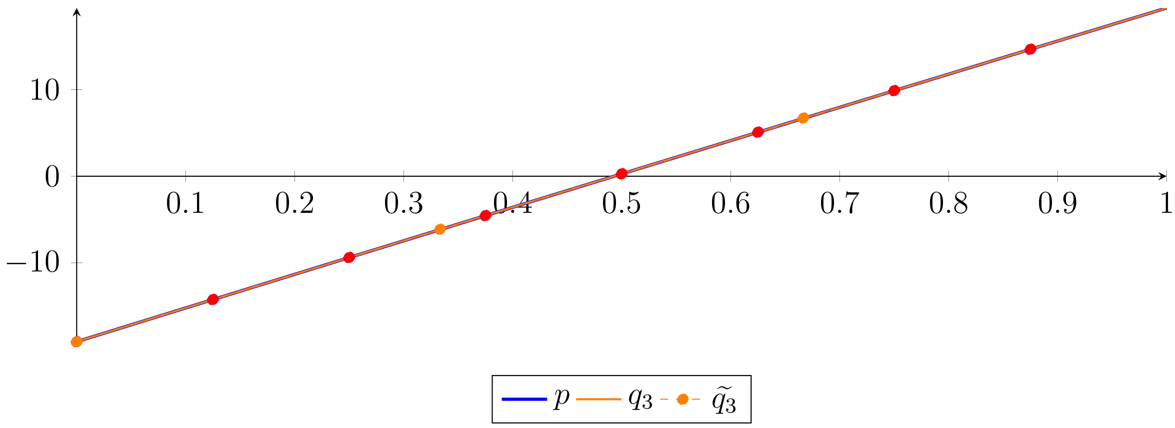
$$\begin{aligned} p &= -1.25947 \cdot 10^{-16} X^8 - 1.9937 \cdot 10^{-13} X^7 - 8.53073 \cdot 10^{-11} X^6 + 8.72839 \cdot 10^{-09} X^5 \\ &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= -1.96643{\cdot}10^{-303}X^8 + 1.82947{\cdot}10^{-302}X^7 - 4.89395{\cdot}10^{-302}X^6 + 5.49554{\cdot}10^{-302}X^5 \\ &\quad - 2.47838{\cdot}10^{-302}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902{\cdot}10^{-07}$.

Bounding polynomials M and m:

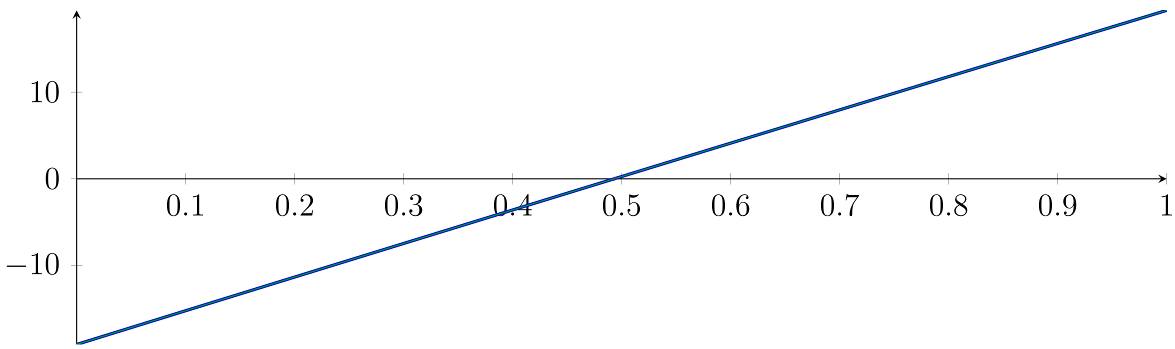
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

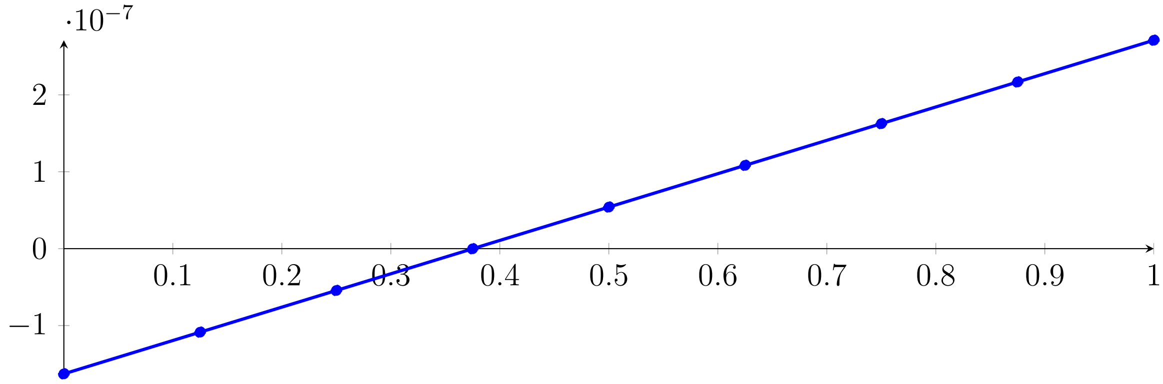
Longest intersection interval: $1.1252{\cdot}10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

225.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

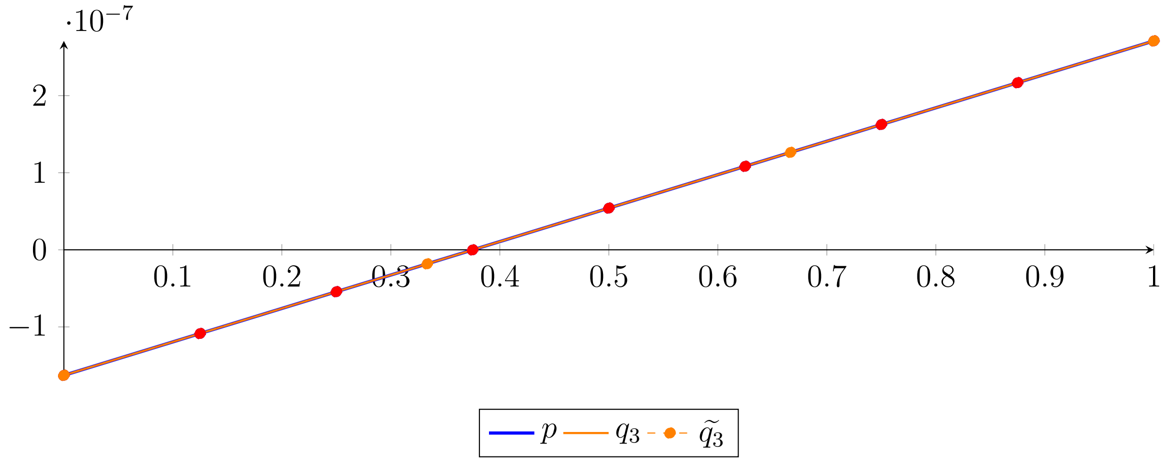
$$p = -3.2361\cdot10^{-80}X^8 - 4.56398\cdot10^{-69}X^7 - 1.74524\cdot10^{-58}X^6 + 1.52857\cdot10^{-48}X^5 + 1.62178\cdot10^{-37}X^4 - 3.43646\cdot10^{-28}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,8}(X) - 1.08585\cdot10^{-07}B_{1,8}(X) - 5.43592\cdot10^{-08}B_{2,8}(X) - 1.33711\cdot10^{-10}B_{3,8}(X) + 5.40918\cdot10^{-08}B_{4,8}(X) + 1.08317\cdot10^{-07}B_{5,8}(X) + 1.62543\cdot10^{-07}B_{6,8}(X) + 2.16768\cdot10^{-07}B_{7,8}(X) + 2.70994\cdot10^{-07}B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = -3.43646\cdot10^{-28}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,3} - 1.82089\cdot10^{-08}B_{1,3} + 1.26393\cdot10^{-07}B_{2,3} + 2.70994\cdot10^{-07}B_{3,3}$$

$$\tilde{q}_3 = 8.86066\cdot10^{-312}X^8 + 7.41744\cdot10^{-311}X^7 - 5.24902\cdot10^{-310}X^6 + 1.01267\cdot10^{-309}X^5 - 7.8921\cdot10^{-310}X^4 - 3.43646\cdot10^{-28}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,8} - 1.08585\cdot10^{-07}B_{1,8} - 5.43592\cdot10^{-08}B_{2,8} - 1.33711\cdot10^{-10}B_{3,8} + 5.40918\cdot10^{-08}B_{4,8} + 1.08317\cdot10^{-07}B_{5,8} + 1.62543\cdot10^{-07}B_{6,8} + 2.16768\cdot10^{-07}B_{7,8} + 2.70994\cdot10^{-07}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 3.47524\cdot10^{-39}$.

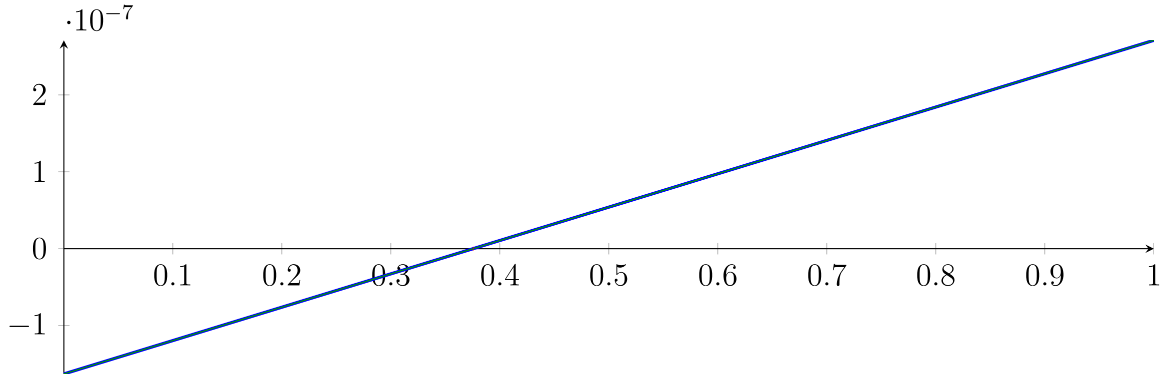
Bounding polynomials M and m:

$$M = -3.43646\cdot10^{-28}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$m = -3.43646\cdot10^{-28}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$

Root of M and m:

$$N(M) = \{-1.6133\cdot10^{11}, 0.375292, 7.82468\cdot10^{9}\} \quad N(m) = \{-1.6133\cdot10^{11}, 0.375292, 7.82468\cdot10^{9}\}$$

Intersection intervals:



$[0.375292, 0.375292]$

Longest intersection interval: $1.60221 \cdot 10^{-32}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

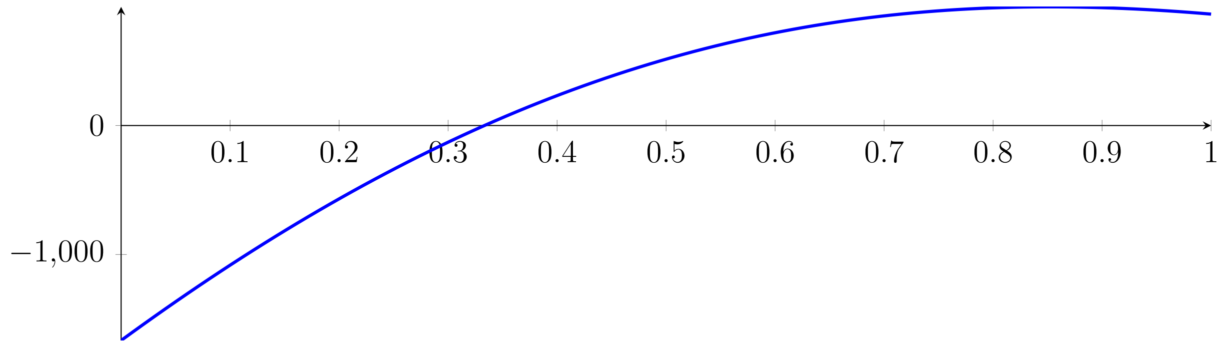
225.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 4!
p(0) = -6.9178e-12 - p(1) -6.9178e-12

225.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

with precision $\varepsilon = 1 \cdot 10^{-32}$.

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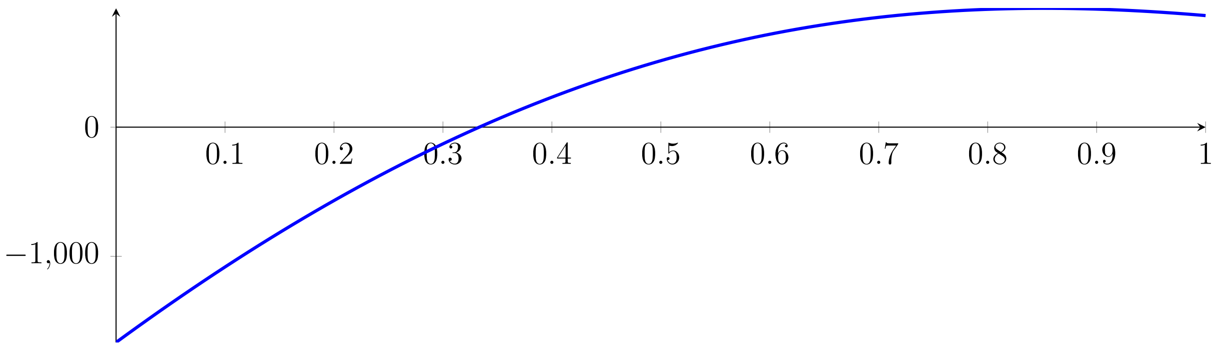
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

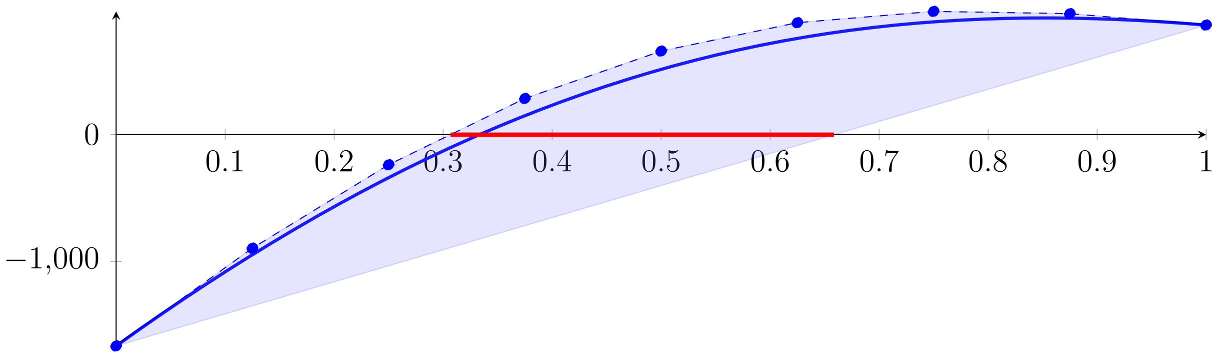
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



226.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

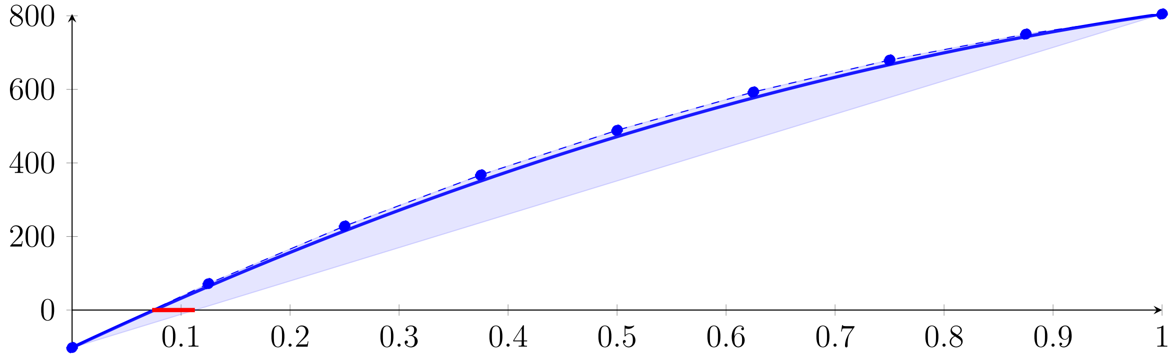
$$[0.306796, 0.658588]$$

Longest intersection interval: 0.351792
$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

226.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5\\&\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17\\&= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X)\\&\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

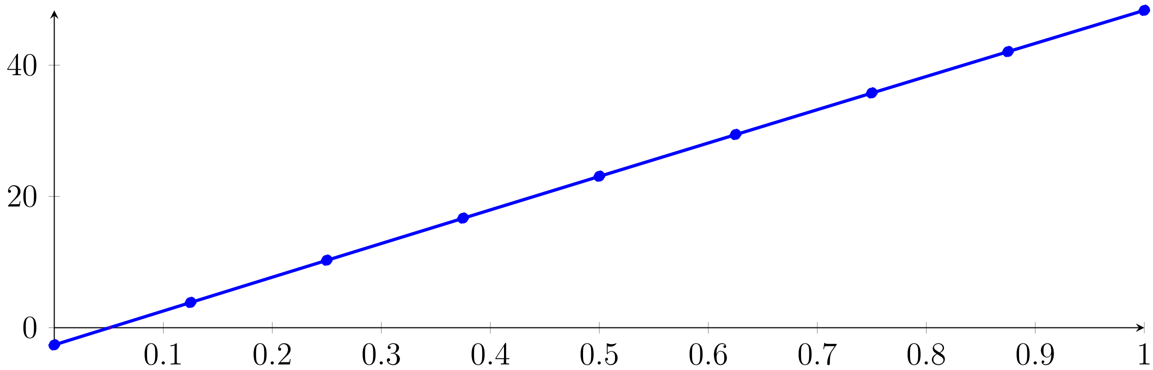
$$[0.0734515, 0.112637]$$

Longest intersection interval: 0.0391855
$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

226.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -1.30406\cdot 10^{-15}X^8 - 1.54459\cdot 10^{-12}X^7 - 4.9583\cdot 10^{-10}X^6 + 3.66751\cdot 10^{-08}X^5\\&\quad + 3.25466\cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683\\&= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X)\\&\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

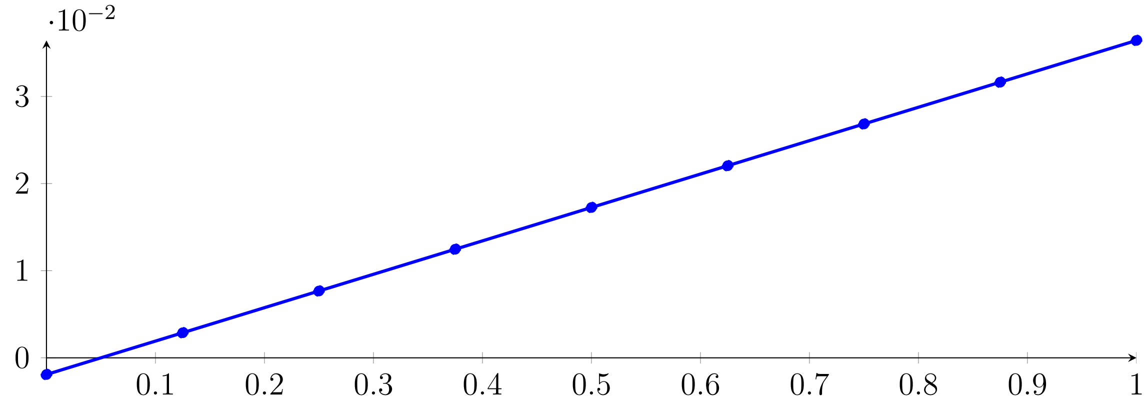
$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

226.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -1.20583\cdot10^{-40}X^8 - 1.92397\cdot10^{-34}X^7 - 8.32342\cdot10^{-29}X^6 + 8.24755\cdot10^{-24}X^5 \\ &\quad + 9.89972\cdot10^{-18}X^4 - 2.37322\cdot10^{-13}X^3 - 4.12146\cdot10^{-07}X^2 + 0.0383444X - 0.00191378 \\ &= -0.00191378B_{0,8}(X) + 0.00287927B_{1,8}(X) + 0.00767231B_{2,8}(X) \\ &\quad + 0.0124653B_{3,8}(X) + 0.0172583B_{4,8}(X) + 0.0220513B_{5,8}(X) \\ &\quad + 0.0268443B_{6,8}(X) + 0.0316373B_{7,8}(X) + 0.0364302B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

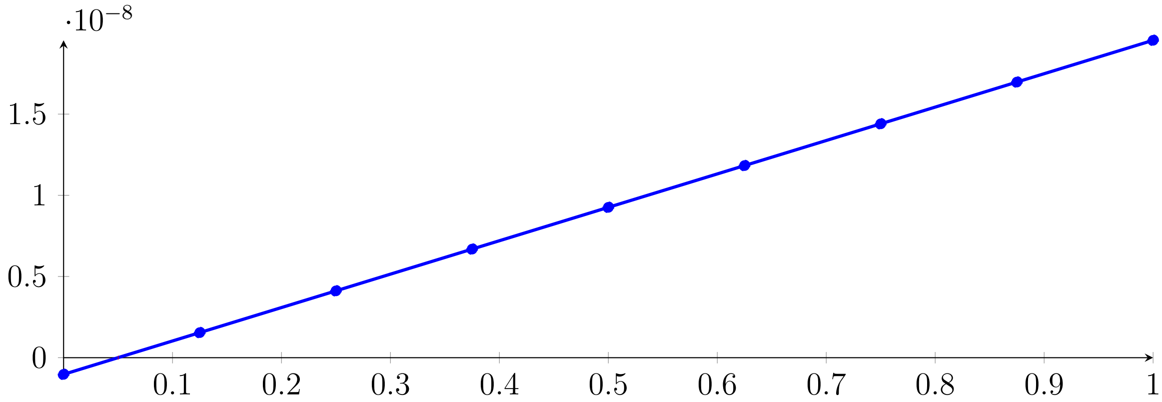
$$[0.0499103, 0.0499109]$$

Longest intersection interval: $5.36469\cdot10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

226.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -8.27263\cdot10^{-91}X^8 - 2.46044\cdot10^{-78}X^7 - 1.98413\cdot10^{-66}X^6 + 3.66478\cdot10^{-55}X^5 + 8.19978 \\ &\quad \cdot10^{-43}X^4 - 3.66412\cdot10^{-32}X^3 - 1.18615\cdot10^{-19}X^2 + 2.05706\cdot10^{-08}X - 1.02667\cdot10^{-09} \\ &= -1.02667\cdot10^{-09}B_{0,8}(X) + 1.54465\cdot10^{-09}B_{1,8}(X) + 4.11597\cdot10^{-09}B_{2,8}(X) \\ &\quad + 6.68729\cdot10^{-09}B_{3,8}(X) + 9.25862\cdot10^{-09}B_{4,8}(X) + 1.18299\cdot10^{-08}B_{5,8}(X) \\ &\quad + 1.44013\cdot10^{-08}B_{6,8}(X) + 1.69726\cdot10^{-08}B_{7,8}(X) + 1.95439\cdot10^{-08}B_{8,8}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

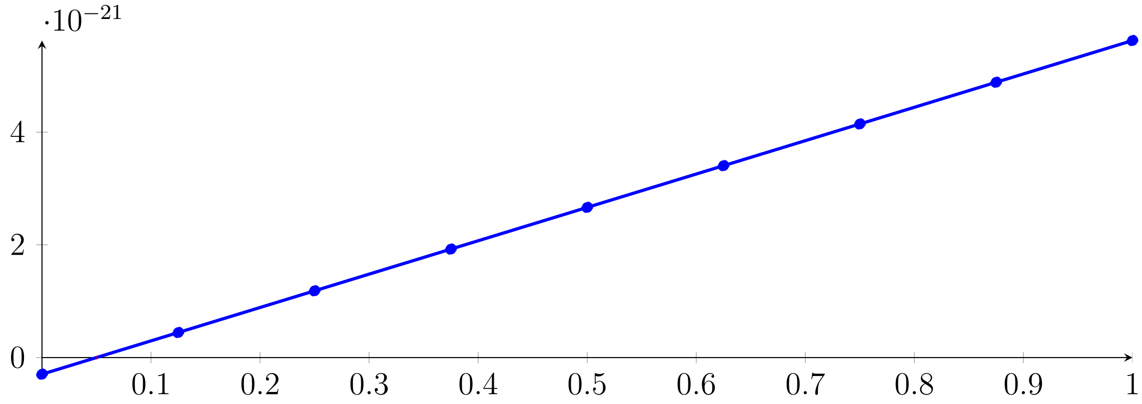
$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87793 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

226.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -3.89305 \cdot 10^{-191} X^8 - 4.02327 \cdot 10^{-166} X^7 - 1.12734 \cdot 10^{-141} X^6 + 7.23523 \cdot 10^{-118} X^5 + 5.62504 \\ &\quad \cdot 10^{-93} X^4 - 8.73397 \cdot 10^{-70} X^3 - 9.82433 \cdot 10^{-45} X^2 + 5.92008 \cdot 10^{-21} X - 2.9547 \cdot 10^{-22} \\ &= -2.9547 \cdot 10^{-22} B_{0,8}(X) + 4.4454 \cdot 10^{-22} B_{1,8}(X) + 1.18455 \cdot 10^{-21} B_{2,8}(X) \\ &\quad + 1.92456 \cdot 10^{-21} B_{3,8}(X) + 2.66457 \cdot 10^{-21} B_{4,8}(X) + 3.40458 \cdot 10^{-21} B_{5,8}(X) \\ &\quad + 4.14459 \cdot 10^{-21} B_{6,8}(X) + 4.8846 \cdot 10^{-21} B_{7,8}(X) + 5.62461 \cdot 10^{-21} B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

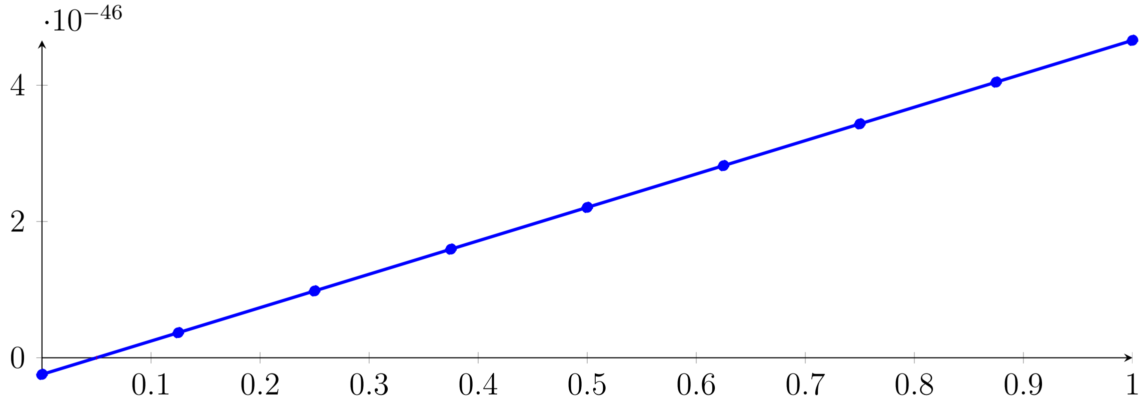
$$[0.0499098, 0.0499098]$$

Longest intersection interval: $8.28251 \cdot 10^{-26}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

226.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.13027 \cdot 10^{-352} X^8 - 1.07575 \cdot 10^{-341} X^7 - 3.63937 \cdot 10^{-292} X^6 + 2.82008 \cdot 10^{-243} X^5 + 2.64711 \\ &\quad \cdot 10^{-193} X^4 - 4.96246 \cdot 10^{-145} X^3 - 6.73948 \cdot 10^{-95} X^2 + 4.90331 \cdot 10^{-46} X - 2.44723 \cdot 10^{-47} \\ &= -2.44723 \cdot 10^{-47} B_{0,8}(X) + 3.6819 \cdot 10^{-47} B_{1,8}(X) + 9.81104 \cdot 10^{-47} B_{2,8}(X) \\ &\quad + 1.59402 \cdot 10^{-46} B_{3,8}(X) + 2.20693 \cdot 10^{-46} B_{4,8}(X) + 2.81984 \cdot 10^{-46} B_{5,8}(X) \\ &\quad + 3.43276 \cdot 10^{-46} B_{6,8}(X) + 4.04567 \cdot 10^{-46} B_{7,8}(X) + 4.65858 \cdot 10^{-46} B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $6.85999 \cdot 10^{-51}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

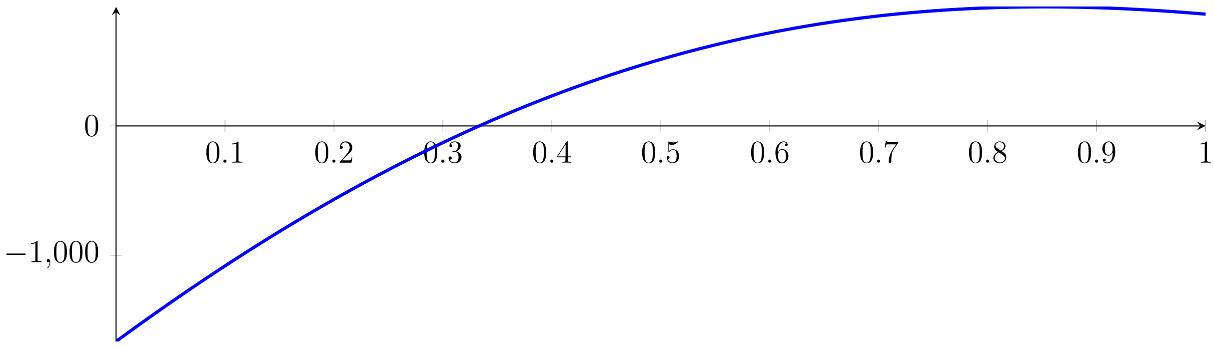
226.8 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 8!

226.9 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-64}$.

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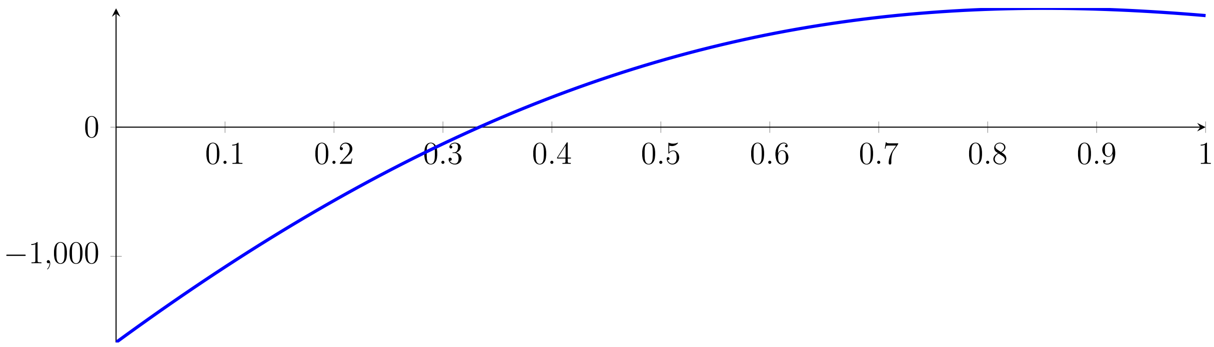
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

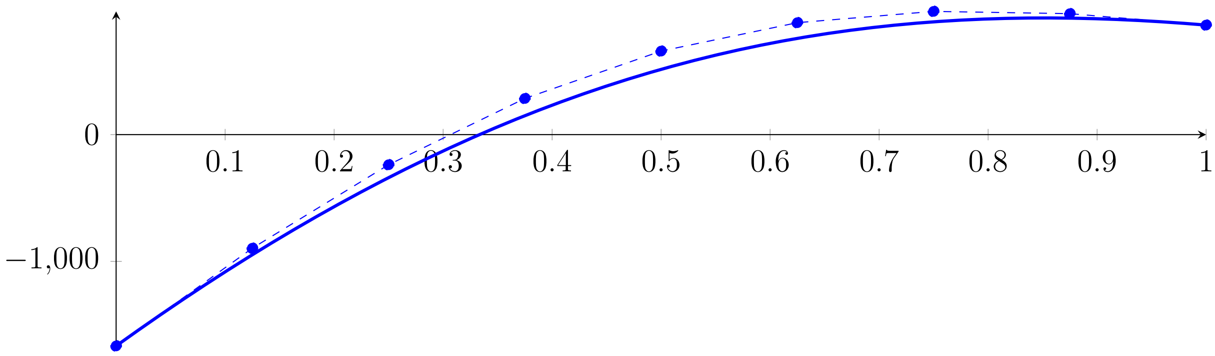
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



227.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

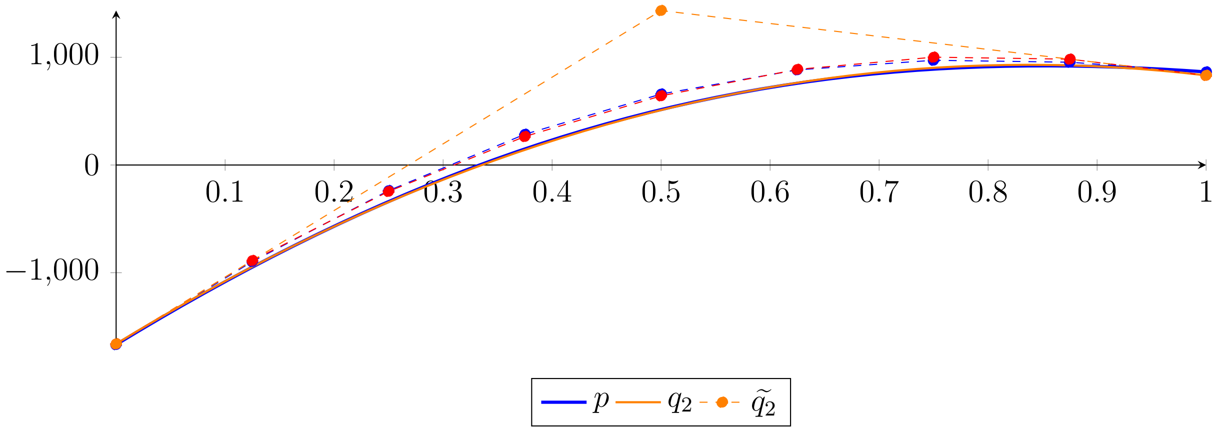
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 6.38292 \cdot 10^{-300}X^8 - 2.86993 \cdot 10^{-299}X^7 + 5.15342 \cdot 10^{-299}X^6 - 4.6928 \cdot 10^{-299}X^5 \\ &\quad + 2.29296 \cdot 10^{-299}X^4 - 6.02693 \cdot 10^{-300}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

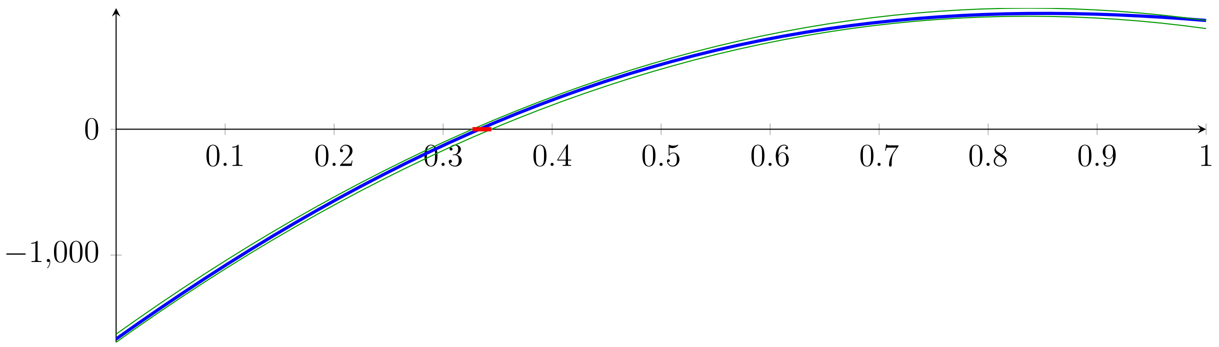
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

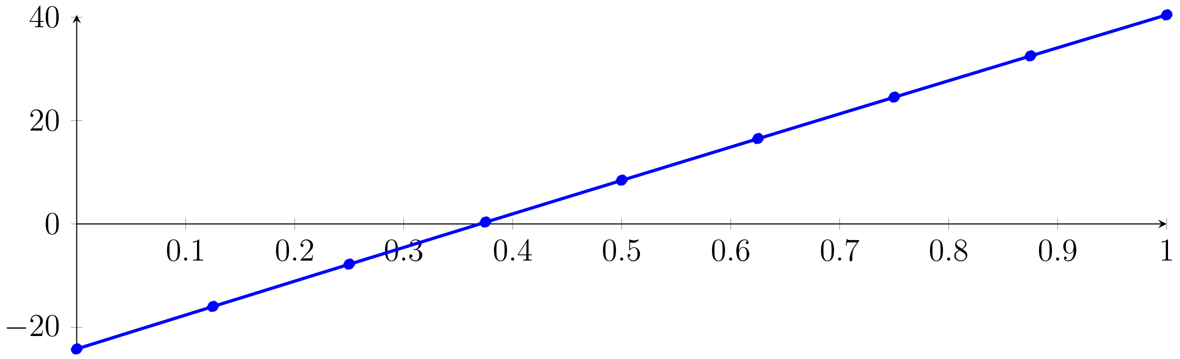
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

227.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

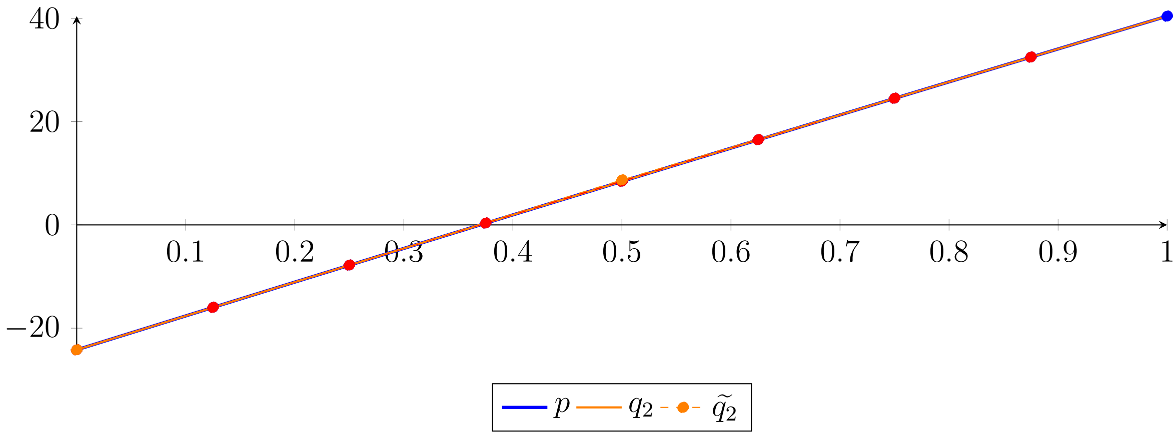
$$\begin{aligned} p = &-8.16249 \cdot 10^{-15} X^8 - 7.66571 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ &+ 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ = &-24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &+ 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$
\begin{aligned}
q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\
&= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{q}_2 &= 4.27533\cdot 10^{-302}X^8 - 2.42468\cdot 10^{-301}X^7 + 4.90537\cdot 10^{-301}X^6 - 4.6286\cdot 10^{-301}X^5 \\
&\quad + 2.17546\cdot 10^{-301}X^4 - 5.36695\cdot 10^{-302}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\
&= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\
&\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}
\end{aligned}
$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894\cdot 10^{-05}$.
Bounding polynomials M and m:

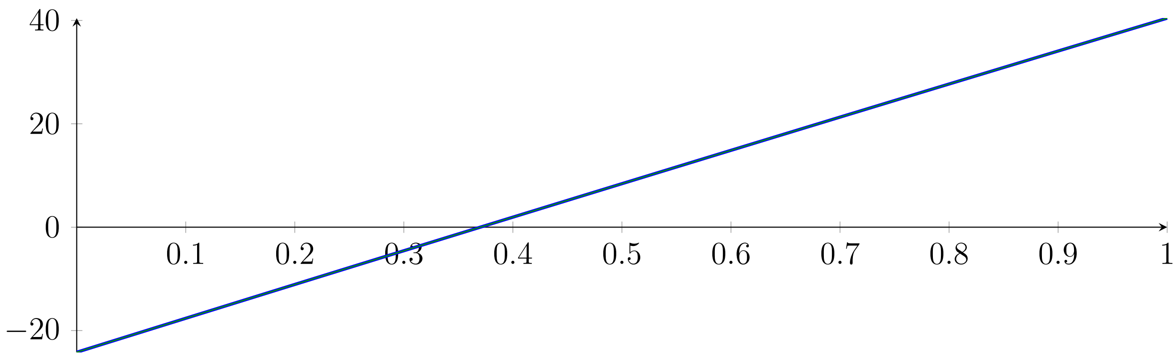
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



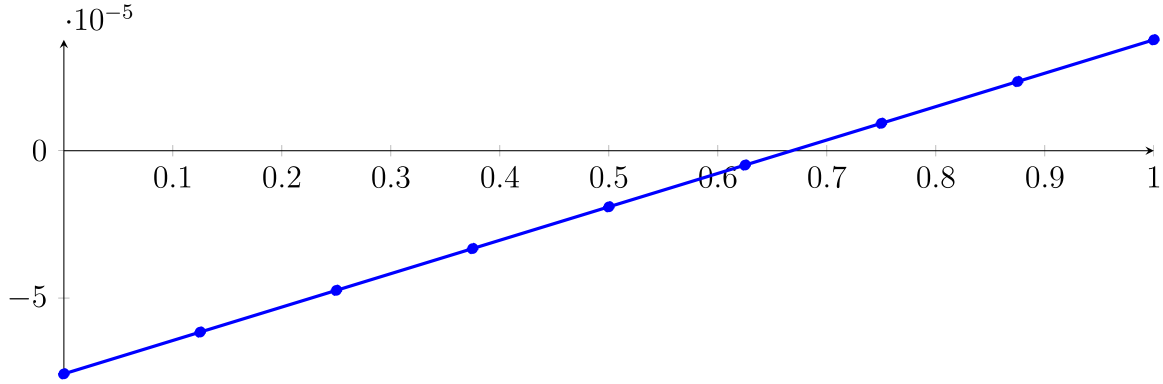
$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588\cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

227.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

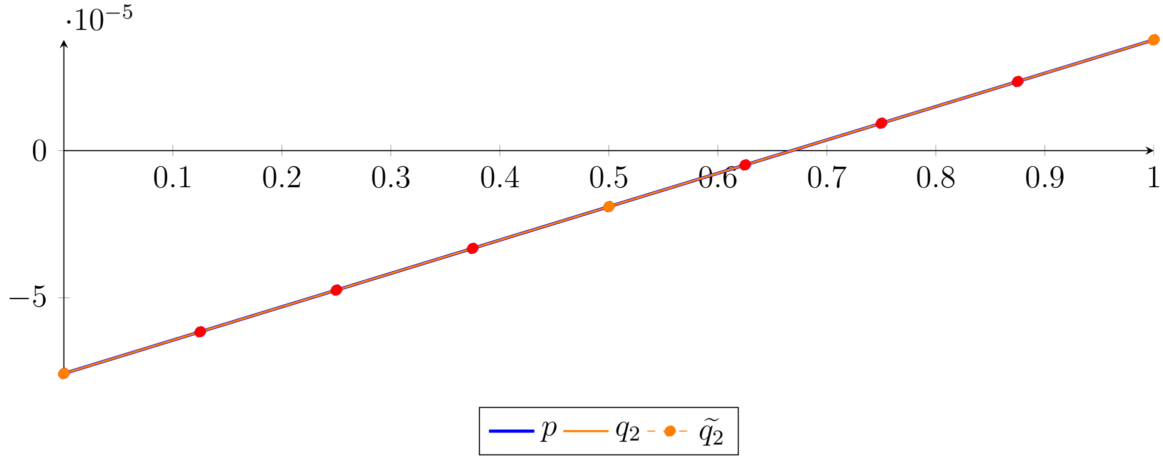
$$\begin{aligned}p &= -7.04578\cdot 10^{-61}X^8 - 3.80201\cdot 10^{-52}X^7 - 5.5627\cdot 10^{-44}X^6 + 1.86413\cdot 10^{-36}X^5 + 7.56737\\ &\quad\cdot 10^{-28}X^4 - 6.13517\cdot 10^{-21}X^3 - 3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05}\\ &= -7.5732\cdot 10^{-05}B_{0,8}(X) - 6.15596\cdot 10^{-05}B_{1,8}(X) - 4.73873\cdot 10^{-05}B_{2,8}(X)\\ &\quad - 3.32149\cdot 10^{-05}B_{3,8}(X) - 1.90425\cdot 10^{-05}B_{4,8}(X) - 4.87016\cdot 10^{-06}B_{5,8}(X)\\ &\quad + 9.3022\cdot 10^{-06}B_{6,8}(X) + 2.34746\cdot 10^{-05}B_{7,8}(X) + 3.76469\cdot 10^{-05}B_{8,8}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05}\\ &= -7.5732\cdot 10^{-05}B_{0,2} - 1.90425\cdot 10^{-05}B_{1,2} + 3.76469\cdot 10^{-05}B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_2 &= -3.62396\cdot 10^{-308}X^8 + 2.32992\cdot 10^{-308}X^7 + 2.61753\cdot 10^{-307}X^6 - 5.97049\cdot 10^{-307}X^5 + 5.13401\\ &\quad\cdot 10^{-307}X^4 - 1.82868\cdot 10^{-307}X^3 - 3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05}\\ &= -7.5732\cdot 10^{-05}B_{0,8} - 6.15596\cdot 10^{-05}B_{1,8} - 4.73873\cdot 10^{-05}B_{2,8} - 3.32149\cdot 10^{-05}B_{3,8} - 1.90425\\ &\quad\cdot 10^{-05}B_{4,8} - 4.87016\cdot 10^{-06}B_{5,8} + 9.3022\cdot 10^{-06}B_{6,8} + 2.34746\cdot 10^{-05}B_{7,8} + 3.76469\cdot 10^{-05}B_{8,8}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.06758\cdot 10^{-22}$.

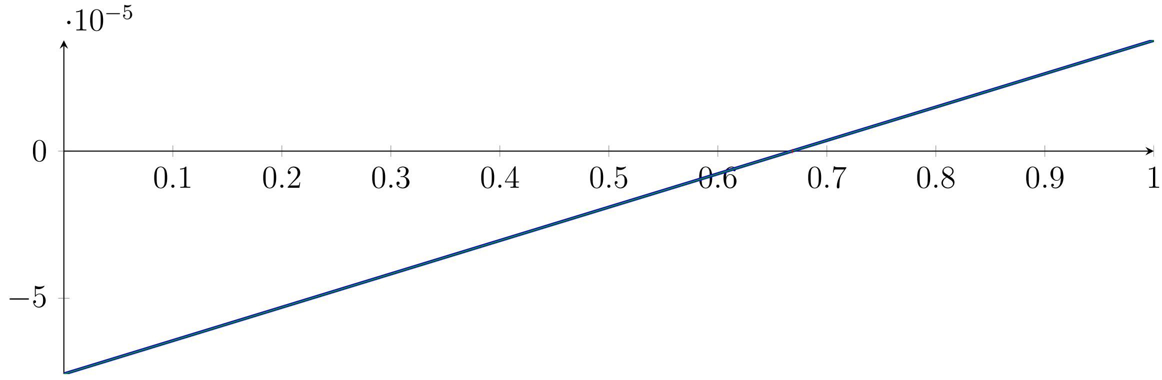
Bounding polynomials M and m:

$$M = -3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05}$$
$$m = -3.6034\cdot 10^{-12}X^2 + 0.000113379X - 7.5732\cdot 10^{-05}$$

Root of M and m:

$$N(M) = \{0.667955, 3.14644\cdot 10^{7}\} \qquad N(m) = \{0.667955, 3.14644\cdot 10^{7}\}$$

Intersection intervals:



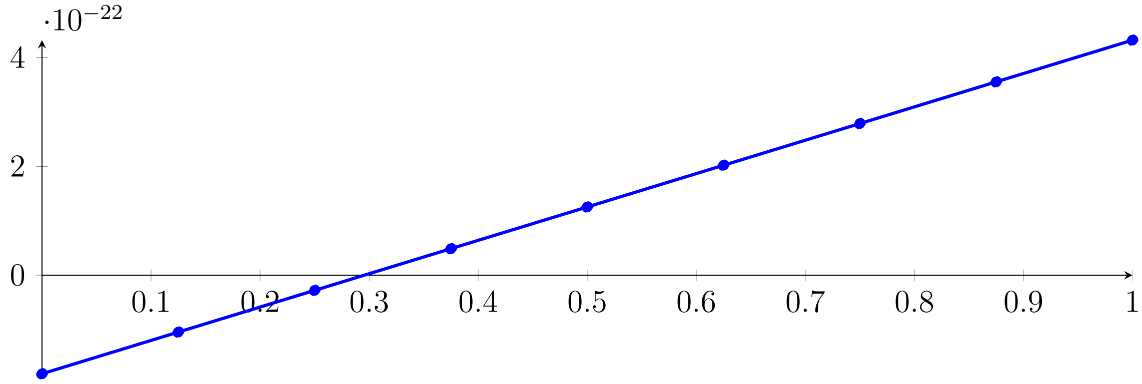
[0.667955, 0.667955]

Longest intersection interval: $5.41121 \cdot 10^{-18}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

227.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

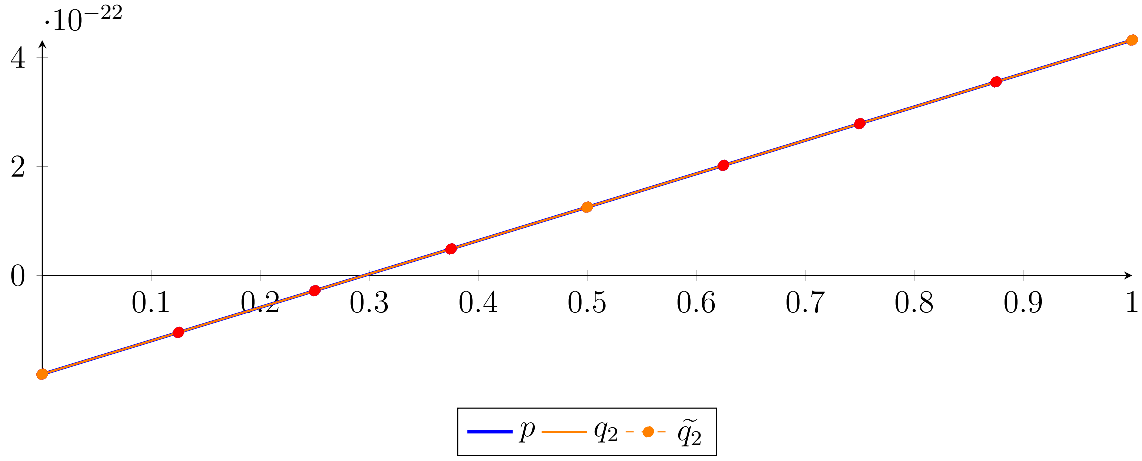
$$\begin{aligned} p = &-5.17944 \cdot 10^{-199} X^8 - 5.16502 \cdot 10^{-173} X^7 - 1.39653 \cdot 10^{-147} X^6 + 8.64863 \cdot 10^{-123} X^5 + 6.48817 \\ &\cdot 10^{-97} X^4 - 9.72096 \cdot 10^{-73} X^3 - 1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22} \\ = &-1.81261 \cdot 10^{-22} B_{0,8}(X) - 1.04571 \cdot 10^{-22} B_{1,8}(X) - 2.78818 \cdot 10^{-23} B_{2,8}(X) \\ &+ 4.88078 \cdot 10^{-23} B_{3,8}(X) + 1.25497 \cdot 10^{-22} B_{4,8}(X) + 2.02187 \cdot 10^{-22} B_{5,8}(X) \\ &+ 2.78877 \cdot 10^{-22} B_{6,8}(X) + 3.55566 \cdot 10^{-22} B_{7,8}(X) + 4.32256 \cdot 10^{-22} B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 = &-1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22} \\ = &-1.81261 \cdot 10^{-22} B_{0,2} + 1.25497 \cdot 10^{-22} B_{1,2} + 4.32256 \cdot 10^{-22} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 = &\, 5.39888 \cdot 10^{-325} X^8 - 2.84119 \cdot 10^{-324} X^7 + 5.35011 \cdot 10^{-324} X^6 - 4.57499 \cdot 10^{-324} X^5 + 1.82797 \\ &\cdot 10^{-324} X^4 - 3.72306 \cdot 10^{-325} X^3 - 1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22} \\ = &-1.81261 \cdot 10^{-22} B_{0,8} - 1.04571 \cdot 10^{-22} B_{1,8} - 2.78818 \cdot 10^{-23} B_{2,8} + 4.88078 \cdot 10^{-23} B_{3,8} + 1.25497 \\ &\cdot 10^{-22} B_{4,8} + 2.02187 \cdot 10^{-22} B_{5,8} + 2.78877 \cdot 10^{-22} B_{6,8} + 3.55566 \cdot 10^{-22} B_{7,8} + 4.32256 \cdot 10^{-22} B_{8,8} \end{aligned}$$



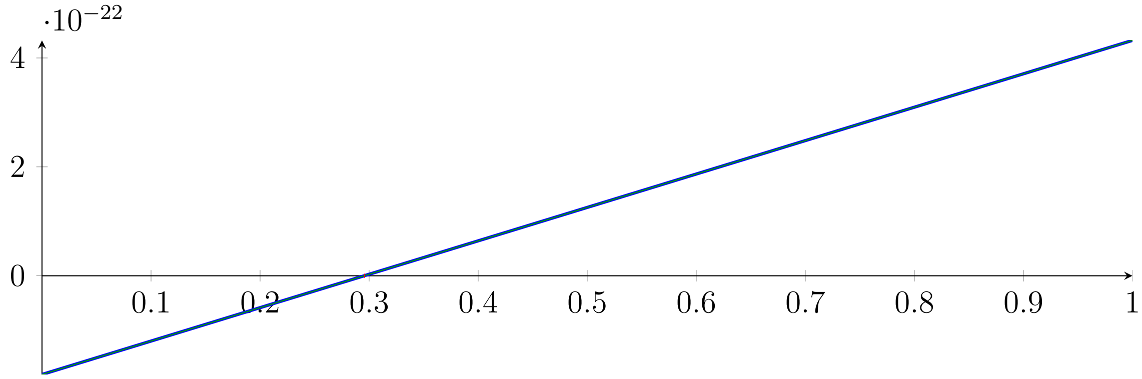
The maximum difference of the Bézier coefficients is $\delta = 4.86048 \cdot 10^{-74}$.
Bounding polynomials M and m:

$$M = -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22}$$
$$m = -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22}$$

Root of M and m:

$$N(M) = \{0.295446, 5.81467 \cdot 10^{24}\} \qquad N(m) = \{0.295446, 5.81467 \cdot 10^{24}\}$$

Intersection intervals:



$$[0.295446, 0.295446]$$

Longest intersection interval: $1.58446 \cdot 10^{-52}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

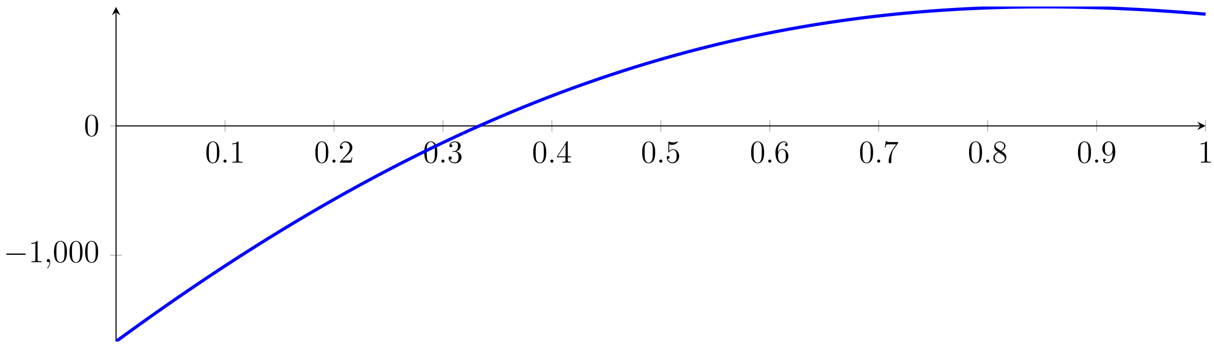
227.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

227.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

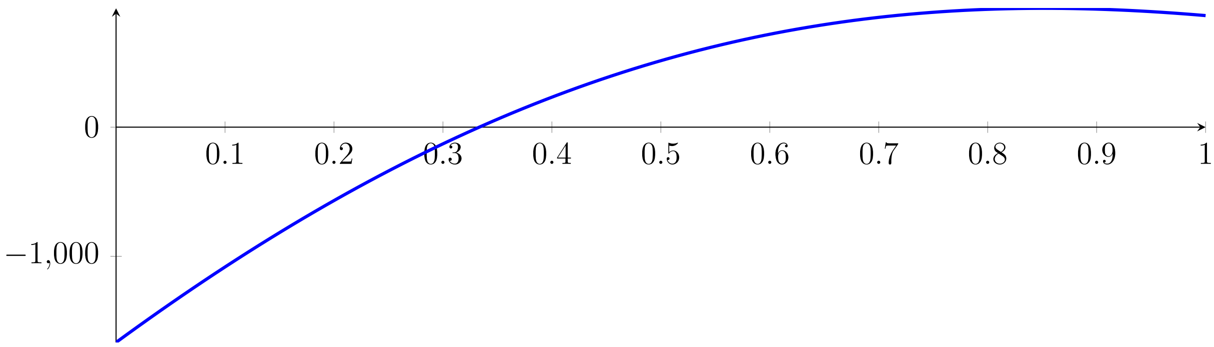
with precision $\varepsilon = 1 \cdot 10^{-64}$.

228 Running CubeClip on f_8 with epsilon 64

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

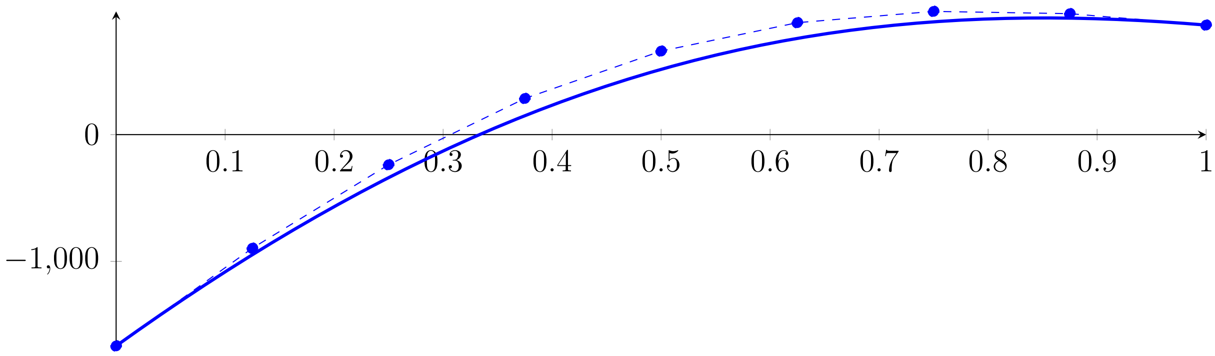
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



228.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

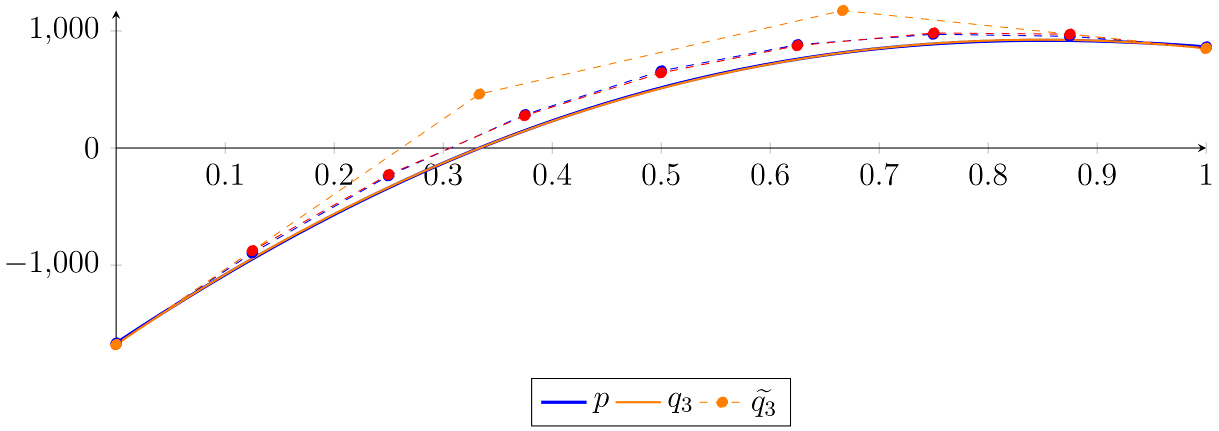
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.80826{\cdot}10^{-300}X^8 + 1.16606{\cdot}10^{-299}X^7 - 2.1197{\cdot}10^{-299}X^6 + 2.15003{\cdot}10^{-299}X^5 \\ &\quad - 1.20234{\cdot}10^{-299}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

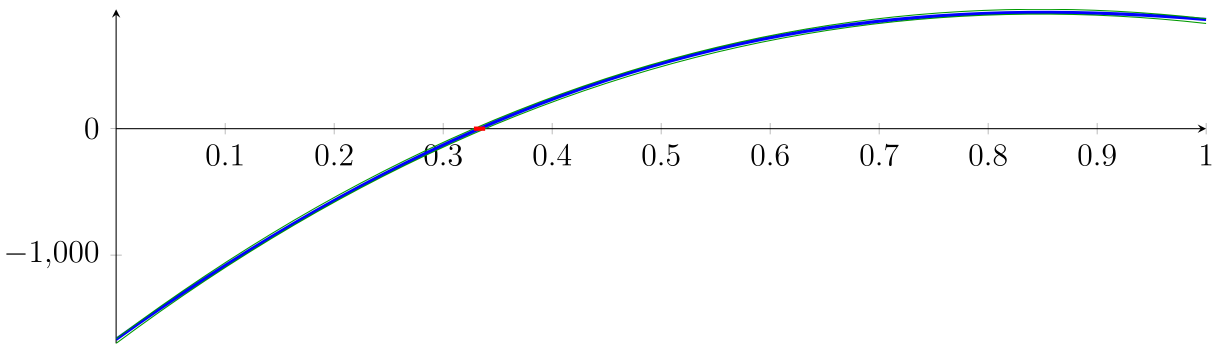
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$[0.328258, 0.338551]$

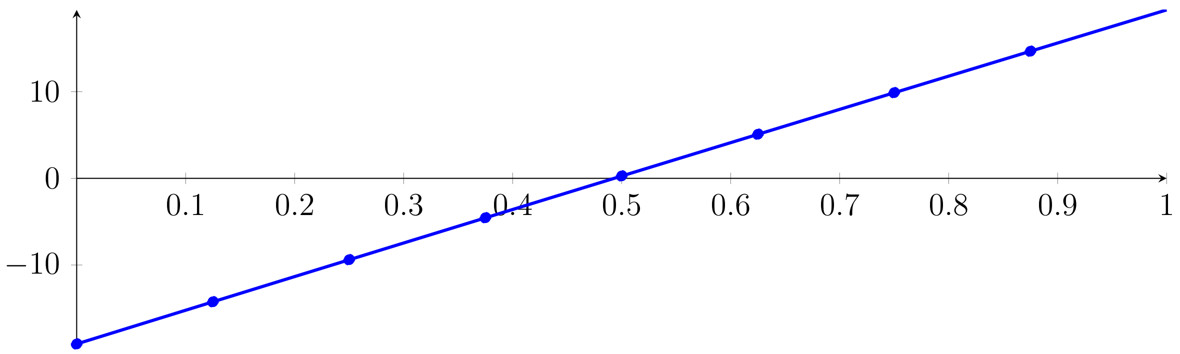
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

228.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

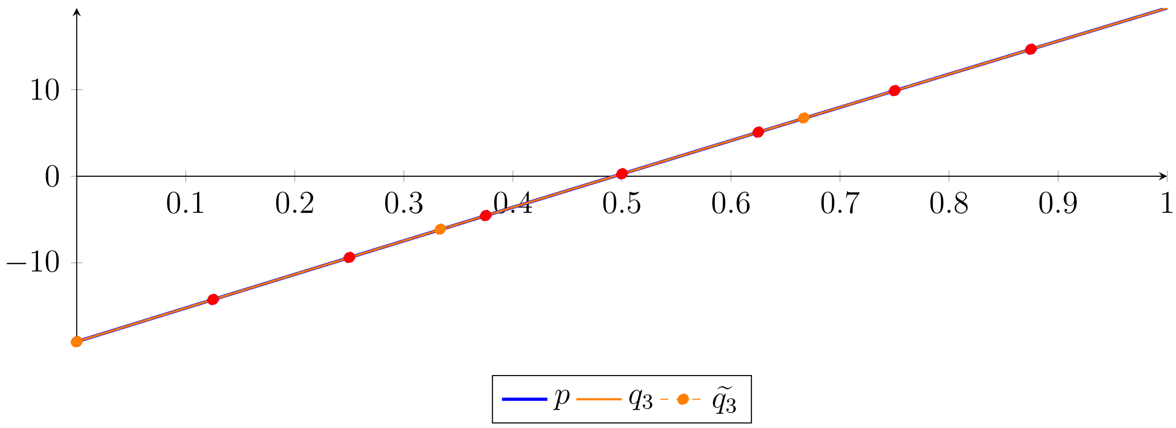
$$\begin{aligned} p &= -1.25947 \cdot 10^{-16} X^8 - 1.9937 \cdot 10^{-13} X^7 - 8.53073 \cdot 10^{-11} X^6 + 8.72839 \cdot 10^{-09} X^5 \\ &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= -1.96643\cdot10^{-303}X^8 + 1.82947\cdot10^{-302}X^7 - 4.89395\cdot10^{-302}X^6 + 5.49554\cdot10^{-302}X^5 \\ &\quad - 2.47838\cdot10^{-302}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902\cdot10^{-07}$.

Bounding polynomials M and m:

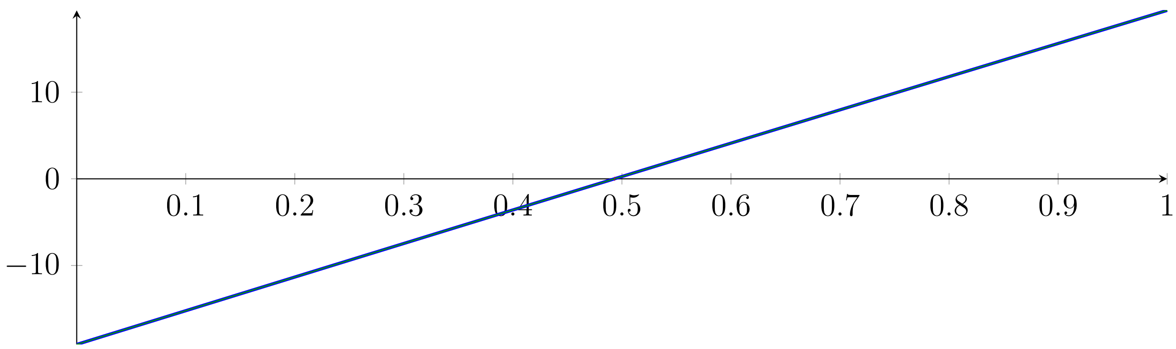
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

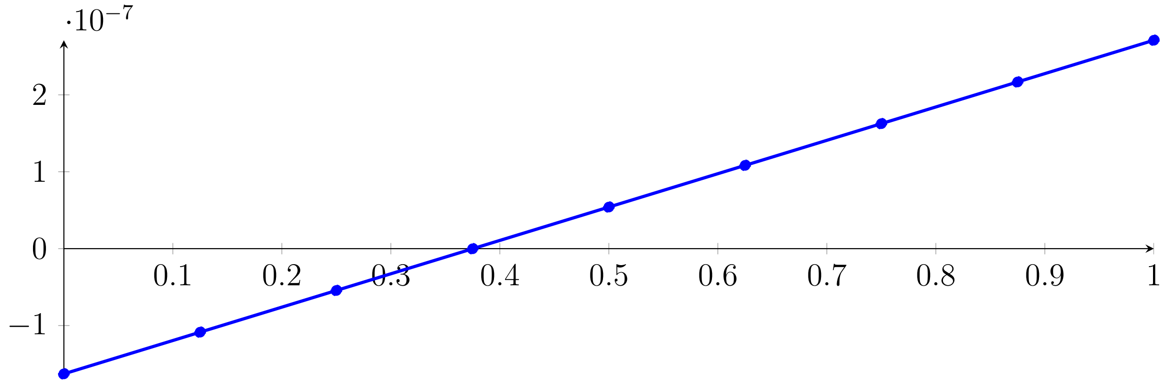
Longest intersection interval: $1.1252\cdot10^{-08}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

228.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

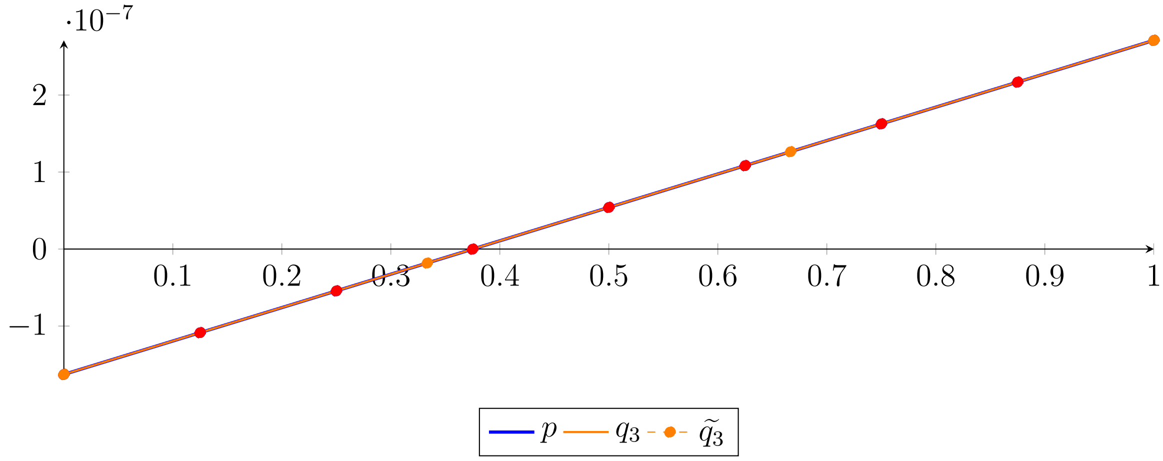
$$p = -3.2361\cdot10^{-80}X^8 - 4.56398\cdot10^{-69}X^7 - 1.74524\cdot10^{-58}X^6 + 1.52857\cdot10^{-48}X^5 + 1.62178\cdot10^{-37}X^4 - 3.43646\cdot10^{-28}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,8}(X) - 1.08585\cdot10^{-07}B_{1,8}(X) - 5.43592\cdot10^{-08}B_{2,8}(X) - 1.33711\cdot10^{-10}B_{3,8}(X) + 5.40918\cdot10^{-08}B_{4,8}(X) + 1.08317\cdot10^{-07}B_{5,8}(X) + 1.62543\cdot10^{-07}B_{6,8}(X) + 2.16768\cdot10^{-07}B_{7,8}(X) + 2.70994\cdot10^{-07}B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = -3.43646\cdot10^{-28}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,3} - 1.82089\cdot10^{-08}B_{1,3} + 1.26393\cdot10^{-07}B_{2,3} + 2.70994\cdot10^{-07}B_{3,3}$$

$$\tilde{q}_3 = 8.86066\cdot10^{-312}X^8 + 7.41744\cdot10^{-311}X^7 - 5.24902\cdot10^{-310}X^6 + 1.01267\cdot10^{-309}X^5 - 7.8921\cdot10^{-310}X^4 - 3.43646\cdot10^{-28}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$= -1.6281\cdot10^{-07}B_{0,8} - 1.08585\cdot10^{-07}B_{1,8} - 5.43592\cdot10^{-08}B_{2,8} - 1.33711\cdot10^{-10}B_{3,8} + 5.40918\cdot10^{-08}B_{4,8} + 1.08317\cdot10^{-07}B_{5,8} + 1.62543\cdot10^{-07}B_{6,8} + 2.16768\cdot10^{-07}B_{7,8} + 2.70994\cdot10^{-07}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 3.47524\cdot10^{-39}$.

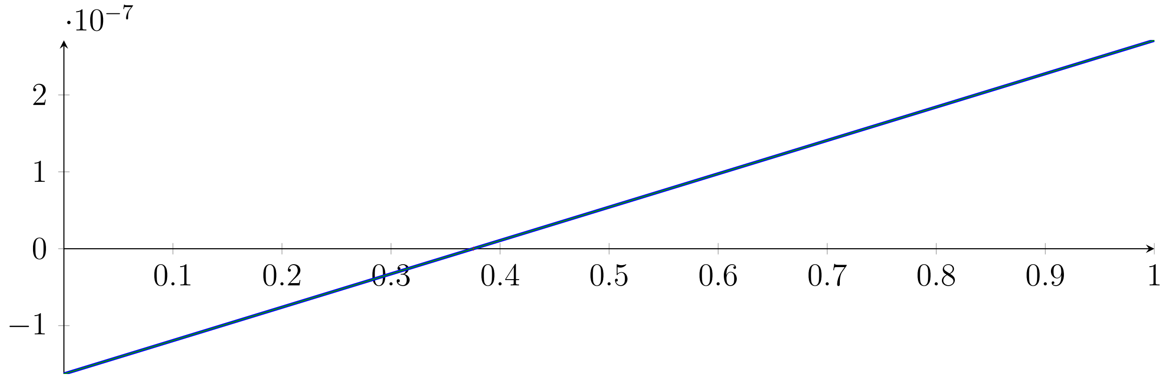
Bounding polynomials M and m:

$$M = -3.43646\cdot10^{-28}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$
$$m = -3.43646\cdot10^{-28}X^3 - 5.27516\cdot10^{-17}X^2 + 4.33804\cdot10^{-07}X - 1.6281\cdot10^{-07}$$

Root of M and m:

$$N(M) = \{-1.6133\cdot10^{11}, 0.375292, 7.82468\cdot10^{9}\} \quad N(m) = \{-1.6133\cdot10^{11}, 0.375292, 7.82468\cdot10^{9}\}$$

Intersection intervals:



$[0.375292, 0.375292]$

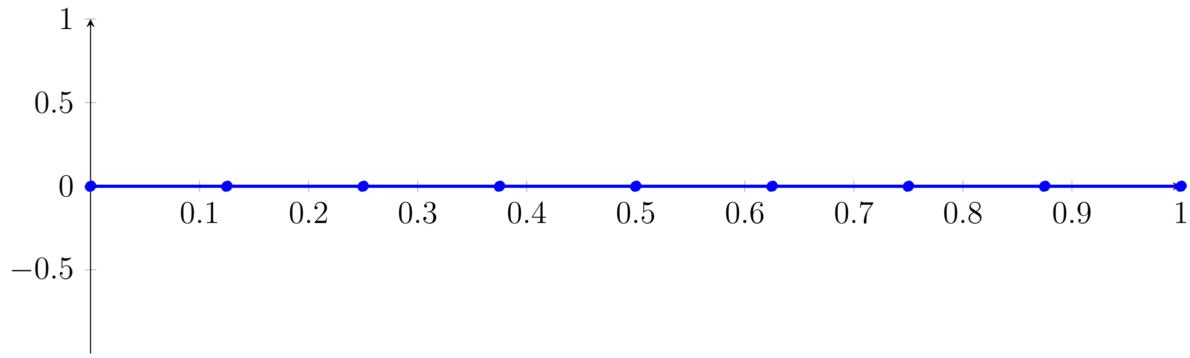
Longest intersection interval: $1.60221 \cdot 10^{-32}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

228.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

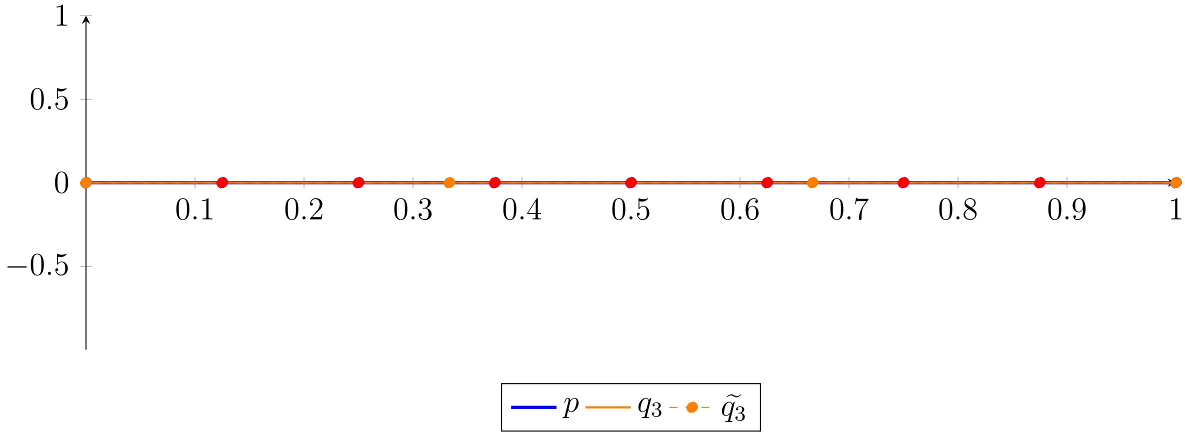
$$\begin{aligned}p &= 3.5617 \cdot 10^{-318} X^8 - 1.23705 \cdot 10^{-291} X^7 - 2.95243 \cdot 10^{-249} X^6 + 1.61394 \cdot 10^{-207} X^5 + 1.06875 \\ &\quad \cdot 10^{-164} X^4 - 1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12} \\ &= -6.9178 \cdot 10^{-12} B_{0,8}(X) - 6.9178 \cdot 10^{-12} B_{1,8}(X) - 6.9178 \cdot 10^{-12} B_{2,8}(X) \\ &\quad - 6.9178 \cdot 10^{-12} B_{3,8}(X) - 6.9178 \cdot 10^{-12} B_{4,8}(X) - 6.9178 \cdot 10^{-12} B_{5,8}(X) \\ &\quad - 6.9178 \cdot 10^{-12} B_{6,8}(X) - 6.9178 \cdot 10^{-12} B_{7,8}(X) - 6.9178 \cdot 10^{-12} B_{8,8}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= -1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12} \\ &= -6.9178 \cdot 10^{-12} B_{0,3} - 6.9178 \cdot 10^{-12} B_{1,3} - 6.9178 \cdot 10^{-12} B_{2,3} - 6.9178 \cdot 10^{-12} B_{3,3}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= -3.69134 \cdot 10^{-315} X^8 + 1.58227 \cdot 10^{-314} X^7 - 4.11376 \cdot 10^{-316} X^6 - 5.57319 \cdot 10^{-314} X^5 + 6.99255 \\ &\quad \cdot 10^{-314} X^4 - 1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12} \\ &= -6.9178 \cdot 10^{-12} B_{0,8} - 6.9178 \cdot 10^{-12} B_{1,8} - 6.9178 \cdot 10^{-12} B_{2,8} - 6.9178 \cdot 10^{-12} B_{3,8} - 6.9178 \\ &\quad \cdot 10^{-12} B_{4,8} - 6.9178 \cdot 10^{-12} B_{5,8} - 6.9178 \cdot 10^{-12} B_{6,8} - 6.9178 \cdot 10^{-12} B_{7,8} - 6.9178 \cdot 10^{-12} B_{8,8}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.29017 \cdot 10^{-166}$.

Bounding polynomials M and m:

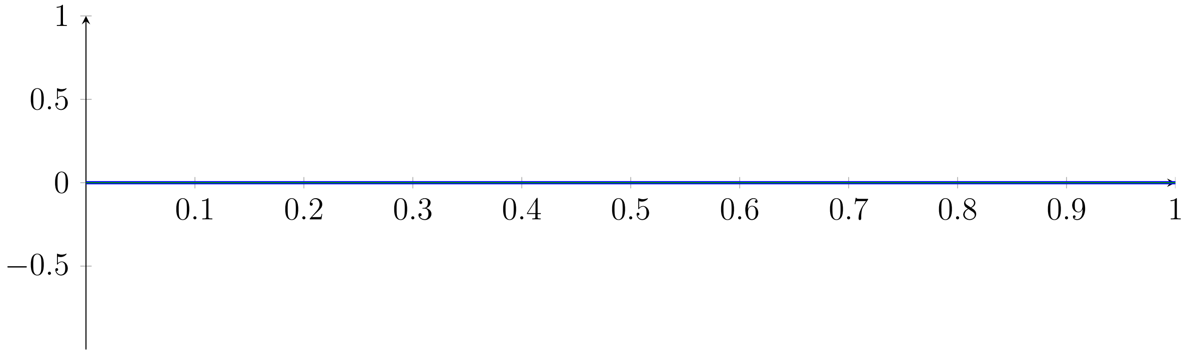
$$M = -1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12}$$

$$m = -1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12}$$

Root of M and m:

$N(M) = \{-1.00692 \cdot 10^{43}, 2.13634 \cdot 10^{17}, 4.88366 \cdot 10^{41}\}$ $N(m) = \{-1.00692 \cdot 10^{43}, 2.13634 \cdot 10^{17}, 4.88366 \cdot 10^{41}\}$

Intersection intervals:

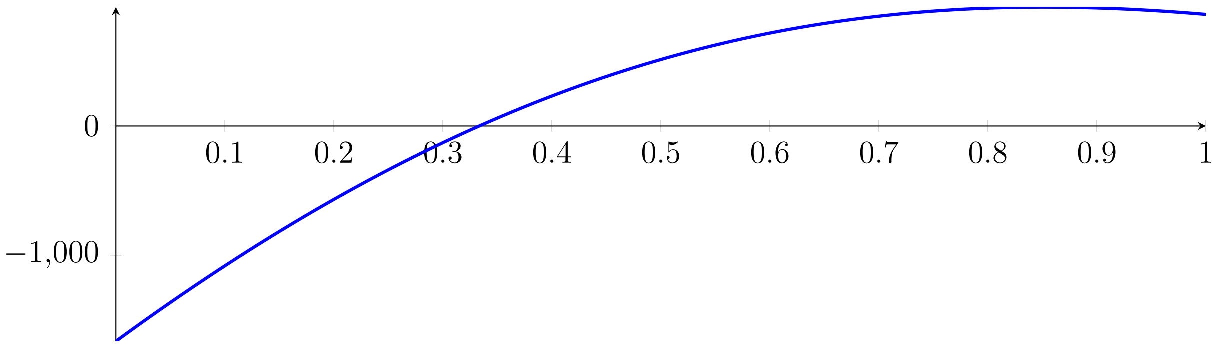


No intersection intervals with the x axis.

228.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

with precision $\varepsilon = 1 \cdot 10^{-64}$.

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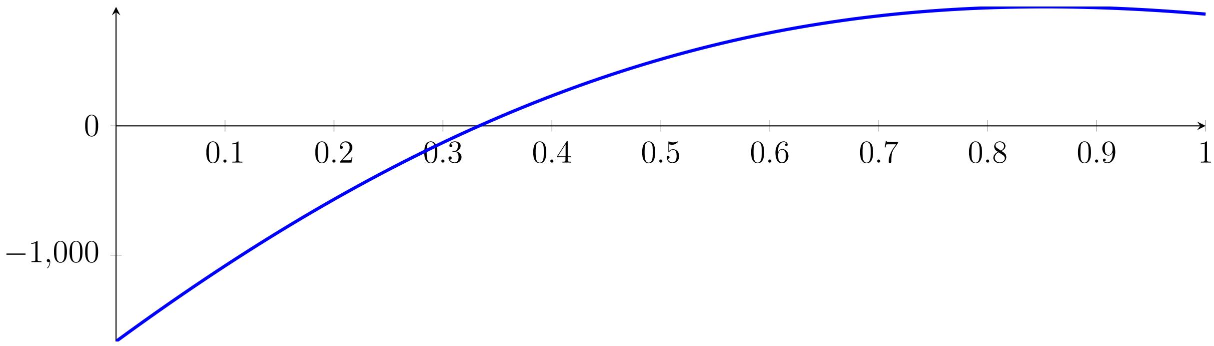
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

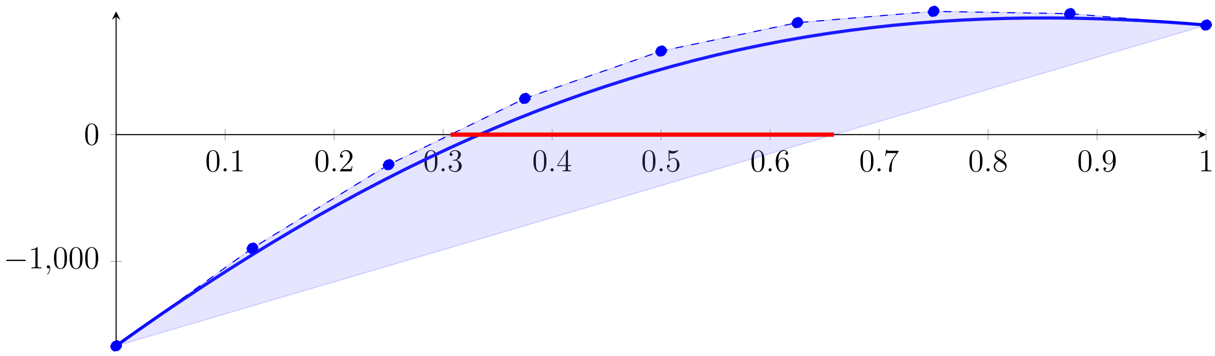
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



229.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

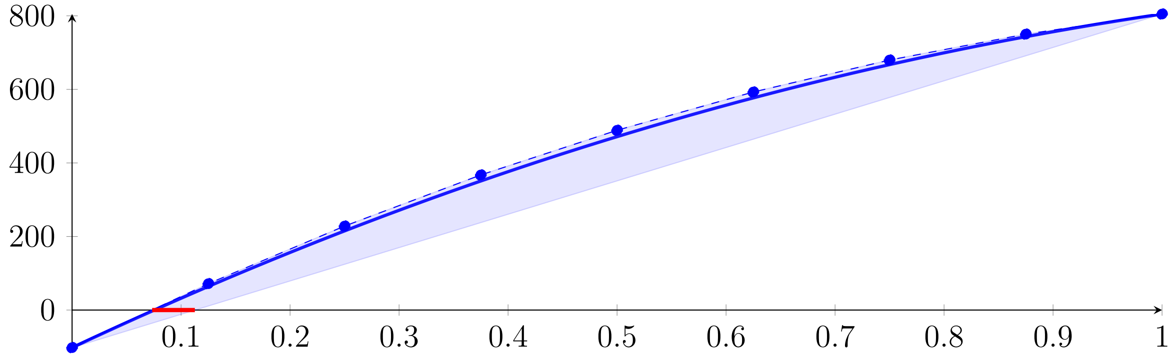
Longest intersection interval: 0.351792

$\Longrightarrow$ Selective recursion: interval 1: $[0.306796, 0.658588]$,

229.2 Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\ &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\ &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\ &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

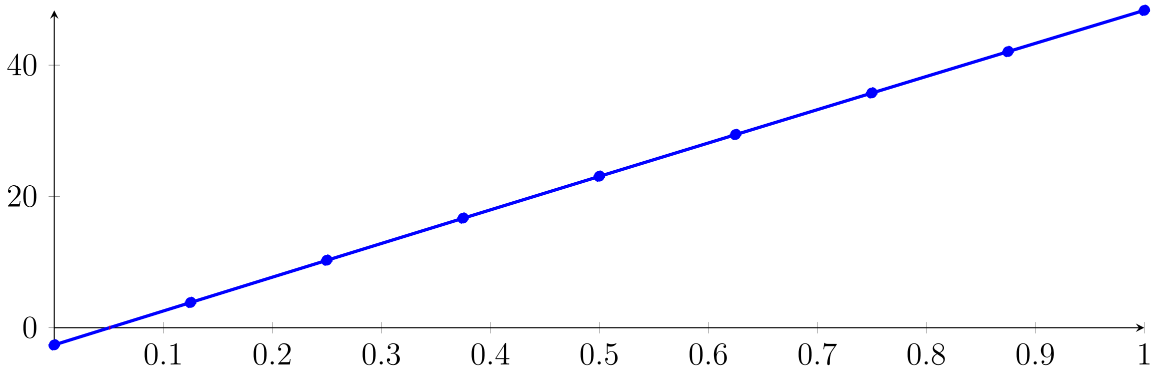
$$[0.0734515, 0.112637]$$

Longest intersection interval: 0.0391855
$\Longrightarrow$ Selective recursion: interval 1: $[0.332635, 0.34642]$,

229.3 Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.30406\cdot 10^{-15}X^8 - 1.54459\cdot 10^{-12}X^7 - 4.9583\cdot 10^{-10}X^6 + 3.66751\cdot 10^{-08}X^5 \\ &\quad + 3.25466\cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\ &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\ &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

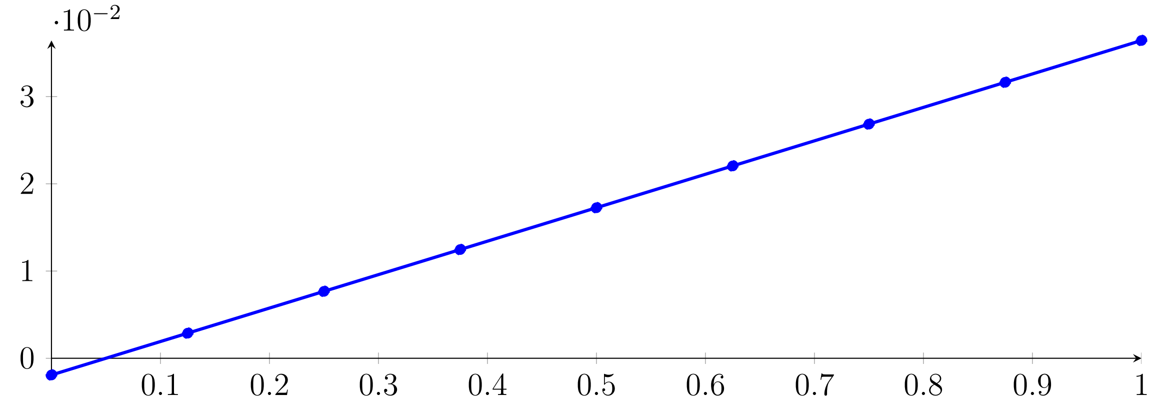
$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333343]$,

229.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.20583\cdot 10^{-40} X^8 - 1.92397\cdot 10^{-34} X^7 - 8.32342\cdot 10^{-29} X^6 + 8.24755\cdot 10^{-24} X^5 \\ &\quad + 9.89972\cdot 10^{-18} X^4 - 2.37322\cdot 10^{-13} X^3 - 4.12146\cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\ &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\ &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\ &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

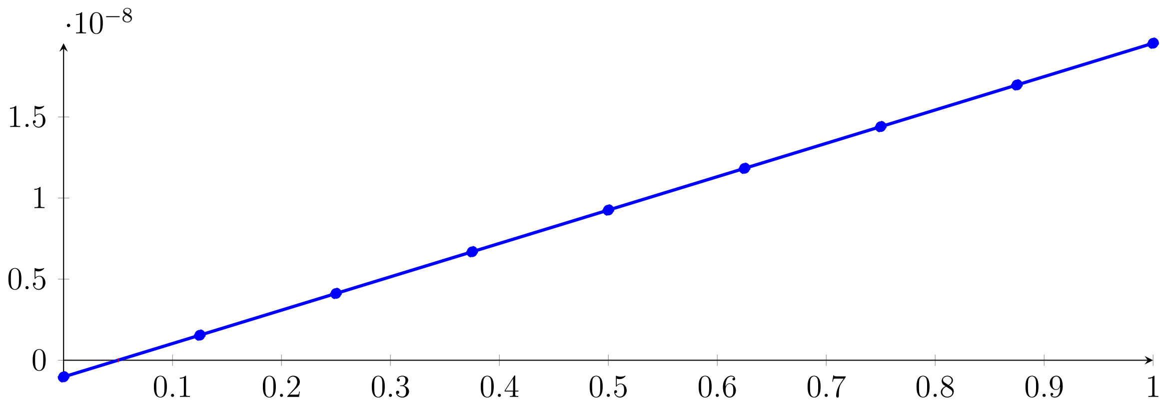
$$[0.0499103, 0.0499109]$$

Longest intersection interval: $5.36469\cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

229.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -8.27263\cdot 10^{-91} X^8 - 2.46044\cdot 10^{-78} X^7 - 1.98413\cdot 10^{-66} X^6 + 3.66478\cdot 10^{-55} X^5 + 8.19978 \\ &\quad \cdot 10^{-43} X^4 - 3.66412\cdot 10^{-32} X^3 - 1.18615\cdot 10^{-19} X^2 + 2.05706\cdot 10^{-08} X - 1.02667\cdot 10^{-09} \\ &= -1.02667\cdot 10^{-09} B_{0,8}(X) + 1.54465\cdot 10^{-09} B_{1,8}(X) + 4.11597\cdot 10^{-09} B_{2,8}(X) \\ &\quad + 6.68729\cdot 10^{-09} B_{3,8}(X) + 9.25862\cdot 10^{-09} B_{4,8}(X) + 1.18299\cdot 10^{-08} B_{5,8}(X) \\ &\quad + 1.44013\cdot 10^{-08} B_{6,8}(X) + 1.69726\cdot 10^{-08} B_{7,8}(X) + 1.95439\cdot 10^{-08} B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

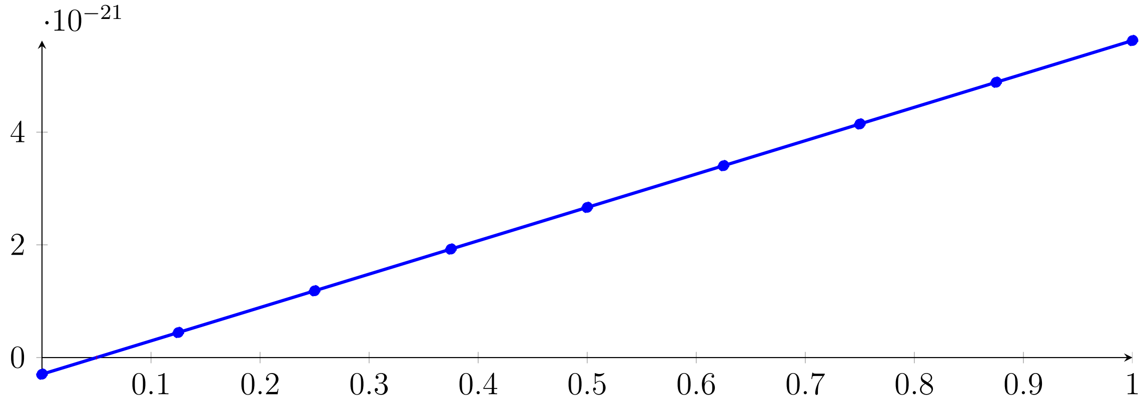
$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87793 \cdot 10^{-13}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

229.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -3.89305 \cdot 10^{-191} X^8 - 4.02327 \cdot 10^{-166} X^7 - 1.12734 \cdot 10^{-141} X^6 + 7.23523 \cdot 10^{-118} X^5 + 5.62504 \\ &\quad \cdot 10^{-93} X^4 - 8.73397 \cdot 10^{-70} X^3 - 9.82433 \cdot 10^{-45} X^2 + 5.92008 \cdot 10^{-21} X - 2.9547 \cdot 10^{-22} \\ &= -2.9547 \cdot 10^{-22} B_{0,8}(X) + 4.4454 \cdot 10^{-22} B_{1,8}(X) + 1.18455 \cdot 10^{-21} B_{2,8}(X) \\ &\quad + 1.92456 \cdot 10^{-21} B_{3,8}(X) + 2.66457 \cdot 10^{-21} B_{4,8}(X) + 3.40458 \cdot 10^{-21} B_{5,8}(X) \\ &\quad + 4.14459 \cdot 10^{-21} B_{6,8}(X) + 4.8846 \cdot 10^{-21} B_{7,8}(X) + 5.62461 \cdot 10^{-21} B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

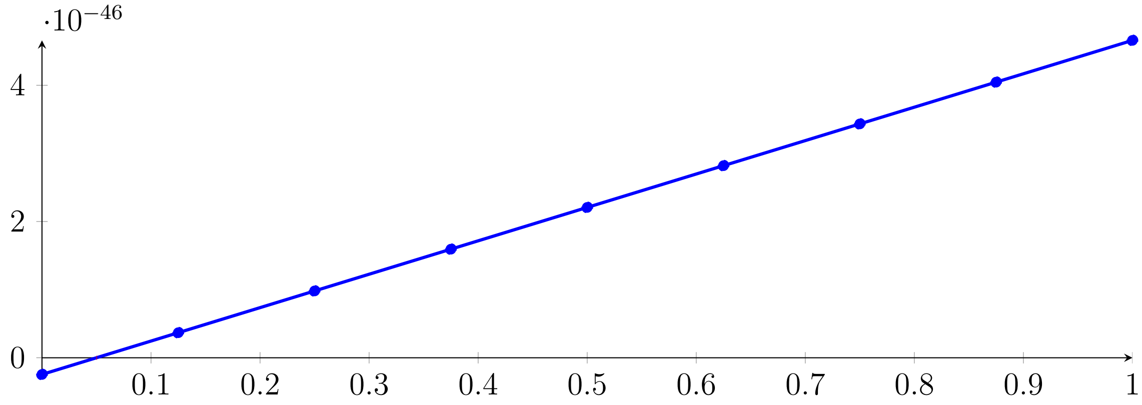
$$[0.0499098, 0.0499098]$$

Longest intersection interval: $8.28251 \cdot 10^{-26}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

229.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.13027 \cdot 10^{-352} X^8 - 1.07575 \cdot 10^{-341} X^7 - 3.63937 \cdot 10^{-292} X^6 + 2.82008 \cdot 10^{-243} X^5 + 2.64711 \\ &\quad \cdot 10^{-193} X^4 - 4.96246 \cdot 10^{-145} X^3 - 6.73948 \cdot 10^{-95} X^2 + 4.90331 \cdot 10^{-46} X - 2.44723 \cdot 10^{-47} \\ &= -2.44723 \cdot 10^{-47} B_{0,8}(X) + 3.6819 \cdot 10^{-47} B_{1,8}(X) + 9.81104 \cdot 10^{-47} B_{2,8}(X) \\ &\quad + 1.59402 \cdot 10^{-46} B_{3,8}(X) + 2.20693 \cdot 10^{-46} B_{4,8}(X) + 2.81984 \cdot 10^{-46} B_{5,8}(X) \\ &\quad + 3.43276 \cdot 10^{-46} B_{6,8}(X) + 4.04567 \cdot 10^{-46} B_{7,8}(X) + 4.65858 \cdot 10^{-46} B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

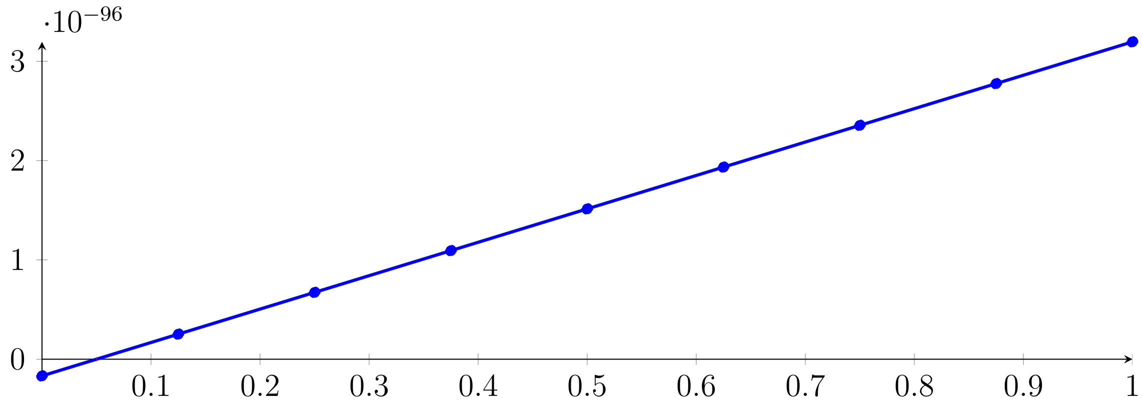
$$[0.0499098, 0.0499098]$$

Longest intersection interval: $6.85999 \cdot 10^{-51}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

229.8 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.12504 \cdot 10^{-402} X^8 - 2.5001 \cdot 10^{-402} X^7 + 2.6251 \cdot 10^{-402} X^6 - 7.00027 \cdot 10^{-402} X^5 + 5.86228 \\ &\quad \cdot 10^{-394} X^4 - 1.60202 \cdot 10^{-295} X^3 - 3.17156 \cdot 10^{-195} X^2 + 3.36366 \cdot 10^{-96} X - 1.6788 \cdot 10^{-97} \\ &= -1.6788 \cdot 10^{-97} B_{0,8}(X) + 2.52578 \cdot 10^{-97} B_{1,8}(X) + 6.73036 \cdot 10^{-97} B_{2,8}(X) \\ &\quad + 1.09349 \cdot 10^{-96} B_{3,8}(X) + 1.51395 \cdot 10^{-96} B_{4,8}(X) + 1.93441 \cdot 10^{-96} B_{5,8}(X) \\ &\quad + 2.35487 \cdot 10^{-96} B_{6,8}(X) + 2.77533 \cdot 10^{-96} B_{7,8}(X) + 3.19578 \cdot 10^{-96} B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $4.70595 \cdot 10^{-101}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

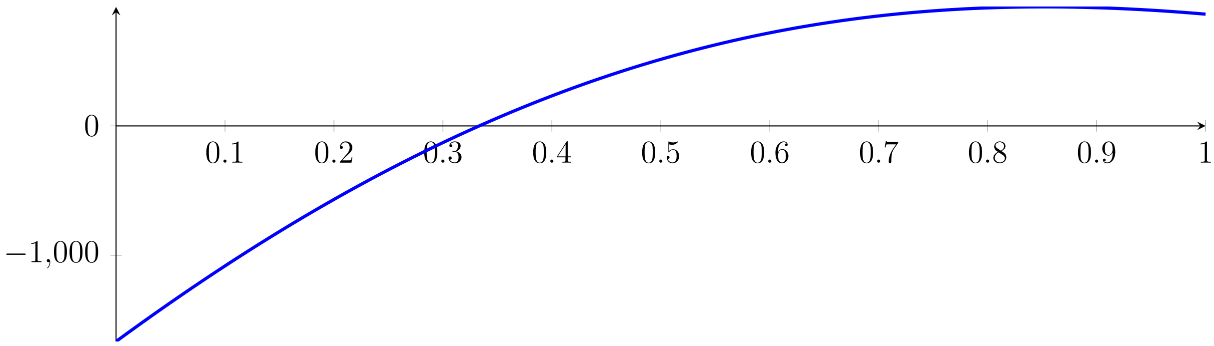
229.9 Recursion Branch 1 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 9!

229.10 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-128}$.

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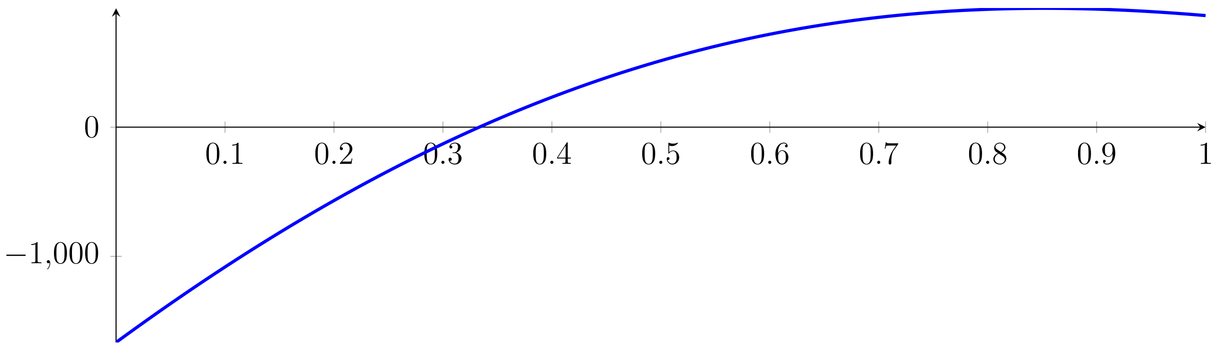
$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called `QuadClip` with input polynomial on interval $[0, 1]$:

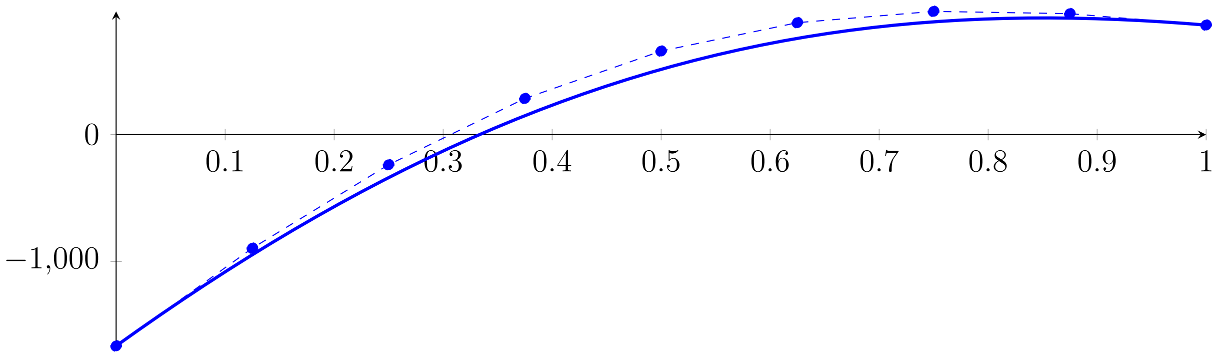
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



230.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

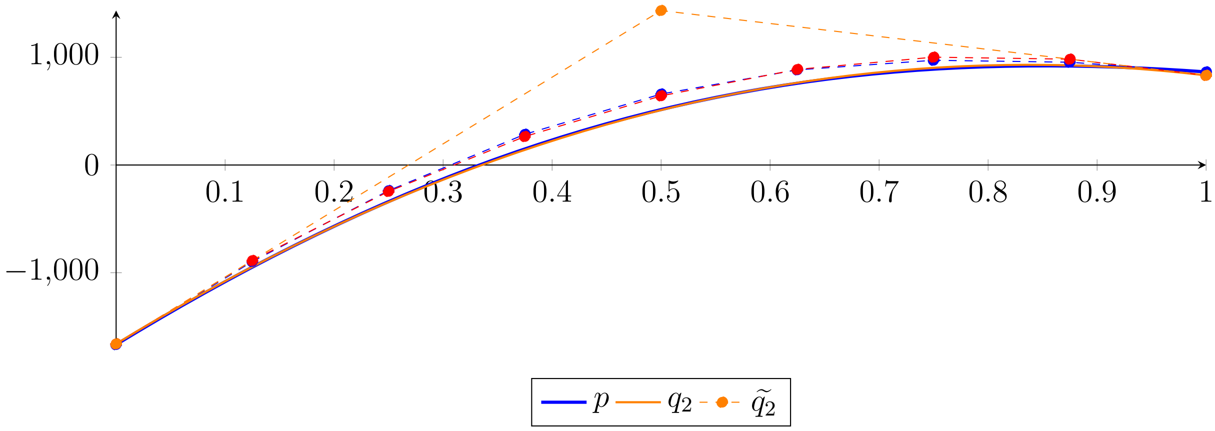
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 6.38292 \cdot 10^{-300}X^8 - 2.86993 \cdot 10^{-299}X^7 + 5.15342 \cdot 10^{-299}X^6 - 4.6928 \cdot 10^{-299}X^5 \\ &\quad + 2.29296 \cdot 10^{-299}X^4 - 6.02693 \cdot 10^{-300}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

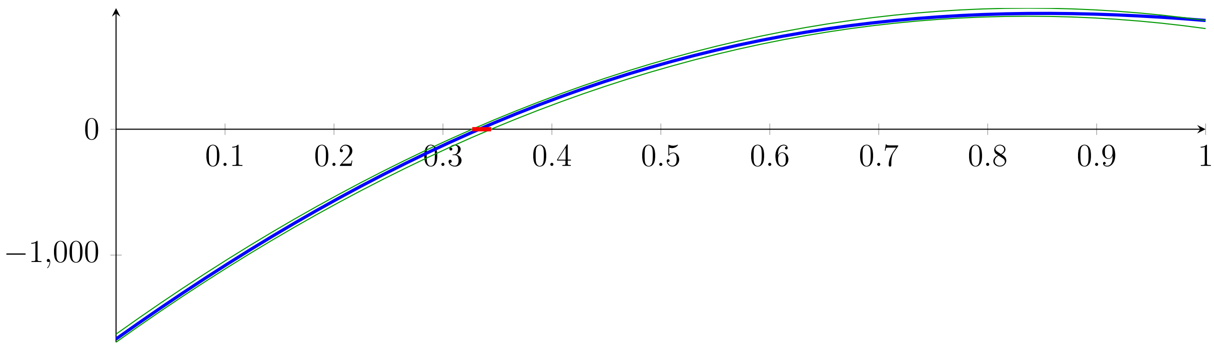
Bounding polynomials M and m:

$$M = -3695.78X^2 + 6187.64X - 1627.86$$
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m:

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$[0.326917, 0.344255]$

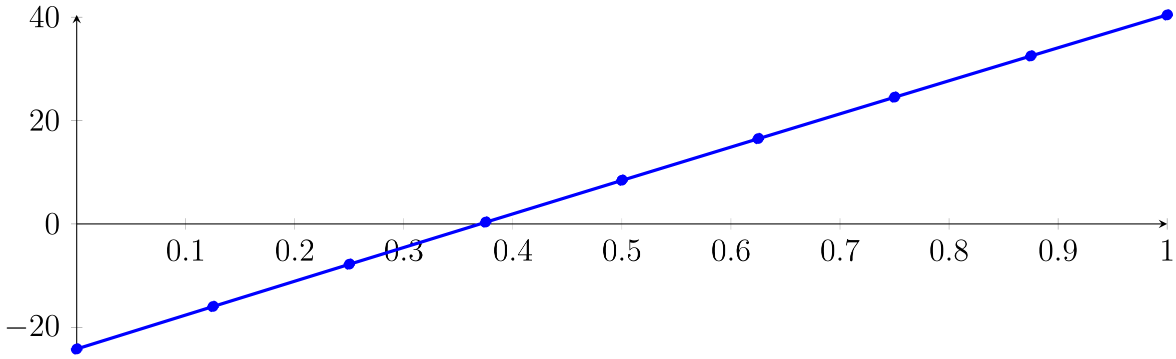
Longest intersection interval: 0.0173372

$\Longrightarrow$ Selective recursion: interval 1: $[0.326917, 0.344255]$,

230.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

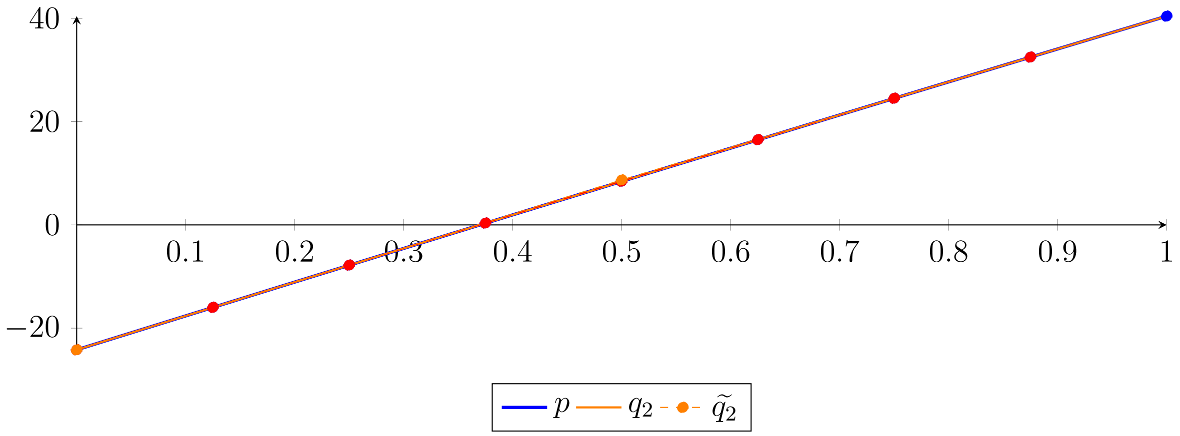
$$\begin{aligned} p = &-8.16249 \cdot 10^{-15} X^8 - 7.66571 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ &+ 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ = &-24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &+ 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 4.27533\cdot 10^{-302}X^8 - 2.42468\cdot 10^{-301}X^7 + 4.90537\cdot 10^{-301}X^6 - 4.6286\cdot 10^{-301}X^5 \\ &\quad + 2.17546\cdot 10^{-301}X^4 - 5.36695\cdot 10^{-302}X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894\cdot 10^{-05}$.
Bounding polynomials M and m:

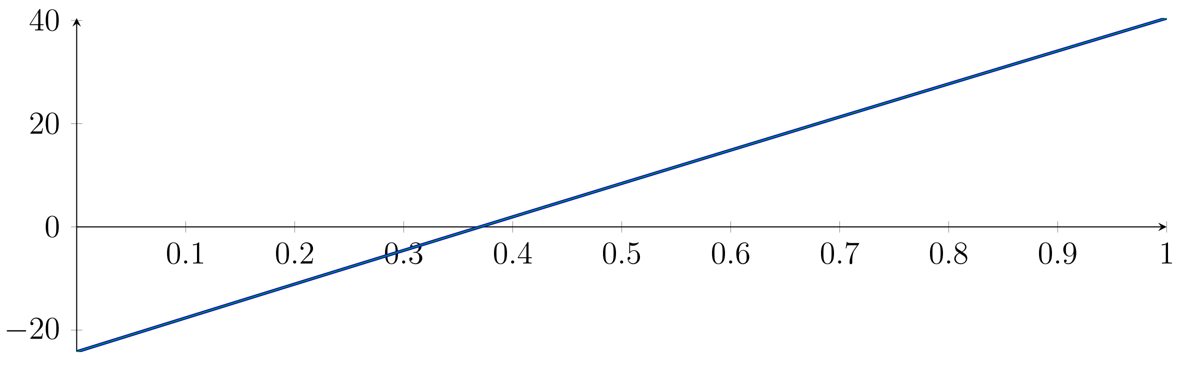
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m:

$$N(M) = \{0.370068, 55.2832\} \qquad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



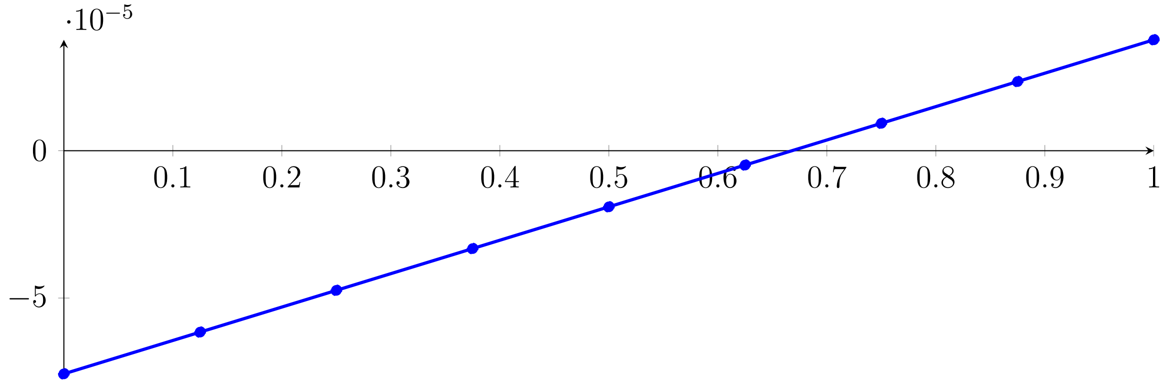
$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588\cdot 10^{-06}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

230.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

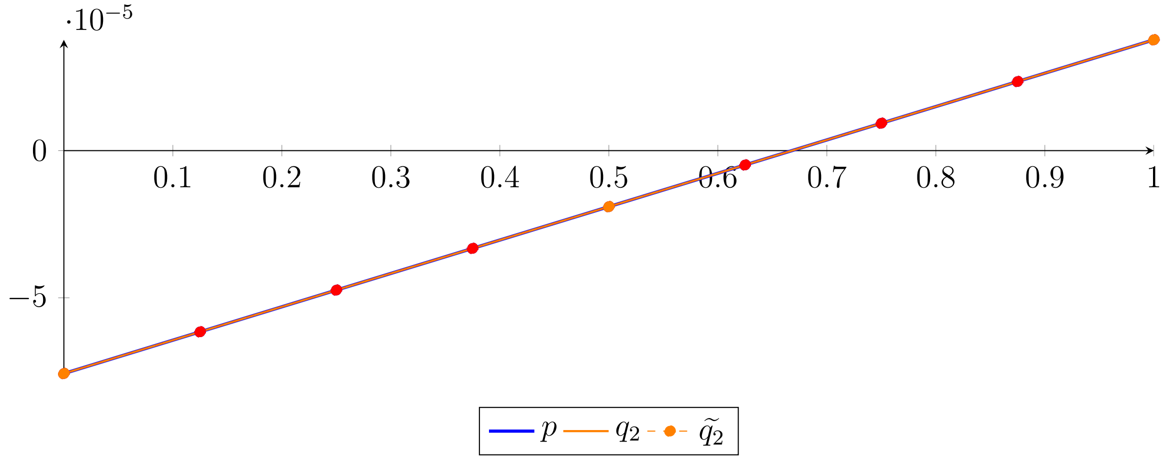
$$p = -7.04578{\cdot}10^{-61}X^8 - 3.80201{\cdot}10^{-52}X^7 - 5.5627{\cdot}10^{-44}X^6 + 1.86413{\cdot}10^{-36}X^5 + 7.56737{\cdot}10^{-28}X^4 - 6.13517{\cdot}10^{-21}X^3 - 3.6034{\cdot}10^{-12}X^2 + 0.000113379X - 7.5732{\cdot}10^{-05}$$
$$= -7.5732{\cdot}10^{-05}B_{0,8}(X) - 6.15596{\cdot}10^{-05}B_{1,8}(X) - 4.73873{\cdot}10^{-05}B_{2,8}(X) - 3.32149{\cdot}10^{-05}B_{3,8}(X) - 1.90425{\cdot}10^{-05}B_{4,8}(X) - 4.87016{\cdot}10^{-06}B_{5,8}(X) + 9.3022{\cdot}10^{-06}B_{6,8}(X) + 2.34746{\cdot}10^{-05}B_{7,8}(X) + 3.76469{\cdot}10^{-05}B_{8,8}(X)$$



Degree reduction and raising:

$$q_2 = -3.6034{\cdot}10^{-12}X^2 + 0.000113379X - 7.5732{\cdot}10^{-05}$$
$$= -7.5732{\cdot}10^{-05}B_{0,2} - 1.90425{\cdot}10^{-05}B_{1,2} + 3.76469{\cdot}10^{-05}B_{2,2}$$

$$\widetilde{q}_2 = -3.62396{\cdot}10^{-308}X^8 + 2.32992{\cdot}10^{-308}X^7 + 2.61753{\cdot}10^{-307}X^6 - 5.97049{\cdot}10^{-307}X^5 + 5.13401{\cdot}10^{-307}X^4 - 1.82868{\cdot}10^{-307}X^3 - 3.6034{\cdot}10^{-12}X^2 + 0.000113379X - 7.5732{\cdot}10^{-05}$$
$$= -7.5732{\cdot}10^{-05}B_{0,8} - 6.15596{\cdot}10^{-05}B_{1,8} - 4.73873{\cdot}10^{-05}B_{2,8} - 3.32149{\cdot}10^{-05}B_{3,8} - 1.90425{\cdot}10^{-05}B_{4,8} - 4.87016{\cdot}10^{-06}B_{5,8} + 9.3022{\cdot}10^{-06}B_{6,8} + 2.34746{\cdot}10^{-05}B_{7,8} + 3.76469{\cdot}10^{-05}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 3.06758{\cdot}10^{-22}$.

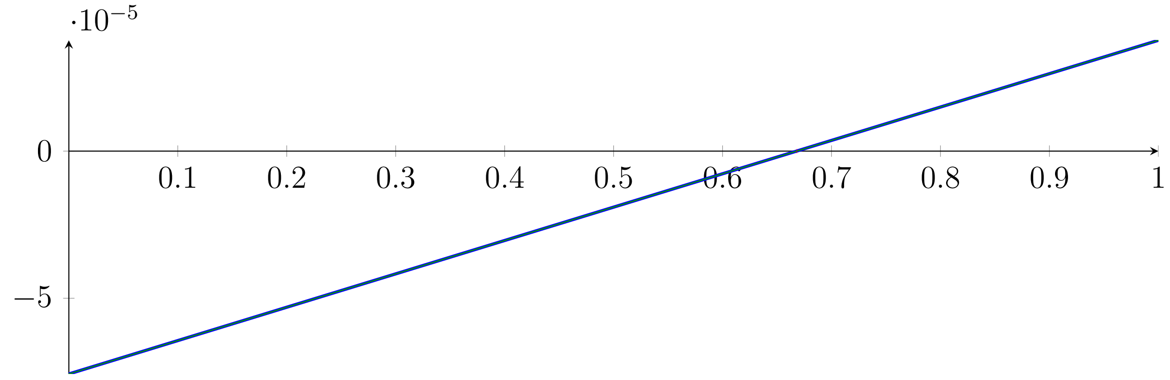
Bounding polynomials M and m:

$$M = -3.6034{\cdot}10^{-12}X^2 + 0.000113379X - 7.5732{\cdot}10^{-05}$$
$$m = -3.6034{\cdot}10^{-12}X^2 + 0.000113379X - 7.5732{\cdot}10^{-05}$$

Root of M and m:

$$N(M) = \{0.667955, 3.14644{\cdot}10^{7}\} \qquad N(m) = \{0.667955, 3.14644{\cdot}10^{7}\}$$

Intersection intervals:



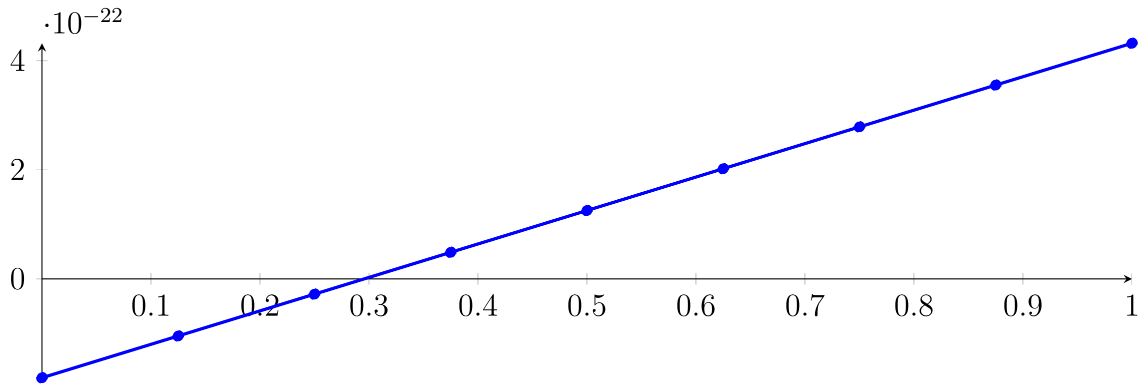
$[0.667955, 0.667955]$

Longest intersection interval: $5.41121 \cdot 10^{-18}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

230.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

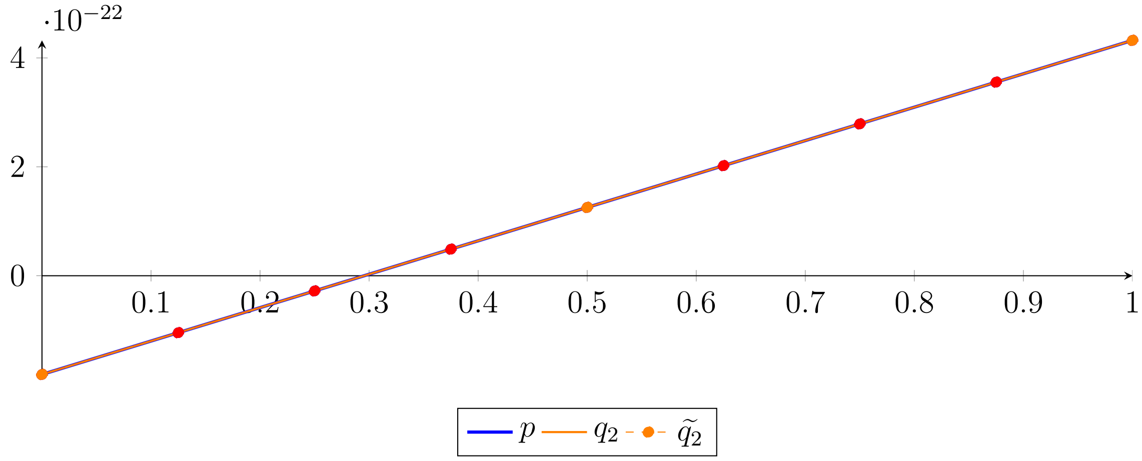
$$p = -5.17944 \cdot 10^{-199} X^8 - 5.16502 \cdot 10^{-173} X^7 - 1.39653 \cdot 10^{-147} X^6 + 8.64863 \cdot 10^{-123} X^5 + 6.48817 \cdot 10^{-97} X^4 - 9.72096 \cdot 10^{-73} X^3 - 1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22}$$
$$= -1.81261 \cdot 10^{-22} B_{0,8}(X) - 1.04571 \cdot 10^{-22} B_{1,8}(X) - 2.78818 \cdot 10^{-23} B_{2,8}(X) + 4.88078 \cdot 10^{-23} B_{3,8}(X) + 1.25497 \cdot 10^{-22} B_{4,8}(X) + 2.02187 \cdot 10^{-22} B_{5,8}(X) + 2.78877 \cdot 10^{-22} B_{6,8}(X) + 3.55566 \cdot 10^{-22} B_{7,8}(X) + 4.32256 \cdot 10^{-22} B_{8,8}(X)$$



Degree reduction and raising:

$$q_2 = -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22}$$
$$= -1.81261 \cdot 10^{-22} B_{0,2} + 1.25497 \cdot 10^{-22} B_{1,2} + 4.32256 \cdot 10^{-22} B_{2,2}$$

$$\widetilde{q}_2 = 5.39888 \cdot 10^{-325} X^8 - 2.84119 \cdot 10^{-324} X^7 + 5.35011 \cdot 10^{-324} X^6 - 4.57499 \cdot 10^{-324} X^5 + 1.82797 \cdot 10^{-324} X^4 - 3.72306 \cdot 10^{-325} X^3 - 1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22}$$
$$= -1.81261 \cdot 10^{-22} B_{0,8} - 1.04571 \cdot 10^{-22} B_{1,8} - 2.78818 \cdot 10^{-23} B_{2,8} + 4.88078 \cdot 10^{-23} B_{3,8} + 1.25497 \cdot 10^{-22} B_{4,8} + 2.02187 \cdot 10^{-22} B_{5,8} + 2.78877 \cdot 10^{-22} B_{6,8} + 3.55566 \cdot 10^{-22} B_{7,8} + 4.32256 \cdot 10^{-22} B_{8,8}$$



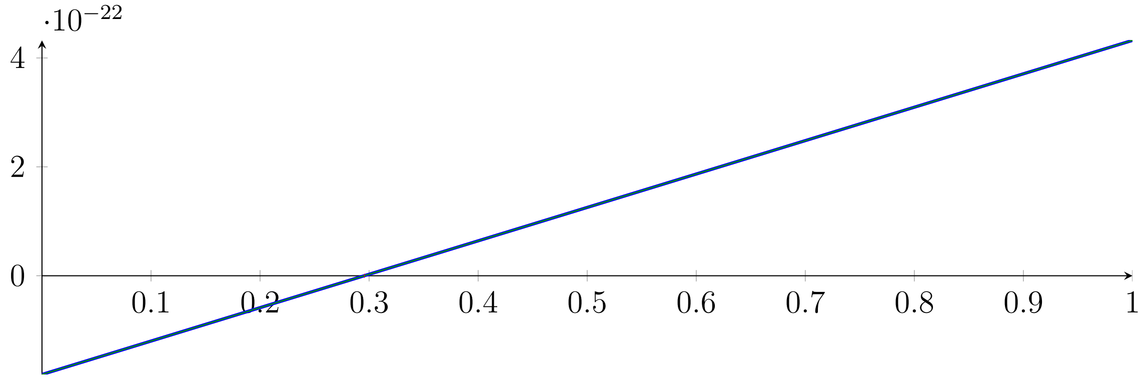
The maximum difference of the Bézier coefficients is $\delta = 4.86048 \cdot 10^{-74}$.
Bounding polynomials M and m:

$$M = -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22}$$
$$m = -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22}$$

Root of M and m:

$$N(M) = \{0.295446, 5.81467 \cdot 10^{24}\} \qquad N(m) = \{0.295446, 5.81467 \cdot 10^{24}\}$$

Intersection intervals:



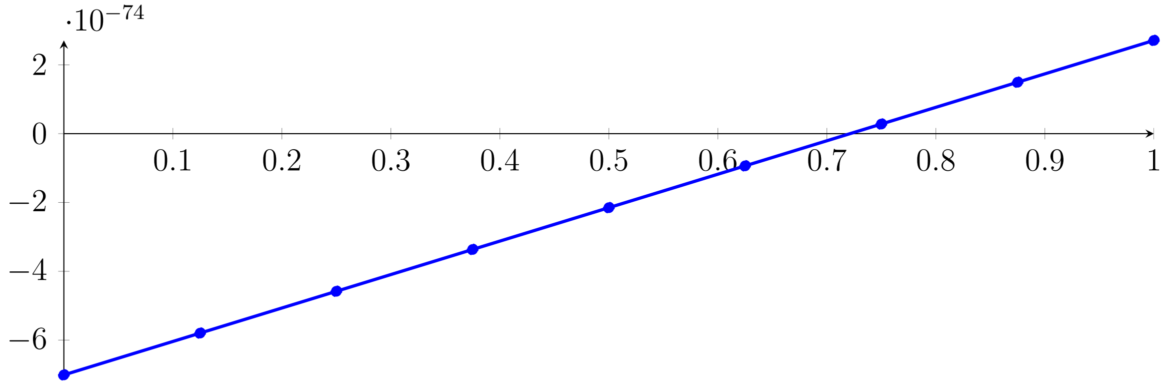
$$[0.295446, 0.295446]$$

Longest intersection interval: $1.58446 \cdot 10^{-52}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

230.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

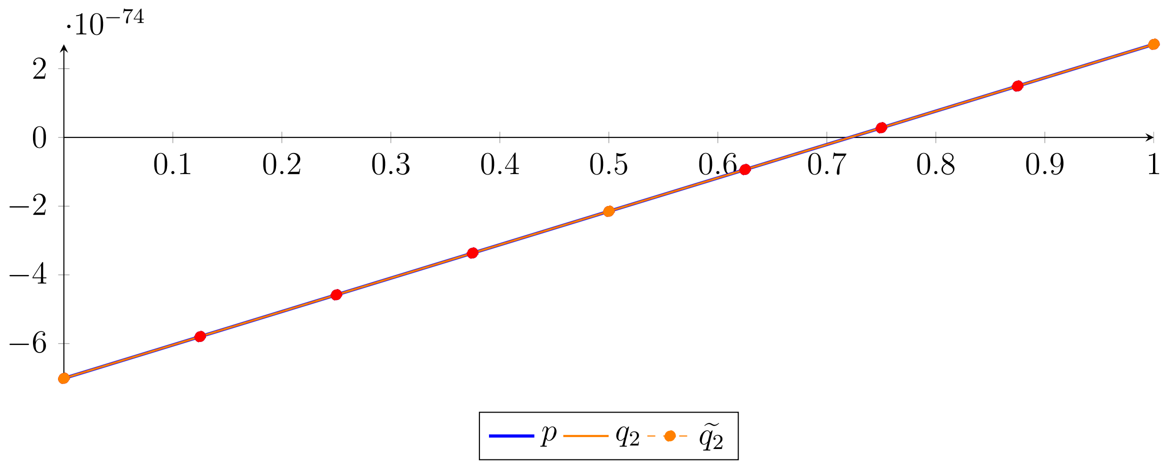
$$\begin{aligned} p &= 6.29674 \cdot 10^{-381} X^8 - 2.51869 \cdot 10^{-380} X^7 - 8.81543 \cdot 10^{-380} X^6 + 4.08933 \cdot 10^{-304} X^4 \\ &\quad - 3.86684 \cdot 10^{-228} X^3 - 2.6489 \cdot 10^{-150} X^2 + 9.72096 \cdot 10^{-74} X - 7.01115 \cdot 10^{-74} \\ &= -7.01115 \cdot 10^{-74} B_{0,8}(X) - 5.79603 \cdot 10^{-74} B_{1,8}(X) - 4.58091 \cdot 10^{-74} B_{2,8}(X) \\ &\quad - 3.36579 \cdot 10^{-74} B_{3,8}(X) - 2.15067 \cdot 10^{-74} B_{4,8}(X) - 9.35553 \cdot 10^{-75} B_{5,8}(X) \\ &\quad + 2.79566 \cdot 10^{-75} B_{6,8}(X) + 1.49469 \cdot 10^{-74} B_{7,8}(X) + 2.70981 \cdot 10^{-74} B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.6489\cdot10^{-150}X^2 + 9.72096\cdot10^{-74}X - 7.01115\cdot10^{-74}$$
$$= -7.01115\cdot10^{-74}B_{0,2} - 2.15067\cdot10^{-74}B_{1,2} + 2.70981\cdot10^{-74}B_{2,2}$$

$$\widetilde{q}_2 = -4.77607\cdot10^{-377}X^8 + 8.70965\cdot10^{-377}X^7 + 1.35581\cdot10^{-376}X^6 - 4.8194\cdot10^{-376}X^5 + 4.61929\cdot10^{-376}X^4 - 1.70887\cdot10^{-376}X^3 - 2.6489\cdot10^{-150}X^2 + 9.72096\cdot10^{-74}X - 7.01115\cdot10^{-74}$$
$$= -7.01115\cdot10^{-74}B_{0,8} - 5.79603\cdot10^{-74}B_{1,8} - 4.58091\cdot10^{-74}B_{2,8} - 3.36579\cdot10^{-74}B_{3,8} - 2.15067\cdot10^{-74}B_{4,8} - 9.35553\cdot10^{-75}B_{5,8} + 2.79566\cdot10^{-75}B_{6,8} + 1.49469\cdot10^{-74}B_{7,8} + 2.70981\cdot10^{-74}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 1.93342\cdot10^{-229}$.

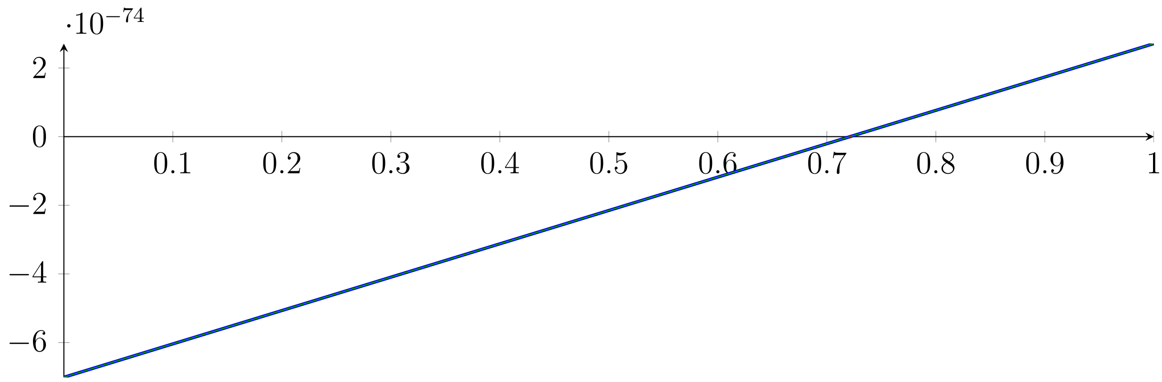
Bounding polynomials M and m:

$$M = -2.6489\cdot10^{-150}X^2 + 9.72096\cdot10^{-74}X - 7.01115\cdot10^{-74}$$
$$m = -2.6489\cdot10^{-150}X^2 + 9.72096\cdot10^{-74}X - 7.01115\cdot10^{-74}$$

Root of M and m:

$$N(M) = \{0.721241, 3.6698\cdot10^{76}\} \qquad N(m) = \{0.721241, 3.6698\cdot10^{76}\}$$

Intersection intervals:



$$[0.721241, 0.721241]$$

Longest intersection interval: $3.97784 \cdot 10^{-156}$
$\implies$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

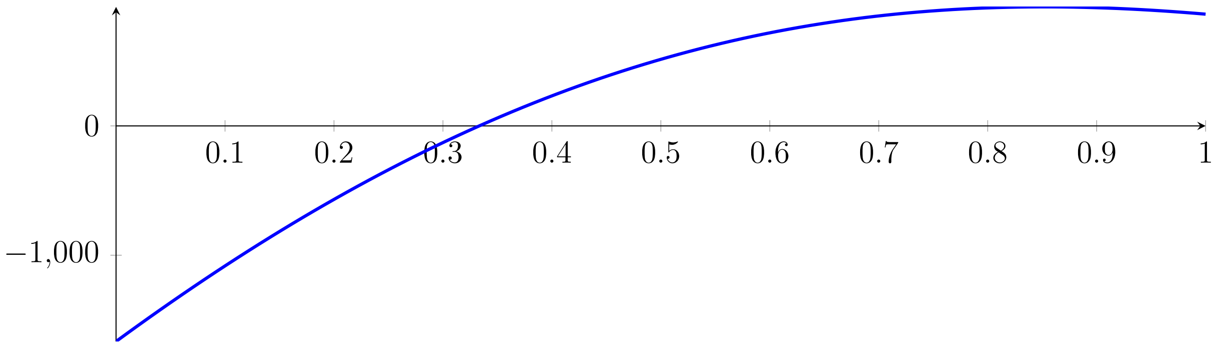
230.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

230.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



Result: Root Intervals

$$[0.333333, 0.333333]$$

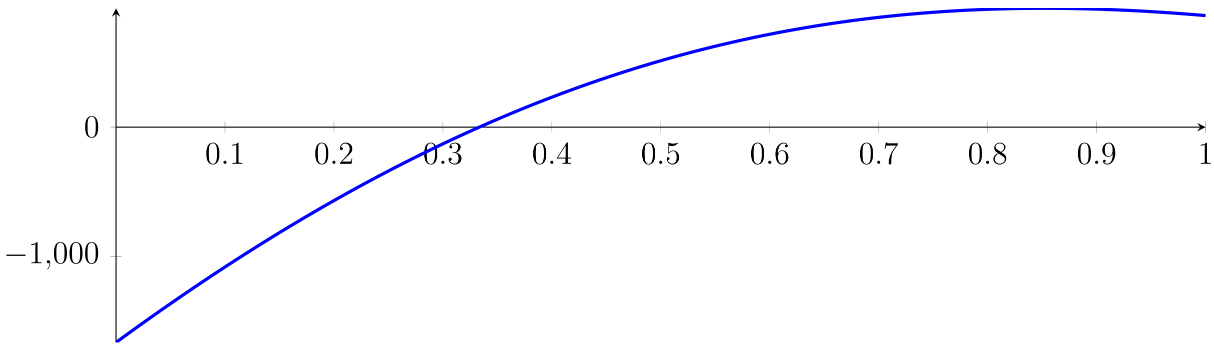
with precision $\varepsilon = 1 \cdot 10^{-128}$.

231 Running CubeClip on f_8 with epsilon 128

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

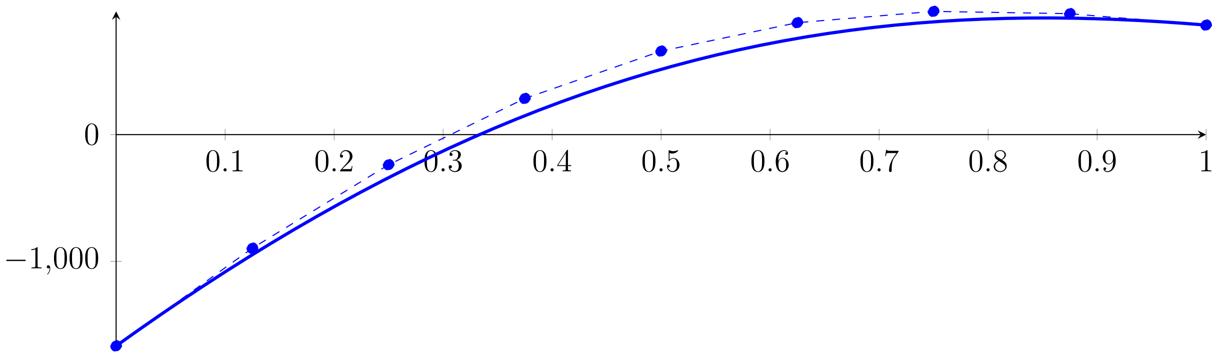
$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$



231.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

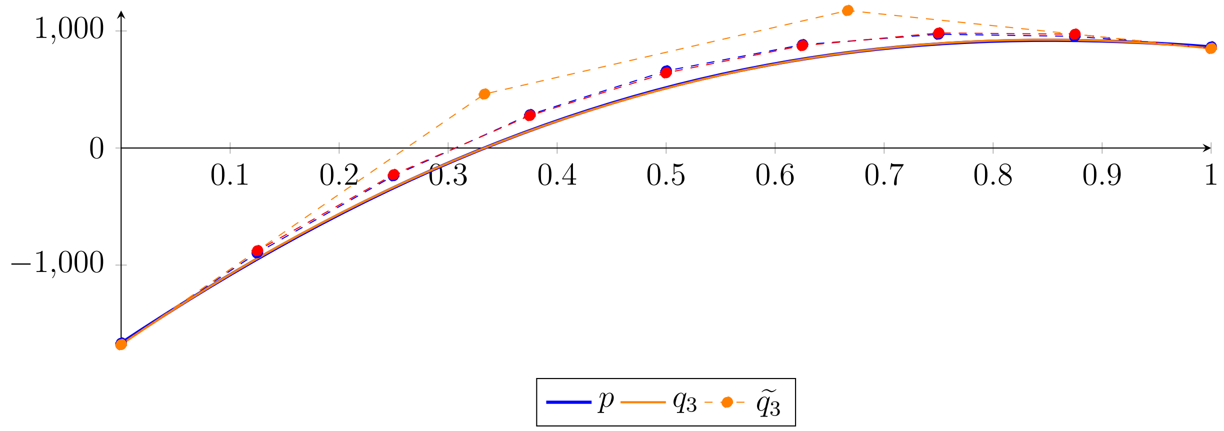
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.80826 \cdot 10^{-300}X^8 + 1.16606 \cdot 10^{-299}X^7 - 2.1197 \cdot 10^{-299}X^6 + 2.15003 \cdot 10^{-299}X^5 \\ &\quad - 1.20234 \cdot 10^{-299}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

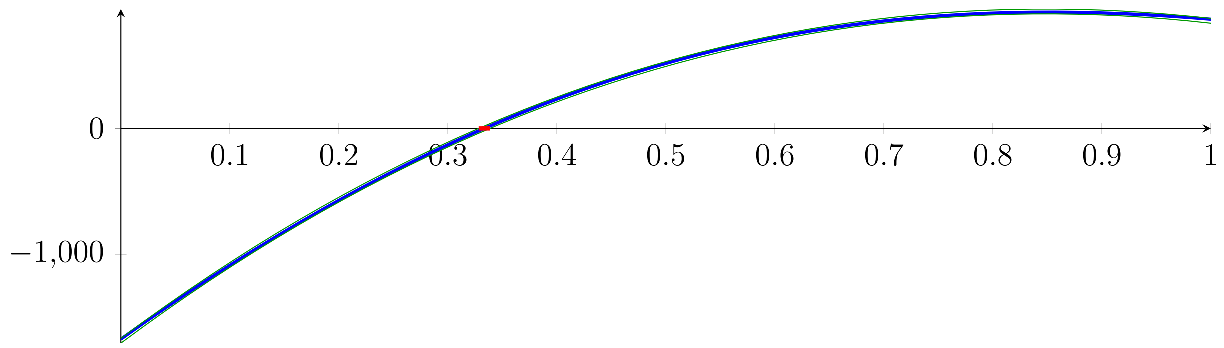
Bounding polynomials M and m:

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m:

$$N(M) = \{0.328258, 1.40284, 9.2594\} \qquad N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$[0.328258, 0.338551]$

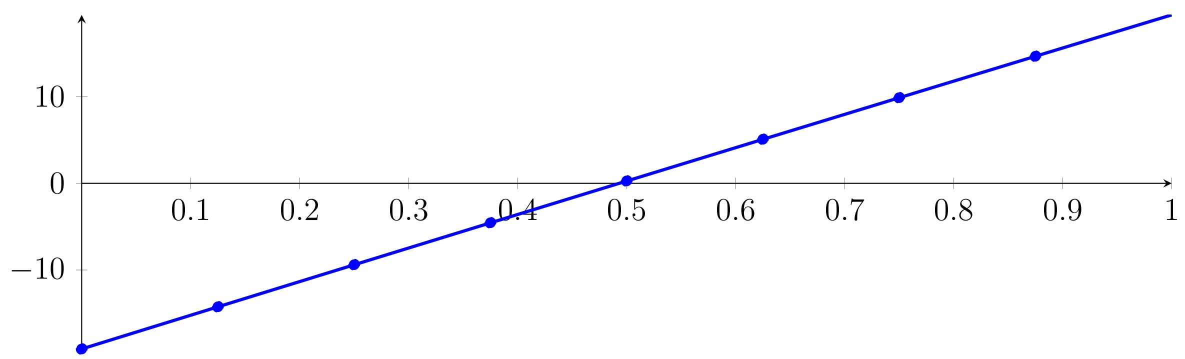
Longest intersection interval: 0.0102926

$\Longrightarrow$ Selective recursion: interval 1: $[0.328258, 0.338551]$,

231.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

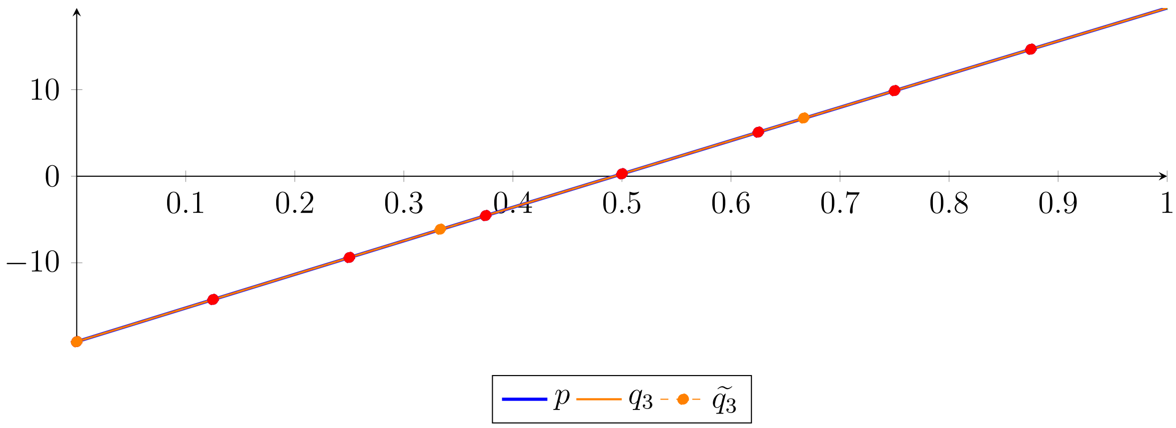
$$p = -1.25947 \cdot 10^{-16}X^8 - 1.9937 \cdot 10^{-13}X^7 - 8.53073 \cdot 10^{-11}X^6 + 8.72839 \cdot 10^{-09}X^5$$
$$+ 1.00963 \cdot 10^{-05}X^4 - 0.000261161X^3 - 0.416284X^2 + 38.9643X - 19.1124$$
$$= -19.1124B_{0,8}(X) - 14.2419B_{1,8}(X) - 9.38619B_{2,8}(X) - 4.54539B_{3,8}(X)$$
$$+ 0.280537B_{4,8}(X) + 5.09158B_{5,8}(X) + 9.88774B_{6,8}(X) + 14.669B_{7,8}(X) + 19.4354B_{8,8}(X)$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124\\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= -1.96643{\cdot}10^{-303}X^8 + 1.82947{\cdot}10^{-302}X^7 - 4.89395{\cdot}10^{-302}X^6 + 5.49554{\cdot}10^{-302}X^5\\ &\quad - 2.47838{\cdot}10^{-302}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124\\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}\\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902{\cdot}10^{-07}$.
Bounding polynomials M and m:

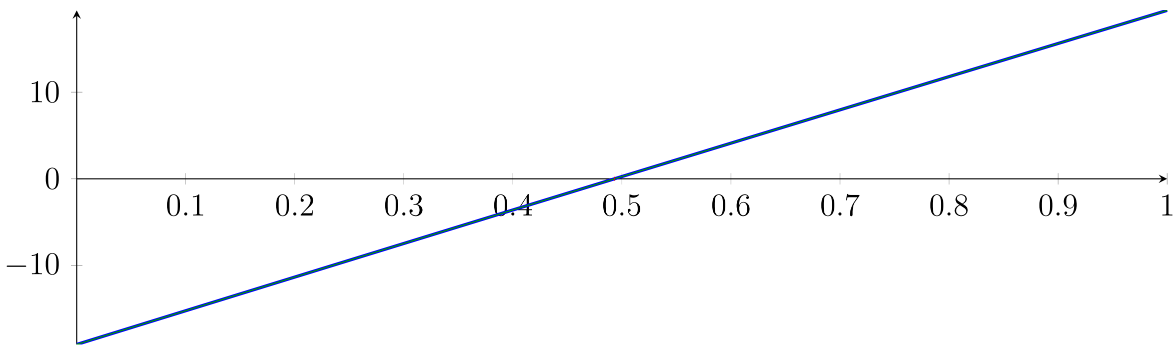
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m:

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \qquad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



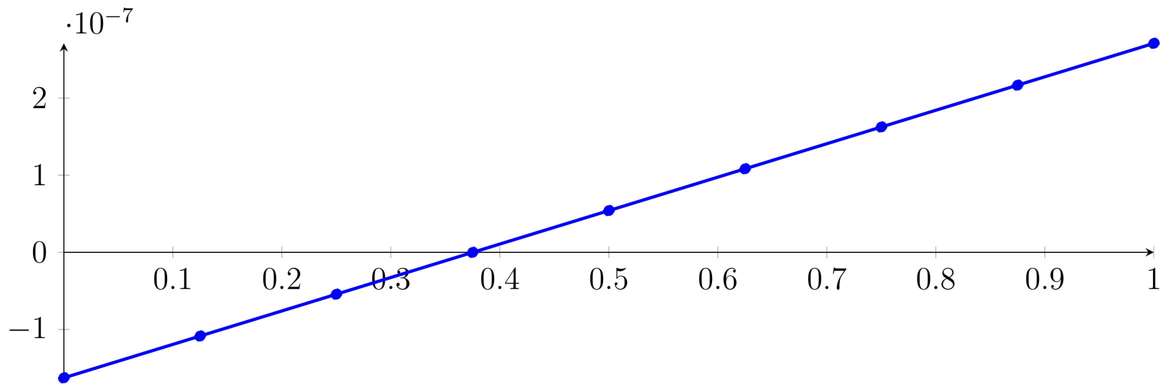
$$[0.493109, 0.493109]$$

Longest intersection interval: $1.1252{\cdot}10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

231.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

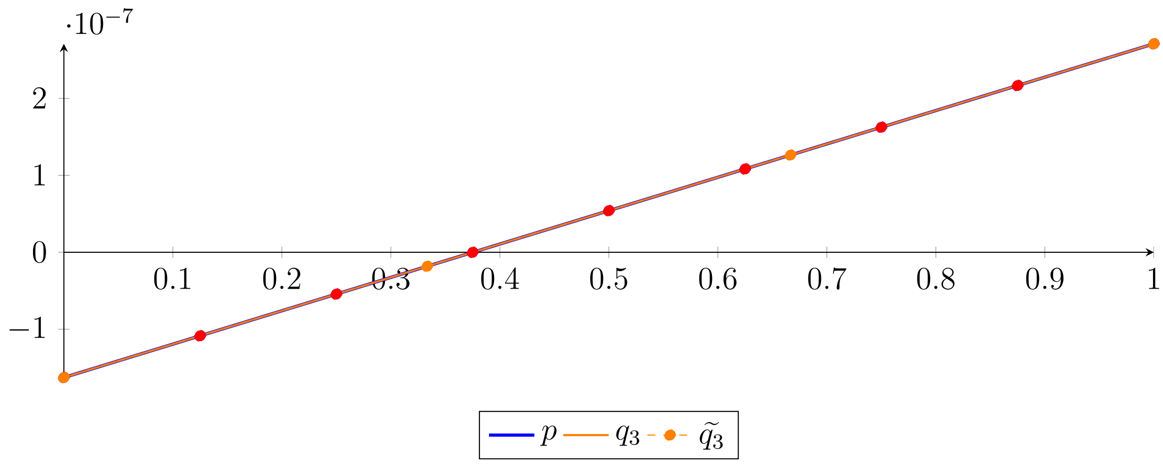
$$p = -3.2361\cdot 10^{-80}X^8 - 4.56398\cdot 10^{-69}X^7 - 1.74524\cdot 10^{-58}X^6 + 1.52857\cdot 10^{-48}X^5 + 1.62178 \cdot 10^{-37}X^4 - 3.43646\cdot 10^{-28}X^3 - 5.27516\cdot 10^{-17}X^2 + 4.33804\cdot 10^{-07}X - 1.6281\cdot 10^{-07}$$
$$= -1.6281\cdot 10^{-07}B_{0,8}(X) - 1.08585\cdot 10^{-07}B_{1,8}(X) - 5.43592\cdot 10^{-08}B_{2,8}(X) - 1.33711\cdot 10^{-10}B_{3,8}(X) + 5.40918\cdot 10^{-08}B_{4,8}(X) + 1.08317\cdot 10^{-07}B_{5,8}(X) + 1.62543\cdot 10^{-07}B_{6,8}(X) + 2.16768\cdot 10^{-07}B_{7,8}(X) + 2.70994\cdot 10^{-07}B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = -3.43646\cdot 10^{-28}X^3 - 5.27516\cdot 10^{-17}X^2 + 4.33804\cdot 10^{-07}X - 1.6281\cdot 10^{-07}$$
$$= -1.6281\cdot 10^{-07}B_{0,3} - 1.82089\cdot 10^{-08}B_{1,3} + 1.26393\cdot 10^{-07}B_{2,3} + 2.70994\cdot 10^{-07}B_{3,3}$$

$$\tilde{q}_3 = 8.86066\cdot 10^{-312}X^8 + 7.41744\cdot 10^{-311}X^7 - 5.24902\cdot 10^{-310}X^6 + 1.01267\cdot 10^{-309}X^5 - 7.8921 \cdot 10^{-310}X^4 - 3.43646\cdot 10^{-28}X^3 - 5.27516\cdot 10^{-17}X^2 + 4.33804\cdot 10^{-07}X - 1.6281\cdot 10^{-07}$$
$$= -1.6281\cdot 10^{-07}B_{0,8} - 1.08585\cdot 10^{-07}B_{1,8} - 5.43592\cdot 10^{-08}B_{2,8} - 1.33711\cdot 10^{-10}B_{3,8} + 5.40918 \cdot 10^{-08}B_{4,8} + 1.08317\cdot 10^{-07}B_{5,8} + 1.62543\cdot 10^{-07}B_{6,8} + 2.16768\cdot 10^{-07}B_{7,8} + 2.70994\cdot 10^{-07}B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 3.47524\cdot 10^{-39}$.

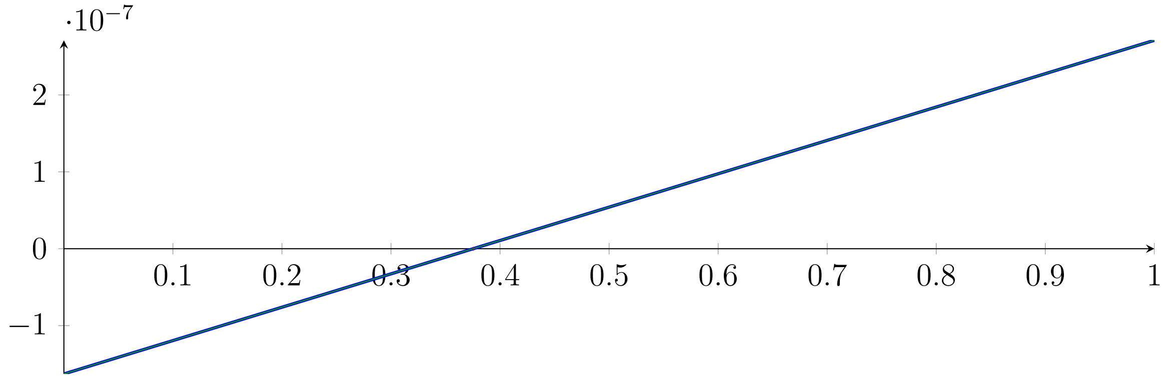
Bounding polynomials M and m:

$$M = -3.43646\cdot 10^{-28}X^3 - 5.27516\cdot 10^{-17}X^2 + 4.33804\cdot 10^{-07}X - 1.6281\cdot 10^{-07}$$
$$m = -3.43646\cdot 10^{-28}X^3 - 5.27516\cdot 10^{-17}X^2 + 4.33804\cdot 10^{-07}X - 1.6281\cdot 10^{-07}$$

Root of M and m:

$$N(M) = \{-1.6133\cdot 10^{11}, 0.375292, 7.82468\cdot 10^{9}\} \quad N(m) = \{-1.6133\cdot 10^{11}, 0.375292, 7.82468\cdot 10^{9}\}$$

Intersection intervals:



$[0.375292, 0.375292]$

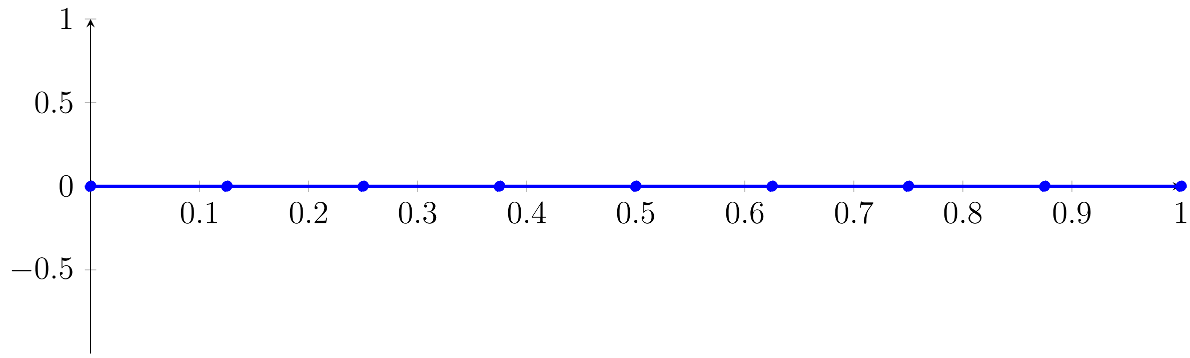
Longest intersection interval: $1.60221 \cdot 10^{-32}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

231.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

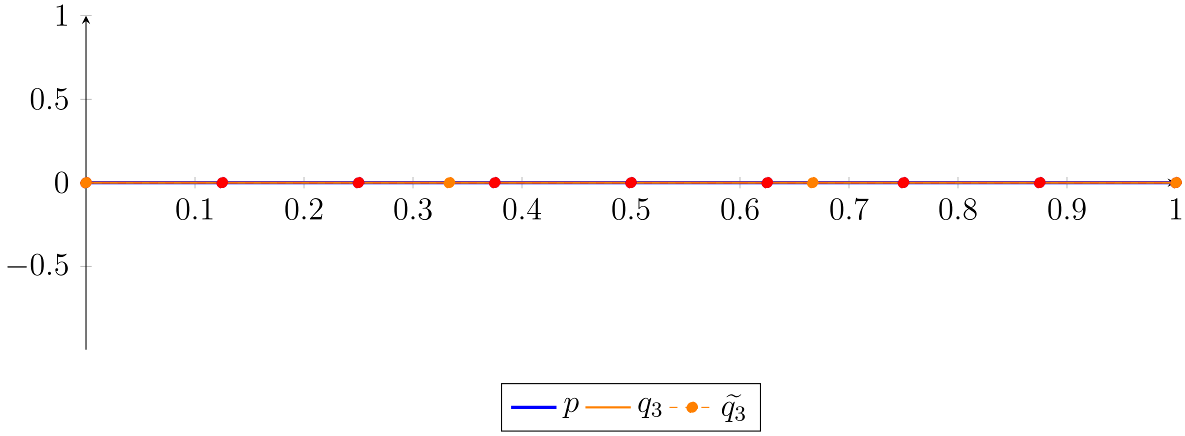
$$p = 3.5617 \cdot 10^{-318} X^8 - 1.23705 \cdot 10^{-291} X^7 - 2.95243 \cdot 10^{-249} X^6 + 1.61394 \cdot 10^{-207} X^5 + 1.06875 \cdot 10^{-164} X^4 - 1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12}$$
$$= -6.9178 \cdot 10^{-12} B_{0,8}(X) - 6.9178 \cdot 10^{-12} B_{1,8}(X) - 6.9178 \cdot 10^{-12} B_{2,8}(X) - 6.9178 \cdot 10^{-12} B_{3,8}(X) - 6.9178 \cdot 10^{-12} B_{4,8}(X) - 6.9178 \cdot 10^{-12} B_{5,8}(X) - 6.9178 \cdot 10^{-12} B_{6,8}(X) - 6.9178 \cdot 10^{-12} B_{7,8}(X) - 6.9178 \cdot 10^{-12} B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = -1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12}$$
$$= -6.9178 \cdot 10^{-12} B_{0,3} - 6.9178 \cdot 10^{-12} B_{1,3} - 6.9178 \cdot 10^{-12} B_{2,3} - 6.9178 \cdot 10^{-12} B_{3,3}$$

$$\widetilde{q}_3 = -3.69134 \cdot 10^{-315} X^8 + 1.58227 \cdot 10^{-314} X^7 - 4.11376 \cdot 10^{-316} X^6 - 5.57319 \cdot 10^{-314} X^5 + 6.99255 \cdot 10^{-314} X^4 - 1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12}$$
$$= -6.9178 \cdot 10^{-12} B_{0,8} - 6.9178 \cdot 10^{-12} B_{1,8} - 6.9178 \cdot 10^{-12} B_{2,8} - 6.9178 \cdot 10^{-12} B_{3,8} - 6.9178 \cdot 10^{-12} B_{4,8} - 6.9178 \cdot 10^{-12} B_{5,8} - 6.9178 \cdot 10^{-12} B_{6,8} - 6.9178 \cdot 10^{-12} B_{7,8} - 6.9178 \cdot 10^{-12} B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.29017{\cdot}10^{-166}$.

Bounding polynomials M and m:

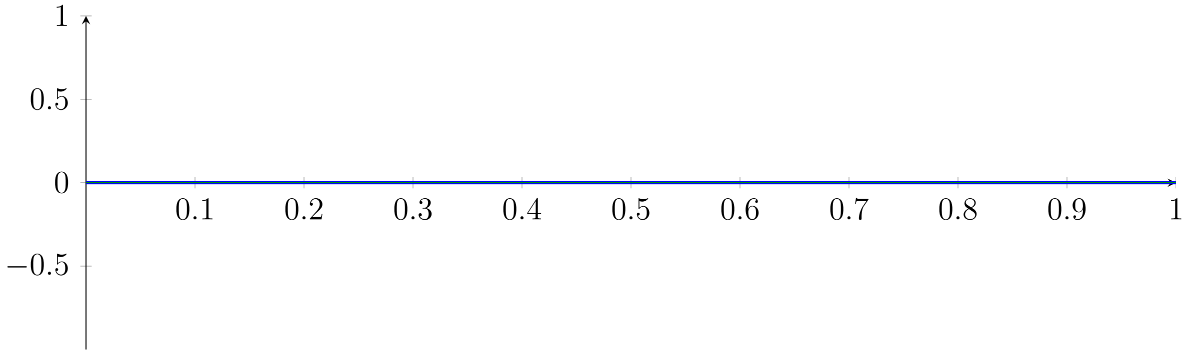
$$M = -1.41343{\cdot}10^{-123}X^3 - 1.35418{\cdot}10^{-80}X^2 + 6.95048{\cdot}10^{-39}X - 6.9178{\cdot}10^{-12}$$

$$m = -1.41343{\cdot}10^{-123}X^3 - 1.35418{\cdot}10^{-80}X^2 + 6.95048{\cdot}10^{-39}X - 6.9178{\cdot}10^{-12}$$

Root of M and m:

$N(M) = \{-1.00692{\cdot}10^{43}, 2.13634{\cdot}10^{17}, 4.88366{\cdot}10^{41}\}$ $N(m) = \{-1.00692{\cdot}10^{43}, 2.13634{\cdot}10^{17}, 4.88366{\cdot}10^{41}\}$

Intersection intervals:

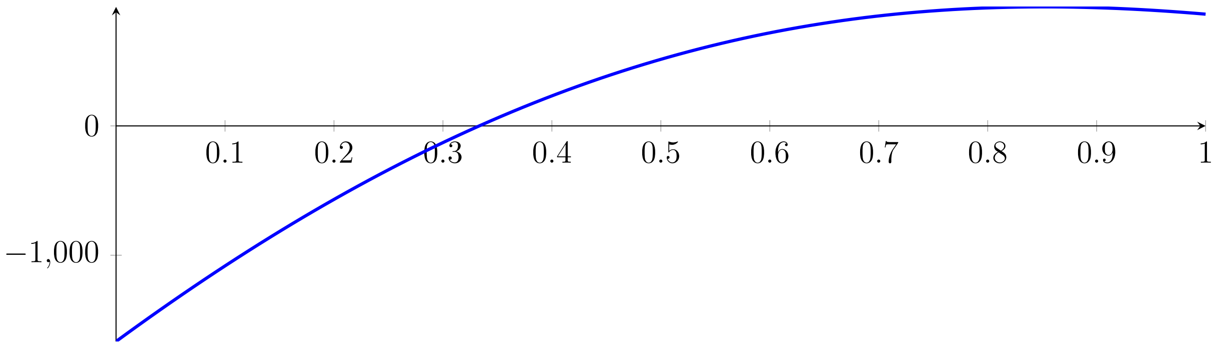


No intersection intervals with the x axis.

231.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

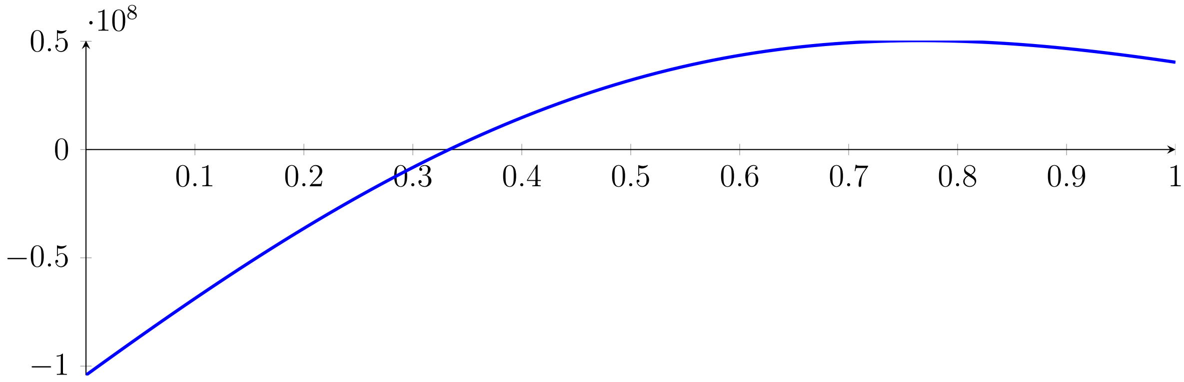
with precision $\varepsilon = 1 \cdot 10^{-128}$.

232 Running BezClip on f_{16} with epsilon 2

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called BezClip with input polynomial on interval $[0, 1]$:

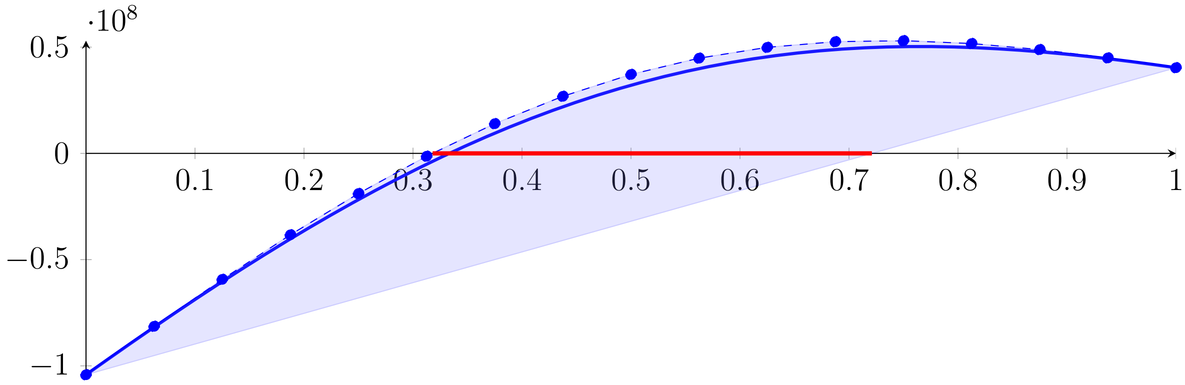
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} \\ &+ 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \end{aligned}$$



232.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} \\ &+ 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ = &-1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ &\cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ &+ 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ &\cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ &+ 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

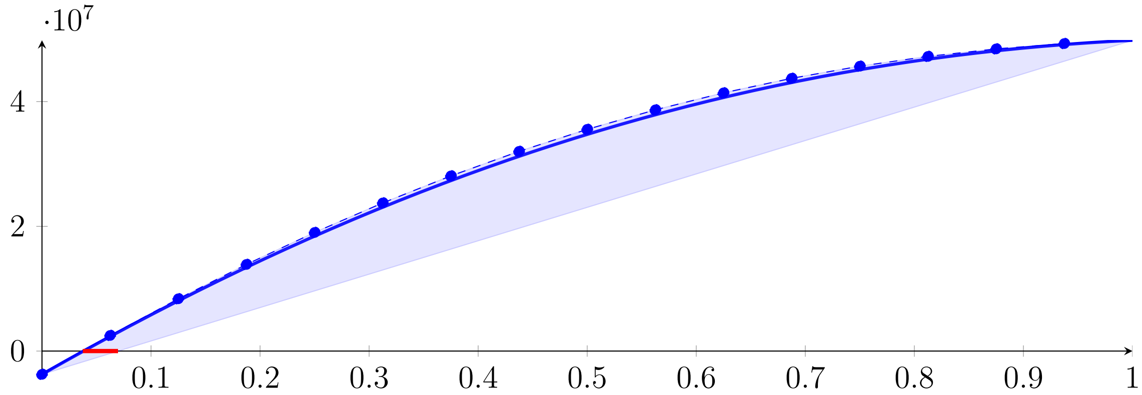
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

232.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -4.83858\cdot 10^{-07}X^{16} - 5.37355\cdot 10^{-05}X^{15} - 0.00254146X^{14} - 0.064977X^{13} - 0.909205X^{12} \\ &\quad - 5.03924X^{11} + 36.4549X^{10} + 692.921X^{9} + 1886.97X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5} \\ &\quad + 3.91945\cdot 10^{6}X^{4} - 7.54561\cdot 10^{6}X^{3} - 4.29225\cdot 10^{7}X^{2} + 9.97982\cdot 10^{7}X - 3.73501\cdot 10^{6} \\ &= -3.73501\cdot 10^{6}B_{0,16}(X) + 2.50237\cdot 10^{6}B_{1,16}(X) + 8.38207\cdot 10^{6}B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^{7}B_{3,16}(X) + 1.90167\cdot 10^{7}B_{4,16}(X) + 2.37512\cdot 10^{7}B_{5,16}(X) + 2.80875\cdot 10^{7}B_{6,16}(X) \\ &\quad + 3.20213\cdot 10^{7}B_{7,16}(X) + 3.55508\cdot 10^{7}B_{8,16}(X) + 3.86768\cdot 10^{7}B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^{7}B_{10,16}(X) + 4.37336\cdot 10^{7}B_{11,16}(X) + 4.56783\cdot 10^{7}B_{12,16}(X) + 4.7247\cdot 10^{7}B_{13,16}(X) \\ &\quad + 4.84524\cdot 10^{7}B_{14,16}(X) + 4.93093\cdot 10^{7}B_{15,16}(X) + 4.98343\cdot 10^{7}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

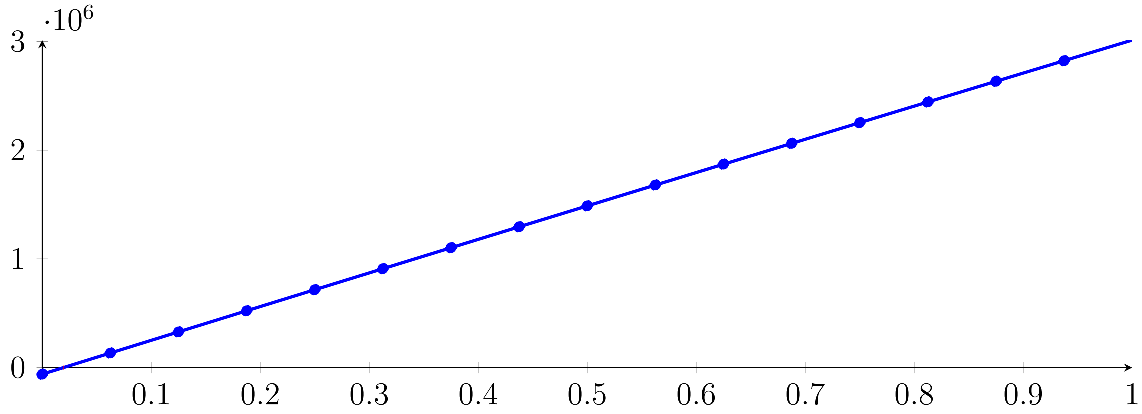
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

232.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -6.78234\cdot 10^{-31}X^{16} - 2.34473\cdot 10^{-27}X^{15} - 3.45581\cdot 10^{-24}X^{14} - 2.75921\cdot 10^{-21}X^{13} - 1.21242\cdot 10^{-18}X^{12} \\ &\quad - 2.17572\cdot 10^{-16}X^{11} + 4.23544\cdot 10^{-14}X^{10} + 2.70025\cdot 10^{-11}X^{9} + 2.51306\cdot 10^{-09}X^{8} - 9.23474\cdot 10^{-07}X^{7} \\ &\quad - 0.000177459X^{6} + 0.0161038X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot 10^{6}X - 60508.5 \\ &= -60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X) \\ &\quad + 910202B_{5,16}(X) + 1.1032\cdot 10^{6}B_{6,16}(X) + 1.29582\cdot 10^{6}B_{7,16}(X) + 1.48805\cdot 10^{6}B_{8,16}(X) \\ &\quad + 1.6799\cdot 10^{6}B_{9,16}(X) + 1.87137\cdot 10^{6}B_{10,16}(X) + 2.06245\cdot 10^{6}B_{11,16}(X) + 2.25315\cdot 10^{6}B_{12,16}(X) \\ &\quad + 2.44346\cdot 10^{6}B_{13,16}(X) + 2.63339\cdot 10^{6}B_{14,16}(X) + 2.82293\cdot 10^{6}B_{15,16}(X) + 3.01209\cdot 10^{6}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

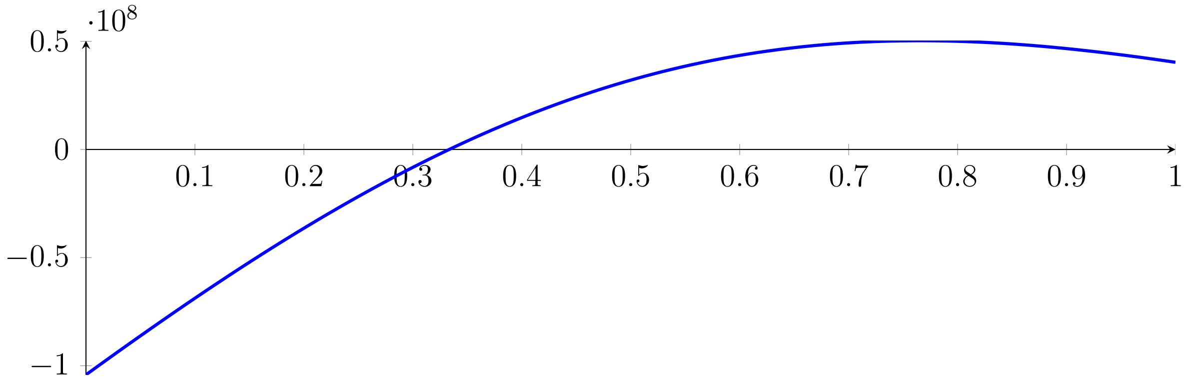
232.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Found root in interval $[0.333333, 0.333337]$ at recursion depth 4!

232.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333337]$$

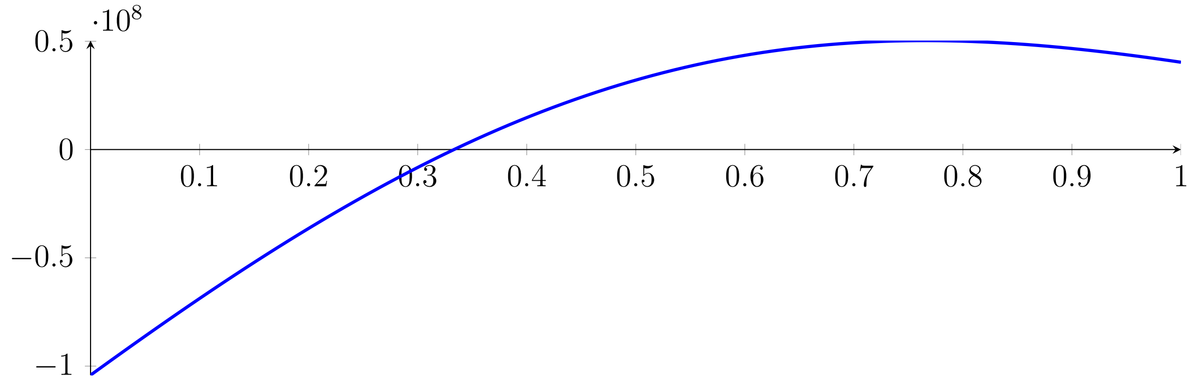
with precision $\varepsilon = 0.01$.

233 Running QuadClip on f_{16} with epsilon 2

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

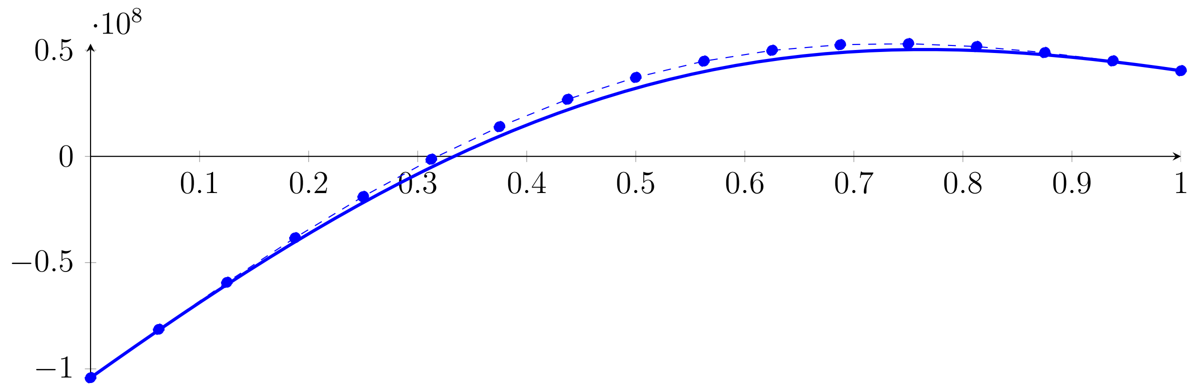
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



233.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

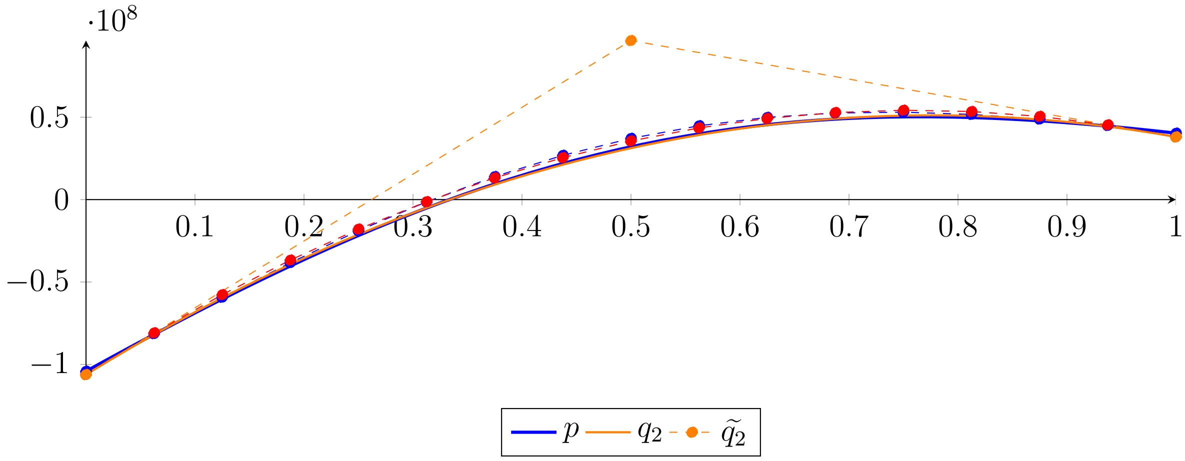
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} \\ &+ 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ = &-1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ &\cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ &+ 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ &\cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ &+ 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.06197{\cdot}10^8$$
$$= -1.06197{\cdot}10^8 B_{0,2} + 9.65115{\cdot}10^7 B_{1,2} + 3.80384{\cdot}10^7 B_{2,2}$$

$$\widetilde{q_2} = 3.74473{\cdot}10^{-288} X^{16} - 3.08889{\cdot}10^{-287} X^{15} + 1.17443{\cdot}10^{-286} X^{14} - 2.71603{\cdot}10^{-286} X^{13} + 4.23554{\cdot}10^{-286} X^{12} - 4.66107{\cdot}10^{-286} X^{11} + 3.6845{\cdot}10^{-286} X^{10} - 2.09811{\cdot}10^{-286} X^9 + 8.58708{\cdot}10^{-287} X^8 - 2.54063{\cdot}10^{-287} X^7 + 5.57852{\cdot}10^{-288} X^6 - 9.21534{\cdot}10^{-289} X^5 + 1.02122{\cdot}10^{-289} X^4 - 5.95624{\cdot}10^{-291} X^3 - 2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.06197{\cdot}10^8$$
$$= -1.06197{\cdot}10^8 B_{0,16} - 8.08583{\cdot}10^7 B_{1,16} - 5.76963{\cdot}10^7 B_{2,16} - 3.67107{\cdot}10^7 B_{3,16} - 1.79017{\cdot}10^7 B_{4,16} - 1.26924{\cdot}10^6 B_{5,16} + 1.31867{\cdot}10^7 B_{6,16} + 2.54662{\cdot}10^7 B_{7,16} + 3.55692{\cdot}10^7 B_{8,16} + 4.34956{\cdot}10^7 B_{9,16} + 4.92456{\cdot}10^7 B_{10,16} + 5.2819{\cdot}10^7 B_{11,16} + 5.42159{\cdot}10^7 B_{12,16} + 5.34363{\cdot}10^7 B_{13,16} + 5.04802{\cdot}10^7 B_{14,16} + 4.53476{\cdot}10^7 B_{15,16} + 3.80384{\cdot}10^7 B_{16,16}$$



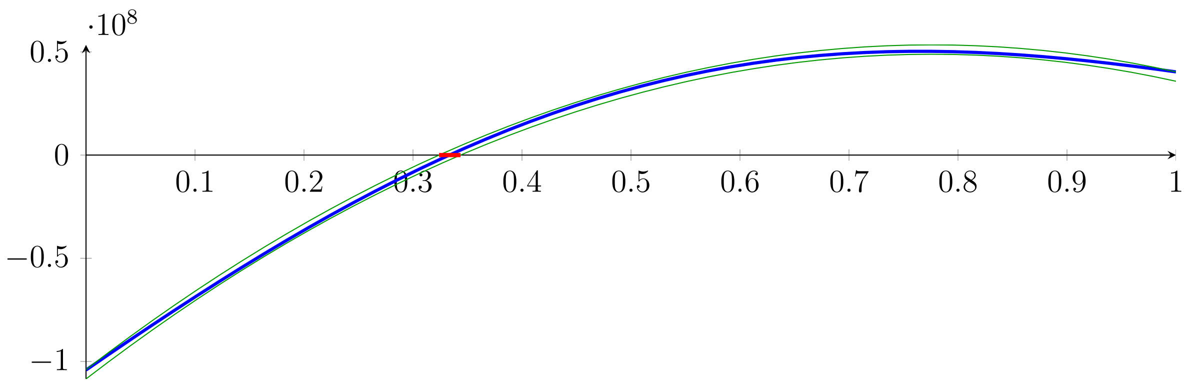
The maximum difference of the Bézier coefficients is $\delta = 2.27233{\cdot}10^6$.
Bounding polynomials M and m:

$$M = -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.03924{\cdot}10^8$$
$$m = -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.08469{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



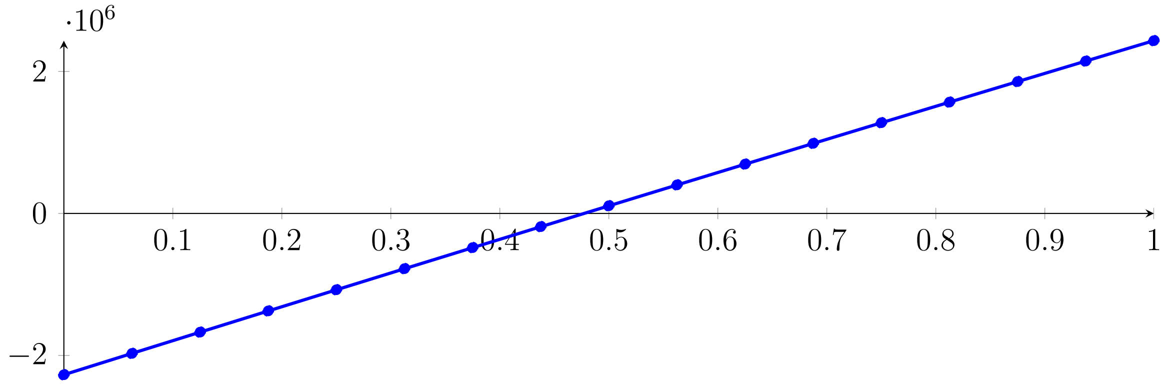
$$[0.323946, 0.343615]$$

Longest intersection interval: 0.0196686
$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

233.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

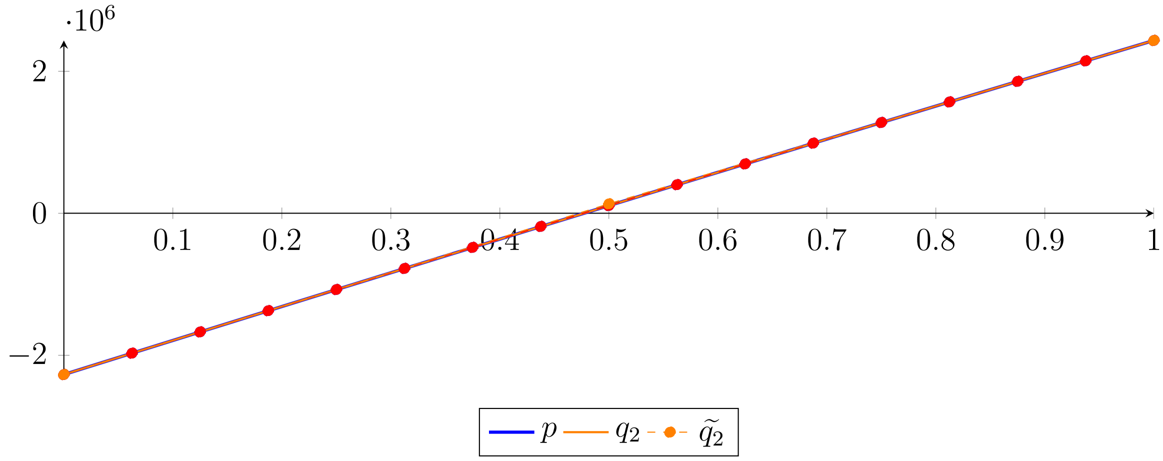
$$p = -5.0162{\cdot}10^{-28}X^{16} - 1.14383{\cdot}10^{-24}X^{15} - 1.11125{\cdot}10^{-21}X^{14} - 5.84096{\cdot}10^{-19}X^{13} - 1.684{\cdot}10^{-16}X^{12} - 1.94707{\cdot}10^{-14}X^{11} + 2.73227{\cdot}10^{-12}X^{10} + 1.09727{\cdot}10^{-09}X^{9} + 6.37314{\cdot}10^{-08}X^{8} - 1.68645{\cdot}10^{-05}X^{7} - 0.00205892X^{6} + 0.132721X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874{\cdot}10^{6}X - 2.2715{\cdot}10^{6}$$
$$= -2.2715{\cdot}10^{6}B_{0,16}(X) - 1.97096{\cdot}10^{6}B_{1,16}(X) - 1.67127{\cdot}10^{6}B_{2,16}(X) - 1.37244{\cdot}10^{6}B_{3,16}(X) - 1.07448{\cdot}10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806{\cdot}10^{6}B_{12,16}(X) + 1.5682{\cdot}10^{6}B_{13,16}(X) + 1.85747{\cdot}10^{6}B_{14,16}(X) + 2.14586{\cdot}10^{6}B_{15,16}(X) + 2.43338{\cdot}10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265X^{2} + 4.80923{\cdot}10^{6}X - 2.27154{\cdot}10^{6}$$
$$= -2.27154{\cdot}10^{6}B_{0,2} + 133069B_{1,2} + 2.43342{\cdot}10^{6}B_{2,2}$$

$$\widetilde{q}_2 = -8.18461{\cdot}10^{-291}X^{16} + 4.13745{\cdot}10^{-290}X^{15} - 5.8254{\cdot}10^{-290}X^{14} - 6.80919{\cdot}10^{-290}X^{13} + 3.48059{\cdot}10^{-289}X^{12} - 5.47245{\cdot}10^{-289}X^{11} + 4.8261{\cdot}10^{-289}X^{10} - 2.67867{\cdot}10^{-289}X^{9} + 1.00314{\cdot}10^{-289}X^{8} - 2.84766{\cdot}10^{-290}X^{7} + 6.85072{\cdot}10^{-291}X^{6} - 1.12501{\cdot}10^{-291}X^{5} + 1.24731{\cdot}10^{-293}X^{4} + 2.25944{\cdot}10^{-293}X^{3} - 104265X^{2} + 4.80923{\cdot}10^{6}X - 2.27154{\cdot}10^{6}$$
$$= -2.27154{\cdot}10^{6}B_{0,16} - 1.97097{\cdot}10^{6}B_{1,16} - 1.67126{\cdot}10^{6}B_{2,16} - 1.37242{\cdot}10^{6}B_{3,16} - 1.07445{\cdot}10^{6}B_{4,16} - 777350B_{5,16} - 481118B_{6,16} - 185754B_{7,16} + 108740B_{8,16} + 402366B_{9,16} + 695123B_{10,16} + 987011B_{11,16} + 1.27803{\cdot}10^{6}B_{12,16} + 1.56818{\cdot}10^{6}B_{13,16} + 1.85746{\cdot}10^{6}B_{14,16} + 2.14587{\cdot}10^{6}B_{15,16} + 2.43342{\cdot}10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.
Bounding polynomials M and m:

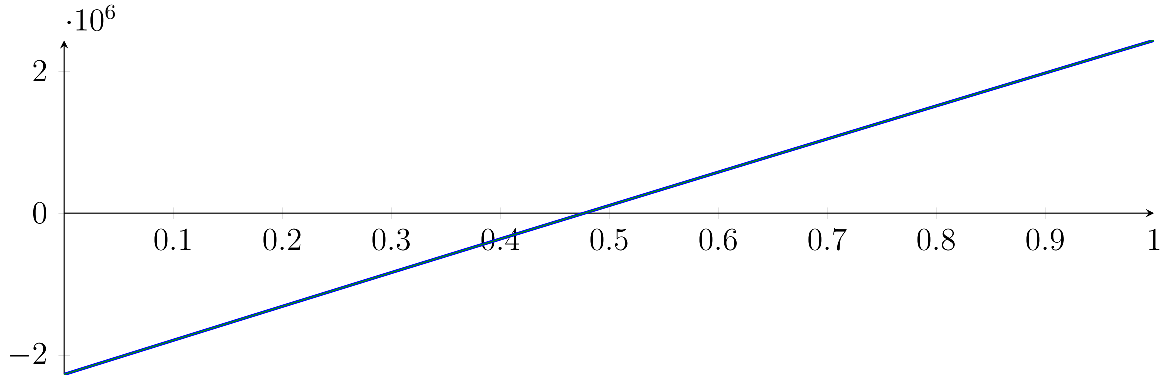
$$M = -104265X^2 + 4.80923{\cdot}10^6X - 2.2715{\cdot}10^6$$

$$m = -104265X^2 + 4.80923{\cdot}10^6X - 2.27159{\cdot}10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301{\cdot}10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

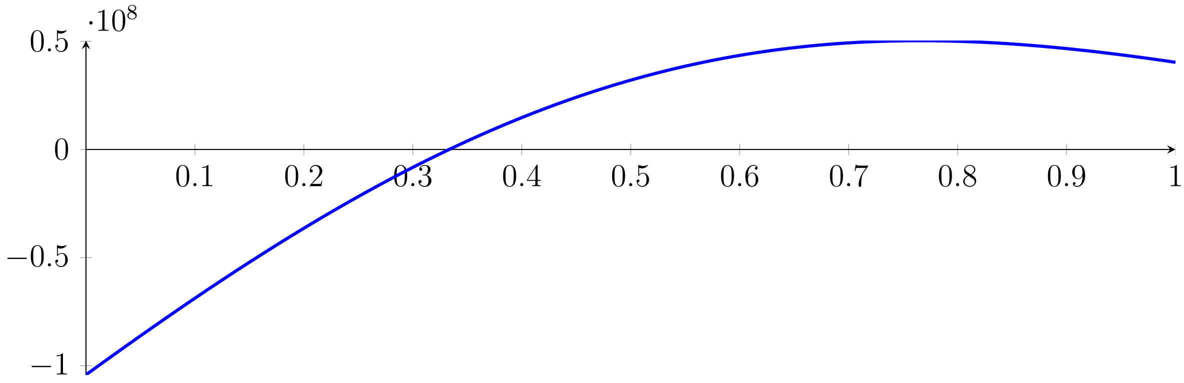
233.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

233.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

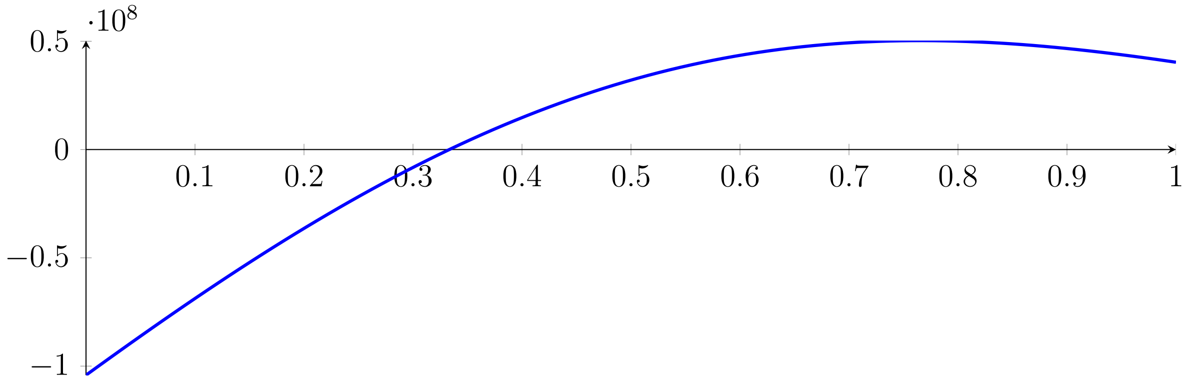
with precision $\varepsilon = 0.01$.

234 Running CubeClip on f_{16} with epsilon 2

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8$$

Called CubeClip with input polynomial on interval $[0, 1]$:

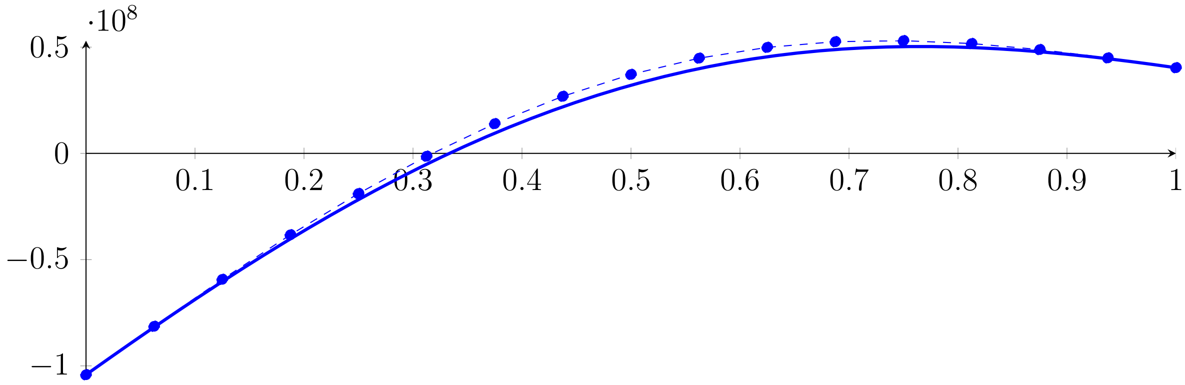
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8$$



234.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

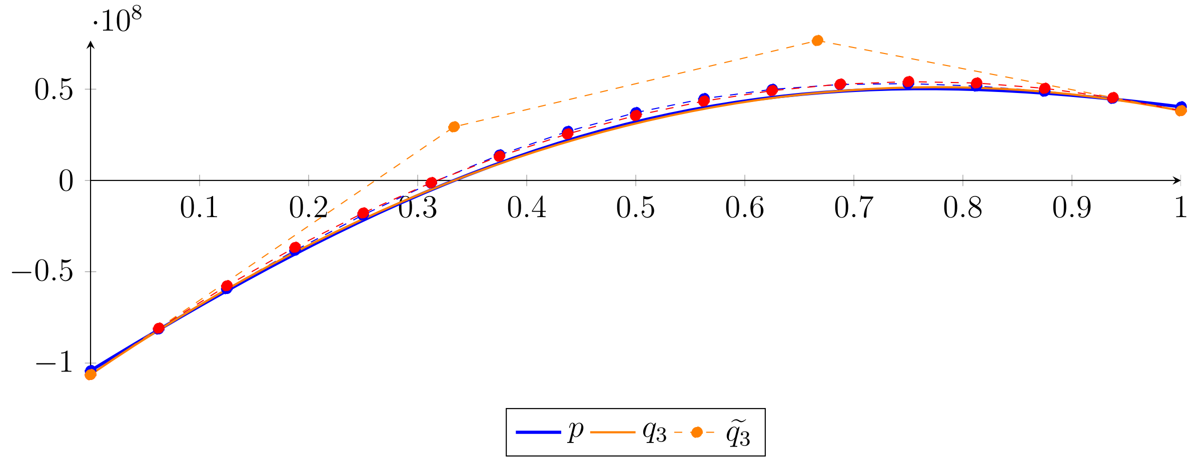
$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\ = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\ & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\ & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\ & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\ & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 2.75812\cdot 10^{6}X^{3} - 2.65319\cdot 10^{8}X^{2} + 4.07072\cdot 10^{8}X - 1.06335\cdot 10^{8} \\ &= -1.06335\cdot 10^{8}B_{0,3} + 2.93558\cdot 10^{7}B_{1,3} + 7.66068\cdot 10^{7}B_{2,3} + 3.81764\cdot 10^{7}B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 1.89955\cdot 10^{-288}X^{16} - 1.4851\cdot 10^{-287}X^{15} + 5.12896\cdot 10^{-287}X^{14} - 1.02224\cdot 10^{-286}X^{13} \\ &\quad + 1.29027\cdot 10^{-286}X^{12} - 1.07058\cdot 10^{-286}X^{11} + 6.00739\cdot 10^{-287}X^{10} - 2.54352\cdot 10^{-287}X^{9} \\ &\quad + 1.12089\cdot 10^{-287}X^{8} - 5.8673\cdot 10^{-288}X^{7} + 2.52637\cdot 10^{-288}X^{6} - 6.79624\cdot 10^{-289}X^{5} \\ &\quad + 9.36341\cdot 10^{-290}X^{4} + 2.75812\cdot 10^{6}X^{3} - 2.65319\cdot 10^{8}X^{2} + 4.07072\cdot 10^{8}X - 1.06335\cdot 10^{8} \\ &= -1.06335\cdot 10^{8}B_{0,16} - 8.08928\cdot 10^{7}B_{1,16} - 5.76618\cdot 10^{7}B_{2,16} - 3.66369\cdot 10^{7}B_{3,16} - 1.78131 \\ &\quad \cdot 10^{7}B_{4,16} - 1.18551\cdot 10^{6}B_{5,16} + 1.32508\cdot 10^{7}B_{6,16} + 2.55007\cdot 10^{7}B_{7,16} + 3.55692\cdot 10^{7}B_{8,16} \\ &\quad + 4.34611\cdot 10^{7}B_{9,16} + 4.91815\cdot 10^{7}B_{10,16} + 5.27353\cdot 10^{7}B_{11,16} + 5.41273\cdot 10^{7}B_{12,16} \\ &\quad + 5.33624\cdot 10^{7}B_{13,16} + 5.04457\cdot 10^{7}B_{14,16} + 4.53821\cdot 10^{7}B_{15,16} + 3.81764\cdot 10^{7}B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806\cdot 10^{6}$.

Bounding polynomials M and m:

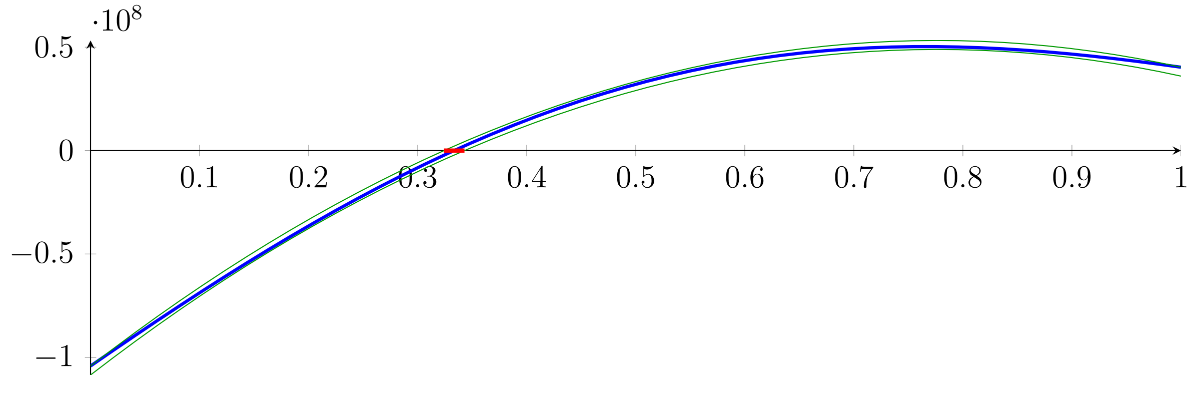
$$M = 2.75812\cdot 10^{6}X^{3} - 2.65319\cdot 10^{8}X^{2} + 4.07072\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

$$m = 2.75812\cdot 10^{6}X^{3} - 2.65319\cdot 10^{8}X^{2} + 4.07072\cdot 10^{8}X - 1.08503\cdot 10^{8}$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

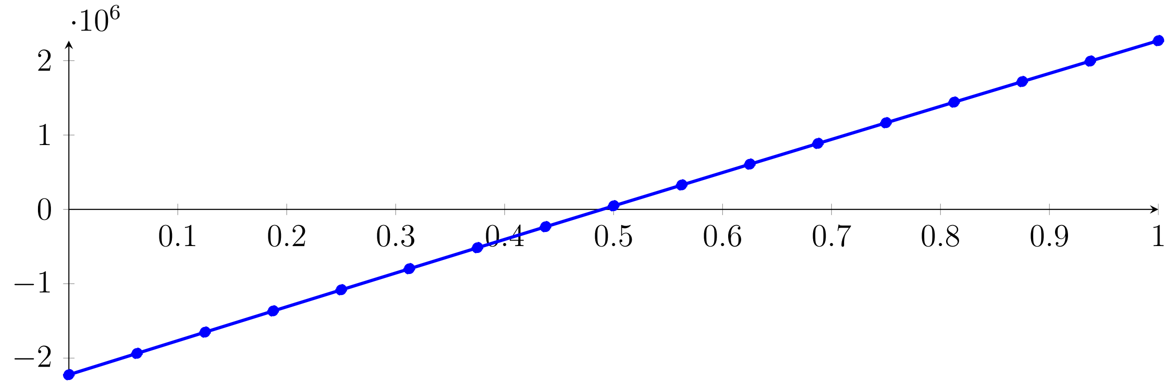
Longest intersection interval: 0.0187703

$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

234.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

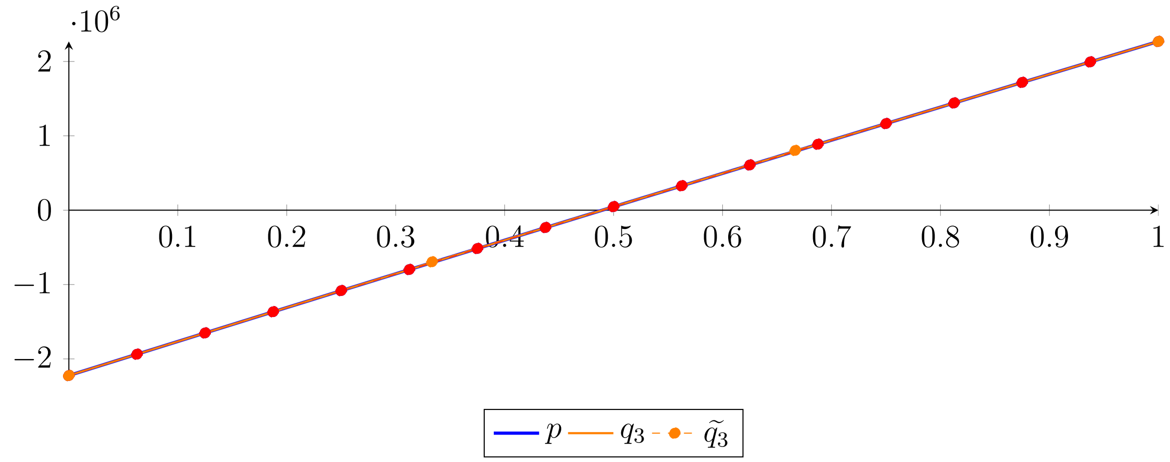
$$p = -2.37433 \cdot 10^{-28} X^{16} - 5.67363 \cdot 10^{-25} X^{15} - 5.77631 \cdot 10^{-22} X^{14} - 3.1818 \cdot 10^{-19} X^{13} - 9.6142 \cdot 10^{-17} X^{12} - 1.16549 \cdot 10^{-14} X^{11} + 1.71065 \cdot 10^{-12} X^{10} + 7.20611 \cdot 10^{-10} X^{9} + 4.39147 \cdot 10^{-08} X^{8} - 1.21542 \cdot 10^{-05} X^{7} - 0.00155624 X^{6} + 0.10496 X^{5} + 18.6215 X^{4} - 738.202 X^{3} - 93855.7 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$
$$= -2.22335 \cdot 10^{6} B_{0,16}(X) - 1.93666 \cdot 10^{6} B_{1,16}(X) - 1.65074 \cdot 10^{6} B_{2,16}(X) - 1.36561 \cdot 10^{6} B_{3,16}(X) - 1.08127 \cdot 10^{6} B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^{6} B_{12,16}(X) + 1.44233 \cdot 10^{6} B_{13,16}(X) + 1.71876 \cdot 10^{6} B_{14,16}(X) + 1.99439 \cdot 10^{6} B_{15,16}(X) + 2.26922 \cdot 10^{6} B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -700.673 X^{3} - 93879.9 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$
$$= -2.22335 \cdot 10^{6} B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^{6} B_{3,3}$$

$$\widetilde{q}_3 = 4.41595 \cdot 10^{-291} X^{16} - 1.48598 \cdot 10^{-290} X^{15} - 2.81102 \cdot 10^{-290} X^{14} + 2.65129 \cdot 10^{-289} X^{13} - 7.34593 \cdot 10^{-289} X^{12} + 1.17654 \cdot 10^{-288} X^{11} - 1.23119 \cdot 10^{-288} X^{10} + 8.63493 \cdot 10^{-289} X^{9} - 3.92909 \cdot 10^{-289} X^{8} + 1.01265 \cdot 10^{-289} X^{7} - 6.13644 \cdot 10^{-291} X^{6} - 3.92664 \cdot 10^{-291} X^{5} + 9.04488 \cdot 10^{-292} X^{4} - 700.673 X^{3} - 93879.9 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$
$$= -2.22335 \cdot 10^{6} B_{0,16} - 1.93666 \cdot 10^{6} B_{1,16} - 1.65074 \cdot 10^{6} B_{2,16} - 1.36561 \cdot 10^{6} B_{3,16} - 1.08126 \cdot 10^{6} B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^{6} B_{12,16} + 1.44233 \cdot 10^{6} B_{13,16} + 1.71876 \cdot 10^{6} B_{14,16} + 1.99439 \cdot 10^{6} B_{15,16} + 2.26922 \cdot 10^{6} B_{16,16}$$



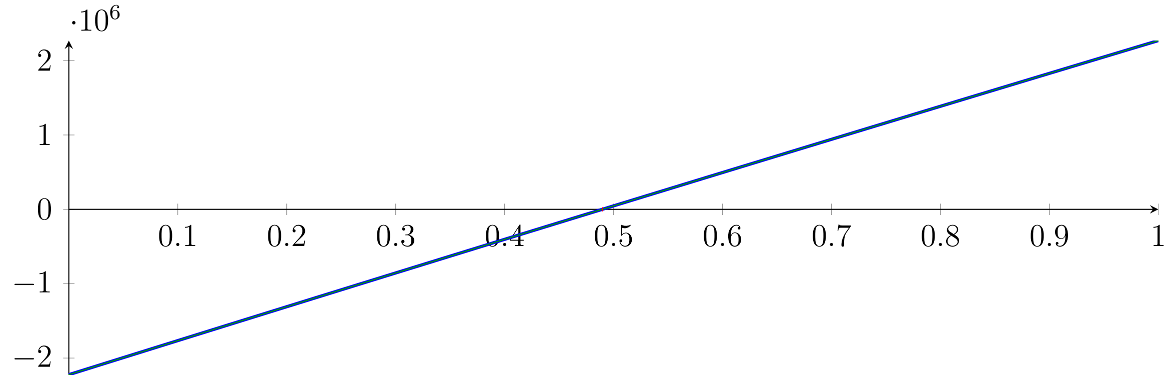
The maximum difference of the Bézier coefficients is $\delta = 0.270074$.
Bounding polynomials M and m:

$$M = -700.673X^3 - 93879.9X^2 + 4.58715{\cdot}10^6X - 2.22335{\cdot}10^6$$
$$m = -700.673X^3 - 93879.9X^2 + 4.58715{\cdot}10^6X - 2.22335{\cdot}10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174{\cdot}10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

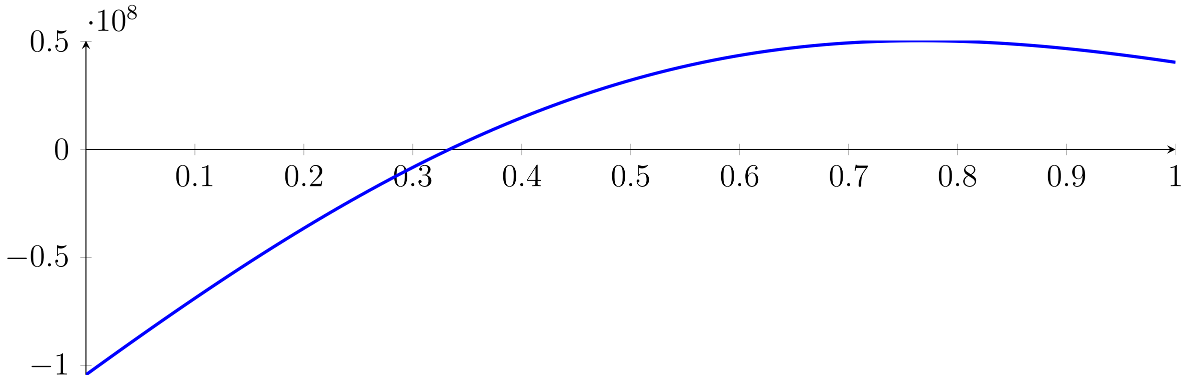
234.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

234.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} + 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

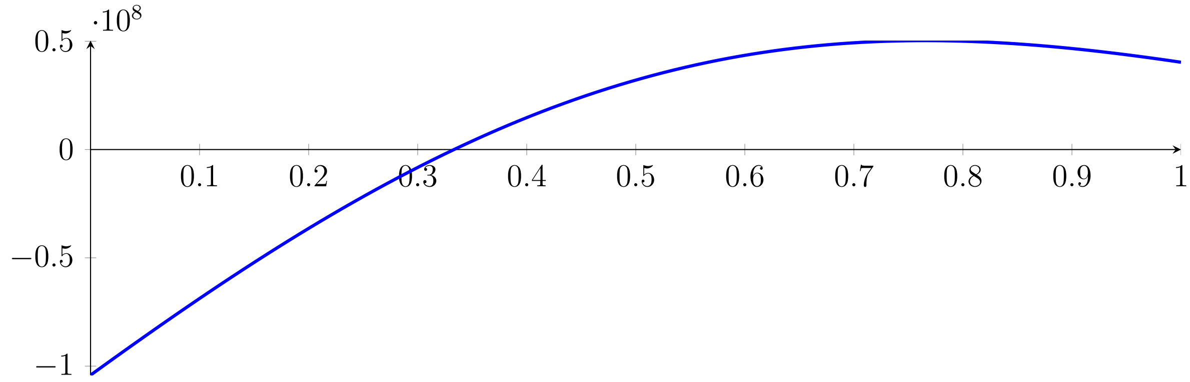
with precision $\varepsilon = 0.01$.

235 Running BezClip on f_{16} with epsilon 4

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^6 X^9 - 2.52262\cdot 10^6 X^8 - 1.43506\cdot 10^7 X^7 + 138542X^6 + 7.92823\cdot 10^7 X^5 + 3.97396\cdot 10^7 X^4 - 2.40625\cdot 10^8 X^3 - 8.33333\cdot 10^7 X^2 + 3.64583\cdot 10^8 X - 1.04167\cdot 10^8$$

Called BezClip with input polynomial on interval $[0, 1]$:

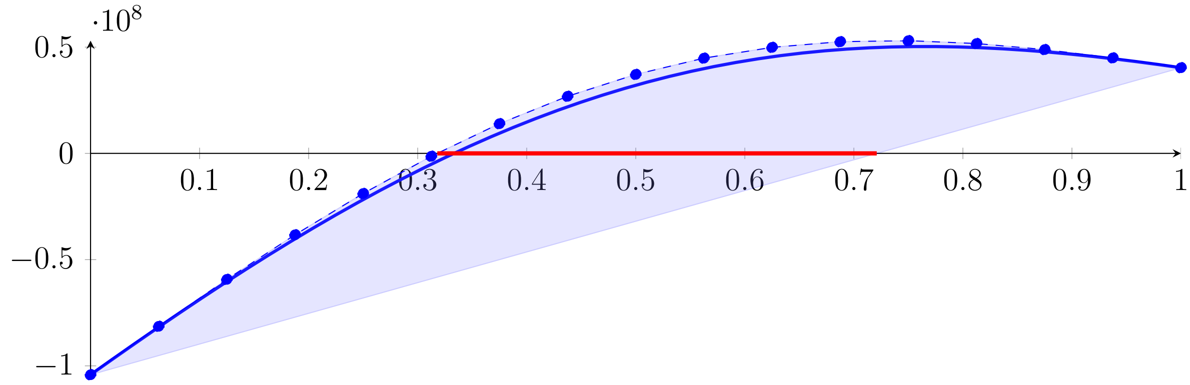
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^6 X^9 - 2.52262\cdot 10^6 X^8 - 1.43506\cdot 10^7 X^7 + 138542X^6 + 7.92823\cdot 10^7 X^5 + 3.97396\cdot 10^7 X^4 - 2.40625\cdot 10^8 X^3 - 8.33333\cdot 10^7 X^2 + 3.64583\cdot 10^8 X - 1.04167\cdot 10^8$$



235.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &\quad + 1.12172\cdot 10^6 X^9 - 2.52262\cdot 10^6 X^8 - 1.43506\cdot 10^7 X^7 + 138542X^6 + 7.92823\cdot 10^7 X^5 \\ &\quad + 3.97396\cdot 10^7 X^4 - 2.40625\cdot 10^8 X^3 - 8.33333\cdot 10^7 X^2 + 3.64583\cdot 10^8 X - 1.04167\cdot 10^8 \\ &= -1.04167\cdot 10^8 B_{0,16}(X) - 8.13802\cdot 10^7 B_{1,16}(X) - 5.92882\cdot 10^7 B_{2,16}(X) - 3.83203 \\ &\quad \cdot 10^7 B_{3,16}(X) - 1.88844\cdot 10^7 B_{4,16}(X) - 1.34837\cdot 10^6 B_{5,16}(X) + 1.39781\cdot 10^7 B_{6,16}(X) \\ &\quad + 2.68604\cdot 10^7 B_{7,16}(X) + 3.71532\cdot 10^7 B_{8,16}(X) + 4.48105\cdot 10^7 B_{9,16}(X) + 4.98901 \\ &\quad \cdot 10^7 B_{10,16}(X) + 5.25504\cdot 10^7 B_{11,16}(X) + 5.30407\cdot 10^7 B_{12,16}(X) + 5.16832\cdot 10^7 B_{13,16}(X) \\ &\quad + 4.88488\cdot 10^7 B_{14,16}(X) + 4.49297\cdot 10^7 B_{15,16}(X) + 4.03108\cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

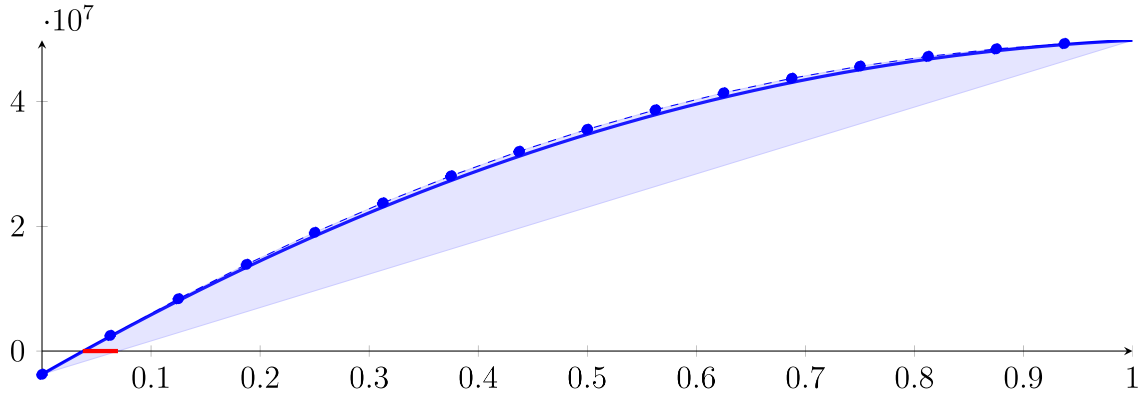
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

235.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p = &-4.83858\cdot10^{-07}X^{16} - 5.37355\cdot10^{-05}X^{15} - 0.00254146X^{14} - 0.064977X^{13} - 0.909205X^{12}\\ &- 5.03924X^{11} + 36.4549X^{10} + 692.921X^{9} + 1886.97X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5}\\ &+ 3.91945\cdot10^{6}X^{4} - 7.54561\cdot10^{6}X^{3} - 4.29225\cdot10^{7}X^{2} + 9.97982\cdot10^{7}X - 3.73501\cdot10^{6}\\ = &-3.73501\cdot10^{6}B_{0,16}(X) + 2.50237\cdot10^{6}B_{1,16}(X) + 8.38207\cdot10^{6}B_{2,16}(X) + 1.38906\\ &\cdot10^{7}B_{3,16}(X) + 1.90167\cdot10^{7}B_{4,16}(X) + 2.37512\cdot10^{7}B_{5,16}(X) + 2.80875\cdot10^{7}B_{6,16}(X)\\ &+ 3.20213\cdot10^{7}B_{7,16}(X) + 3.55508\cdot10^{7}B_{8,16}(X) + 3.86768\cdot10^{7}B_{9,16}(X) + 4.14025\\ &\cdot10^{7}B_{10,16}(X) + 4.37336\cdot10^{7}B_{11,16}(X) + 4.56783\cdot10^{7}B_{12,16}(X) + 4.7247\cdot10^{7}B_{13,16}(X)\\ &+ 4.84524\cdot10^{7}B_{14,16}(X) + 4.93093\cdot10^{7}B_{15,16}(X) + 4.98343\cdot10^{7}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

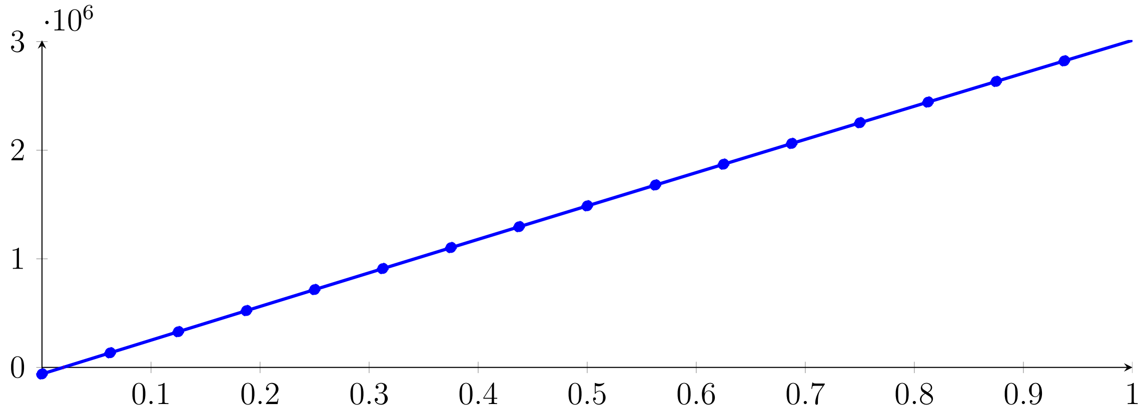
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

235.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p = &-6.78234\cdot10^{-31}X^{16} - 2.34473\cdot10^{-27}X^{15} - 3.45581\cdot10^{-24}X^{14} - 2.75921\cdot10^{-21}X^{13} - 1.21242\cdot10^{-18}X^{12}\\ &- 2.17572\cdot10^{-16}X^{11} + 4.23544\cdot10^{-14}X^{10} + 2.70025\cdot10^{-11}X^{9} + 2.51306\cdot10^{-09}X^{8} - 9.23474\cdot10^{-07}X^{7}\\ &- 0.000177459X^{6} + 0.0161038X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot10^{6}X - 60508.5\\ = &-60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X)\\ &+ 910202B_{5,16}(X) + 1.1032\cdot10^{6}B_{6,16}(X) + 1.29582\cdot10^{6}B_{7,16}(X) + 1.48805\cdot10^{6}B_{8,16}(X)\\ &+ 1.6799\cdot10^{6}B_{9,16}(X) + 1.87137\cdot10^{6}B_{10,16}(X) + 2.06245\cdot10^{6}B_{11,16}(X) + 2.25315\cdot10^{6}B_{12,16}(X)\\ &+ 2.44346\cdot10^{6}B_{13,16}(X) + 2.63339\cdot10^{6}B_{14,16}(X) + 2.82293\cdot10^{6}B_{15,16}(X) + 3.01209\cdot10^{6}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

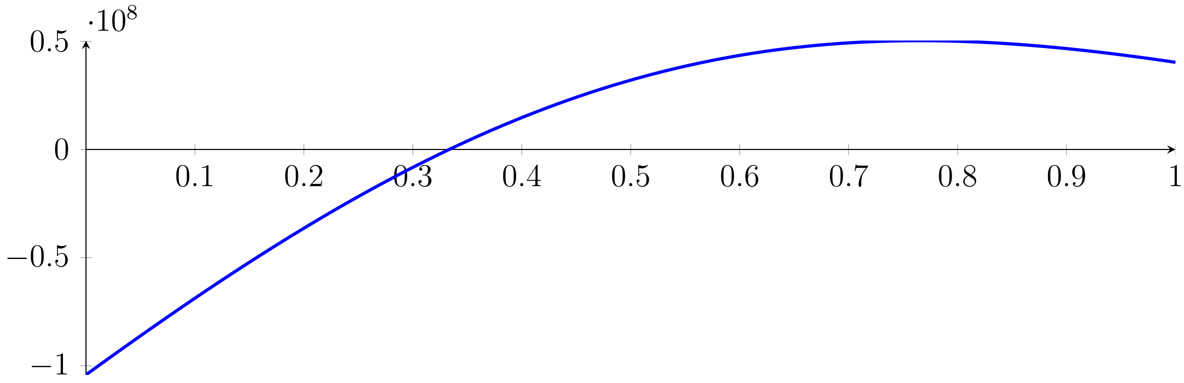
235.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Found root in interval $[0.333333, 0.333337]$ at recursion depth 4!

235.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^{6}X^{9} - 2.52262 \cdot 10^{6}X^{8} - 1.43506 \cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \cdot 10^{7}X^{5} + 3.97396 \cdot 10^{7}X^{4} - 2.40625 \cdot 10^{8}X^{3} - 8.33333 \cdot 10^{7}X^{2} + 3.64583 \cdot 10^{8}X - 1.04167 \cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333337]$$

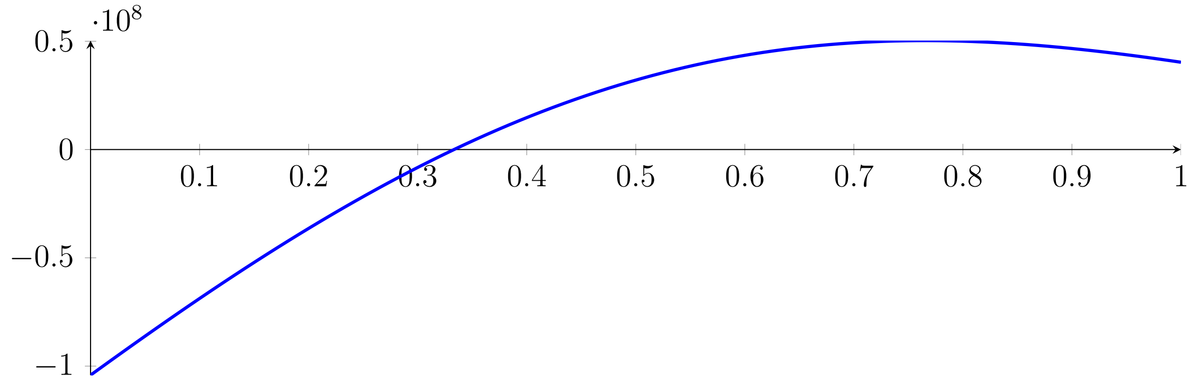
with precision $\varepsilon = 0.0001$.

236 Running QuadClip on f_{16} with epsilon 4

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} + 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8}$$

Called QuadClip with input polynomial on interval $[0,1]$:

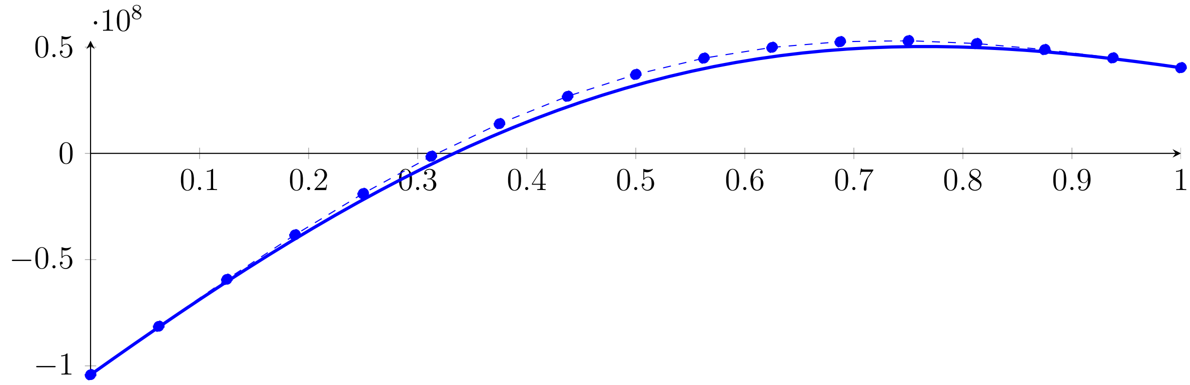
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} \\ &+ 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8} \end{aligned}$$



236.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

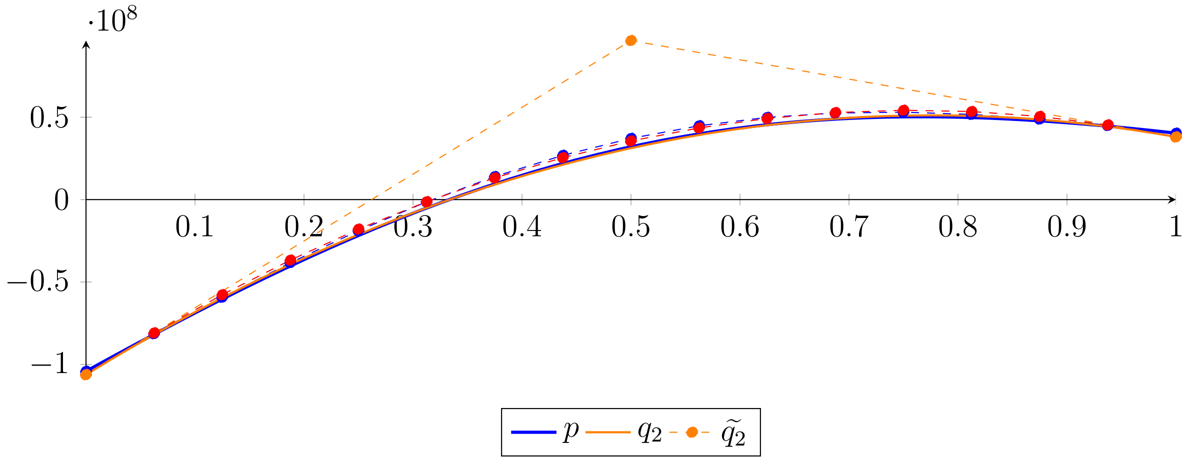
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} \\ &+ 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8} \\ = &-1.04167{\cdot}10^{8}B_{0,16}(X) - 8.13802{\cdot}10^{7}B_{1,16}(X) - 5.92882{\cdot}10^{7}B_{2,16}(X) - 3.83203 \\ &{\cdot}10^{7}B_{3,16}(X) - 1.88844{\cdot}10^{7}B_{4,16}(X) - 1.34837{\cdot}10^{6}B_{5,16}(X) + 1.39781{\cdot}10^{7}B_{6,16}(X) \\ &+ 2.68604{\cdot}10^{7}B_{7,16}(X) + 3.71532{\cdot}10^{7}B_{8,16}(X) + 4.48105{\cdot}10^{7}B_{9,16}(X) + 4.98901 \\ &{\cdot}10^{7}B_{10,16}(X) + 5.25504{\cdot}10^{7}B_{11,16}(X) + 5.30407{\cdot}10^{7}B_{12,16}(X) + 5.16832{\cdot}10^{7}B_{13,16}(X) \\ &+ 4.88488{\cdot}10^{7}B_{14,16}(X) + 4.49297{\cdot}10^{7}B_{15,16}(X) + 4.03108{\cdot}10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -2.61181\cdot 10^{8} X^2 + 4.05417\cdot 10^{8} X - 1.06197\cdot 10^{8} \\ &= -1.06197\cdot 10^{8} B_{0,2} + 9.65115\cdot 10^{7} B_{1,2} + 3.80384\cdot 10^{7} B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q_2} &= 3.74473\cdot 10^{-288} X^{16} - 3.08889\cdot 10^{-287} X^{15} + 1.17443\cdot 10^{-286} X^{14} - 2.71603\cdot 10^{-286} X^{13} \\ &\quad + 4.23554\cdot 10^{-286} X^{12} - 4.66107\cdot 10^{-286} X^{11} + 3.6845\cdot 10^{-286} X^{10} - 2.09811\cdot 10^{-286} X^{9} \\ &\quad + 8.58708\cdot 10^{-287} X^{8} - 2.54063\cdot 10^{-287} X^{7} + 5.57852\cdot 10^{-288} X^{6} - 9.21534\cdot 10^{-289} X^{5} \\ &\quad + 1.02122\cdot 10^{-289} X^{4} - 5.95624\cdot 10^{-291} X^{3} - 2.61181\cdot 10^{8} X^{2} + 4.05417\cdot 10^{8} X - 1.06197\cdot 10^{8} \\ &= -1.06197\cdot 10^{8} B_{0,16} - 8.08583\cdot 10^{7} B_{1,16} - 5.76963\cdot 10^{7} B_{2,16} - 3.67107\cdot 10^{7} B_{3,16} - 1.79017 \\ &\quad \cdot 10^{7} B_{4,16} - 1.26924\cdot 10^{6} B_{5,16} + 1.31867\cdot 10^{7} B_{6,16} + 2.54662\cdot 10^{7} B_{7,16} + 3.55692\cdot 10^{7} B_{8,16} \\ &\quad + 4.34956\cdot 10^{7} B_{9,16} + 4.92456\cdot 10^{7} B_{10,16} + 5.2819\cdot 10^{7} B_{11,16} + 5.42159\cdot 10^{7} B_{12,16} \\ &\quad + 5.34363\cdot 10^{7} B_{13,16} + 5.04802\cdot 10^{7} B_{14,16} + 4.53476\cdot 10^{7} B_{15,16} + 3.80384\cdot 10^{7} B_{16,16} \end{aligned}$$



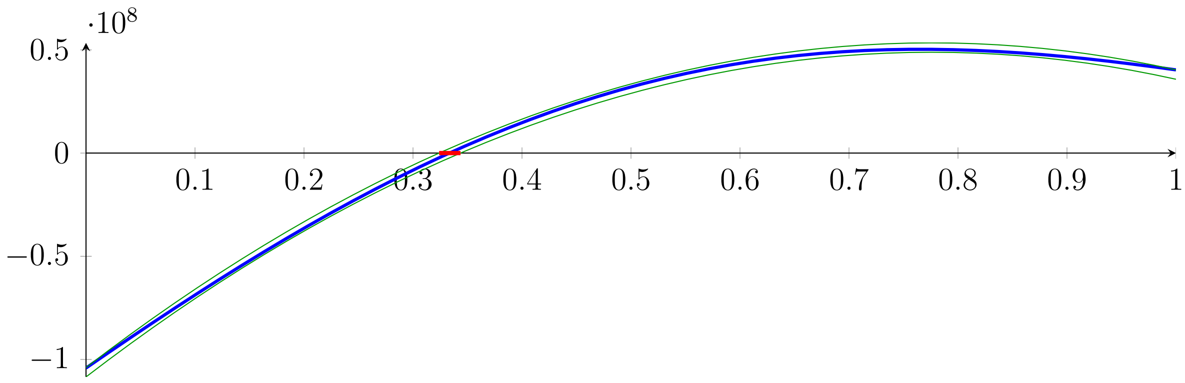
The maximum difference of the Bézier coefficients is $\delta = 2.27233\cdot 10^{6}$.
Bounding polynomials M and m:

$$M = -2.61181\cdot 10^{8} X^2 + 4.05417\cdot 10^{8} X - 1.03924\cdot 10^{8}$$
$$m = -2.61181\cdot 10^{8} X^2 + 4.05417\cdot 10^{8} X - 1.08469\cdot 10^{8}$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



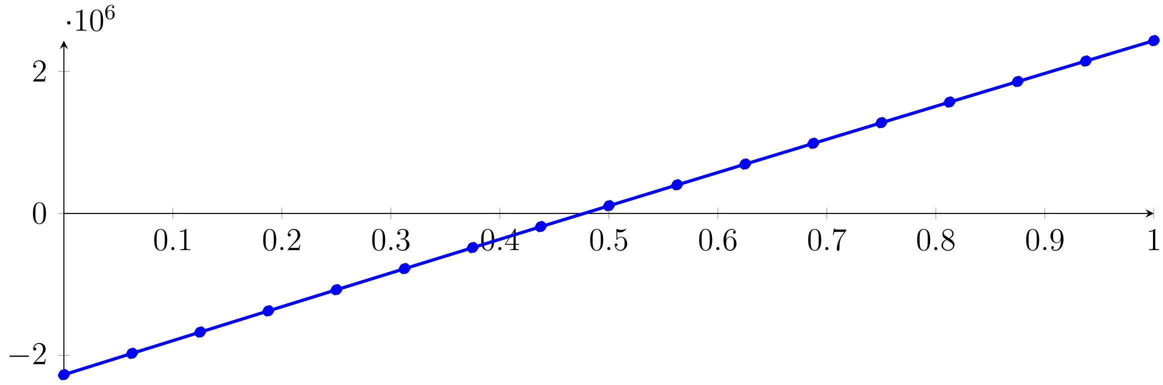
$$[0.323946, 0.343615]$$

Longest intersection interval: 0.0196686
$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

236.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

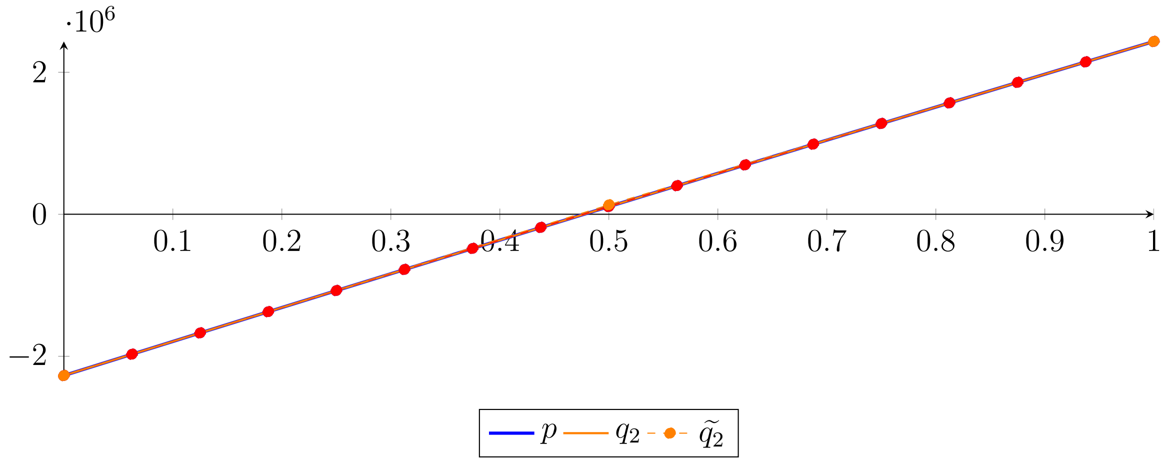
$$p = -5.0162\cdot 10^{-28}X^{16} - 1.14383\cdot 10^{-24}X^{15} - 1.11125\cdot 10^{-21}X^{14} - 5.84096\cdot 10^{-19}X^{13} - 1.684\cdot 10^{-16}X^{12} - 1.94707\cdot 10^{-14}X^{11} + 2.73227\cdot 10^{-12}X^{10} + 1.09727\cdot 10^{-09}X^{9} + 6.37314\cdot 10^{-08}X^{8} - 1.68645\cdot 10^{-05}X^{7} - 0.00205892X^{6} + 0.132721X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874\cdot 10^{6}X - 2.2715\cdot 10^{6}$$
$$= -2.2715\cdot 10^{6}B_{0,16}(X) - 1.97096\cdot 10^{6}B_{1,16}(X) - 1.67127\cdot 10^{6}B_{2,16}(X) - 1.37244\cdot 10^{6}B_{3,16}(X) - 1.07448\cdot 10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806\cdot 10^{6}B_{12,16}(X) + 1.5682\cdot 10^{6}B_{13,16}(X) + 1.85747\cdot 10^{6}B_{14,16}(X) + 2.14586\cdot 10^{6}B_{15,16}(X) + 2.43338\cdot 10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265X^{2} + 4.80923\cdot 10^{6}X - 2.27154\cdot 10^{6}$$
$$= -2.27154\cdot 10^{6}B_{0,2} + 133069B_{1,2} + 2.43342\cdot 10^{6}B_{2,2}$$

$$\widetilde{q}_2 = -8.18461\cdot 10^{-291}X^{16} + 4.13745\cdot 10^{-290}X^{15} - 5.8254\cdot 10^{-290}X^{14} - 6.80919\cdot 10^{-290}X^{13} + 3.48059\cdot 10^{-289}X^{12} - 5.47245\cdot 10^{-289}X^{11} + 4.8261\cdot 10^{-289}X^{10} - 2.67867\cdot 10^{-289}X^{9} + 1.00314\cdot 10^{-289}X^{8} - 2.84766\cdot 10^{-290}X^{7} + 6.85072\cdot 10^{-291}X^{6} - 1.12501\cdot 10^{-291}X^{5} + 1.24731\cdot 10^{-293}X^{4} + 2.25944\cdot 10^{-293}X^{3} - 104265X^{2} + 4.80923\cdot 10^{6}X - 2.27154\cdot 10^{6}$$
$$= -2.27154\cdot 10^{6}B_{0,16} - 1.97097\cdot 10^{6}B_{1,16} - 1.67126\cdot 10^{6}B_{2,16} - 1.37242\cdot 10^{6}B_{3,16} - 1.07445\cdot 10^{6}B_{4,16} - 777350B_{5,16} - 481118B_{6,16} - 185754B_{7,16} + 108740B_{8,16} + 402366B_{9,16} + 695123B_{10,16} + 987011B_{11,16} + 1.27803\cdot 10^{6}B_{12,16} + 1.56818\cdot 10^{6}B_{13,16} + 1.85746\cdot 10^{6}B_{14,16} + 2.14587\cdot 10^{6}B_{15,16} + 2.43342\cdot 10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.
Bounding polynomials M and m:

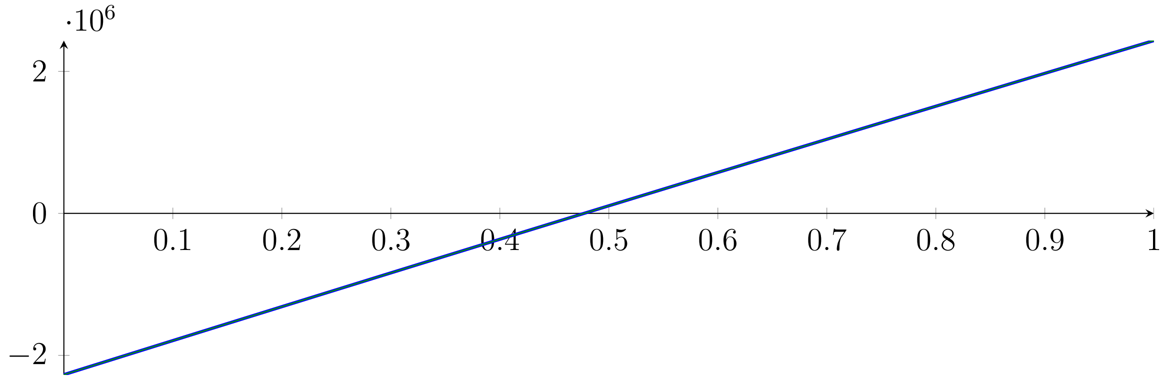
$$M = -104265X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

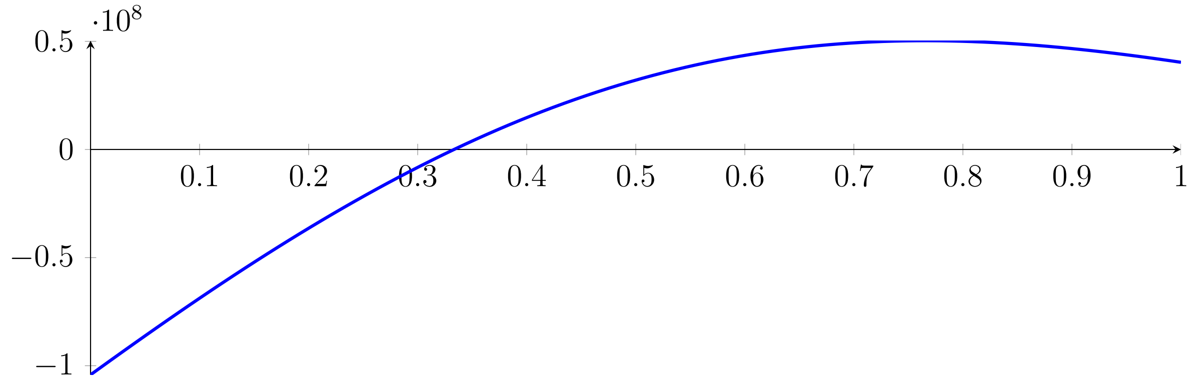
236.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

236.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

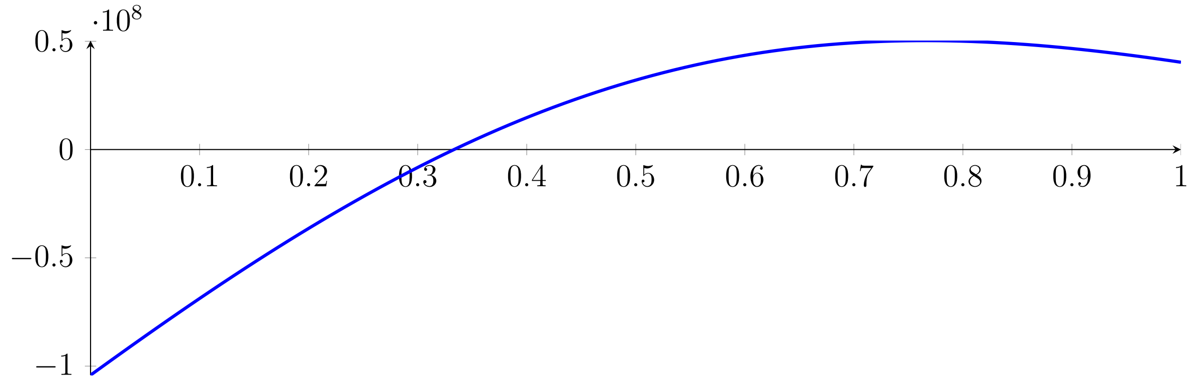
with precision $\varepsilon = 0.0001$.

237 Running CubeClip on f_{16} with epsilon 4

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} + 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

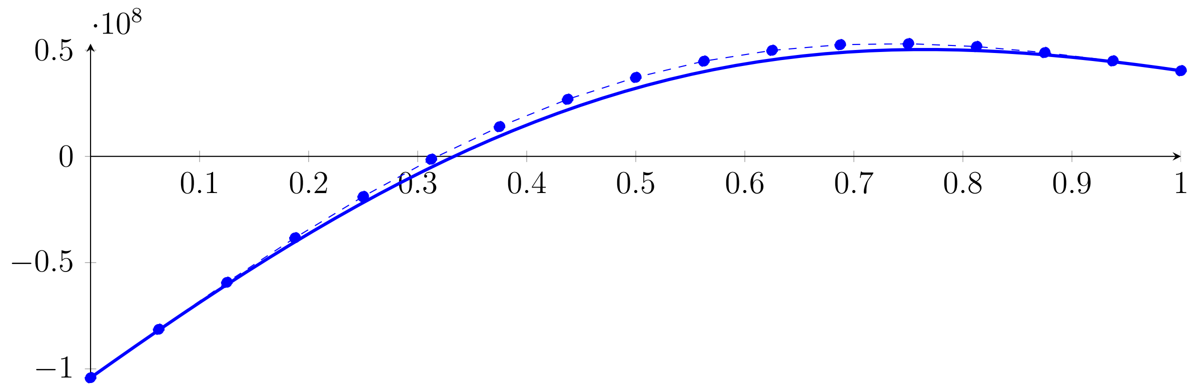
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} \\ &+ 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8} \end{aligned}$$



237.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

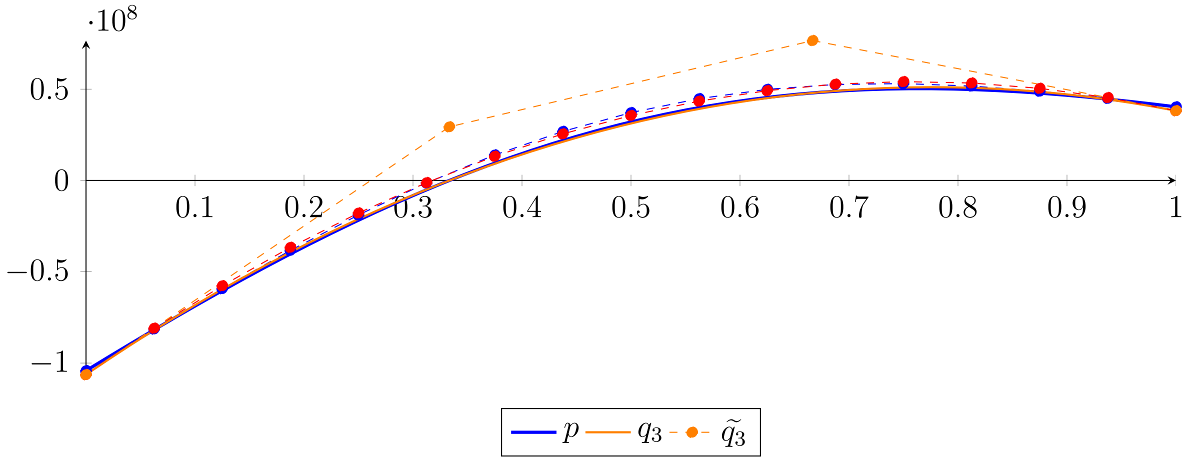
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} \\ &+ 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8} \\ = &-1.04167{\cdot}10^{8}B_{0,16}(X) - 8.13802{\cdot}10^{7}B_{1,16}(X) - 5.92882{\cdot}10^{7}B_{2,16}(X) - 3.83203 \\ &{\cdot}10^{7}B_{3,16}(X) - 1.88844{\cdot}10^{7}B_{4,16}(X) - 1.34837{\cdot}10^{6}B_{5,16}(X) + 1.39781{\cdot}10^{7}B_{6,16}(X) \\ &+ 2.68604{\cdot}10^{7}B_{7,16}(X) + 3.71532{\cdot}10^{7}B_{8,16}(X) + 4.48105{\cdot}10^{7}B_{9,16}(X) + 4.98901 \\ &{\cdot}10^{7}B_{10,16}(X) + 5.25504{\cdot}10^{7}B_{11,16}(X) + 5.30407{\cdot}10^{7}B_{12,16}(X) + 5.16832{\cdot}10^{7}B_{13,16}(X) \\ &+ 4.88488{\cdot}10^{7}B_{14,16}(X) + 4.49297{\cdot}10^{7}B_{15,16}(X) + 4.03108{\cdot}10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.06335{\cdot}10^8$$
$$= -1.06335{\cdot}10^8 B_{0,3} + 2.93558{\cdot}10^7 B_{1,3} + 7.66068{\cdot}10^7 B_{2,3} + 3.81764{\cdot}10^7 B_{3,3}$$

$$\widetilde{q}_3 = 1.89955{\cdot}10^{-288} X^{16} - 1.4851{\cdot}10^{-287} X^{15} + 5.12896{\cdot}10^{-287} X^{14} - 1.02224{\cdot}10^{-286} X^{13} + 1.29027{\cdot}10^{-286} X^{12} - 1.07058{\cdot}10^{-286} X^{11} + 6.00739{\cdot}10^{-287} X^{10} - 2.54352{\cdot}10^{-287} X^9 + 1.12089{\cdot}10^{-287} X^8 - 5.8673{\cdot}10^{-288} X^7 + 2.52637{\cdot}10^{-288} X^6 - 6.79624{\cdot}10^{-289} X^5 + 9.36341{\cdot}10^{-290} X^4 + 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.06335{\cdot}10^8$$
$$= -1.06335{\cdot}10^8 B_{0,16} - 8.08928{\cdot}10^7 B_{1,16} - 5.76618{\cdot}10^7 B_{2,16} - 3.66369{\cdot}10^7 B_{3,16} - 1.78131{\cdot}10^7 B_{4,16} - 1.18551{\cdot}10^6 B_{5,16} + 1.32508{\cdot}10^7 B_{6,16} + 2.55007{\cdot}10^7 B_{7,16} + 3.55692{\cdot}10^7 B_{8,16} + 4.34611{\cdot}10^7 B_{9,16} + 4.91815{\cdot}10^7 B_{10,16} + 5.27353{\cdot}10^7 B_{11,16} + 5.41273{\cdot}10^7 B_{12,16} + 5.33624{\cdot}10^7 B_{13,16} + 5.04457{\cdot}10^7 B_{14,16} + 4.53821{\cdot}10^7 B_{15,16} + 3.81764{\cdot}10^7 B_{16,16}$$



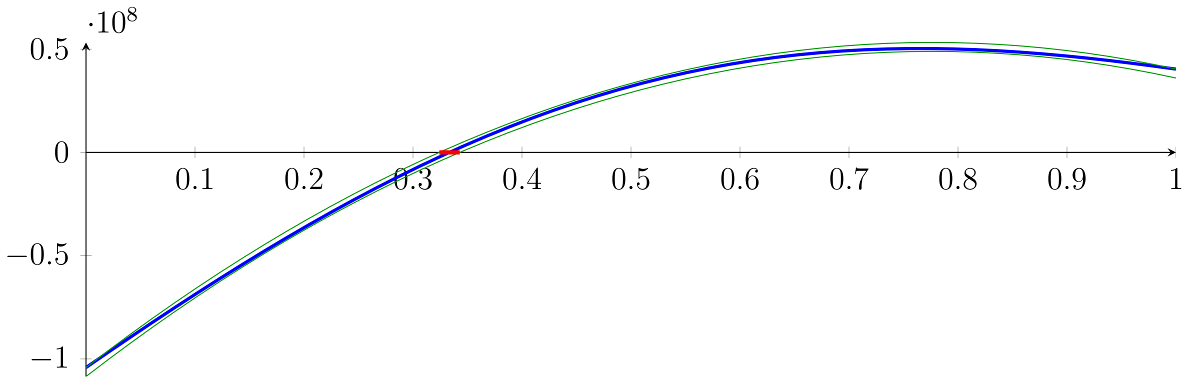
The maximum difference of the Bézier coefficients is $\delta = 2.16806{\cdot}10^6$.
Bounding polynomials M and m:

$$M = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.04167{\cdot}10^8$$
$$m = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.08503{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



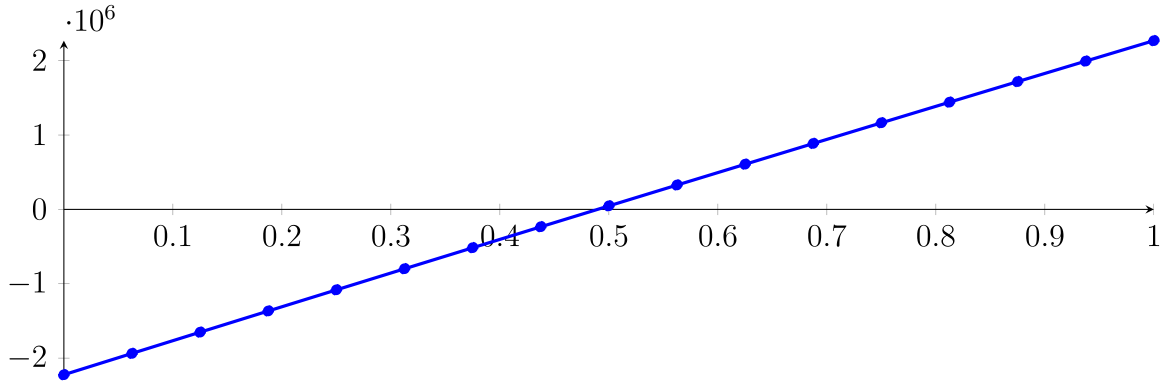
$$[0.324143, 0.342913]$$

Longest intersection interval: 0.0187703
$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

237.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

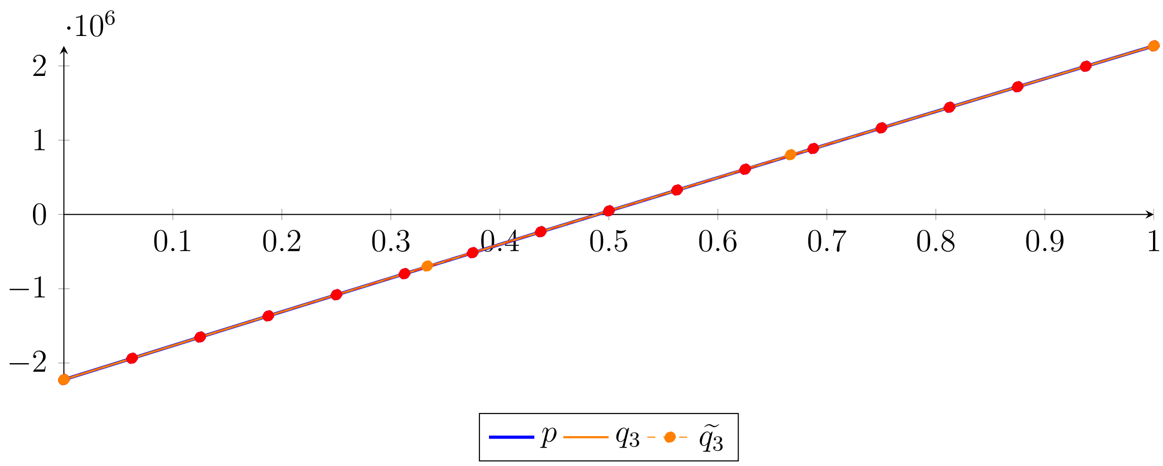
$$p = -2.37433 \cdot 10^{-28} X^{16} - 5.67363 \cdot 10^{-25} X^{15} - 5.77631 \cdot 10^{-22} X^{14} - 3.1818 \cdot 10^{-19} X^{13} - 9.6142 \cdot 10^{-17} X^{12} - 1.16549 \cdot 10^{-14} X^{11} + 1.71065 \cdot 10^{-12} X^{10} + 7.20611 \cdot 10^{-10} X^{9} + 4.39147 \cdot 10^{-08} X^{8} - 1.21542 \cdot 10^{-05} X^{7} - 0.00155624 X^{6} + 0.10496 X^{5} + 18.6215 X^{4} - 738.202 X^{3} - 93855.7 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$
$$= -2.22335 \cdot 10^{6} B_{0,16}(X) - 1.93666 \cdot 10^{6} B_{1,16}(X) - 1.65074 \cdot 10^{6} B_{2,16}(X) - 1.36561 \cdot 10^{6} B_{3,16}(X) - 1.08127 \cdot 10^{6} B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^{6} B_{12,16}(X) + 1.44233 \cdot 10^{6} B_{13,16}(X) + 1.71876 \cdot 10^{6} B_{14,16}(X) + 1.99439 \cdot 10^{6} B_{15,16}(X) + 2.26922 \cdot 10^{6} B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -700.673 X^{3} - 93879.9 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$
$$= -2.22335 \cdot 10^{6} B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^{6} B_{3,3}$$

$$\widetilde{q}_3 = 4.41595 \cdot 10^{-291} X^{16} - 1.48598 \cdot 10^{-290} X^{15} - 2.81102 \cdot 10^{-290} X^{14} + 2.65129 \cdot 10^{-289} X^{13} - 7.34593 \cdot 10^{-289} X^{12} + 1.17654 \cdot 10^{-288} X^{11} - 1.23119 \cdot 10^{-288} X^{10} + 8.63493 \cdot 10^{-289} X^{9} - 3.92909 \cdot 10^{-289} X^{8} + 1.01265 \cdot 10^{-289} X^{7} - 6.13644 \cdot 10^{-291} X^{6} - 3.92664 \cdot 10^{-291} X^{5} + 9.04488 \cdot 10^{-292} X^{4} - 700.673 X^{3} - 93879.9 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$
$$= -2.22335 \cdot 10^{6} B_{0,16} - 1.93666 \cdot 10^{6} B_{1,16} - 1.65074 \cdot 10^{6} B_{2,16} - 1.36561 \cdot 10^{6} B_{3,16} - 1.08126 \cdot 10^{6} B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^{6} B_{12,16} + 1.44233 \cdot 10^{6} B_{13,16} + 1.71876 \cdot 10^{6} B_{14,16} + 1.99439 \cdot 10^{6} B_{15,16} + 2.26922 \cdot 10^{6} B_{16,16}$$



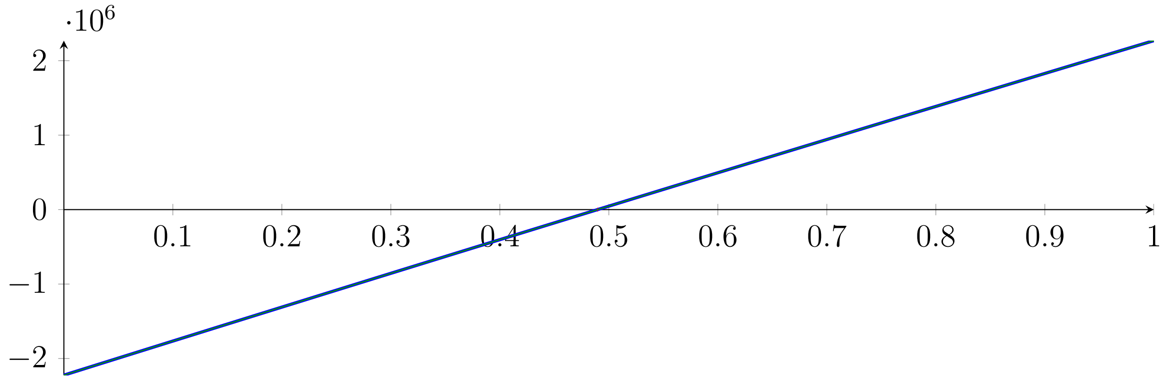
The maximum difference of the Bézier coefficients is $\delta = 0.270074$.
Bounding polynomials M and m:

$$M = -700.673X^3 - 93879.9X^2 + 4.58715\cdot10^6X - 2.22335\cdot10^6$$
$$m = -700.673X^3 - 93879.9X^2 + 4.58715\cdot10^6X - 2.22335\cdot10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174\cdot10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

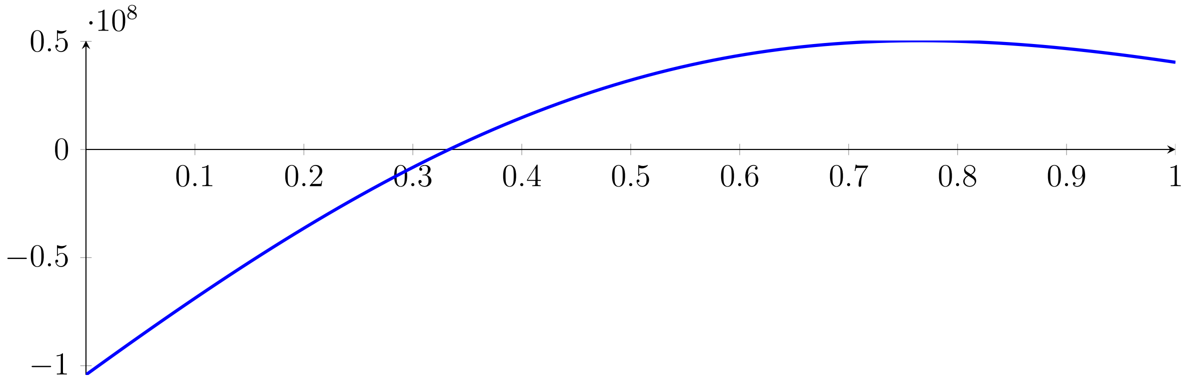
237.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

237.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

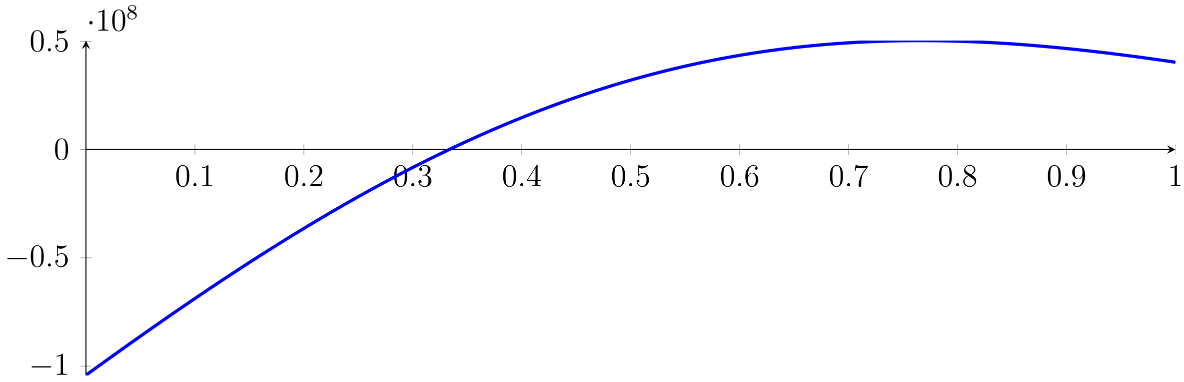
with precision $\varepsilon = 0.0001$.

238 Running `BezClip` on f_{16} with epsilon 8

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called `BezClip` with input polynomial on interval $[0,1]$:

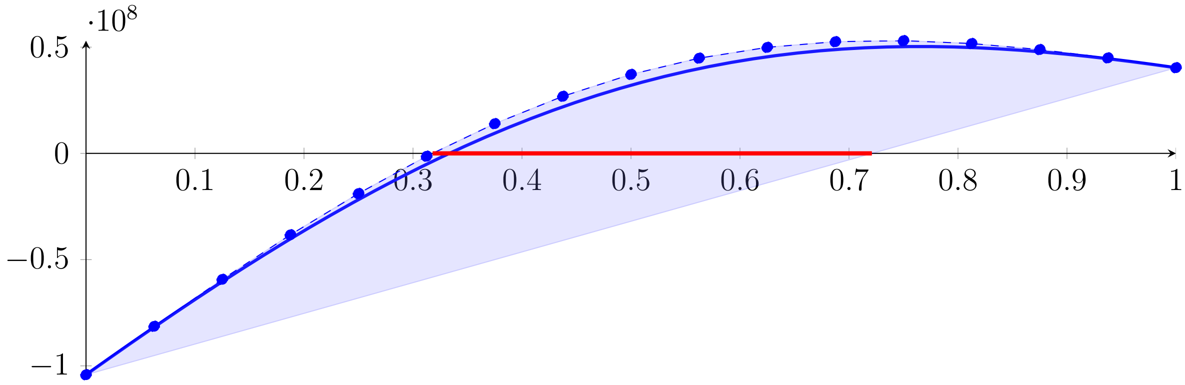
$$\begin{aligned}p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10}\\ &+ 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5}\\ &+ 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}\end{aligned}$$



238.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10}\\ &+ 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5}\\ &+ 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}\\ = &-1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203\\ &\cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X)\\ &+ 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901\\ &\cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X)\\ &+ 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

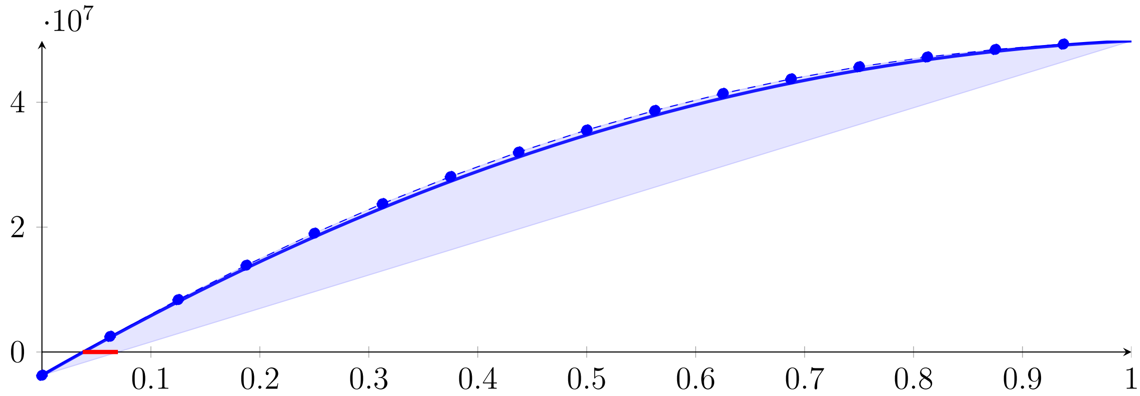
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

238.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -4.83858\cdot 10^{-07} X^{16} - 5.37355\cdot 10^{-05} X^{15} - 0.00254146 X^{14} - 0.064977 X^{13} - 0.909205 X^{12} \\ & - 5.03924 X^{11} + 36.4549 X^{10} + 692.921 X^{9} + 1886.97 X^{8} - 25792 X^{7} - 149671 X^{6} + 492605 X^{5} \\ & + 3.91945\cdot 10^{6} X^{4} - 7.54561\cdot 10^{6} X^{3} - 4.29225\cdot 10^{7} X^{2} + 9.97982\cdot 10^{7} X - 3.73501\cdot 10^{6} \\ = & -3.73501\cdot 10^{6} B_{0,16}(X) + 2.50237\cdot 10^{6} B_{1,16}(X) + 8.38207\cdot 10^{6} B_{2,16}(X) + 1.38906 \\ & \cdot 10^{7} B_{3,16}(X) + 1.90167\cdot 10^{7} B_{4,16}(X) + 2.37512\cdot 10^{7} B_{5,16}(X) + 2.80875\cdot 10^{7} B_{6,16}(X) \\ & + 3.20213\cdot 10^{7} B_{7,16}(X) + 3.55508\cdot 10^{7} B_{8,16}(X) + 3.86768\cdot 10^{7} B_{9,16}(X) + 4.14025 \\ & \cdot 10^{7} B_{10,16}(X) + 4.37336\cdot 10^{7} B_{11,16}(X) + 4.56783\cdot 10^{7} B_{12,16}(X) + 4.7247\cdot 10^{7} B_{13,16}(X) \\ & + 4.84524\cdot 10^{7} B_{14,16}(X) + 4.93093\cdot 10^{7} B_{15,16}(X) + 4.98343\cdot 10^{7} B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

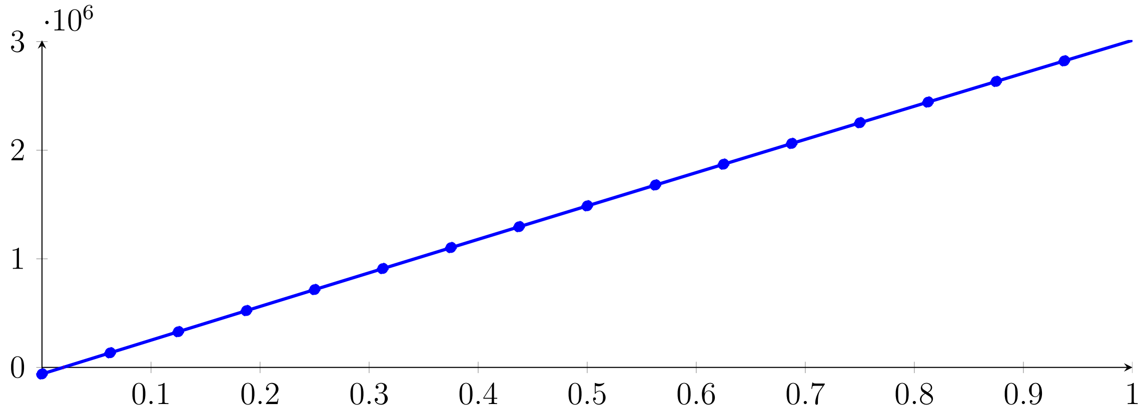
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

238.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -6.78234\cdot 10^{-31} X^{16} - 2.34473\cdot 10^{-27} X^{15} - 3.45581\cdot 10^{-24} X^{14} - 2.75921\cdot 10^{-21} X^{13} - 1.21242\cdot 10^{-18} X^{12} \\ & - 2.17572\cdot 10^{-16} X^{11} + 4.23544\cdot 10^{-14} X^{10} + 2.70025\cdot 10^{-11} X^{9} + 2.51306\cdot 10^{-09} X^{8} - 9.23474\cdot 10^{-07} X^{7} \\ & - 0.000177459 X^{6} + 0.0161038 X^{5} + 4.36155 X^{4} - 234.216 X^{3} - 45622.2 X^{2} + 3.11845\cdot 10^{6} X - 60508.5 \\ = & -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ & + 910202 B_{5,16}(X) + 1.1032\cdot 10^{6} B_{6,16}(X) + 1.29582\cdot 10^{6} B_{7,16}(X) + 1.48805\cdot 10^{6} B_{8,16}(X) \\ & + 1.6799\cdot 10^{6} B_{9,16}(X) + 1.87137\cdot 10^{6} B_{10,16}(X) + 2.06245\cdot 10^{6} B_{11,16}(X) + 2.25315\cdot 10^{6} B_{12,16}(X) \\ & + 2.44346\cdot 10^{6} B_{13,16}(X) + 2.63339\cdot 10^{6} B_{14,16}(X) + 2.82293\cdot 10^{6} B_{15,16}(X) + 3.01209\cdot 10^{6} B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

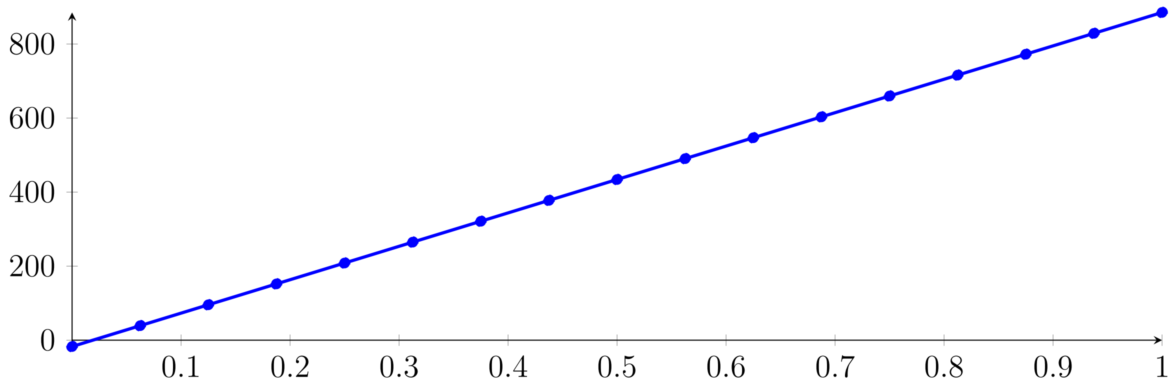
$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

238.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-1.65599\cdot 10^{-87}X^{16} - 1.97733\cdot 10^{-80}X^{15} - 1.00659\cdot 10^{-73}X^{14} - 2.77601\cdot 10^{-67}X^{13} \\ &- 4.21367\cdot 10^{-61}X^{12} - 2.61333\cdot 10^{-55}X^{11} + 1.75275\cdot 10^{-49}X^{10} + 3.8646\cdot 10^{-43}X^{9} \\ &+ 1.2441\cdot 10^{-37}X^{8} - 1.57525\cdot 10^{-31}X^{7} - 1.04661\cdot 10^{-25}X^{6} + 3.27355\cdot 10^{-20}X^{5} \\ &+ 3.06701\cdot 10^{-14}X^{4} - 5.67777\cdot 10^{-09}X^{3} - 0.00382618X^{2} + 902.448X - 17.178 \\ = &-17.178B_{0,16}(X) + 39.225B_{1,16}(X) + 95.6279B_{2,16}(X) + 152.031B_{3,16}(X) + 208.434B_{4,16}(X) \\ &+ 264.837B_{5,16}(X) + 321.24B_{6,16}(X) + 377.642B_{7,16}(X) + 434.045B_{8,16}(X) \\ &+ 490.448B_{9,16}(X) + 546.851B_{10,16}(X) + 603.253B_{11,16}(X) + 659.656B_{12,16}(X) \\ &+ 716.059B_{13,16}(X) + 772.461B_{14,16}(X) + 828.864B_{15,16}(X) + 885.266B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045\cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

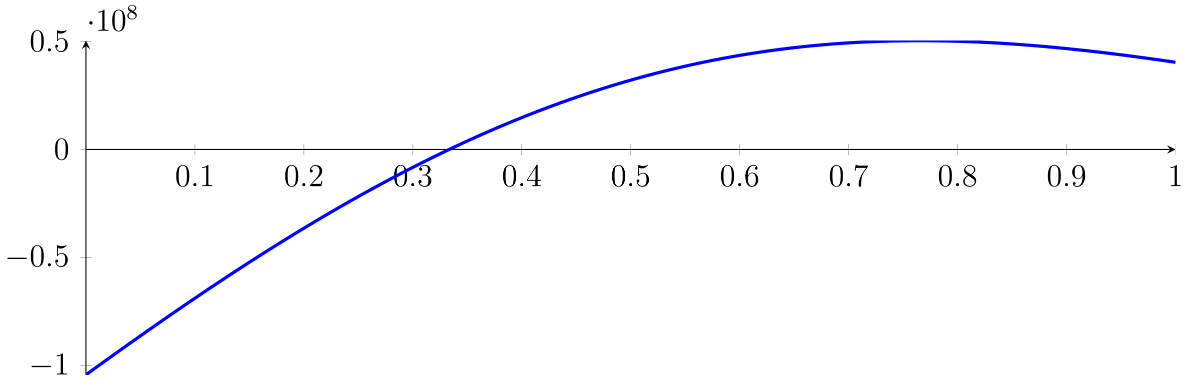
238.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

238.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

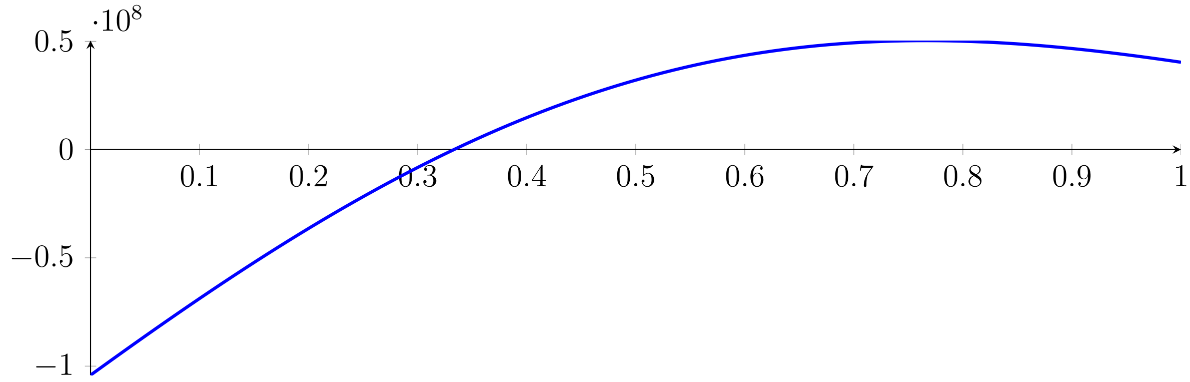
with precision $\varepsilon = 1\cdot10^{-08}$.

239 Running QuadClip on f_{16} with epsilon 8

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called QuadClip with input polynomial on interval $[0,1]$:

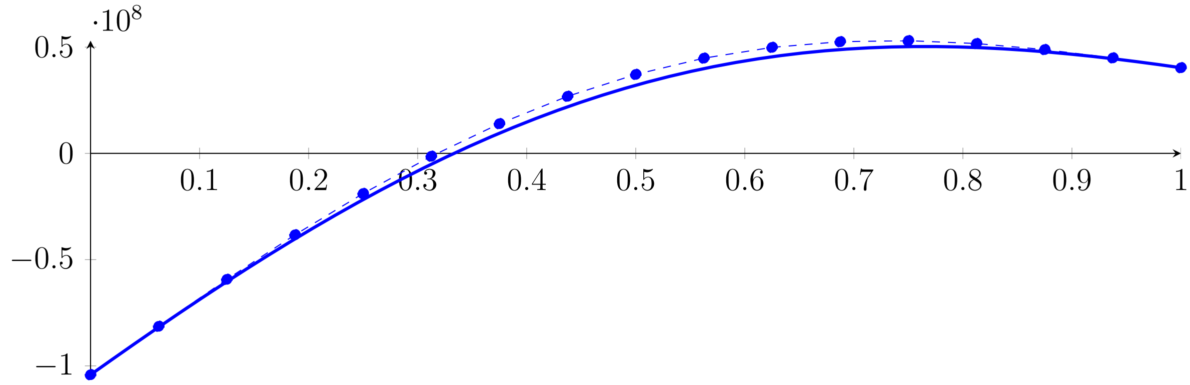
$$\begin{aligned}p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10}\\ &+ 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5}\\ &+ 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}\end{aligned}$$



239.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

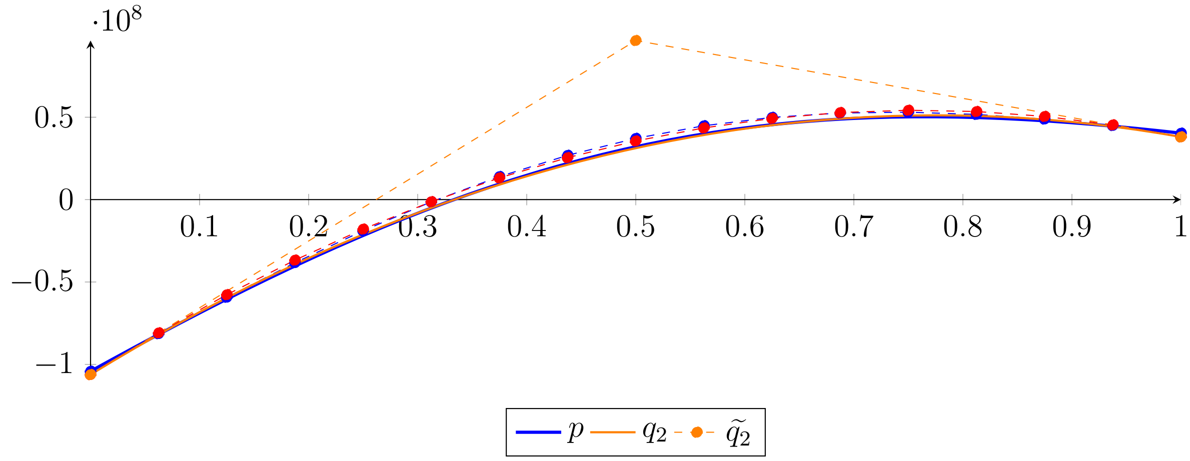
$$\begin{aligned}p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10}\\ &+ 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5}\\ &+ 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}\\ = &-1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203\\ &\cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X)\\ &+ 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901\\ &\cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X)\\ &+ 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X)\end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181\cdot 10^8 X^2 + 4.05417\cdot 10^8 X - 1.06197\cdot 10^8$$
$$= -1.06197\cdot 10^8 B_{0,2} + 9.65115\cdot 10^7 B_{1,2} + 3.80384\cdot 10^7 B_{2,2}$$

$$\begin{aligned}\widetilde{q_2} = {} & 3.74473\cdot 10^{-288} X^{16} - 3.08889\cdot 10^{-287} X^{15} + 1.17443\cdot 10^{-286} X^{14} - 2.71603\cdot 10^{-286} X^{13} \\ & + 4.23554\cdot 10^{-286} X^{12} - 4.66107\cdot 10^{-286} X^{11} + 3.6845\cdot 10^{-286} X^{10} - 2.09811\cdot 10^{-286} X^9 \\ & + 8.58708\cdot 10^{-287} X^8 - 2.54063\cdot 10^{-287} X^7 + 5.57852\cdot 10^{-288} X^6 - 9.21534\cdot 10^{-289} X^5 \\ & + 1.02122\cdot 10^{-289} X^4 - 5.95624\cdot 10^{-291} X^3 - 2.61181\cdot 10^8 X^2 + 4.05417\cdot 10^8 X - 1.06197\cdot 10^8 \\ = {} & -1.06197\cdot 10^8 B_{0,16} - 8.08583\cdot 10^7 B_{1,16} - 5.76963\cdot 10^7 B_{2,16} - 3.67107\cdot 10^7 B_{3,16} - 1.79017 \\ & \cdot 10^7 B_{4,16} - 1.26924\cdot 10^6 B_{5,16} + 1.31867\cdot 10^7 B_{6,16} + 2.54662\cdot 10^7 B_{7,16} + 3.55692\cdot 10^7 B_{8,16} \\ & + 4.34956\cdot 10^7 B_{9,16} + 4.92456\cdot 10^7 B_{10,16} + 5.2819\cdot 10^7 B_{11,16} + 5.42159\cdot 10^7 B_{12,16} \\ & + 5.34363\cdot 10^7 B_{13,16} + 5.04802\cdot 10^7 B_{14,16} + 4.53476\cdot 10^7 B_{15,16} + 3.80384\cdot 10^7 B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233\cdot 10^6$.

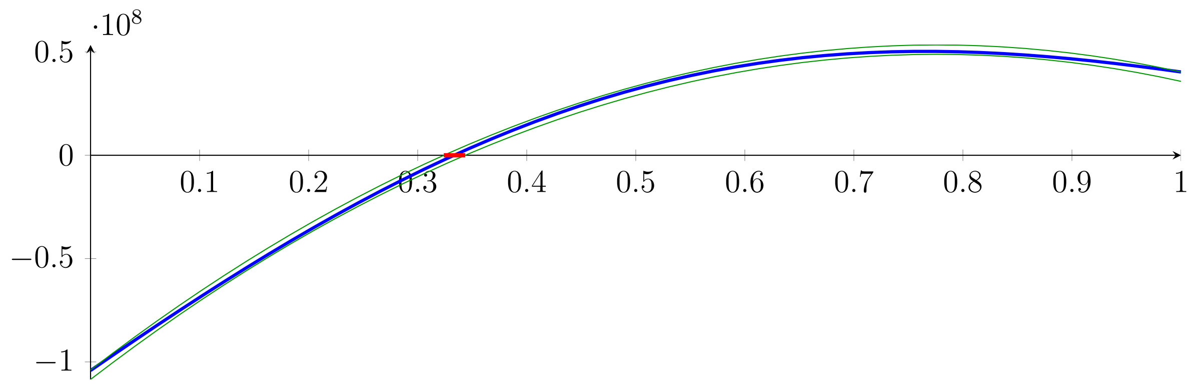
Bounding polynomials M and m:

$$M = -2.61181\cdot 10^8 X^2 + 4.05417\cdot 10^8 X - 1.03924\cdot 10^8$$
$$m = -2.61181\cdot 10^8 X^2 + 4.05417\cdot 10^8 X - 1.08469\cdot 10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

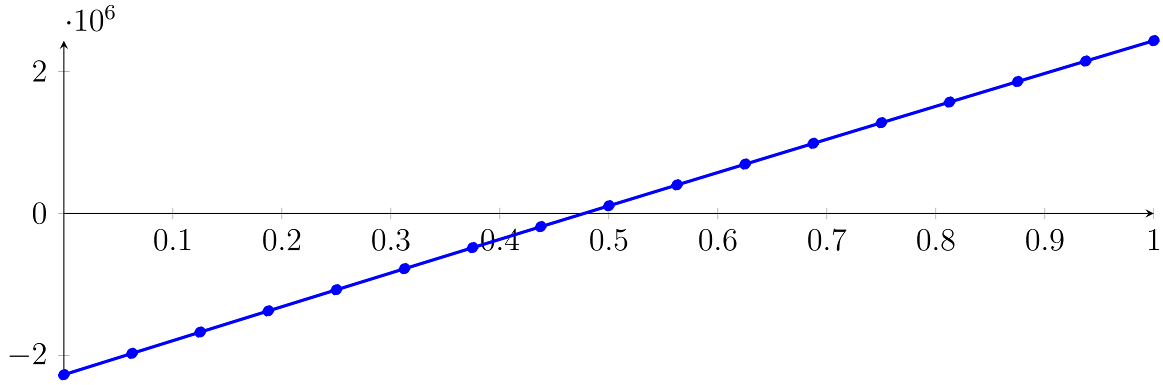
Longest intersection interval: 0.0196686

$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

239.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

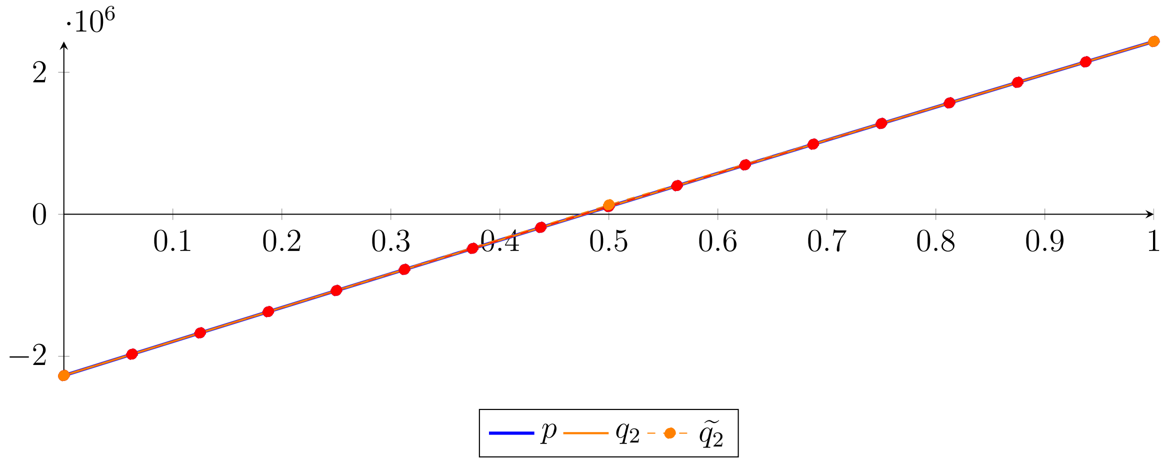
$$p = -5.0162\cdot10^{-28}X^{16} - 1.14383\cdot10^{-24}X^{15} - 1.11125\cdot10^{-21}X^{14} - 5.84096\cdot10^{-19}X^{13} - 1.684\cdot10^{-16}X^{12} - 1.94707\cdot10^{-14}X^{11} + 2.73227\cdot10^{-12}X^{10} + 1.09727\cdot10^{-09}X^{9} + 6.37314\cdot10^{-08}X^{8} - 1.68645\cdot10^{-05}X^{7} - 0.00205892X^{6} + 0.132721X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874\cdot10^{6}X - 2.2715\cdot10^{6}$$
$$= -2.2715\cdot10^{6}B_{0,16}(X) - 1.97096\cdot10^{6}B_{1,16}(X) - 1.67127\cdot10^{6}B_{2,16}(X) - 1.37244\cdot10^{6}B_{3,16}(X) - 1.07448\cdot10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806\cdot10^{6}B_{12,16}(X) + 1.5682\cdot10^{6}B_{13,16}(X) + 1.85747\cdot10^{6}B_{14,16}(X) + 2.14586\cdot10^{6}B_{15,16}(X) + 2.43338\cdot10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265X^{2} + 4.80923\cdot10^{6}X - 2.27154\cdot10^{6}$$
$$= -2.27154\cdot10^{6}B_{0,2} + 133069B_{1,2} + 2.43342\cdot10^{6}B_{2,2}$$

$$\widetilde{q}_2 = -8.18461\cdot10^{-291}X^{16} + 4.13745\cdot10^{-290}X^{15} - 5.8254\cdot10^{-290}X^{14} - 6.80919\cdot10^{-290}X^{13} + 3.48059\cdot10^{-289}X^{12} - 5.47245\cdot10^{-289}X^{11} + 4.8261\cdot10^{-289}X^{10} - 2.67867\cdot10^{-289}X^{9} + 1.00314\cdot10^{-289}X^{8} - 2.84766\cdot10^{-290}X^{7} + 6.85072\cdot10^{-291}X^{6} - 1.12501\cdot10^{-291}X^{5} + 1.24731\cdot10^{-293}X^{4} + 2.25944\cdot10^{-293}X^{3} - 104265X^{2} + 4.80923\cdot10^{6}X - 2.27154\cdot10^{6}$$
$$= -2.27154\cdot10^{6}B_{0,16} - 1.97097\cdot10^{6}B_{1,16} - 1.67126\cdot10^{6}B_{2,16} - 1.37242\cdot10^{6}B_{3,16} - 1.07445\cdot10^{6}B_{4,16} - 777350B_{5,16} - 481118B_{6,16} - 185754B_{7,16} + 108740B_{8,16} + 402366B_{9,16} + 695123B_{10,16} + 987011B_{11,16} + 1.27803\cdot10^{6}B_{12,16} + 1.56818\cdot10^{6}B_{13,16} + 1.85746\cdot10^{6}B_{14,16} + 2.14587\cdot10^{6}B_{15,16} + 2.43342\cdot10^{6}B_{16,16}$$



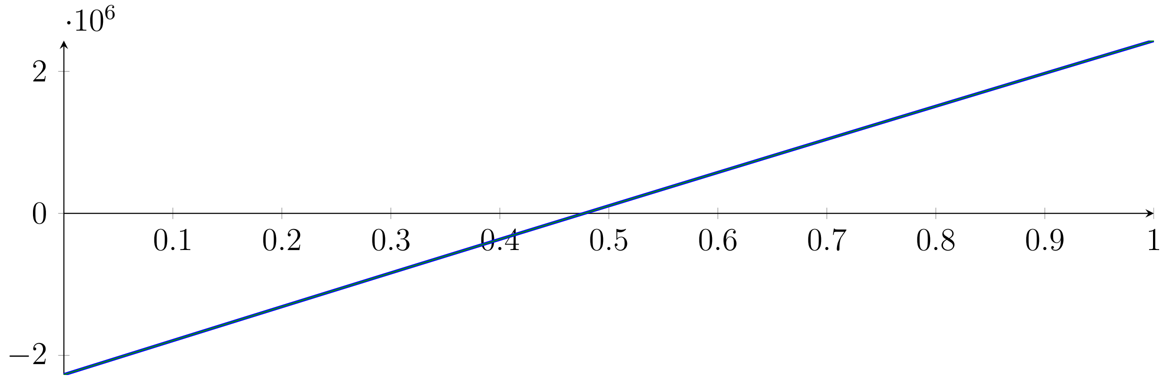
The maximum difference of the Bézier coefficients is $\delta = 40.5742$.
Bounding polynomials M and m:

$$M = -104265X^2 + 4.80923\cdot10^6X - 2.2715\cdot10^6$$
$$m = -104265X^2 + 4.80923\cdot10^6X - 2.27159\cdot10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



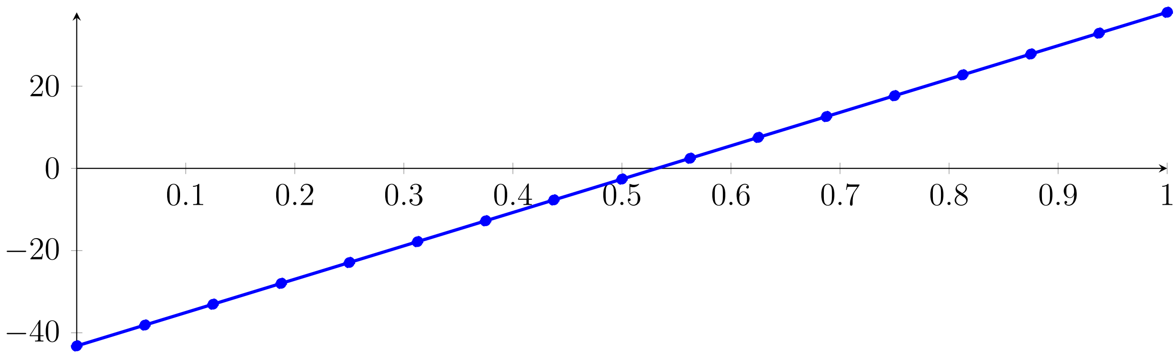
$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301\cdot10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

239.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

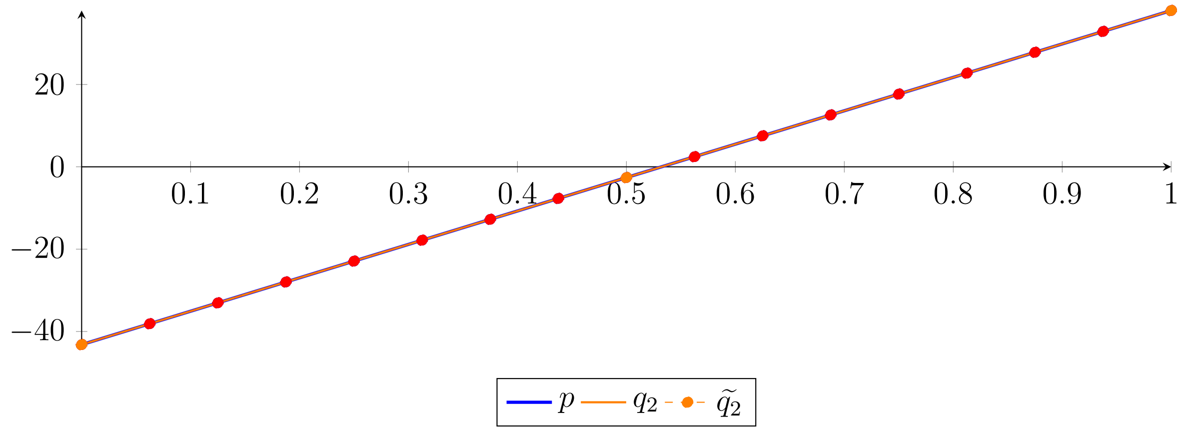
$$\begin{aligned}
p = & -3.02667\cdot10^{-104}X^{16} - 4.019\cdot10^{-96}X^{15} - 2.27522\cdot10^{-88}X^{14} - 6.97783\cdot10^{-81}X^{13} \\
& - 1.17785\cdot10^{-73}X^{12} - 8.12373\cdot10^{-67}X^{11} + 6.05916\cdot10^{-60}X^{10} + 1.48569\cdot10^{-52}X^{9} \\
& + 5.31875\cdot10^{-46}X^{8} - 7.48919\cdot10^{-39}X^{7} - 5.53349\cdot10^{-32}X^{6} + 1.92471\cdot10^{-25}X^{5} \\
& + 2.00536\cdot10^{-18}X^{4} - 4.12844\cdot10^{-12}X^{3} - 3.09388\cdot10^{-05}X^{2} + 81.1506X - 43.1911 \\
= & -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\
& - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X) \\
& + 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X) \\
& + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X)
\end{aligned}$$



Degree reduction and raising:

$$q_2 = -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$
$$= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2}$$

$$\widetilde{q}_2 = -1.88281 \cdot 10^{-295} X^{16} + 1.09893 \cdot 10^{-294} X^{15} - 2.26419 \cdot 10^{-294} X^{14} + 8.08223 \cdot 10^{-295} X^{13} + 4.92626 \cdot 10^{-294} X^{12} - 1.07605 \cdot 10^{-293} X^{11} + 1.11858 \cdot 10^{-293} X^{10} - 6.87288 \cdot 10^{-294} X^9 + 2.54873 \cdot 10^{-294} X^8 - 5.2305 \cdot 10^{-295} X^7 + 3.18923 \cdot 10^{-296} X^6 + 1.34092 \cdot 10^{-296} X^5 - 4.89549 \cdot 10^{-297} X^4 + 5.89947 \cdot 10^{-298} X^3 - 3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$
$$= -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} - 12.7597B_{6,16} - 7.68778B_{7,16} - 2.61587B_{8,16} + 2.45604B_{9,16} + 7.52795B_{10,16} + 12.5999B_{11,16} + 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.06422 \cdot 10^{-13}$.

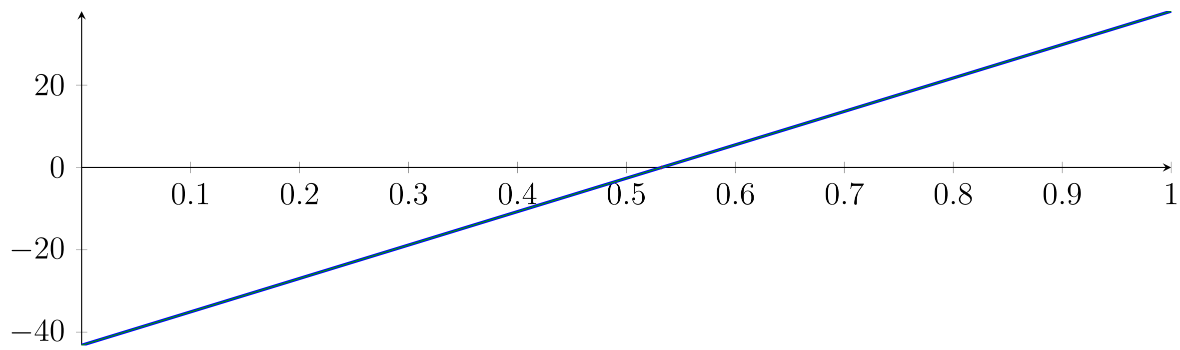
Bounding polynomials M and m:

$$M = -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$
$$m = -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$

Root of M and m:

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\} \qquad N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

Longest intersection interval: $5.08738 \cdot 10^{-15}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

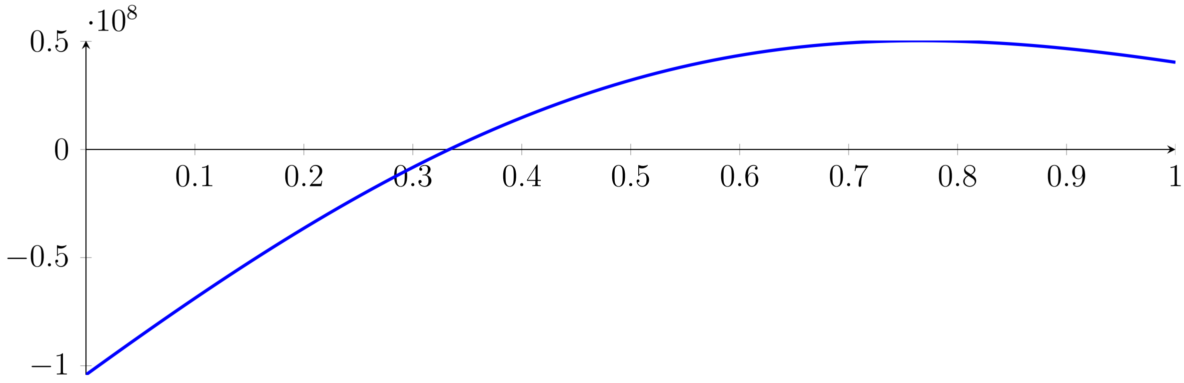
239.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

239.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} + 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

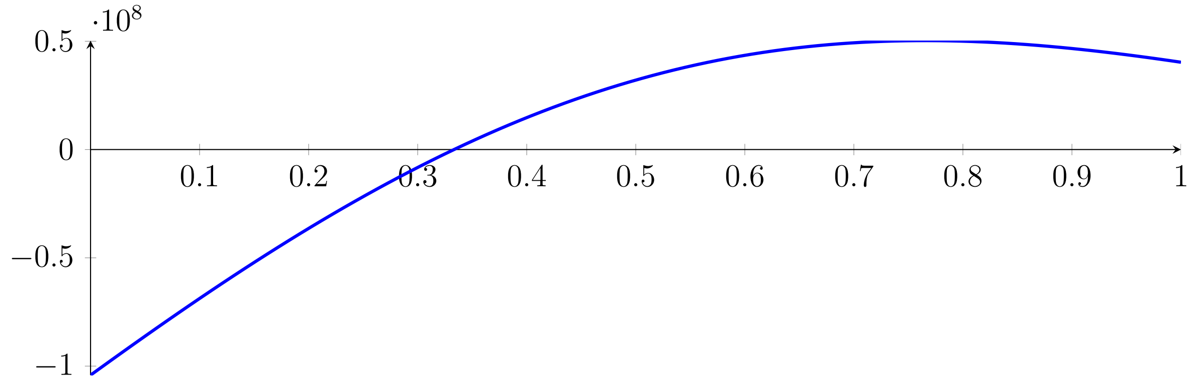
with precision $\varepsilon = 1{\cdot}10^{-08}$.

240 Running CubeClip on f_{16} with epsilon 8

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

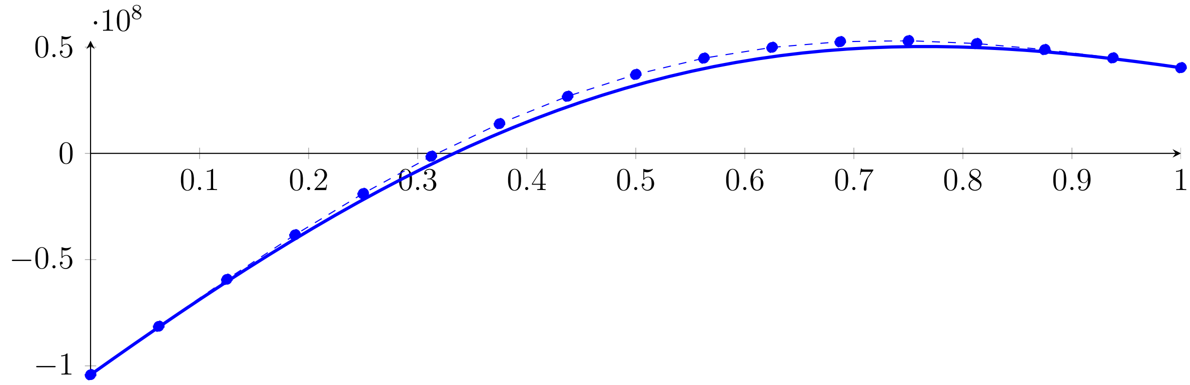
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} \\ &+ 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \end{aligned}$$



240.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

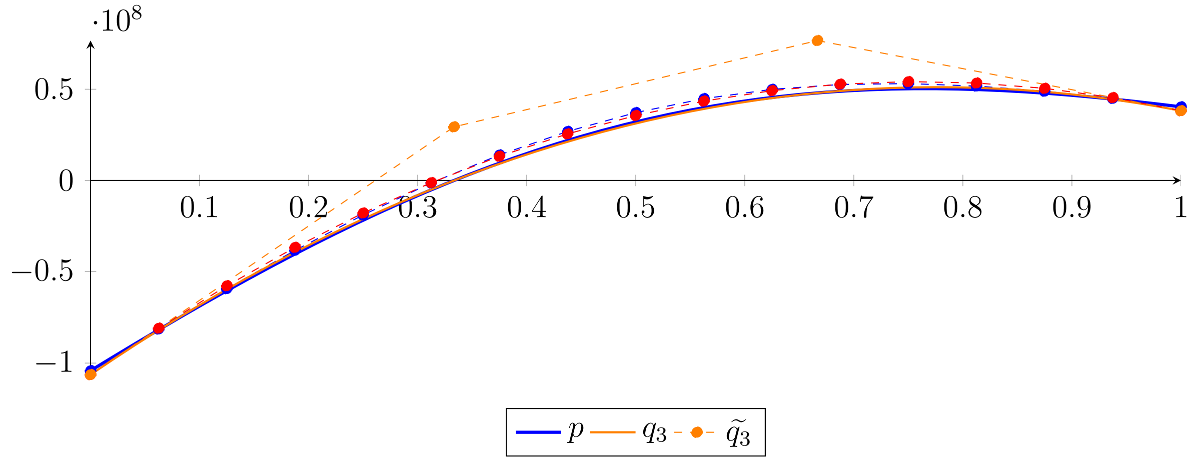
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} \\ &+ 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \\ = &-1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203 \\ &\cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X) \\ &+ 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901 \\ &\cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X) \\ &+ 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$
\begin{aligned}
q_3 &= 2.75812\cdot10^{6}X^{3} - 2.65319\cdot10^{8}X^{2} + 4.07072\cdot10^{8}X - 1.06335\cdot10^{8}\\
&= -1.06335\cdot10^{8}B_{0,3} + 2.93558\cdot10^{7}B_{1,3} + 7.66068\cdot10^{7}B_{2,3} + 3.81764\cdot10^{7}B_{3,3}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{q}_3 &= 1.89955\cdot10^{-288}X^{16} - 1.4851\cdot10^{-287}X^{15} + 5.12896\cdot10^{-287}X^{14} - 1.02224\cdot10^{-286}X^{13}\\
&\quad + 1.29027\cdot10^{-286}X^{12} - 1.07058\cdot10^{-286}X^{11} + 6.00739\cdot10^{-287}X^{10} - 2.54352\cdot10^{-287}X^{9}\\
&\quad + 1.12089\cdot10^{-287}X^{8} - 5.8673\cdot10^{-288}X^{7} + 2.52637\cdot10^{-288}X^{6} - 6.79624\cdot10^{-289}X^{5}\\
&\quad + 9.36341\cdot10^{-290}X^{4} + 2.75812\cdot10^{6}X^{3} - 2.65319\cdot10^{8}X^{2} + 4.07072\cdot10^{8}X - 1.06335\cdot10^{8}\\
&= -1.06335\cdot10^{8}B_{0,16} - 8.08928\cdot10^{7}B_{1,16} - 5.76618\cdot10^{7}B_{2,16} - 3.66369\cdot10^{7}B_{3,16} - 1.78131\\
&\quad \cdot10^{7}B_{4,16} - 1.18551\cdot10^{6}B_{5,16} + 1.32508\cdot10^{7}B_{6,16} + 2.55007\cdot10^{7}B_{7,16} + 3.55692\cdot10^{7}B_{8,16}\\
&\quad + 4.34611\cdot10^{7}B_{9,16} + 4.91815\cdot10^{7}B_{10,16} + 5.27353\cdot10^{7}B_{11,16} + 5.41273\cdot10^{7}B_{12,16}\\
&\quad + 5.33624\cdot10^{7}B_{13,16} + 5.04457\cdot10^{7}B_{14,16} + 4.53821\cdot10^{7}B_{15,16} + 3.81764\cdot10^{7}B_{16,16}
\end{aligned}
$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806\cdot10^{6}$.

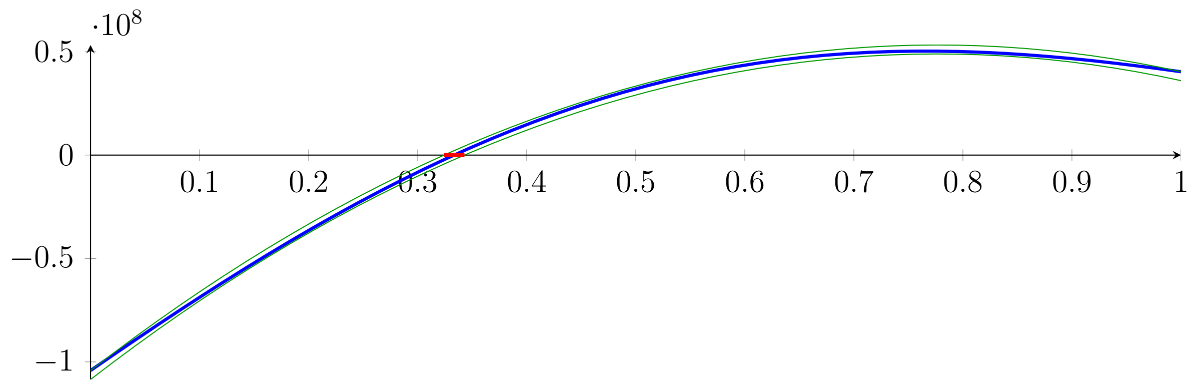
Bounding polynomials M and m:

$$
\begin{aligned}
M &= 2.75812\cdot10^{6}X^{3} - 2.65319\cdot10^{8}X^{2} + 4.07072\cdot10^{8}X - 1.04167\cdot10^{8}\\
m &= 2.75812\cdot10^{6}X^{3} - 2.65319\cdot10^{8}X^{2} + 4.07072\cdot10^{8}X - 1.08503\cdot10^{8}
\end{aligned}
$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

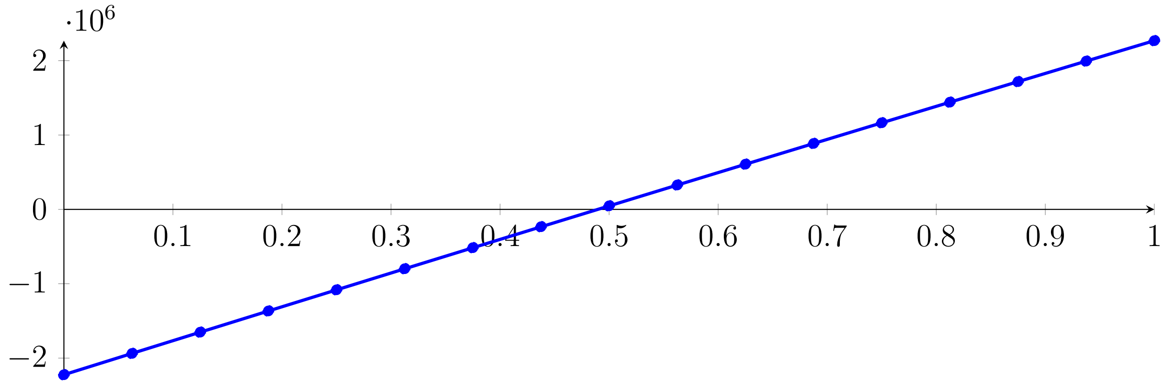
Longest intersection interval: 0.0187703

$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

240.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

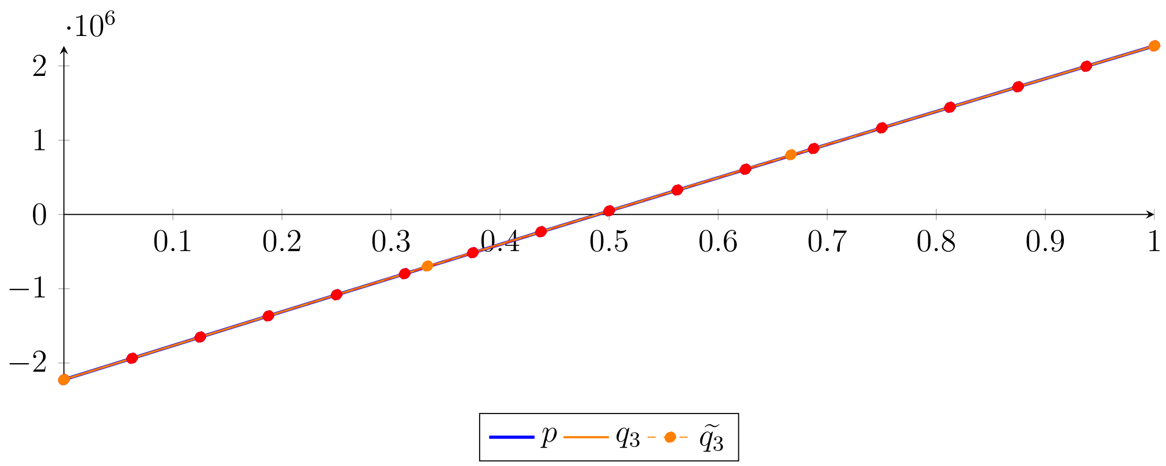
$$p = -2.37433\cdot10^{-28}X^{16} - 5.67363\cdot10^{-25}X^{15} - 5.77631\cdot10^{-22}X^{14} - 3.1818\cdot10^{-19}X^{13} - 9.6142\cdot10^{-17}X^{12} - 1.16549\cdot10^{-14}X^{11} + 1.71065\cdot10^{-12}X^{10} + 7.20611\cdot10^{-10}X^{9} + 4.39147\cdot10^{-08}X^{8} - 1.21542\cdot10^{-05}X^{7} - 0.00155624X^{6} + 0.10496X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,16}(X) - 1.93666\cdot10^{6}B_{1,16}(X) - 1.65074\cdot10^{6}B_{2,16}(X) - 1.36561\cdot10^{6}B_{3,16}(X) - 1.08127\cdot10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X) + 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot10^{6}B_{12,16}(X) + 1.44233\cdot10^{6}B_{13,16}(X) + 1.71876\cdot10^{6}B_{14,16}(X) + 1.99439\cdot10^{6}B_{15,16}(X) + 2.26922\cdot10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot10^{6}B_{3,3}$$

$$\widetilde{q_3} = 4.41595\cdot10^{-291}X^{16} - 1.48598\cdot10^{-290}X^{15} - 2.81102\cdot10^{-290}X^{14} + 2.65129\cdot10^{-289}X^{13} - 7.34593\cdot10^{-289}X^{12} + 1.17654\cdot10^{-288}X^{11} - 1.23119\cdot10^{-288}X^{10} + 8.63493\cdot10^{-289}X^{9} - 3.92909\cdot10^{-289}X^{8} + 1.01265\cdot10^{-289}X^{7} - 6.13644\cdot10^{-291}X^{6} - 3.92664\cdot10^{-291}X^{5} + 9.04488\cdot10^{-292}X^{4} - 700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,16} - 1.93666\cdot10^{6}B_{1,16} - 1.65074\cdot10^{6}B_{2,16} - 1.36561\cdot10^{6}B_{3,16} - 1.08126\cdot10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.4B_{8,16} + 328650B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot10^{6}B_{12,16} + 1.44233\cdot10^{6}B_{13,16} + 1.71876\cdot10^{6}B_{14,16} + 1.99439\cdot10^{6}B_{15,16} + 2.26922\cdot10^{6}B_{16,16}$$



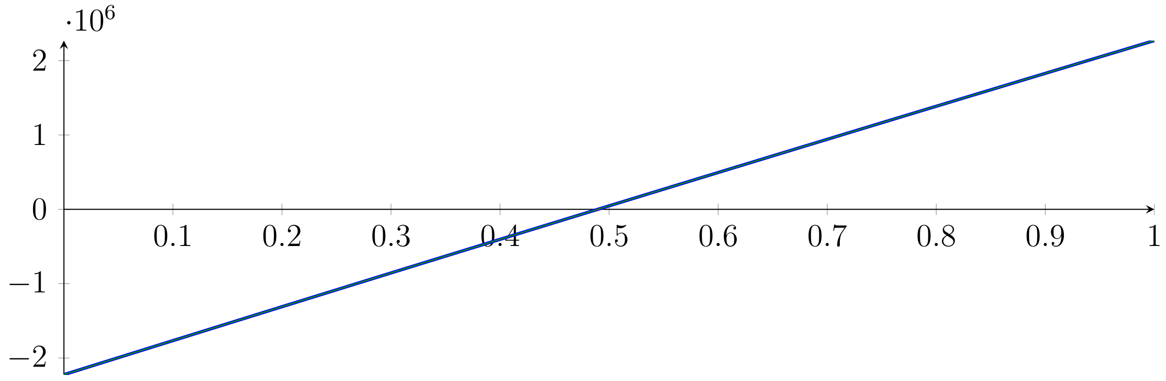
The maximum difference of the Bézier coefficients is $\delta = 0.270074$.
Bounding polynomials M and m:

$$M = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$
$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174 \cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

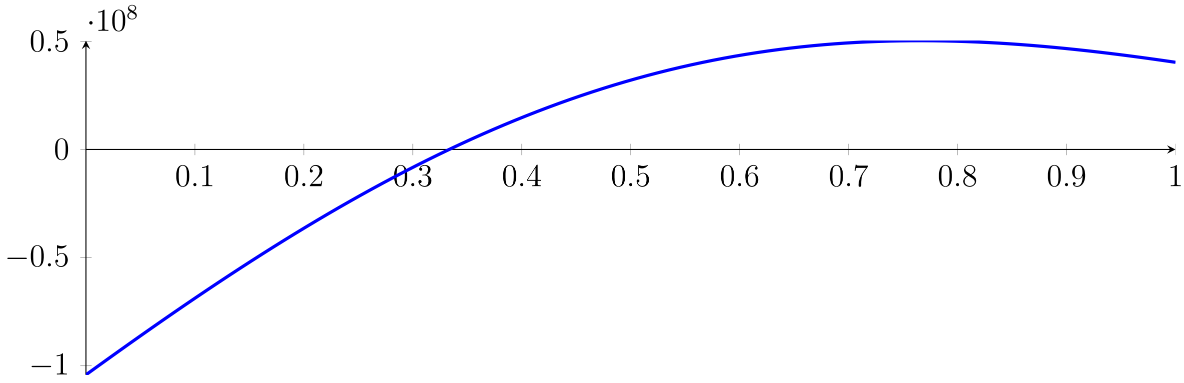
240.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

240.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1\cdot10^{-08}$.

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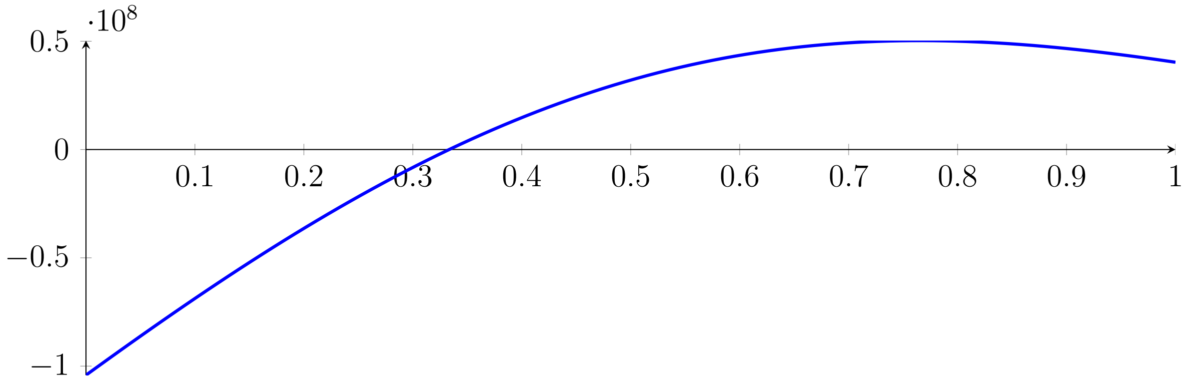
$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called BezClip with input polynomial on interval $[0, 1]$:

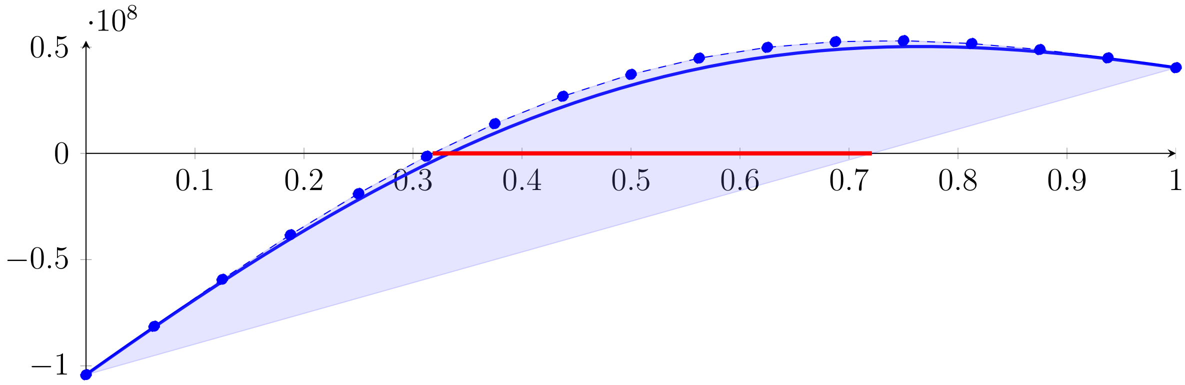
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



241.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &\quad + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} \\ &\quad + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ &= -1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

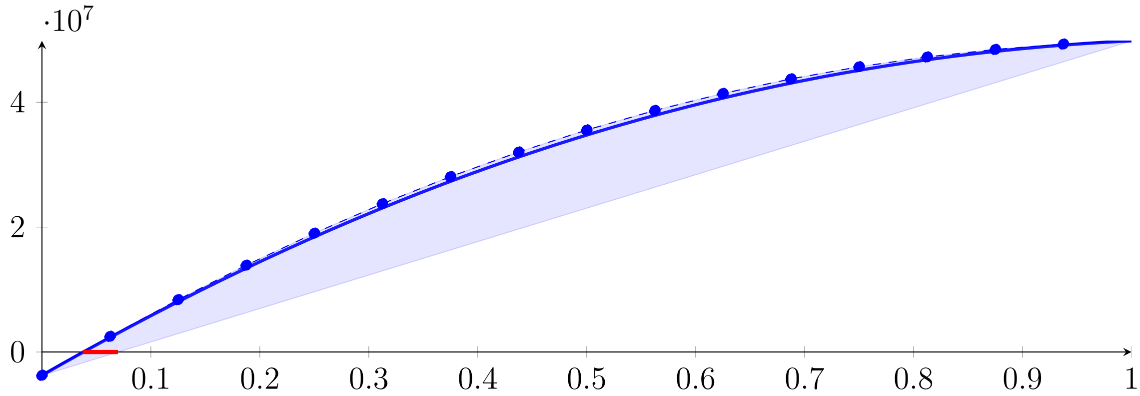
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

241.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -4.83858\cdot 10^{-07}X^{16} - 5.37355\cdot 10^{-05}X^{15} - 0.00254146X^{14} - 0.064977X^{13} - 0.909205X^{12}\\ &\quad - 5.03924X^{11} + 36.4549X^{10} + 692.921X^{9} + 1886.97X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5}\\ &\quad + 3.91945\cdot 10^{6}X^{4} - 7.54561\cdot 10^{6}X^{3} - 4.29225\cdot 10^{7}X^{2} + 9.97982\cdot 10^{7}X - 3.73501\cdot 10^{6}\\ &= -3.73501\cdot 10^{6}B_{0,16}(X) + 2.50237\cdot 10^{6}B_{1,16}(X) + 8.38207\cdot 10^{6}B_{2,16}(X) + 1.38906\\ &\quad \cdot 10^{7}B_{3,16}(X) + 1.90167\cdot 10^{7}B_{4,16}(X) + 2.37512\cdot 10^{7}B_{5,16}(X) + 2.80875\cdot 10^{7}B_{6,16}(X)\\ &\quad + 3.20213\cdot 10^{7}B_{7,16}(X) + 3.55508\cdot 10^{7}B_{8,16}(X) + 3.86768\cdot 10^{7}B_{9,16}(X) + 4.14025\\ &\quad \cdot 10^{7}B_{10,16}(X) + 4.37336\cdot 10^{7}B_{11,16}(X) + 4.56783\cdot 10^{7}B_{12,16}(X) + 4.7247\cdot 10^{7}B_{13,16}(X)\\ &\quad + 4.84524\cdot 10^{7}B_{14,16}(X) + 4.93093\cdot 10^{7}B_{15,16}(X) + 4.98343\cdot 10^{7}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

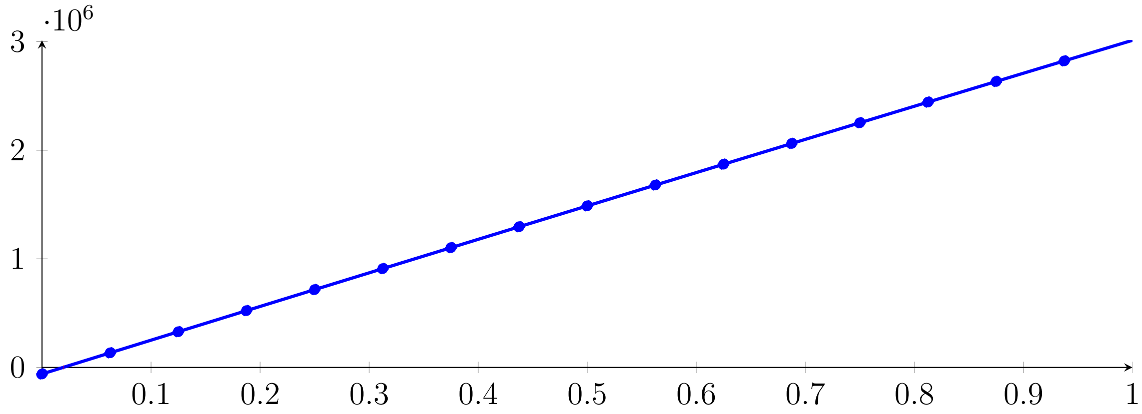
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

241.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p &= -6.78234\cdot 10^{-31}X^{16} - 2.34473\cdot 10^{-27}X^{15} - 3.45581\cdot 10^{-24}X^{14} - 2.75921\cdot 10^{-21}X^{13} - 1.21242\cdot 10^{-18}X^{12}\\ &\quad - 2.17572\cdot 10^{-16}X^{11} + 4.23544\cdot 10^{-14}X^{10} + 2.70025\cdot 10^{-11}X^{9} + 2.51306\cdot 10^{-09}X^{8} - 9.23474\cdot 10^{-07}X^{7}\\ &\quad - 0.000177459X^{6} + 0.0161038X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot 10^{6}X - 60508.5\\ &= -60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X)\\ &\quad + 910202B_{5,16}(X) + 1.1032\cdot 10^{6}B_{6,16}(X) + 1.29582\cdot 10^{6}B_{7,16}(X) + 1.48805\cdot 10^{6}B_{8,16}(X)\\ &\quad + 1.6799\cdot 10^{6}B_{9,16}(X) + 1.87137\cdot 10^{6}B_{10,16}(X) + 2.06245\cdot 10^{6}B_{11,16}(X) + 2.25315\cdot 10^{6}B_{12,16}(X)\\ &\quad + 2.44346\cdot 10^{6}B_{13,16}(X) + 2.63339\cdot 10^{6}B_{14,16}(X) + 2.82293\cdot 10^{6}B_{15,16}(X) + 3.01209\cdot 10^{6}B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

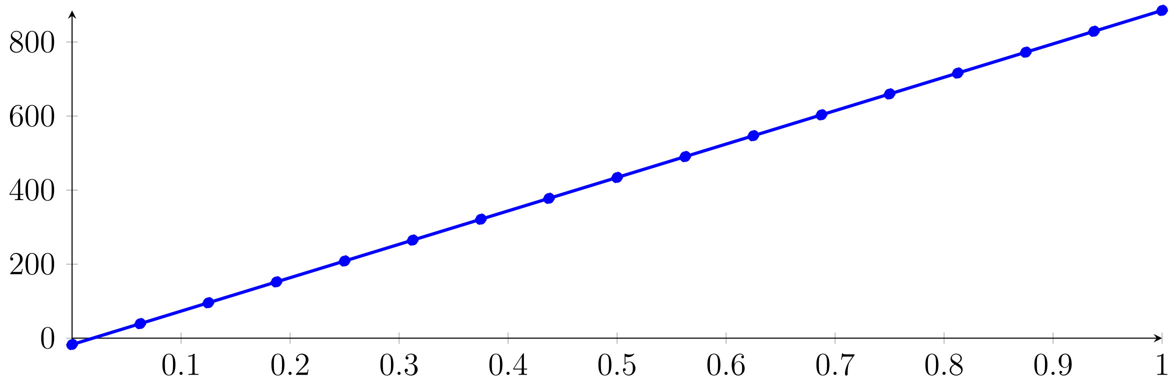
$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

241.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}p = &-1.65599\cdot10^{-87}X^{16} - 1.97733\cdot10^{-80}X^{15} - 1.00659\cdot10^{-73}X^{14} - 2.77601\cdot10^{-67}X^{13}\\ &- 4.21367\cdot10^{-61}X^{12} - 2.61333\cdot10^{-55}X^{11} + 1.75275\cdot10^{-49}X^{10} + 3.8646\cdot10^{-43}X^{9}\\ &+ 1.2441\cdot10^{-37}X^{8} - 1.57525\cdot10^{-31}X^{7} - 1.04661\cdot10^{-25}X^{6} + 3.27355\cdot10^{-20}X^{5}\\ &+ 3.06701\cdot10^{-14}X^{4} - 5.67777\cdot10^{-09}X^{3} - 0.00382618X^{2} + 902.448X - 17.178\\ = &-17.178B_{0,16}(X) + 39.225B_{1,16}(X) + 95.6279B_{2,16}(X) + 152.031B_{3,16}(X) + 208.434B_{4,16}(X)\\ &+ 264.837B_{5,16}(X) + 321.24B_{6,16}(X) + 377.642B_{7,16}(X) + 434.045B_{8,16}(X)\\ &+ 490.448B_{9,16}(X) + 546.851B_{10,16}(X) + 603.253B_{11,16}(X) + 659.656B_{12,16}(X)\\ &+ 716.059B_{13,16}(X) + 772.461B_{14,16}(X) + 828.864B_{15,16}(X) + 885.266B_{16,16}(X)\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

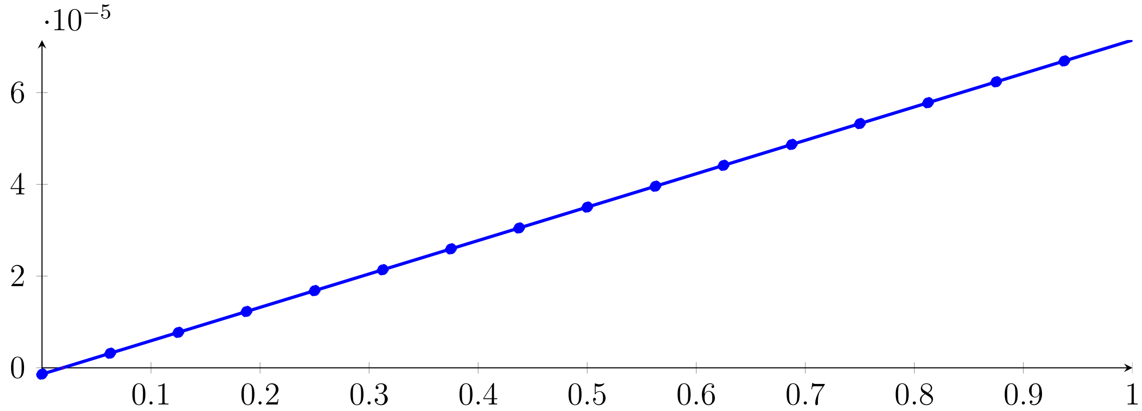
$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045\cdot10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

241.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -5.36315\cdot10^{-201}X^{16} - 7.93495\cdot10^{-187}X^{15} - 5.0052\cdot10^{-173}X^{14} - 1.71037\cdot10^{-159}X^{13} - 3.21686\cdot10^{-146}X^{12} - 2.47211\cdot10^{-133}X^{11} + 2.05446\cdot10^{-120}X^{10} + 5.61285\cdot10^{-107}X^{9} + 2.23891\cdot10^{-94}X^{8} - 3.51264\cdot10^{-81}X^{7} - 2.89181\cdot10^{-68}X^{6} + 1.12075\cdot10^{-55}X^{5} + 1.30109\cdot10^{-42}X^{4} - 2.98449\cdot10^{-30}X^{3} - 2.49207\cdot10^{-17}X^{2} + 7.28316\cdot10^{-05}X - 1.38634\cdot10^{-06}$$
$$= -1.38634\cdot10^{-06}B_{0,16}(X) + 3.16564\cdot10^{-06}B_{1,16}(X) + 7.71761\cdot10^{-06}B_{2,16}(X) + 1.22696\cdot10^{-05}B_{3,16}(X) + 1.68216\cdot10^{-05}B_{4,16}(X) + 2.13735\cdot10^{-05}B_{5,16}(X) + 2.59255\cdot10^{-05}B_{6,16}(X) + 3.04775\cdot10^{-05}B_{7,16}(X) + 3.50295\cdot10^{-05}B_{8,16}(X) + 3.95814\cdot10^{-05}B_{9,16}(X) + 4.41334\cdot10^{-05}B_{10,16}(X) + 4.86854\cdot10^{-05}B_{11,16}(X) + 5.32374\cdot10^{-05}B_{12,16}(X) + 5.77893\cdot10^{-05}B_{13,16}(X) + 6.23413\cdot10^{-05}B_{14,16}(X) + 6.68933\cdot10^{-05}B_{15,16}(X) + 7.14453\cdot10^{-05}B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.51314\cdot10^{-15}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

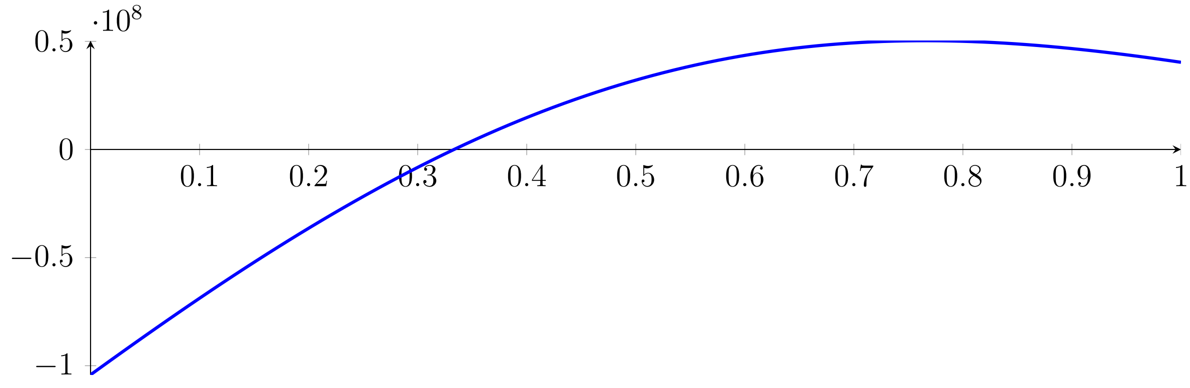
241.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

241.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} + 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1{\cdot}10^{-16}$.

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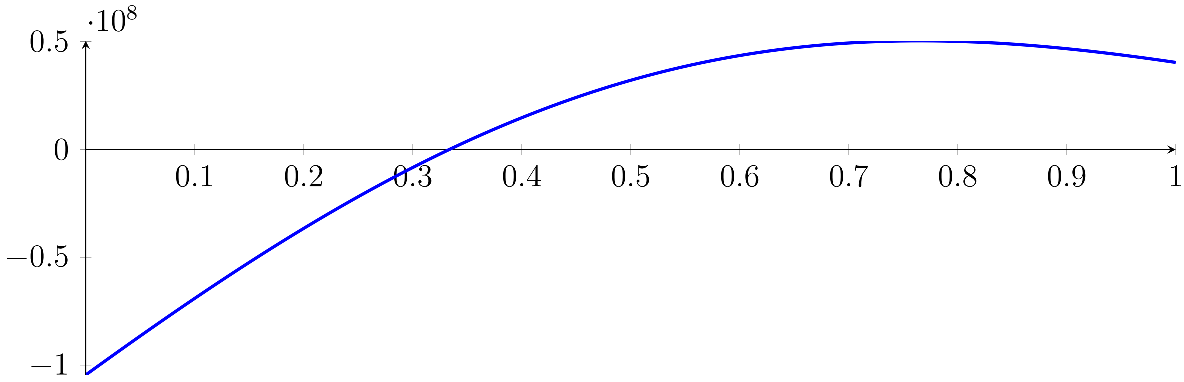
$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} + 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

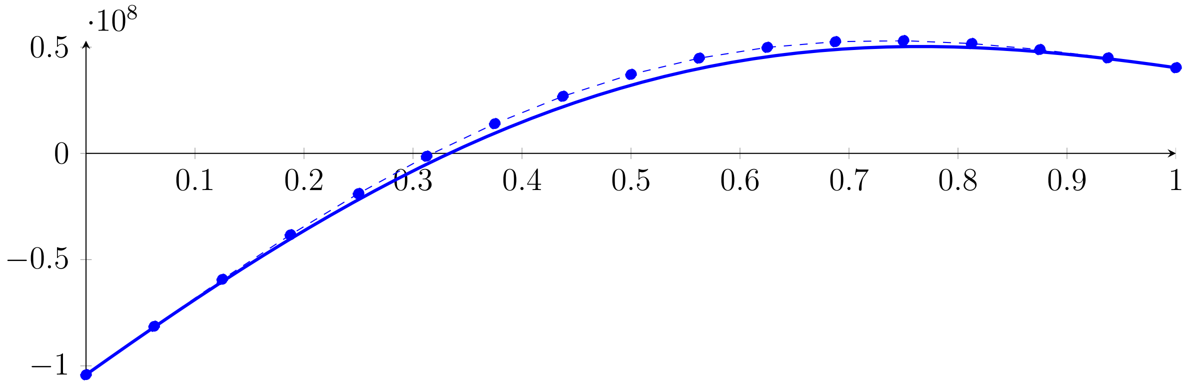
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} + 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8}$$



242.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

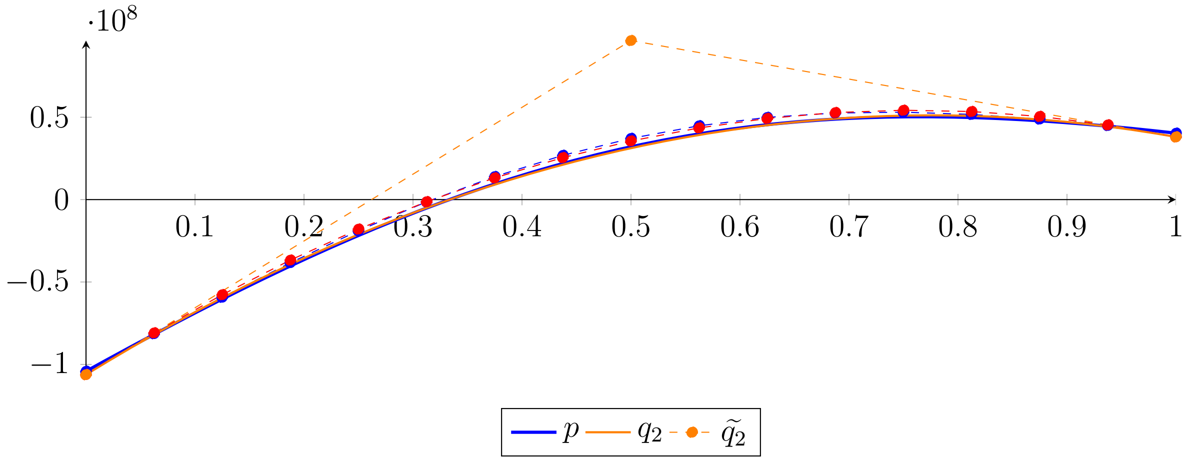
$$\begin{aligned} p &= -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &\quad + 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} \\ &\quad + 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8} \\ &= -1.04167{\cdot}10^{8}B_{0,16}(X) - 8.13802{\cdot}10^{7}B_{1,16}(X) - 5.92882{\cdot}10^{7}B_{2,16}(X) - 3.83203 \\ &\quad {\cdot}10^{7}B_{3,16}(X) - 1.88844{\cdot}10^{7}B_{4,16}(X) - 1.34837{\cdot}10^{6}B_{5,16}(X) + 1.39781{\cdot}10^{7}B_{6,16}(X) \\ &\quad + 2.68604{\cdot}10^{7}B_{7,16}(X) + 3.71532{\cdot}10^{7}B_{8,16}(X) + 4.48105{\cdot}10^{7}B_{9,16}(X) + 4.98901 \\ &\quad {\cdot}10^{7}B_{10,16}(X) + 5.25504{\cdot}10^{7}B_{11,16}(X) + 5.30407{\cdot}10^{7}B_{12,16}(X) + 5.16832{\cdot}10^{7}B_{13,16}(X) \\ &\quad + 4.88488{\cdot}10^{7}B_{14,16}(X) + 4.49297{\cdot}10^{7}B_{15,16}(X) + 4.03108{\cdot}10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$
$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\widetilde{q_2} = 3.74473 \cdot 10^{-288} X^{16} - 3.08889 \cdot 10^{-287} X^{15} + 1.17443 \cdot 10^{-286} X^{14} - 2.71603 \cdot 10^{-286} X^{13} + 4.23554 \cdot 10^{-286} X^{12} - 4.66107 \cdot 10^{-286} X^{11} + 3.6845 \cdot 10^{-286} X^{10} - 2.09811 \cdot 10^{-286} X^9 + 8.58708 \cdot 10^{-287} X^8 - 2.54063 \cdot 10^{-287} X^7 + 5.57852 \cdot 10^{-288} X^6 - 9.21534 \cdot 10^{-289} X^5 + 1.02122 \cdot 10^{-289} X^4 - 5.95624 \cdot 10^{-291} X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$
$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017 \cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16} + 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16} + 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

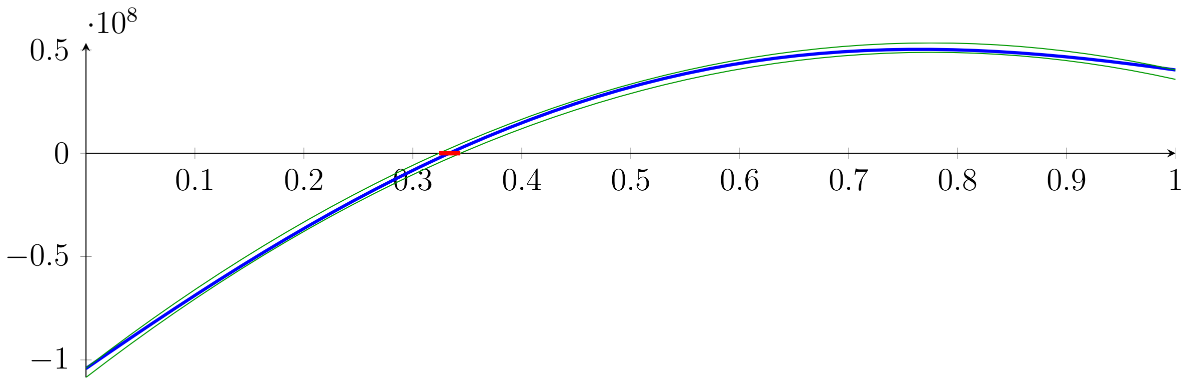
Bounding polynomials M and m:

$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$
$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

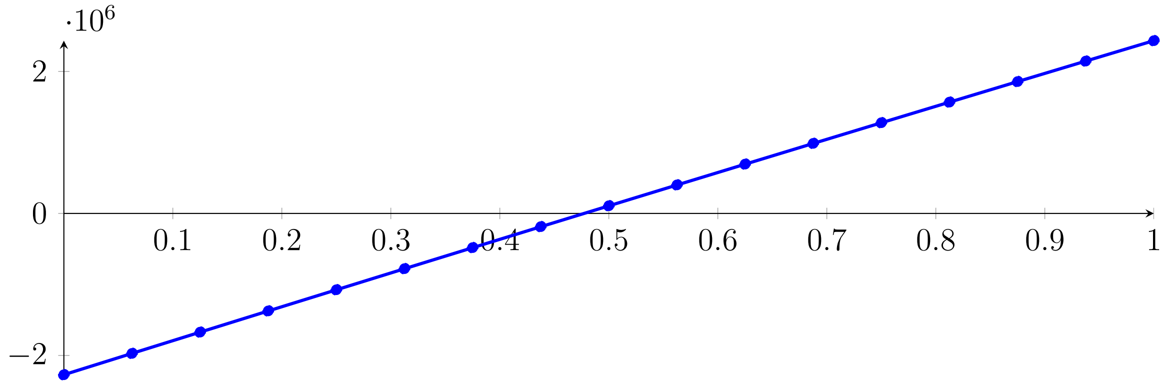
Longest intersection interval: 0.0196686

$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

242.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

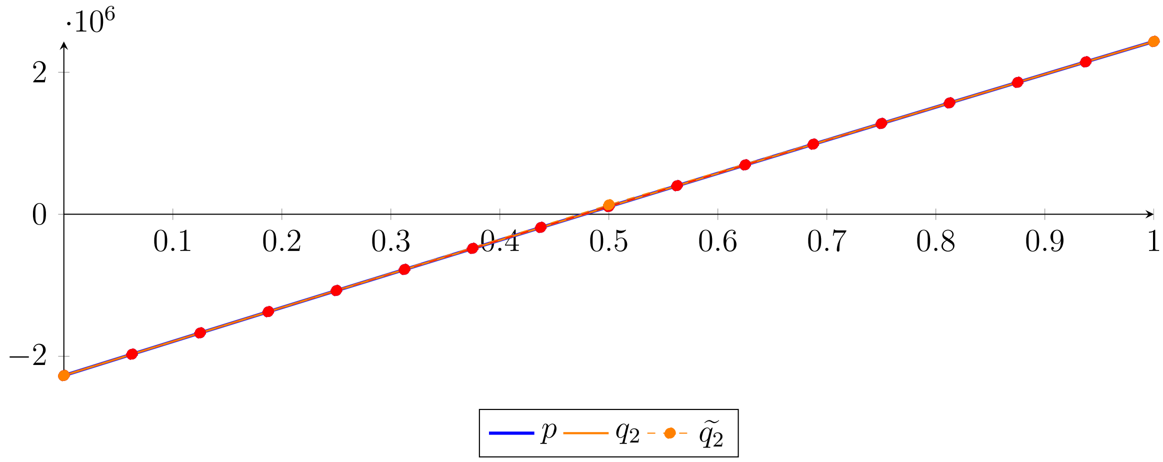
$$p = -5.0162{\cdot}10^{-28}X^{16} - 1.14383{\cdot}10^{-24}X^{15} - 1.11125{\cdot}10^{-21}X^{14} - 5.84096{\cdot}10^{-19}X^{13} - 1.684{\cdot}10^{-16}X^{12} - 1.94707{\cdot}10^{-14}X^{11} + 2.73227{\cdot}10^{-12}X^{10} + 1.09727{\cdot}10^{-09}X^{9} + 6.37314{\cdot}10^{-08}X^{8} - 1.68645{\cdot}10^{-05}X^{7} - 0.00205892X^{6} + 0.132721X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874{\cdot}10^{6}X - 2.2715{\cdot}10^{6}$$
$$= -2.2715{\cdot}10^{6}B_{0,16}(X) - 1.97096{\cdot}10^{6}B_{1,16}(X) - 1.67127{\cdot}10^{6}B_{2,16}(X) - 1.37244{\cdot}10^{6}B_{3,16}(X) - 1.07448{\cdot}10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806{\cdot}10^{6}B_{12,16}(X) + 1.5682{\cdot}10^{6}B_{13,16}(X) + 1.85747{\cdot}10^{6}B_{14,16}(X) + 2.14586{\cdot}10^{6}B_{15,16}(X) + 2.43338{\cdot}10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265X^2 + 4.80923{\cdot}10^{6}X - 2.27154{\cdot}10^{6}$$
$$= -2.27154{\cdot}10^{6}B_{0,2} + 133069B_{1,2} + 2.43342{\cdot}10^{6}B_{2,2}$$

$$\widetilde{q}_2 = -8.18461{\cdot}10^{-291}X^{16} + 4.13745{\cdot}10^{-290}X^{15} - 5.8254{\cdot}10^{-290}X^{14} - 6.80919{\cdot}10^{-290}X^{13} + 3.48059{\cdot}10^{-289}X^{12} - 5.47245{\cdot}10^{-289}X^{11} + 4.8261{\cdot}10^{-289}X^{10} - 2.67867{\cdot}10^{-289}X^{9} + 1.00314{\cdot}10^{-289}X^{8} - 2.84766{\cdot}10^{-290}X^{7} + 6.85072{\cdot}10^{-291}X^{6} - 1.12501{\cdot}10^{-291}X^{5} + 1.24731{\cdot}10^{-293}X^{4} + 2.25944{\cdot}10^{-293}X^{3} - 104265X^{2} + 4.80923{\cdot}10^{6}X - 2.27154{\cdot}10^{6}$$
$$= -2.27154{\cdot}10^{6}B_{0,16} - 1.97097{\cdot}10^{6}B_{1,16} - 1.67126{\cdot}10^{6}B_{2,16} - 1.37242{\cdot}10^{6}B_{3,16} - 1.07445{\cdot}10^{6}B_{4,16} - 777350B_{5,16} - 481118B_{6,16} - 185754B_{7,16} + 108740B_{8,16} + 402366B_{9,16} + 695123B_{10,16} + 987011B_{11,16} + 1.27803{\cdot}10^{6}B_{12,16} + 1.56818{\cdot}10^{6}B_{13,16} + 1.85746{\cdot}10^{6}B_{14,16} + 2.14587{\cdot}10^{6}B_{15,16} + 2.43342{\cdot}10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.
Bounding polynomials M and m:

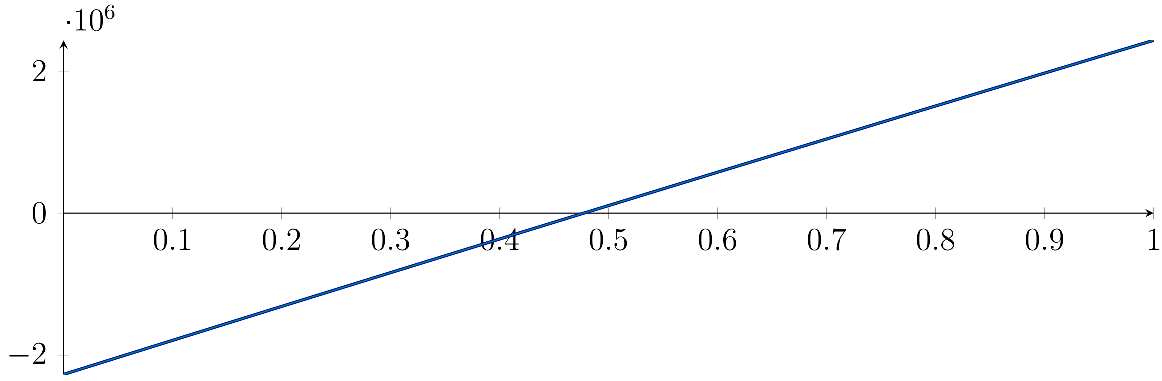
$$M = -104265X^2 + 4.80923\cdot 10^6 X - 2.2715\cdot 10^6$$

$$m = -104265X^2 + 4.80923\cdot 10^6 X - 2.27159\cdot 10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



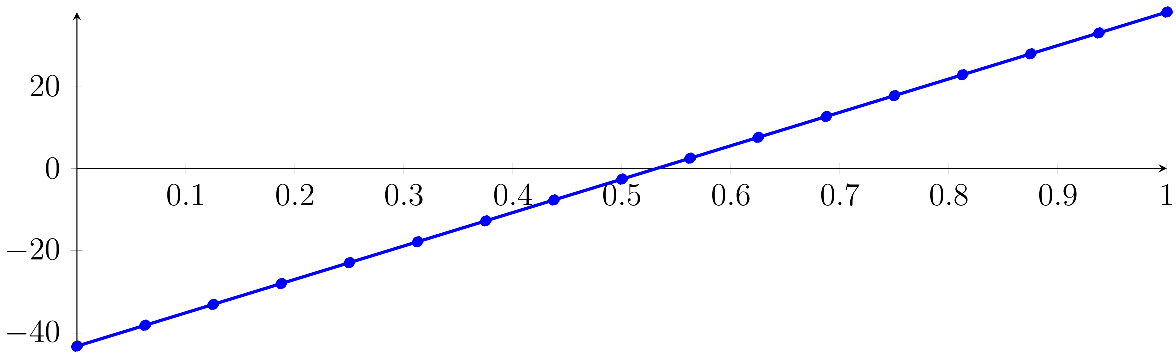
$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301\cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

242.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

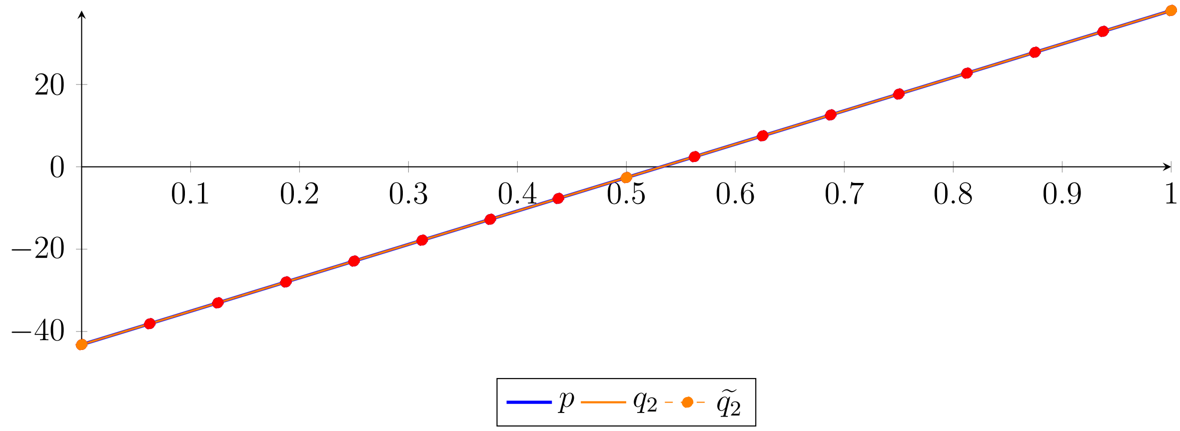
$$\begin{aligned}
p = & -3.02667\cdot 10^{-104}X^{16} - 4.019\cdot 10^{-96}X^{15} - 2.27522\cdot 10^{-88}X^{14} - 6.97783\cdot 10^{-81}X^{13} \\
& - 1.17785\cdot 10^{-73}X^{12} - 8.12373\cdot 10^{-67}X^{11} + 6.05916\cdot 10^{-60}X^{10} + 1.48569\cdot 10^{-52}X^{9} \\
& + 5.31875\cdot 10^{-46}X^{8} - 7.48919\cdot 10^{-39}X^{7} - 5.53349\cdot 10^{-32}X^{6} + 1.92471\cdot 10^{-25}X^{5} \\
& + 2.00536\cdot 10^{-18}X^{4} - 4.12844\cdot 10^{-12}X^{3} - 3.09388\cdot 10^{-05}X^{2} + 81.1506X - 43.1911 \\
= & -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\
& - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X) \\
& + 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X) \\
& + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X)
\end{aligned}$$



Degree reduction and raising:

$$q_2 = -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911$$
$$= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2}$$

$$\tilde{q}_2 = -1.88281{\cdot}10^{-295}X^{16} + 1.09893{\cdot}10^{-294}X^{15} - 2.26419{\cdot}10^{-294}X^{14} + 8.08223{\cdot}10^{-295}X^{13} + 4.92626{\cdot}10^{-294}X^{12} - 1.07605{\cdot}10^{-293}X^{11} + 1.11858{\cdot}10^{-293}X^{10} - 6.87288{\cdot}10^{-294}X^9 + 2.54873{\cdot}10^{-294}X^8 - 5.2305{\cdot}10^{-295}X^7 + 3.18923{\cdot}10^{-296}X^6 + 1.34092{\cdot}10^{-296}X^5 - 4.89549{\cdot}10^{-297}X^4 + 5.89947{\cdot}10^{-298}X^3 - 3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911$$
$$= -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} - 12.7597B_{6,16} - 7.68778B_{7,16} - 2.61587B_{8,16} + 2.45604B_{9,16} + 7.52795B_{10,16} + 12.5999B_{11,16} + 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.06422{\cdot}10^{-13}$.

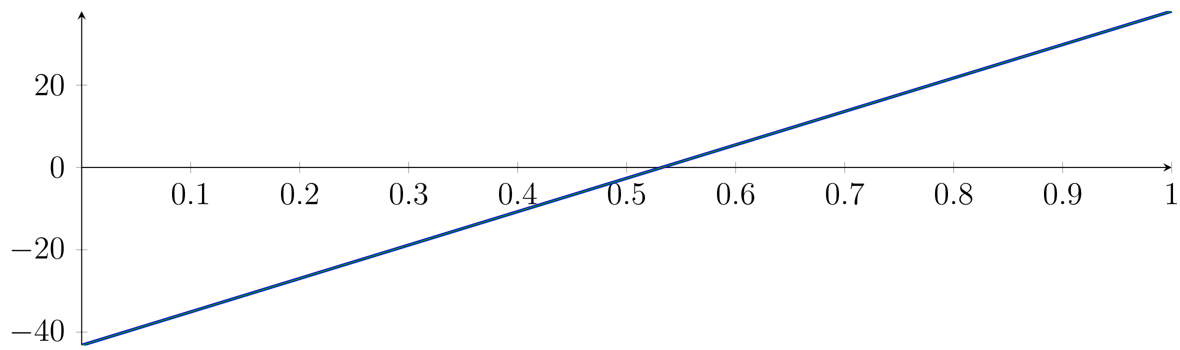
Bounding polynomials M and m:

$$M = -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911$$
$$m = -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911$$

Root of M and m:

$$N(M) = \{0.532235, 2.62293{\cdot}10^6\} \qquad N(m) = \{0.532235, 2.62293{\cdot}10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

Longest intersection interval: $5.08738{\cdot}10^{-15}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

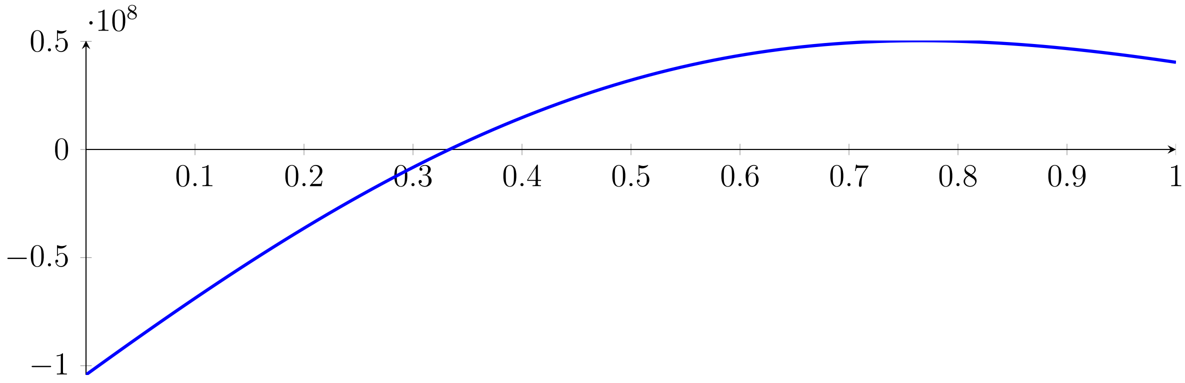
242.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

242.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

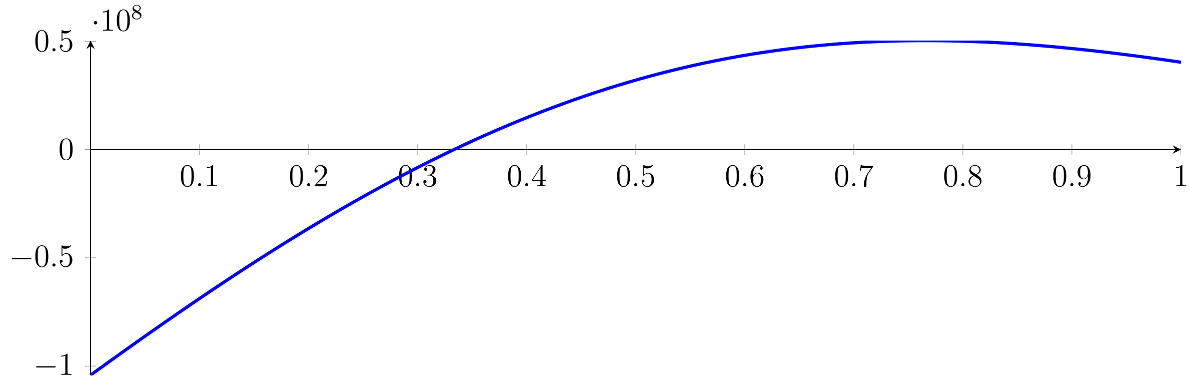
with precision $\varepsilon = 1\cdot10^{-16}$.

243 Running CubeClip on f_{16} with epsilon 16

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^6X^9 - 2.52262{\cdot}10^6X^8 - 1.43506{\cdot}10^7X^7 + 138542X^6 + 7.92823{\cdot}10^7X^5 + 3.97396{\cdot}10^7X^4 - 2.40625{\cdot}10^8X^3 - 8.33333{\cdot}10^7X^2 + 3.64583{\cdot}10^8X - 1.04167{\cdot}10^8$$

Called CubeClip with input polynomial on interval $[0, 1]$:

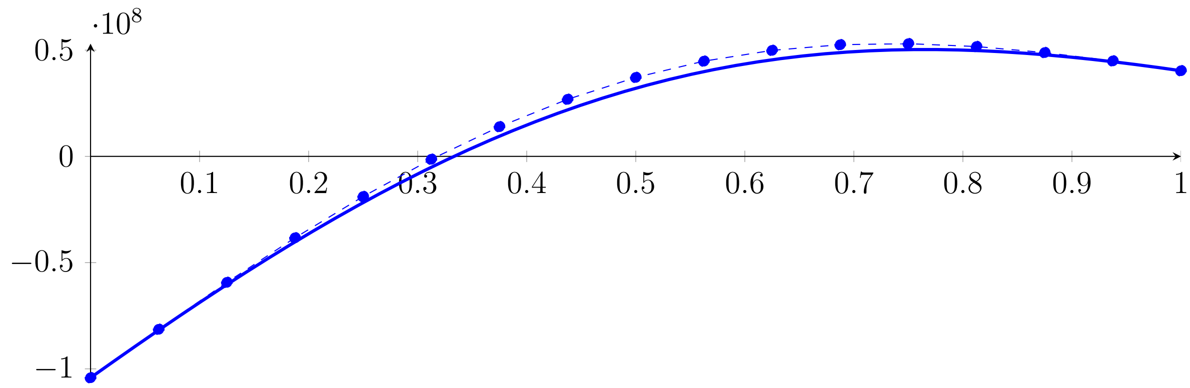
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172{\cdot}10^6X^9 - 2.52262{\cdot}10^6X^8 - 1.43506{\cdot}10^7X^7 + 138542X^6 + 7.92823{\cdot}10^7X^5 \\ &+ 3.97396{\cdot}10^7X^4 - 2.40625{\cdot}10^8X^3 - 8.33333{\cdot}10^7X^2 + 3.64583{\cdot}10^8X - 1.04167{\cdot}10^8 \end{aligned}$$



243.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

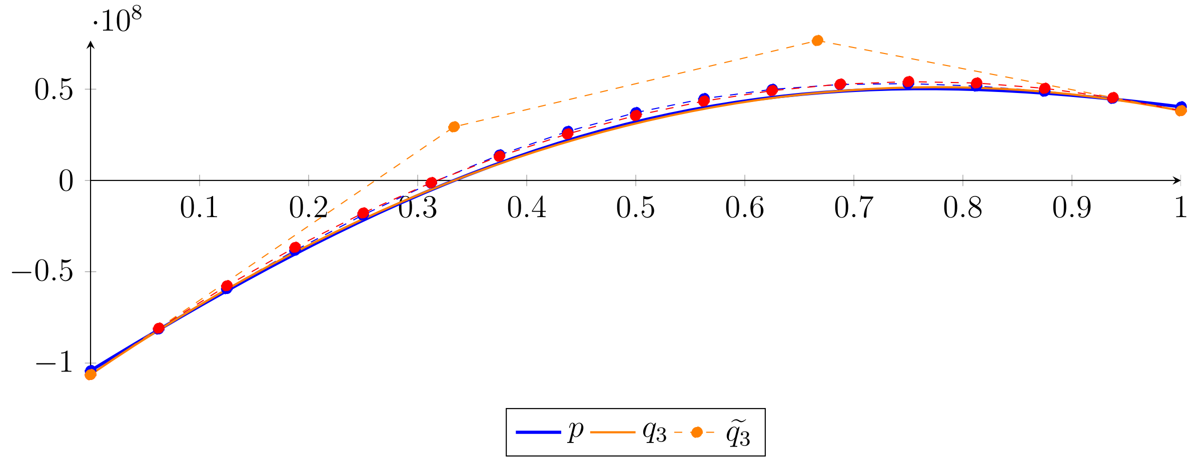
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172{\cdot}10^6X^9 - 2.52262{\cdot}10^6X^8 - 1.43506{\cdot}10^7X^7 + 138542X^6 + 7.92823{\cdot}10^7X^5 \\ &+ 3.97396{\cdot}10^7X^4 - 2.40625{\cdot}10^8X^3 - 8.33333{\cdot}10^7X^2 + 3.64583{\cdot}10^8X - 1.04167{\cdot}10^8 \\ = &-1.04167{\cdot}10^8B_{0,16}(X) - 8.13802{\cdot}10^7B_{1,16}(X) - 5.92882{\cdot}10^7B_{2,16}(X) - 3.83203 \\ &{\cdot}10^7B_{3,16}(X) - 1.88844{\cdot}10^7B_{4,16}(X) - 1.34837{\cdot}10^6B_{5,16}(X) + 1.39781{\cdot}10^7B_{6,16}(X) \\ &+ 2.68604{\cdot}10^7B_{7,16}(X) + 3.71532{\cdot}10^7B_{8,16}(X) + 4.48105{\cdot}10^7B_{9,16}(X) + 4.98901 \\ &{\cdot}10^7B_{10,16}(X) + 5.25504{\cdot}10^7B_{11,16}(X) + 5.30407{\cdot}10^7B_{12,16}(X) + 5.16832{\cdot}10^7B_{13,16}(X) \\ &+ 4.88488{\cdot}10^7B_{14,16}(X) + 4.49297{\cdot}10^7B_{15,16}(X) + 4.03108{\cdot}10^7B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.06335{\cdot}10^8 \\ &= -1.06335{\cdot}10^8 B_{0,3} + 2.93558{\cdot}10^7 B_{1,3} + 7.66068{\cdot}10^7 B_{2,3} + 3.81764{\cdot}10^7 B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 1.89955{\cdot}10^{-288} X^{16} - 1.4851{\cdot}10^{-287} X^{15} + 5.12896{\cdot}10^{-287} X^{14} - 1.02224{\cdot}10^{-286} X^{13} \\ &\quad + 1.29027{\cdot}10^{-286} X^{12} - 1.07058{\cdot}10^{-286} X^{11} + 6.00739{\cdot}10^{-287} X^{10} - 2.54352{\cdot}10^{-287} X^9 \\ &\quad + 1.12089{\cdot}10^{-287} X^8 - 5.8673{\cdot}10^{-288} X^7 + 2.52637{\cdot}10^{-288} X^6 - 6.79624{\cdot}10^{-289} X^5 \\ &\quad + 9.36341{\cdot}10^{-290} X^4 + 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.06335{\cdot}10^8 \\ &= -1.06335{\cdot}10^8 B_{0,16} - 8.08928{\cdot}10^7 B_{1,16} - 5.76618{\cdot}10^7 B_{2,16} - 3.66369{\cdot}10^7 B_{3,16} - 1.78131 \\ &\quad {\cdot}10^7 B_{4,16} - 1.18551{\cdot}10^6 B_{5,16} + 1.32508{\cdot}10^7 B_{6,16} + 2.55007{\cdot}10^7 B_{7,16} + 3.55692{\cdot}10^7 B_{8,16} \\ &\quad + 4.34611{\cdot}10^7 B_{9,16} + 4.91815{\cdot}10^7 B_{10,16} + 5.27353{\cdot}10^7 B_{11,16} + 5.41273{\cdot}10^7 B_{12,16} \\ &\quad + 5.33624{\cdot}10^7 B_{13,16} + 5.04457{\cdot}10^7 B_{14,16} + 4.53821{\cdot}10^7 B_{15,16} + 3.81764{\cdot}10^7 B_{16,16} \end{aligned}$$



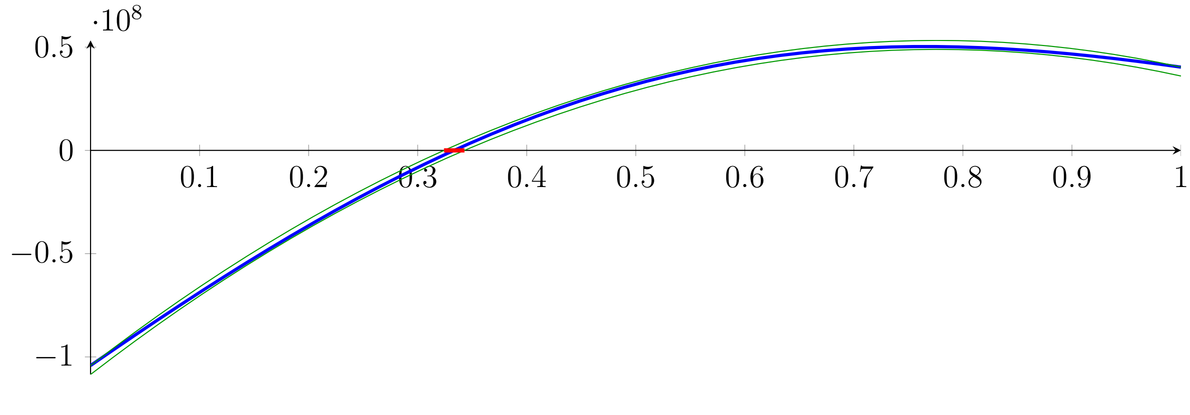
The maximum difference of the Bézier coefficients is $\delta = 2.16806{\cdot}10^6$.
Bounding polynomials M and m:

$$M = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.04167{\cdot}10^8$$
$$m = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.08503{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



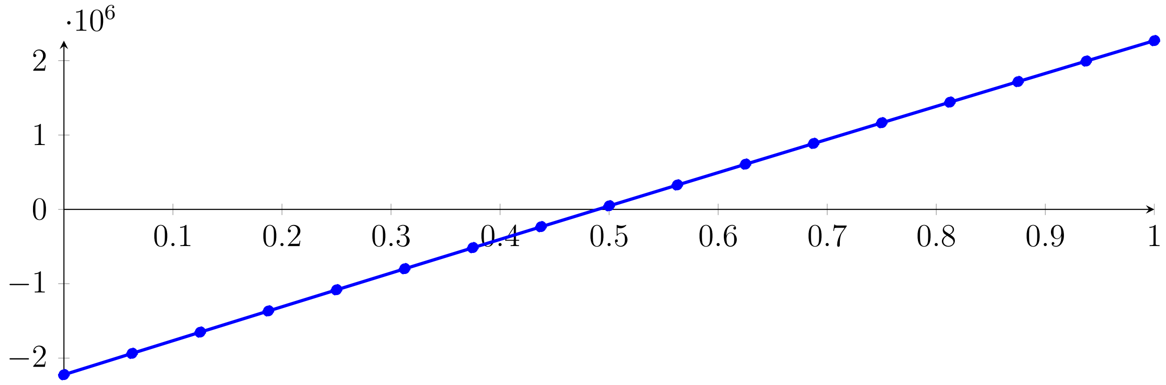
$$[0.324143, 0.342913]$$

Longest intersection interval: 0.0187703
$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

243.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

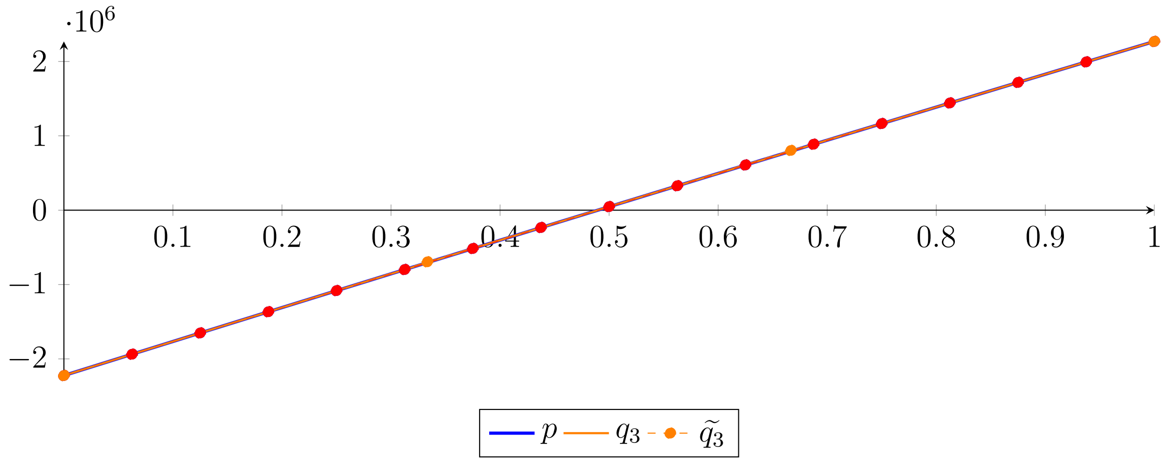
$$p = -2.37433 \cdot 10^{-28} X^{16} - 5.67363 \cdot 10^{-25} X^{15} - 5.77631 \cdot 10^{-22} X^{14} - 3.1818 \cdot 10^{-19} X^{13} - 9.6142 \cdot 10^{-17} X^{12} - 1.16549 \cdot 10^{-14} X^{11} + 1.71065 \cdot 10^{-12} X^{10} + 7.20611 \cdot 10^{-10} X^{9} + 4.39147 \cdot 10^{-08} X^{8} - 1.21542 \cdot 10^{-05} X^{7} - 0.00155624 X^{6} + 0.10496 X^{5} + 18.6215 X^{4} - 738.202 X^{3} - 93855.7 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$
$$= -2.22335 \cdot 10^{6} B_{0,16}(X) - 1.93666 \cdot 10^{6} B_{1,16}(X) - 1.65074 \cdot 10^{6} B_{2,16}(X) - 1.36561 \cdot 10^{6} B_{3,16}(X) - 1.08127 \cdot 10^{6} B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^{6} B_{12,16}(X) + 1.44233 \cdot 10^{6} B_{13,16}(X) + 1.71876 \cdot 10^{6} B_{14,16}(X) + 1.99439 \cdot 10^{6} B_{15,16}(X) + 2.26922 \cdot 10^{6} B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -700.673 X^{3} - 93879.9 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$
$$= -2.22335 \cdot 10^{6} B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^{6} B_{3,3}$$

$$\widetilde{q}_3 = 4.41595 \cdot 10^{-291} X^{16} - 1.48598 \cdot 10^{-290} X^{15} - 2.81102 \cdot 10^{-290} X^{14} + 2.65129 \cdot 10^{-289} X^{13} - 7.34593 \cdot 10^{-289} X^{12} + 1.17654 \cdot 10^{-288} X^{11} - 1.23119 \cdot 10^{-288} X^{10} + 8.63493 \cdot 10^{-289} X^{9} - 3.92909 \cdot 10^{-289} X^{8} + 1.01265 \cdot 10^{-289} X^{7} - 6.13644 \cdot 10^{-291} X^{6} - 3.92664 \cdot 10^{-291} X^{5} + 9.04488 \cdot 10^{-292} X^{4} - 700.673 X^{3} - 93879.9 X^{2} + 4.58715 \cdot 10^{6} X - 2.22335 \cdot 10^{6}$$
$$= -2.22335 \cdot 10^{6} B_{0,16} - 1.93666 \cdot 10^{6} B_{1,16} - 1.65074 \cdot 10^{6} B_{2,16} - 1.36561 \cdot 10^{6} B_{3,16} - 1.08126 \cdot 10^{6} B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^{6} B_{12,16} + 1.44233 \cdot 10^{6} B_{13,16} + 1.71876 \cdot 10^{6} B_{14,16} + 1.99439 \cdot 10^{6} B_{15,16} + 2.26922 \cdot 10^{6} B_{16,16}$$



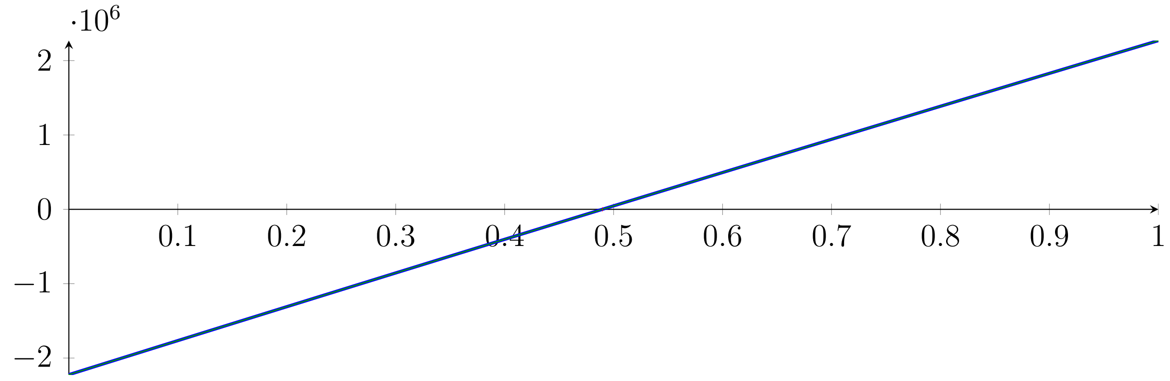
The maximum difference of the Bézier coefficients is $\delta = 0.270074$.
Bounding polynomials M and m:

$$M = -700.673X^3 - 93879.9X^2 + 4.58715\cdot 10^6 X - 2.22335\cdot 10^6$$
$$m = -700.673X^3 - 93879.9X^2 + 4.58715\cdot 10^6 X - 2.22335\cdot 10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



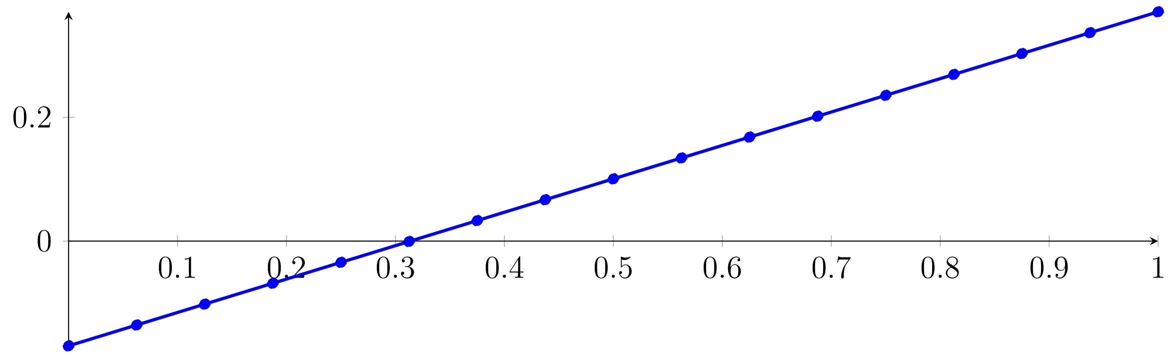
$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174\cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

243.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

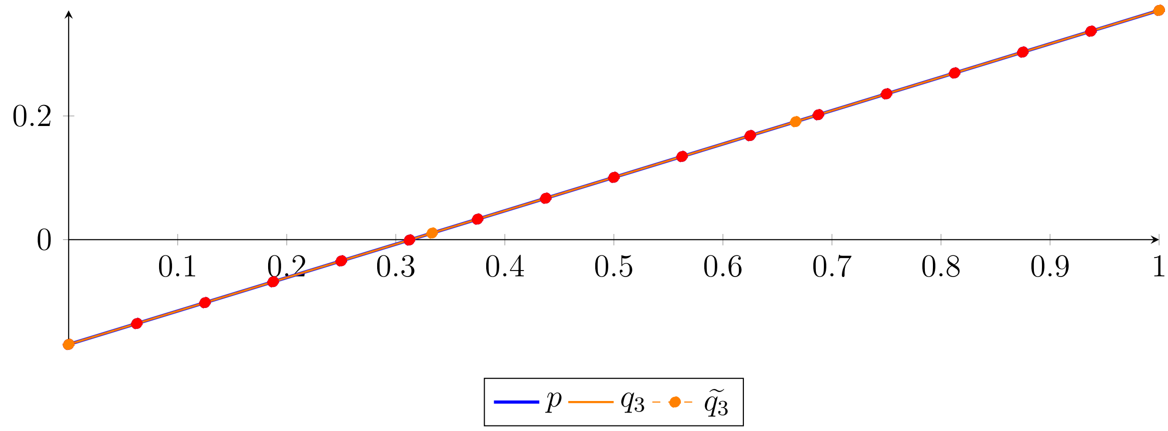
$$\begin{aligned} p = &-4.49274\cdot 10^{-139}X^{16} - 8.96277\cdot 10^{-129}X^{15} - 7.623\cdot 10^{-119}X^{14} - 3.51238\cdot 10^{-109}X^{13} \\ &- 8.90739\cdot 10^{-100}X^{12} - 9.22984\cdot 10^{-91}X^{11} + 1.03426\cdot 10^{-81}X^{10} + 3.80998\cdot 10^{-72}X^{9} \\ &+ 2.04919\cdot 10^{-63}X^{8} - 4.33497\cdot 10^{-54}X^{7} - 4.81204\cdot 10^{-45}X^{6} + 2.51462\cdot 10^{-36}X^{5} \\ &+ 3.93622\cdot 10^{-27}X^{4} - 1.21745\cdot 10^{-18}X^{3} - 1.37072\cdot 10^{-09}X^{2} + 0.540148X - 0.169396 \\ = &-0.169396B_{0,16}(X) - 0.135637B_{1,16}(X) - 0.101877B_{2,16}(X) - 0.068118B_{3,16}(X) \\ &- 0.0343588B_{4,16}(X) - 0.000599488B_{5,16}(X) + 0.0331598B_{6,16}(X) \\ &+ 0.066919B_{7,16}(X) + 0.100678B_{8,16}(X) + 0.134438B_{9,16}(X) + 0.168197B_{10,16}(X) \\ &+ 0.201956B_{11,16}(X) + 0.235715B_{12,16}(X) + 0.269475B_{13,16}(X) \\ &+ 0.303234B_{14,16}(X) + 0.336993B_{15,16}(X) + 0.370752B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -1.21745{\cdot}10^{-18}X^3 - 1.37072{\cdot}10^{-09}X^2 + 0.540148X - 0.169396$$
$$= -0.169396B_{0,3} + 0.0106536B_{1,3} + 0.190703B_{2,3} + 0.370752B_{3,3}$$

$$\tilde{q}_3 = 8.03185{\cdot}10^{-297}X^{16} - 6.20841{\cdot}10^{-296}X^{15} + 2.13274{\cdot}10^{-295}X^{14} - 4.26614{\cdot}10^{-295}X^{13} + 5.47461{\cdot}10^{-295}X^{12} - 4.70265{\cdot}10^{-295}X^{11} + 2.78551{\cdot}10^{-295}X^{10} - 1.22442{\cdot}10^{-295}X^9 + 4.88954{\cdot}10^{-296}X^8 - 2.11494{\cdot}10^{-296}X^7 + 8.20665{\cdot}10^{-297}X^6 - 2.15458{\cdot}10^{-297}X^5 + 3.01517{\cdot}10^{-298}X^4 - 1.21745{\cdot}10^{-18}X^3 - 1.37072{\cdot}10^{-09}X^2 + 0.540148X - 0.169396$$
$$= -0.169396B_{0,16} - 0.135637B_{1,16} - 0.101877B_{2,16} - 0.068118B_{3,16} - 0.0343588B_{4,16} - 0.000599488B_{5,16} + 0.0331598B_{6,16} + 0.066919B_{7,16} + 0.100678B_{8,16} + 0.134438B_{9,16} + 0.168197B_{10,16} + 0.201956B_{11,16} + 0.235715B_{12,16} + 0.269475B_{13,16} + 0.303234B_{14,16} + 0.336993B_{15,16} + 0.370752B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 5.62317{\cdot}10^{-29}$.

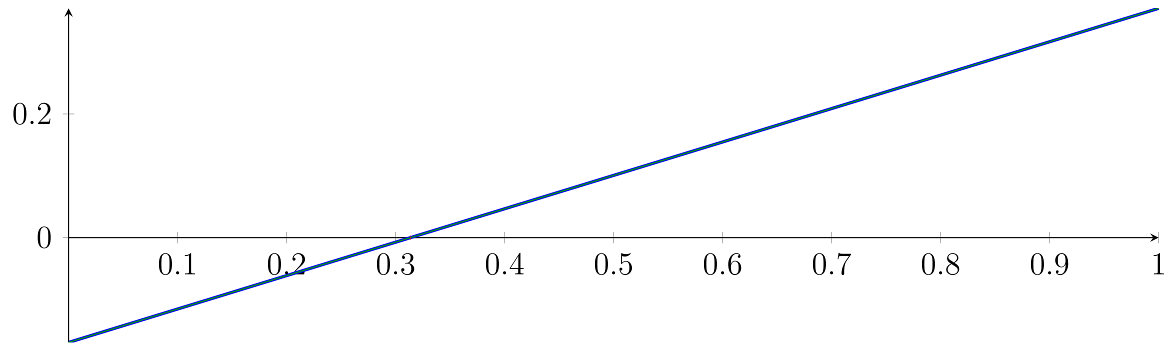
Bounding polynomials M and m:

$$M = -1.21745{\cdot}10^{-18}X^3 - 1.37072{\cdot}10^{-09}X^2 + 0.540148X - 0.169396$$
$$m = -1.21745{\cdot}10^{-18}X^3 - 1.37072{\cdot}10^{-09}X^2 + 0.540148X - 0.169396$$

Root of M and m:

$$N(M) = \{-1.43506{\cdot}10^9, 0.31361, 3.09167{\cdot}10^8\} \quad N(m) = \{-1.43506{\cdot}10^9, 0.31361, 3.09167{\cdot}10^8\}$$

Intersection intervals:



$[0.31361, 0.31361]$

Longest intersection interval: $2.08208{\cdot}10^{-28}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

243.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

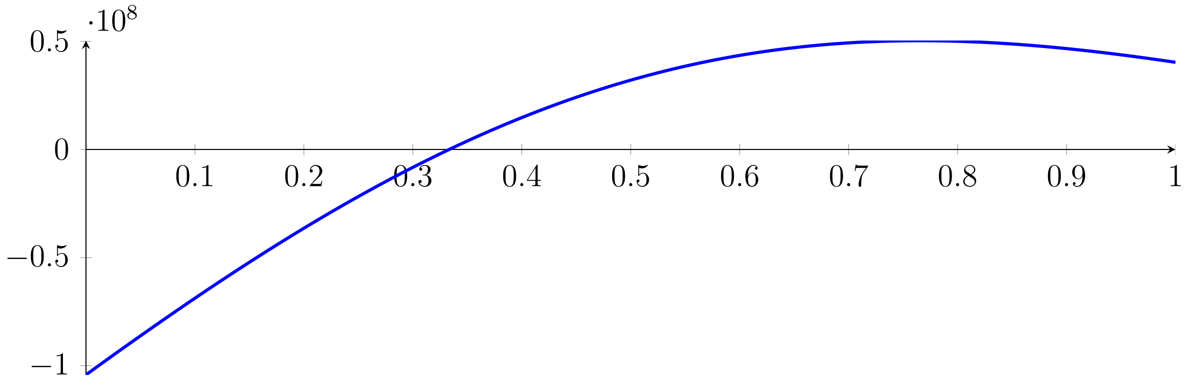
Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 4!

p(0) = -8.88188e-08 - p(1) -8.88188e-08

243.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} + 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8}$$



Result: Root Intervals

with precision $\varepsilon = 1{\cdot}10^{-16}$.

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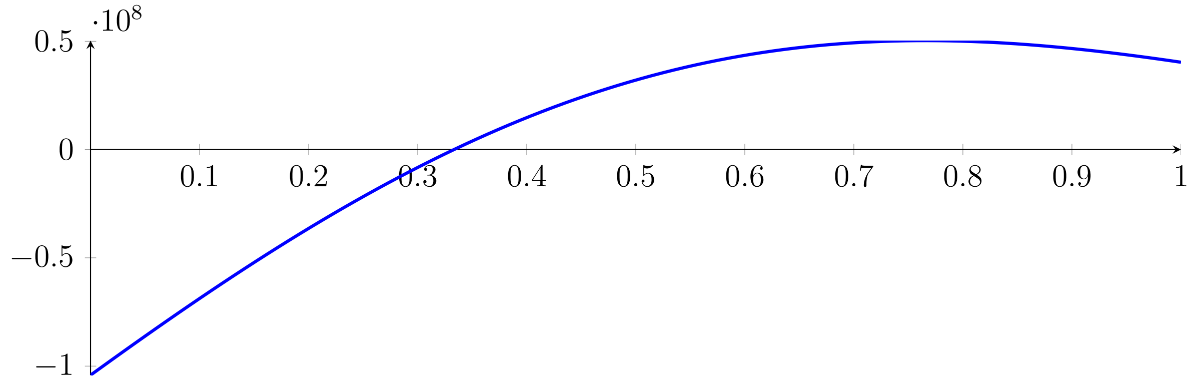
$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called BezClip with input polynomial on interval $[0, 1]$:

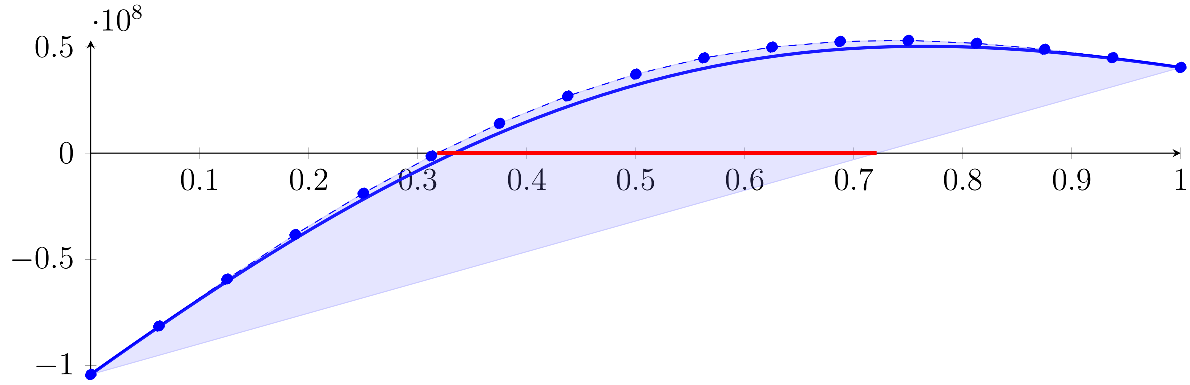
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



244.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &\quad + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} \\ &\quad + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ &= -1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

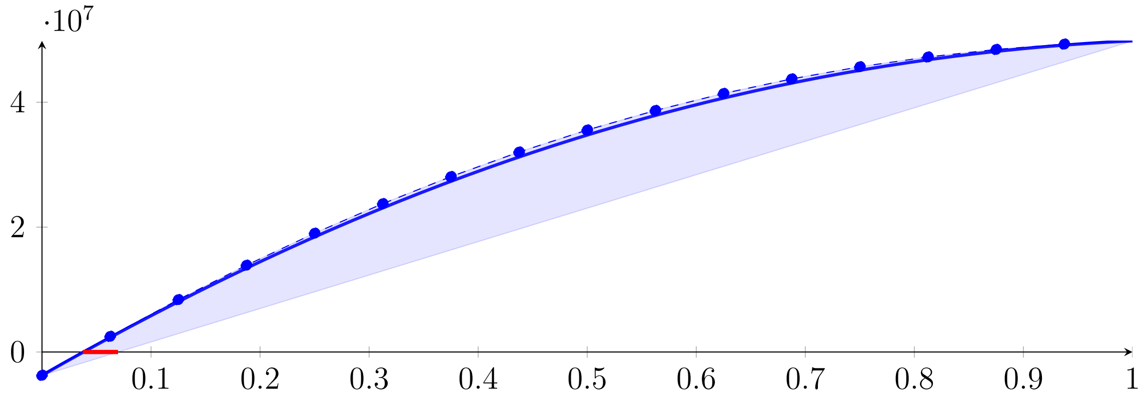
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

244.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-4.83858\cdot 10^{-07}X^{16} - 5.37355\cdot 10^{-05}X^{15} - 0.00254146X^{14} - 0.064977X^{13} - 0.909205X^{12} \\ &- 5.03924X^{11} + 36.4549X^{10} + 692.921X^{9} + 1886.97X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5} \\ &+ 3.91945\cdot 10^{6}X^{4} - 7.54561\cdot 10^{6}X^{3} - 4.29225\cdot 10^{7}X^{2} + 9.97982\cdot 10^{7}X - 3.73501\cdot 10^{6} \\ = &-3.73501\cdot 10^{6}B_{0,16}(X) + 2.50237\cdot 10^{6}B_{1,16}(X) + 8.38207\cdot 10^{6}B_{2,16}(X) + 1.38906 \\ &\cdot 10^{7}B_{3,16}(X) + 1.90167\cdot 10^{7}B_{4,16}(X) + 2.37512\cdot 10^{7}B_{5,16}(X) + 2.80875\cdot 10^{7}B_{6,16}(X) \\ &+ 3.20213\cdot 10^{7}B_{7,16}(X) + 3.55508\cdot 10^{7}B_{8,16}(X) + 3.86768\cdot 10^{7}B_{9,16}(X) + 4.14025 \\ &\cdot 10^{7}B_{10,16}(X) + 4.37336\cdot 10^{7}B_{11,16}(X) + 4.56783\cdot 10^{7}B_{12,16}(X) + 4.7247\cdot 10^{7}B_{13,16}(X) \\ &+ 4.84524\cdot 10^{7}B_{14,16}(X) + 4.93093\cdot 10^{7}B_{15,16}(X) + 4.98343\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

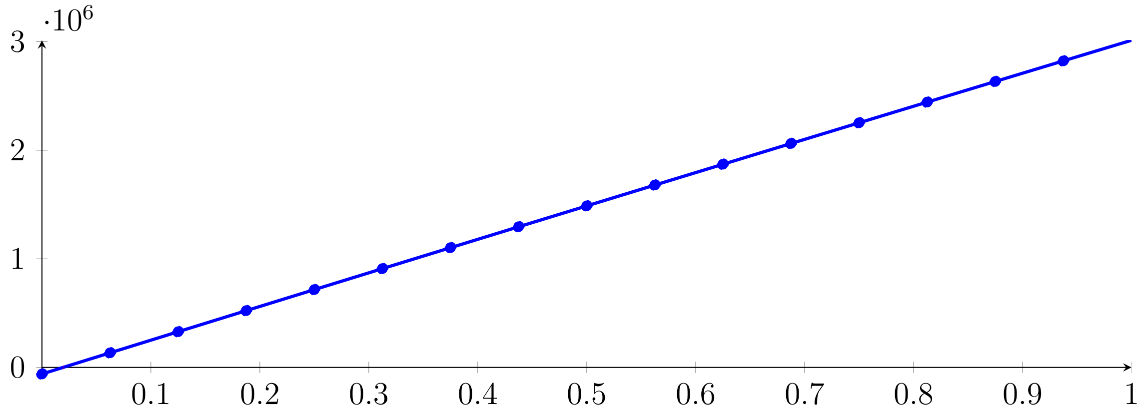
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

244.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-6.78234\cdot 10^{-31}X^{16} - 2.34473\cdot 10^{-27}X^{15} - 3.45581\cdot 10^{-24}X^{14} - 2.75921\cdot 10^{-21}X^{13} - 1.21242\cdot 10^{-18}X^{12} \\ &- 2.17572\cdot 10^{-16}X^{11} + 4.23544\cdot 10^{-14}X^{10} + 2.70025\cdot 10^{-11}X^{9} + 2.51306\cdot 10^{-09}X^{8} - 9.23474\cdot 10^{-07}X^{7} \\ &- 0.000177459X^{6} + 0.0161038X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot 10^{6}X - 60508.5 \\ = &-60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X) \\ &+ 910202B_{5,16}(X) + 1.1032\cdot 10^{6}B_{6,16}(X) + 1.29582\cdot 10^{6}B_{7,16}(X) + 1.48805\cdot 10^{6}B_{8,16}(X) \\ &+ 1.6799\cdot 10^{6}B_{9,16}(X) + 1.87137\cdot 10^{6}B_{10,16}(X) + 2.06245\cdot 10^{6}B_{11,16}(X) + 2.25315\cdot 10^{6}B_{12,16}(X) \\ &+ 2.44346\cdot 10^{6}B_{13,16}(X) + 2.63339\cdot 10^{6}B_{14,16}(X) + 2.82293\cdot 10^{6}B_{15,16}(X) + 3.01209\cdot 10^{6}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

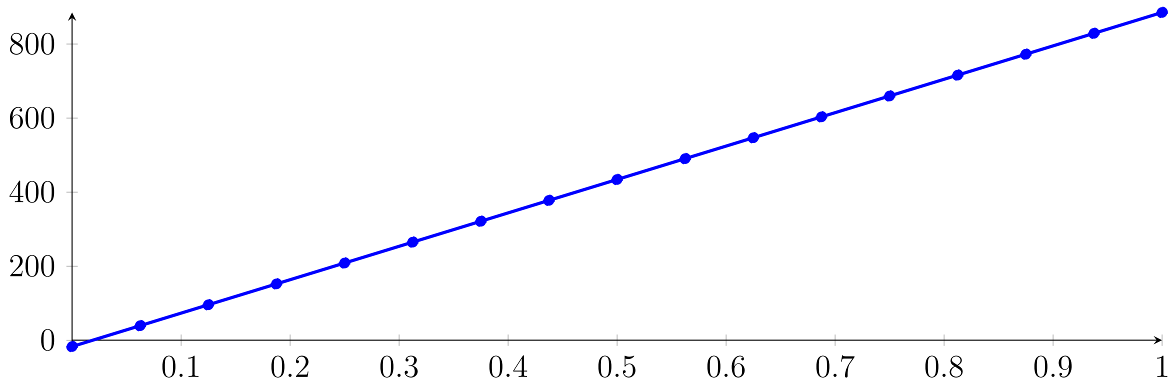
$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

244.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
p = &-1.65599\cdot 10^{-87}X^{16} - 1.97733\cdot 10^{-80}X^{15} - 1.00659\cdot 10^{-73}X^{14} - 2.77601\cdot 10^{-67}X^{13}\\
&- 4.21367\cdot 10^{-61}X^{12} - 2.61333\cdot 10^{-55}X^{11} + 1.75275\cdot 10^{-49}X^{10} + 3.8646\cdot 10^{-43}X^{9}\\
&+ 1.2441\cdot 10^{-37}X^{8} - 1.57525\cdot 10^{-31}X^{7} - 1.04661\cdot 10^{-25}X^{6} + 3.27355\cdot 10^{-20}X^{5}\\
&+ 3.06701\cdot 10^{-14}X^{4} - 5.67777\cdot 10^{-09}X^{3} - 0.00382618X^{2} + 902.448X - 17.178\\
= &-17.178B_{0,16}(X) + 39.225B_{1,16}(X) + 95.6279B_{2,16}(X) + 152.031B_{3,16}(X) + 208.434B_{4,16}(X)\\
&+ 264.837B_{5,16}(X) + 321.24B_{6,16}(X) + 377.642B_{7,16}(X) + 434.045B_{8,16}(X)\\
&+ 490.448B_{9,16}(X) + 546.851B_{10,16}(X) + 603.253B_{11,16}(X) + 659.656B_{12,16}(X)\\
&+ 716.059B_{13,16}(X) + 772.461B_{14,16}(X) + 828.864B_{15,16}(X) + 885.266B_{16,16}(X)
\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

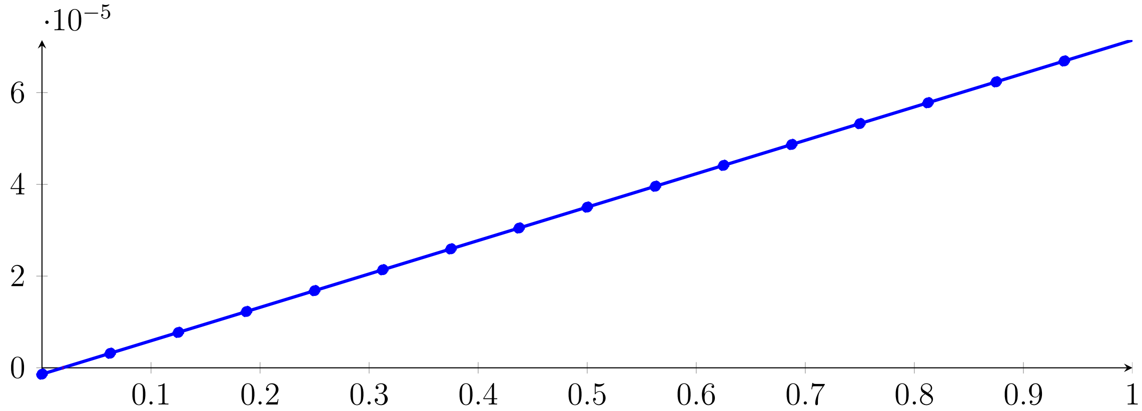
$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045\cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

244.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -5.36315 \cdot 10^{-201} X^{16} - 7.93495 \cdot 10^{-187} X^{15} - 5.0052 \cdot 10^{-173} X^{14} - 1.71037 \cdot 10^{-159} X^{13} - 3.21686 \cdot 10^{-146} X^{12} - 2.47211 \cdot 10^{-133} X^{11} + 2.05446 \cdot 10^{-120} X^{10} + 5.61285 \cdot 10^{-107} X^{9} + 2.23891 \cdot 10^{-94} X^{8} - 3.51264 \cdot 10^{-81} X^{7} - 2.89181 \cdot 10^{-68} X^{6} + 1.12075 \cdot 10^{-55} X^{5} + 1.30109 \cdot 10^{-42} X^{4} - 2.98449 \cdot 10^{-30} X^{3} - 2.49207 \cdot 10^{-17} X^{2} + 7.28316 \cdot 10^{-05} X - 1.38634 \cdot 10^{-06}$$
$$= -1.38634 \cdot 10^{-06} B_{0,16}(X) + 3.16564 \cdot 10^{-06} B_{1,16}(X) + 7.71761 \cdot 10^{-06} B_{2,16}(X) + 1.22696 \cdot 10^{-05} B_{3,16}(X) + 1.68216 \cdot 10^{-05} B_{4,16}(X) + 2.13735 \cdot 10^{-05} B_{5,16}(X) + 2.59255 \cdot 10^{-05} B_{6,16}(X) + 3.04775 \cdot 10^{-05} B_{7,16}(X) + 3.50295 \cdot 10^{-05} B_{8,16}(X) + 3.95814 \cdot 10^{-05} B_{9,16}(X) + 4.41334 \cdot 10^{-05} B_{10,16}(X) + 4.86854 \cdot 10^{-05} B_{11,16}(X) + 5.32374 \cdot 10^{-05} B_{12,16}(X) + 5.77893 \cdot 10^{-05} B_{13,16}(X) + 6.23413 \cdot 10^{-05} B_{14,16}(X) + 6.68933 \cdot 10^{-05} B_{15,16}(X) + 7.14453 \cdot 10^{-05} B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

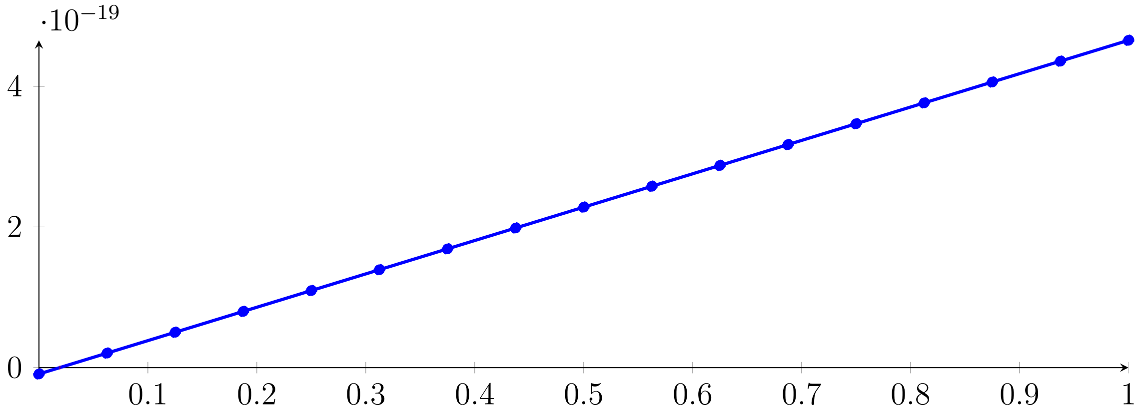
$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.51314 \cdot 10^{-15}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

244.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.70149 \cdot 10^{-323} X^{16} + 1.97819 \cdot 10^{-322} X^{15} - 4.34527 \cdot 10^{-322} X^{14} + 3.97182 \cdot 10^{-322} X^{13} - 1.87464 \cdot 10^{-316} X^{12} - 2.21189 \cdot 10^{-289} X^{11} + 2.82229 \cdot 10^{-262} X^{10} + 1.18385 \cdot 10^{-234} X^{9} + 7.25038 \cdot 10^{-208} X^{8} - 1.74649 \cdot 10^{-180} X^{7} - 2.20756 \cdot 10^{-153} X^{6} + 1.31359 \cdot 10^{-126} X^{5} + 2.34136 \cdot 10^{-99} X^{4} - 8.24597 \cdot 10^{-73} X^{3} - 1.05716 \cdot 10^{-45} X^{2} + 4.74362 \cdot 10^{-19} X - 9.02941 \cdot 10^{-21}$$
$$= -9.02941 \cdot 10^{-21} B_{0,16}(X) + 2.06182 \cdot 10^{-20} B_{1,16}(X) + 5.02659 \cdot 10^{-20} B_{2,16}(X) + 7.99135 \cdot 10^{-20} B_{3,16}(X) + 1.09561 \cdot 10^{-19} B_{4,16}(X) + 1.39209 \cdot 10^{-19} B_{5,16}(X) + 1.68856 \cdot 10^{-19} B_{6,16}(X) + 1.98504 \cdot 10^{-19} B_{7,16}(X) + 2.28152 \cdot 10^{-19} B_{8,16}(X) + 2.57799 \cdot 10^{-19} B_{9,16}(X) + 2.87447 \cdot 10^{-19} B_{10,16}(X) + 3.17095 \cdot 10^{-19} B_{11,16}(X) + 3.46742 \cdot 10^{-19} B_{12,16}(X) + 3.7639 \cdot 10^{-19} B_{13,16}(X) + 4.06038 \cdot 10^{-19} B_{14,16}(X) + 4.35685 \cdot 10^{-19} B_{15,16}(X) + 4.65333 \cdot 10^{-19} B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $4.2421 \cdot 10^{-29}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

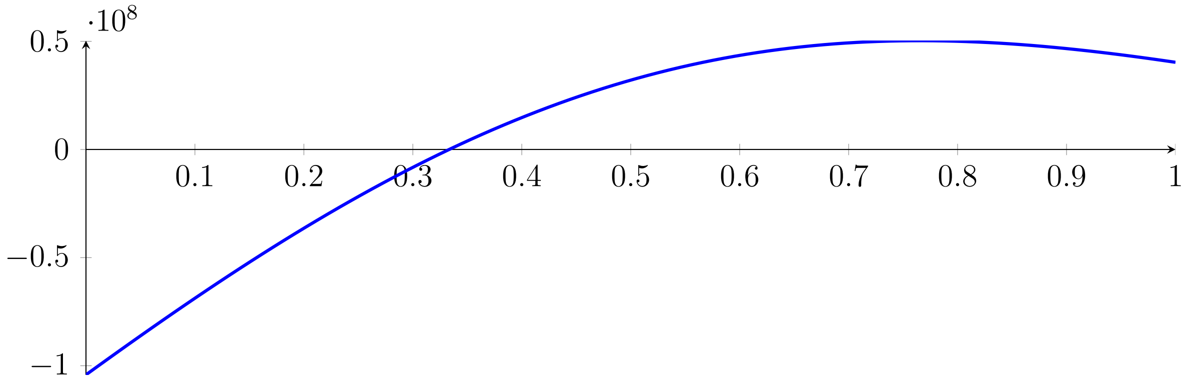
244.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 7!

244.8 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1\cdot 10^{-32}$.

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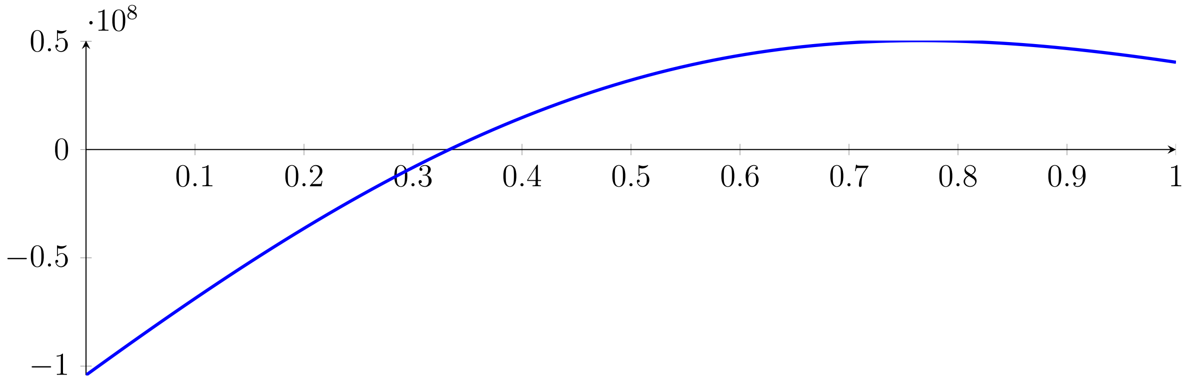
$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} + 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8}$$

Called QuadClip with input polynomial on interval $[0,1]$:

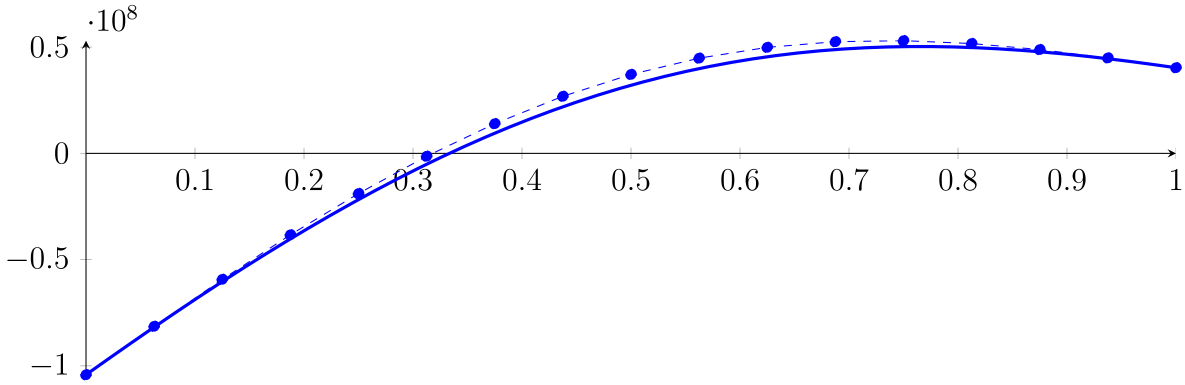
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} \\ &+ 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8} \end{aligned}$$



245.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

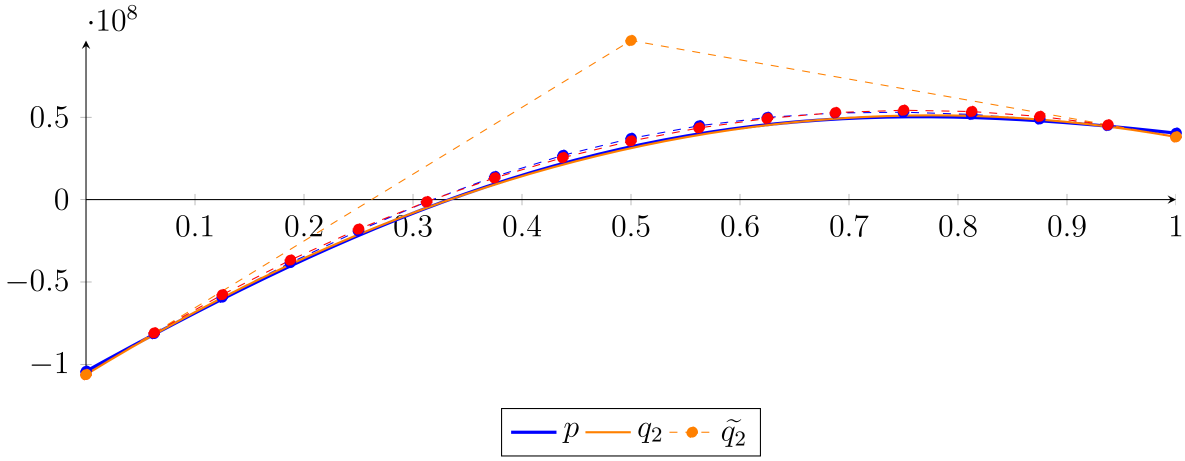
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} \\ &+ 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8} \\ = &-1.04167{\cdot}10^{8}B_{0,16}(X) - 8.13802{\cdot}10^{7}B_{1,16}(X) - 5.92882{\cdot}10^{7}B_{2,16}(X) - 3.83203 \\ &{\cdot}10^{7}B_{3,16}(X) - 1.88844{\cdot}10^{7}B_{4,16}(X) - 1.34837{\cdot}10^{6}B_{5,16}(X) + 1.39781{\cdot}10^{7}B_{6,16}(X) \\ &+ 2.68604{\cdot}10^{7}B_{7,16}(X) + 3.71532{\cdot}10^{7}B_{8,16}(X) + 4.48105{\cdot}10^{7}B_{9,16}(X) + 4.98901 \\ &{\cdot}10^{7}B_{10,16}(X) + 5.25504{\cdot}10^{7}B_{11,16}(X) + 5.30407{\cdot}10^{7}B_{12,16}(X) + 5.16832{\cdot}10^{7}B_{13,16}(X) \\ &+ 4.88488{\cdot}10^{7}B_{14,16}(X) + 4.49297{\cdot}10^{7}B_{15,16}(X) + 4.03108{\cdot}10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181\cdot 10^8 X^2 + 4.05417\cdot 10^8 X - 1.06197\cdot 10^8$$
$$= -1.06197\cdot 10^8 B_{0,2} + 9.65115\cdot 10^7 B_{1,2} + 3.80384\cdot 10^7 B_{2,2}$$

$$\widetilde{q_2} = 3.74473\cdot 10^{-288} X^{16} - 3.08889\cdot 10^{-287} X^{15} + 1.17443\cdot 10^{-286} X^{14} - 2.71603\cdot 10^{-286} X^{13} + 4.23554\cdot 10^{-286} X^{12} - 4.66107\cdot 10^{-286} X^{11} + 3.6845\cdot 10^{-286} X^{10} - 2.09811\cdot 10^{-286} X^9 + 8.58708\cdot 10^{-287} X^8 - 2.54063\cdot 10^{-287} X^7 + 5.57852\cdot 10^{-288} X^6 - 9.21534\cdot 10^{-289} X^5 + 1.02122\cdot 10^{-289} X^4 - 5.95624\cdot 10^{-291} X^3 - 2.61181\cdot 10^8 X^2 + 4.05417\cdot 10^8 X - 1.06197\cdot 10^8$$
$$= -1.06197\cdot 10^8 B_{0,16} - 8.08583\cdot 10^7 B_{1,16} - 5.76963\cdot 10^7 B_{2,16} - 3.67107\cdot 10^7 B_{3,16} - 1.79017\cdot 10^7 B_{4,16} - 1.26924\cdot 10^6 B_{5,16} + 1.31867\cdot 10^7 B_{6,16} + 2.54662\cdot 10^7 B_{7,16} + 3.55692\cdot 10^7 B_{8,16} + 4.34956\cdot 10^7 B_{9,16} + 4.92456\cdot 10^7 B_{10,16} + 5.2819\cdot 10^7 B_{11,16} + 5.42159\cdot 10^7 B_{12,16} + 5.34363\cdot 10^7 B_{13,16} + 5.04802\cdot 10^7 B_{14,16} + 4.53476\cdot 10^7 B_{15,16} + 3.80384\cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233\cdot 10^6$.

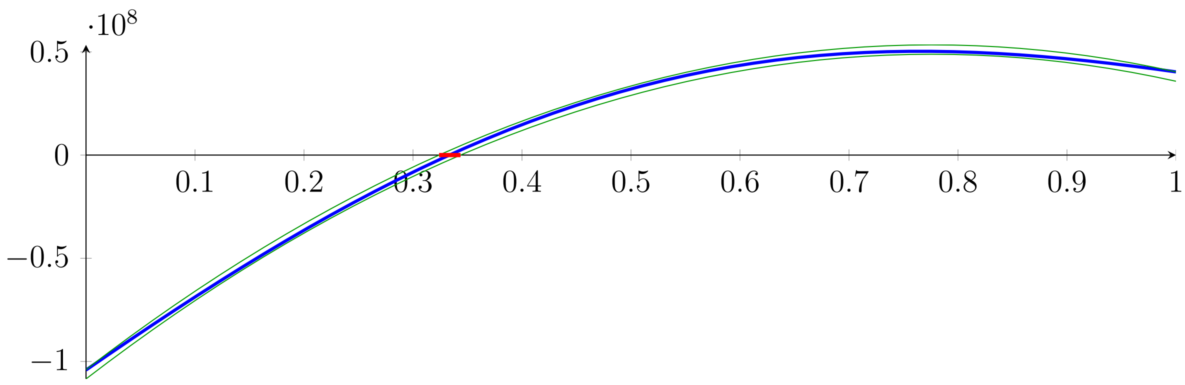
Bounding polynomials M and m:

$$M = -2.61181\cdot 10^8 X^2 + 4.05417\cdot 10^8 X - 1.03924\cdot 10^8$$
$$m = -2.61181\cdot 10^8 X^2 + 4.05417\cdot 10^8 X - 1.08469\cdot 10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

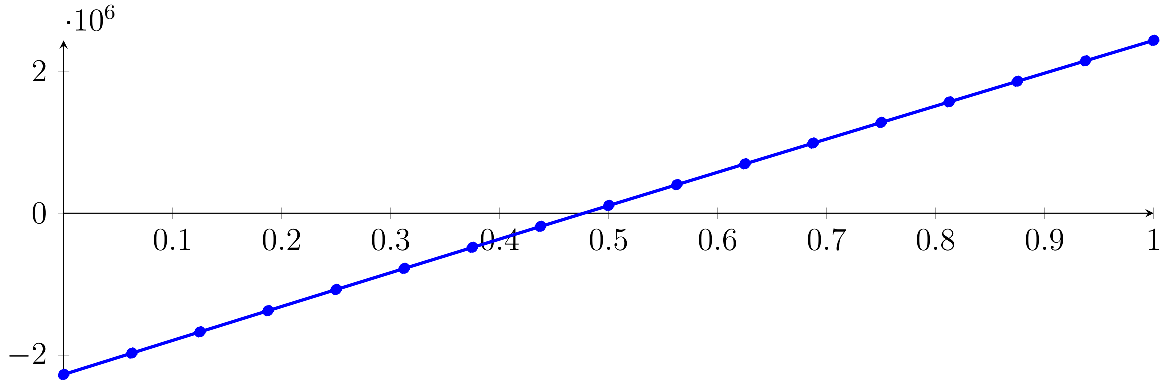
Longest intersection interval: 0.0196686

$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

245.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

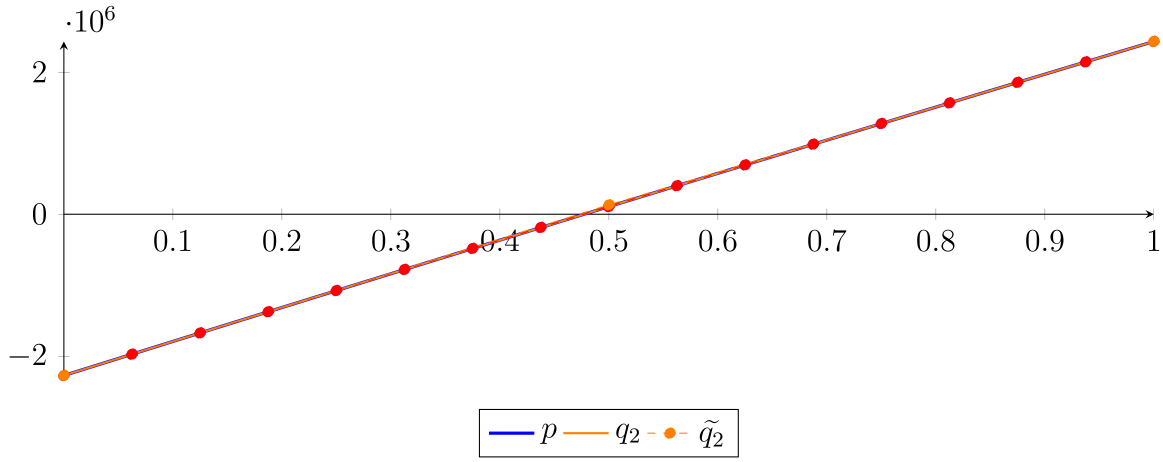
$$p = -5.0162\cdot10^{-28}X^{16} - 1.14383\cdot10^{-24}X^{15} - 1.11125\cdot10^{-21}X^{14} - 5.84096\cdot10^{-19}X^{13} - 1.684\cdot10^{-16}X^{12} - 1.94707\cdot10^{-14}X^{11} + 2.73227\cdot10^{-12}X^{10} + 1.09727\cdot10^{-09}X^{9} + 6.37314\cdot10^{-08}X^{8} - 1.68645\cdot10^{-05}X^{7} - 0.00205892X^{6} + 0.132721X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874\cdot10^{6}X - 2.2715\cdot10^{6}$$
$$= -2.2715\cdot10^{6}B_{0,16}(X) - 1.97096\cdot10^{6}B_{1,16}(X) - 1.67127\cdot10^{6}B_{2,16}(X) - 1.37244\cdot10^{6}B_{3,16}(X) - 1.07448\cdot10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806\cdot10^{6}B_{12,16}(X) + 1.5682\cdot10^{6}B_{13,16}(X) + 1.85747\cdot10^{6}B_{14,16}(X) + 2.14586\cdot10^{6}B_{15,16}(X) + 2.43338\cdot10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265X^{2} + 4.80923\cdot10^{6}X - 2.27154\cdot10^{6}$$
$$= -2.27154\cdot10^{6}B_{0,2} + 133069B_{1,2} + 2.43342\cdot10^{6}B_{2,2}$$

$$\widetilde{q}_2 = -8.18461\cdot10^{-291}X^{16} + 4.13745\cdot10^{-290}X^{15} - 5.8254\cdot10^{-290}X^{14} - 6.80919\cdot10^{-290}X^{13} + 3.48059\cdot10^{-289}X^{12} - 5.47245\cdot10^{-289}X^{11} + 4.8261\cdot10^{-289}X^{10} - 2.67867\cdot10^{-289}X^{9} + 1.00314\cdot10^{-289}X^{8} - 2.84766\cdot10^{-290}X^{7} + 6.85072\cdot10^{-291}X^{6} - 1.12501\cdot10^{-291}X^{5} + 1.24731\cdot10^{-293}X^{4} + 2.25944\cdot10^{-293}X^{3} - 104265X^{2} + 4.80923\cdot10^{6}X - 2.27154\cdot10^{6}$$
$$= -2.27154\cdot10^{6}B_{0,16} - 1.97097\cdot10^{6}B_{1,16} - 1.67126\cdot10^{6}B_{2,16} - 1.37242\cdot10^{6}B_{3,16} - 1.07445\cdot10^{6}B_{4,16} - 777350B_{5,16} - 481118B_{6,16} - 185754B_{7,16} + 108740B_{8,16} + 402366B_{9,16} + 695123B_{10,16} + 987011B_{11,16} + 1.27803\cdot10^{6}B_{12,16} + 1.56818\cdot10^{6}B_{13,16} + 1.85746\cdot10^{6}B_{14,16} + 2.14587\cdot10^{6}B_{15,16} + 2.43342\cdot10^{6}B_{16,16}$$



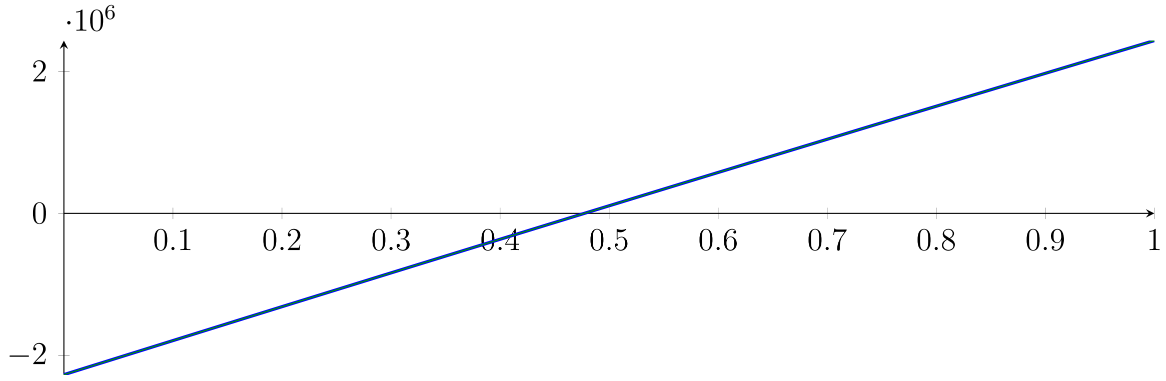
The maximum difference of the Bézier coefficients is $\delta = 40.5742$.
Bounding polynomials M and m:

$$M = -104265X^2 + 4.80923{\cdot}10^6X - 2.2715{\cdot}10^6$$
$$m = -104265X^2 + 4.80923{\cdot}10^6X - 2.27159{\cdot}10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



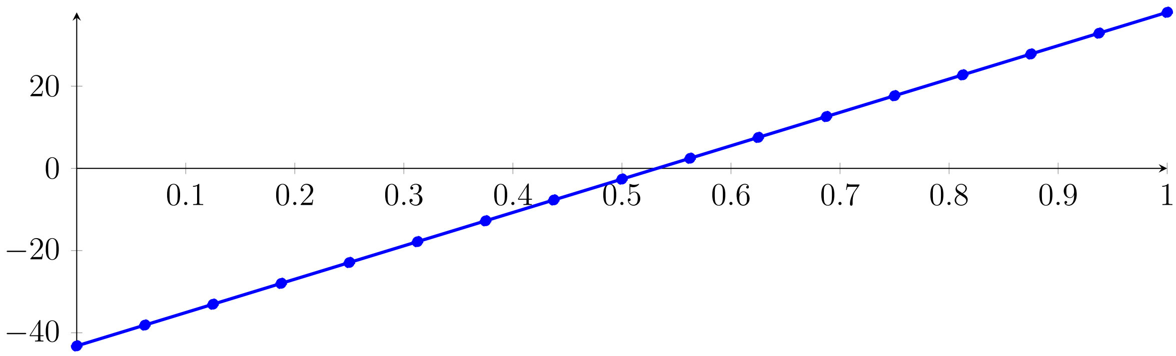
$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301{\cdot}10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

245.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

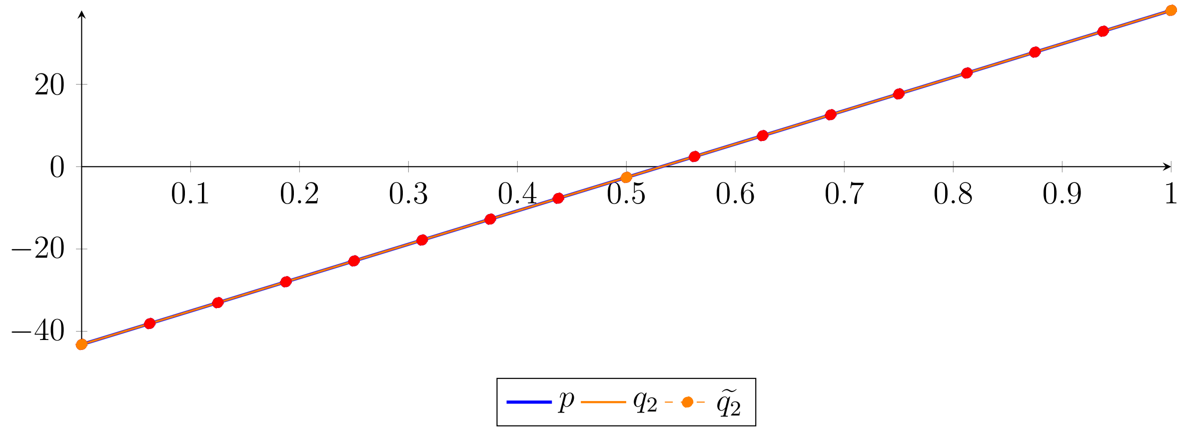
$$\begin{aligned}
p = &-3.02667{\cdot}10^{-104}X^{16} - 4.019{\cdot}10^{-96}X^{15} - 2.27522{\cdot}10^{-88}X^{14} - 6.97783{\cdot}10^{-81}X^{13} \\
&- 1.17785{\cdot}10^{-73}X^{12} - 8.12373{\cdot}10^{-67}X^{11} + 6.05916{\cdot}10^{-60}X^{10} + 1.48569{\cdot}10^{-52}X^9 \\
&+ 5.31875{\cdot}10^{-46}X^8 - 7.48919{\cdot}10^{-39}X^7 - 5.53349{\cdot}10^{-32}X^6 + 1.92471{\cdot}10^{-25}X^5 \\
&+ 2.00536{\cdot}10^{-18}X^4 - 4.12844{\cdot}10^{-12}X^3 - 3.09388{\cdot}10^{-05}X^2 + 81.1506X - 43.1911 \\
= &-43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\
&- 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X) \\
&+ 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X) \\
&+ 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X)
\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_2 &= -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911\\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_2 &= -1.88281{\cdot}10^{-295}X^{16} + 1.09893{\cdot}10^{-294}X^{15} - 2.26419{\cdot}10^{-294}X^{14} + 8.08223{\cdot}10^{-295}X^{13}\\ &\quad + 4.92626{\cdot}10^{-294}X^{12} - 1.07605{\cdot}10^{-293}X^{11} + 1.11858{\cdot}10^{-293}X^{10} - 6.87288{\cdot}10^{-294}X^{9}\\ &\quad + 2.54873{\cdot}10^{-294}X^{8} - 5.2305{\cdot}10^{-295}X^{7} + 3.18923{\cdot}10^{-296}X^{6} + 1.34092{\cdot}10^{-296}X^{5}\\ &\quad - 4.89549{\cdot}10^{-297}X^{4} + 5.89947{\cdot}10^{-298}X^{3} - 3.09389{\cdot}10^{-05}X^{2} + 81.1506X - 43.1911\\ &= -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16}\\ &\quad - 12.7597B_{6,16} - 7.68778B_{7,16} - 2.61587B_{8,16} + 2.45604B_{9,16} + 7.52795B_{10,16} + 12.5999B_{11,16}\\ &\quad + 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.06422{\cdot}10^{-13}$.

Bounding polynomials M and m:

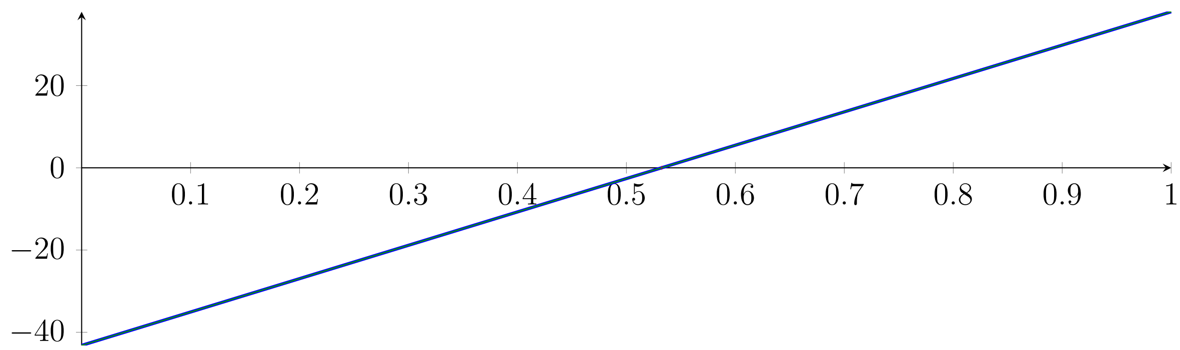
$$M = -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911$$

$$m = -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911$$

Root of M and m:

$$N(M) = \{0.532235, 2.62293{\cdot}10^{6}\} \qquad N(m) = \{0.532235, 2.62293{\cdot}10^{6}\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

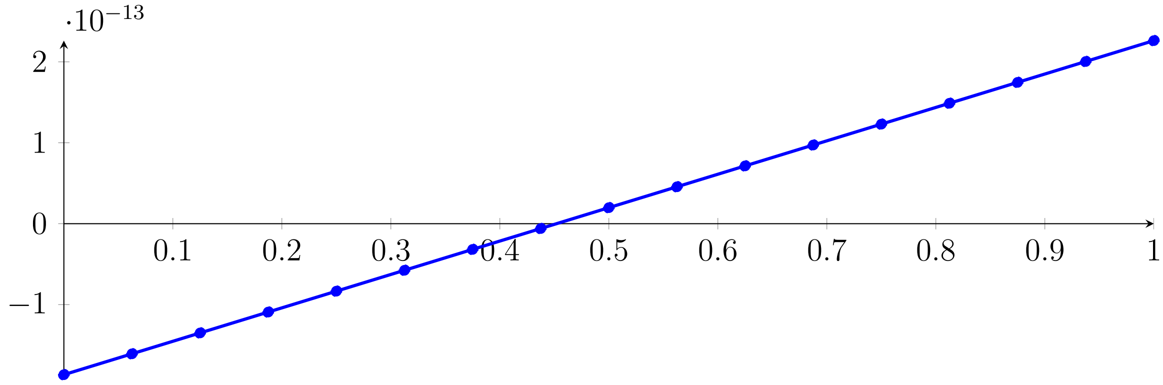
Longest intersection interval: $5.08738{\cdot}10^{-15}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

245.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

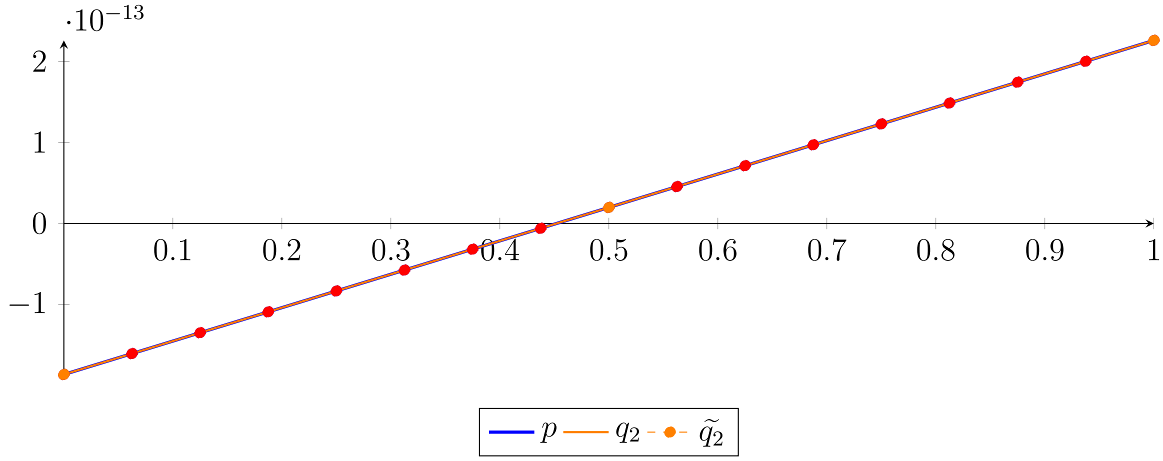
$$p = -3.84502\cdot10^{-319}X^{16} - 1.59047\cdot10^{-310}X^{15} - 1.76985\cdot10^{-288}X^{14} - 1.06694\cdot10^{-266}X^{13} - 3.54011\cdot10^{-245}X^{12} - 4.79942\cdot10^{-224}X^{11} + 7.03641\cdot10^{-203}X^{10} + 3.39135\cdot10^{-181}X^{9} + 2.3865\cdot10^{-160}X^{8} - 6.60529\cdot10^{-139}X^{7} - 9.59319\cdot10^{-118}X^{6} + 6.55895\cdot10^{-97}X^{5} + 1.34328\cdot10^{-75}X^{4} - 5.43584\cdot10^{-55}X^{3} - 8.00741\cdot10^{-34}X^{2} + 4.12843\cdot10^{-13}X - 1.86598\cdot10^{-13}$$
$$= -1.86598\cdot10^{-13}B_{0,16}(X) - 1.60795\cdot10^{-13}B_{1,16}(X) - 1.34993\cdot10^{-13}B_{2,16}(X) - 1.0919\cdot10^{-13}B_{3,16}(X) - 8.33872\cdot10^{-14}B_{4,16}(X) - 5.75845\cdot10^{-14}B_{5,16}(X) - 3.17818\cdot10^{-14}B_{6,16}(X) - 5.97912\cdot10^{-15}B_{7,16}(X) + 1.98236\cdot10^{-14}B_{8,16}(X) + 4.56263\cdot10^{-14}B_{9,16}(X) + 7.1429\cdot10^{-14}B_{10,16}(X) + 9.72317\cdot10^{-14}B_{11,16}(X) + 1.23034\cdot10^{-13}B_{12,16}(X) + 1.48837\cdot10^{-13}B_{13,16}(X) + 1.7464\cdot10^{-13}B_{14,16}(X) + 2.00443\cdot10^{-13}B_{15,16}(X) + 2.26245\cdot10^{-13}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -8.00741\cdot10^{-34}X^{2} + 4.12843\cdot10^{-13}X - 1.86598\cdot10^{-13}$$
$$= -1.86598\cdot10^{-13}B_{0,2} + 1.98236\cdot10^{-14}B_{1,2} + 2.26245\cdot10^{-13}B_{2,2}$$

$$\widetilde{q}_2 = -8.08289\cdot10^{-310}X^{16} + 4.33931\cdot10^{-309}X^{15} - 8.19282\cdot10^{-309}X^{14} + 3.30456\cdot10^{-309}X^{13} + 1.10762\cdot10^{-308}X^{12} - 2.01579\cdot10^{-308}X^{11} + 1.55412\cdot10^{-308}X^{10} - 6.55981\cdot10^{-309}X^{9} + 2.12881\cdot10^{-309}X^{8} - 1.08538\cdot10^{-309}X^{7} + 5.4154\cdot10^{-310}X^{6} - 1.38567\cdot10^{-310}X^{5} + 9.27245\cdot10^{-312}X^{4} + 1.80627\cdot10^{-312}X^{3} - 8.00741\cdot10^{-34}X^{2} + 4.12843\cdot10^{-13}X - 1.86598\cdot10^{-13}$$
$$= -1.86598\cdot10^{-13}B_{0,16} - 1.60795\cdot10^{-13}B_{1,16} - 1.34993\cdot10^{-13}B_{2,16} - 1.0919\cdot10^{-13}B_{3,16} - 8.33872\cdot10^{-14}B_{4,16} - 5.75845\cdot10^{-14}B_{5,16} - 3.17818\cdot10^{-14}B_{6,16} - 5.97912\cdot10^{-15}B_{7,16} + 1.98236\cdot10^{-14}B_{8,16} + 4.56263\cdot10^{-14}B_{9,16} + 7.1429\cdot10^{-14}B_{10,16} + 9.72317\cdot10^{-14}B_{11,16} + 1.23034\cdot10^{-13}B_{12,16} + 1.48837\cdot10^{-13}B_{13,16} + 1.7464\cdot10^{-13}B_{14,16} + 2.00443\cdot10^{-13}B_{15,16} + 2.26245\cdot10^{-13}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.71792{\cdot}10^{-56}$.
Bounding polynomials M and m:

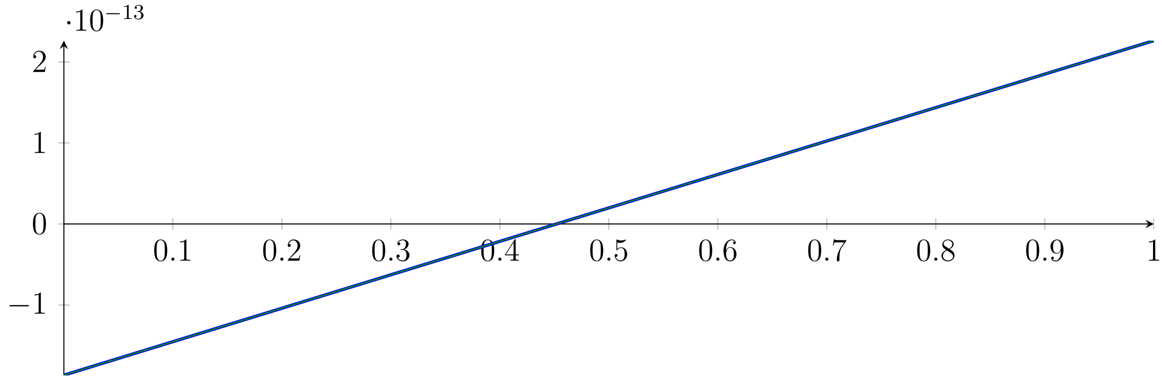
$$M = -8.00741{\cdot}10^{-34}X^2 + 4.12843{\cdot}10^{-13}X - 1.86598{\cdot}10^{-13}$$

$$m = -8.00741{\cdot}10^{-34}X^2 + 4.12843{\cdot}10^{-13}X - 1.86598{\cdot}10^{-13}$$

Root of M and m:

$$N(M) = \{0.451983, 5.15577{\cdot}10^{20}\} \qquad N(m) = \{0.451983, 5.15577{\cdot}10^{20}\}$$

Intersection intervals:



$$[0.451983, 0.451983]$$

Longest intersection interval: $1.31668{\cdot}10^{-43}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

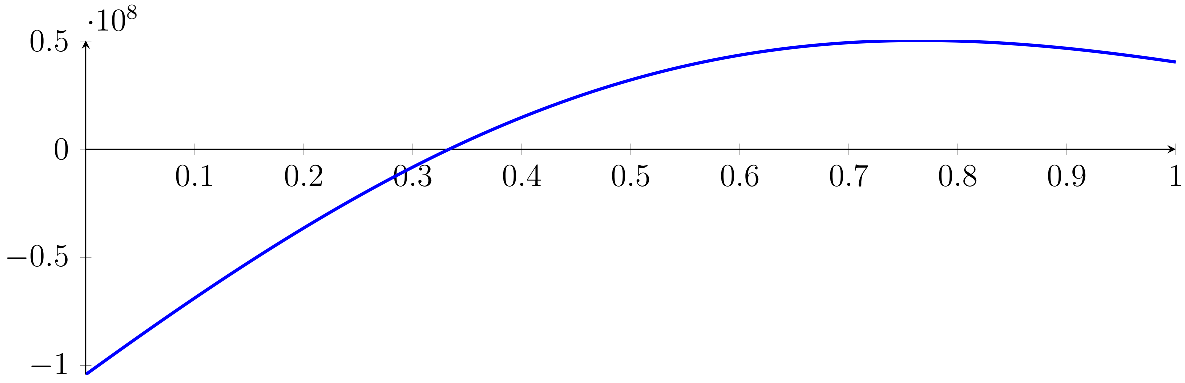
245.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

245.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1\cdot10^{-32}$.

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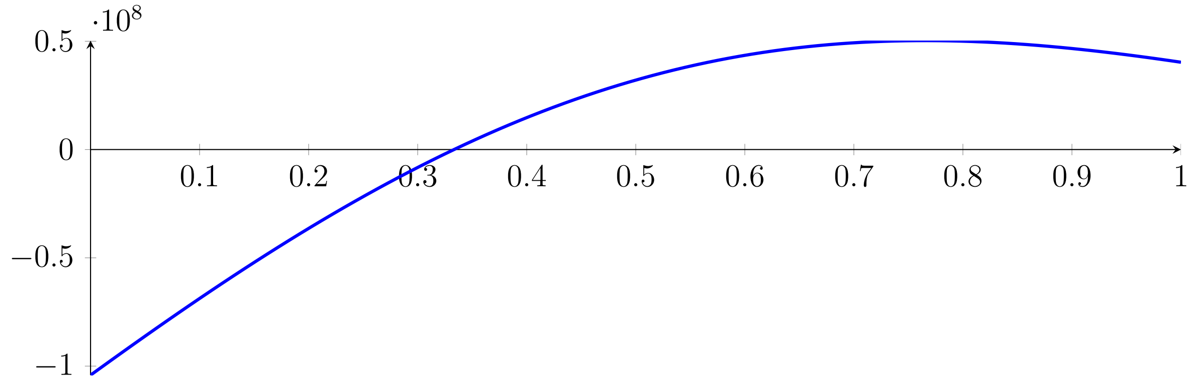
$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^{6}X^{9} - 2.52262 \cdot 10^{6}X^{8} - 1.43506 \cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \cdot 10^{7}X^{5} + 3.97396 \cdot 10^{7}X^{4} - 2.40625 \cdot 10^{8}X^{3} - 8.33333 \cdot 10^{7}X^{2} + 3.64583 \cdot 10^{8}X - 1.04167 \cdot 10^{8}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

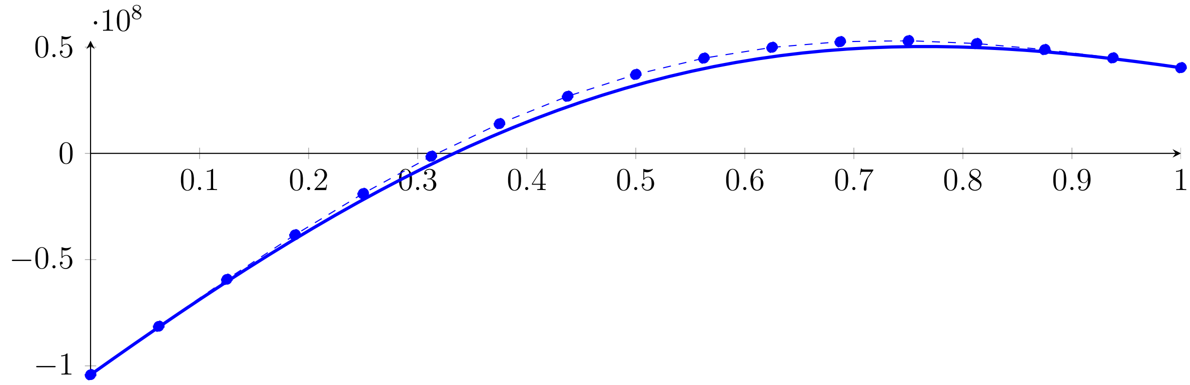
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^{6}X^{9} - 2.52262 \cdot 10^{6}X^{8} - 1.43506 \cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \cdot 10^{7}X^{5} + 3.97396 \cdot 10^{7}X^{4} - 2.40625 \cdot 10^{8}X^{3} - 8.33333 \cdot 10^{7}X^{2} + 3.64583 \cdot 10^{8}X - 1.04167 \cdot 10^{8}$$



246.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

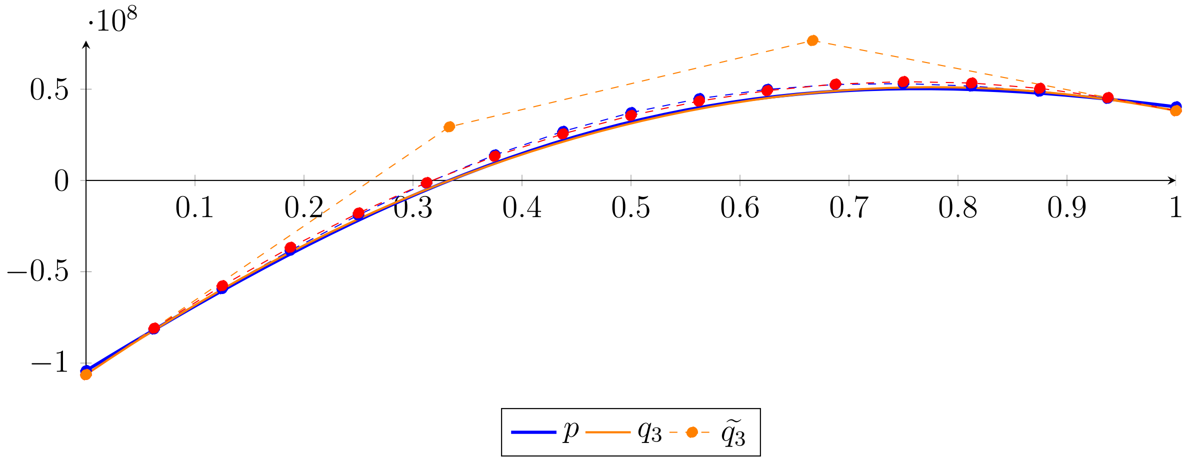
$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^{6}X^{9} - 2.52262 \cdot 10^{6}X^{8} - 1.43506 \cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \cdot 10^{7}X^{5} \\ & + 3.97396 \cdot 10^{7}X^{4} - 2.40625 \cdot 10^{8}X^{3} - 8.33333 \cdot 10^{7}X^{2} + 3.64583 \cdot 10^{8}X - 1.04167 \cdot 10^{8} \\ = & -1.04167 \cdot 10^{8}B_{0,16}(X) - 8.13802 \cdot 10^{7}B_{1,16}(X) - 5.92882 \cdot 10^{7}B_{2,16}(X) - 3.83203 \\ & \cdot 10^{7}B_{3,16}(X) - 1.88844 \cdot 10^{7}B_{4,16}(X) - 1.34837 \cdot 10^{6}B_{5,16}(X) + 1.39781 \cdot 10^{7}B_{6,16}(X) \\ & + 2.68604 \cdot 10^{7}B_{7,16}(X) + 3.71532 \cdot 10^{7}B_{8,16}(X) + 4.48105 \cdot 10^{7}B_{9,16}(X) + 4.98901 \\ & \cdot 10^{7}B_{10,16}(X) + 5.25504 \cdot 10^{7}B_{11,16}(X) + 5.30407 \cdot 10^{7}B_{12,16}(X) + 5.16832 \cdot 10^{7}B_{13,16}(X) \\ & + 4.88488 \cdot 10^{7}B_{14,16}(X) + 4.49297 \cdot 10^{7}B_{15,16}(X) + 4.03108 \cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812\cdot10^6X^3 - 2.65319\cdot10^8X^2 + 4.07072\cdot10^8X - 1.06335\cdot10^8$$
$$= -1.06335\cdot10^8B_{0,3} + 2.93558\cdot10^7B_{1,3} + 7.66068\cdot10^7B_{2,3} + 3.81764\cdot10^7B_{3,3}$$

$$\widetilde{q}_3 = 1.89955\cdot10^{-288}X^{16} - 1.4851\cdot10^{-287}X^{15} + 5.12896\cdot10^{-287}X^{14} - 1.02224\cdot10^{-286}X^{13} + 1.29027\cdot10^{-286}X^{12} - 1.07058\cdot10^{-286}X^{11} + 6.00739\cdot10^{-287}X^{10} - 2.54352\cdot10^{-287}X^9 + 1.12089\cdot10^{-287}X^8 - 5.8673\cdot10^{-288}X^7 + 2.52637\cdot10^{-288}X^6 - 6.79624\cdot10^{-289}X^5 + 9.36341\cdot10^{-290}X^4 + 2.75812\cdot10^6X^3 - 2.65319\cdot10^8X^2 + 4.07072\cdot10^8X - 1.06335\cdot10^8$$
$$= -1.06335\cdot10^8B_{0,16} - 8.08928\cdot10^7B_{1,16} - 5.76618\cdot10^7B_{2,16} - 3.66369\cdot10^7B_{3,16} - 1.78131\cdot10^7B_{4,16} - 1.18551\cdot10^6B_{5,16} + 1.32508\cdot10^7B_{6,16} + 2.55007\cdot10^7B_{7,16} + 3.55692\cdot10^7B_{8,16} + 4.34611\cdot10^7B_{9,16} + 4.91815\cdot10^7B_{10,16} + 5.27353\cdot10^7B_{11,16} + 5.41273\cdot10^7B_{12,16} + 5.33624\cdot10^7B_{13,16} + 5.04457\cdot10^7B_{14,16} + 4.53821\cdot10^7B_{15,16} + 3.81764\cdot10^7B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806\cdot10^6$.

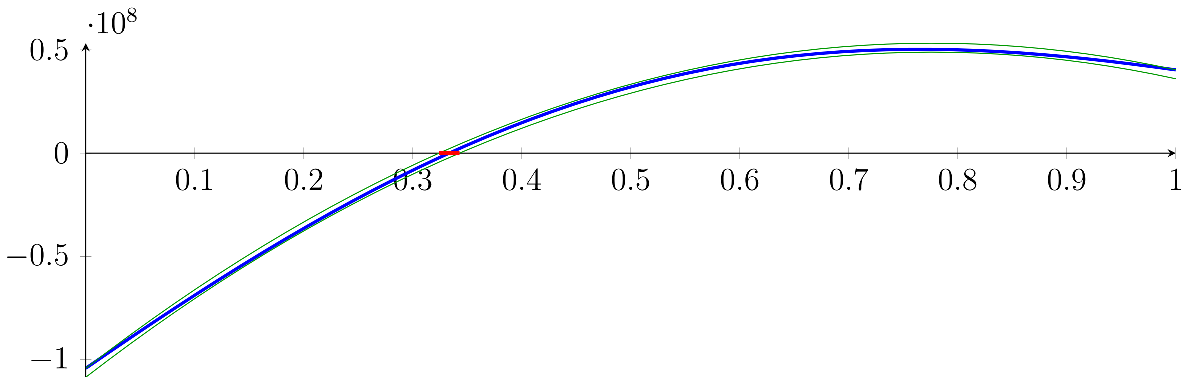
Bounding polynomials M and m:

$$M = 2.75812\cdot10^6X^3 - 2.65319\cdot10^8X^2 + 4.07072\cdot10^8X - 1.04167\cdot10^8$$
$$m = 2.75812\cdot10^6X^3 - 2.65319\cdot10^8X^2 + 4.07072\cdot10^8X - 1.08503\cdot10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

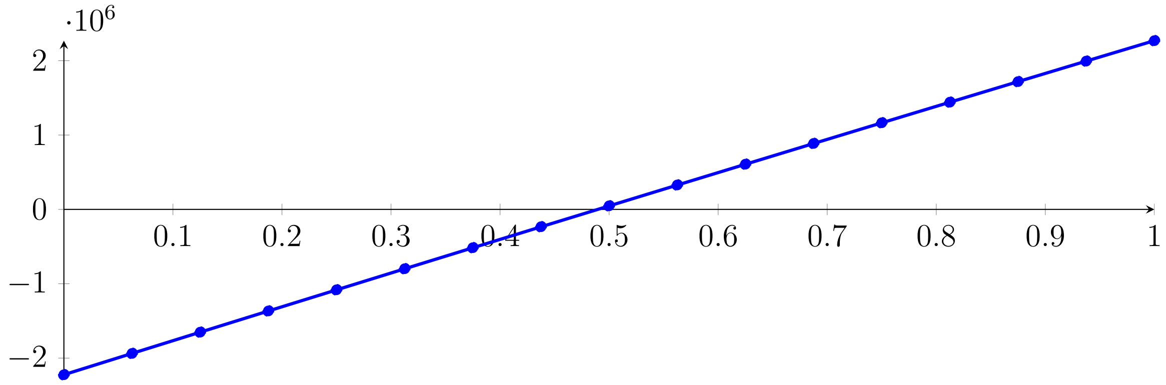
Longest intersection interval: 0.0187703

$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

246.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

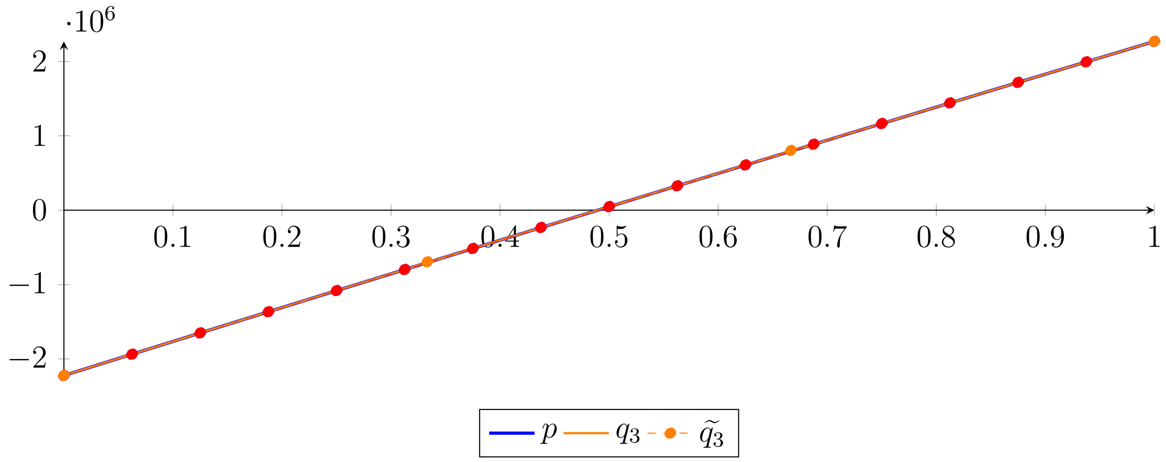
$$\begin{aligned}p &= -2.37433\cdot 10^{-28}X^{16} - 5.67363\cdot 10^{-25}X^{15} - 5.77631\cdot 10^{-22}X^{14} - 3.1818\cdot 10^{-19}X^{13} - 9.6142\cdot 10^{-17}X^{12} \\ &\quad - 1.16549\cdot 10^{-14}X^{11} + 1.71065\cdot 10^{-12}X^{10} + 7.20611\cdot 10^{-10}X^{9} + 4.39147\cdot 10^{-08}X^{8} - 1.21542\cdot 10^{-05}X^{7} \\ &\quad - 0.00155624X^{6} + 0.10496X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6} \\ &= -2.22335\cdot 10^{6}B_{0,16}(X) - 1.93666\cdot 10^{6}B_{1,16}(X) - 1.65074\cdot 10^{6}B_{2,16}(X) - 1.36561\cdot 10^{6}B_{3,16}(X) \\ &\quad - 1.08127\cdot 10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X) \\ &\quad + 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot 10^{6}B_{12,16}(X) + 1.44233 \\ &\quad \cdot 10^{6}B_{13,16}(X) + 1.71876\cdot 10^{6}B_{14,16}(X) + 1.99439\cdot 10^{6}B_{15,16}(X) + 2.26922\cdot 10^{6}B_{16,16}(X)\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}q_3 &= -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6} \\ &= -2.22335\cdot 10^{6}B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot 10^{6}B_{3,3}\end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= 4.41595\cdot 10^{-291}X^{16} - 1.48598\cdot 10^{-290}X^{15} - 2.81102\cdot 10^{-290}X^{14} + 2.65129\cdot 10^{-289}X^{13} \\ &\quad - 7.34593\cdot 10^{-289}X^{12} + 1.17654\cdot 10^{-288}X^{11} - 1.23119\cdot 10^{-288}X^{10} + 8.63493\cdot 10^{-289}X^{9} \\ &\quad - 3.92909\cdot 10^{-289}X^{8} + 1.01265\cdot 10^{-289}X^{7} - 6.13644\cdot 10^{-291}X^{6} - 3.92664\cdot 10^{-291}X^{5} \\ &\quad + 9.04488\cdot 10^{-292}X^{4} - 700.673X^{3} - 93879.9X^{2} + 4.58715\cdot 10^{6}X - 2.22335\cdot 10^{6} \\ &= -2.22335\cdot 10^{6}B_{0,16} - 1.93666\cdot 10^{6}B_{1,16} - 1.65074\cdot 10^{6}B_{2,16} - 1.36561\cdot 10^{6}B_{3,16} \\ &\quad - 1.08126\cdot 10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.4B_{8,16} \\ &\quad + 328650B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot 10^{6}B_{12,16} + 1.44233 \\ &\quad \cdot 10^{6}B_{13,16} + 1.71876\cdot 10^{6}B_{14,16} + 1.99439\cdot 10^{6}B_{15,16} + 2.26922\cdot 10^{6}B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m:

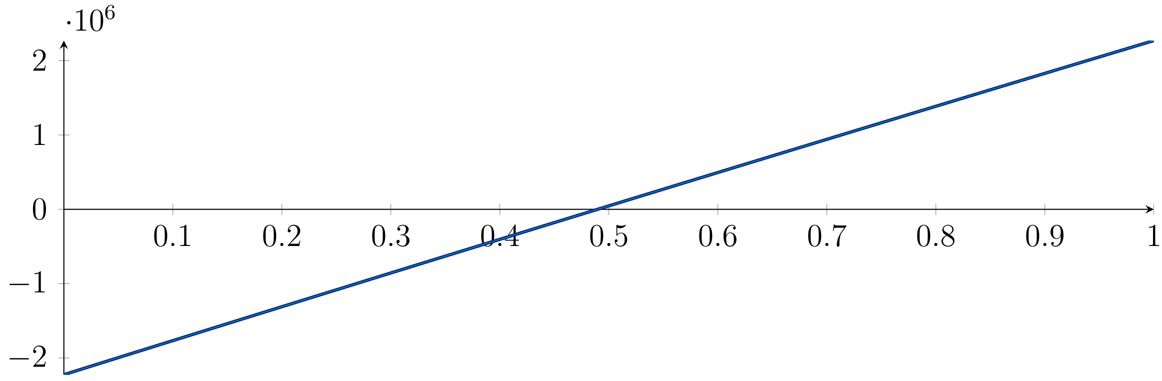
$$M = -700.673X^3 - 93879.9X^2 + 4.58715{\cdot}10^6 X - 2.22335{\cdot}10^6$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715{\cdot}10^6 X - 2.22335{\cdot}10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

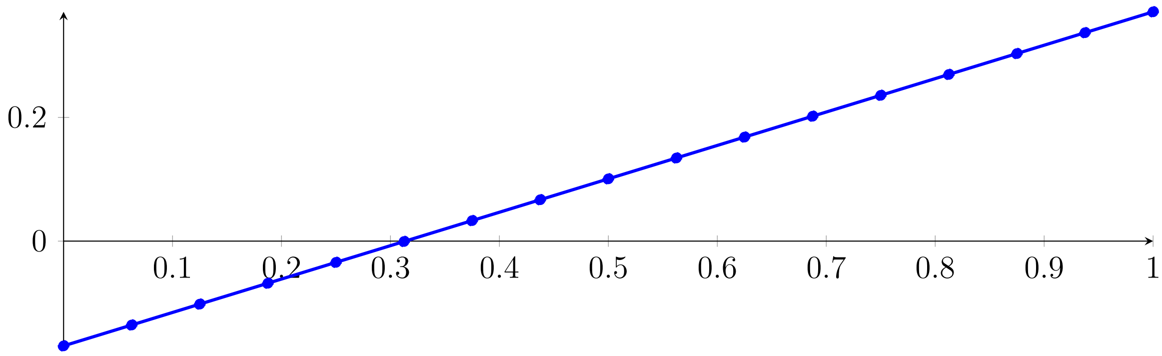
Longest intersection interval: $1.20174{\cdot}10^{-07}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

246.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

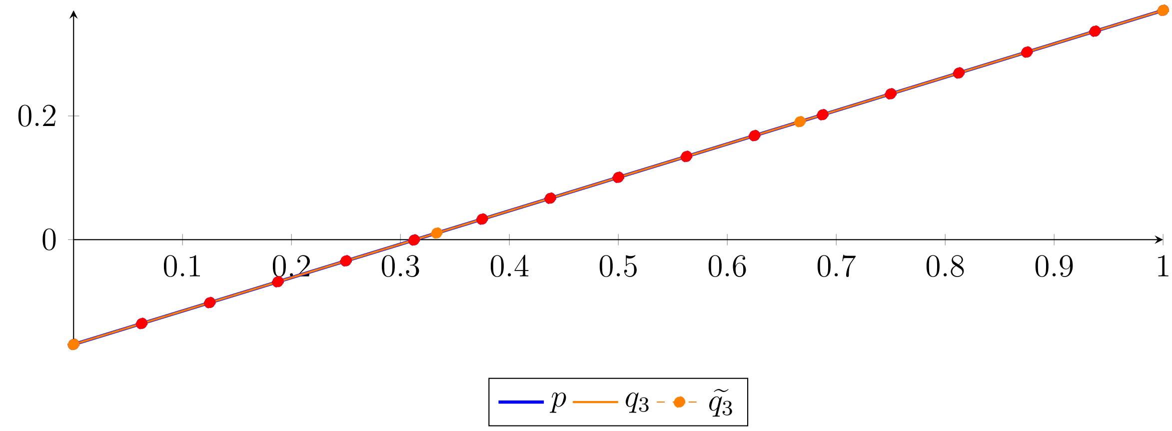
$$\begin{aligned} p = & -4.49274{\cdot}10^{-139}X^{16} - 8.96277{\cdot}10^{-129}X^{15} - 7.623{\cdot}10^{-119}X^{14} - 3.51238{\cdot}10^{-109}X^{13} \\ & - 8.90739{\cdot}10^{-100}X^{12} - 9.22984{\cdot}10^{-91}X^{11} + 1.03426{\cdot}10^{-81}X^{10} + 3.80998{\cdot}10^{-72}X^9 \\ & + 2.04919{\cdot}10^{-63}X^8 - 4.33497{\cdot}10^{-54}X^7 - 4.81204{\cdot}10^{-45}X^6 + 2.51462{\cdot}10^{-36}X^5 \\ & + 3.93622{\cdot}10^{-27}X^4 - 1.21745{\cdot}10^{-18}X^3 - 1.37072{\cdot}10^{-09}X^2 + 0.540148X - 0.169396 \\ = & -0.169396B_{0,16}(X) - 0.135637B_{1,16}(X) - 0.101877B_{2,16}(X) - 0.068118B_{3,16}(X) \\ & - 0.0343588B_{4,16}(X) - 0.000599488B_{5,16}(X) + 0.0331598B_{6,16}(X) \\ & + 0.066919B_{7,16}(X) + 0.100678B_{8,16}(X) + 0.134438B_{9,16}(X) + 0.168197B_{10,16}(X) \\ & + 0.201956B_{11,16}(X) + 0.235715B_{12,16}(X) + 0.269475B_{13,16}(X) \\ & + 0.303234B_{14,16}(X) + 0.336993B_{15,16}(X) + 0.370752B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -1.21745{\cdot}10^{-18}X^3 - 1.37072{\cdot}10^{-09}X^2 + 0.540148X - 0.169396$$
$$= -0.169396B_{0,3} + 0.0106536B_{1,3} + 0.190703B_{2,3} + 0.370752B_{3,3}$$

$$\widetilde{q_3} = 8.03185{\cdot}10^{-297}X^{16} - 6.20841{\cdot}10^{-296}X^{15} + 2.13274{\cdot}10^{-295}X^{14} - 4.26614{\cdot}10^{-295}X^{13} + 5.47461{\cdot}10^{-295}X^{12} - 4.70265{\cdot}10^{-295}X^{11} + 2.78551{\cdot}10^{-295}X^{10} - 1.22442{\cdot}10^{-295}X^{9} + 4.88954{\cdot}10^{-296}X^{8} - 2.11494{\cdot}10^{-296}X^{7} + 8.20665{\cdot}10^{-297}X^{6} - 2.15458{\cdot}10^{-297}X^{5} + 3.01517{\cdot}10^{-298}X^{4} - 1.21745{\cdot}10^{-18}X^{3} - 1.37072{\cdot}10^{-09}X^{2} + 0.540148X - 0.169396$$
$$= -0.169396B_{0,16} - 0.135637B_{1,16} - 0.101877B_{2,16} - 0.068118B_{3,16} - 0.0343588B_{4,16} - 0.000599488B_{5,16} + 0.0331598B_{6,16} + 0.066919B_{7,16} + 0.100678B_{8,16} + 0.134438B_{9,16} + 0.168197B_{10,16} + 0.201956B_{11,16} + 0.235715B_{12,16} + 0.269475B_{13,16} + 0.303234B_{14,16} + 0.336993B_{15,16} + 0.370752B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 5.62317{\cdot}10^{-29}$.

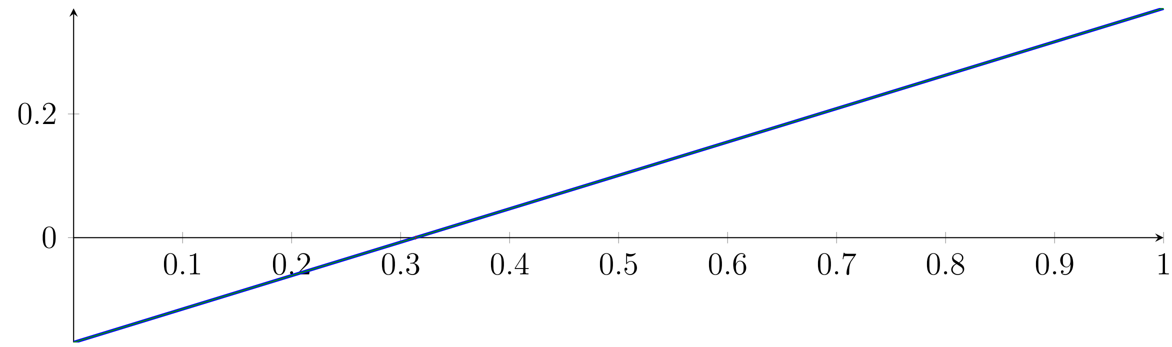
Bounding polynomials M and m:

$$M = -1.21745{\cdot}10^{-18}X^3 - 1.37072{\cdot}10^{-09}X^2 + 0.540148X - 0.169396$$
$$m = -1.21745{\cdot}10^{-18}X^3 - 1.37072{\cdot}10^{-09}X^2 + 0.540148X - 0.169396$$

Root of M and m:

$$N(M) = \{-1.43506{\cdot}10^{9}, 0.31361, 3.09167{\cdot}10^{8}\} \quad N(m) = \{-1.43506{\cdot}10^{9}, 0.31361, 3.09167{\cdot}10^{8}\}$$

Intersection intervals:



$$[0.31361, 0.31361]$$

Longest intersection interval: $2.08208{\cdot}10^{-28}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

246.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

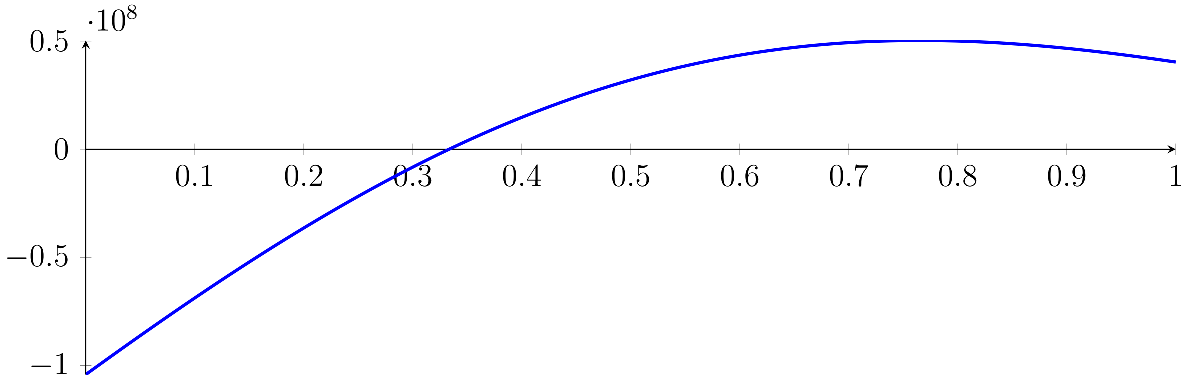
Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 4!

p(0) = -8.88188e-08 - p(1) -8.88188e-08

246.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

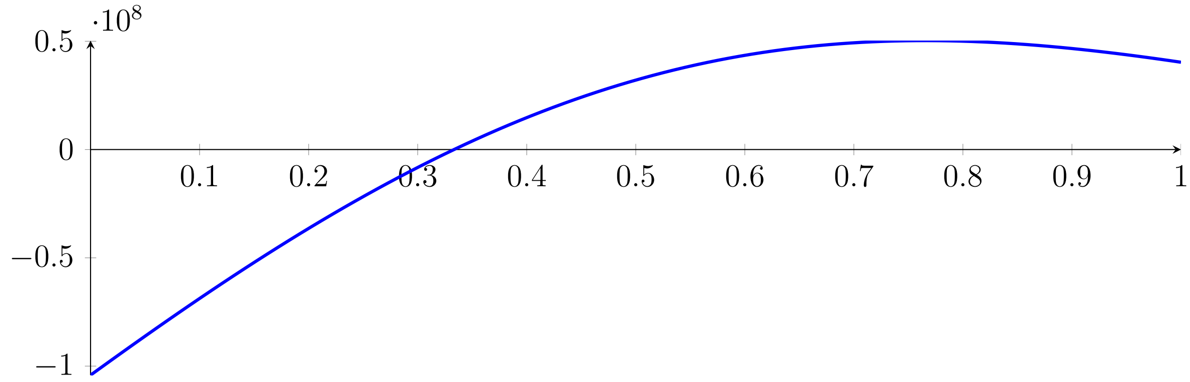
with precision $\varepsilon = 1\cdot10^{-32}$.

247 Running BezClip on f_{16} with epsilon 64

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called BezClip with input polynomial on interval $[0,1]$:

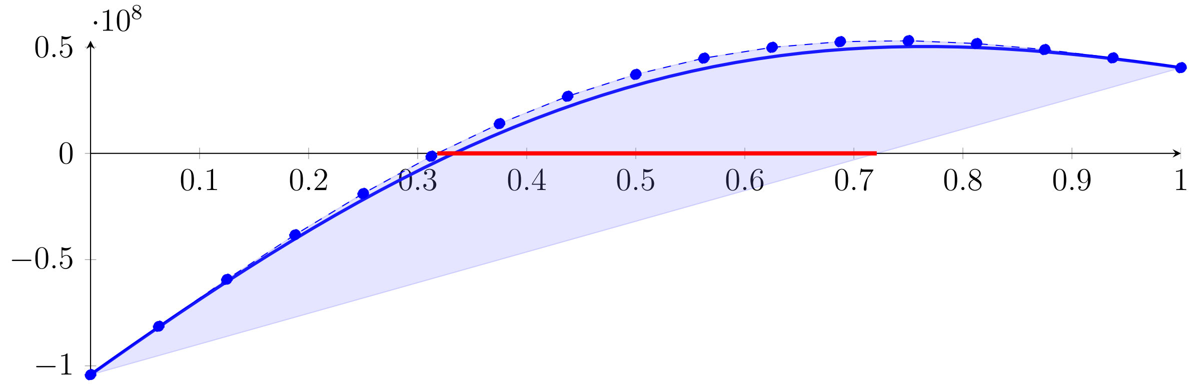
$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} \\ & + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \end{aligned}$$



247.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} \\ & + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ = & -1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ & \cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ & + 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ & \cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ & + 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

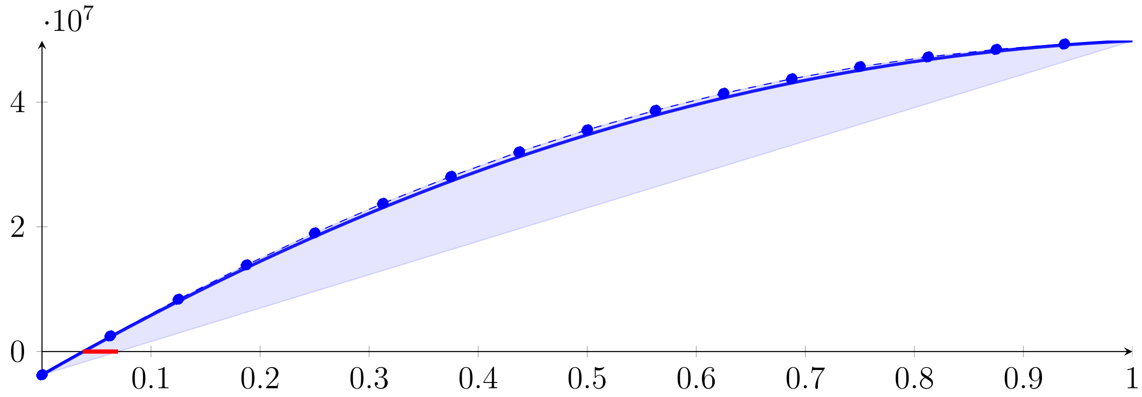
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

247.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -4.83858\cdot 10^{-07}X^{16} - 5.37355\cdot 10^{-05}X^{15} - 0.00254146X^{14} - 0.064977X^{13} - 0.909205X^{12} \\ & - 5.03924X^{11} + 36.4549X^{10} + 692.921X^{9} + 1886.97X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5} \\ & + 3.91945\cdot 10^{6}X^{4} - 7.54561\cdot 10^{6}X^{3} - 4.29225\cdot 10^{7}X^{2} + 9.97982\cdot 10^{7}X - 3.73501\cdot 10^{6} \\ = & -3.73501\cdot 10^{6}B_{0,16}(X) + 2.50237\cdot 10^{6}B_{1,16}(X) + 8.38207\cdot 10^{6}B_{2,16}(X) + 1.38906 \\ & \cdot 10^{7}B_{3,16}(X) + 1.90167\cdot 10^{7}B_{4,16}(X) + 2.37512\cdot 10^{7}B_{5,16}(X) + 2.80875\cdot 10^{7}B_{6,16}(X) \\ & + 3.20213\cdot 10^{7}B_{7,16}(X) + 3.55508\cdot 10^{7}B_{8,16}(X) + 3.86768\cdot 10^{7}B_{9,16}(X) + 4.14025 \\ & \cdot 10^{7}B_{10,16}(X) + 4.37336\cdot 10^{7}B_{11,16}(X) + 4.56783\cdot 10^{7}B_{12,16}(X) + 4.7247\cdot 10^{7}B_{13,16}(X) \\ & + 4.84524\cdot 10^{7}B_{14,16}(X) + 4.93093\cdot 10^{7}B_{15,16}(X) + 4.98343\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

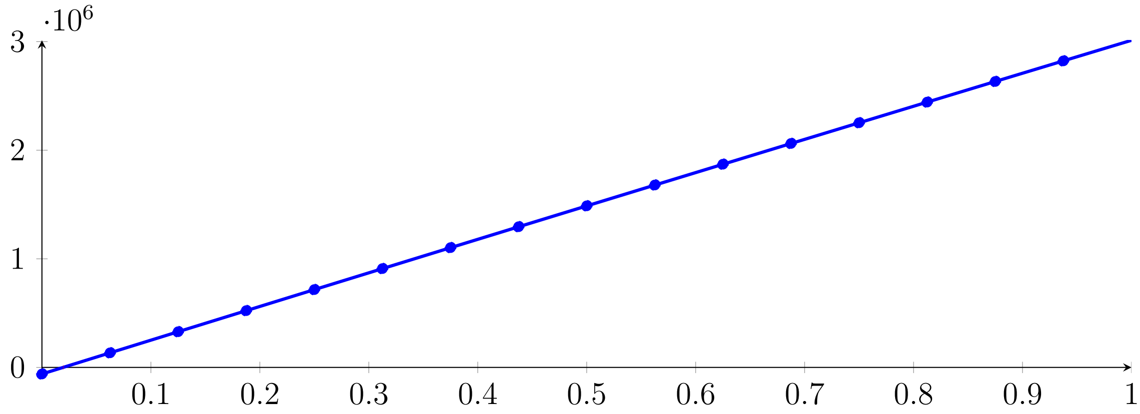
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

247.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -6.78234\cdot 10^{-31}X^{16} - 2.34473\cdot 10^{-27}X^{15} - 3.45581\cdot 10^{-24}X^{14} - 2.75921\cdot 10^{-21}X^{13} - 1.21242\cdot 10^{-18}X^{12} \\ & - 2.17572\cdot 10^{-16}X^{11} + 4.23544\cdot 10^{-14}X^{10} + 2.70025\cdot 10^{-11}X^{9} + 2.51306\cdot 10^{-09}X^{8} - 9.23474\cdot 10^{-07}X^{7} \\ & - 0.000177459X^{6} + 0.0161038X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot 10^{6}X - 60508.5 \\ = & -60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X) \\ & + 910202B_{5,16}(X) + 1.1032\cdot 10^{6}B_{6,16}(X) + 1.29582\cdot 10^{6}B_{7,16}(X) + 1.48805\cdot 10^{6}B_{8,16}(X) \\ & + 1.6799\cdot 10^{6}B_{9,16}(X) + 1.87137\cdot 10^{6}B_{10,16}(X) + 2.06245\cdot 10^{6}B_{11,16}(X) + 2.25315\cdot 10^{6}B_{12,16}(X) \\ & + 2.44346\cdot 10^{6}B_{13,16}(X) + 2.63339\cdot 10^{6}B_{14,16}(X) + 2.82293\cdot 10^{6}B_{15,16}(X) + 3.01209\cdot 10^{6}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

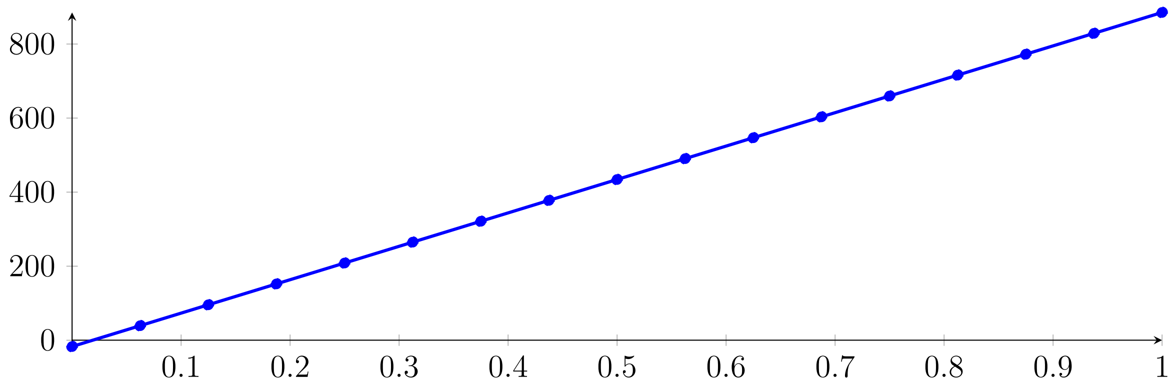
$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

247.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-1.65599\cdot 10^{-87}X^{16} - 1.97733\cdot 10^{-80}X^{15} - 1.00659\cdot 10^{-73}X^{14} - 2.77601\cdot 10^{-67}X^{13} \\ &- 4.21367\cdot 10^{-61}X^{12} - 2.61333\cdot 10^{-55}X^{11} + 1.75275\cdot 10^{-49}X^{10} + 3.8646\cdot 10^{-43}X^{9} \\ &+ 1.2441\cdot 10^{-37}X^{8} - 1.57525\cdot 10^{-31}X^{7} - 1.04661\cdot 10^{-25}X^{6} + 3.27355\cdot 10^{-20}X^{5} \\ &+ 3.06701\cdot 10^{-14}X^{4} - 5.67777\cdot 10^{-09}X^{3} - 0.00382618X^{2} + 902.448X - 17.178 \\ = &-17.178B_{0,16}(X) + 39.225B_{1,16}(X) + 95.6279B_{2,16}(X) + 152.031B_{3,16}(X) + 208.434B_{4,16}(X) \\ &+ 264.837B_{5,16}(X) + 321.24B_{6,16}(X) + 377.642B_{7,16}(X) + 434.045B_{8,16}(X) \\ &+ 490.448B_{9,16}(X) + 546.851B_{10,16}(X) + 603.253B_{11,16}(X) + 659.656B_{12,16}(X) \\ &+ 716.059B_{13,16}(X) + 772.461B_{14,16}(X) + 828.864B_{15,16}(X) + 885.266B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

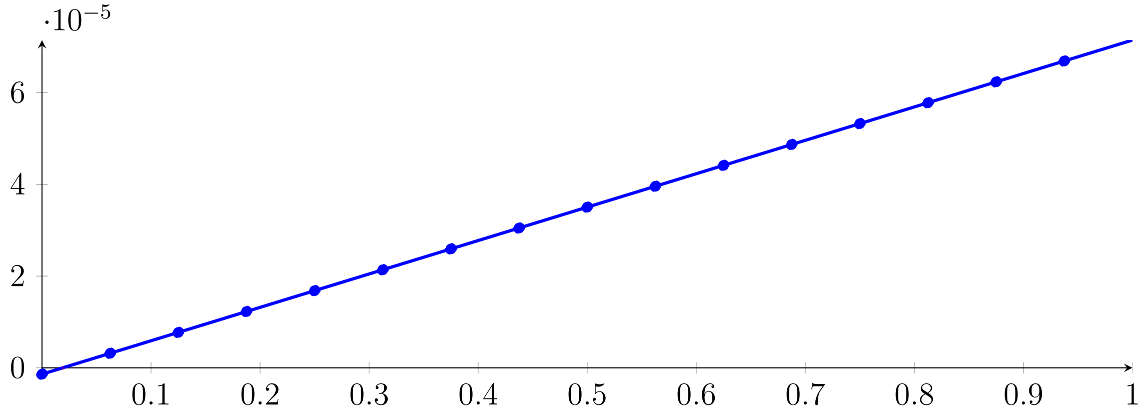
$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045\cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

247.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -5.36315\cdot10^{-201}X^{16} - 7.93495\cdot10^{-187}X^{15} - 5.0052\cdot10^{-173}X^{14} - 1.71037\cdot10^{-159}X^{13} - 3.21686\cdot10^{-146}X^{12} - 2.47211\cdot10^{-133}X^{11} + 2.05446\cdot10^{-120}X^{10} + 5.61285\cdot10^{-107}X^{9} + 2.23891\cdot10^{-94}X^{8} - 3.51264\cdot10^{-81}X^{7} - 2.89181\cdot10^{-68}X^{6} + 1.12075\cdot10^{-55}X^{5} + 1.30109\cdot10^{-42}X^{4} - 2.98449\cdot10^{-30}X^{3} - 2.49207\cdot10^{-17}X^{2} + 7.28316\cdot10^{-05}X - 1.38634\cdot10^{-06}$$
$$= -1.38634\cdot10^{-06}B_{0,16}(X) + 3.16564\cdot10^{-06}B_{1,16}(X) + 7.71761\cdot10^{-06}B_{2,16}(X) + 1.22696\cdot10^{-05}B_{3,16}(X) + 1.68216\cdot10^{-05}B_{4,16}(X) + 2.13735\cdot10^{-05}B_{5,16}(X) + 2.59255\cdot10^{-05}B_{6,16}(X) + 3.04775\cdot10^{-05}B_{7,16}(X) + 3.50295\cdot10^{-05}B_{8,16}(X) + 3.95814\cdot10^{-05}B_{9,16}(X) + 4.41334\cdot10^{-05}B_{10,16}(X) + 4.86854\cdot10^{-05}B_{11,16}(X) + 5.32374\cdot10^{-05}B_{12,16}(X) + 5.77893\cdot10^{-05}B_{13,16}(X) + 6.23413\cdot10^{-05}B_{14,16}(X) + 6.68933\cdot10^{-05}B_{15,16}(X) + 7.14453\cdot10^{-05}B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

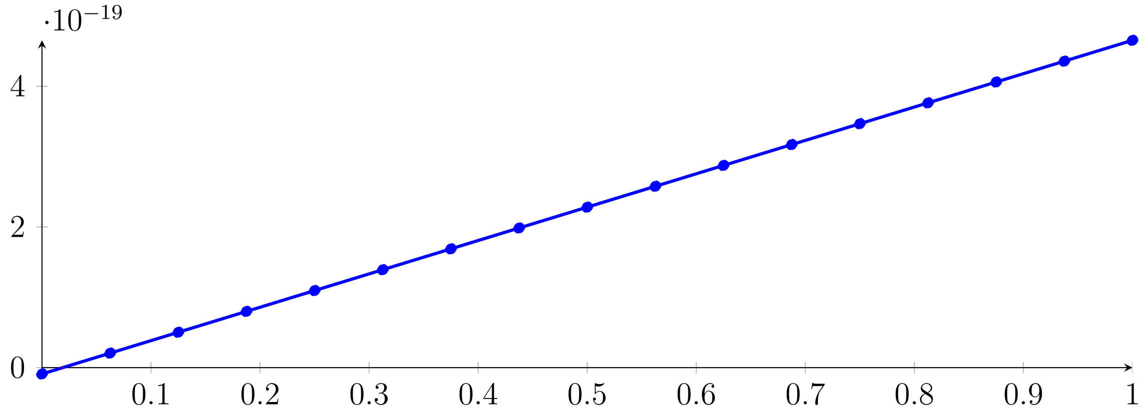
$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.51314\cdot10^{-15}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

247.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.70149\cdot10^{-323}X^{16} + 1.97819\cdot10^{-322}X^{15} - 4.34527\cdot10^{-322}X^{14} + 3.97182\cdot10^{-322}X^{13} - 1.87464\cdot10^{-316}X^{12} - 2.21189\cdot10^{-289}X^{11} + 2.82229\cdot10^{-262}X^{10} + 1.18385\cdot10^{-234}X^{9} + 7.25038\cdot10^{-208}X^{8} - 1.74649\cdot10^{-180}X^{7} - 2.20756\cdot10^{-153}X^{6} + 1.31359\cdot10^{-126}X^{5} + 2.34136\cdot10^{-99}X^{4} - 8.24597\cdot10^{-73}X^{3} - 1.05716\cdot10^{-45}X^{2} + 4.74362\cdot10^{-19}X - 9.02941\cdot10^{-21}$$
$$= -9.02941\cdot10^{-21}B_{0,16}(X) + 2.06182\cdot10^{-20}B_{1,16}(X) + 5.02659\cdot10^{-20}B_{2,16}(X) + 7.99135\cdot10^{-20}B_{3,16}(X) + 1.09561\cdot10^{-19}B_{4,16}(X) + 1.39209\cdot10^{-19}B_{5,16}(X) + 1.68856\cdot10^{-19}B_{6,16}(X) + 1.98504\cdot10^{-19}B_{7,16}(X) + 2.28152\cdot10^{-19}B_{8,16}(X) + 2.57799\cdot10^{-19}B_{9,16}(X) + 2.87447\cdot10^{-19}B_{10,16}(X) + 3.17095\cdot10^{-19}B_{11,16}(X) + 3.46742\cdot10^{-19}B_{12,16}(X) + 3.7639\cdot10^{-19}B_{13,16}(X) + 4.06038\cdot10^{-19}B_{14,16}(X) + 4.35685\cdot10^{-19}B_{15,16}(X) + 4.65333\cdot10^{-19}B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

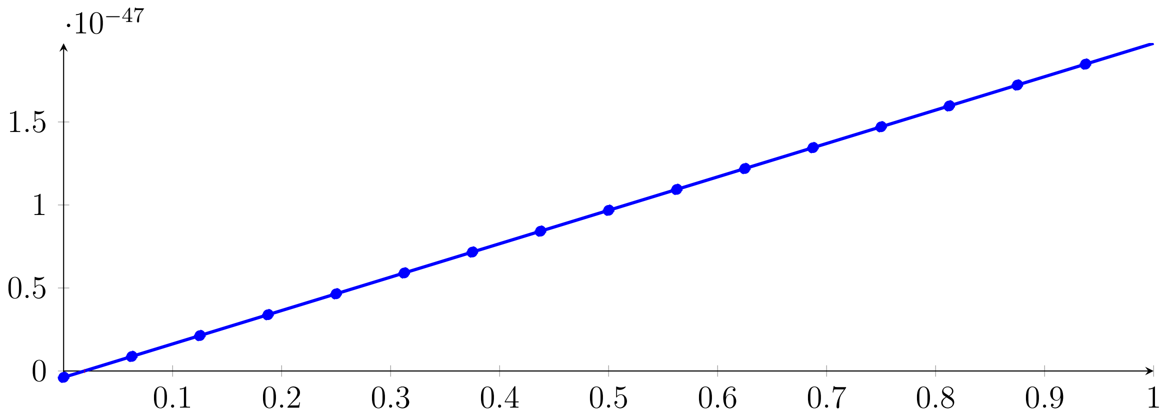
$$[0.0190348, 0.0190348]$$

Longest intersection interval: $4.2421 \cdot 10^{-29}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

247.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.52489 \cdot 10^{-352} X^{16} + 4.56006 \cdot 10^{-351} X^{15} - 3.36159 \cdot 10^{-350} X^{14} + 2.87148 \cdot 10^{-350} X^{13} \\ &\quad - 1.35884 \cdot 10^{-349} X^{12} + 1.13318 \cdot 10^{-349} X^{11} - 1.84828 \cdot 10^{-349} X^{10} + 7.52411 \cdot 10^{-350} X^{9} \\ &\quad - 2.19453 \cdot 10^{-350} X^{8} + 2.78671 \cdot 10^{-351} X^{7} - 1.28647 \cdot 10^{-323} X^{6} + 1.80453 \cdot 10^{-268} X^{5} + 7.58214 \\ &\quad \cdot 10^{-213} X^{4} - 6.29484 \cdot 10^{-158} X^{3} - 1.90241 \cdot 10^{-102} X^{2} + 2.01229 \cdot 10^{-47} X - 3.83037 \cdot 10^{-49} \\ &= -3.83037 \cdot 10^{-49} B_{0,16}(X) + 8.74646 \cdot 10^{-49} B_{1,16}(X) + 2.13233 \cdot 10^{-48} B_{2,16}(X) + 3.39001 \\ &\quad \cdot 10^{-48} B_{3,16}(X) + 4.6477 \cdot 10^{-48} B_{4,16}(X) + 5.90538 \cdot 10^{-48} B_{5,16}(X) + 7.16306 \cdot 10^{-48} B_{6,16}(X) \\ &\quad + 8.42074 \cdot 10^{-48} B_{7,16}(X) + 9.67843 \cdot 10^{-48} B_{8,16}(X) + 1.09361 \cdot 10^{-47} B_{9,16}(X) + 1.21938 \\ &\quad \cdot 10^{-47} B_{10,16}(X) + 1.34515 \cdot 10^{-47} B_{11,16}(X) + 1.47092 \cdot 10^{-47} B_{12,16}(X) + 1.59668 \cdot 10^{-47} B_{13,16}(X) \\ &\quad + 1.72245 \cdot 10^{-47} B_{14,16}(X) + 1.84822 \cdot 10^{-47} B_{15,16}(X) + 1.97399 \cdot 10^{-47} B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $1.79954 \cdot 10^{-57}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

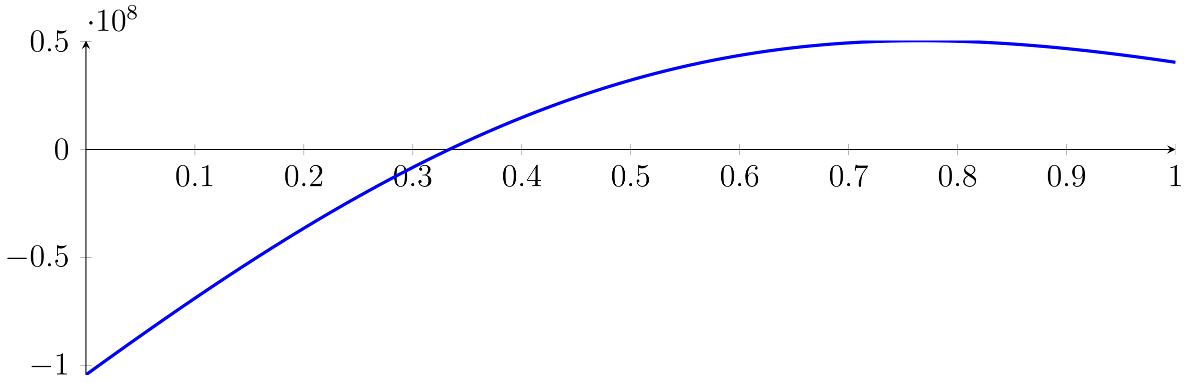
247.8 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 8!

247.9 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

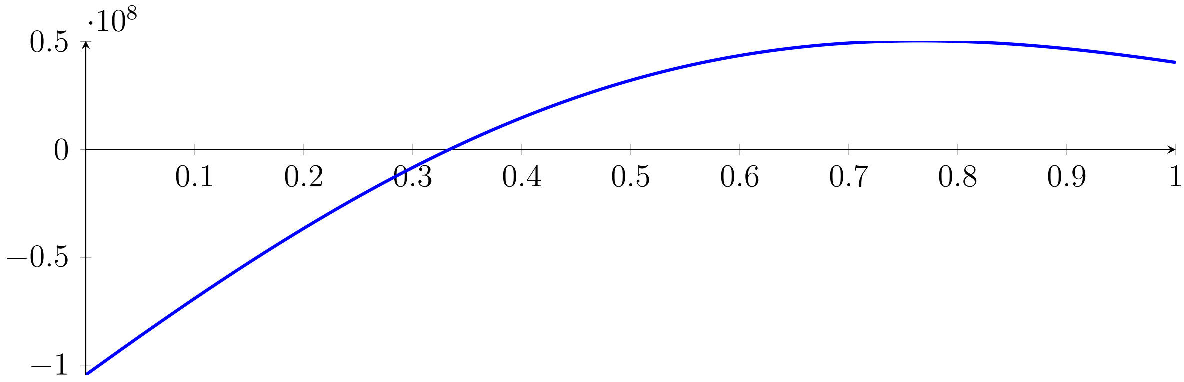
with precision $\varepsilon = 1\cdot10^{-64}$.

248 Running `QuadClip` on f_{16} with epsilon 64

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$

Called `QuadClip` with input polynomial on interval $[0, 1]$:

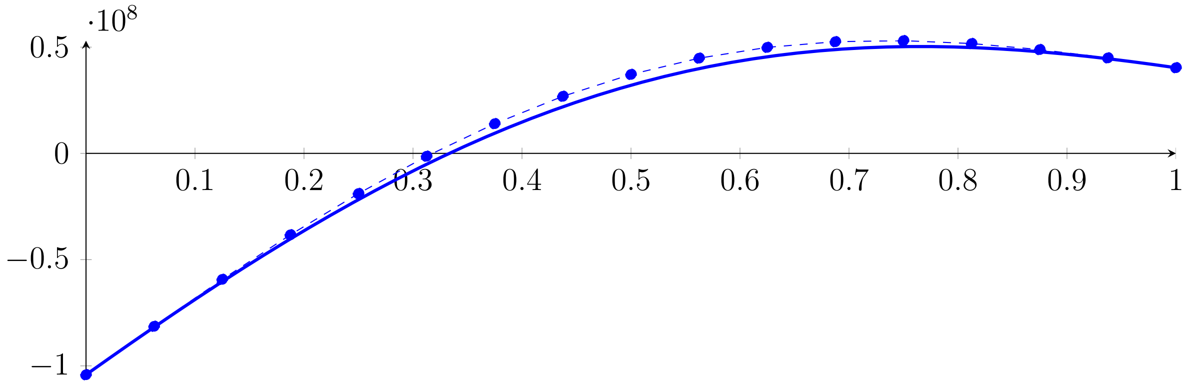
$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} \\ & + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \end{aligned}$$



248.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

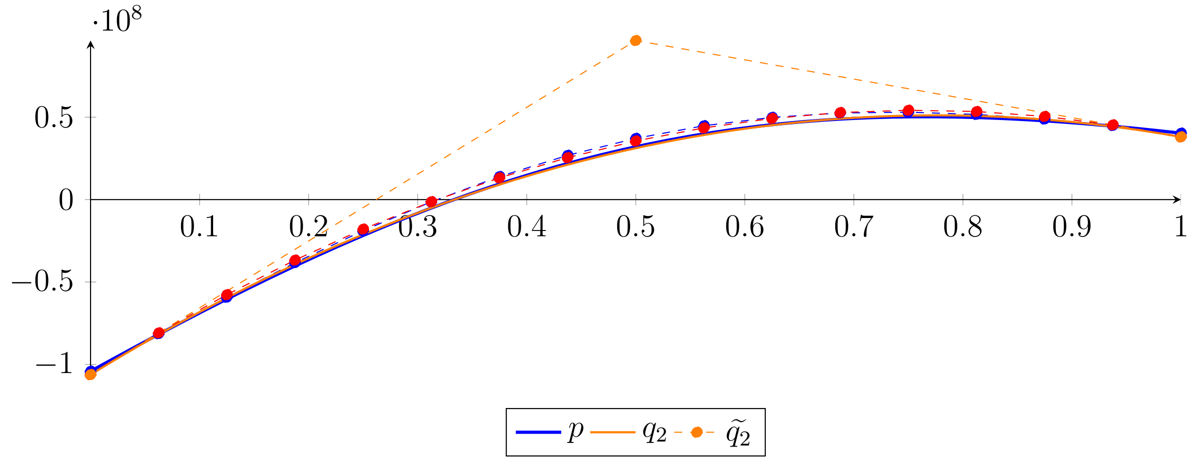
$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} \\ & + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8} \\ = & -1.04167\cdot 10^{8}B_{0,16}(X) - 8.13802\cdot 10^{7}B_{1,16}(X) - 5.92882\cdot 10^{7}B_{2,16}(X) - 3.83203 \\ & \cdot 10^{7}B_{3,16}(X) - 1.88844\cdot 10^{7}B_{4,16}(X) - 1.34837\cdot 10^{6}B_{5,16}(X) + 1.39781\cdot 10^{7}B_{6,16}(X) \\ & + 2.68604\cdot 10^{7}B_{7,16}(X) + 3.71532\cdot 10^{7}B_{8,16}(X) + 4.48105\cdot 10^{7}B_{9,16}(X) + 4.98901 \\ & \cdot 10^{7}B_{10,16}(X) + 5.25504\cdot 10^{7}B_{11,16}(X) + 5.30407\cdot 10^{7}B_{12,16}(X) + 5.16832\cdot 10^{7}B_{13,16}(X) \\ & + 4.88488\cdot 10^{7}B_{14,16}(X) + 4.49297\cdot 10^{7}B_{15,16}(X) + 4.03108\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.06197{\cdot}10^8$$
$$= -1.06197{\cdot}10^8 B_{0,2} + 9.65115{\cdot}10^7 B_{1,2} + 3.80384{\cdot}10^7 B_{2,2}$$

$$\widetilde{q_2} = 3.74473{\cdot}10^{-288} X^{16} - 3.08889{\cdot}10^{-287} X^{15} + 1.17443{\cdot}10^{-286} X^{14} - 2.71603{\cdot}10^{-286} X^{13} + 4.23554{\cdot}10^{-286} X^{12} - 4.66107{\cdot}10^{-286} X^{11} + 3.6845{\cdot}10^{-286} X^{10} - 2.09811{\cdot}10^{-286} X^9 + 8.58708{\cdot}10^{-287} X^8 - 2.54063{\cdot}10^{-287} X^7 + 5.57852{\cdot}10^{-288} X^6 - 9.21534{\cdot}10^{-289} X^5 + 1.02122{\cdot}10^{-289} X^4 - 5.95624{\cdot}10^{-291} X^3 - 2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.06197{\cdot}10^8$$
$$= -1.06197{\cdot}10^8 B_{0,16} - 8.08583{\cdot}10^7 B_{1,16} - 5.76963{\cdot}10^7 B_{2,16} - 3.67107{\cdot}10^7 B_{3,16} - 1.79017{\cdot}10^7 B_{4,16} - 1.26924{\cdot}10^6 B_{5,16} + 1.31867{\cdot}10^7 B_{6,16} + 2.54662{\cdot}10^7 B_{7,16} + 3.55692{\cdot}10^7 B_{8,16} + 4.34956{\cdot}10^7 B_{9,16} + 4.92456{\cdot}10^7 B_{10,16} + 5.2819{\cdot}10^7 B_{11,16} + 5.42159{\cdot}10^7 B_{12,16} + 5.34363{\cdot}10^7 B_{13,16} + 5.04802{\cdot}10^7 B_{14,16} + 4.53476{\cdot}10^7 B_{15,16} + 3.80384{\cdot}10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233{\cdot}10^6$.

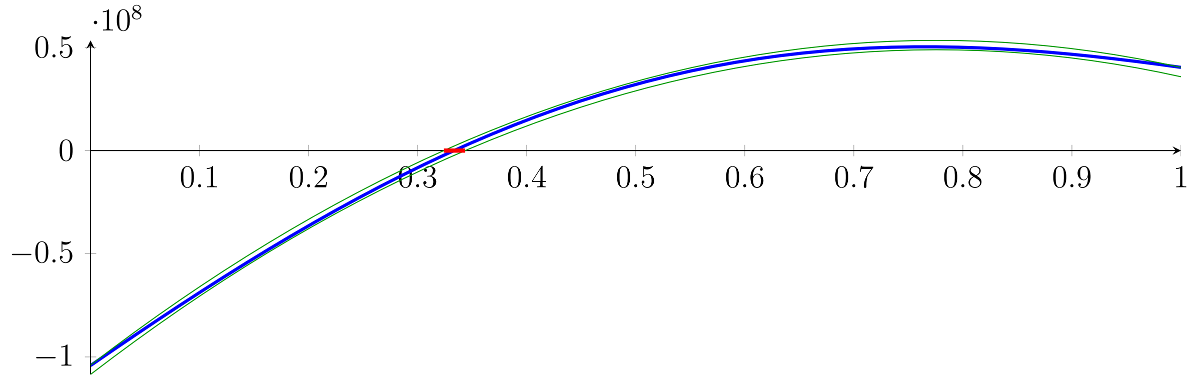
Bounding polynomials M and m:

$$M = -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.03924{\cdot}10^8$$
$$m = -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.08469{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

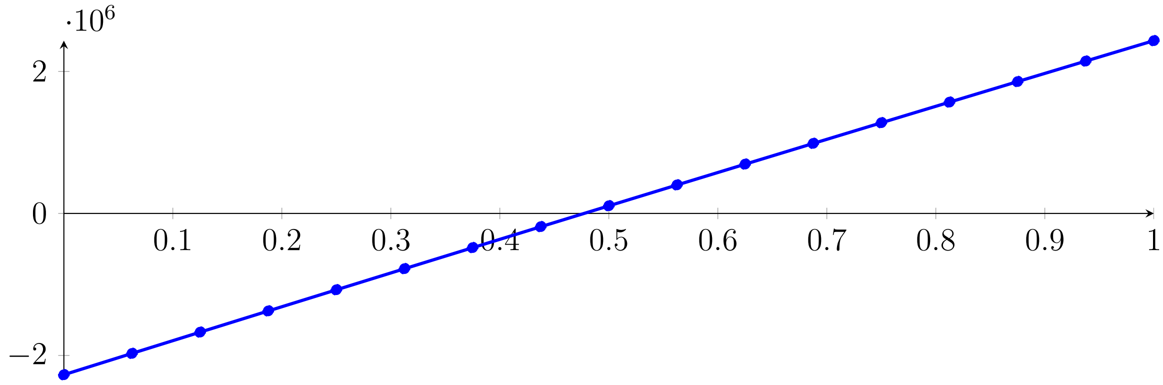
Longest intersection interval: 0.0196686

$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

248.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

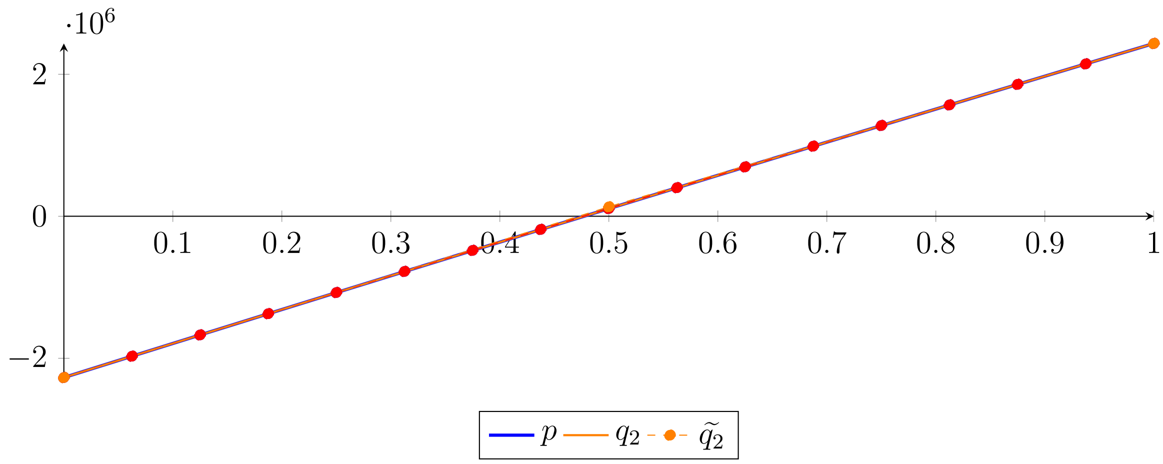
$$p = -5.0162{\cdot}10^{-28}X^{16} - 1.14383{\cdot}10^{-24}X^{15} - 1.11125{\cdot}10^{-21}X^{14} - 5.84096{\cdot}10^{-19}X^{13} - 1.684{\cdot}10^{-16}X^{12} - 1.94707{\cdot}10^{-14}X^{11} + 2.73227{\cdot}10^{-12}X^{10} + 1.09727{\cdot}10^{-09}X^{9} + 6.37314{\cdot}10^{-08}X^{8} - 1.68645{\cdot}10^{-05}X^{7} - 0.00205892X^{6} + 0.132721X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874{\cdot}10^{6}X - 2.2715{\cdot}10^{6}$$
$$= -2.2715{\cdot}10^{6}B_{0,16}(X) - 1.97096{\cdot}10^{6}B_{1,16}(X) - 1.67127{\cdot}10^{6}B_{2,16}(X) - 1.37244{\cdot}10^{6}B_{3,16}(X) - 1.07448{\cdot}10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806{\cdot}10^{6}B_{12,16}(X) + 1.5682{\cdot}10^{6}B_{13,16}(X) + 1.85747{\cdot}10^{6}B_{14,16}(X) + 2.14586{\cdot}10^{6}B_{15,16}(X) + 2.43338{\cdot}10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265X^{2} + 4.80923{\cdot}10^{6}X - 2.27154{\cdot}10^{6}$$
$$= -2.27154{\cdot}10^{6}B_{0,2} + 133069B_{1,2} + 2.43342{\cdot}10^{6}B_{2,2}$$

$$\widetilde{q}_2 = -8.18461{\cdot}10^{-291}X^{16} + 4.13745{\cdot}10^{-290}X^{15} - 5.8254{\cdot}10^{-290}X^{14} - 6.80919{\cdot}10^{-290}X^{13} + 3.48059{\cdot}10^{-289}X^{12} - 5.47245{\cdot}10^{-289}X^{11} + 4.8261{\cdot}10^{-289}X^{10} - 2.67867{\cdot}10^{-289}X^{9} + 1.00314{\cdot}10^{-289}X^{8} - 2.84766{\cdot}10^{-290}X^{7} + 6.85072{\cdot}10^{-291}X^{6} - 1.12501{\cdot}10^{-291}X^{5} + 1.24731{\cdot}10^{-293}X^{4} + 2.25944{\cdot}10^{-293}X^{3} - 104265X^{2} + 4.80923{\cdot}10^{6}X - 2.27154{\cdot}10^{6}$$
$$= -2.27154{\cdot}10^{6}B_{0,16} - 1.97097{\cdot}10^{6}B_{1,16} - 1.67126{\cdot}10^{6}B_{2,16} - 1.37242{\cdot}10^{6}B_{3,16} - 1.07445{\cdot}10^{6}B_{4,16} - 777350B_{5,16} - 481118B_{6,16} - 185754B_{7,16} + 108740B_{8,16} + 402366B_{9,16} + 695123B_{10,16} + 987011B_{11,16} + 1.27803{\cdot}10^{6}B_{12,16} + 1.56818{\cdot}10^{6}B_{13,16} + 1.85746{\cdot}10^{6}B_{14,16} + 2.14587{\cdot}10^{6}B_{15,16} + 2.43342{\cdot}10^{6}B_{16,16}$$



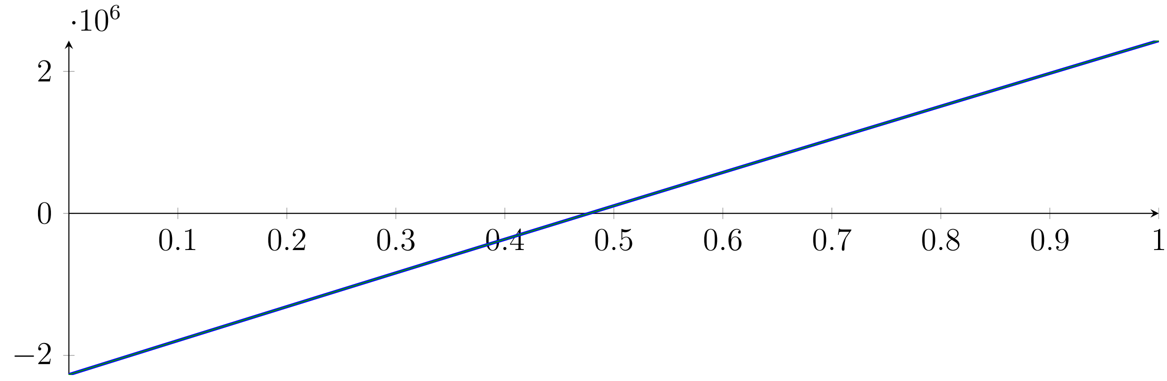
The maximum difference of the Bézier coefficients is $\delta = 40.5742$.
Bounding polynomials M and m:

$$M = -104265X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



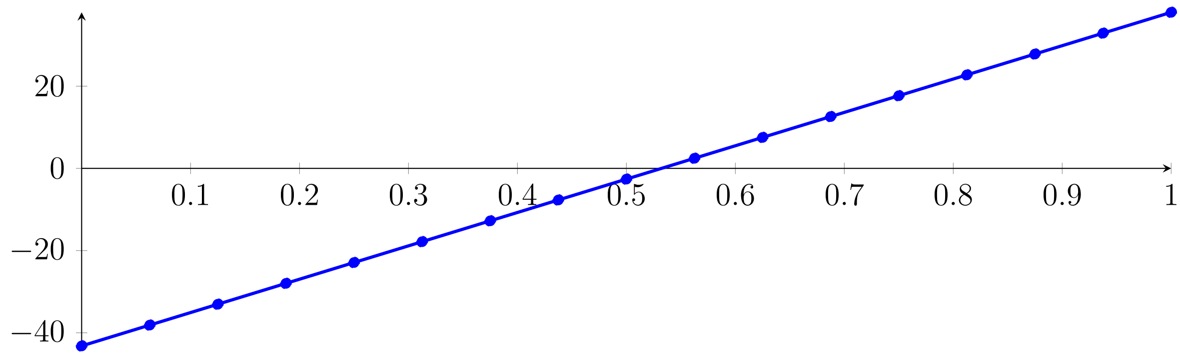
$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301 \cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

248.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

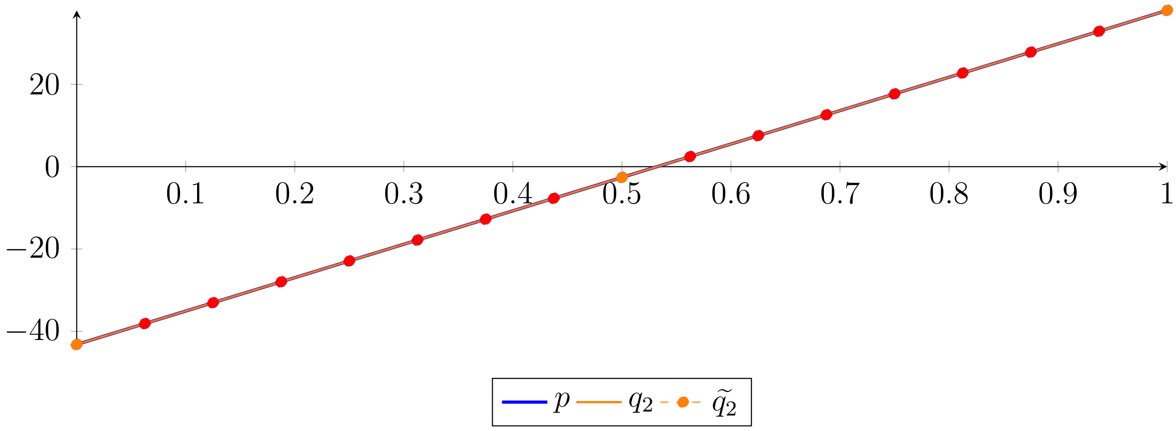
$$\begin{aligned} p = & -3.02667 \cdot 10^{-104} X^{16} - 4.019 \cdot 10^{-96} X^{15} - 2.27522 \cdot 10^{-88} X^{14} - 6.97783 \cdot 10^{-81} X^{13} \\ & - 1.17785 \cdot 10^{-73} X^{12} - 8.12373 \cdot 10^{-67} X^{11} + 6.05916 \cdot 10^{-60} X^{10} + 1.48569 \cdot 10^{-52} X^{9} \\ & + 5.31875 \cdot 10^{-46} X^{8} - 7.48919 \cdot 10^{-39} X^{7} - 5.53349 \cdot 10^{-32} X^{6} + 1.92471 \cdot 10^{-25} X^{5} \\ & + 2.00536 \cdot 10^{-18} X^{4} - 4.12844 \cdot 10^{-12} X^{3} - 3.09388 \cdot 10^{-05} X^{2} + 81.1506X - 43.1911 \\ = & -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\ & - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X) \\ & + 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X) \\ & + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= -1.88281{\cdot}10^{-295}X^{16} + 1.09893{\cdot}10^{-294}X^{15} - 2.26419{\cdot}10^{-294}X^{14} + 8.08223{\cdot}10^{-295}X^{13} \\ &\quad + 4.92626{\cdot}10^{-294}X^{12} - 1.07605{\cdot}10^{-293}X^{11} + 1.11858{\cdot}10^{-293}X^{10} - 6.87288{\cdot}10^{-294}X^{9} \\ &\quad + 2.54873{\cdot}10^{-294}X^{8} - 5.2305{\cdot}10^{-295}X^{7} + 3.18923{\cdot}10^{-296}X^{6} + 1.34092{\cdot}10^{-296}X^{5} \\ &\quad - 4.89549{\cdot}10^{-297}X^{4} + 5.89947{\cdot}10^{-298}X^{3} - 3.09389{\cdot}10^{-05}X^{2} + 81.1506X - 43.1911 \\ &= -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} \\ &\quad - 12.7597B_{6,16} - 7.68778B_{7,16} - 2.61587B_{8,16} + 2.45604B_{9,16} + 7.52795B_{10,16} + 12.5999B_{11,16} \\ &\quad + 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.06422{\cdot}10^{-13}$.

Bounding polynomials M and m:

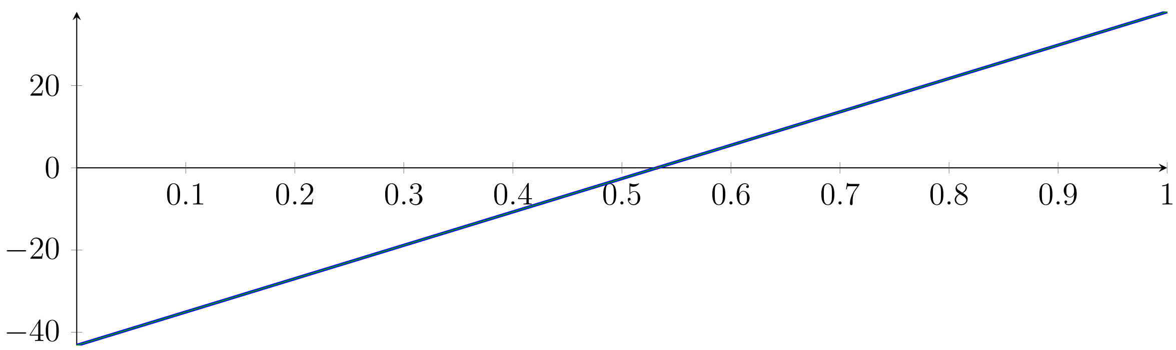
$$M = -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911$$

$$m = -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911$$

Root of M and m:

$$N(M) = \{0.532235, 2.62293{\cdot}10^{6}\} \qquad N(m) = \{0.532235, 2.62293{\cdot}10^{6}\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

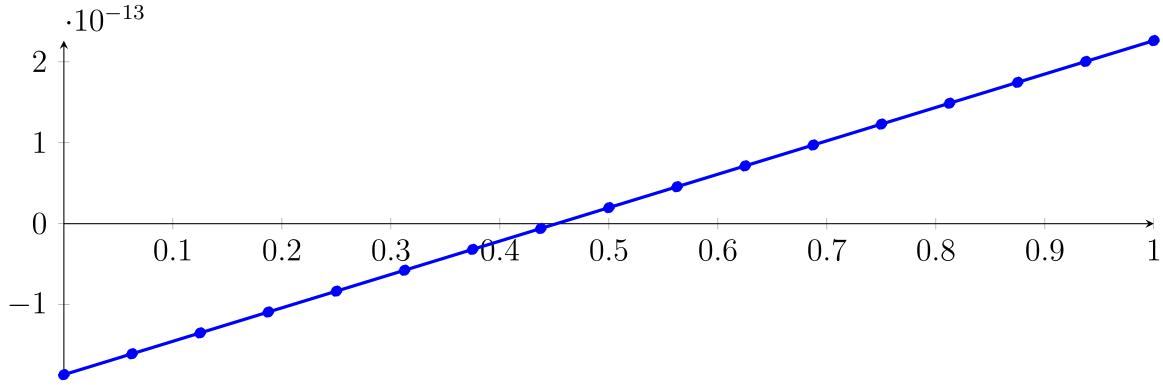
Longest intersection interval: $5.08738{\cdot}10^{-15}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

248.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

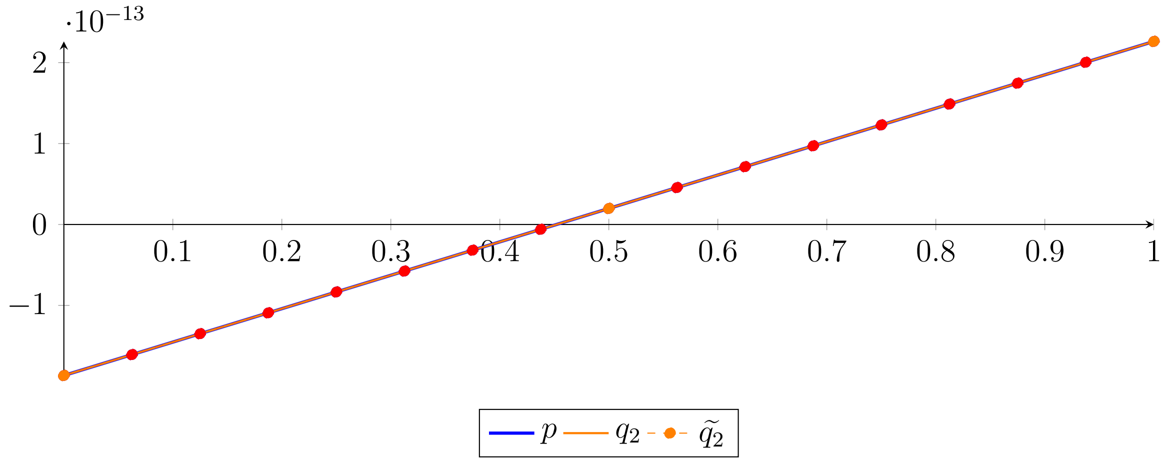
$$p = -3.84502\cdot10^{-319}X^{16} - 1.59047\cdot10^{-310}X^{15} - 1.76985\cdot10^{-288}X^{14} - 1.06694\cdot10^{-266}X^{13} - 3.54011\cdot10^{-245}X^{12} - 4.79942\cdot10^{-224}X^{11} + 7.03641\cdot10^{-203}X^{10} + 3.39135\cdot10^{-181}X^{9} + 2.3865\cdot10^{-160}X^{8} - 6.60529\cdot10^{-139}X^{7} - 9.59319\cdot10^{-118}X^{6} + 6.55895\cdot10^{-97}X^{5} + 1.34328\cdot10^{-75}X^{4} - 5.43584\cdot10^{-55}X^{3} - 8.00741\cdot10^{-34}X^{2} + 4.12843\cdot10^{-13}X - 1.86598\cdot10^{-13}$$
$$= -1.86598\cdot10^{-13}B_{0,16}(X) - 1.60795\cdot10^{-13}B_{1,16}(X) - 1.34993\cdot10^{-13}B_{2,16}(X) - 1.0919\cdot10^{-13}B_{3,16}(X) - 8.33872\cdot10^{-14}B_{4,16}(X) - 5.75845\cdot10^{-14}B_{5,16}(X) - 3.17818\cdot10^{-14}B_{6,16}(X) - 5.97912\cdot10^{-15}B_{7,16}(X) + 1.98236\cdot10^{-14}B_{8,16}(X) + 4.56263\cdot10^{-14}B_{9,16}(X) + 7.1429\cdot10^{-14}B_{10,16}(X) + 9.72317\cdot10^{-14}B_{11,16}(X) + 1.23034\cdot10^{-13}B_{12,16}(X) + 1.48837\cdot10^{-13}B_{13,16}(X) + 1.7464\cdot10^{-13}B_{14,16}(X) + 2.00443\cdot10^{-13}B_{15,16}(X) + 2.26245\cdot10^{-13}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -8.00741\cdot10^{-34}X^{2} + 4.12843\cdot10^{-13}X - 1.86598\cdot10^{-13}$$
$$= -1.86598\cdot10^{-13}B_{0,2} + 1.98236\cdot10^{-14}B_{1,2} + 2.26245\cdot10^{-13}B_{2,2}$$

$$\widetilde{q}_2 = -8.08289\cdot10^{-310}X^{16} + 4.33931\cdot10^{-309}X^{15} - 8.19282\cdot10^{-309}X^{14} + 3.30456\cdot10^{-309}X^{13} + 1.10762\cdot10^{-308}X^{12} - 2.01579\cdot10^{-308}X^{11} + 1.55412\cdot10^{-308}X^{10} - 6.55981\cdot10^{-309}X^{9} + 2.12881\cdot10^{-309}X^{8} - 1.08538\cdot10^{-309}X^{7} + 5.4154\cdot10^{-310}X^{6} - 1.38567\cdot10^{-310}X^{5} + 9.27245\cdot10^{-312}X^{4} + 1.80627\cdot10^{-312}X^{3} - 8.00741\cdot10^{-34}X^{2} + 4.12843\cdot10^{-13}X - 1.86598\cdot10^{-13}$$
$$= -1.86598\cdot10^{-13}B_{0,16} - 1.60795\cdot10^{-13}B_{1,16} - 1.34993\cdot10^{-13}B_{2,16} - 1.0919\cdot10^{-13}B_{3,16} - 8.33872\cdot10^{-14}B_{4,16} - 5.75845\cdot10^{-14}B_{5,16} - 3.17818\cdot10^{-14}B_{6,16} - 5.97912\cdot10^{-15}B_{7,16} + 1.98236\cdot10^{-14}B_{8,16} + 4.56263\cdot10^{-14}B_{9,16} + 7.1429\cdot10^{-14}B_{10,16} + 9.72317\cdot10^{-14}B_{11,16} + 1.23034\cdot10^{-13}B_{12,16} + 1.48837\cdot10^{-13}B_{13,16} + 1.7464\cdot10^{-13}B_{14,16} + 2.00443\cdot10^{-13}B_{15,16} + 2.26245\cdot10^{-13}B_{16,16}$$



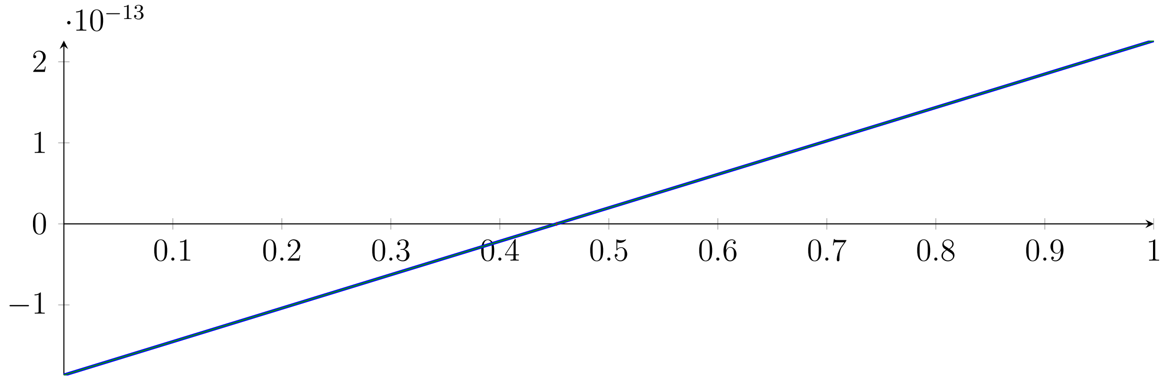
The maximum difference of the Bézier coefficients is $\delta = 2.71792 \cdot 10^{-56}$.
Bounding polynomials M and m:

$$M = -8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13}$$
$$m = -8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13}$$

Root of M and m:

$$N(M) = \{0.451983, 5.15577 \cdot 10^{20}\} \qquad N(m) = \{0.451983, 5.15577 \cdot 10^{20}\}$$

Intersection intervals:



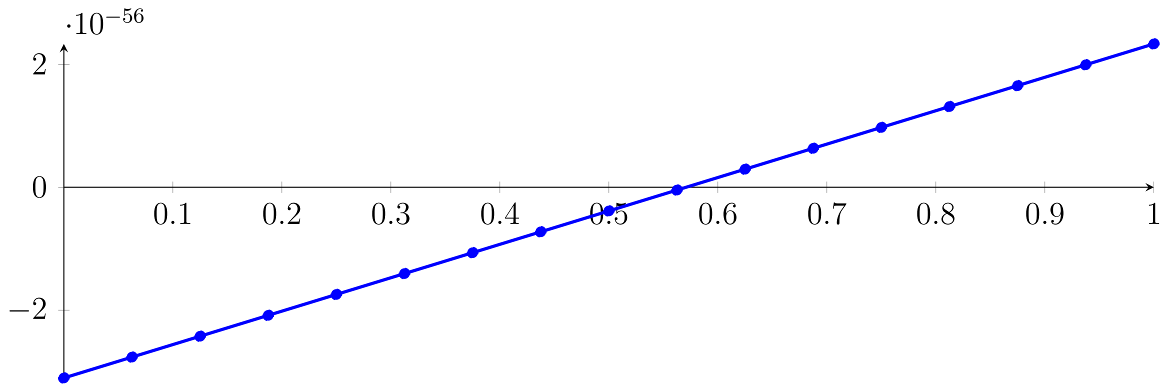
$$[0.451983, 0.451983]$$

Longest intersection interval: $1.31668 \cdot 10^{-43}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

248.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

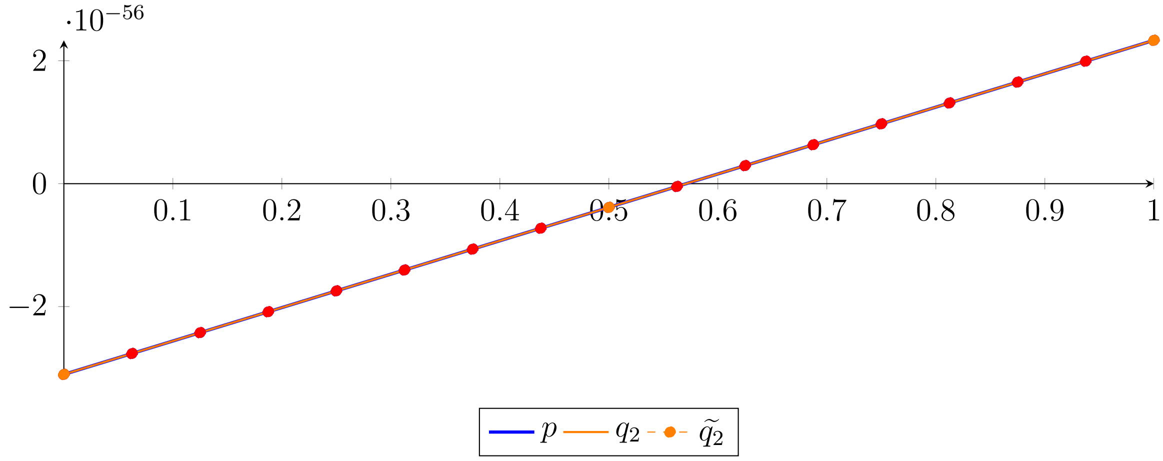
$$\begin{aligned} p = {} & 5.66252 \cdot 10^{-361} X^{16} + 8.30503 \cdot 10^{-360} X^{15} - 8.71157 \cdot 10^{-361} X^{14} + 3.25232 \cdot 10^{-359} X^{13} \\ & + 1.65157 \cdot 10^{-358} X^{12} - 1.58551 \cdot 10^{-358} X^{11} - 7.2669 \cdot 10^{-359} X^{10} - 4.15252 \cdot 10^{-359} X^{9} \\ & + 9.34316 \cdot 10^{-359} X^{8} + 1.45338 \cdot 10^{-359} X^{6} + 2.59562 \cdot 10^{-311} X^{5} + 4.03733 \cdot 10^{-247} X^{4} \\ & - 1.24083 \cdot 10^{-183} X^{3} - 1.38821 \cdot 10^{-119} X^{2} + 5.43584 \cdot 10^{-56} X - 3.10342 \cdot 10^{-56} \\ = {} & -3.10342 \cdot 10^{-56} B_{0,16}(X) - 2.76368 \cdot 10^{-56} B_{1,16}(X) - 2.42394 \cdot 10^{-56} B_{2,16}(X) - 2.0842 \\ & \cdot 10^{-56} B_{3,16}(X) - 1.74446 \cdot 10^{-56} B_{4,16}(X) - 1.40472 \cdot 10^{-56} B_{5,16}(X) - 1.06498 \cdot 10^{-56} B_{6,16}(X) \\ & - 7.25243 \cdot 10^{-57} B_{7,16}(X) - 3.85503 \cdot 10^{-57} B_{8,16}(X) - 4.57628 \cdot 10^{-58} B_{9,16}(X) + 2.93977 \\ & \cdot 10^{-57} B_{10,16}(X) + 6.33717 \cdot 10^{-57} B_{11,16}(X) + 9.73457 \cdot 10^{-57} B_{12,16}(X) + 1.3132 \cdot 10^{-56} B_{13,16}(X) \\ & + 1.65294 \cdot 10^{-56} B_{14,16}(X) + 1.99268 \cdot 10^{-56} B_{15,16}(X) + 2.33242 \cdot 10^{-56} B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.38821\cdot10^{-119}X^2 + 5.43584\cdot10^{-56}X - 3.10342\cdot10^{-56}$$
$$= -3.10342\cdot10^{-56}B_{0,2} - 3.85503\cdot10^{-57}B_{1,2} + 2.33242\cdot10^{-56}B_{2,2}$$

$$\widetilde{q}_2 = -1.35612\cdot10^{-352}X^{16} + 8.15544\cdot10^{-352}X^{15} - 1.72777\cdot10^{-351}X^{14} + 5.92647\cdot10^{-352}X^{13} + 4.18743\cdot10^{-351}X^{12} - 9.40291\cdot10^{-351}X^{11} + 1.01181\cdot10^{-350}X^{10} - 6.40657\cdot10^{-351}X^{9} + 2.39508\cdot10^{-351}X^{8} - 4.50359\cdot10^{-352}X^{7} - 2.71062\cdot10^{-354}X^{6} + 2.21064\cdot10^{-353}X^{5} - 5.44842\cdot10^{-354}X^{4} + 4.71019\cdot10^{-355}X^{3} - 1.38821\cdot10^{-119}X^{2} + 5.43584\cdot10^{-56}X - 3.10342\cdot10^{-56}$$
$$= -3.10342\cdot10^{-56}B_{0,16} - 2.76368\cdot10^{-56}B_{1,16} - 2.42394\cdot10^{-56}B_{2,16} - 2.0842\cdot10^{-56}B_{3,16} - 1.74446\cdot10^{-56}B_{4,16} - 1.40472\cdot10^{-56}B_{5,16} - 1.06498\cdot10^{-56}B_{6,16} - 7.25243\cdot10^{-57}B_{7,16} - 3.85503\cdot10^{-57}B_{8,16} - 4.57628\cdot10^{-58}B_{9,16} + 2.93977\cdot10^{-57}B_{10,16} + 6.33717\cdot10^{-57}B_{11,16} + 9.73457\cdot10^{-57}B_{12,16} + 1.3132\cdot10^{-56}B_{13,16} + 1.65294\cdot10^{-56}B_{14,16} + 1.99268\cdot10^{-56}B_{15,16} + 2.33242\cdot10^{-56}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 6.20413\cdot10^{-185}$.

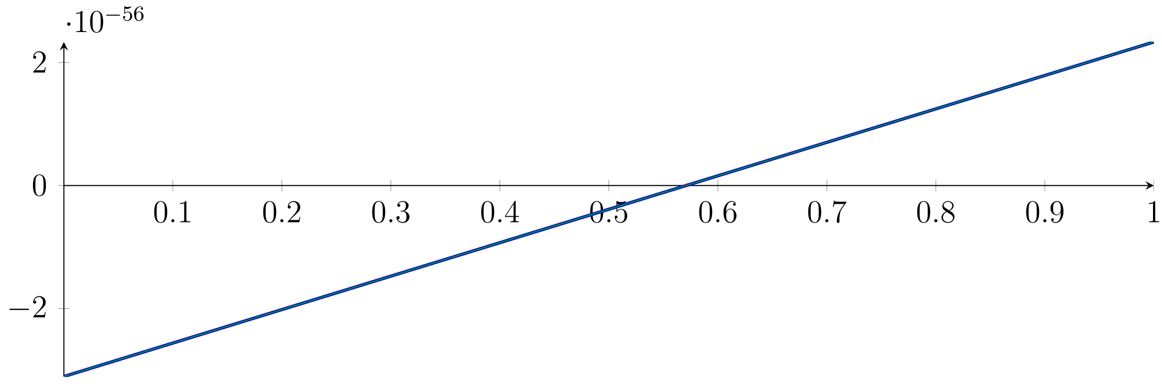
Bounding polynomials M and m:

$$M = -1.38821\cdot10^{-119}X^2 + 5.43584\cdot10^{-56}X - 3.10342\cdot10^{-56}$$
$$m = -1.38821\cdot10^{-119}X^2 + 5.43584\cdot10^{-56}X - 3.10342\cdot10^{-56}$$

Root of M and m:

$$N(M) = \{0.570919, 3.91572\cdot10^{63}\} \qquad N(m) = \{0.570919, 3.91572\cdot10^{63}\}$$

Intersection intervals:



$[0.570919, 0.570919]$

Longest intersection interval: $2.28268\cdot10^{-129}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

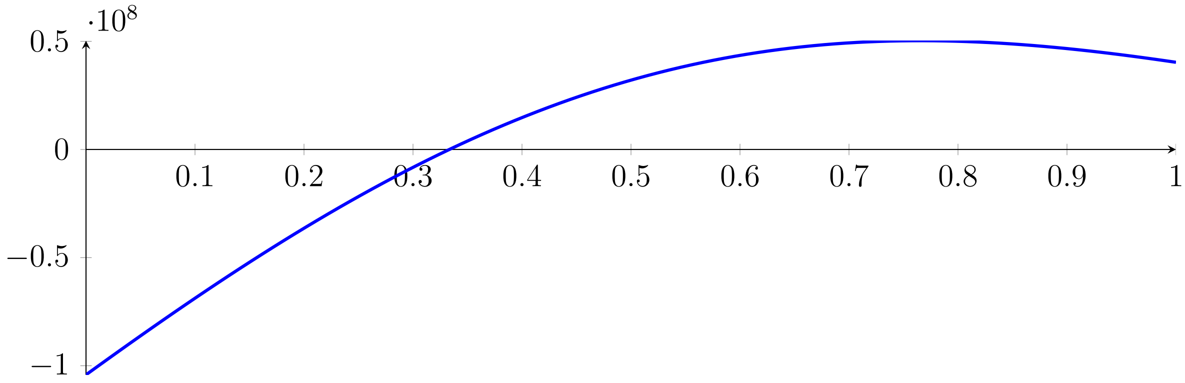
248.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

248.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^{6}X^{9} - 2.52262 \cdot 10^{6}X^{8} - 1.43506 \cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \cdot 10^{7}X^{5} + 3.97396 \cdot 10^{7}X^{4} - 2.40625 \cdot 10^{8}X^{3} - 8.33333 \cdot 10^{7}X^{2} + 3.64583 \cdot 10^{8}X - 1.04167 \cdot 10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

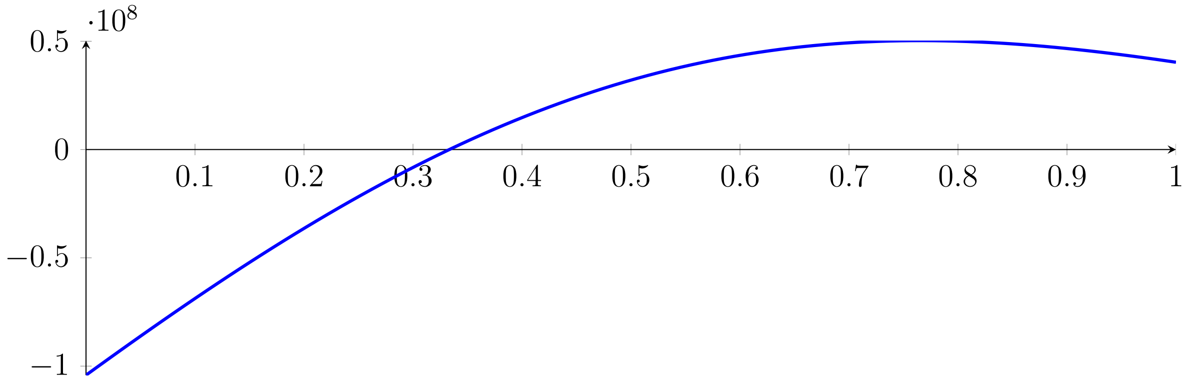
with precision $\varepsilon = 1 \cdot 10^{-64}$.

249 Running CubeClip on f_{16} with epsilon 64

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

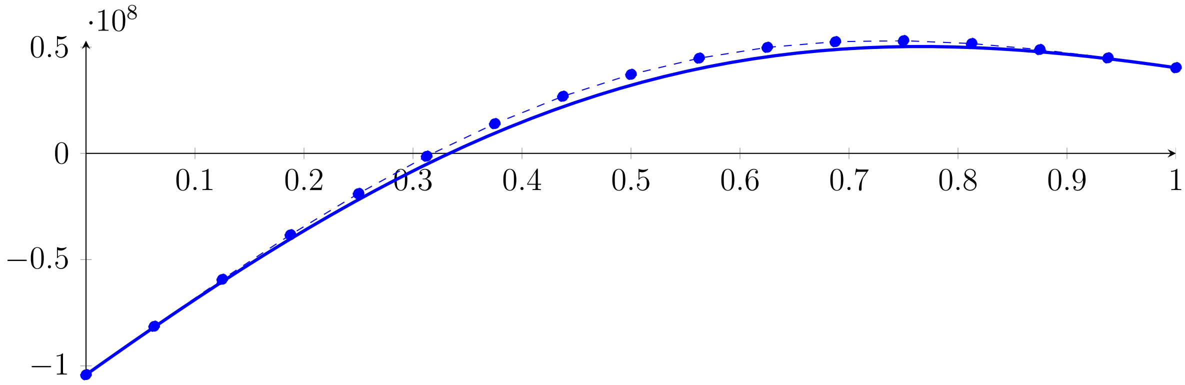
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



249.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

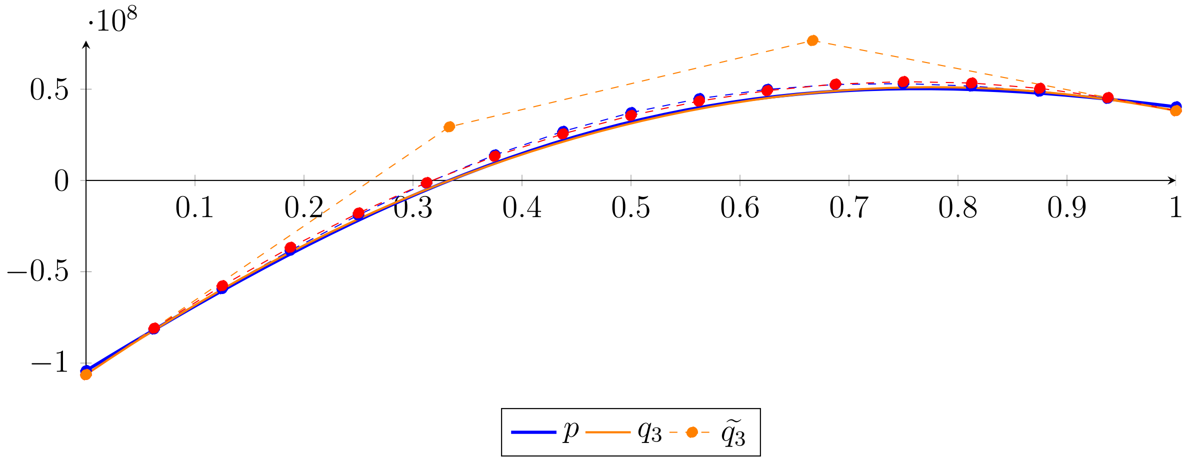
$$\begin{aligned} p &= -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &\quad + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} \\ &\quad + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \\ &= -1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.06335{\cdot}10^8$$
$$= -1.06335{\cdot}10^8 B_{0,3} + 2.93558{\cdot}10^7 B_{1,3} + 7.66068{\cdot}10^7 B_{2,3} + 3.81764{\cdot}10^7 B_{3,3}$$

$$\widetilde{q}_3 = 1.89955{\cdot}10^{-288} X^{16} - 1.4851{\cdot}10^{-287} X^{15} + 5.12896{\cdot}10^{-287} X^{14} - 1.02224{\cdot}10^{-286} X^{13} + 1.29027{\cdot}10^{-286} X^{12} - 1.07058{\cdot}10^{-286} X^{11} + 6.00739{\cdot}10^{-287} X^{10} - 2.54352{\cdot}10^{-287} X^9 + 1.12089{\cdot}10^{-287} X^8 - 5.8673{\cdot}10^{-288} X^7 + 2.52637{\cdot}10^{-288} X^6 - 6.79624{\cdot}10^{-289} X^5 + 9.36341{\cdot}10^{-290} X^4 + 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.06335{\cdot}10^8$$
$$= -1.06335{\cdot}10^8 B_{0,16} - 8.08928{\cdot}10^7 B_{1,16} - 5.76618{\cdot}10^7 B_{2,16} - 3.66369{\cdot}10^7 B_{3,16} - 1.78131{\cdot}10^7 B_{4,16} - 1.18551{\cdot}10^6 B_{5,16} + 1.32508{\cdot}10^7 B_{6,16} + 2.55007{\cdot}10^7 B_{7,16} + 3.55692{\cdot}10^7 B_{8,16} + 4.34611{\cdot}10^7 B_{9,16} + 4.91815{\cdot}10^7 B_{10,16} + 5.27353{\cdot}10^7 B_{11,16} + 5.41273{\cdot}10^7 B_{12,16} + 5.33624{\cdot}10^7 B_{13,16} + 5.04457{\cdot}10^7 B_{14,16} + 4.53821{\cdot}10^7 B_{15,16} + 3.81764{\cdot}10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806{\cdot}10^6$.

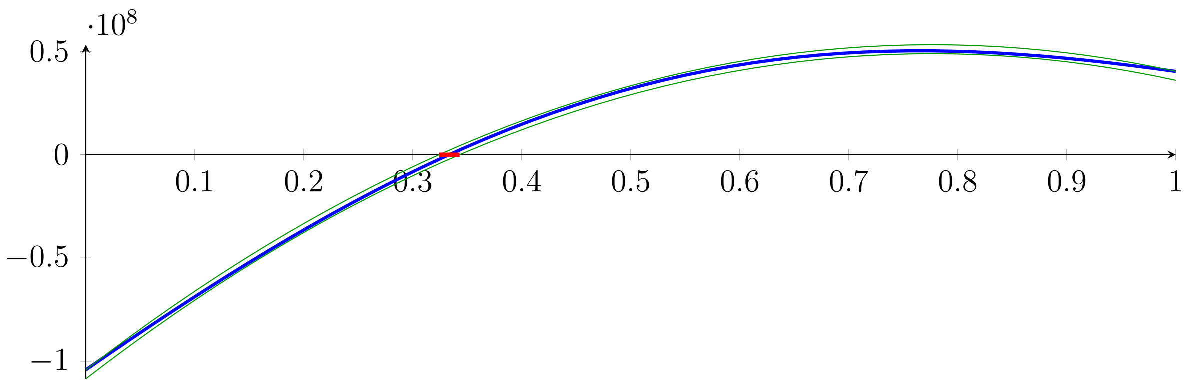
Bounding polynomials M and m:

$$M = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.04167{\cdot}10^8$$
$$m = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.08503{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

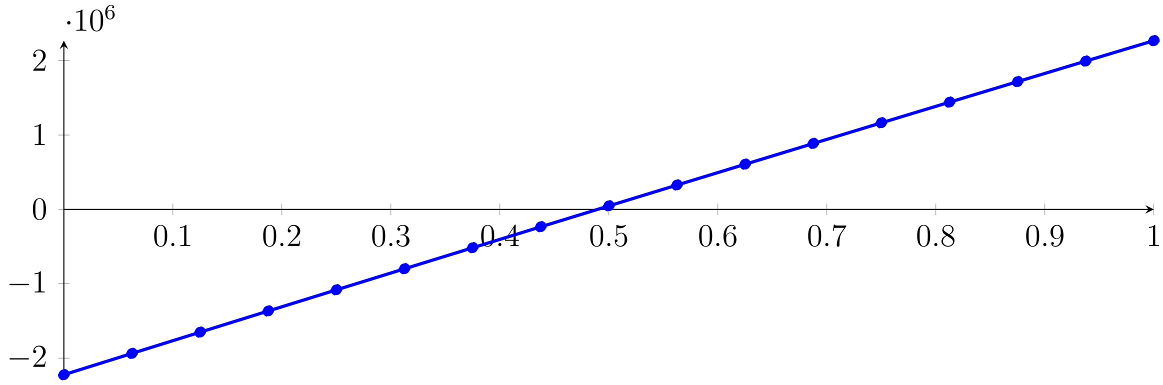
Longest intersection interval: 0.0187703

$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

249.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

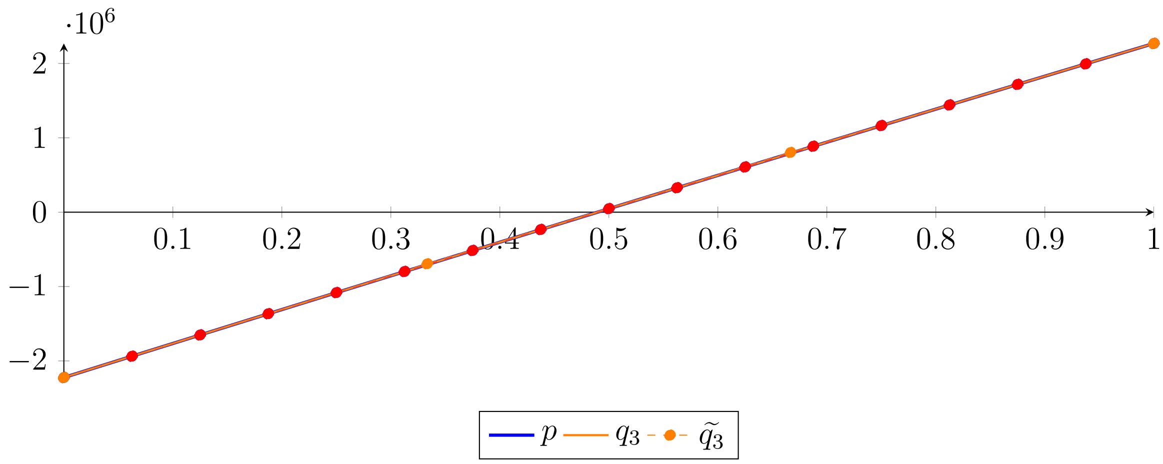
$$p = -2.37433\cdot10^{-28}X^{16} - 5.67363\cdot10^{-25}X^{15} - 5.77631\cdot10^{-22}X^{14} - 3.1818\cdot10^{-19}X^{13} - 9.6142\cdot10^{-17}X^{12} - 1.16549\cdot10^{-14}X^{11} + 1.71065\cdot10^{-12}X^{10} + 7.20611\cdot10^{-10}X^{9} + 4.39147\cdot10^{-08}X^{8} - 1.21542\cdot10^{-05}X^{7} - 0.00155624X^{6} + 0.10496X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,16}(X) - 1.93666\cdot10^{6}B_{1,16}(X) - 1.65074\cdot10^{6}B_{2,16}(X) - 1.36561\cdot10^{6}B_{3,16}(X) - 1.08127\cdot10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X) + 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot10^{6}B_{12,16}(X) + 1.44233\cdot10^{6}B_{13,16}(X) + 1.71876\cdot10^{6}B_{14,16}(X) + 1.99439\cdot10^{6}B_{15,16}(X) + 2.26922\cdot10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot10^{6}B_{3,3}$$

$$\widetilde{q}_3 = 4.41595\cdot10^{-291}X^{16} - 1.48598\cdot10^{-290}X^{15} - 2.81102\cdot10^{-290}X^{14} + 2.65129\cdot10^{-289}X^{13} - 7.34593\cdot10^{-289}X^{12} + 1.17654\cdot10^{-288}X^{11} - 1.23119\cdot10^{-288}X^{10} + 8.63493\cdot10^{-289}X^{9} - 3.92909\cdot10^{-289}X^{8} + 1.01265\cdot10^{-289}X^{7} - 6.13644\cdot10^{-291}X^{6} - 3.92664\cdot10^{-291}X^{5} + 9.04488\cdot10^{-292}X^{4} - 700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,16} - 1.93666\cdot10^{6}B_{1,16} - 1.65074\cdot10^{6}B_{2,16} - 1.36561\cdot10^{6}B_{3,16} - 1.08126\cdot10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.4B_{8,16} + 328650B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot10^{6}B_{12,16} + 1.44233\cdot10^{6}B_{13,16} + 1.71876\cdot10^{6}B_{14,16} + 1.99439\cdot10^{6}B_{15,16} + 2.26922\cdot10^{6}B_{16,16}$$



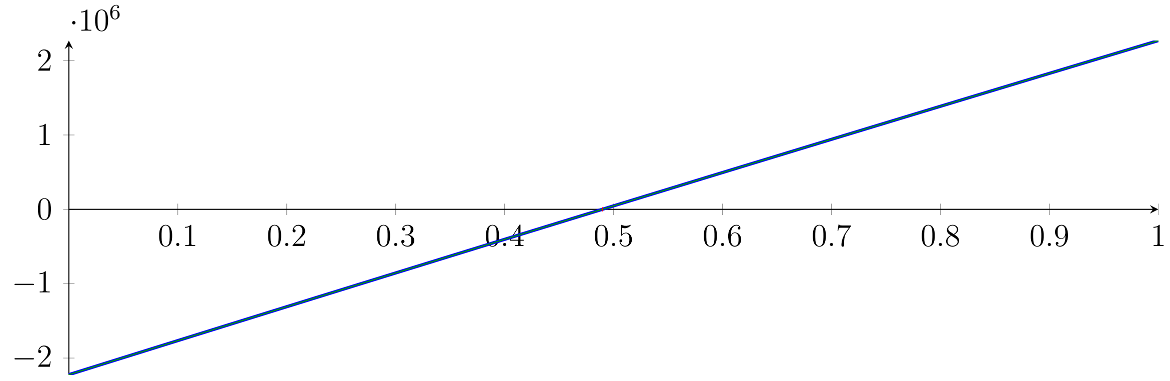
The maximum difference of the Bézier coefficients is $\delta = 0.270074$.
Bounding polynomials M and m:

$$M = -700.673X^3 - 93879.9X^2 + 4.58715\cdot 10^6 X - 2.22335\cdot 10^6$$
$$m = -700.673X^3 - 93879.9X^2 + 4.58715\cdot 10^6 X - 2.22335\cdot 10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



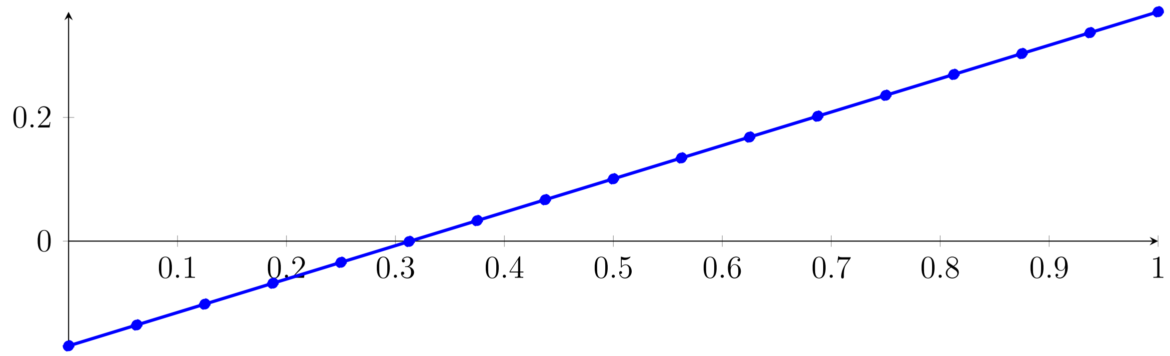
$[0.489616, 0.489616]$

Longest intersection interval: $1.20174\cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

249.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

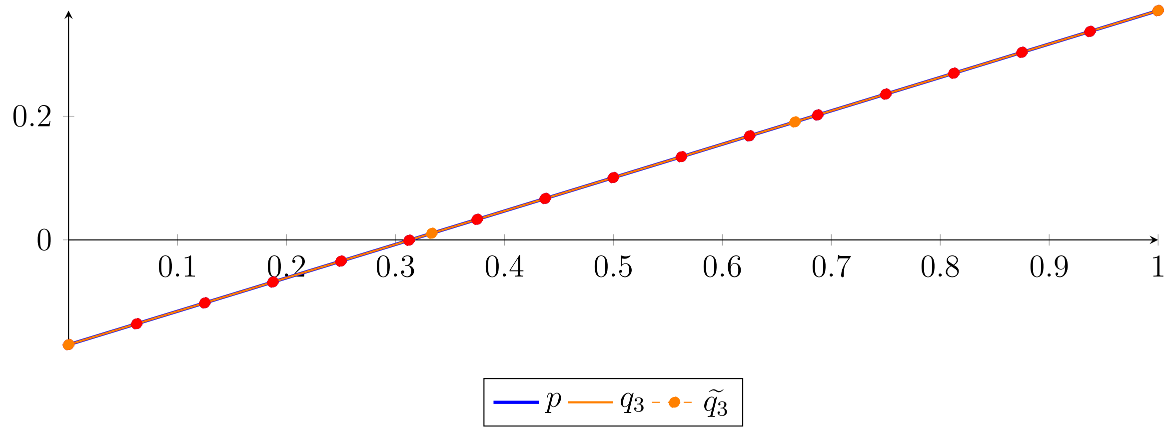
$$\begin{aligned} p = &-4.49274\cdot 10^{-139}X^{16} - 8.96277\cdot 10^{-129}X^{15} - 7.623\cdot 10^{-119}X^{14} - 3.51238\cdot 10^{-109}X^{13} \\ &- 8.90739\cdot 10^{-100}X^{12} - 9.22984\cdot 10^{-91}X^{11} + 1.03426\cdot 10^{-81}X^{10} + 3.80998\cdot 10^{-72}X^9 \\ &+ 2.04919\cdot 10^{-63}X^8 - 4.33497\cdot 10^{-54}X^7 - 4.81204\cdot 10^{-45}X^6 + 2.51462\cdot 10^{-36}X^5 \\ &+ 3.93622\cdot 10^{-27}X^4 - 1.21745\cdot 10^{-18}X^3 - 1.37072\cdot 10^{-09}X^2 + 0.540148X - 0.169396 \\ = &-0.169396B_{0,16}(X) - 0.135637B_{1,16}(X) - 0.101877B_{2,16}(X) - 0.068118B_{3,16}(X) \\ &- 0.0343588B_{4,16}(X) - 0.000599488B_{5,16}(X) + 0.0331598B_{6,16}(X) \\ &+ 0.066919B_{7,16}(X) + 0.100678B_{8,16}(X) + 0.134438B_{9,16}(X) + 0.168197B_{10,16}(X) \\ &+ 0.201956B_{11,16}(X) + 0.235715B_{12,16}(X) + 0.269475B_{13,16}(X) \\ &+ 0.303234B_{14,16}(X) + 0.336993B_{15,16}(X) + 0.370752B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.21745\cdot 10^{-18}X^3 - 1.37072\cdot 10^{-09}X^2 + 0.540148X - 0.169396 \\ &= -0.169396B_{0,3} + 0.0106536B_{1,3} + 0.190703B_{2,3} + 0.370752B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= 8.03185\cdot 10^{-297}X^{16} - 6.20841\cdot 10^{-296}X^{15} + 2.13274\cdot 10^{-295}X^{14} - 4.26614\cdot 10^{-295}X^{13} \\ &\quad + 5.47461\cdot 10^{-295}X^{12} - 4.70265\cdot 10^{-295}X^{11} + 2.78551\cdot 10^{-295}X^{10} - 1.22442\cdot 10^{-295}X^{9} \\ &\quad + 4.88954\cdot 10^{-296}X^{8} - 2.11494\cdot 10^{-296}X^{7} + 8.20665\cdot 10^{-297}X^{6} - 2.15458\cdot 10^{-297}X^{5} \\ &\quad + 3.01517\cdot 10^{-298}X^{4} - 1.21745\cdot 10^{-18}X^{3} - 1.37072\cdot 10^{-09}X^{2} + 0.540148X - 0.169396 \\ &= -0.169396B_{0,16} - 0.135637B_{1,16} - 0.101877B_{2,16} - 0.068118B_{3,16} - 0.0343588B_{4,16} \\ &\quad - 0.000599488B_{5,16} + 0.0331598B_{6,16} + 0.066919B_{7,16} + 0.100678B_{8,16} \\ &\quad + 0.134438B_{9,16} + 0.168197B_{10,16} + 0.201956B_{11,16} + 0.235715B_{12,16} \\ &\quad + 0.269475B_{13,16} + 0.303234B_{14,16} + 0.336993B_{15,16} + 0.370752B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.62317\cdot 10^{-29}$.
Bounding polynomials M and m:

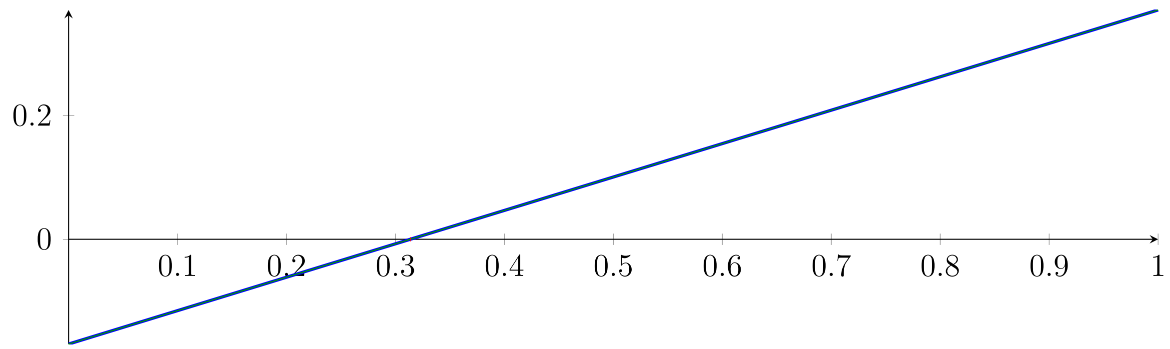
$$M = -1.21745\cdot 10^{-18}X^3 - 1.37072\cdot 10^{-09}X^2 + 0.540148X - 0.169396$$

$$m = -1.21745\cdot 10^{-18}X^3 - 1.37072\cdot 10^{-09}X^2 + 0.540148X - 0.169396$$

Root of M and m:

$$N(M) = \{-1.43506\cdot 10^9, 0.31361, 3.09167\cdot 10^8\} \quad N(m) = \{-1.43506\cdot 10^9, 0.31361, 3.09167\cdot 10^8\}$$

Intersection intervals:



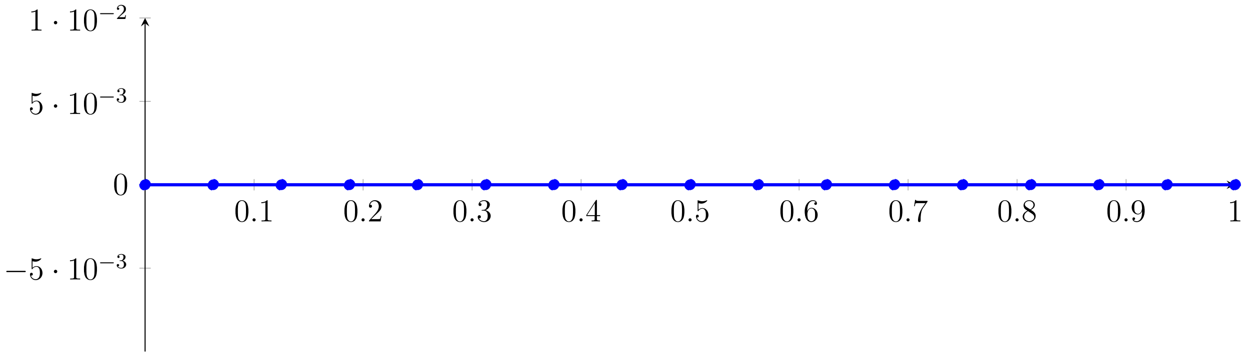
$$[0.31361, 0.31361]$$

Longest intersection interval: $2.08208\cdot 10^{-28}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

249.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

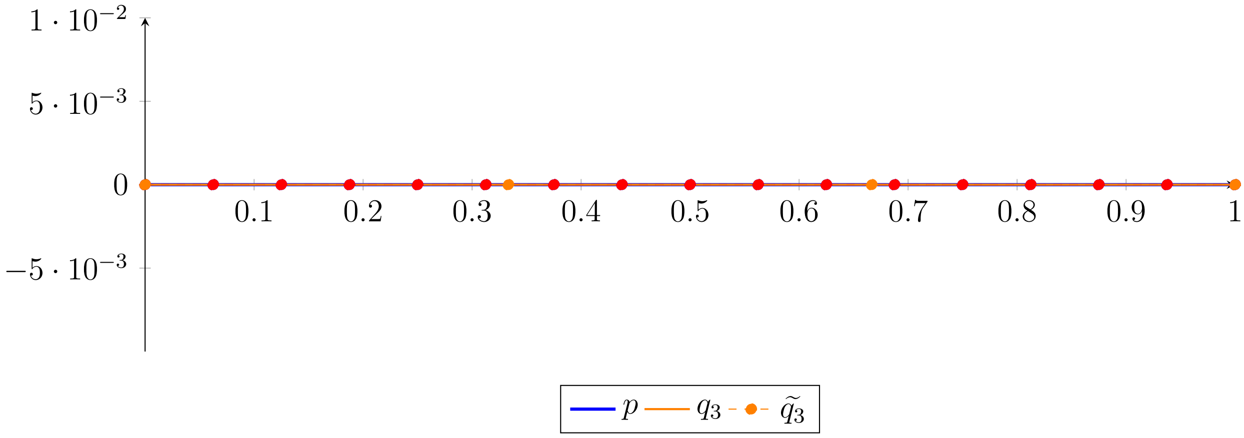
$$p = 3.09811\cdot10^{-312}X^{16} + 1.13739\cdot10^{-311}X^{15} + 1.09495\cdot10^{-310}X^{14} - 3.56495\cdot10^{-311}X^{13} - 1.48688\cdot10^{-309}X^{12} + 1.48302\cdot10^{-309}X^{11} + 7.222\cdot10^{-310}X^{10} + 1.21378\cdot10^{-310}X^{9} + 7.23716\cdot10^{-285}X^{8} - 7.35315\cdot10^{-248}X^{7} - 3.92029\cdot10^{-211}X^{6} + 9.83929\cdot10^{-175}X^{5} + 7.39728\cdot10^{-138}X^{4} - 1.09887\cdot10^{-101}X^{3} - 5.94215\cdot10^{-65}X^{2} + 1.12463\cdot10^{-28}X - 8.88188\cdot10^{-08}$$
$$= -8.88188\cdot10^{-08}B_{0,16}(X) - 8.88188\cdot10^{-08}B_{1,16}(X) - 8.88188\cdot10^{-08}B_{2,16}(X) - 8.88188\cdot10^{-08}B_{3,16}(X) - 8.88188\cdot10^{-08}B_{4,16}(X) - 8.88188\cdot10^{-08}B_{5,16}(X) - 8.88188\cdot10^{-08}B_{6,16}(X) - 8.88188\cdot10^{-08}B_{7,16}(X) - 8.88188\cdot10^{-08}B_{8,16}(X) - 8.88188\cdot10^{-08}B_{9,16}(X) - 8.88188\cdot10^{-08}B_{10,16}(X) - 8.88188\cdot10^{-08}B_{11,16}(X) - 8.88188\cdot10^{-08}B_{12,16}(X) - 8.88188\cdot10^{-08}B_{13,16}(X) - 8.88188\cdot10^{-08}B_{14,16}(X) - 8.88188\cdot10^{-08}B_{15,16}(X) - 8.88188\cdot10^{-08}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -1.09887\cdot10^{-101}X^{3} - 5.94215\cdot10^{-65}X^{2} + 1.12463\cdot10^{-28}X - 8.88188\cdot10^{-08}$$
$$= -8.88188\cdot10^{-08}B_{0,3} - 8.88188\cdot10^{-08}B_{1,3} - 8.88188\cdot10^{-08}B_{2,3} - 8.88188\cdot10^{-08}B_{3,3}$$

$$\widetilde{q}_3 = -6.98397\cdot10^{-303}X^{16} + 5.61515\cdot10^{-302}X^{15} - 2.01778\cdot10^{-301}X^{14} + 4.27019\cdot10^{-301}X^{13} - 5.92096\cdot10^{-301}X^{12} + 5.69601\cdot10^{-301}X^{11} - 3.9714\cdot10^{-301}X^{10} + 2.10656\cdot10^{-301}X^{9} - 8.95545\cdot10^{-302}X^{8} + 3.10786\cdot10^{-302}X^{7} - 8.35303\cdot10^{-303}X^{6} + 1.57296\cdot10^{-303}X^{5} - 1.80277\cdot10^{-304}X^{4} - 1.09887\cdot10^{-101}X^{3} - 5.94215\cdot10^{-65}X^{2} + 1.12463\cdot10^{-28}X - 8.88188\cdot10^{-08}$$
$$= -8.88188\cdot10^{-08}B_{0,16} - 8.88188\cdot10^{-08}B_{1,16} - 8.88188\cdot10^{-08}B_{2,16} - 8.88188\cdot10^{-08}B_{3,16} - 8.88188\cdot10^{-08}B_{4,16} - 8.88188\cdot10^{-08}B_{5,16} - 8.88188\cdot10^{-08}B_{6,16} - 8.88188\cdot10^{-08}B_{7,16} - 8.88188\cdot10^{-08}B_{8,16} - 8.88188\cdot10^{-08}B_{9,16} - 8.88188\cdot10^{-08}B_{10,16} - 8.88188\cdot10^{-08}B_{11,16} - 8.88188\cdot10^{-08}B_{12,16} - 8.88188\cdot10^{-08}B_{13,16} - 8.88188\cdot10^{-08}B_{14,16} - 8.88188\cdot10^{-08}B_{15,16} - 8.88188\cdot10^{-08}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 1.05675{\cdot}10^{-139}$.
Bounding polynomials M and m:

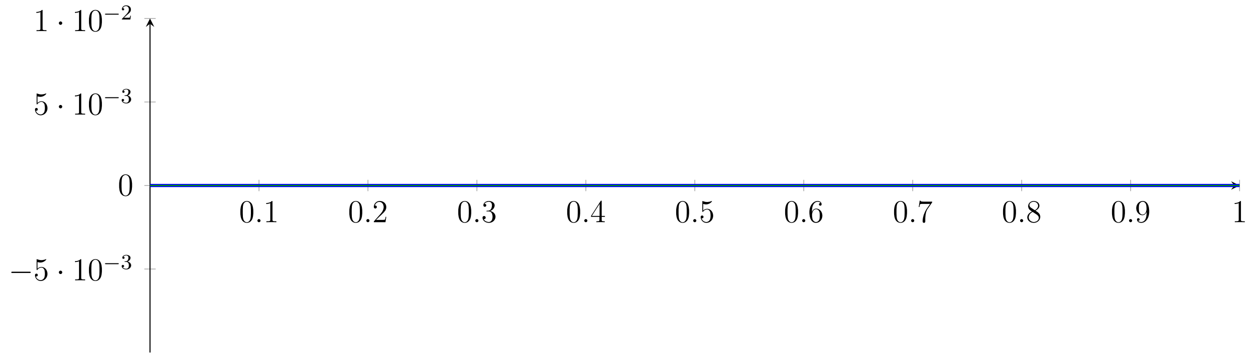
$$M = -1.09887{\cdot}10^{-101}X^3 - 5.94215{\cdot}10^{-65}X^2 + 1.12463{\cdot}10^{-28}X - 8.88188{\cdot}10^{-08}$$

$$m = -1.09887{\cdot}10^{-101}X^3 - 5.94215{\cdot}10^{-65}X^2 + 1.12463{\cdot}10^{-28}X - 8.88188{\cdot}10^{-08}$$

Root of M and m:

$N(M) = \{-6.89243{\cdot}10^{36}, 1.34848{\cdot}10^{12}, 1.48489{\cdot}10^{36}\}$ $N(m) = \{-6.89243{\cdot}10^{36}, 1.34848{\cdot}10^{12}, 1.48489{\cdot}10^{36}\}$

Intersection intervals:

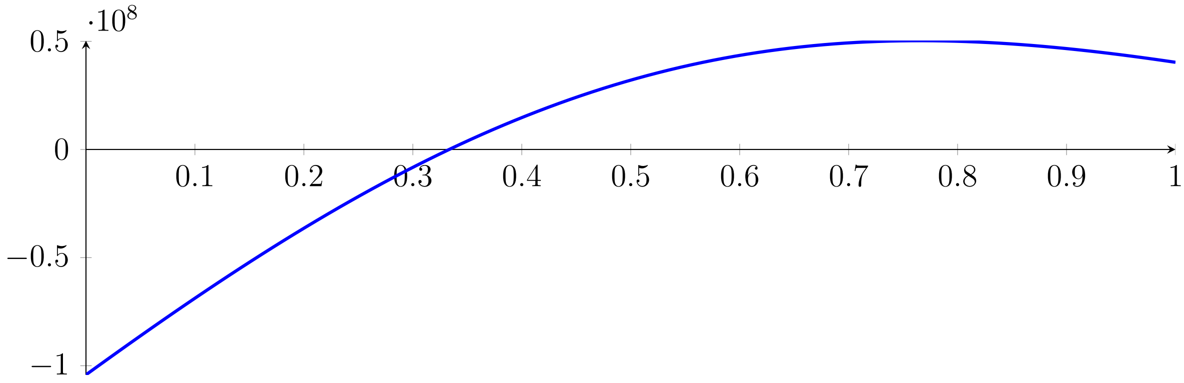


No intersection intervals with the x axis.

249.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}$$



Result: Root Intervals

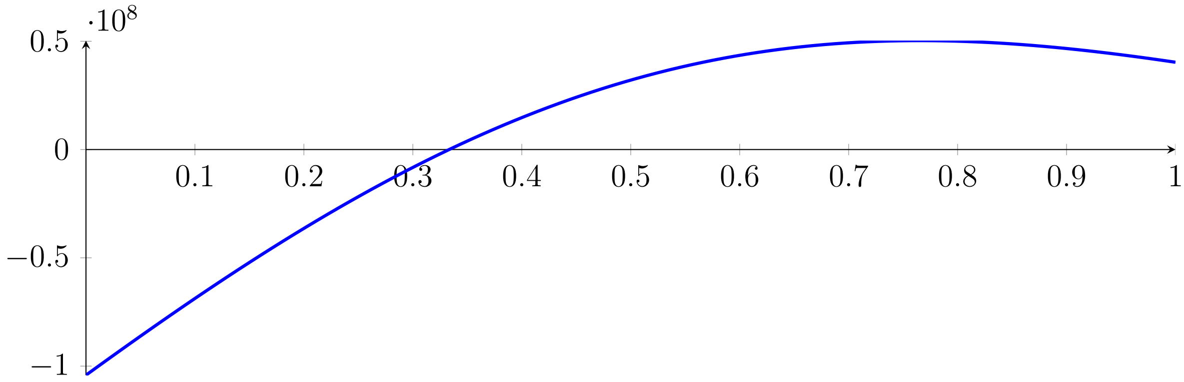
with precision $\varepsilon = 1\cdot 10^{-64}$.

250 Running `BezClip` on f_{16} with epsilon 128

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^{6}X^{9} - 2.52262 \cdot 10^{6}X^{8} - 1.43506 \cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \cdot 10^{7}X^{5} + 3.97396 \cdot 10^{7}X^{4} - 2.40625 \cdot 10^{8}X^{3} - 8.33333 \cdot 10^{7}X^{2} + 3.64583 \cdot 10^{8}X - 1.04167 \cdot 10^{8}$$

Called `BezClip` with input polynomial on interval $[0, 1]$:

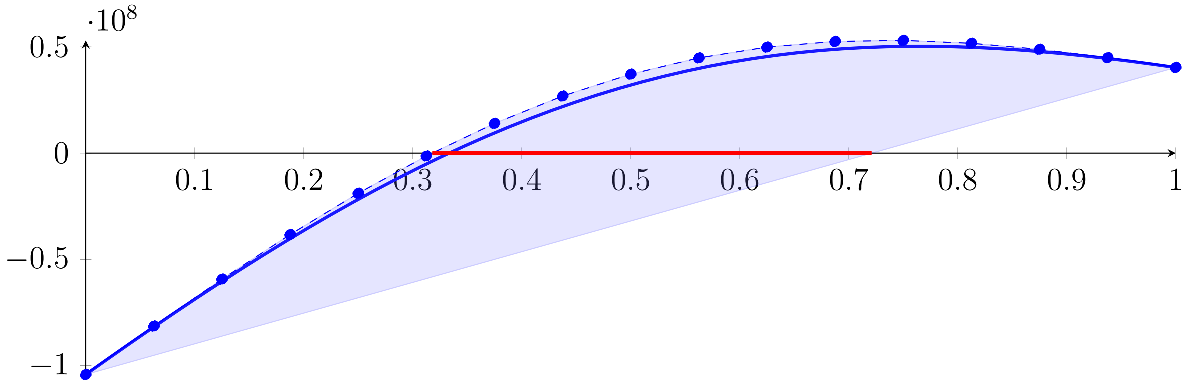
$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^{6}X^{9} - 2.52262 \cdot 10^{6}X^{8} - 1.43506 \cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \cdot 10^{7}X^{5} \\ & + 3.97396 \cdot 10^{7}X^{4} - 2.40625 \cdot 10^{8}X^{3} - 8.33333 \cdot 10^{7}X^{2} + 3.64583 \cdot 10^{8}X - 1.04167 \cdot 10^{8} \end{aligned}$$



250.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^{6}X^{9} - 2.52262 \cdot 10^{6}X^{8} - 1.43506 \cdot 10^{7}X^{7} + 138542X^{6} + 7.92823 \cdot 10^{7}X^{5} \\ & + 3.97396 \cdot 10^{7}X^{4} - 2.40625 \cdot 10^{8}X^{3} - 8.33333 \cdot 10^{7}X^{2} + 3.64583 \cdot 10^{8}X - 1.04167 \cdot 10^{8} \\ = & -1.04167 \cdot 10^{8}B_{0,16}(X) - 8.13802 \cdot 10^{7}B_{1,16}(X) - 5.92882 \cdot 10^{7}B_{2,16}(X) - 3.83203 \\ & \cdot 10^{7}B_{3,16}(X) - 1.88844 \cdot 10^{7}B_{4,16}(X) - 1.34837 \cdot 10^{6}B_{5,16}(X) + 1.39781 \cdot 10^{7}B_{6,16}(X) \\ & + 2.68604 \cdot 10^{7}B_{7,16}(X) + 3.71532 \cdot 10^{7}B_{8,16}(X) + 4.48105 \cdot 10^{7}B_{9,16}(X) + 4.98901 \\ & \cdot 10^{7}B_{10,16}(X) + 5.25504 \cdot 10^{7}B_{11,16}(X) + 5.30407 \cdot 10^{7}B_{12,16}(X) + 5.16832 \cdot 10^{7}B_{13,16}(X) \\ & + 4.88488 \cdot 10^{7}B_{14,16}(X) + 4.49297 \cdot 10^{7}B_{15,16}(X) + 4.03108 \cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

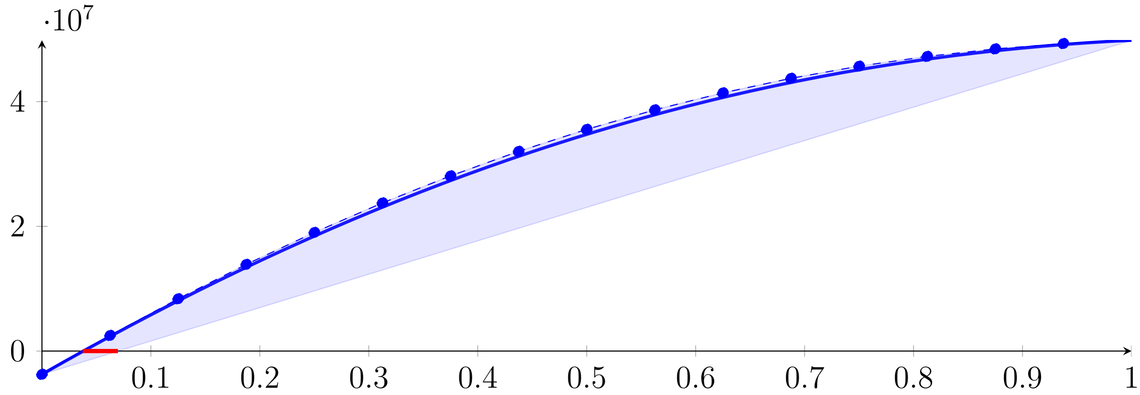
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\implies$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

250.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -4.83858\cdot 10^{-07}X^{16} - 5.37355\cdot 10^{-05}X^{15} - 0.00254146X^{14} - 0.064977X^{13} - 0.909205X^{12} \\ & - 5.03924X^{11} + 36.4549X^{10} + 692.921X^{9} + 1886.97X^{8} - 25792X^{7} - 149671X^{6} + 492605X^{5} \\ & + 3.91945\cdot 10^{6}X^{4} - 7.54561\cdot 10^{6}X^{3} - 4.29225\cdot 10^{7}X^{2} + 9.97982\cdot 10^{7}X - 3.73501\cdot 10^{6} \\ = & -3.73501\cdot 10^{6}B_{0,16}(X) + 2.50237\cdot 10^{6}B_{1,16}(X) + 8.38207\cdot 10^{6}B_{2,16}(X) + 1.38906 \\ & \cdot 10^{7}B_{3,16}(X) + 1.90167\cdot 10^{7}B_{4,16}(X) + 2.37512\cdot 10^{7}B_{5,16}(X) + 2.80875\cdot 10^{7}B_{6,16}(X) \\ & + 3.20213\cdot 10^{7}B_{7,16}(X) + 3.55508\cdot 10^{7}B_{8,16}(X) + 3.86768\cdot 10^{7}B_{9,16}(X) + 4.14025 \\ & \cdot 10^{7}B_{10,16}(X) + 4.37336\cdot 10^{7}B_{11,16}(X) + 4.56783\cdot 10^{7}B_{12,16}(X) + 4.7247\cdot 10^{7}B_{13,16}(X) \\ & + 4.84524\cdot 10^{7}B_{14,16}(X) + 4.93093\cdot 10^{7}B_{15,16}(X) + 4.98343\cdot 10^{7}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

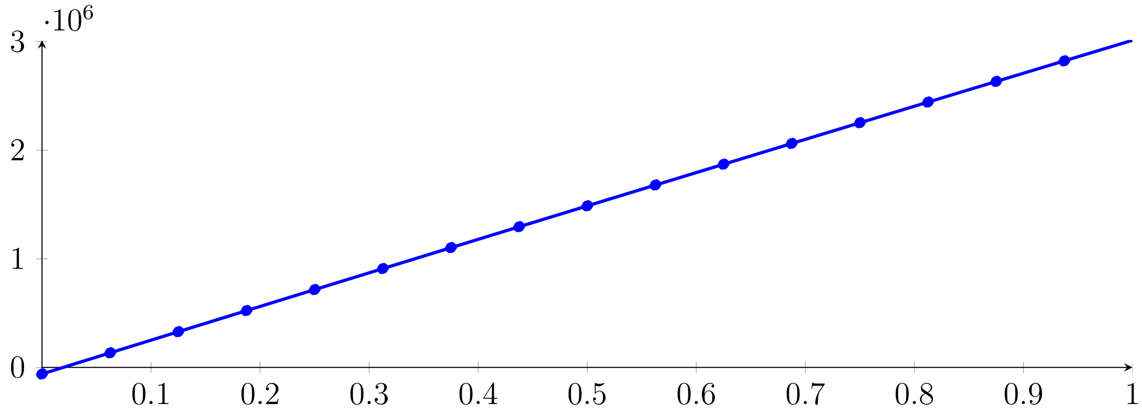
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\implies$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

250.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = & -6.78234\cdot 10^{-31}X^{16} - 2.34473\cdot 10^{-27}X^{15} - 3.45581\cdot 10^{-24}X^{14} - 2.75921\cdot 10^{-21}X^{13} - 1.21242\cdot 10^{-18}X^{12} \\ & - 2.17572\cdot 10^{-16}X^{11} + 4.23544\cdot 10^{-14}X^{10} + 2.70025\cdot 10^{-11}X^{9} + 2.51306\cdot 10^{-09}X^{8} - 9.23474\cdot 10^{-07}X^{7} \\ & - 0.000177459X^{6} + 0.0161038X^{5} + 4.36155X^{4} - 234.216X^{3} - 45622.2X^{2} + 3.11845\cdot 10^{6}X - 60508.5 \\ = & -60508.5B_{0,16}(X) + 134395B_{1,16}(X) + 328918B_{2,16}(X) + 523060B_{3,16}(X) + 716822B_{4,16}(X) \\ & + 910202B_{5,16}(X) + 1.1032\cdot 10^{6}B_{6,16}(X) + 1.29582\cdot 10^{6}B_{7,16}(X) + 1.48805\cdot 10^{6}B_{8,16}(X) \\ & + 1.6799\cdot 10^{6}B_{9,16}(X) + 1.87137\cdot 10^{6}B_{10,16}(X) + 2.06245\cdot 10^{6}B_{11,16}(X) + 2.25315\cdot 10^{6}B_{12,16}(X) \\ & + 2.44346\cdot 10^{6}B_{13,16}(X) + 2.63339\cdot 10^{6}B_{14,16}(X) + 2.82293\cdot 10^{6}B_{15,16}(X) + 3.01209\cdot 10^{6}B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

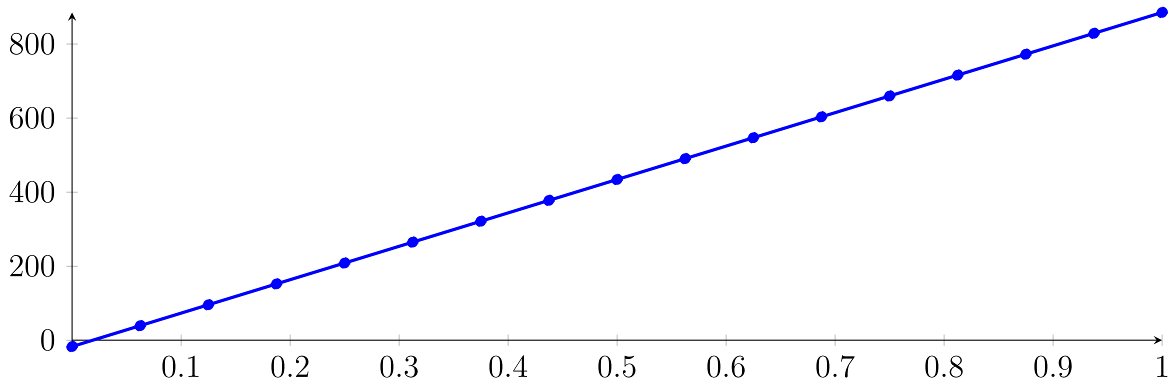
$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

250.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = &-1.65599\cdot 10^{-87}X^{16} - 1.97733\cdot 10^{-80}X^{15} - 1.00659\cdot 10^{-73}X^{14} - 2.77601\cdot 10^{-67}X^{13} \\ &- 4.21367\cdot 10^{-61}X^{12} - 2.61333\cdot 10^{-55}X^{11} + 1.75275\cdot 10^{-49}X^{10} + 3.8646\cdot 10^{-43}X^{9} \\ &+ 1.2441\cdot 10^{-37}X^{8} - 1.57525\cdot 10^{-31}X^{7} - 1.04661\cdot 10^{-25}X^{6} + 3.27355\cdot 10^{-20}X^{5} \\ &+ 3.06701\cdot 10^{-14}X^{4} - 5.67777\cdot 10^{-09}X^{3} - 0.00382618X^{2} + 902.448X - 17.178 \\ = &-17.178B_{0,16}(X) + 39.225B_{1,16}(X) + 95.6279B_{2,16}(X) + 152.031B_{3,16}(X) + 208.434B_{4,16}(X) \\ &+ 264.837B_{5,16}(X) + 321.24B_{6,16}(X) + 377.642B_{7,16}(X) + 434.045B_{8,16}(X) \\ &+ 490.448B_{9,16}(X) + 546.851B_{10,16}(X) + 603.253B_{11,16}(X) + 659.656B_{12,16}(X) \\ &+ 716.059B_{13,16}(X) + 772.461B_{14,16}(X) + 828.864B_{15,16}(X) + 885.266B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

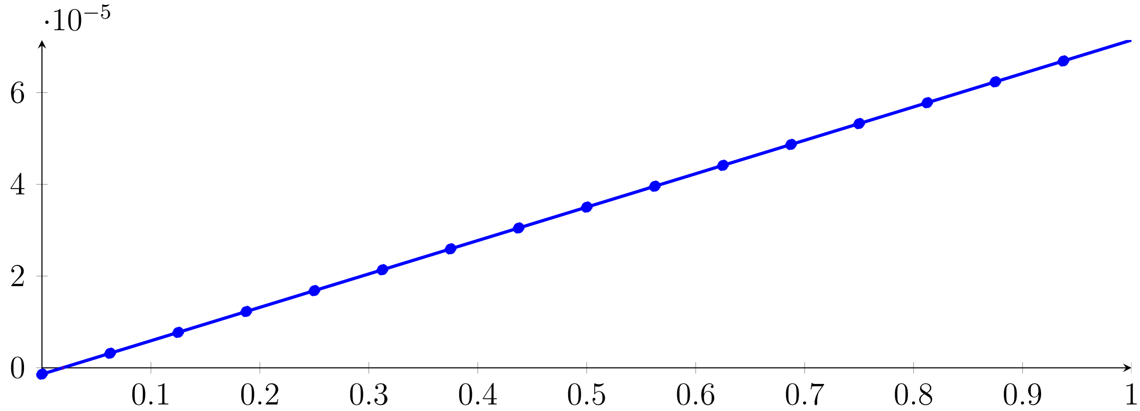
$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045\cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

250.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -5.36315\cdot10^{-201}X^{16} - 7.93495\cdot10^{-187}X^{15} - 5.0052\cdot10^{-173}X^{14} - 1.71037\cdot10^{-159}X^{13} - 3.21686\cdot10^{-146}X^{12} - 2.47211\cdot10^{-133}X^{11} + 2.05446\cdot10^{-120}X^{10} + 5.61285\cdot10^{-107}X^{9} + 2.23891\cdot10^{-94}X^{8} - 3.51264\cdot10^{-81}X^{7} - 2.89181\cdot10^{-68}X^{6} + 1.12075\cdot10^{-55}X^{5} + 1.30109\cdot10^{-42}X^{4} - 2.98449\cdot10^{-30}X^{3} - 2.49207\cdot10^{-17}X^{2} + 7.28316\cdot10^{-05}X - 1.38634\cdot10^{-06}$$
$$= -1.38634\cdot10^{-06}B_{0,16}(X) + 3.16564\cdot10^{-06}B_{1,16}(X) + 7.71761\cdot10^{-06}B_{2,16}(X) + 1.22696\cdot10^{-05}B_{3,16}(X) + 1.68216\cdot10^{-05}B_{4,16}(X) + 2.13735\cdot10^{-05}B_{5,16}(X) + 2.59255\cdot10^{-05}B_{6,16}(X) + 3.04775\cdot10^{-05}B_{7,16}(X) + 3.50295\cdot10^{-05}B_{8,16}(X) + 3.95814\cdot10^{-05}B_{9,16}(X) + 4.41334\cdot10^{-05}B_{10,16}(X) + 4.86854\cdot10^{-05}B_{11,16}(X) + 5.32374\cdot10^{-05}B_{12,16}(X) + 5.77893\cdot10^{-05}B_{13,16}(X) + 6.23413\cdot10^{-05}B_{14,16}(X) + 6.68933\cdot10^{-05}B_{15,16}(X) + 7.14453\cdot10^{-05}B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

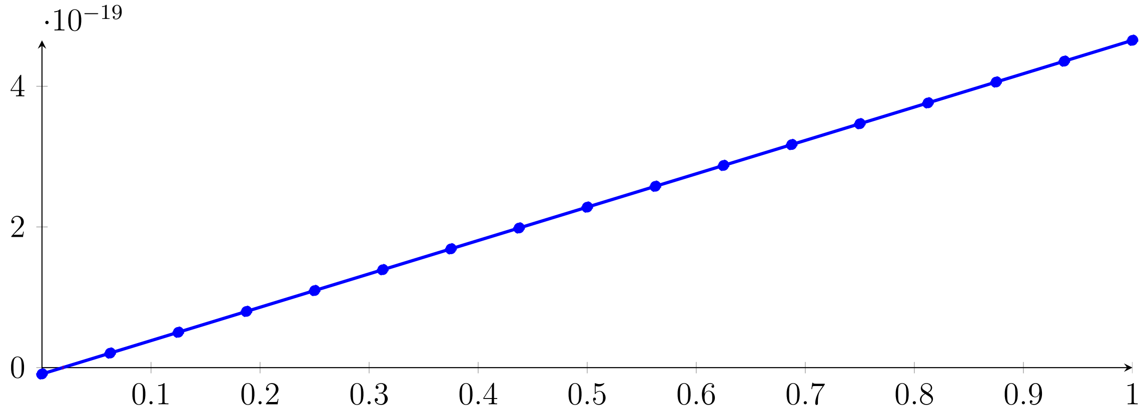
$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.51314\cdot10^{-15}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

250.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.70149\cdot10^{-323}X^{16} + 1.97819\cdot10^{-322}X^{15} - 4.34527\cdot10^{-322}X^{14} + 3.97182\cdot10^{-322}X^{13} - 1.87464\cdot10^{-316}X^{12} - 2.21189\cdot10^{-289}X^{11} + 2.82229\cdot10^{-262}X^{10} + 1.18385\cdot10^{-234}X^{9} + 7.25038\cdot10^{-208}X^{8} - 1.74649\cdot10^{-180}X^{7} - 2.20756\cdot10^{-153}X^{6} + 1.31359\cdot10^{-126}X^{5} + 2.34136\cdot10^{-99}X^{4} - 8.24597\cdot10^{-73}X^{3} - 1.05716\cdot10^{-45}X^{2} + 4.74362\cdot10^{-19}X - 9.02941\cdot10^{-21}$$
$$= -9.02941\cdot10^{-21}B_{0,16}(X) + 2.06182\cdot10^{-20}B_{1,16}(X) + 5.02659\cdot10^{-20}B_{2,16}(X) + 7.99135\cdot10^{-20}B_{3,16}(X) + 1.09561\cdot10^{-19}B_{4,16}(X) + 1.39209\cdot10^{-19}B_{5,16}(X) + 1.68856\cdot10^{-19}B_{6,16}(X) + 1.98504\cdot10^{-19}B_{7,16}(X) + 2.28152\cdot10^{-19}B_{8,16}(X) + 2.57799\cdot10^{-19}B_{9,16}(X) + 2.87447\cdot10^{-19}B_{10,16}(X) + 3.17095\cdot10^{-19}B_{11,16}(X) + 3.46742\cdot10^{-19}B_{12,16}(X) + 3.7639\cdot10^{-19}B_{13,16}(X) + 4.06038\cdot10^{-19}B_{14,16}(X) + 4.35685\cdot10^{-19}B_{15,16}(X) + 4.65333\cdot10^{-19}B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

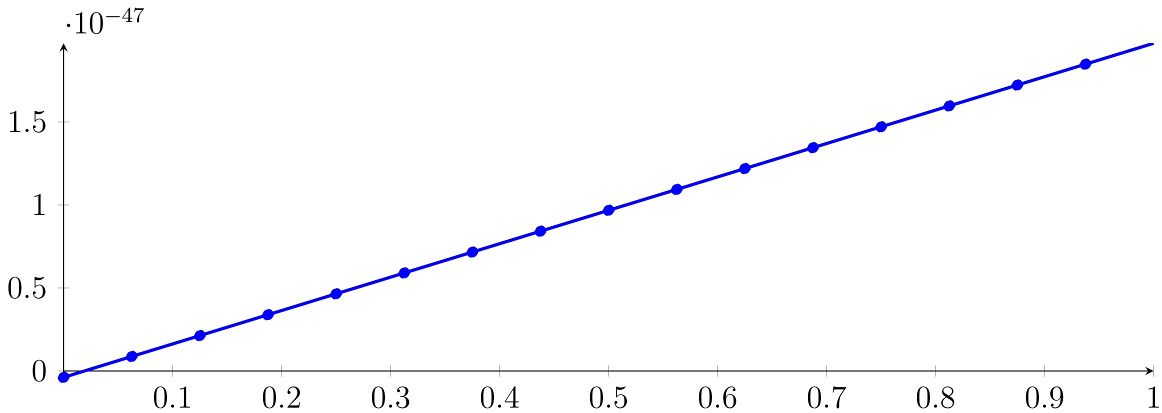
$$[0.0190348, 0.0190348]$$

Longest intersection interval: $4.2421 \cdot 10^{-29}$
$\implies$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

250.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = {} & 1.52489 \cdot 10^{-352} X^{16} + 4.56006 \cdot 10^{-351} X^{15} - 3.36159 \cdot 10^{-350} X^{14} + 2.87148 \cdot 10^{-350} X^{13} \\ & - 1.35884 \cdot 10^{-349} X^{12} + 1.13318 \cdot 10^{-349} X^{11} - 1.84828 \cdot 10^{-349} X^{10} + 7.52411 \cdot 10^{-350} X^{9} \\ & - 2.19453 \cdot 10^{-350} X^{8} + 2.78671 \cdot 10^{-351} X^{7} - 1.28647 \cdot 10^{-323} X^{6} + 1.80453 \cdot 10^{-268} X^{5} + 7.58214 \\ & \cdot 10^{-213} X^{4} - 6.29484 \cdot 10^{-158} X^{3} - 1.90241 \cdot 10^{-102} X^{2} + 2.01229 \cdot 10^{-47} X - 3.83037 \cdot 10^{-49} \\ = {} & -3.83037 \cdot 10^{-49} B_{0,16}(X) + 8.74646 \cdot 10^{-49} B_{1,16}(X) + 2.13233 \cdot 10^{-48} B_{2,16}(X) + 3.39001 \\ & \cdot 10^{-48} B_{3,16}(X) + 4.6477 \cdot 10^{-48} B_{4,16}(X) + 5.90538 \cdot 10^{-48} B_{5,16}(X) + 7.16306 \cdot 10^{-48} B_{6,16}(X) \\ & + 8.42074 \cdot 10^{-48} B_{7,16}(X) + 9.67843 \cdot 10^{-48} B_{8,16}(X) + 1.09361 \cdot 10^{-47} B_{9,16}(X) + 1.21938 \\ & \cdot 10^{-47} B_{10,16}(X) + 1.34515 \cdot 10^{-47} B_{11,16}(X) + 1.47092 \cdot 10^{-47} B_{12,16}(X) + 1.59668 \cdot 10^{-47} B_{13,16}(X) \\ & + 1.72245 \cdot 10^{-47} B_{14,16}(X) + 1.84822 \cdot 10^{-47} B_{15,16}(X) + 1.97399 \cdot 10^{-47} B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

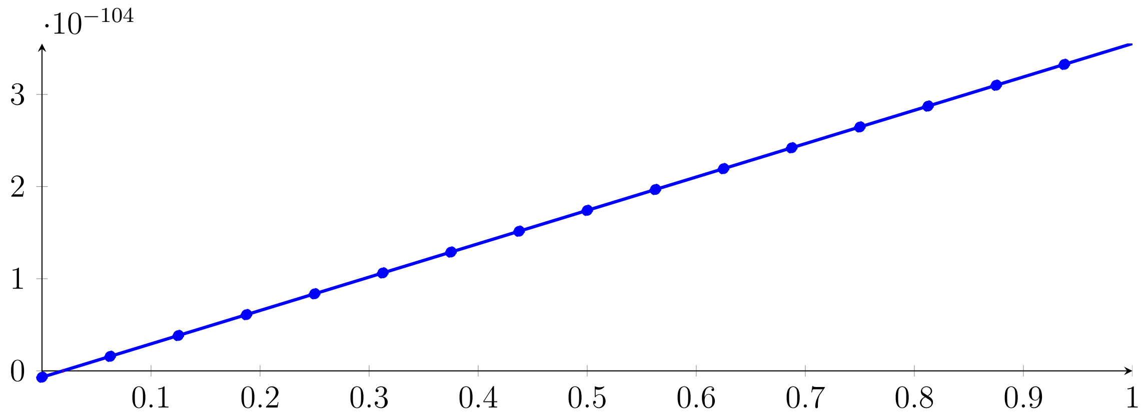
$$[0.0190348, 0.0190348]$$

Longest intersection interval: $1.79954 \cdot 10^{-57}$
$\implies$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

250.8 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 1.13315\cdot10^{-409}X^{16} + 3.67576\cdot10^{-408}X^{15} + 3.40877\cdot10^{-408}X^{14} - 2.82512\cdot10^{-407}X^{13} + 5.53724\cdot10^{-407}X^{12} - 2.84772\cdot10^{-407}X^{11} + 1.69055\cdot10^{-406}X^{10} + 8.70138\cdot10^{-407}X^{9} - 4.59486\cdot10^{-407}X^{8} + 2.21974\cdot10^{-407}X^{7} - 1.24305\cdot10^{-407}X^{6} + 1.41256\cdot10^{-409}X^{4} - 3.66835\cdot10^{-328}X^{3} - 6.16067\cdot10^{-216}X^{2} + 3.6212\cdot10^{-104}X - 6.8929\cdot10^{-106}$$
$$= -6.8929\cdot10^{-106}B_{0,16}(X) + 1.57396\cdot10^{-105}B_{1,16}(X) + 3.83722\cdot10^{-105}B_{2,16}(X) + 6.10047\cdot10^{-105}B_{3,16}(X) + 8.36372\cdot10^{-105}B_{4,16}(X) + 1.0627\cdot10^{-104}B_{5,16}(X) + 1.28902\cdot10^{-104}B_{6,16}(X) + 1.51535\cdot10^{-104}B_{7,16}(X) + 1.74167\cdot10^{-104}B_{8,16}(X) + 1.968\cdot10^{-104}B_{9,16}(X) + 2.19432\cdot10^{-104}B_{10,16}(X) + 2.42065\cdot10^{-104}B_{11,16}(X) + 2.64697\cdot10^{-104}B_{12,16}(X) + 2.8733\cdot10^{-104}B_{13,16}(X) + 3.09963\cdot10^{-104}B_{14,16}(X) + 3.32595\cdot10^{-104}B_{15,16}(X) + 3.55228\cdot10^{-104}B_{16,16}(X)$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $3.23835\cdot10^{-114}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

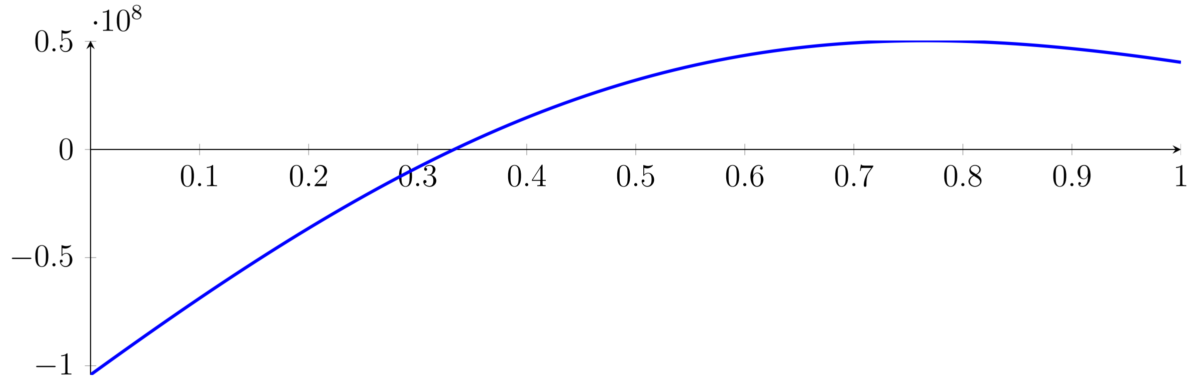
250.9 Recursion Branch 1 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 9!

250.10 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

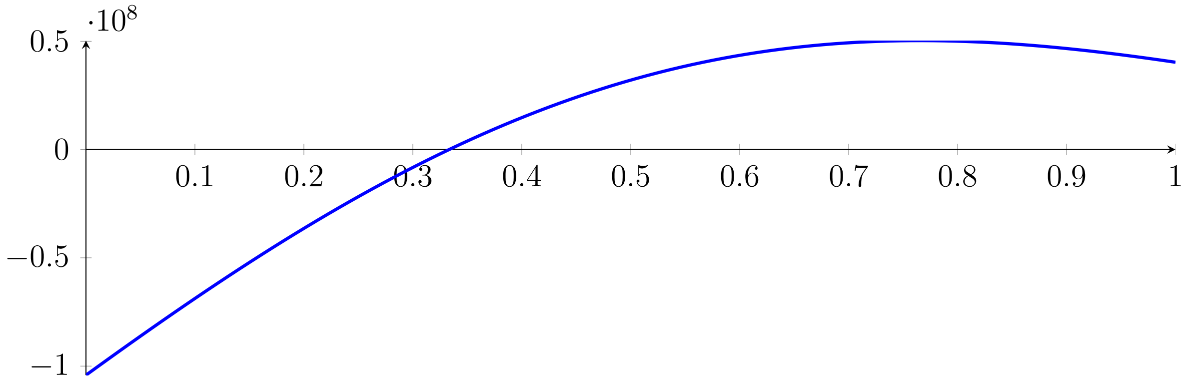
with precision $\varepsilon = 1\cdot10^{-128}$.

251 Running `QuadClip` on f_{16} with epsilon 128

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called `QuadClip` with input polynomial on interval $[0,1]$:

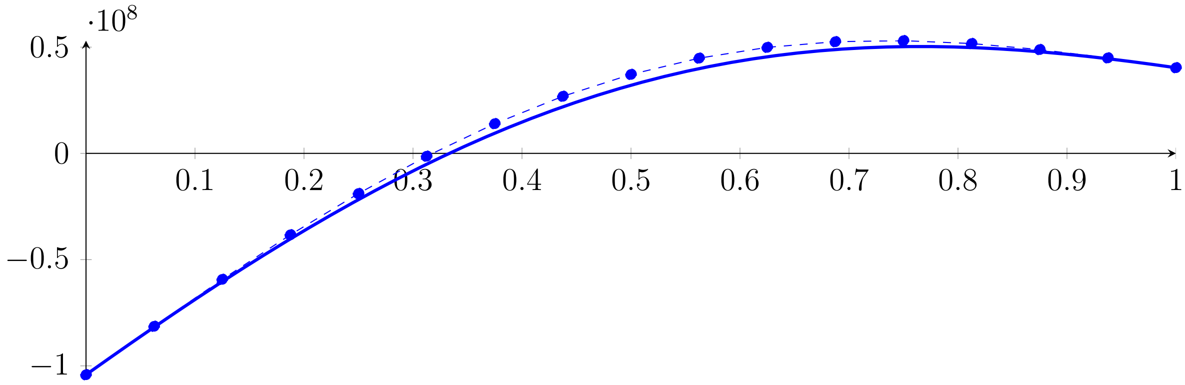
$$\begin{aligned}p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10}\\ &+ 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5}\\ &+ 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}\end{aligned}$$



251.1 Recursion Branch 1 for Input Interval $[0,1]$

Normalized monomial und Bézier representations and the Bézier polygon:

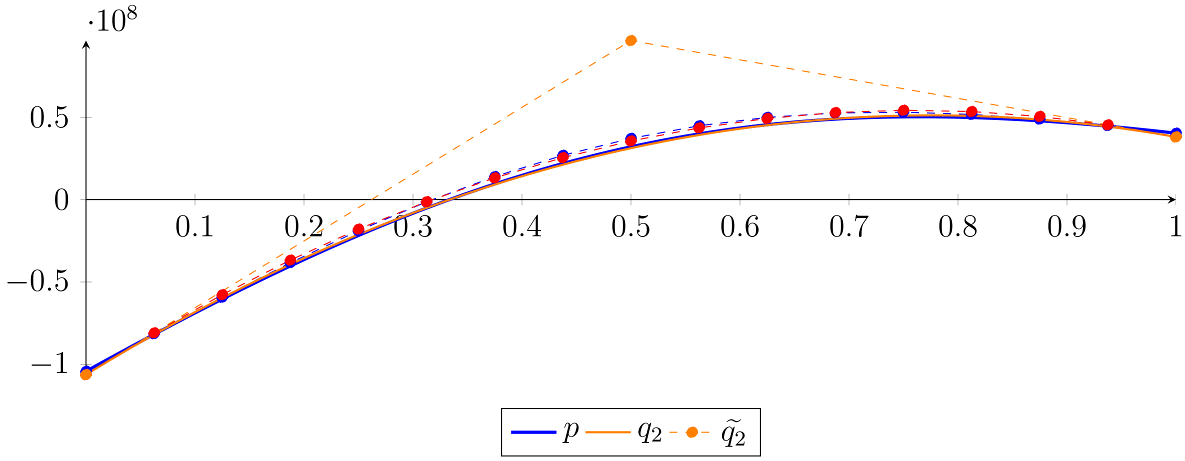
$$\begin{aligned}p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10}\\ &+ 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5}\\ &+ 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}\\ = &-1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203\\ &\cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X)\\ &+ 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901\\ &\cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X)\\ &+ 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X)\end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$
$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\widetilde{q_2} = 3.74473 \cdot 10^{-288} X^{16} - 3.08889 \cdot 10^{-287} X^{15} + 1.17443 \cdot 10^{-286} X^{14} - 2.71603 \cdot 10^{-286} X^{13} + 4.23554 \cdot 10^{-286} X^{12} - 4.66107 \cdot 10^{-286} X^{11} + 3.6845 \cdot 10^{-286} X^{10} - 2.09811 \cdot 10^{-286} X^9 + 8.58708 \cdot 10^{-287} X^8 - 2.54063 \cdot 10^{-287} X^7 + 5.57852 \cdot 10^{-288} X^6 - 9.21534 \cdot 10^{-289} X^5 + 1.02122 \cdot 10^{-289} X^4 - 5.95624 \cdot 10^{-291} X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$
$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017 \cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16} + 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16} + 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

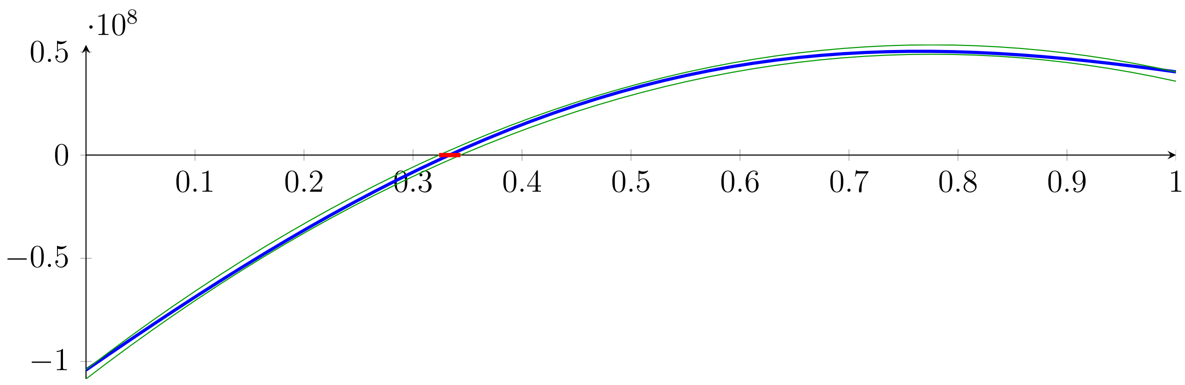
Bounding polynomials M and m:

$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$
$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

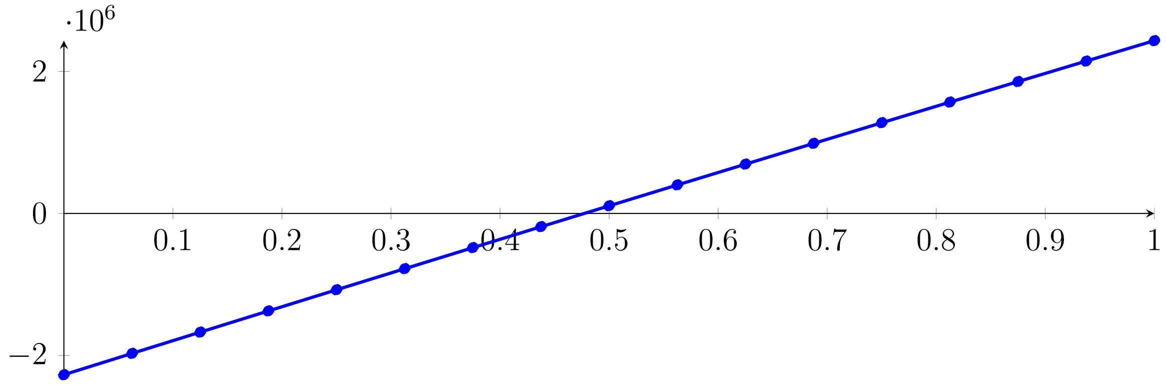
Longest intersection interval: 0.0196686

$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

251.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

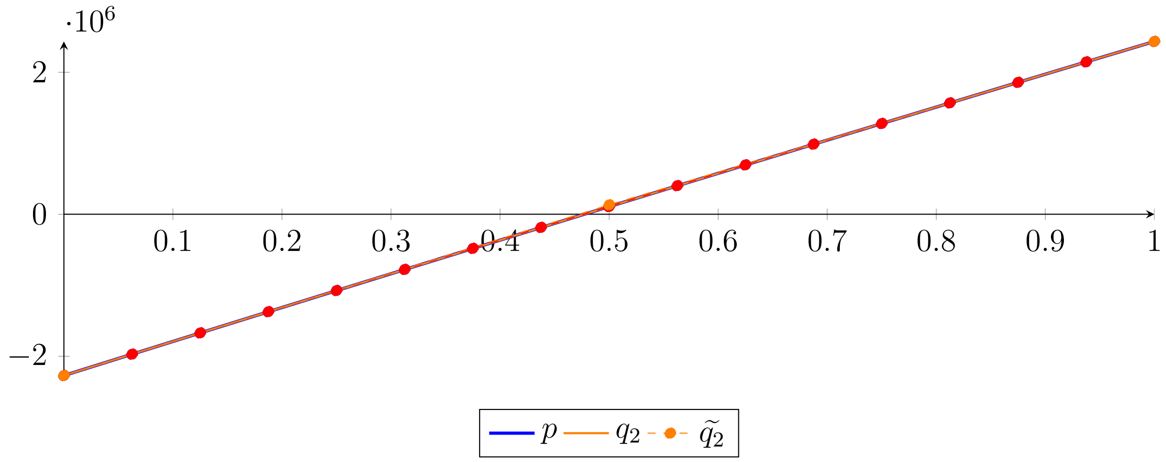
$$p = -5.0162{\cdot}10^{-28}X^{16} - 1.14383{\cdot}10^{-24}X^{15} - 1.11125{\cdot}10^{-21}X^{14} - 5.84096{\cdot}10^{-19}X^{13} - 1.684{\cdot}10^{-16}X^{12} - 1.94707{\cdot}10^{-14}X^{11} + 2.73227{\cdot}10^{-12}X^{10} + 1.09727{\cdot}10^{-09}X^{9} + 6.37314{\cdot}10^{-08}X^{8} - 1.68645{\cdot}10^{-05}X^{7} - 0.00205892X^{6} + 0.132721X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874{\cdot}10^{6}X - 2.2715{\cdot}10^{6}$$
$$= -2.2715{\cdot}10^{6}B_{0,16}(X) - 1.97096{\cdot}10^{6}B_{1,16}(X) - 1.67127{\cdot}10^{6}B_{2,16}(X) - 1.37244{\cdot}10^{6}B_{3,16}(X) - 1.07448{\cdot}10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806{\cdot}10^{6}B_{12,16}(X) + 1.5682{\cdot}10^{6}B_{13,16}(X) + 1.85747{\cdot}10^{6}B_{14,16}(X) + 2.14586{\cdot}10^{6}B_{15,16}(X) + 2.43338{\cdot}10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265X^{2} + 4.80923{\cdot}10^{6}X - 2.27154{\cdot}10^{6}$$
$$= -2.27154{\cdot}10^{6}B_{0,2} + 133069B_{1,2} + 2.43342{\cdot}10^{6}B_{2,2}$$

$$\widetilde{q}_2 = -8.18461{\cdot}10^{-291}X^{16} + 4.13745{\cdot}10^{-290}X^{15} - 5.8254{\cdot}10^{-290}X^{14} - 6.80919{\cdot}10^{-290}X^{13} + 3.48059{\cdot}10^{-289}X^{12} - 5.47245{\cdot}10^{-289}X^{11} + 4.8261{\cdot}10^{-289}X^{10} - 2.67867{\cdot}10^{-289}X^{9} + 1.00314{\cdot}10^{-289}X^{8} - 2.84766{\cdot}10^{-290}X^{7} + 6.85072{\cdot}10^{-291}X^{6} - 1.12501{\cdot}10^{-291}X^{5} + 1.24731{\cdot}10^{-293}X^{4} + 2.25944{\cdot}10^{-293}X^{3} - 104265X^{2} + 4.80923{\cdot}10^{6}X - 2.27154{\cdot}10^{6}$$
$$= -2.27154{\cdot}10^{6}B_{0,16} - 1.97097{\cdot}10^{6}B_{1,16} - 1.67126{\cdot}10^{6}B_{2,16} - 1.37242{\cdot}10^{6}B_{3,16} - 1.07445{\cdot}10^{6}B_{4,16} - 777350B_{5,16} - 481118B_{6,16} - 185754B_{7,16} + 108740B_{8,16} + 402366B_{9,16} + 695123B_{10,16} + 987011B_{11,16} + 1.27803{\cdot}10^{6}B_{12,16} + 1.56818{\cdot}10^{6}B_{13,16} + 1.85746{\cdot}10^{6}B_{14,16} + 2.14587{\cdot}10^{6}B_{15,16} + 2.43342{\cdot}10^{6}B_{16,16}$$



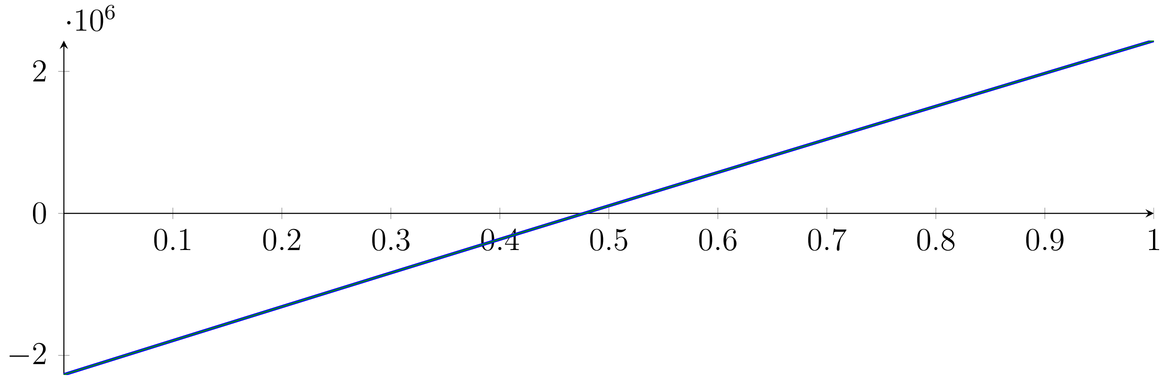
The maximum difference of the Bézier coefficients is $\delta = 40.5742$.
Bounding polynomials M and m:

$$M = -104265X^2 + 4.80923\cdot 10^6 X - 2.2715\cdot 10^6$$
$$m = -104265X^2 + 4.80923\cdot 10^6 X - 2.27159\cdot 10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



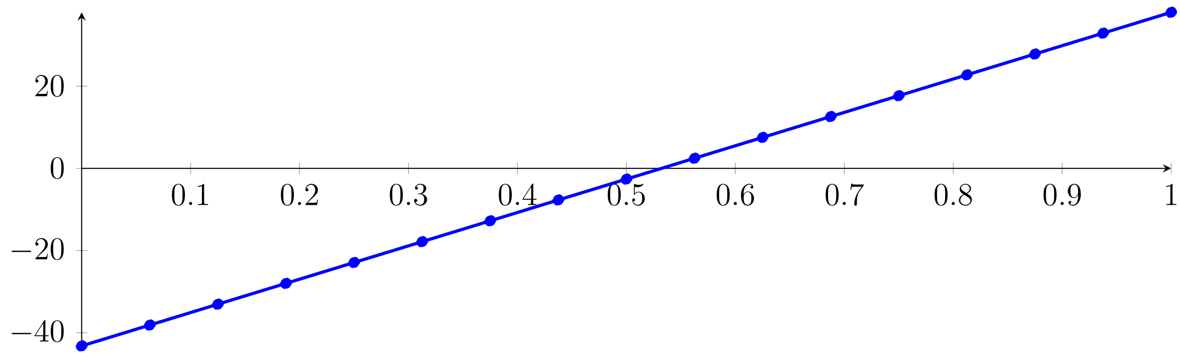
$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301\cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

251.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

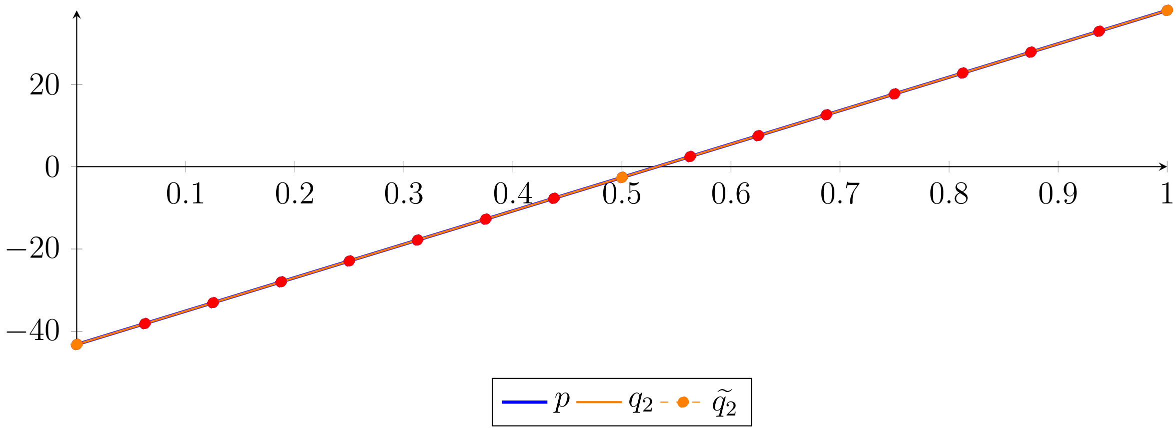
$$\begin{aligned} p = &-3.02667\cdot 10^{-104}X^{16} - 4.019\cdot 10^{-96}X^{15} - 2.27522\cdot 10^{-88}X^{14} - 6.97783\cdot 10^{-81}X^{13}\\ &- 1.17785\cdot 10^{-73}X^{12} - 8.12373\cdot 10^{-67}X^{11} + 6.05916\cdot 10^{-60}X^{10} + 1.48569\cdot 10^{-52}X^{9}\\ &+ 5.31875\cdot 10^{-46}X^{8} - 7.48919\cdot 10^{-39}X^{7} - 5.53349\cdot 10^{-32}X^{6} + 1.92471\cdot 10^{-25}X^{5}\\ &+ 2.00536\cdot 10^{-18}X^{4} - 4.12844\cdot 10^{-12}X^{3} - 3.09388\cdot 10^{-05}X^{2} + 81.1506X - 43.1911\\ = &-43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X)\\ &- 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X)\\ &+ 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X)\\ &+ 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09389\cdot 10^{-05}X^2 + 81.1506X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 = &-1.88281\cdot 10^{-295}X^{16} + 1.09893\cdot 10^{-294}X^{15} - 2.26419\cdot 10^{-294}X^{14} + 8.08223\cdot 10^{-295}X^{13} \\ &+ 4.92626\cdot 10^{-294}X^{12} - 1.07605\cdot 10^{-293}X^{11} + 1.11858\cdot 10^{-293}X^{10} - 6.87288\cdot 10^{-294}X^9 \\ &+ 2.54873\cdot 10^{-294}X^8 - 5.2305\cdot 10^{-295}X^7 + 3.18923\cdot 10^{-296}X^6 + 1.34092\cdot 10^{-296}X^5 \\ &- 4.89549\cdot 10^{-297}X^4 + 5.89947\cdot 10^{-298}X^3 - 3.09389\cdot 10^{-05}X^2 + 81.1506X - 43.1911 \\ = &-43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} \\ &- 12.7597B_{6,16} - 7.68778B_{7,16} - 2.61587B_{8,16} + 2.45604B_{9,16} + 7.52795B_{10,16} + 12.5999B_{11,16} \\ &+ 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.06422\cdot 10^{-13}$.
Bounding polynomials M and m:

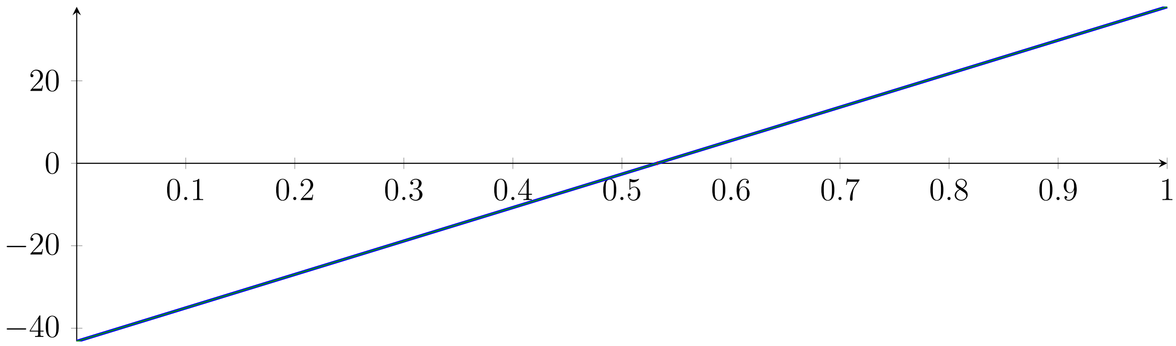
$$M = -3.09389\cdot 10^{-05}X^2 + 81.1506X - 43.1911$$

$$m = -3.09389\cdot 10^{-05}X^2 + 81.1506X - 43.1911$$

Root of M and m:

$$N(M) = \{0.532235, 2.62293\cdot 10^6\} \qquad N(m) = \{0.532235, 2.62293\cdot 10^6\}$$

Intersection intervals:



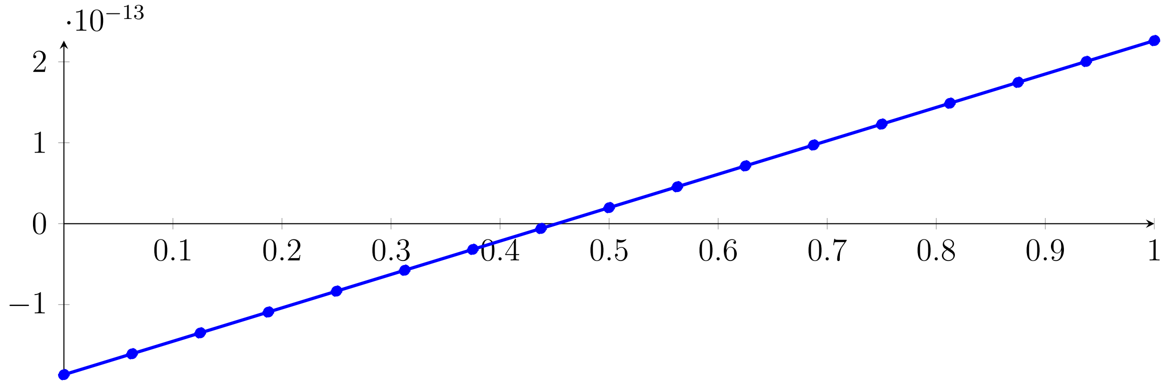
$$[0.532235, 0.532235]$$

Longest intersection interval: $5.08738\cdot 10^{-15}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

251.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

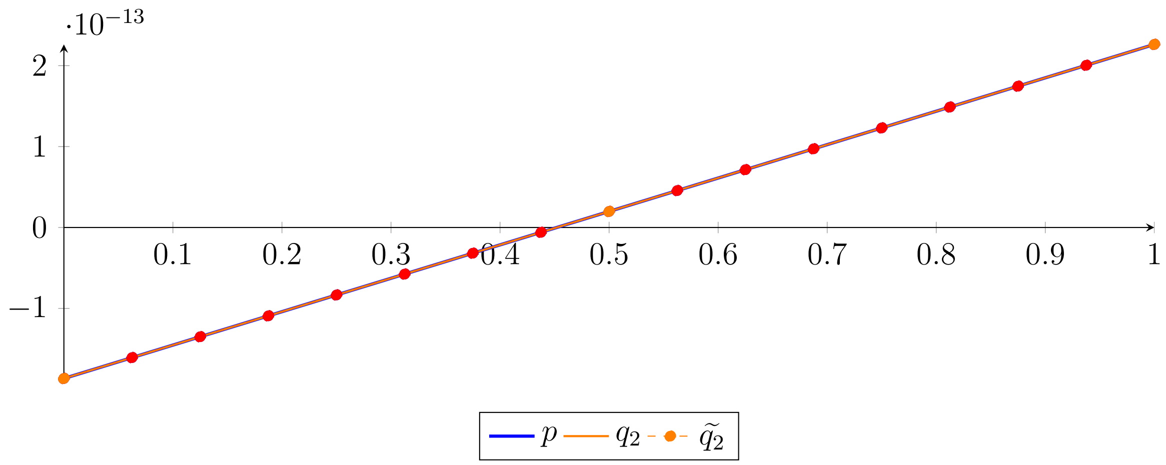
$$\begin{aligned} p = & -3.84502\cdot 10^{-319}X^{16} - 1.59047\cdot 10^{-310}X^{15} - 1.76985\cdot 10^{-288}X^{14} - 1.06694\cdot 10^{-266}X^{13} \\ & - 3.54011\cdot 10^{-245}X^{12} - 4.79942\cdot 10^{-224}X^{11} + 7.03641\cdot 10^{-203}X^{10} + 3.39135\cdot 10^{-181}X^{9} \\ & + 2.3865\cdot 10^{-160}X^{8} - 6.60529\cdot 10^{-139}X^{7} - 9.59319\cdot 10^{-118}X^{6} + 6.55895\cdot 10^{-97}X^{5} + 1.34328 \\ & \cdot 10^{-75}X^{4} - 5.43584\cdot 10^{-55}X^{3} - 8.00741\cdot 10^{-34}X^{2} + 4.12843\cdot 10^{-13}X - 1.86598\cdot 10^{-13} \\ = & -1.86598\cdot 10^{-13}B_{0,16}(X) - 1.60795\cdot 10^{-13}B_{1,16}(X) - 1.34993\cdot 10^{-13}B_{2,16}(X) - 1.0919 \\ & \cdot 10^{-13}B_{3,16}(X) - 8.33872\cdot 10^{-14}B_{4,16}(X) - 5.75845\cdot 10^{-14}B_{5,16}(X) - 3.17818\cdot 10^{-14}B_{6,16}(X) \\ & - 5.97912\cdot 10^{-15}B_{7,16}(X) + 1.98236\cdot 10^{-14}B_{8,16}(X) + 4.56263\cdot 10^{-14}B_{9,16}(X) + 7.1429 \\ & \cdot 10^{-14}B_{10,16}(X) + 9.72317\cdot 10^{-14}B_{11,16}(X) + 1.23034\cdot 10^{-13}B_{12,16}(X) + 1.48837 \\ & \cdot 10^{-13}B_{13,16}(X) + 1.7464\cdot 10^{-13}B_{14,16}(X) + 2.00443\cdot 10^{-13}B_{15,16}(X) + 2.26245\cdot 10^{-13}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 = & -8.00741\cdot 10^{-34}X^{2} + 4.12843\cdot 10^{-13}X - 1.86598\cdot 10^{-13} \\ = & -1.86598\cdot 10^{-13}B_{0,2} + 1.98236\cdot 10^{-14}B_{1,2} + 2.26245\cdot 10^{-13}B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 = & -8.08289\cdot 10^{-310}X^{16} + 4.33931\cdot 10^{-309}X^{15} - 8.19282\cdot 10^{-309}X^{14} + 3.30456\cdot 10^{-309}X^{13} \\ & + 1.10762\cdot 10^{-308}X^{12} - 2.01579\cdot 10^{-308}X^{11} + 1.55412\cdot 10^{-308}X^{10} - 6.55981\cdot 10^{-309}X^{9} \\ & + 2.12881\cdot 10^{-309}X^{8} - 1.08538\cdot 10^{-309}X^{7} + 5.4154\cdot 10^{-310}X^{6} - 1.38567\cdot 10^{-310}X^{5} + 9.27245 \\ & \cdot 10^{-312}X^{4} + 1.80627\cdot 10^{-312}X^{3} - 8.00741\cdot 10^{-34}X^{2} + 4.12843\cdot 10^{-13}X - 1.86598\cdot 10^{-13} \\ = & -1.86598\cdot 10^{-13}B_{0,16} - 1.60795\cdot 10^{-13}B_{1,16} - 1.34993\cdot 10^{-13}B_{2,16} - 1.0919\cdot 10^{-13}B_{3,16} - 8.33872 \\ & \cdot 10^{-14}B_{4,16} - 5.75845\cdot 10^{-14}B_{5,16} - 3.17818\cdot 10^{-14}B_{6,16} - 5.97912\cdot 10^{-15}B_{7,16} + 1.98236\cdot 10^{-14}B_{8,16} \\ & + 4.56263\cdot 10^{-14}B_{9,16} + 7.1429\cdot 10^{-14}B_{10,16} + 9.72317\cdot 10^{-14}B_{11,16} + 1.23034\cdot 10^{-13}B_{12,16} \\ & + 1.48837\cdot 10^{-13}B_{13,16} + 1.7464\cdot 10^{-13}B_{14,16} + 2.00443\cdot 10^{-13}B_{15,16} + 2.26245\cdot 10^{-13}B_{16,16} \end{aligned}$$



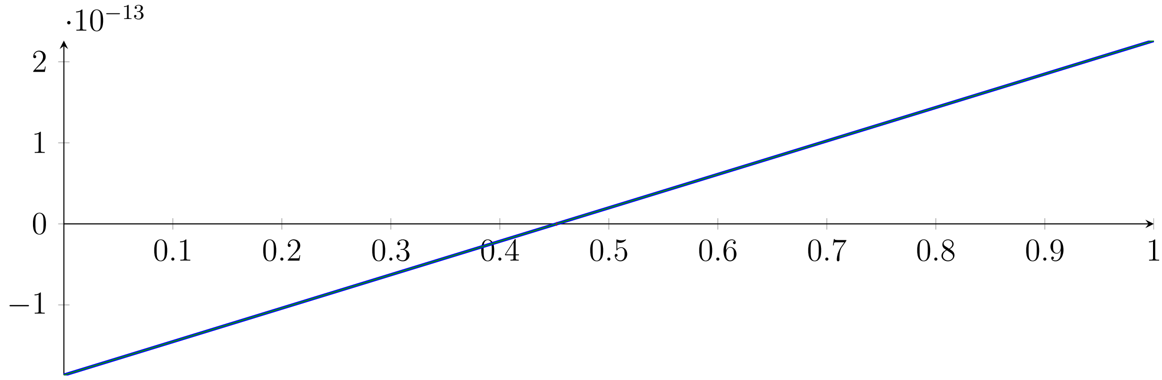
The maximum difference of the Bézier coefficients is $\delta = 2.71792\cdot 10^{-56}$.
Bounding polynomials M and m:

$$M = -8.00741\cdot 10^{-34}X^2 + 4.12843\cdot 10^{-13}X - 1.86598\cdot 10^{-13}$$
$$m = -8.00741\cdot 10^{-34}X^2 + 4.12843\cdot 10^{-13}X - 1.86598\cdot 10^{-13}$$

Root of M and m:

$$N(M) = \{0.451983, 5.15577\cdot 10^{20}\} \qquad N(m) = \{0.451983, 5.15577\cdot 10^{20}\}$$

Intersection intervals:



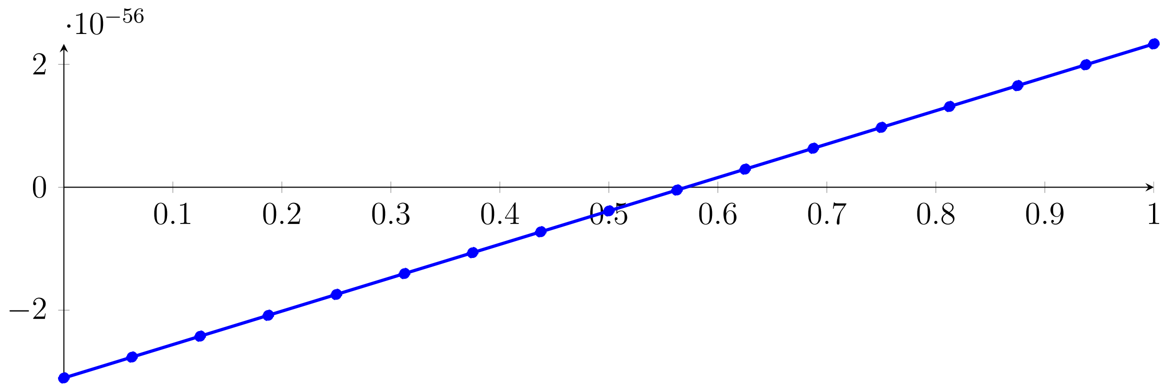
$$[0.451983, 0.451983]$$

Longest intersection interval: $1.31668\cdot 10^{-43}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

251.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

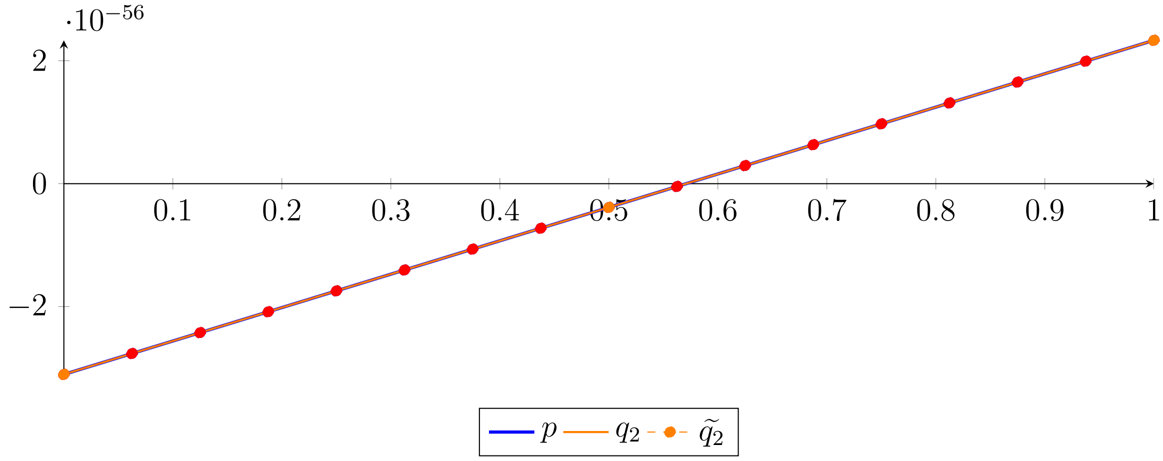
$$\begin{aligned} p = & 5.66252\cdot 10^{-361}X^{16} + 8.30503\cdot 10^{-360}X^{15} - 8.71157\cdot 10^{-361}X^{14} + 3.25232\cdot 10^{-359}X^{13} \\ & + 1.65157\cdot 10^{-358}X^{12} - 1.58551\cdot 10^{-358}X^{11} - 7.2669\cdot 10^{-359}X^{10} - 4.15252\cdot 10^{-359}X^{9} \\ & + 9.34316\cdot 10^{-359}X^{8} + 1.45338\cdot 10^{-359}X^{6} + 2.59562\cdot 10^{-311}X^{5} + 4.03733\cdot 10^{-247}X^{4} \\ & - 1.24083\cdot 10^{-183}X^{3} - 1.38821\cdot 10^{-119}X^{2} + 5.43584\cdot 10^{-56}X - 3.10342\cdot 10^{-56} \\ = & -3.10342\cdot 10^{-56}B_{0,16}(X) - 2.76368\cdot 10^{-56}B_{1,16}(X) - 2.42394\cdot 10^{-56}B_{2,16}(X) - 2.0842 \\ & \cdot 10^{-56}B_{3,16}(X) - 1.74446\cdot 10^{-56}B_{4,16}(X) - 1.40472\cdot 10^{-56}B_{5,16}(X) - 1.06498\cdot 10^{-56}B_{6,16}(X) \\ & - 7.25243\cdot 10^{-57}B_{7,16}(X) - 3.85503\cdot 10^{-57}B_{8,16}(X) - 4.57628\cdot 10^{-58}B_{9,16}(X) + 2.93977 \\ & \cdot 10^{-57}B_{10,16}(X) + 6.33717\cdot 10^{-57}B_{11,16}(X) + 9.73457\cdot 10^{-57}B_{12,16}(X) + 1.3132\cdot 10^{-56}B_{13,16}(X) \\ & + 1.65294\cdot 10^{-56}B_{14,16}(X) + 1.99268\cdot 10^{-56}B_{15,16}(X) + 2.33242\cdot 10^{-56}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.38821{\cdot}10^{-119}X^2 + 5.43584{\cdot}10^{-56}X - 3.10342{\cdot}10^{-56}$$
$$= -3.10342{\cdot}10^{-56}B_{0,2} - 3.85503{\cdot}10^{-57}B_{1,2} + 2.33242{\cdot}10^{-56}B_{2,2}$$

$$\widetilde{q}_2 = -1.35612{\cdot}10^{-352}X^{16} + 8.15544{\cdot}10^{-352}X^{15} - 1.72777{\cdot}10^{-351}X^{14} + 5.92647{\cdot}10^{-352}X^{13} + 4.18743{\cdot}10^{-351}X^{12} - 9.40291{\cdot}10^{-351}X^{11} + 1.01181{\cdot}10^{-350}X^{10} - 6.40657{\cdot}10^{-351}X^9 + 2.39508{\cdot}10^{-351}X^8 - 4.50359{\cdot}10^{-352}X^7 - 2.71062{\cdot}10^{-354}X^6 + 2.21064{\cdot}10^{-353}X^5 - 5.44842{\cdot}10^{-354}X^4 + 4.71019{\cdot}10^{-355}X^3 - 1.38821{\cdot}10^{-119}X^2 + 5.43584{\cdot}10^{-56}X - 3.10342{\cdot}10^{-56}$$
$$= -3.10342{\cdot}10^{-56}B_{0,16} - 2.76368{\cdot}10^{-56}B_{1,16} - 2.42394{\cdot}10^{-56}B_{2,16} - 2.0842{\cdot}10^{-56}B_{3,16} - 1.74446{\cdot}10^{-56}B_{4,16} - 1.40472{\cdot}10^{-56}B_{5,16} - 1.06498{\cdot}10^{-56}B_{6,16} - 7.25243{\cdot}10^{-57}B_{7,16} - 3.85503{\cdot}10^{-57}B_{8,16} - 4.57628{\cdot}10^{-58}B_{9,16} + 2.93977{\cdot}10^{-57}B_{10,16} + 6.33717{\cdot}10^{-57}B_{11,16} + 9.73457{\cdot}10^{-57}B_{12,16} + 1.3132{\cdot}10^{-56}B_{13,16} + 1.65294{\cdot}10^{-56}B_{14,16} + 1.99268{\cdot}10^{-56}B_{15,16} + 2.33242{\cdot}10^{-56}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 6.20413{\cdot}10^{-185}$.

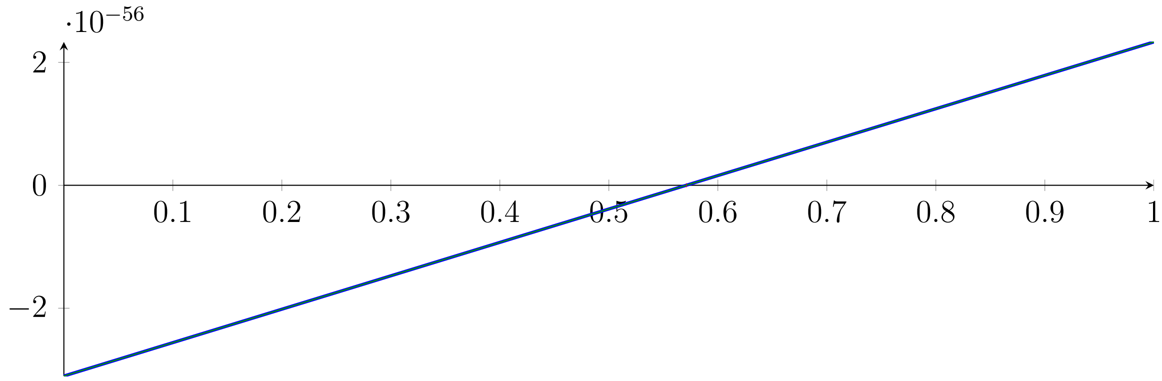
Bounding polynomials M and m:

$$M = -1.38821{\cdot}10^{-119}X^2 + 5.43584{\cdot}10^{-56}X - 3.10342{\cdot}10^{-56}$$
$$m = -1.38821{\cdot}10^{-119}X^2 + 5.43584{\cdot}10^{-56}X - 3.10342{\cdot}10^{-56}$$

Root of M and m:

$$N(M) = \{0.570919, 3.91572{\cdot}10^{63}\} \qquad N(m) = \{0.570919, 3.91572{\cdot}10^{63}\}$$

Intersection intervals:



$$[0.570919, 0.570919]$$

Longest intersection interval: $2.28268{\cdot}10^{-129}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

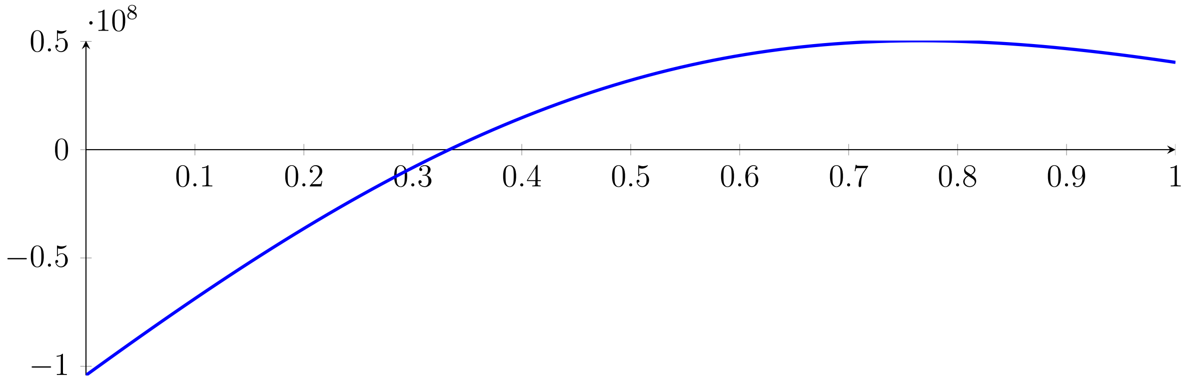
251.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

251.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

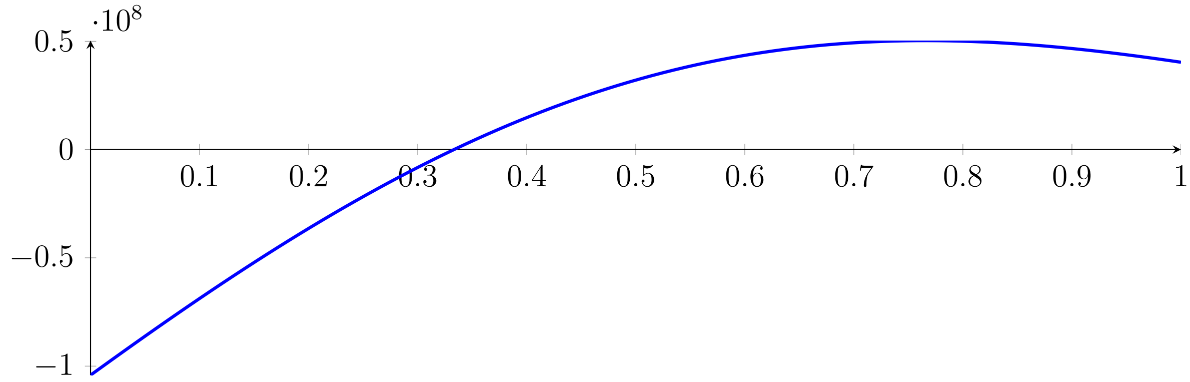
with precision $\varepsilon = 1\cdot10^{-128}$.

252 Running CubeClip on f_{16} with epsilon 128

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot 10^6 X^9 - 2.52262\cdot 10^6 X^8 - 1.43506\cdot 10^7 X^7 + 138542X^6 + 7.92823\cdot 10^7 X^5 + 3.97396\cdot 10^7 X^4 - 2.40625\cdot 10^8 X^3 - 8.33333\cdot 10^7 X^2 + 3.64583\cdot 10^8 X - 1.04167\cdot 10^8$$

Called CubeClip with input polynomial on interval $[0, 1]$:

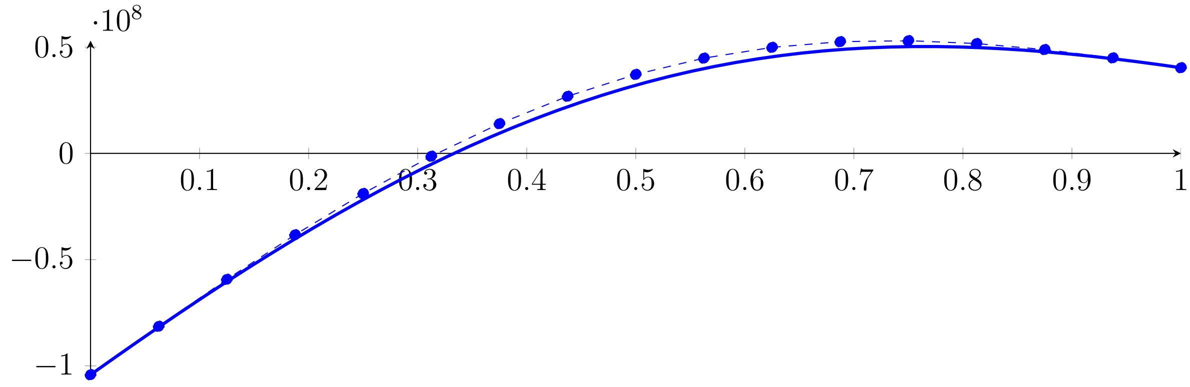
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172\cdot 10^6 X^9 - 2.52262\cdot 10^6 X^8 - 1.43506\cdot 10^7 X^7 + 138542X^6 + 7.92823\cdot 10^7 X^5 \\ &+ 3.97396\cdot 10^7 X^4 - 2.40625\cdot 10^8 X^3 - 8.33333\cdot 10^7 X^2 + 3.64583\cdot 10^8 X - 1.04167\cdot 10^8 \end{aligned}$$



252.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

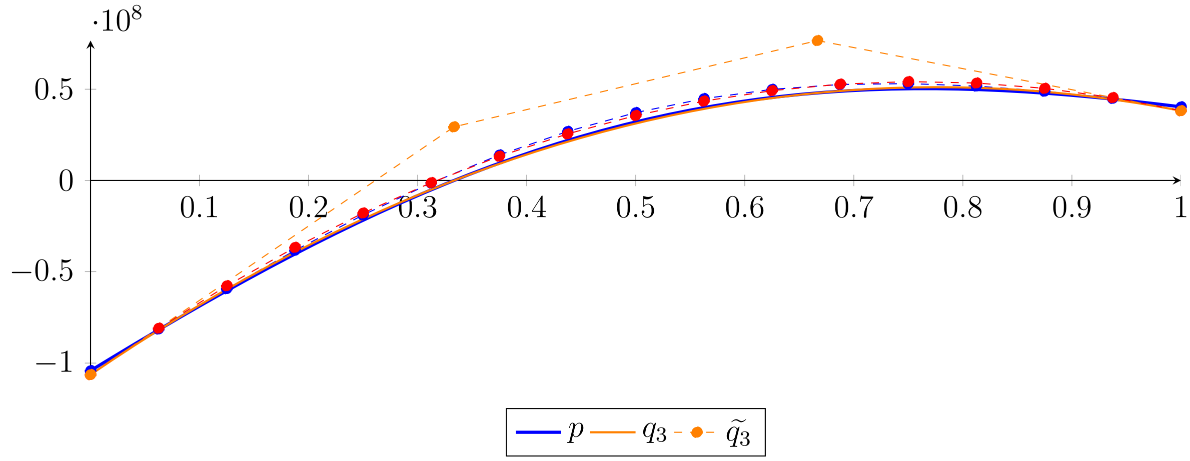
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172\cdot 10^6 X^9 - 2.52262\cdot 10^6 X^8 - 1.43506\cdot 10^7 X^7 + 138542X^6 + 7.92823\cdot 10^7 X^5 \\ &+ 3.97396\cdot 10^7 X^4 - 2.40625\cdot 10^8 X^3 - 8.33333\cdot 10^7 X^2 + 3.64583\cdot 10^8 X - 1.04167\cdot 10^8 \\ = &-1.04167\cdot 10^8 B_{0,16}(X) - 8.13802\cdot 10^7 B_{1,16}(X) - 5.92882\cdot 10^7 B_{2,16}(X) - 3.83203 \\ &\cdot 10^7 B_{3,16}(X) - 1.88844\cdot 10^7 B_{4,16}(X) - 1.34837\cdot 10^6 B_{5,16}(X) + 1.39781\cdot 10^7 B_{6,16}(X) \\ &+ 2.68604\cdot 10^7 B_{7,16}(X) + 3.71532\cdot 10^7 B_{8,16}(X) + 4.48105\cdot 10^7 B_{9,16}(X) + 4.98901 \\ &\cdot 10^7 B_{10,16}(X) + 5.25504\cdot 10^7 B_{11,16}(X) + 5.30407\cdot 10^7 B_{12,16}(X) + 5.16832\cdot 10^7 B_{13,16}(X) \\ &+ 4.88488\cdot 10^7 B_{14,16}(X) + 4.49297\cdot 10^7 B_{15,16}(X) + 4.03108\cdot 10^7 B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.06335{\cdot}10^8$$
$$= -1.06335{\cdot}10^8 B_{0,3} + 2.93558{\cdot}10^7 B_{1,3} + 7.66068{\cdot}10^7 B_{2,3} + 3.81764{\cdot}10^7 B_{3,3}$$

$$\widetilde{q}_3 = 1.89955{\cdot}10^{-288} X^{16} - 1.4851{\cdot}10^{-287} X^{15} + 5.12896{\cdot}10^{-287} X^{14} - 1.02224{\cdot}10^{-286} X^{13} + 1.29027{\cdot}10^{-286} X^{12} - 1.07058{\cdot}10^{-286} X^{11} + 6.00739{\cdot}10^{-287} X^{10} - 2.54352{\cdot}10^{-287} X^9 + 1.12089{\cdot}10^{-287} X^8 - 5.8673{\cdot}10^{-288} X^7 + 2.52637{\cdot}10^{-288} X^6 - 6.79624{\cdot}10^{-289} X^5 + 9.36341{\cdot}10^{-290} X^4 + 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.06335{\cdot}10^8$$
$$= -1.06335{\cdot}10^8 B_{0,16} - 8.08928{\cdot}10^7 B_{1,16} - 5.76618{\cdot}10^7 B_{2,16} - 3.66369{\cdot}10^7 B_{3,16} - 1.78131{\cdot}10^7 B_{4,16} - 1.18551{\cdot}10^6 B_{5,16} + 1.32508{\cdot}10^7 B_{6,16} + 2.55007{\cdot}10^7 B_{7,16} + 3.55692{\cdot}10^7 B_{8,16} + 4.34611{\cdot}10^7 B_{9,16} + 4.91815{\cdot}10^7 B_{10,16} + 5.27353{\cdot}10^7 B_{11,16} + 5.41273{\cdot}10^7 B_{12,16} + 5.33624{\cdot}10^7 B_{13,16} + 5.04457{\cdot}10^7 B_{14,16} + 4.53821{\cdot}10^7 B_{15,16} + 3.81764{\cdot}10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806{\cdot}10^6$.

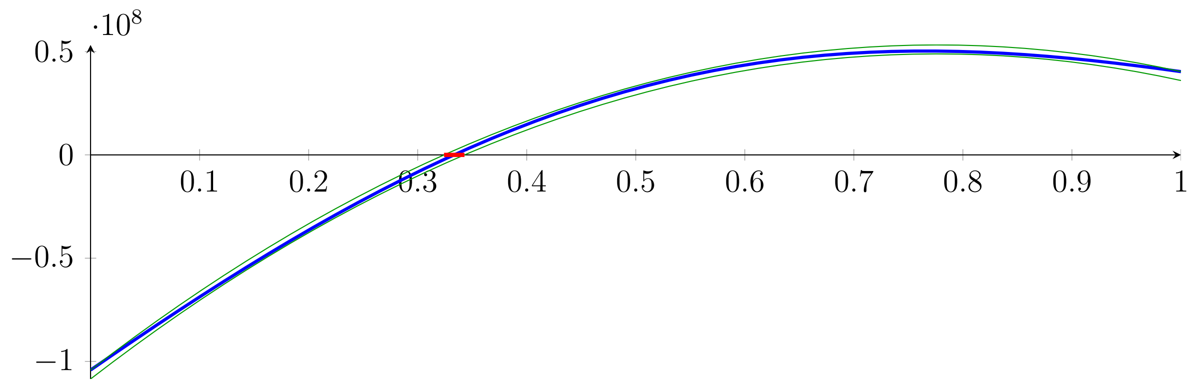
Bounding polynomials M and m:

$$M = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.04167{\cdot}10^8$$
$$m = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.08503{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

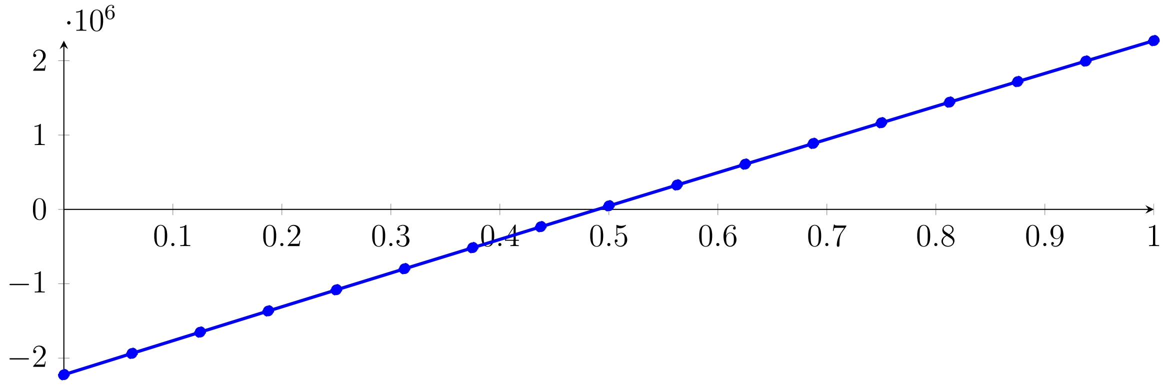
Longest intersection interval: 0.0187703

$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

252.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

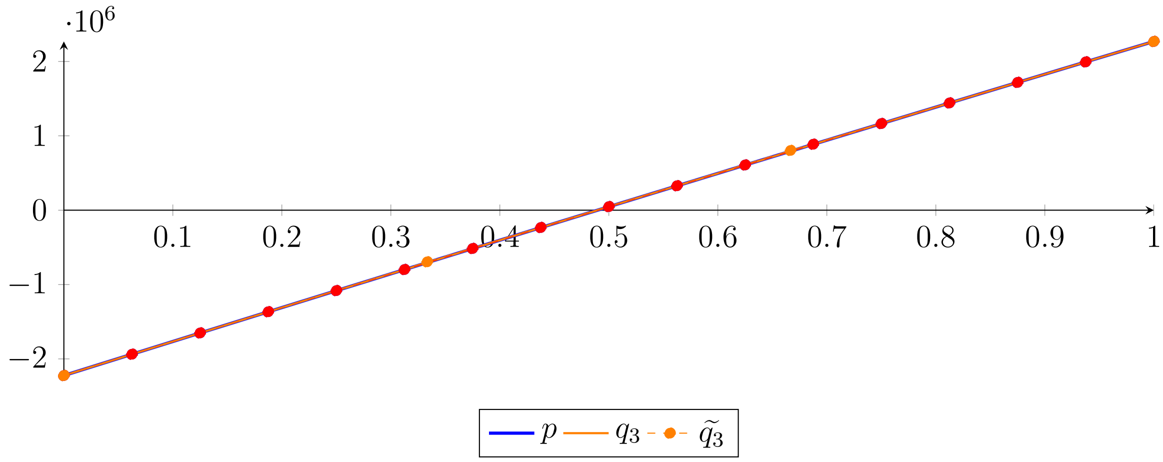
$$p = -2.37433\cdot10^{-28}X^{16} - 5.67363\cdot10^{-25}X^{15} - 5.77631\cdot10^{-22}X^{14} - 3.1818\cdot10^{-19}X^{13} - 9.6142\cdot10^{-17}X^{12} - 1.16549\cdot10^{-14}X^{11} + 1.71065\cdot10^{-12}X^{10} + 7.20611\cdot10^{-10}X^{9} + 4.39147\cdot10^{-08}X^{8} - 1.21542\cdot10^{-05}X^{7} - 0.00155624X^{6} + 0.10496X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,16}(X) - 1.93666\cdot10^{6}B_{1,16}(X) - 1.65074\cdot10^{6}B_{2,16}(X) - 1.36561\cdot10^{6}B_{3,16}(X) - 1.08127\cdot10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X) + 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot10^{6}B_{12,16}(X) + 1.44233\cdot10^{6}B_{13,16}(X) + 1.71876\cdot10^{6}B_{14,16}(X) + 1.99439\cdot10^{6}B_{15,16}(X) + 2.26922\cdot10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot10^{6}B_{3,3}$$

$$\widetilde{q}_3 = 4.41595\cdot10^{-291}X^{16} - 1.48598\cdot10^{-290}X^{15} - 2.81102\cdot10^{-290}X^{14} + 2.65129\cdot10^{-289}X^{13} - 7.34593\cdot10^{-289}X^{12} + 1.17654\cdot10^{-288}X^{11} - 1.23119\cdot10^{-288}X^{10} + 8.63493\cdot10^{-289}X^{9} - 3.92909\cdot10^{-289}X^{8} + 1.01265\cdot10^{-289}X^{7} - 6.13644\cdot10^{-291}X^{6} - 3.92664\cdot10^{-291}X^{5} + 9.04488\cdot10^{-292}X^{4} - 700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$
$$= -2.22335\cdot10^{6}B_{0,16} - 1.93666\cdot10^{6}B_{1,16} - 1.65074\cdot10^{6}B_{2,16} - 1.36561\cdot10^{6}B_{3,16} - 1.08126\cdot10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.4B_{8,16} + 328650B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot10^{6}B_{12,16} + 1.44233\cdot10^{6}B_{13,16} + 1.71876\cdot10^{6}B_{14,16} + 1.99439\cdot10^{6}B_{15,16} + 2.26922\cdot10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.
Bounding polynomials M and m:

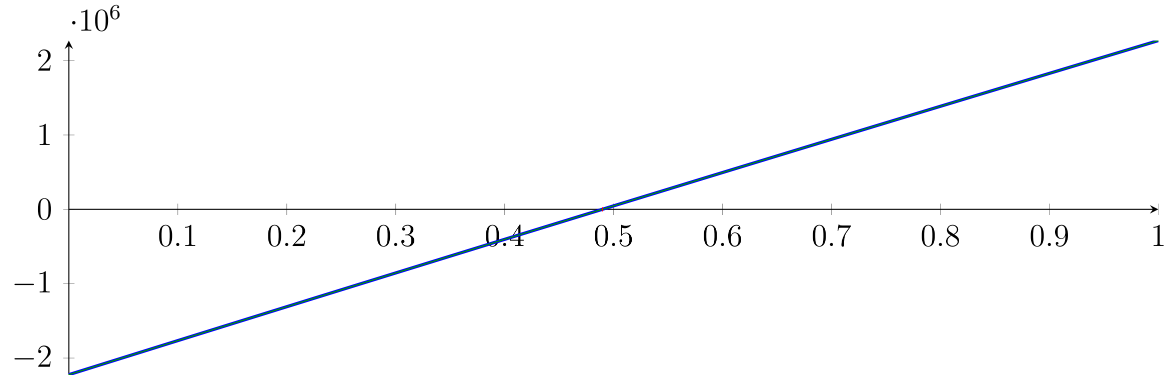
$$M = -700.673X^3 - 93879.9X^2 + 4.58715\cdot 10^6 X - 2.22335\cdot 10^6$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715\cdot 10^6 X - 2.22335\cdot 10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



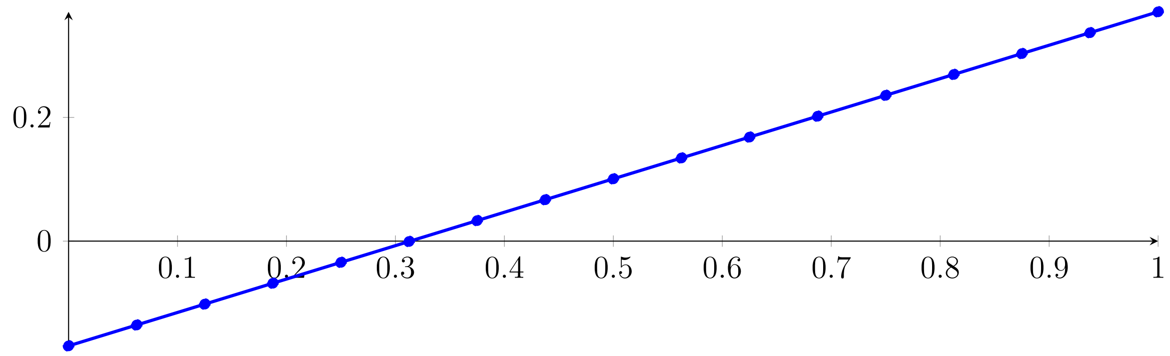
$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174\cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

252.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

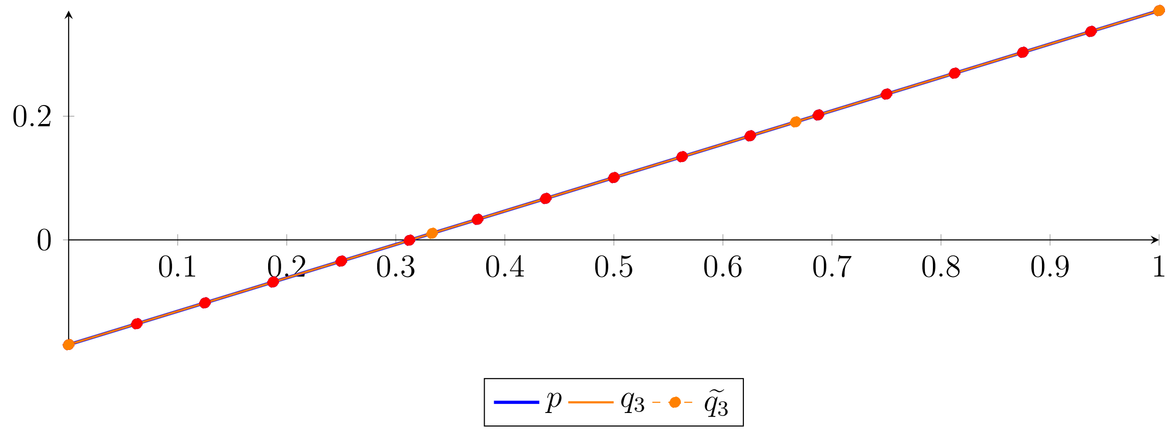
$$\begin{aligned} p = &-4.49274\cdot 10^{-139}X^{16} - 8.96277\cdot 10^{-129}X^{15} - 7.623\cdot 10^{-119}X^{14} - 3.51238\cdot 10^{-109}X^{13} \\ &- 8.90739\cdot 10^{-100}X^{12} - 9.22984\cdot 10^{-91}X^{11} + 1.03426\cdot 10^{-81}X^{10} + 3.80998\cdot 10^{-72}X^9 \\ &+ 2.04919\cdot 10^{-63}X^8 - 4.33497\cdot 10^{-54}X^7 - 4.81204\cdot 10^{-45}X^6 + 2.51462\cdot 10^{-36}X^5 \\ &+ 3.93622\cdot 10^{-27}X^4 - 1.21745\cdot 10^{-18}X^3 - 1.37072\cdot 10^{-09}X^2 + 0.540148X - 0.169396 \\ = &-0.169396B_{0,16}(X) - 0.135637B_{1,16}(X) - 0.101877B_{2,16}(X) - 0.068118B_{3,16}(X) \\ &- 0.0343588B_{4,16}(X) - 0.000599488B_{5,16}(X) + 0.0331598B_{6,16}(X) \\ &+ 0.066919B_{7,16}(X) + 0.100678B_{8,16}(X) + 0.134438B_{9,16}(X) + 0.168197B_{10,16}(X) \\ &+ 0.201956B_{11,16}(X) + 0.235715B_{12,16}(X) + 0.269475B_{13,16}(X) \\ &+ 0.303234B_{14,16}(X) + 0.336993B_{15,16}(X) + 0.370752B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.21745\cdot 10^{-18}X^3 - 1.37072\cdot 10^{-09}X^2 + 0.540148X - 0.169396 \\ &= -0.169396B_{0,3} + 0.0106536B_{1,3} + 0.190703B_{2,3} + 0.370752B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q_3} &= 8.03185\cdot 10^{-297}X^{16} - 6.20841\cdot 10^{-296}X^{15} + 2.13274\cdot 10^{-295}X^{14} - 4.26614\cdot 10^{-295}X^{13} \\ &\quad + 5.47461\cdot 10^{-295}X^{12} - 4.70265\cdot 10^{-295}X^{11} + 2.78551\cdot 10^{-295}X^{10} - 1.22442\cdot 10^{-295}X^{9} \\ &\quad + 4.88954\cdot 10^{-296}X^{8} - 2.11494\cdot 10^{-296}X^{7} + 8.20665\cdot 10^{-297}X^{6} - 2.15458\cdot 10^{-297}X^{5} \\ &\quad + 3.01517\cdot 10^{-298}X^{4} - 1.21745\cdot 10^{-18}X^{3} - 1.37072\cdot 10^{-09}X^{2} + 0.540148X - 0.169396 \\ &= -0.169396B_{0,16} - 0.135637B_{1,16} - 0.101877B_{2,16} - 0.068118B_{3,16} - 0.0343588B_{4,16} \\ &\quad - 0.000599488B_{5,16} + 0.0331598B_{6,16} + 0.066919B_{7,16} + 0.100678B_{8,16} \\ &\quad + 0.134438B_{9,16} + 0.168197B_{10,16} + 0.201956B_{11,16} + 0.235715B_{12,16} \\ &\quad + 0.269475B_{13,16} + 0.303234B_{14,16} + 0.336993B_{15,16} + 0.370752B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.62317\cdot 10^{-29}$.
Bounding polynomials M and m:

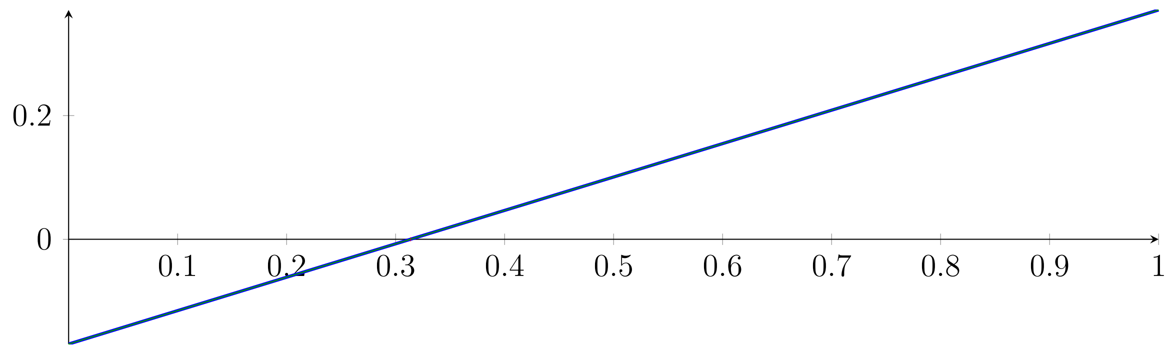
$$M = -1.21745\cdot 10^{-18}X^3 - 1.37072\cdot 10^{-09}X^2 + 0.540148X - 0.169396$$

$$m = -1.21745\cdot 10^{-18}X^3 - 1.37072\cdot 10^{-09}X^2 + 0.540148X - 0.169396$$

Root of M and m:

$$N(M) = \{-1.43506\cdot 10^{9}, 0.31361, 3.09167\cdot 10^{8}\} \quad N(m) = \{-1.43506\cdot 10^{9}, 0.31361, 3.09167\cdot 10^{8}\}$$

Intersection intervals:



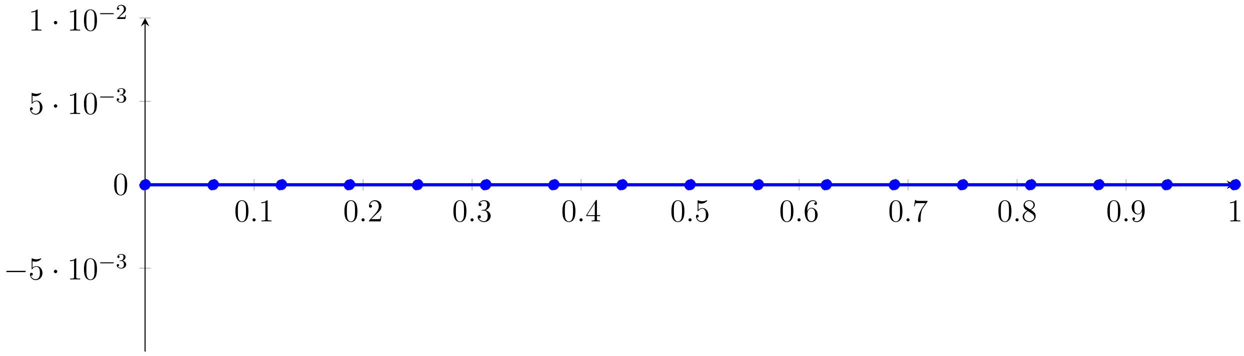
$$[0.31361, 0.31361]$$

Longest intersection interval: $2.08208\cdot 10^{-28}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

252.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

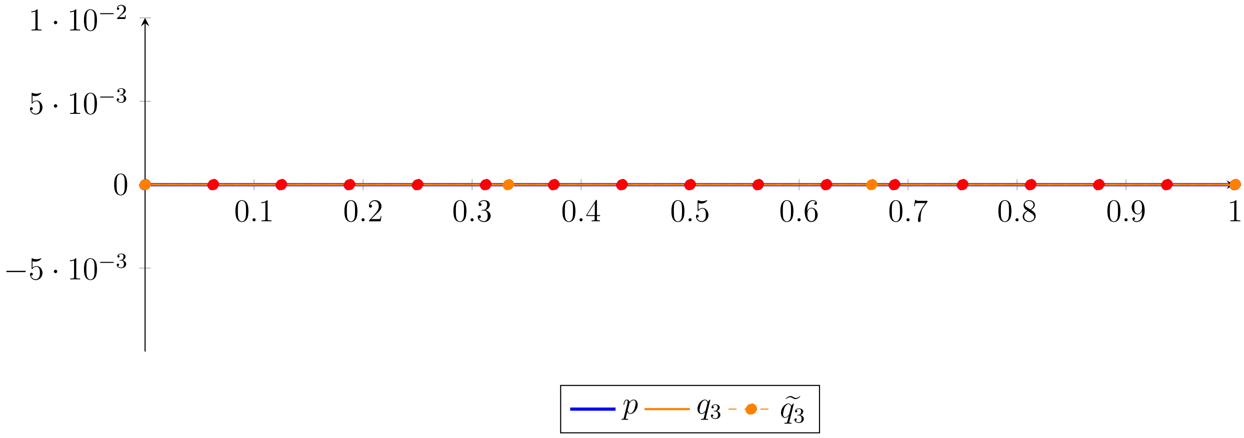
$$p = 3.09811{\cdot}10^{-312}X^{16} + 1.13739{\cdot}10^{-311}X^{15} + 1.09495{\cdot}10^{-310}X^{14} - 3.56495{\cdot}10^{-311}X^{13} - 1.48688{\cdot}10^{-309}X^{12} + 1.48302{\cdot}10^{-309}X^{11} + 7.222{\cdot}10^{-310}X^{10} + 1.21378{\cdot}10^{-310}X^{9} + 7.23716{\cdot}10^{-285}X^{8} - 7.35315{\cdot}10^{-248}X^{7} - 3.92029{\cdot}10^{-211}X^{6} + 9.83929{\cdot}10^{-175}X^{5} + 7.39728{\cdot}10^{-138}X^{4} - 1.09887{\cdot}10^{-101}X^{3} - 5.94215{\cdot}10^{-65}X^{2} + 1.12463{\cdot}10^{-28}X - 8.88188{\cdot}10^{-08}$$
$$= -8.88188{\cdot}10^{-08}B_{0,16}(X) - 8.88188{\cdot}10^{-08}B_{1,16}(X) - 8.88188{\cdot}10^{-08}B_{2,16}(X) - 8.88188{\cdot}10^{-08}B_{3,16}(X) - 8.88188{\cdot}10^{-08}B_{4,16}(X) - 8.88188{\cdot}10^{-08}B_{5,16}(X) - 8.88188{\cdot}10^{-08}B_{6,16}(X) - 8.88188{\cdot}10^{-08}B_{7,16}(X) - 8.88188{\cdot}10^{-08}B_{8,16}(X) - 8.88188{\cdot}10^{-08}B_{9,16}(X) - 8.88188{\cdot}10^{-08}B_{10,16}(X) - 8.88188{\cdot}10^{-08}B_{11,16}(X) - 8.88188{\cdot}10^{-08}B_{12,16}(X) - 8.88188{\cdot}10^{-08}B_{13,16}(X) - 8.88188{\cdot}10^{-08}B_{14,16}(X) - 8.88188{\cdot}10^{-08}B_{15,16}(X) - 8.88188{\cdot}10^{-08}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -1.09887{\cdot}10^{-101}X^{3} - 5.94215{\cdot}10^{-65}X^{2} + 1.12463{\cdot}10^{-28}X - 8.88188{\cdot}10^{-08}$$
$$= -8.88188{\cdot}10^{-08}B_{0,3} - 8.88188{\cdot}10^{-08}B_{1,3} - 8.88188{\cdot}10^{-08}B_{2,3} - 8.88188{\cdot}10^{-08}B_{3,3}$$

$$\widetilde{q}_3 = -6.98397{\cdot}10^{-303}X^{16} + 5.61515{\cdot}10^{-302}X^{15} - 2.01778{\cdot}10^{-301}X^{14} + 4.27019{\cdot}10^{-301}X^{13} - 5.92096{\cdot}10^{-301}X^{12} + 5.69601{\cdot}10^{-301}X^{11} - 3.9714{\cdot}10^{-301}X^{10} + 2.10656{\cdot}10^{-301}X^{9} - 8.95545{\cdot}10^{-302}X^{8} + 3.10786{\cdot}10^{-302}X^{7} - 8.35303{\cdot}10^{-303}X^{6} + 1.57296{\cdot}10^{-303}X^{5} - 1.80277{\cdot}10^{-304}X^{4} - 1.09887{\cdot}10^{-101}X^{3} - 5.94215{\cdot}10^{-65}X^{2} + 1.12463{\cdot}10^{-28}X - 8.88188{\cdot}10^{-08}$$
$$= -8.88188{\cdot}10^{-08}B_{0,16} - 8.88188{\cdot}10^{-08}B_{1,16} - 8.88188{\cdot}10^{-08}B_{2,16} - 8.88188{\cdot}10^{-08}B_{3,16} - 8.88188{\cdot}10^{-08}B_{4,16} - 8.88188{\cdot}10^{-08}B_{5,16} - 8.88188{\cdot}10^{-08}B_{6,16} - 8.88188{\cdot}10^{-08}B_{7,16} - 8.88188{\cdot}10^{-08}B_{8,16} - 8.88188{\cdot}10^{-08}B_{9,16} - 8.88188{\cdot}10^{-08}B_{10,16} - 8.88188{\cdot}10^{-08}B_{11,16} - 8.88188{\cdot}10^{-08}B_{12,16} - 8.88188{\cdot}10^{-08}B_{13,16} - 8.88188{\cdot}10^{-08}B_{14,16} - 8.88188{\cdot}10^{-08}B_{15,16} - 8.88188{\cdot}10^{-08}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 1.05675 \cdot 10^{-139}$.
Bounding polynomials M and m:

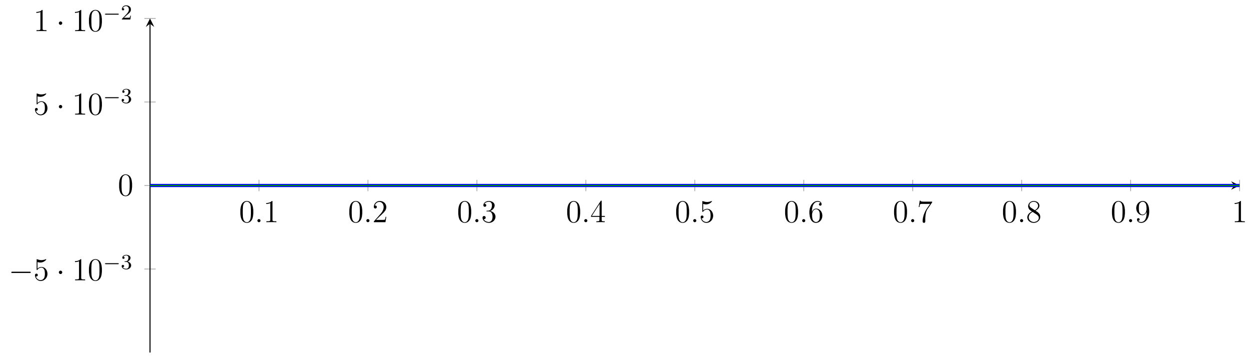
$$M = -1.09887 \cdot 10^{-101} X^3 - 5.94215 \cdot 10^{-65} X^2 + 1.12463 \cdot 10^{-28} X - 8.88188 \cdot 10^{-08}$$

$$m = -1.09887 \cdot 10^{-101} X^3 - 5.94215 \cdot 10^{-65} X^2 + 1.12463 \cdot 10^{-28} X - 8.88188 \cdot 10^{-08}$$

Root of M and m:

$N(M) = \{-6.89243 \cdot 10^{36}, 1.34848 \cdot 10^{12}, 1.48489 \cdot 10^{36}\}$ $N(m) = \{-6.89243 \cdot 10^{36}, 1.34848 \cdot 10^{12}, 1.48489 \cdot 10^{36}\}$

Intersection intervals:

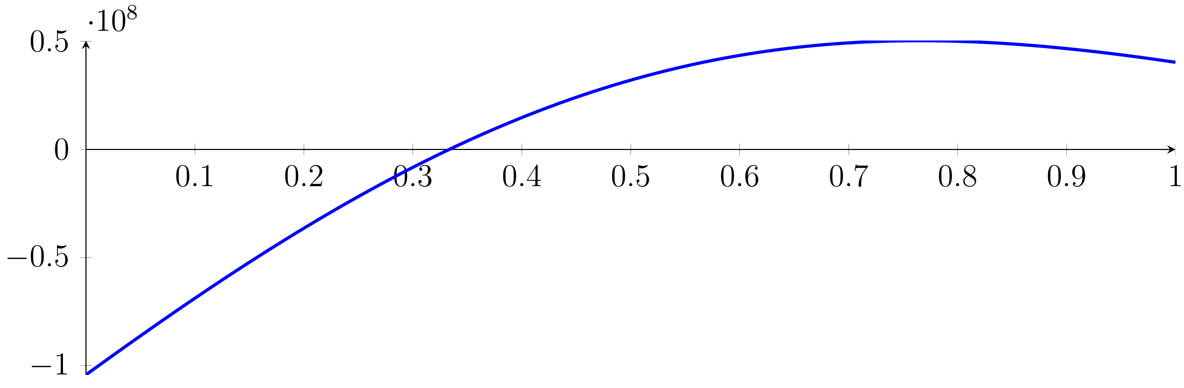


No intersection intervals with the x axis.

252.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

with precision $\varepsilon = 1\cdot10^{-128}$.