

Demo 6

Demo 6 is a speed test, which can be explicitly analysed using this execution protocol. While running the speed test the LaTeX output is deactivated and the root finding process iterated a large number of times to better measure the average execution time. The measurement results are saved as raw data in the file demo6speed.txt.

In this demo or speed test the following polynomials with a single root at $\frac{1}{3}$ are used (Example 9 from Bartoň and Jüttler):

$$\begin{aligned}
 f_2 &:= (t - \frac{1}{3})(3 - t) \\
 f_4 &:= (t - \frac{1}{3})(2 - t)(t + 5)^2 \\
 f_8 &:= (t - \frac{1}{3})(2 - t)^3(t + 5)^4 \\
 f_{16} &:= (t - \frac{1}{3})(2 - t)^5(t + 5)^{10}
 \end{aligned}$$

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58.2	Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]	289
58.3	Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]	289
58.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]	290
58.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	290
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59	Running QuadClip on f_8 with epsilon 64	293
59.1	Recursion Branch 1 for Input Interval [0, 1]	293

59.2	Recursion Branch 1 1 in Interval 1: [0.326917, 0.344255]	294
59.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	296
59.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	297
59.5	Result: 0 Root Intervals	298
60	Running CubeClip on f_8 with epsilon 64	299
60.1	Recursion Branch 1 for Input Interval [0, 1]	299
60.2	Recursion Branch 1 1 in Interval 1: [0.328258, 0.338551]	300
60.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	302
60.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	303
60.5	Result: 1 Root Intervals	304
61	Running BezClip on f_8 with epsilon 128	305
61.1	Recursion Branch 1 for Input Interval [0, 1]	305
61.2	Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]	306
61.3	Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]	306
61.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]	307
61.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	307
61.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	308
61.7	Result: 1 Root Intervals	309
62	Running QuadClip on f_8 with epsilon 128	310
62.1	Recursion Branch 1 for Input Interval [0, 1]	310
62.2	Recursion Branch 1 1 in Interval 1: [0.326917, 0.344255]	311
62.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	313
62.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	314
62.5	Result: 0 Root Intervals	315
63	Running CubeClip on f_8 with epsilon 128	316
63.1	Recursion Branch 1 for Input Interval [0, 1]	316
63.2	Recursion Branch 1 1 in Interval 1: [0.328258, 0.338551]	317
63.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	319
63.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	320
63.5	Result: 1 Root Intervals	321
64	Running BezClip on f_{16} with epsilon 2	322
64.1	Recursion Branch 1 for Input Interval [0, 1]	322
64.2	Recursion Branch 1 1 in Interval 1: [0.317999, 0.720989]	323
64.3	Recursion Branch 1 1 1 in Interval 1: [0.333081, 0.346096]	323
64.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333337]	324
64.5	Result: 1 Root Intervals	325
65	Running QuadClip on f_{16} with epsilon 2	326
65.1	Recursion Branch 1 for Input Interval [0, 1]	326
65.2	Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]	328
65.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	329
65.4	Result: 1 Root Intervals	330
66	Running CubeClip on f_{16} with epsilon 2	331
66.1	Recursion Branch 1 for Input Interval [0, 1]	331
66.2	Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]	333

66.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	334
66.4	Result: 1 Root Intervals	335
67	Running BezClip on f_{16} with epsilon 4	336
67.1	Recursion Branch 1 for Input Interval [0, 1]	336
67.2	Recursion Branch 1 1 in Interval 1: [0.317999, 0.720989]	337
67.3	Recursion Branch 1 1 1 in Interval 1: [0.333081, 0.346096]	337
67.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333337]	338
67.5	Result: 1 Root Intervals	339
68	Running QuadClip on f_{16} with epsilon 4	340
68.1	Recursion Branch 1 for Input Interval [0, 1]	340
68.2	Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]	342
68.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	343
68.4	Result: 1 Root Intervals	344
69	Running CubeClip on f_{16} with epsilon 4	345
69.1	Recursion Branch 1 for Input Interval [0, 1]	345
69.2	Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]	347
69.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	348
69.4	Result: 1 Root Intervals	349
70	Running BezClip on f_{16} with epsilon 8	350
70.1	Recursion Branch 1 for Input Interval [0, 1]	350
70.2	Recursion Branch 1 1 in Interval 1: [0.317999, 0.720989]	351
70.3	Recursion Branch 1 1 1 in Interval 1: [0.333081, 0.346096]	351
70.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333337]	352
70.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	353
70.6	Result: 1 Root Intervals	354
71	Running QuadClip on f_{16} with epsilon 8	355
71.1	Recursion Branch 1 for Input Interval [0, 1]	355
71.2	Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]	357
71.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	358
71.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	359
71.5	Result: 1 Root Intervals	360
72	Running CubeClip on f_{16} with epsilon 8	361
72.1	Recursion Branch 1 for Input Interval [0, 1]	361
72.2	Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]	363
72.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	364
72.4	Result: 1 Root Intervals	365
73	Running BezClip on f_{16} with epsilon 16	366
73.1	Recursion Branch 1 for Input Interval [0, 1]	366
73.2	Recursion Branch 1 1 in Interval 1: [0.317999, 0.720989]	367
73.3	Recursion Branch 1 1 1 in Interval 1: [0.333081, 0.346096]	367
73.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333337]	368
73.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	369
73.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	369
73.7	Result: 1 Root Intervals	370

74	Running QuadClip on f_{16} with epsilon 16	371
74.1	Recursion Branch 1 for Input Interval $[0, 1]$	371
74.2	Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$	373
74.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	374
74.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	375
74.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	377
74.6	Result: 1 Root Intervals	378
75	Running CubeClip on f_{16} with epsilon 16	379
75.1	Recursion Branch 1 for Input Interval $[0, 1]$	379
75.2	Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$	381
75.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	382
75.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	384
75.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	385
75.6	Result: 1 Root Intervals	386
76	Running BezClip on f_{16} with epsilon 32	387
76.1	Recursion Branch 1 for Input Interval $[0, 1]$	387
76.2	Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$	388
76.3	Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$	388
76.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$	389
76.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	390
76.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	390
76.7	Result: 1 Root Intervals	391
77	Running QuadClip on f_{16} with epsilon 32	392
77.1	Recursion Branch 1 for Input Interval $[0, 1]$	392
77.2	Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$	394
77.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	395
77.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	396
77.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	398
77.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	399
77.7	Result: 1 Root Intervals	400
78	Running CubeClip on f_{16} with epsilon 32	401
78.1	Recursion Branch 1 for Input Interval $[0, 1]$	401
78.2	Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$	403
78.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	404
78.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	406
78.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	407
78.6	Result: 1 Root Intervals	408
79	Running BezClip on f_{16} with epsilon 64	409
79.1	Recursion Branch 1 for Input Interval $[0, 1]$	409
79.2	Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$	410
79.3	Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$	410
79.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$	411
79.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	412
79.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	412
79.7	Result: 1 Root Intervals	413

80	Running QuadClip on f_{16} with epsilon 64	414
80.1	Recursion Branch 1 for Input Interval $[0, 1]$	414
80.2	Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$	416
80.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	417
80.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	418
80.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	420
80.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	421
80.7	Result: 1 Root Intervals	422
81	Running CubeClip on f_{16} with epsilon 64	423
81.1	Recursion Branch 1 for Input Interval $[0, 1]$	423
81.2	Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$	425
81.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	426
81.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	428
81.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	429
81.6	Result: 1 Root Intervals	430
82	Running BezClip on f_{16} with epsilon 128	431
82.1	Recursion Branch 1 for Input Interval $[0, 1]$	431
82.2	Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$	432
82.3	Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$	432
82.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$	433
82.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	434
82.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	434
82.7	Result: 1 Root Intervals	435
83	Running QuadClip on f_{16} with epsilon 128	436
83.1	Recursion Branch 1 for Input Interval $[0, 1]$	436
83.2	Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$	438
83.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	439
83.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	440
83.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	442
83.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	443
83.7	Result: 1 Root Intervals	444
84	Running CubeClip on f_{16} with epsilon 128	445
84.1	Recursion Branch 1 for Input Interval $[0, 1]$	445
84.2	Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$	447
84.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	448
84.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	450
84.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	451
84.6	Result: 1 Root Intervals	452
II	Numeric = long double	453
85	Running BezClip on f_2 with epsilon 2	453
85.1	Recursion Branch 1 for Input Interval $[0, 1]$	453
85.2	Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$	454
85.3	Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$	454
85.4	Result: 1 Root Intervals	455

86	Running QuadClip on f_2 with epsilon 2	456
86.1	Recursion Branch 1 for Input Interval $[0, 1]$	456
86.2	Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$	457
86.3	Result: 1 Root Intervals	458
87	Running CubeClip on f_2 with epsilon 2	459
87.1	Recursion Branch 1 for Input Interval $[0, 1]$	459
87.2	Recursion Branch 1 1 on the First Half $[0, 0.5]$	460
87.3	Recursion Branch 1 2 on the Second Half $[0.5, 1]$	461
87.4	Result: 0 Root Intervals	463
88	Running BezClip on f_2 with epsilon 4	464
88.1	Recursion Branch 1 for Input Interval $[0, 1]$	464
88.2	Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$	464
88.3	Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$	465
88.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$	465
88.5	Result: 1 Root Intervals	466
89	Running QuadClip on f_2 with epsilon 4	467
89.1	Recursion Branch 1 for Input Interval $[0, 1]$	467
89.2	Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$	468
89.3	Result: 1 Root Intervals	469
90	Running CubeClip on f_2 with epsilon 4	470
90.1	Recursion Branch 1 for Input Interval $[0, 1]$	470
90.2	Recursion Branch 1 1 on the First Half $[0, 0.5]$	471
90.3	Recursion Branch 1 2 on the Second Half $[0.5, 1]$	472
90.4	Result: 0 Root Intervals	474
91	Running BezClip on f_2 with epsilon 8	475
91.1	Recursion Branch 1 for Input Interval $[0, 1]$	475
91.2	Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$	475
91.3	Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$	476
91.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$	477
91.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	477
91.6	Result: 1 Root Intervals	478
92	Running QuadClip on f_2 with epsilon 8	479
92.1	Recursion Branch 1 for Input Interval $[0, 1]$	479
92.2	Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$	480
92.3	Result: 1 Root Intervals	481
93	Running CubeClip on f_2 with epsilon 8	482
93.1	Recursion Branch 1 for Input Interval $[0, 1]$	482
93.2	Recursion Branch 1 1 on the First Half $[0, 0.5]$	483
93.3	Recursion Branch 1 2 on the Second Half $[0.5, 1]$	484
93.4	Result: 0 Root Intervals	486
94	Running BezClip on f_2 with epsilon 16	487
94.1	Recursion Branch 1 for Input Interval $[0, 1]$	487
94.2	Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$	487
94.3	Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$	488

94.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]	489
94.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	489
94.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	489
94.7	Result: 1 Root Intervals	490
95	Running QuadClip on f_2 with epsilon 16	491
95.1	Recursion Branch 1 for Input Interval [0, 1]	491
95.2	Recursion Branch 1 1 in Interval 1: [0.333333, 0.333333]	492
95.3	Result: 1 Root Intervals	493
96	Running CubeClip on f_2 with epsilon 16	494
96.1	Recursion Branch 1 for Input Interval [0, 1]	494
96.2	Recursion Branch 1 1 on the First Half [0, 0.5]	495
96.3	Recursion Branch 1 2 on the Second Half [0.5, 1]	496
96.4	Result: 0 Root Intervals	498
97	Running BezClip on f_2 with epsilon 32	499
97.1	Recursion Branch 1 for Input Interval [0, 1]	499
97.2	Recursion Branch 1 1 in Interval 1: [0.3, 0.428571]	499
97.3	Recursion Branch 1 1 1 in Interval 1: [0.332927, 0.334552]	500
97.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]	501
97.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	501
97.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	501
97.7	Result: 1 Root Intervals	502
98	Running QuadClip on f_2 with epsilon 32	503
98.1	Recursion Branch 1 for Input Interval [0, 1]	503
98.2	Recursion Branch 1 1 in Interval 1: [0.333333, 0.333333]	504
98.3	Result: 0 Root Intervals	506
99	Running CubeClip on f_2 with epsilon 32	507
99.1	Recursion Branch 1 for Input Interval [0, 1]	507
99.2	Recursion Branch 1 1 on the First Half [0, 0.5]	508
99.3	Recursion Branch 1 2 on the Second Half [0.5, 1]	509
99.4	Result: 0 Root Intervals	511
100	Running BezClip on f_2 with epsilon 64	512
100.1	Recursion Branch 1 for Input Interval [0, 1]	512
100.2	Recursion Branch 1 1 in Interval 1: [0.3, 0.428571]	512
100.3	Recursion Branch 1 1 1 in Interval 1: [0.332927, 0.334552]	513
100.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]	514
100.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	514
100.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	514
100.7	Result: 1 Root Intervals	515
101	Running QuadClip on f_2 with epsilon 64	516
101.1	Recursion Branch 1 for Input Interval [0, 1]	516
101.2	Recursion Branch 1 1 in Interval 1: [0.333333, 0.333333]	517
101.3	Result: 0 Root Intervals	519
102	Running CubeClip on f_2 with epsilon 64	520
102.1	Recursion Branch 1 for Input Interval [0, 1]	520

102.2	Recursion Branch 1 1 on the First Half $[0, 0.5]$	521
102.3	Recursion Branch 1 2 on the Second Half $[0.5, 1]$	522
102.4	Result: 0 Root Intervals	524
103	Running BezClip on f_2 with epsilon 128	525
103.1	Recursion Branch 1 for Input Interval $[0, 1]$	525
103.2	Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$	525
103.3	Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$	526
103.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$	527
103.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	527
103.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	527
103.7	Result: 1 Root Intervals	528
104	Running QuadClip on f_2 with epsilon 128	529
104.1	Recursion Branch 1 for Input Interval $[0, 1]$	529
104.2	Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$	530
104.3	Result: 0 Root Intervals	532
105	Running CubeClip on f_2 with epsilon 128	533
105.1	Recursion Branch 1 for Input Interval $[0, 1]$	533
105.2	Recursion Branch 1 1 on the First Half $[0, 0.5]$	534
105.3	Recursion Branch 1 2 on the Second Half $[0.5, 1]$	535
105.4	Result: 0 Root Intervals	537
106	Running BezClip on f_4 with epsilon 2	538
106.1	Recursion Branch 1 for Input Interval $[0, 1]$	538
106.2	Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$	538
106.3	Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$	539
106.4	Result: 1 Root Intervals	540
107	Running QuadClip on f_4 with epsilon 2	541
107.1	Recursion Branch 1 for Input Interval $[0, 1]$	541
107.2	Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$	542
107.3	Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$	543
107.4	Result: 1 Root Intervals	544
108	Running CubeClip on f_4 with epsilon 2	545
108.1	Recursion Branch 1 for Input Interval $[0, 1]$	545
108.2	Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$	546
108.3	Result: 1 Root Intervals	547
109	Running BezClip on f_4 with epsilon 4	548
109.1	Recursion Branch 1 for Input Interval $[0, 1]$	548
109.2	Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$	548
109.3	Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$	549
109.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	549
109.5	Result: 1 Root Intervals	550
110	Running QuadClip on f_4 with epsilon 4	551
110.1	Recursion Branch 1 for Input Interval $[0, 1]$	551
110.2	Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$	552
110.3	Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$	553

110.4	Result: 1 Root Intervals	554
111	Running CubeClip on f_4 with epsilon 4	555
111.1	Recursion Branch 1 for Input Interval $[0, 1]$	555
111.2	Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$	556
111.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	557
111.4	Result: 1 Root Intervals	558
112	Running BezClip on f_4 with epsilon 8	559
112.1	Recursion Branch 1 for Input Interval $[0, 1]$	559
112.2	Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$	559
112.3	Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$	560
112.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	560
112.5	Result: 1 Root Intervals	561
113	Running QuadClip on f_4 with epsilon 8	562
113.1	Recursion Branch 1 for Input Interval $[0, 1]$	562
113.2	Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$	563
113.3	Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$	564
113.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	565
113.5	Result: 1 Root Intervals	566
114	Running CubeClip on f_4 with epsilon 8	567
114.1	Recursion Branch 1 for Input Interval $[0, 1]$	567
114.2	Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$	568
114.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	569
114.4	Result: 1 Root Intervals	570
115	Running BezClip on f_4 with epsilon 16	571
115.1	Recursion Branch 1 for Input Interval $[0, 1]$	571
115.2	Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$	571
115.3	Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$	572
115.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	573
115.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	573
115.6	Result: 1 Root Intervals	574
116	Running QuadClip on f_4 with epsilon 16	575
116.1	Recursion Branch 1 for Input Interval $[0, 1]$	575
116.2	Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$	576
116.3	Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$	577
116.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	578
116.5	Result: 1 Root Intervals	579
117	Running CubeClip on f_4 with epsilon 16	580
117.1	Recursion Branch 1 for Input Interval $[0, 1]$	580
117.2	Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$	581
117.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	582
117.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	584
117.5	Result: 1 Root Intervals	585
118	Running BezClip on f_4 with epsilon 32	586
118.1	Recursion Branch 1 for Input Interval $[0, 1]$	586

118.2	Recursion Branch 1 1 in Interval 1: [0.324834, 0.409836]	586
118.3	Recursion Branch 1 1 1 in Interval 1: [0.333317, 0.333491]	587
118.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	588
118.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	588
118.6	Result: 1 Root Intervals	589
119	Running QuadClip on f_4 with epsilon 32	590
119.1	Recursion Branch 1 for Input Interval [0, 1]	590
119.2	Recursion Branch 1 1 in Interval 1: [0.307477, 0.351097]	591
119.3	Recursion Branch 1 1 1 in Interval 1: [0.333332, 0.333335]	592
119.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	594
119.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	595
119.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	596
119.7	Result: 0 Root Intervals	597
120	Running CubeClip on f_4 with epsilon 32	598
120.1	Recursion Branch 1 for Input Interval [0, 1]	598
120.2	Recursion Branch 1 1 in Interval 1: [0.331524, 0.335136]	599
120.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	600
120.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	602
120.5	Result: 1 Root Intervals	603
121	Running BezClip on f_4 with epsilon 64	604
121.1	Recursion Branch 1 for Input Interval [0, 1]	604
121.2	Recursion Branch 1 1 in Interval 1: [0.324834, 0.409836]	604
121.3	Recursion Branch 1 1 1 in Interval 1: [0.333317, 0.333491]	605
121.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	606
121.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	606
121.6	Result: 1 Root Intervals	607
122	Running QuadClip on f_4 with epsilon 64	608
122.1	Recursion Branch 1 for Input Interval [0, 1]	608
122.2	Recursion Branch 1 1 in Interval 1: [0.307477, 0.351097]	609
122.3	Recursion Branch 1 1 1 in Interval 1: [0.333332, 0.333335]	610
122.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	612
122.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	613
122.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	614
122.7	Result: 0 Root Intervals	615
123	Running CubeClip on f_4 with epsilon 64	616
123.1	Recursion Branch 1 for Input Interval [0, 1]	616
123.2	Recursion Branch 1 1 in Interval 1: [0.331524, 0.335136]	617
123.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	618
123.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	620
123.5	Result: 1 Root Intervals	621
124	Running BezClip on f_4 with epsilon 128	622
124.1	Recursion Branch 1 for Input Interval [0, 1]	622
124.2	Recursion Branch 1 1 in Interval 1: [0.324834, 0.409836]	622
124.3	Recursion Branch 1 1 1 in Interval 1: [0.333317, 0.333491]	623
124.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	624

124.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	624
124.6	Result: 1 Root Intervals	625
125	Running QuadClip on f_4 with epsilon 128	626
125.1	Recursion Branch 1 for Input Interval [0, 1]	626
125.2	Recursion Branch 1 1 in Interval 1: [0.307477, 0.351097]	627
125.3	Recursion Branch 1 1 1 in Interval 1: [0.333332, 0.333335]	628
125.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	630
125.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	631
125.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	632
125.7	Result: 0 Root Intervals	633
126	Running CubeClip on f_4 with epsilon 128	634
126.1	Recursion Branch 1 for Input Interval [0, 1]	634
126.2	Recursion Branch 1 1 in Interval 1: [0.331524, 0.335136]	635
126.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	636
126.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	638
126.5	Result: 1 Root Intervals	639
127	Running BezClip on f_8 with epsilon 2	640
127.1	Recursion Branch 1 for Input Interval [0, 1]	640
127.2	Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]	641
127.3	Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]	641
127.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]	642
127.5	Result: 1 Root Intervals	643
128	Running QuadClip on f_8 with epsilon 2	644
128.1	Recursion Branch 1 for Input Interval [0, 1]	644
128.2	Recursion Branch 1 1 in Interval 1: [0.326917, 0.344255]	645
128.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	646
128.4	Result: 1 Root Intervals	647
129	Running CubeClip on f_8 with epsilon 2	648
129.1	Recursion Branch 1 for Input Interval [0, 1]	648
129.2	Recursion Branch 1 1 in Interval 1: [0.328258, 0.338551]	649
129.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	650
129.4	Result: 1 Root Intervals	651
130	Running BezClip on f_8 with epsilon 4	652
130.1	Recursion Branch 1 for Input Interval [0, 1]	652
130.2	Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]	653
130.3	Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]	653
130.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]	654
130.5	Result: 1 Root Intervals	655
131	Running QuadClip on f_8 with epsilon 4	656
131.1	Recursion Branch 1 for Input Interval [0, 1]	656
131.2	Recursion Branch 1 1 in Interval 1: [0.326917, 0.344255]	657
131.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	658
131.4	Result: 1 Root Intervals	659
132	Running CubeClip on f_8 with epsilon 4	660

132.1	Recursion Branch 1 for Input Interval $[0, 1]$	660
132.2	Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$	661
132.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	662
132.4	Result: 1 Root Intervals	663
133	Running BezClip on f_8 with epsilon 8	664
133.1	Recursion Branch 1 for Input Interval $[0, 1]$	664
133.2	Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$	665
133.3	Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$	665
133.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$	666
133.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	666
133.6	Result: 1 Root Intervals	667
134	Running QuadClip on f_8 with epsilon 8	668
134.1	Recursion Branch 1 for Input Interval $[0, 1]$	668
134.2	Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$	669
134.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	671
134.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	672
134.5	Result: 1 Root Intervals	673
135	Running CubeClip on f_8 with epsilon 8	674
135.1	Recursion Branch 1 for Input Interval $[0, 1]$	674
135.2	Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$	675
135.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	676
135.4	Result: 1 Root Intervals	677
136	Running BezClip on f_8 with epsilon 16	678
136.1	Recursion Branch 1 for Input Interval $[0, 1]$	678
136.2	Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$	679
136.3	Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$	679
136.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$	680
136.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	680
136.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	681
136.7	Result: 1 Root Intervals	682
137	Running QuadClip on f_8 with epsilon 16	683
137.1	Recursion Branch 1 for Input Interval $[0, 1]$	683
137.2	Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$	684
137.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	686
137.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	687
137.5	Result: 1 Root Intervals	688
138	Running CubeClip on f_8 with epsilon 16	689
138.1	Recursion Branch 1 for Input Interval $[0, 1]$	689
138.2	Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$	690
138.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	692
138.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	693
138.5	Result: 1 Root Intervals	694
139	Running BezClip on f_8 with epsilon 32	695
139.1	Recursion Branch 1 for Input Interval $[0, 1]$	695

139.2	Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]	696
139.3	Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]	696
139.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]	697
139.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	697
139.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	698
139.7	Result: 1 Root Intervals	699
140	Running QuadClip on f_8 with epsilon 32	700
140.1	Recursion Branch 1 for Input Interval [0, 1]	700
140.2	Recursion Branch 1 1 in Interval 1: [0.326917, 0.344255]	701
140.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	703
140.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	704
140.5	Result: 0 Root Intervals	705
141	Running CubeClip on f_8 with epsilon 32	706
141.1	Recursion Branch 1 for Input Interval [0, 1]	706
141.2	Recursion Branch 1 1 in Interval 1: [0.328258, 0.338551]	707
141.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	709
141.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	710
141.5	Result: 0 Root Intervals	711
142	Running BezClip on f_8 with epsilon 64	712
142.1	Recursion Branch 1 for Input Interval [0, 1]	712
142.2	Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]	713
142.3	Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]	713
142.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]	714
142.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	714
142.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	715
142.7	Result: 1 Root Intervals	716
143	Running QuadClip on f_8 with epsilon 64	717
143.1	Recursion Branch 1 for Input Interval [0, 1]	717
143.2	Recursion Branch 1 1 in Interval 1: [0.326917, 0.344255]	718
143.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	720
143.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	721
143.5	Result: 0 Root Intervals	722
144	Running CubeClip on f_8 with epsilon 64	723
144.1	Recursion Branch 1 for Input Interval [0, 1]	723
144.2	Recursion Branch 1 1 in Interval 1: [0.328258, 0.338551]	724
144.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	726
144.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	727
144.5	Result: 0 Root Intervals	728
145	Running BezClip on f_8 with epsilon 128	729
145.1	Recursion Branch 1 for Input Interval [0, 1]	729
145.2	Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]	730
145.3	Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]	730
145.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]	731
145.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	731
145.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	732

145.7	Result: 1 Root Intervals	733
146	Running QuadClip on f_8 with epsilon 128	734
146.1	Recursion Branch 1 for Input Interval $[0, 1]$	734
146.2	Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$	735
146.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	737
146.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	738
146.5	Result: 0 Root Intervals	739
147	Running CubeClip on f_8 with epsilon 128	740
147.1	Recursion Branch 1 for Input Interval $[0, 1]$	740
147.2	Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$	741
147.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	743
147.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	744
147.5	Result: 0 Root Intervals	745
148	Running BezClip on f_{16} with epsilon 2	746
148.1	Recursion Branch 1 for Input Interval $[0, 1]$	746
148.2	Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$	747
148.3	Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$	747
148.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$	748
148.5	Result: 1 Root Intervals	749
149	Running QuadClip on f_{16} with epsilon 2	750
149.1	Recursion Branch 1 for Input Interval $[0, 1]$	750
149.2	Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$	752
149.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	753
149.4	Result: 1 Root Intervals	754
150	Running CubeClip on f_{16} with epsilon 2	755
150.1	Recursion Branch 1 for Input Interval $[0, 1]$	755
150.2	Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$	757
150.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	758
150.4	Result: 1 Root Intervals	759
151	Running BezClip on f_{16} with epsilon 4	760
151.1	Recursion Branch 1 for Input Interval $[0, 1]$	760
151.2	Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$	761
151.3	Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$	761
151.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$	762
151.5	Result: 1 Root Intervals	763
152	Running QuadClip on f_{16} with epsilon 4	764
152.1	Recursion Branch 1 for Input Interval $[0, 1]$	764
152.2	Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$	766
152.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	767
152.4	Result: 1 Root Intervals	768
153	Running CubeClip on f_{16} with epsilon 4	769
153.1	Recursion Branch 1 for Input Interval $[0, 1]$	769
153.2	Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$	771
153.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	772

153.4	Result: 1 Root Intervals	773
154	Running BezClip on f_{16} with epsilon 8	774
154.1	Recursion Branch 1 for Input Interval $[0, 1]$	774
154.2	Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$	775
154.3	Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$	775
154.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$	776
154.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	777
154.6	Result: 1 Root Intervals	778
155	Running QuadClip on f_{16} with epsilon 8	779
155.1	Recursion Branch 1 for Input Interval $[0, 1]$	779
155.2	Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$	781
155.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	782
155.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	783
155.5	Result: 1 Root Intervals	784
156	Running CubeClip on f_{16} with epsilon 8	785
156.1	Recursion Branch 1 for Input Interval $[0, 1]$	785
156.2	Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$	787
156.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	788
156.4	Result: 1 Root Intervals	789
157	Running BezClip on f_{16} with epsilon 16	790
157.1	Recursion Branch 1 for Input Interval $[0, 1]$	790
157.2	Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$	791
157.3	Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$	791
157.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$	792
157.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	793
157.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	793
157.7	Result: 1 Root Intervals	794
158	Running QuadClip on f_{16} with epsilon 16	795
158.1	Recursion Branch 1 for Input Interval $[0, 1]$	795
158.2	Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$	797
158.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	798
158.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	799
158.5	Result: 1 Root Intervals	800
159	Running CubeClip on f_{16} with epsilon 16	801
159.1	Recursion Branch 1 for Input Interval $[0, 1]$	801
159.2	Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$	803
159.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	804
159.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	805
159.5	Result: 0 Root Intervals	806
160	Running BezClip on f_{16} with epsilon 32	807
160.1	Recursion Branch 1 for Input Interval $[0, 1]$	807
160.2	Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$	808
160.3	Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$	808
160.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$	809

160.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	810
160.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	810
160.7	Result: 1 Root Intervals	811
161	Running QuadClip on f_{16} with epsilon 32	812
161.1	Recursion Branch 1 for Input Interval [0, 1]	812
161.2	Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]	814
161.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	815
161.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	817
161.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	818
161.6	Result: 1 Root Intervals	819
162	Running CubeClip on f_{16} with epsilon 32	820
162.1	Recursion Branch 1 for Input Interval [0, 1]	820
162.2	Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]	822
162.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	823
162.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	825
162.5	Result: 0 Root Intervals	827
163	Running BezClip on f_{16} with epsilon 64	828
163.1	Recursion Branch 1 for Input Interval [0, 1]	828
163.2	Recursion Branch 1 1 in Interval 1: [0.317999, 0.720989]	829
163.3	Recursion Branch 1 1 1 in Interval 1: [0.333081, 0.346096]	829
163.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333337]	830
163.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	831
163.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	831
163.7	Result: 1 Root Intervals	832
164	Running QuadClip on f_{16} with epsilon 64	833
164.1	Recursion Branch 1 for Input Interval [0, 1]	833
164.2	Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]	835
164.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	836
164.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	838
164.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	839
164.6	Result: 1 Root Intervals	840
165	Running CubeClip on f_{16} with epsilon 64	841
165.1	Recursion Branch 1 for Input Interval [0, 1]	841
165.2	Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]	843
165.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	844
165.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	846
165.5	Result: 0 Root Intervals	848
166	Running BezClip on f_{16} with epsilon 128	849
166.1	Recursion Branch 1 for Input Interval [0, 1]	849
166.2	Recursion Branch 1 1 in Interval 1: [0.317999, 0.720989]	850
166.3	Recursion Branch 1 1 1 in Interval 1: [0.333081, 0.346096]	850
166.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333337]	851
166.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	852
166.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	852
166.7	Result: 1 Root Intervals	853

167	Running QuadClip on f_{16} with epsilon 128	854
167.1	Recursion Branch 1 for Input Interval $[0, 1]$	854
167.2	Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$	856
167.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	857
167.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	859
167.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	860
167.6	Result: 1 Root Intervals	861
168	Running CubeClip on f_{16} with epsilon 128	862
168.1	Recursion Branch 1 for Input Interval $[0, 1]$	862
168.2	Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$	864
168.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	865
168.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	867
168.5	Result: 0 Root Intervals	869
III	Numeric = MpfFloat with precision 1024	870
169	Running BezClip on f_2 with epsilon 2	870
169.1	Recursion Branch 1 for Input Interval $[0, 1]$	870
169.2	Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$	871
169.3	Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$	871
169.4	Result: 1 Root Intervals	872
170	Running QuadClip on f_2 with epsilon 2	873
170.1	Recursion Branch 1 for Input Interval $[0, 1]$	873
170.2	Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$	874
170.3	Result: 1 Root Intervals	875
171	Running CubeClip on f_2 with epsilon 2	876
171.1	Recursion Branch 1 for Input Interval $[0, 1]$	876
171.2	Result: 0 Root Intervals	878
172	Running BezClip on f_2 with epsilon 4	879
172.1	Recursion Branch 1 for Input Interval $[0, 1]$	879
172.2	Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$	879
172.3	Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$	880
172.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$	880
172.5	Result: 1 Root Intervals	881
173	Running QuadClip on f_2 with epsilon 4	882
173.1	Recursion Branch 1 for Input Interval $[0, 1]$	882
173.2	Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$	883
173.3	Result: 1 Root Intervals	884
174	Running CubeClip on f_2 with epsilon 4	885
174.1	Recursion Branch 1 for Input Interval $[0, 1]$	885
174.2	Result: 0 Root Intervals	887
175	Running BezClip on f_2 with epsilon 8	888
175.1	Recursion Branch 1 for Input Interval $[0, 1]$	888
175.2	Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$	888

175.3	Recursion Branch 1 1 1 in Interval 1: [0.332927, 0.334552]	889
175.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]	890
175.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	890
175.6	Result: 1 Root Intervals	891
176	Running QuadClip on f_2 with epsilon 8	892
176.1	Recursion Branch 1 for Input Interval [0, 1]	892
176.2	Recursion Branch 1 1 in Interval 1: [0.333333, 0.333333]	893
176.3	Result: 1 Root Intervals	894
177	Running CubeClip on f_2 with epsilon 8	895
177.1	Recursion Branch 1 for Input Interval [0, 1]	895
177.2	Result: 0 Root Intervals	897
178	Running BezClip on f_2 with epsilon 16	898
178.1	Recursion Branch 1 for Input Interval [0, 1]	898
178.2	Recursion Branch 1 1 in Interval 1: [0.3, 0.428571]	898
178.3	Recursion Branch 1 1 1 in Interval 1: [0.332927, 0.334552]	899
178.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]	900
178.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	900
178.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	900
178.7	Result: 1 Root Intervals	901
179	Running QuadClip on f_2 with epsilon 16	902
179.1	Recursion Branch 1 for Input Interval [0, 1]	902
179.2	Recursion Branch 1 1 in Interval 1: [0.333333, 0.333333]	903
179.3	Result: 1 Root Intervals	904
180	Running CubeClip on f_2 with epsilon 16	905
180.1	Recursion Branch 1 for Input Interval [0, 1]	905
180.2	Result: 0 Root Intervals	907
181	Running BezClip on f_2 with epsilon 32	908
181.1	Recursion Branch 1 for Input Interval [0, 1]	908
181.2	Recursion Branch 1 1 in Interval 1: [0.3, 0.428571]	908
181.3	Recursion Branch 1 1 1 in Interval 1: [0.332927, 0.334552]	909
181.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]	910
181.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	910
181.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	911
181.7	Recursion Branch 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	911
181.8	Result: 1 Root Intervals	912
182	Running QuadClip on f_2 with epsilon 32	913
182.1	Recursion Branch 1 for Input Interval [0, 1]	913
182.2	Recursion Branch 1 1 in Interval 1: [0.333333, 0.333333]	914
182.3	Result: 1 Root Intervals	915
183	Running CubeClip on f_2 with epsilon 32	916
183.1	Recursion Branch 1 for Input Interval [0, 1]	916
183.2	Result: 0 Root Intervals	918
184	Running BezClip on f_2 with epsilon 64	919

184.1	Recursion Branch 1 for Input Interval $[0, 1]$	919
184.2	Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$	919
184.3	Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$	920
184.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$	921
184.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	921
184.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	922
184.7	Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	922
184.8	Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	922
184.9	Result: 1 Root Intervals	923
185	Running QuadClip on f_2 with epsilon 64	924
185.1	Recursion Branch 1 for Input Interval $[0, 1]$	924
185.2	Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$	925
185.3	Result: 1 Root Intervals	926
186	Running CubeClip on f_2 with epsilon 64	927
186.1	Recursion Branch 1 for Input Interval $[0, 1]$	927
186.2	Result: 0 Root Intervals	929
187	Running BezClip on f_2 with epsilon 128	930
187.1	Recursion Branch 1 for Input Interval $[0, 1]$	930
187.2	Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$	930
187.3	Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$	931
187.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$	932
187.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	932
187.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	933
187.7	Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	933
187.8	Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	934
187.9	Recursion Branch 1 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	934
187.10	Result: 1 Root Intervals	935
188	Running QuadClip on f_2 with epsilon 128	936
188.1	Recursion Branch 1 for Input Interval $[0, 1]$	936
188.2	Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$	937
188.3	Result: 1 Root Intervals	938
189	Running CubeClip on f_2 with epsilon 128	939
189.1	Recursion Branch 1 for Input Interval $[0, 1]$	939
189.2	Result: 0 Root Intervals	941
190	Running BezClip on f_4 with epsilon 2	942
190.1	Recursion Branch 1 for Input Interval $[0, 1]$	942
190.2	Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$	942
190.3	Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$	943
190.4	Result: 1 Root Intervals	944
191	Running QuadClip on f_4 with epsilon 2	945
191.1	Recursion Branch 1 for Input Interval $[0, 1]$	945
191.2	Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$	946
191.3	Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$	947
191.4	Result: 1 Root Intervals	948

192	Running CubeClip on f_4 with epsilon 2	949
192.1	Recursion Branch 1 for Input Interval $[0, 1]$	949
192.2	Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$	950
192.3	Result: 1 Root Intervals	951
193	Running BezClip on f_4 with epsilon 4	952
193.1	Recursion Branch 1 for Input Interval $[0, 1]$	952
193.2	Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$	952
193.3	Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$	953
193.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	953
193.5	Result: 1 Root Intervals	954
194	Running QuadClip on f_4 with epsilon 4	955
194.1	Recursion Branch 1 for Input Interval $[0, 1]$	955
194.2	Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$	956
194.3	Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$	957
194.4	Result: 1 Root Intervals	958
195	Running CubeClip on f_4 with epsilon 4	959
195.1	Recursion Branch 1 for Input Interval $[0, 1]$	959
195.2	Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$	960
195.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	961
195.4	Result: 1 Root Intervals	962
196	Running BezClip on f_4 with epsilon 8	963
196.1	Recursion Branch 1 for Input Interval $[0, 1]$	963
196.2	Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$	963
196.3	Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$	964
196.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	964
196.5	Result: 1 Root Intervals	965
197	Running QuadClip on f_4 with epsilon 8	966
197.1	Recursion Branch 1 for Input Interval $[0, 1]$	966
197.2	Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$	967
197.3	Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$	968
197.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	969
197.5	Result: 1 Root Intervals	970
198	Running CubeClip on f_4 with epsilon 8	971
198.1	Recursion Branch 1 for Input Interval $[0, 1]$	971
198.2	Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$	972
198.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	973
198.4	Result: 1 Root Intervals	974
199	Running BezClip on f_4 with epsilon 16	975
199.1	Recursion Branch 1 for Input Interval $[0, 1]$	975
199.2	Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$	975
199.3	Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$	976
199.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	977
199.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	977
199.6	Result: 1 Root Intervals	978

200	Running QuadClip on f_4 with epsilon 16	979
200.1	Recursion Branch 1 for Input Interval $[0, 1]$	979
200.2	Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$	980
200.3	Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$	981
200.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	982
200.5	Result: 1 Root Intervals	983
201	Running CubeClip on f_4 with epsilon 16	984
201.1	Recursion Branch 1 for Input Interval $[0, 1]$	984
201.2	Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$	985
201.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	986
201.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	988
201.5	Result: 0 Root Intervals	989
202	Running BezClip on f_4 with epsilon 32	990
202.1	Recursion Branch 1 for Input Interval $[0, 1]$	990
202.2	Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$	990
202.3	Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$	991
202.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	992
202.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	992
202.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	993
202.7	Result: 1 Root Intervals	994
203	Running QuadClip on f_4 with epsilon 32	995
203.1	Recursion Branch 1 for Input Interval $[0, 1]$	995
203.2	Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$	996
203.3	Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$	997
203.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	999
203.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1000
203.6	Result: 1 Root Intervals	1001
204	Running CubeClip on f_4 with epsilon 32	1002
204.1	Recursion Branch 1 for Input Interval $[0, 1]$	1002
204.2	Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$	1003
204.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1004
204.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1006
204.5	Result: 0 Root Intervals	1007
205	Running BezClip on f_4 with epsilon 64	1008
205.1	Recursion Branch 1 for Input Interval $[0, 1]$	1008
205.2	Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$	1008
205.3	Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$	1009
205.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1010
205.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1010
205.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1011
205.7	Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1011
205.8	Result: 1 Root Intervals	1012
206	Running QuadClip on f_4 with epsilon 64	1013
206.1	Recursion Branch 1 for Input Interval $[0, 1]$	1013
206.2	Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$	1014

206.3	Recursion Branch 1 1 1 in Interval 1: [0.333332, 0.333335]	1015
206.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1017
206.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1018
206.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1019
206.7	Result: 1 Root Intervals	1020
207	Running CubeClip on f_4 with epsilon 64	1021
207.1	Recursion Branch 1 for Input Interval [0, 1]	1021
207.2	Recursion Branch 1 1 in Interval 1: [0.331524, 0.335136]	1022
207.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1023
207.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1025
207.5	Result: 0 Root Intervals	1027
208	Running BezClip on f_4 with epsilon 128	1028
208.1	Recursion Branch 1 for Input Interval [0, 1]	1028
208.2	Recursion Branch 1 1 in Interval 1: [0.324834, 0.409836]	1028
208.3	Recursion Branch 1 1 1 in Interval 1: [0.333317, 0.333491]	1029
208.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1030
208.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1030
208.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1031
208.7	Recursion Branch 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1031
208.8	Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1032
208.9	Result: 1 Root Intervals	1033
209	Running QuadClip on f_4 with epsilon 128	1034
209.1	Recursion Branch 1 for Input Interval [0, 1]	1034
209.2	Recursion Branch 1 1 in Interval 1: [0.307477, 0.351097]	1035
209.3	Recursion Branch 1 1 1 in Interval 1: [0.333332, 0.333335]	1036
209.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1038
209.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1039
209.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1040
209.7	Result: 1 Root Intervals	1041
210	Running CubeClip on f_4 with epsilon 128	1042
210.1	Recursion Branch 1 for Input Interval [0, 1]	1042
210.2	Recursion Branch 1 1 in Interval 1: [0.331524, 0.335136]	1043
210.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1044
210.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1046
210.5	Result: 0 Root Intervals	1048
211	Running BezClip on f_8 with epsilon 2	1049
211.1	Recursion Branch 1 for Input Interval [0, 1]	1049
211.2	Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]	1050
211.3	Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]	1050
211.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]	1051
211.5	Result: 1 Root Intervals	1052
212	Running QuadClip on f_8 with epsilon 2	1053
212.1	Recursion Branch 1 for Input Interval [0, 1]	1053
212.2	Recursion Branch 1 1 in Interval 1: [0.326917, 0.344255]	1054
212.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1055

212.4	Result: 1 Root Intervals	1056
213	Running CubeClip on f_8 with epsilon 2	1057
213.1	Recursion Branch 1 for Input Interval $[0, 1]$	1057
213.2	Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$	1058
213.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1059
213.4	Result: 1 Root Intervals	1060
214	Running BezClip on f_8 with epsilon 4	1061
214.1	Recursion Branch 1 for Input Interval $[0, 1]$	1061
214.2	Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$	1062
214.3	Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$	1062
214.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$	1063
214.5	Result: 1 Root Intervals	1064
215	Running QuadClip on f_8 with epsilon 4	1065
215.1	Recursion Branch 1 for Input Interval $[0, 1]$	1065
215.2	Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$	1066
215.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1067
215.4	Result: 1 Root Intervals	1068
216	Running CubeClip on f_8 with epsilon 4	1069
216.1	Recursion Branch 1 for Input Interval $[0, 1]$	1069
216.2	Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$	1070
216.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1071
216.4	Result: 1 Root Intervals	1072
217	Running BezClip on f_8 with epsilon 8	1073
217.1	Recursion Branch 1 for Input Interval $[0, 1]$	1073
217.2	Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$	1074
217.3	Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$	1074
217.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333343]$	1075
217.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1075
217.6	Result: 1 Root Intervals	1076
218	Running QuadClip on f_8 with epsilon 8	1077
218.1	Recursion Branch 1 for Input Interval $[0, 1]$	1077
218.2	Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$	1078
218.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1080
218.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1081
218.5	Result: 1 Root Intervals	1082
219	Running CubeClip on f_8 with epsilon 8	1083
219.1	Recursion Branch 1 for Input Interval $[0, 1]$	1083
219.2	Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$	1084
219.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1085
219.4	Result: 1 Root Intervals	1086
220	Running BezClip on f_8 with epsilon 16	1087
220.1	Recursion Branch 1 for Input Interval $[0, 1]$	1087
220.2	Recursion Branch 1 1 in Interval 1: $[0.306796, 0.658588]$	1088
220.3	Recursion Branch 1 1 1 in Interval 1: $[0.332635, 0.34642]$	1088

220.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]	1089
220.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1089
220.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1090
220.7	Result: 1 Root Intervals	1091
221	Running QuadClip on f_8 with epsilon 16	1092
221.1	Recursion Branch 1 for Input Interval [0, 1]	1092
221.2	Recursion Branch 1 1 in Interval 1: [0.326917, 0.344255]	1093
221.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1095
221.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1096
221.5	Result: 1 Root Intervals	1097
222	Running CubeClip on f_8 with epsilon 16	1098
222.1	Recursion Branch 1 for Input Interval [0, 1]	1098
222.2	Recursion Branch 1 1 in Interval 1: [0.328258, 0.338551]	1099
222.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1101
222.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1102
222.5	Result: 0 Root Intervals	1103
223	Running BezClip on f_8 with epsilon 32	1104
223.1	Recursion Branch 1 for Input Interval [0, 1]	1104
223.2	Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]	1105
223.3	Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]	1105
223.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]	1106
223.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1106
223.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1107
223.7	Recursion Branch 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1107
223.8	Result: 1 Root Intervals	1108
224	Running QuadClip on f_8 with epsilon 32	1109
224.1	Recursion Branch 1 for Input Interval [0, 1]	1109
224.2	Recursion Branch 1 1 in Interval 1: [0.326917, 0.344255]	1110
224.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1112
224.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1113
224.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1114
224.6	Result: 1 Root Intervals	1115
225	Running CubeClip on f_8 with epsilon 32	1116
225.1	Recursion Branch 1 for Input Interval [0, 1]	1116
225.2	Recursion Branch 1 1 in Interval 1: [0.328258, 0.338551]	1117
225.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1119
225.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1120
225.5	Result: 0 Root Intervals	1121
226	Running BezClip on f_8 with epsilon 64	1122
226.1	Recursion Branch 1 for Input Interval [0, 1]	1122
226.2	Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]	1123
226.3	Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]	1123
226.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]	1124
226.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1124
226.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1125

226.7	Recursion Branch 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1125
226.8	Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1126
226.9	Result: 1 Root Intervals	1127
227	Running QuadClip on f_8 with epsilon 64	1128
227.1	Recursion Branch 1 for Input Interval [0, 1]	1128
227.2	Recursion Branch 1 1 in Interval 1: [0.326917, 0.344255]	1129
227.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1131
227.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1132
227.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1133
227.6	Result: 1 Root Intervals	1134
228	Running CubeClip on f_8 with epsilon 64	1135
228.1	Recursion Branch 1 for Input Interval [0, 1]	1135
228.2	Recursion Branch 1 1 in Interval 1: [0.328258, 0.338551]	1136
228.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1138
228.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1139
228.5	Result: 0 Root Intervals	1141
229	Running BezClip on f_8 with epsilon 128	1142
229.1	Recursion Branch 1 for Input Interval [0, 1]	1142
229.2	Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]	1143
229.3	Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]	1143
229.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]	1144
229.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1144
229.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1145
229.7	Recursion Branch 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1145
229.8	Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1146
229.9	Recursion Branch 1 1 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1146
229.10	Result: 1 Root Intervals	1147
230	Running QuadClip on f_8 with epsilon 128	1148
230.1	Recursion Branch 1 for Input Interval [0, 1]	1148
230.2	Recursion Branch 1 1 in Interval 1: [0.326917, 0.344255]	1149
230.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1151
230.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1152
230.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1153
230.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1155
230.7	Result: 1 Root Intervals	1156
231	Running CubeClip on f_8 with epsilon 128	1157
231.1	Recursion Branch 1 for Input Interval [0, 1]	1157
231.2	Recursion Branch 1 1 in Interval 1: [0.328258, 0.338551]	1158
231.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1160
231.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1161
231.5	Result: 0 Root Intervals	1163
232	Running BezClip on f_{16} with epsilon 2	1164
232.1	Recursion Branch 1 for Input Interval [0, 1]	1164
232.2	Recursion Branch 1 1 in Interval 1: [0.317999, 0.720989]	1165
232.3	Recursion Branch 1 1 1 in Interval 1: [0.333081, 0.346096]	1165

232.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333337]	1166
232.5	Result: 1 Root Intervals	1167
233	Running QuadClip on f_{16} with epsilon 2	1168
233.1	Recursion Branch 1 for Input Interval [0, 1]	1168
233.2	Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]	1170
233.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1171
233.4	Result: 1 Root Intervals	1172
234	Running CubeClip on f_{16} with epsilon 2	1173
234.1	Recursion Branch 1 for Input Interval [0, 1]	1173
234.2	Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]	1175
234.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1176
234.4	Result: 1 Root Intervals	1177
235	Running BezClip on f_{16} with epsilon 4	1178
235.1	Recursion Branch 1 for Input Interval [0, 1]	1178
235.2	Recursion Branch 1 1 in Interval 1: [0.317999, 0.720989]	1179
235.3	Recursion Branch 1 1 1 in Interval 1: [0.333081, 0.346096]	1179
235.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333337]	1180
235.5	Result: 1 Root Intervals	1181
236	Running QuadClip on f_{16} with epsilon 4	1182
236.1	Recursion Branch 1 for Input Interval [0, 1]	1182
236.2	Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]	1184
236.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1185
236.4	Result: 1 Root Intervals	1186
237	Running CubeClip on f_{16} with epsilon 4	1187
237.1	Recursion Branch 1 for Input Interval [0, 1]	1187
237.2	Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]	1189
237.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1190
237.4	Result: 1 Root Intervals	1191
238	Running BezClip on f_{16} with epsilon 8	1192
238.1	Recursion Branch 1 for Input Interval [0, 1]	1192
238.2	Recursion Branch 1 1 in Interval 1: [0.317999, 0.720989]	1193
238.3	Recursion Branch 1 1 1 in Interval 1: [0.333081, 0.346096]	1193
238.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333337]	1194
238.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1195
238.6	Result: 1 Root Intervals	1196
239	Running QuadClip on f_{16} with epsilon 8	1197
239.1	Recursion Branch 1 for Input Interval [0, 1]	1197
239.2	Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]	1199
239.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1200
239.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1201
239.5	Result: 1 Root Intervals	1202
240	Running CubeClip on f_{16} with epsilon 8	1203
240.1	Recursion Branch 1 for Input Interval [0, 1]	1203
240.2	Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]	1205

240.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1206
240.4	Result: 1 Root Intervals	1207
241	Running BezClip on f_{16} with epsilon 16	1208
241.1	Recursion Branch 1 for Input Interval [0, 1]	1208
241.2	Recursion Branch 1 1 in Interval 1: [0.317999, 0.720989]	1209
241.3	Recursion Branch 1 1 1 in Interval 1: [0.333081, 0.346096]	1209
241.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333337]	1210
241.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1211
241.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1211
241.7	Result: 1 Root Intervals	1212
242	Running QuadClip on f_{16} with epsilon 16	1213
242.1	Recursion Branch 1 for Input Interval [0, 1]	1213
242.2	Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]	1215
242.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1216
242.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1217
242.5	Result: 1 Root Intervals	1218
243	Running CubeClip on f_{16} with epsilon 16	1219
243.1	Recursion Branch 1 for Input Interval [0, 1]	1219
243.2	Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]	1221
243.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1222
243.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1223
243.5	Result: 0 Root Intervals	1224
244	Running BezClip on f_{16} with epsilon 32	1225
244.1	Recursion Branch 1 for Input Interval [0, 1]	1225
244.2	Recursion Branch 1 1 in Interval 1: [0.317999, 0.720989]	1226
244.3	Recursion Branch 1 1 1 in Interval 1: [0.333081, 0.346096]	1226
244.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333337]	1227
244.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1228
244.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1228
244.7	Recursion Branch 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1229
244.8	Result: 1 Root Intervals	1230
245	Running QuadClip on f_{16} with epsilon 32	1231
245.1	Recursion Branch 1 for Input Interval [0, 1]	1231
245.2	Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]	1233
245.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1234
245.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1236
245.5	Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1237
245.6	Result: 1 Root Intervals	1238
246	Running CubeClip on f_{16} with epsilon 32	1239
246.1	Recursion Branch 1 for Input Interval [0, 1]	1239
246.2	Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]	1241
246.3	Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]	1242
246.4	Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]	1243
246.5	Result: 0 Root Intervals	1244

247	Running BezClip on f_{16} with epsilon 64	1245
247.1	Recursion Branch 1 for Input Interval $[0, 1]$	1245
247.2	Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$	1246
247.3	Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$	1246
247.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$	1247
247.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1248
247.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1248
247.7	Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1249
247.8	Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1250
247.9	Result: 1 Root Intervals	1251
248	Running QuadClip on f_{16} with epsilon 64	1252
248.1	Recursion Branch 1 for Input Interval $[0, 1]$	1252
248.2	Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$	1254
248.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1255
248.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1257
248.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1258
248.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1259
248.7	Result: 1 Root Intervals	1260
249	Running CubeClip on f_{16} with epsilon 64	1261
249.1	Recursion Branch 1 for Input Interval $[0, 1]$	1261
249.2	Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$	1263
249.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1264
249.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1266
249.5	Result: 0 Root Intervals	1268
250	Running BezClip on f_{16} with epsilon 128	1269
250.1	Recursion Branch 1 for Input Interval $[0, 1]$	1269
250.2	Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$	1270
250.3	Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$	1270
250.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$	1271
250.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1272
250.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1272
250.7	Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1273
250.8	Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1274
250.9	Recursion Branch 1 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1274
250.10	Result: 1 Root Intervals	1275
251	Running QuadClip on f_{16} with epsilon 128	1276
251.1	Recursion Branch 1 for Input Interval $[0, 1]$	1276
251.2	Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$	1278
251.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1279
251.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1281
251.5	Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1282
251.6	Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1283
251.7	Result: 1 Root Intervals	1284
252	Running CubeClip on f_{16} with epsilon 128	1285
252.1	Recursion Branch 1 for Input Interval $[0, 1]$	1285
252.2	Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$	1287

252.3	Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1288
252.4	Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$	1290
252.5	Result: 0 Root Intervals	1292

Part I

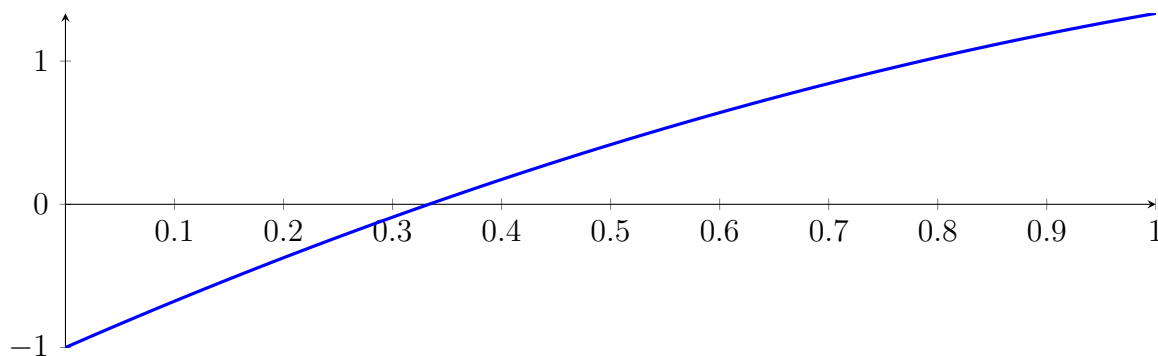
Numeric = double

1 Running BezClip on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

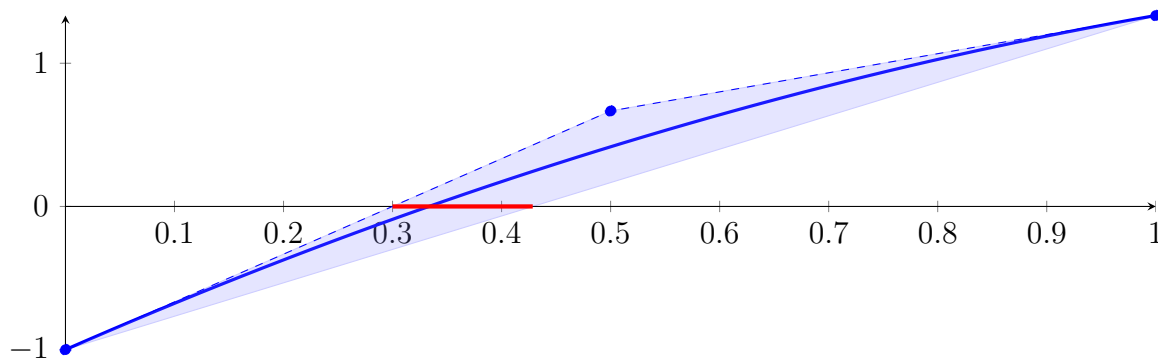
$$p = -1X^2 + 3.33333X - 1$$



1.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

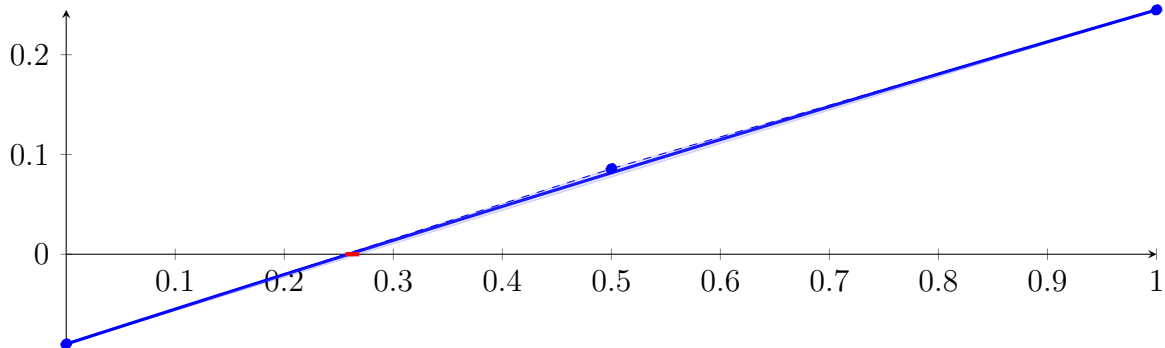
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

1.2 Recursion Branch 1 1 in Interval 1: [0.3, 0.428571]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641

\implies Selective recursion: interval 1: [0.332927, 0.334552],

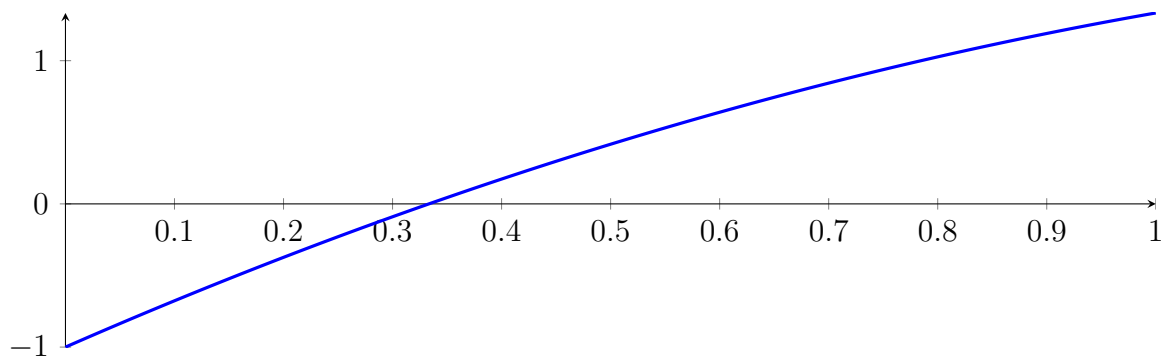
1.3 Recursion Branch 1 1 1 in Interval 1: [0.332927, 0.334552]

Found root in interval [0.332927, 0.334552] at recursion depth 3!

1.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.332927, 0.334552]$$

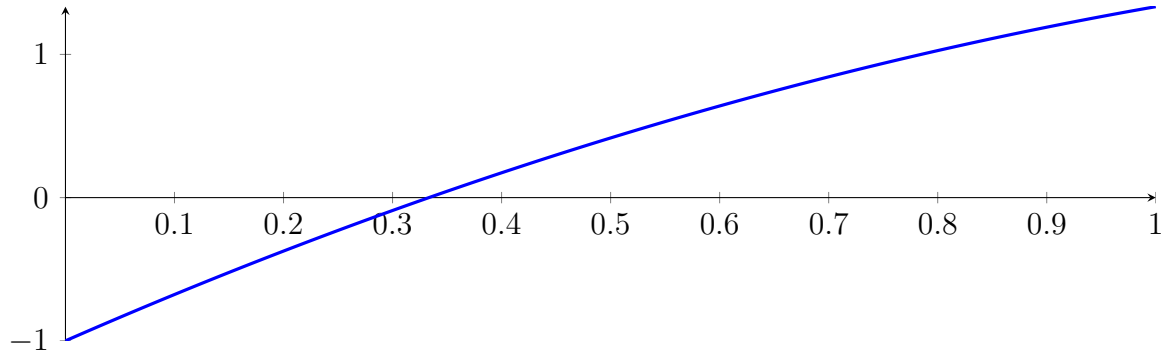
with precision $\varepsilon = 0.01$.

2 Running QuadClip on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

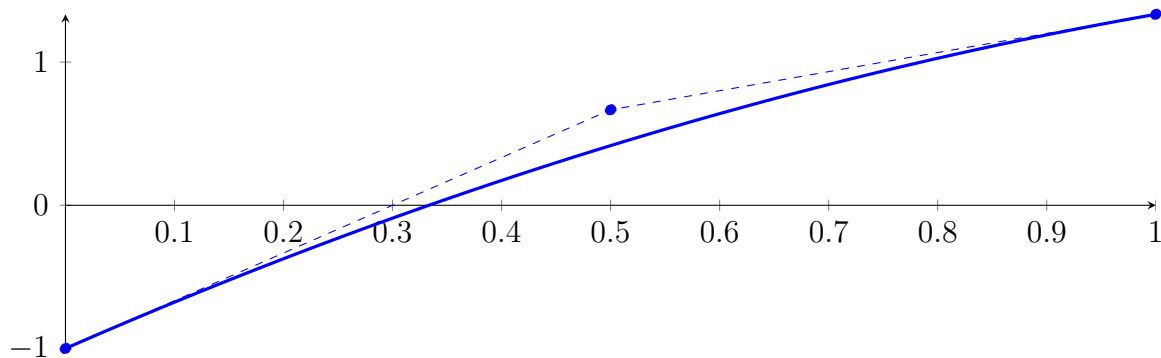
$$p = -1X^2 + 3.33333X - 1$$



2.1 Recursion Branch 1 for Input Interval $[0, 1]$

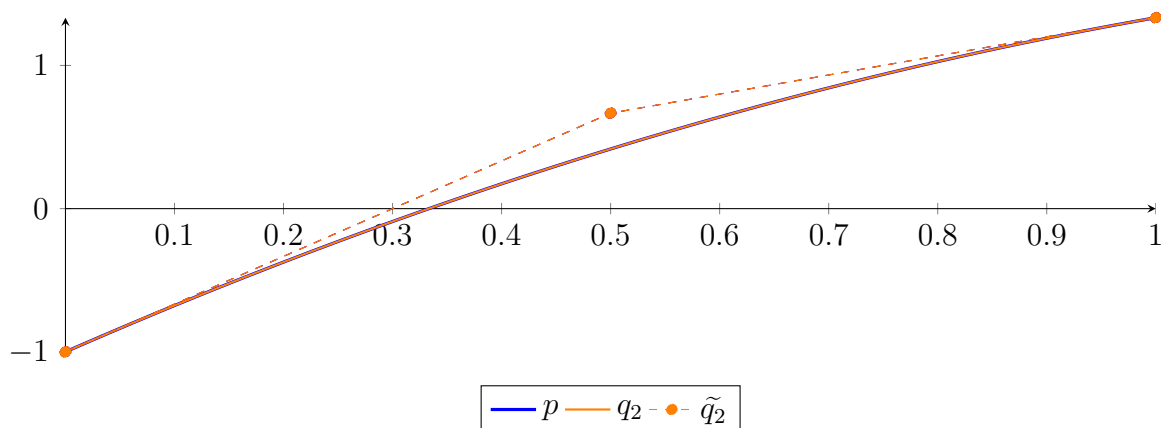
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

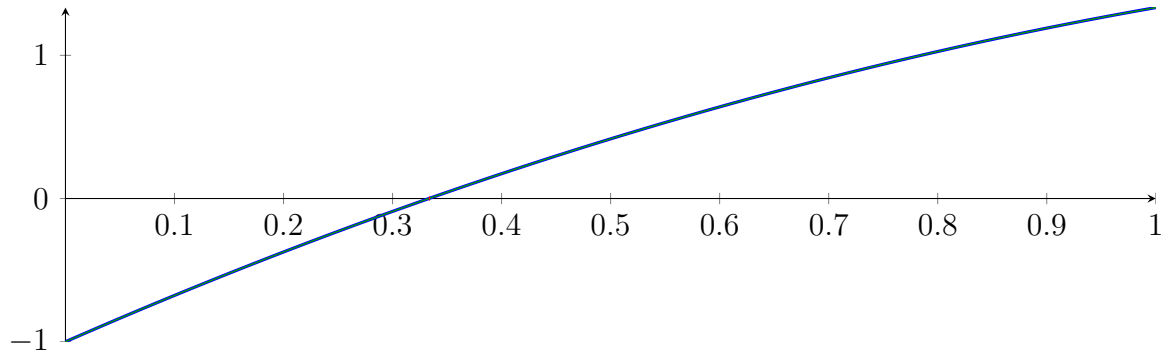
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $4.44089 \cdot 10^{-16}$

\implies Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

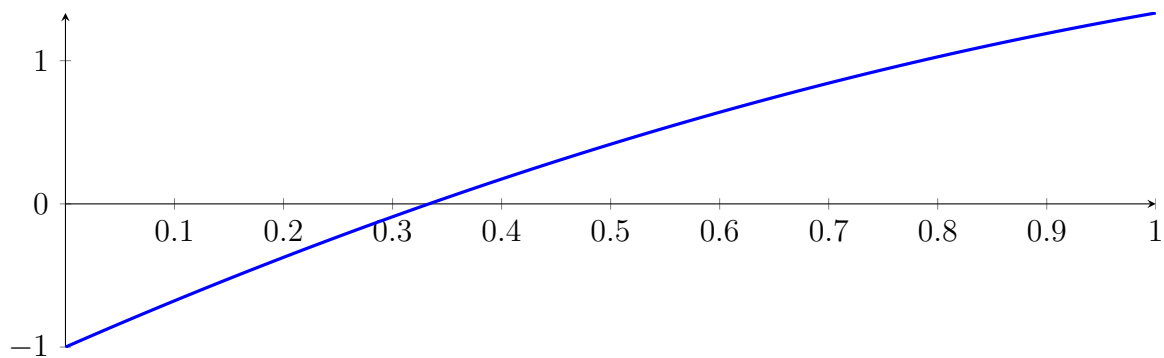
2.2 Recursion Branch 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 2!

2.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

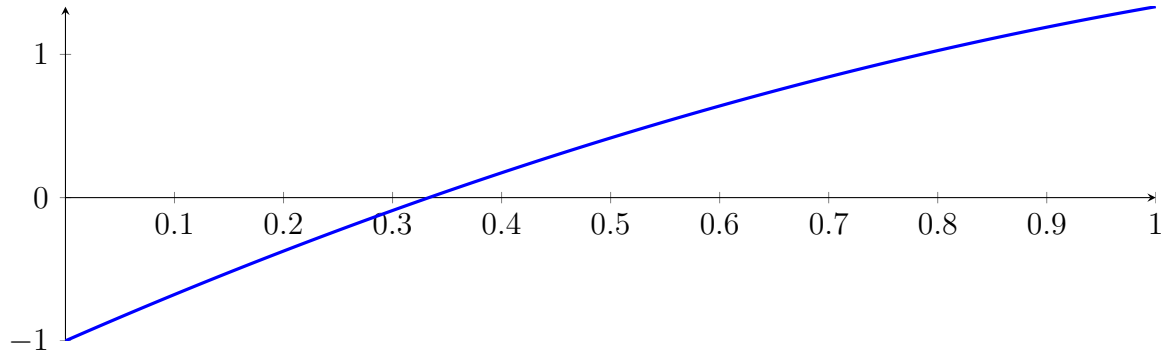
with precision $\varepsilon = 0.01$.

3 Running CubeClip on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

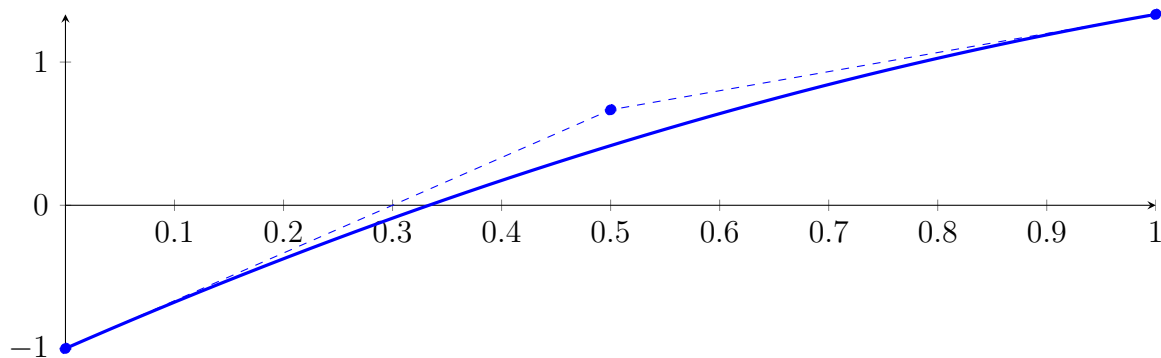
$$p = -1X^2 + 3.33333X - 1$$



3.1 Recursion Branch 1 for Input Interval $[0, 1]$

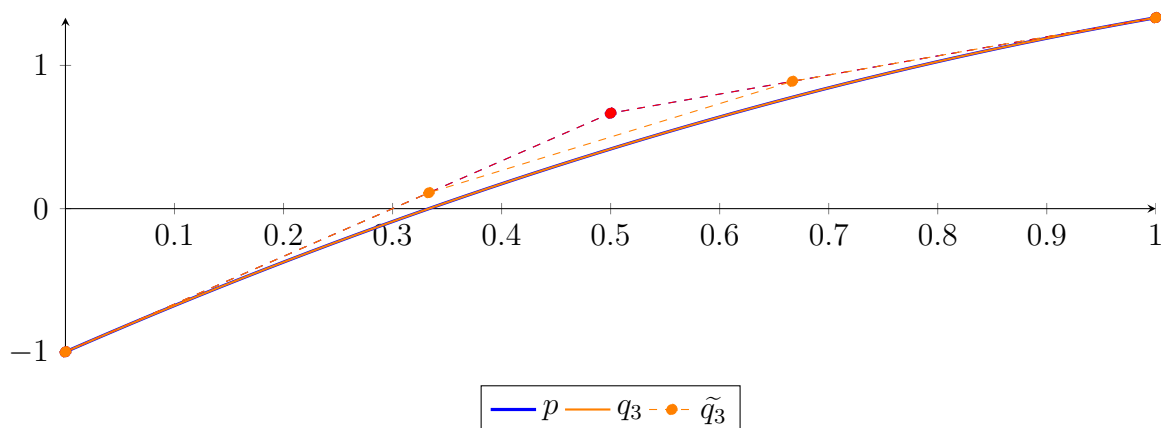
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.90958 \cdot 10^{-14}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.

Bounding polynomials M and m :

$$M = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

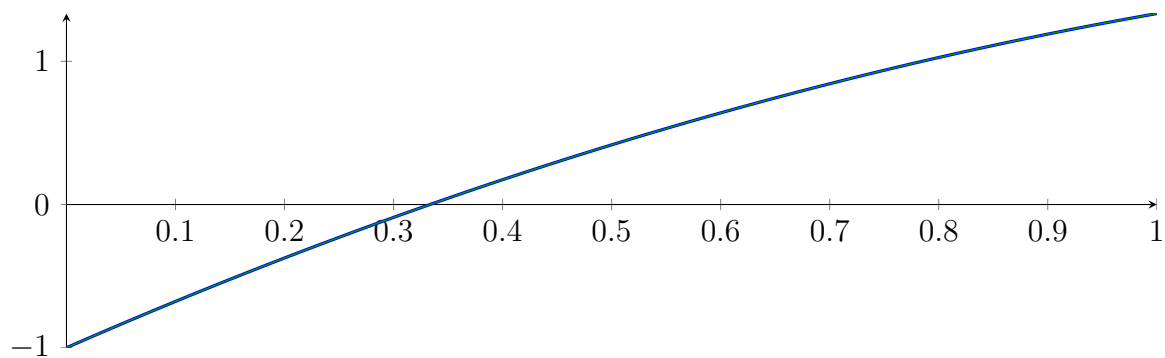
$$m = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-5.17655 \cdot 10^{13}\}$$

$$N(m) = \{-5.17655 \cdot 10^{13}\}$$

Intersection intervals:

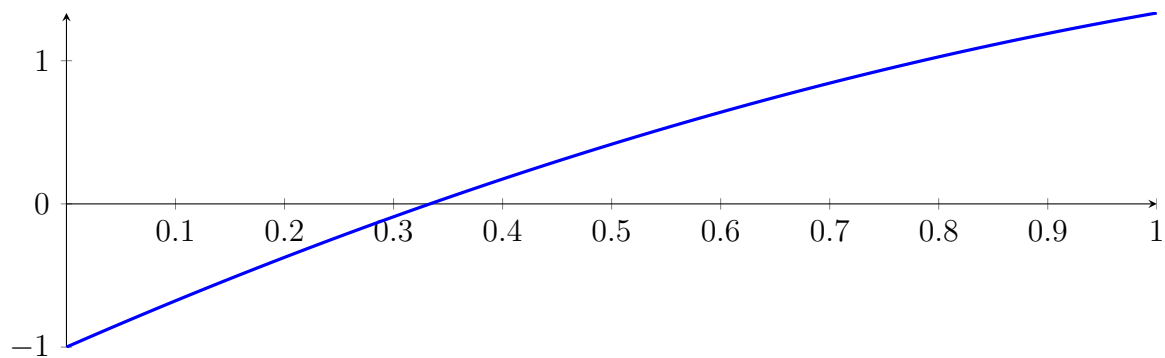


No intersection intervals with the x axis.

3.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

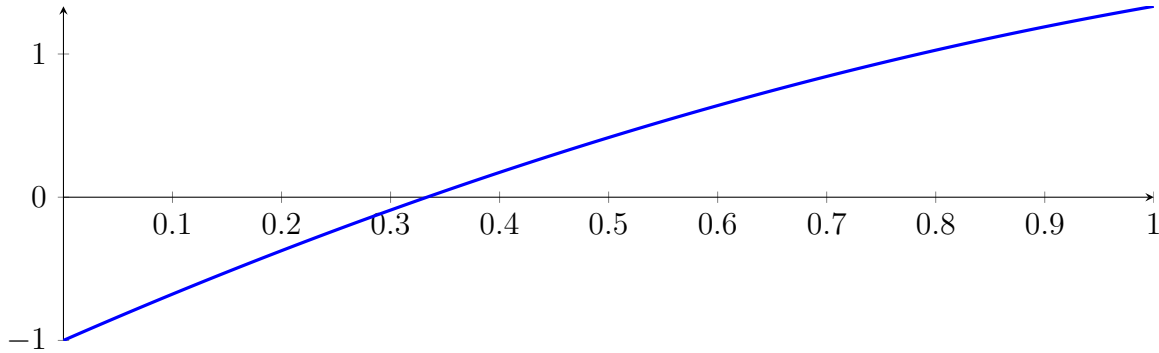
with precision $\varepsilon = 0.01$.

4 Running BezClip on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

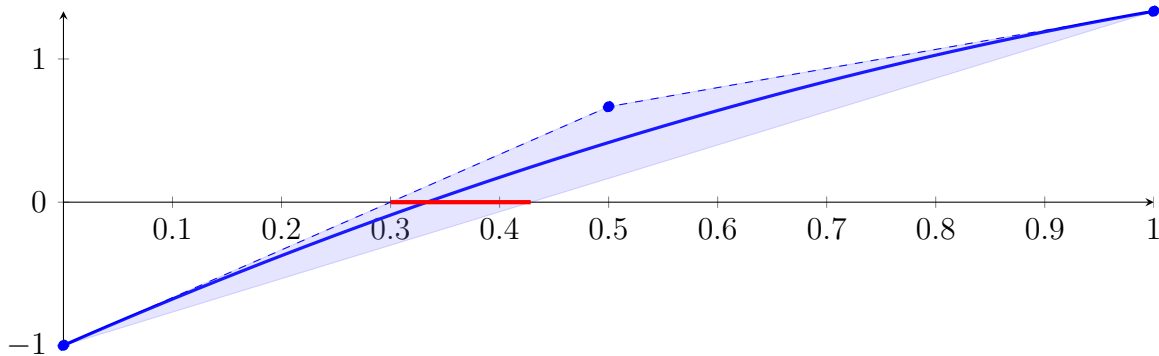
$$p = -1X^2 + 3.33333X - 1$$



4.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

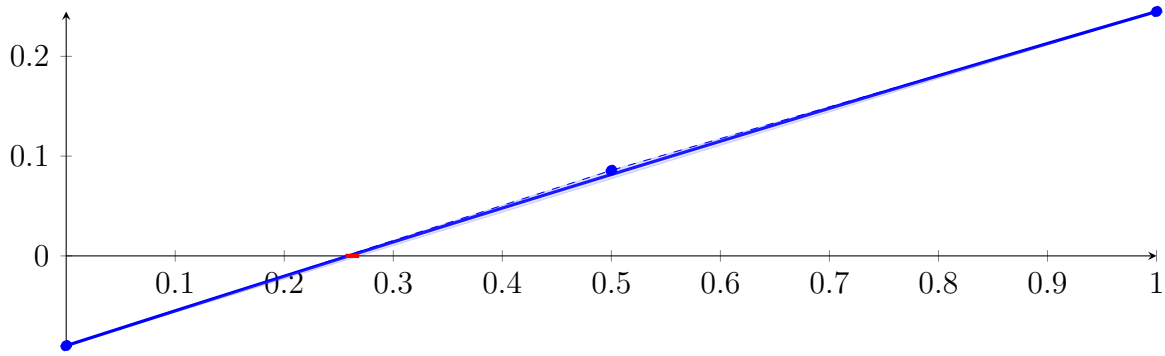
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

4.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

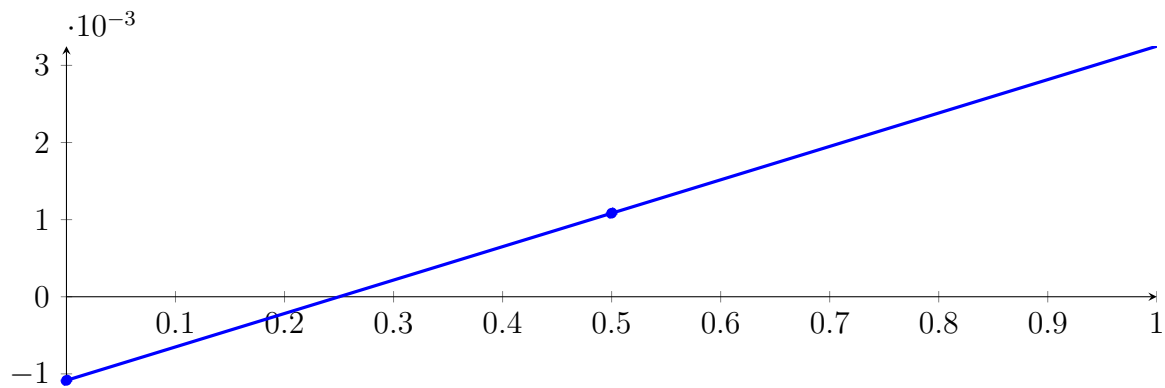
Longest intersection interval: 0.012641

\implies Selective recursion: interval 1: $[0.332927, 0.334552]$,

4.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462

\implies Selective recursion: interval 1: $[0.333333, 0.333334]$,

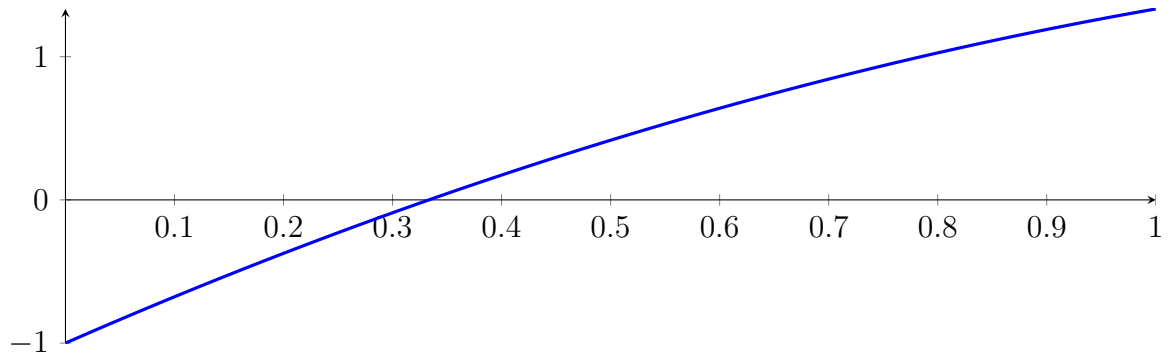
4.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Found root in interval $[0.333333, 0.333334]$ at recursion depth 4!

4.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333334]$$

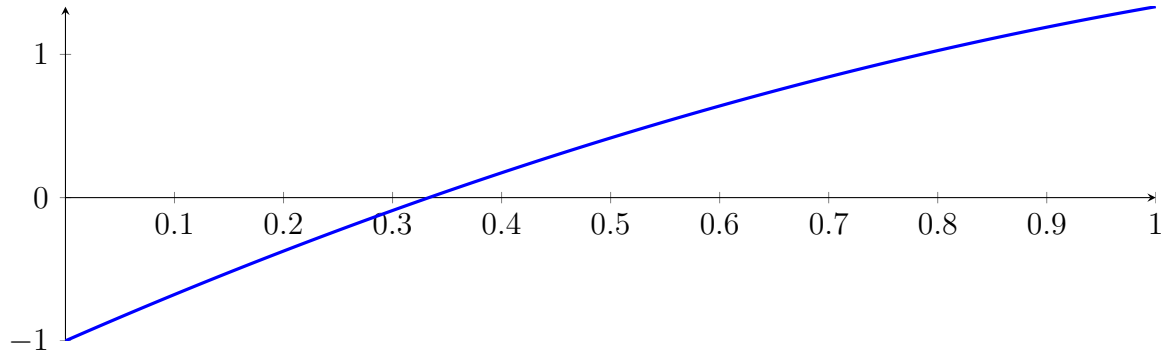
with precision $\varepsilon = 0.0001$.

5 Running QuadClip on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

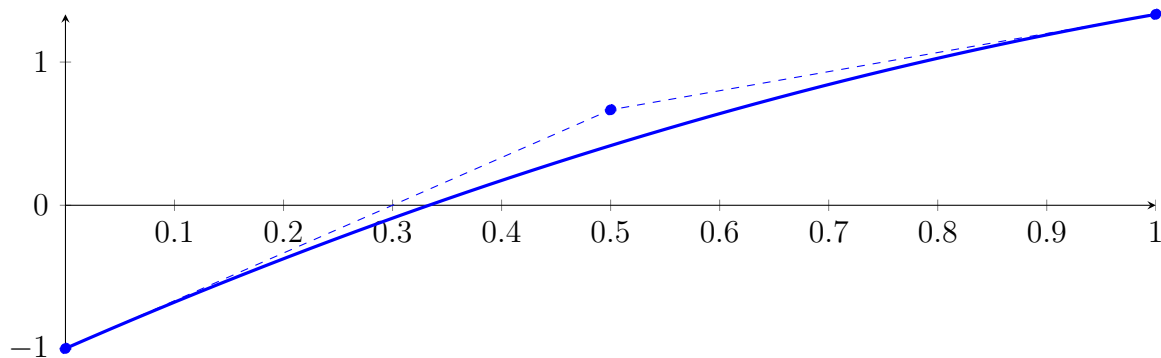
$$p = -1X^2 + 3.33333X - 1$$



5.1 Recursion Branch 1 for Input Interval $[0, 1]$

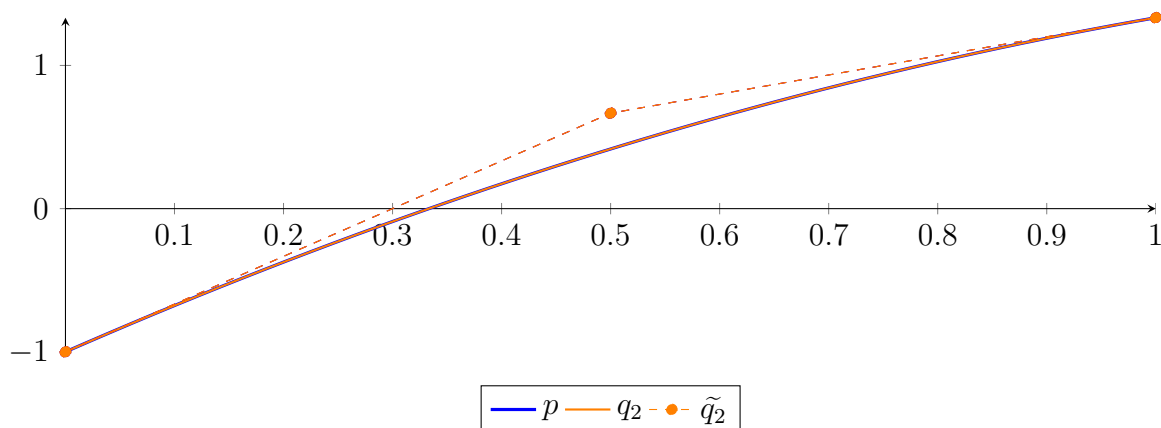
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

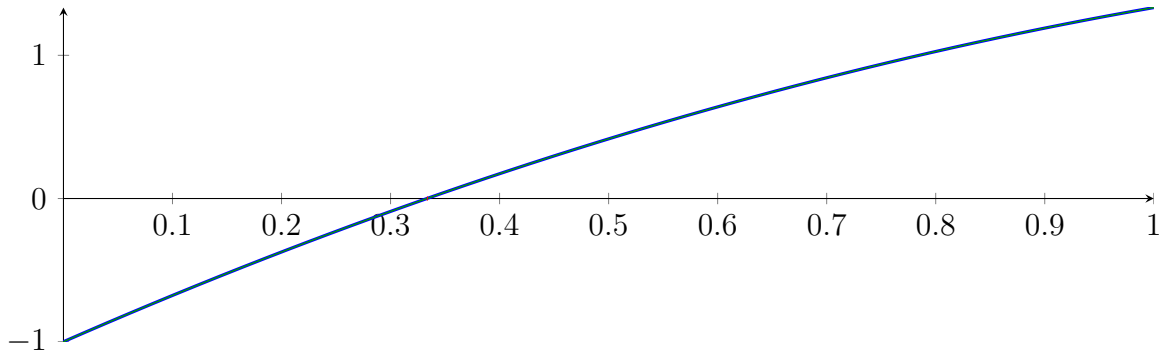
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $4.44089 \cdot 10^{-16}$

\implies Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

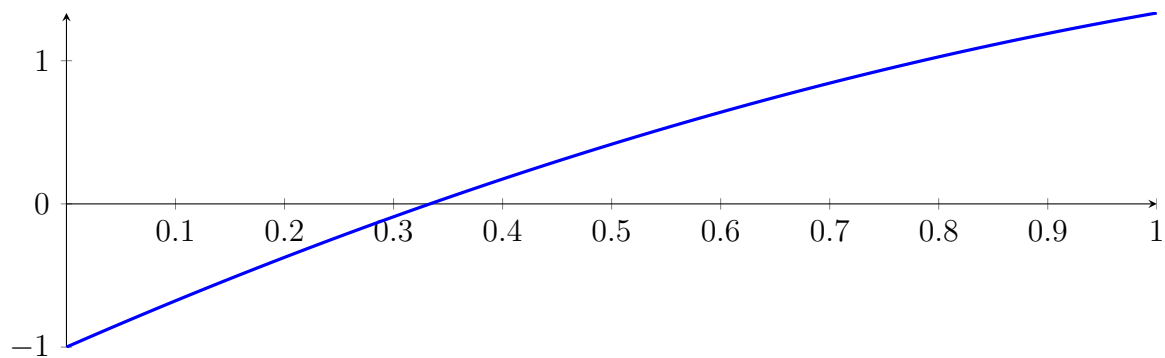
5.2 Recursion Branch 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 2!

5.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

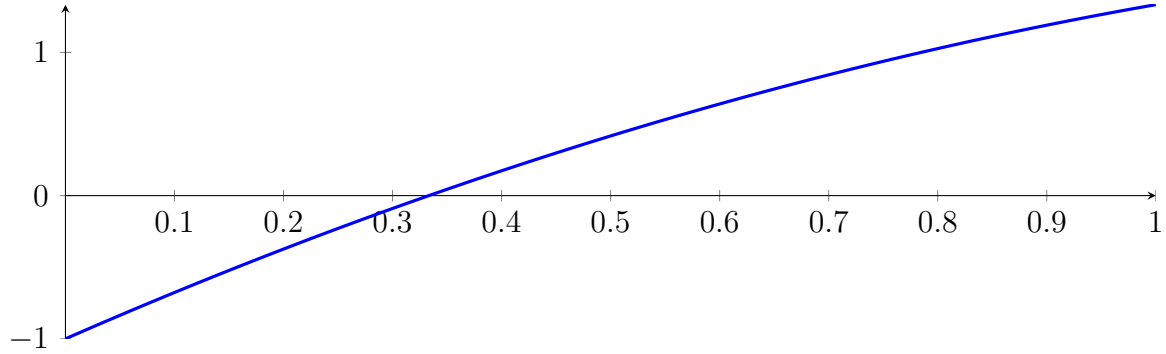
with precision $\varepsilon = 0.0001$.

6 Running CubeClip on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

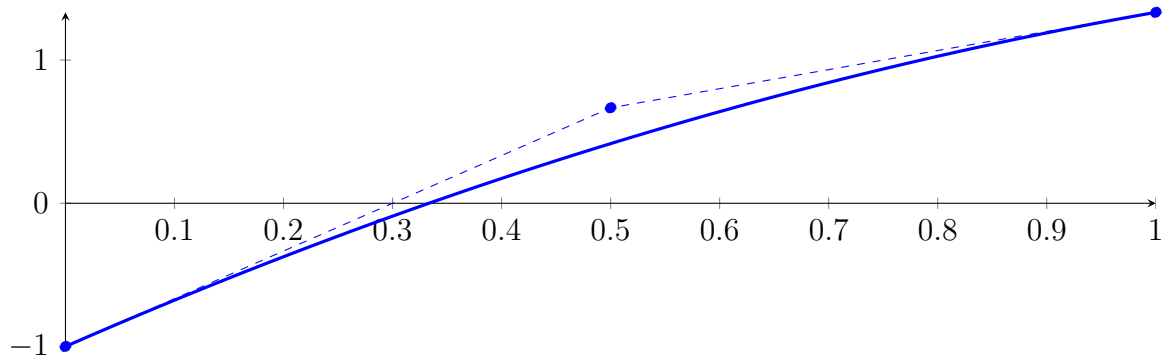
$$p = -1X^2 + 3.33333X - 1$$



6.1 Recursion Branch 1 for Input Interval $[0, 1]$

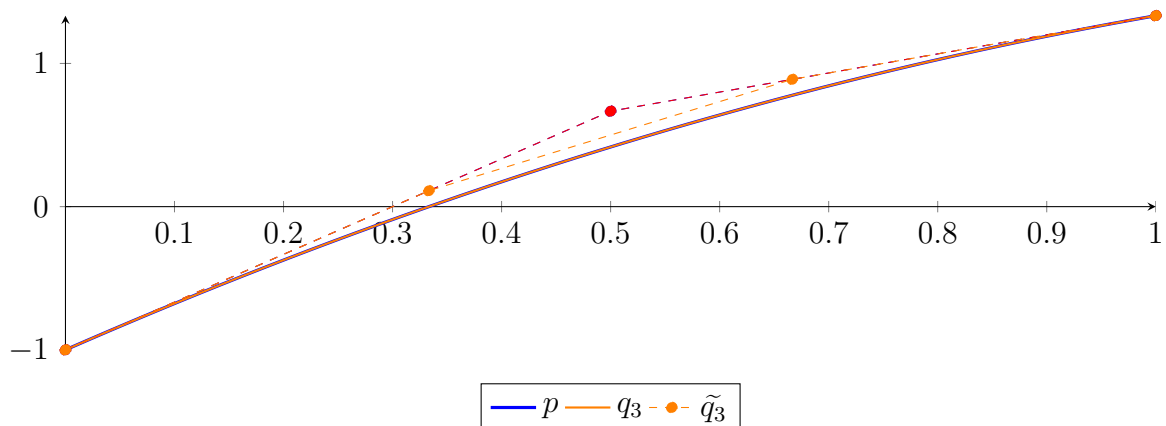
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.90958 \cdot 10^{-14}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.

Bounding polynomials M and m :

$$M = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

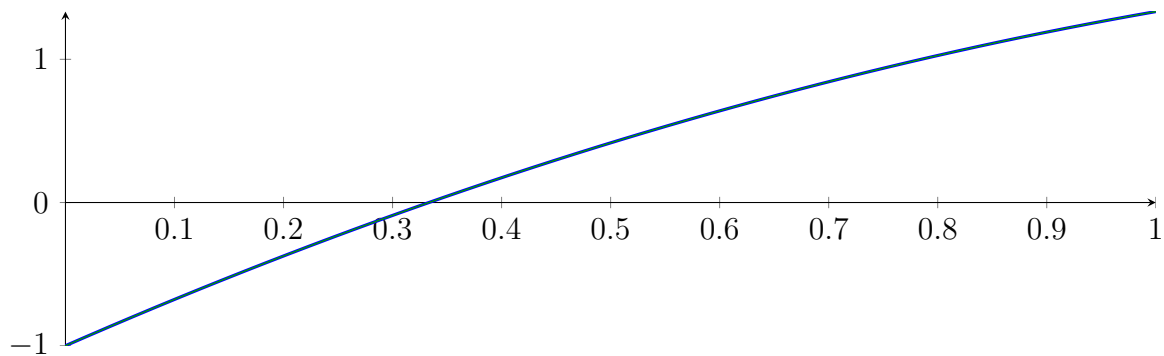
$$m = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-5.17655 \cdot 10^{13}\}$$

$$N(m) = \{-5.17655 \cdot 10^{13}\}$$

Intersection intervals:

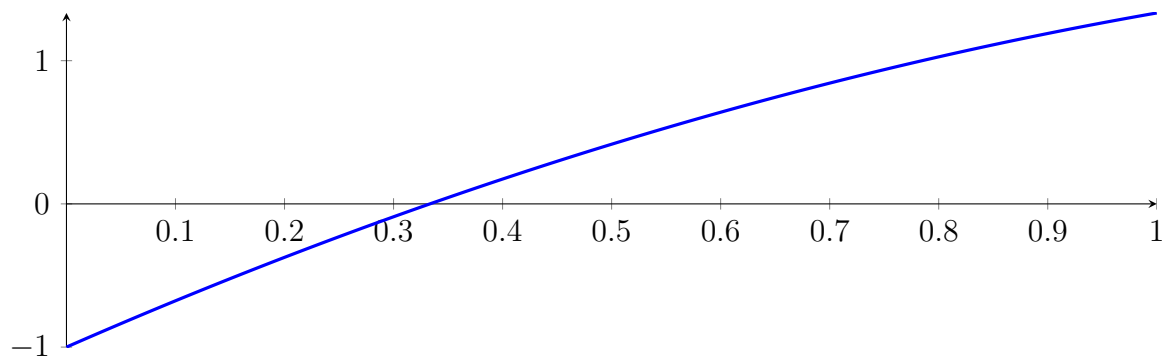


No intersection intervals with the x axis.

6.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

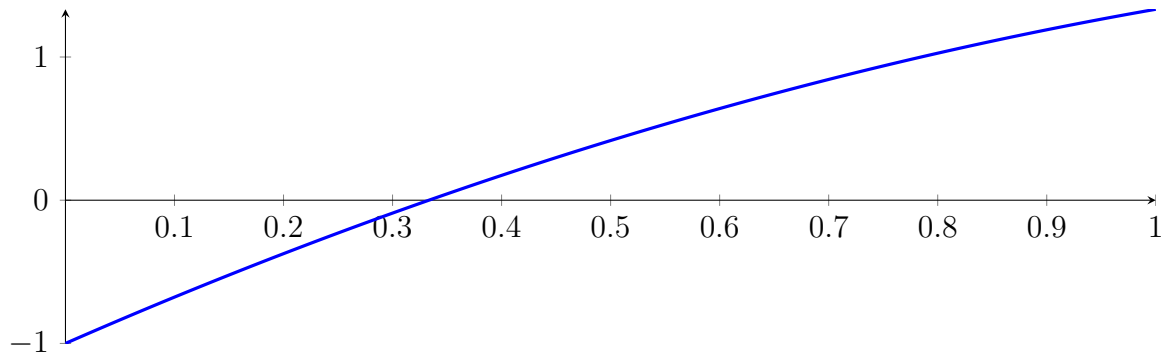
with precision $\varepsilon = 0.0001$.

7 Running BezClip on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

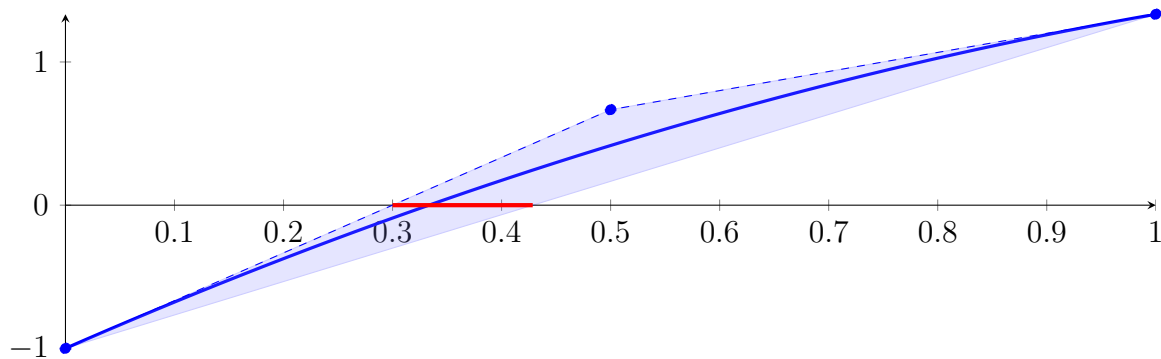
$$p = -1X^2 + 3.33333X - 1$$



7.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

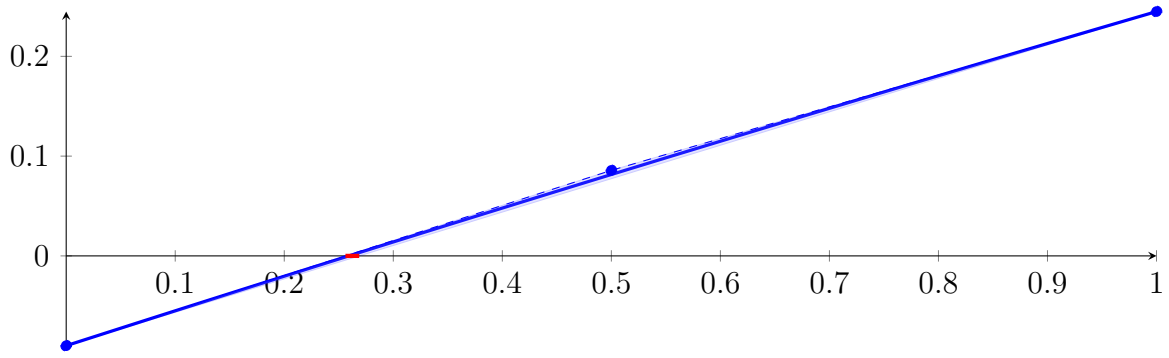
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

7.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

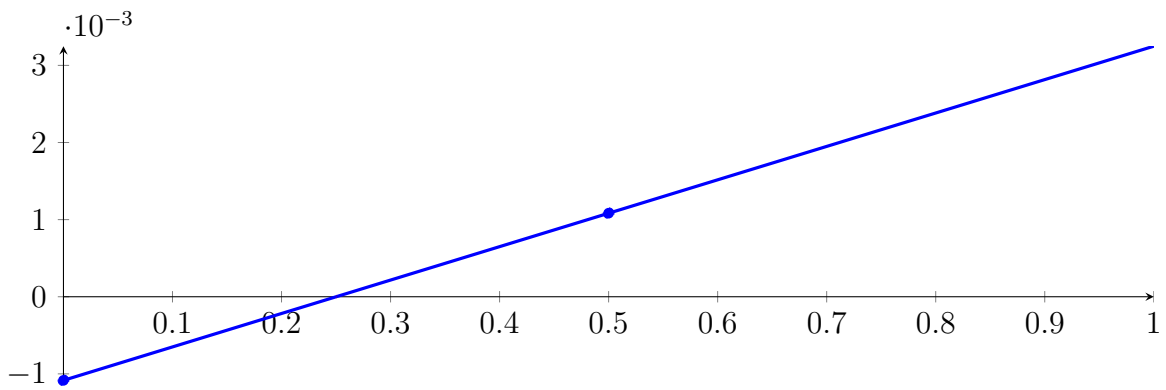
Longest intersection interval: 0.012641

\implies Selective recursion: interval 1: $[0.332927, 0.334552]$,

7.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

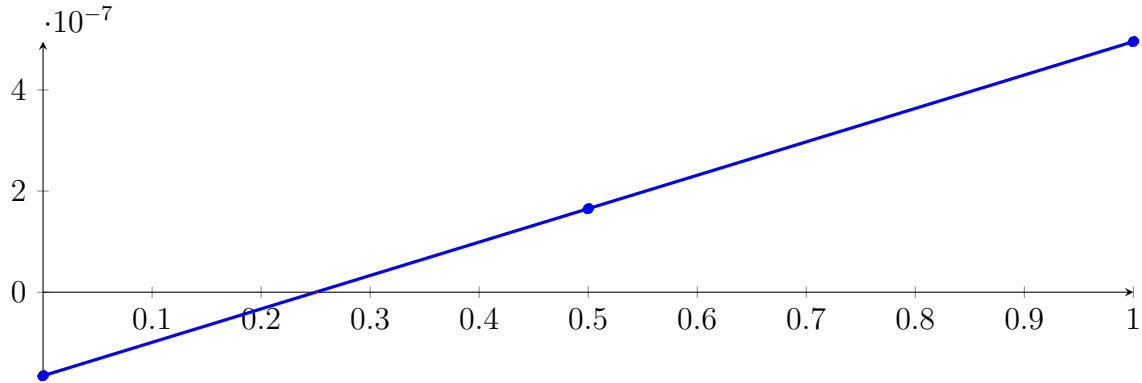
Longest intersection interval: 0.000152462

\implies Selective recursion: interval 1: $[0.333333, 0.333334]$,

7.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07} \\ &= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $2.32306 \cdot 10^{-08}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

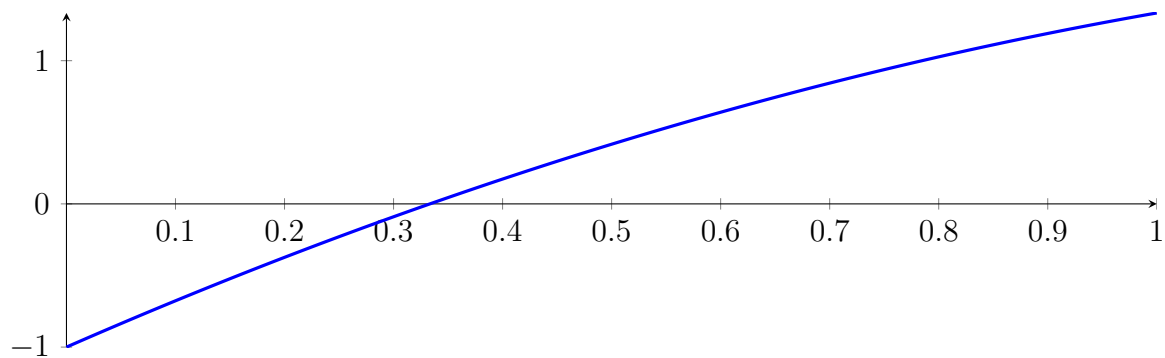
7.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

7.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

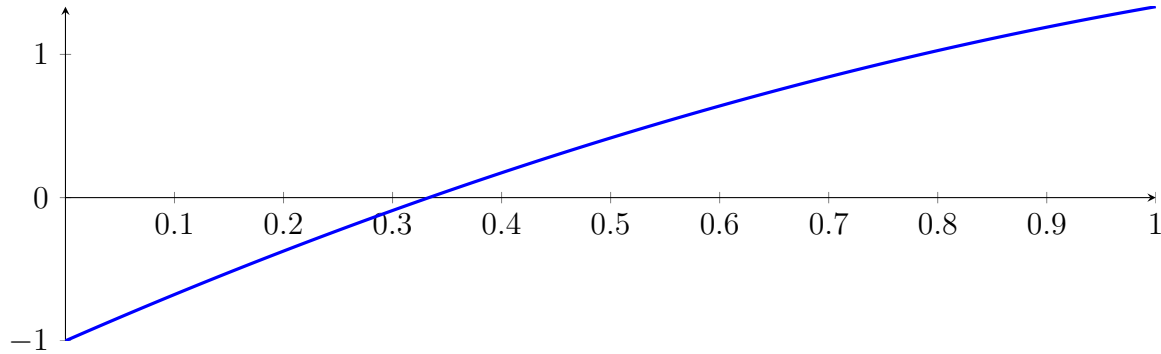
with precision $\varepsilon = 1 \cdot 10^{-08}$.

8 Running QuadClip on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

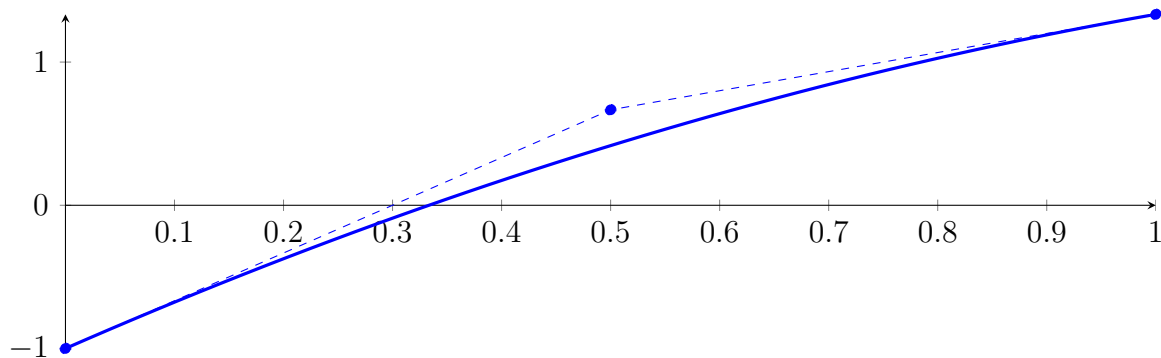
$$p = -1X^2 + 3.33333X - 1$$



8.1 Recursion Branch 1 for Input Interval $[0, 1]$

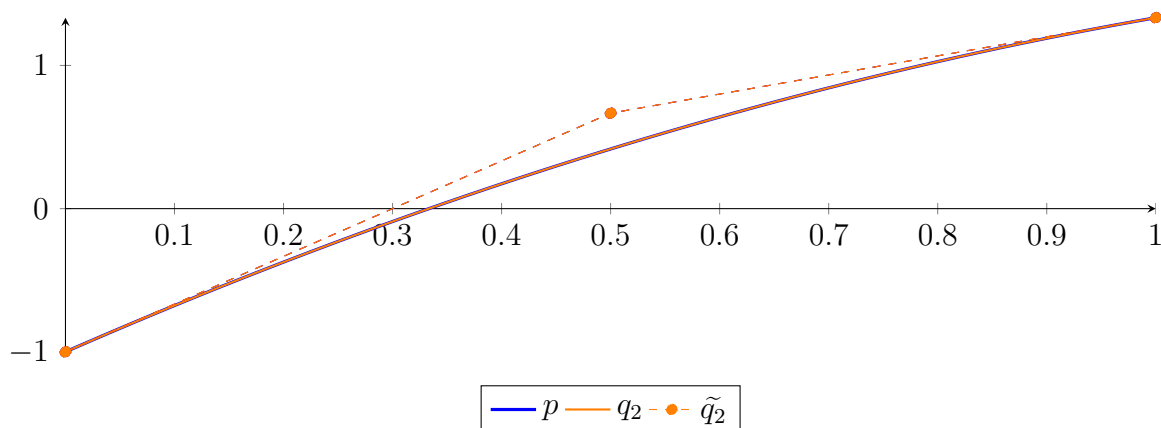
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

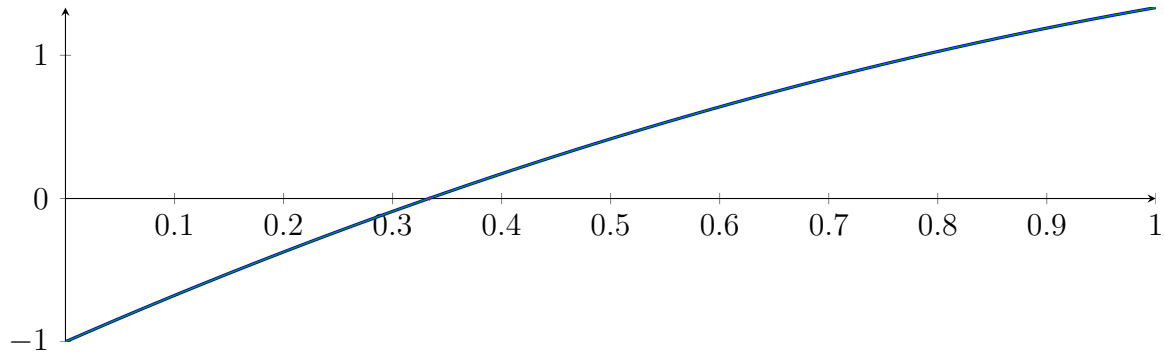
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $4.44089 \cdot 10^{-16}$

\implies Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

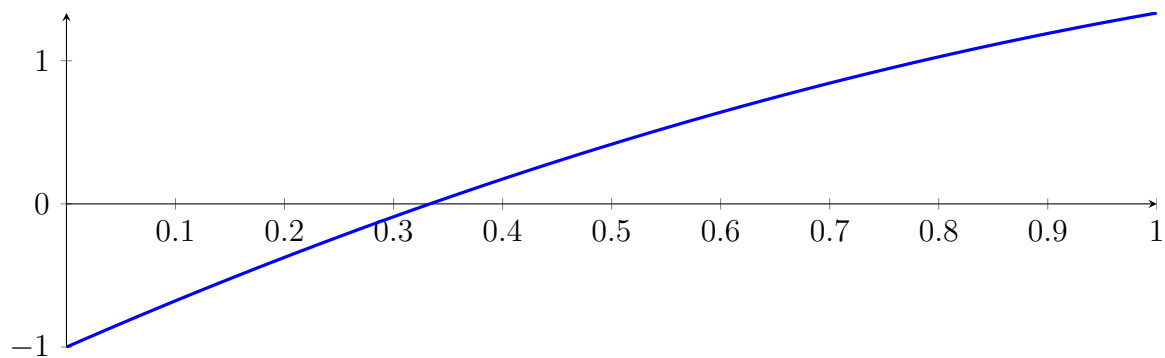
8.2 Recursion Branch 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 2!

8.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

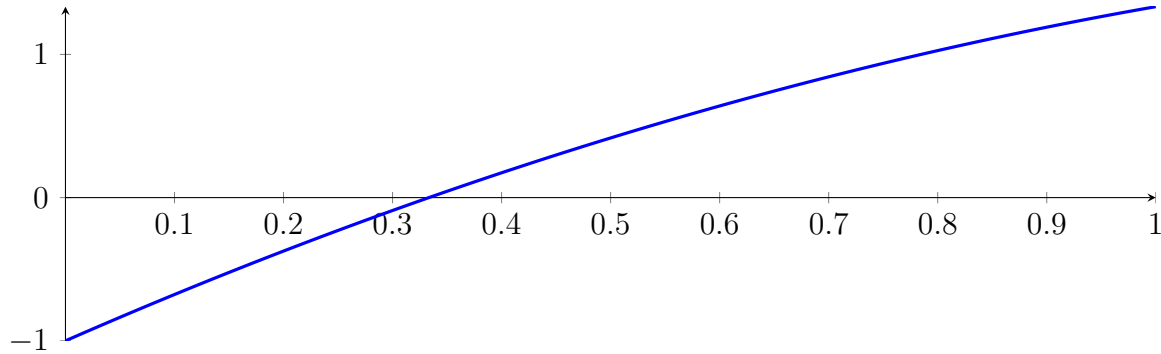
with precision $\varepsilon = 1 \cdot 10^{-08}$.

9 Running CubeClip on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

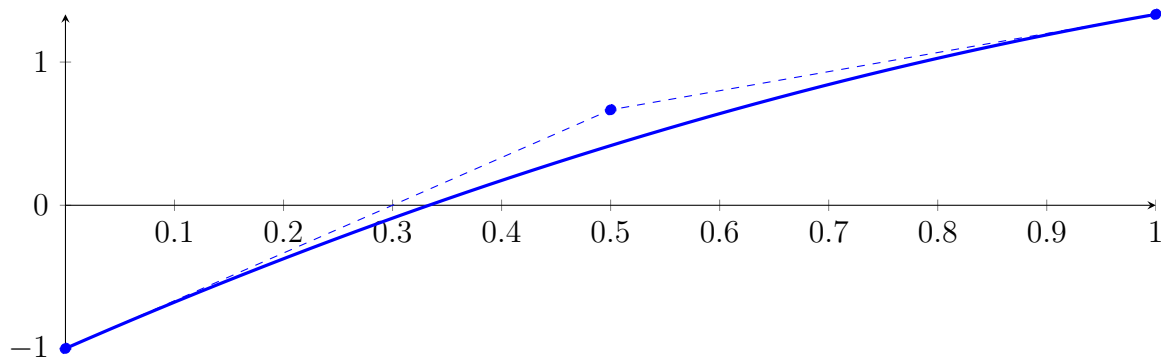
$$p = -1X^2 + 3.33333X - 1$$



9.1 Recursion Branch 1 for Input Interval $[0, 1]$

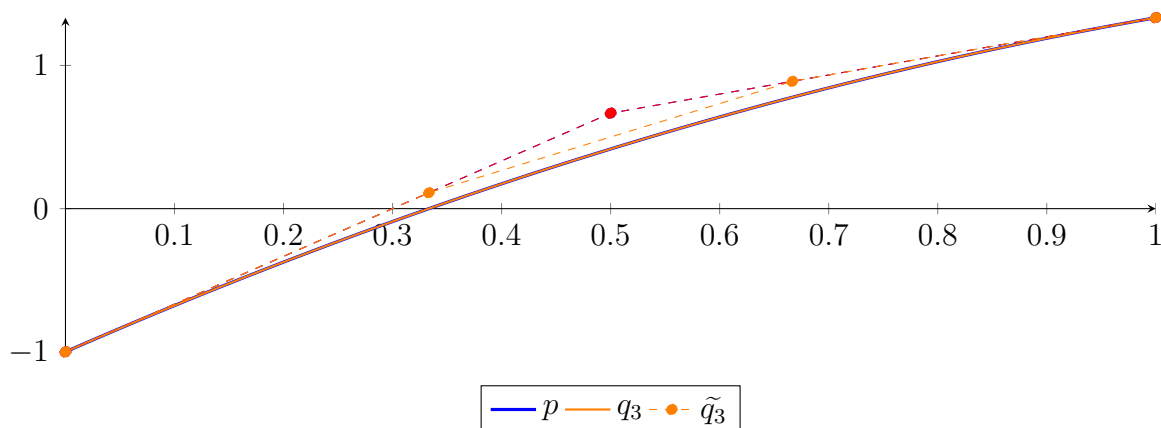
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.90958 \cdot 10^{-14}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.

Bounding polynomials M and m :

$$M = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

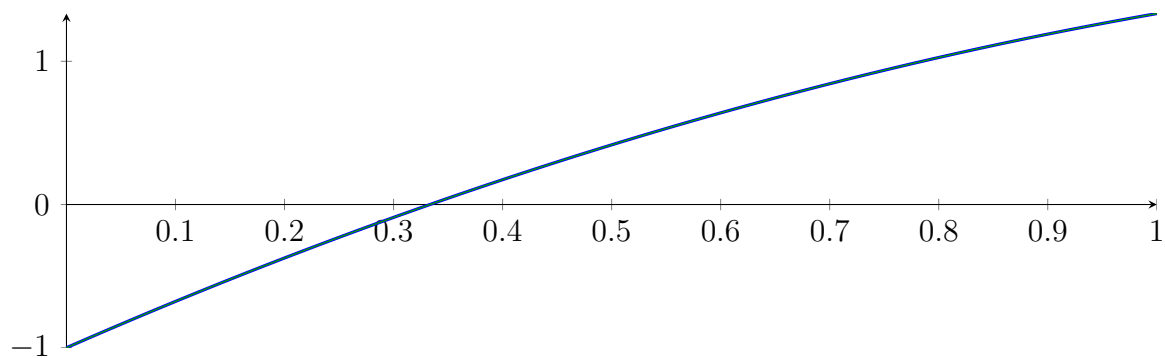
$$m = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-5.17655 \cdot 10^{13}\}$$

$$N(m) = \{-5.17655 \cdot 10^{13}\}$$

Intersection intervals:

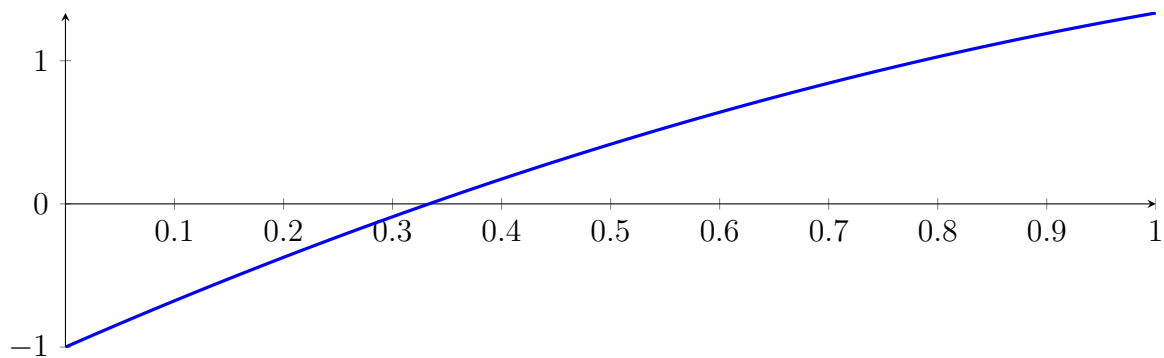


No intersection intervals with the x axis.

9.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

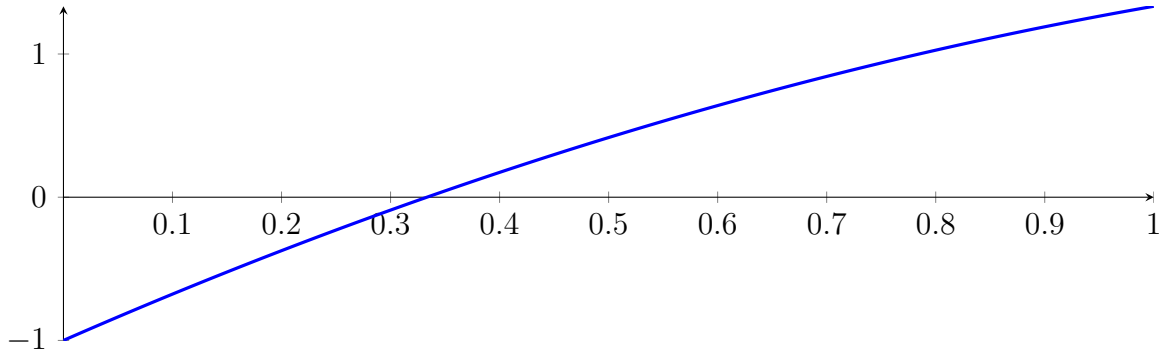
with precision $\varepsilon = 1 \cdot 10^{-08}$.

10 Running BezClip on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

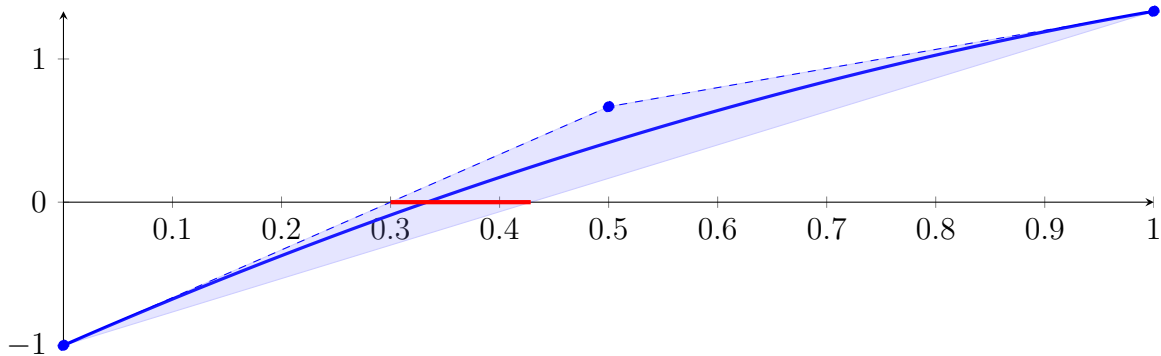
$$p = -1X^2 + 3.33333X - 1$$



10.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

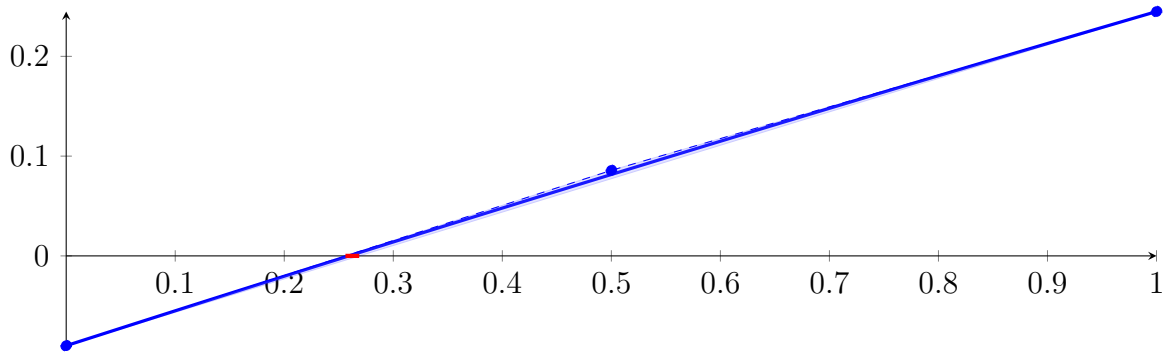
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

10.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

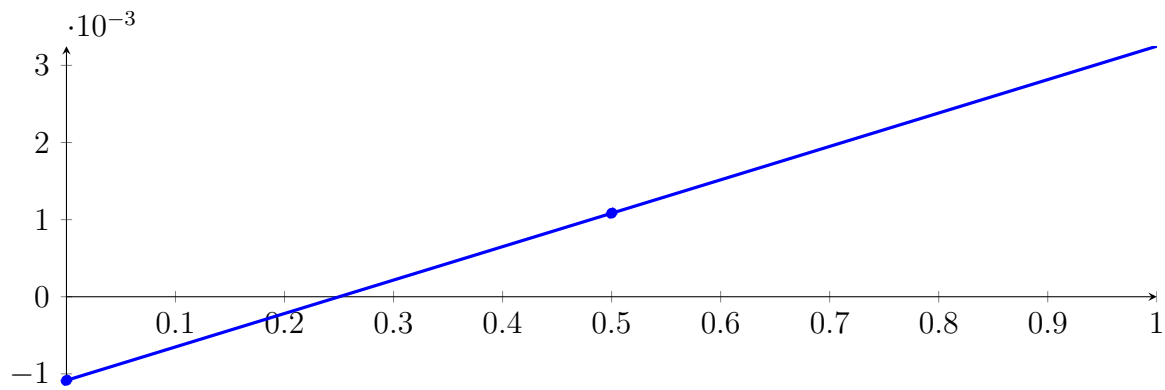
Longest intersection interval: 0.012641

\Rightarrow Selective recursion: interval 1: $[0.332927, 0.334552]$,

10.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

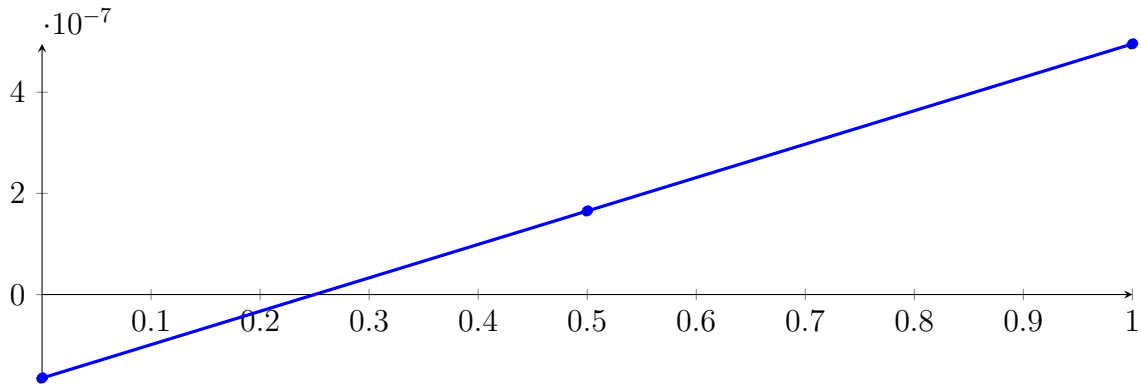
Longest intersection interval: 0.000152462

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333334]$,

10.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07} \\
 &= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

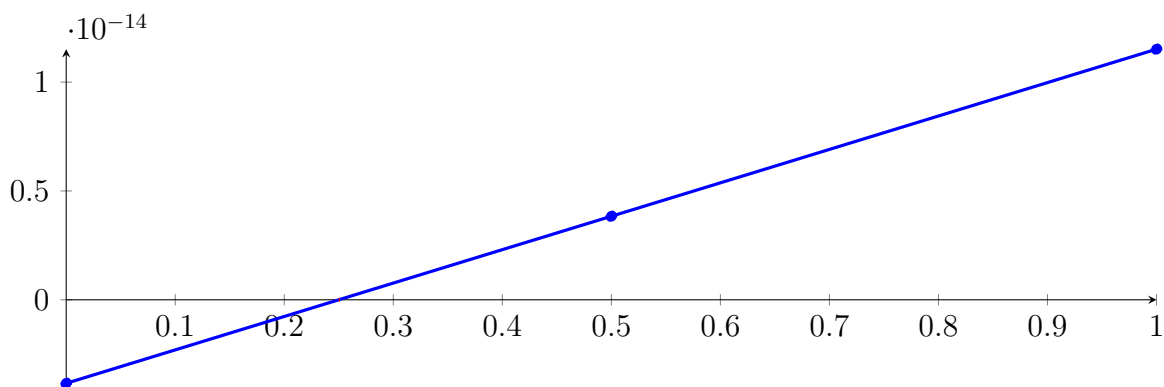
Longest intersection interval: $2.32306 \cdot 10^{-08}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

10.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.31322 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15} \\
 &= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.55112 \cdot 10^{-16}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

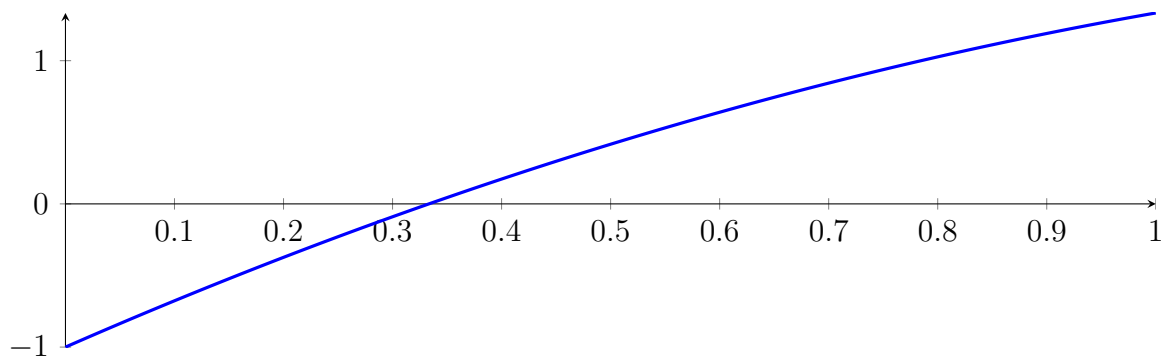
10.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

10.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

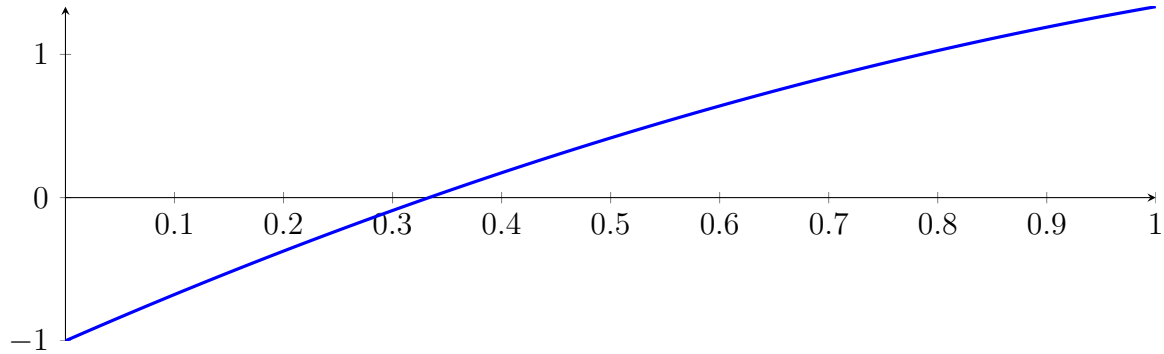
with precision $\varepsilon = 1 \cdot 10^{-16}$.

11 Running QuadClip on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

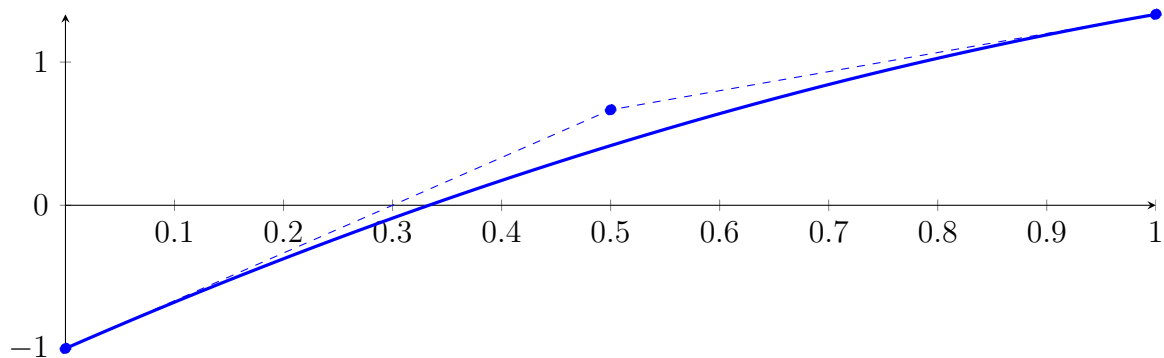
$$p = -1X^2 + 3.33333X - 1$$



11.1 Recursion Branch 1 for Input Interval $[0, 1]$

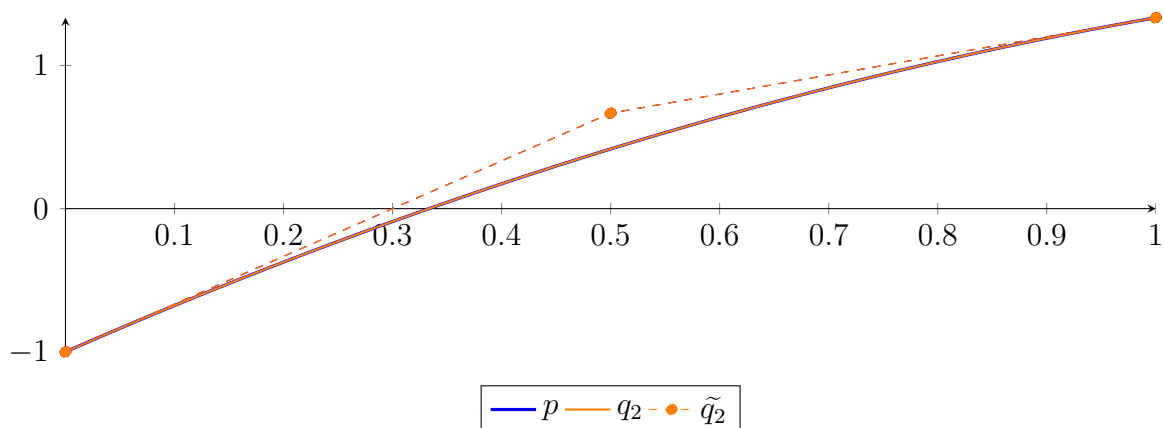
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

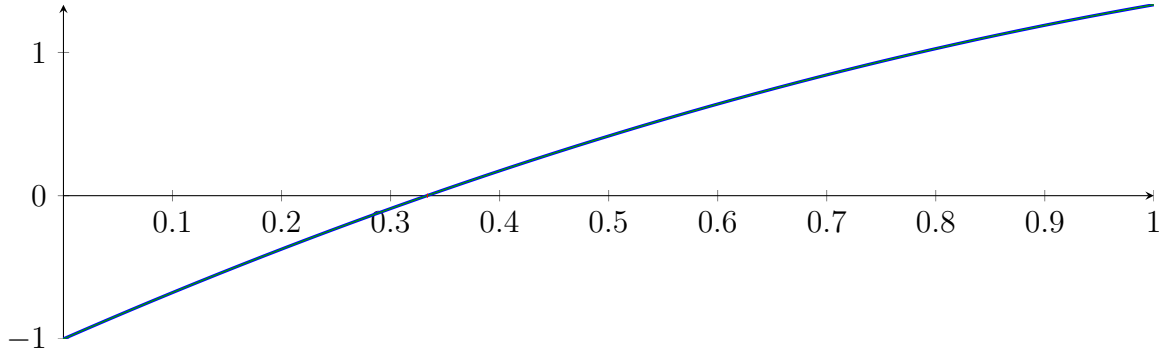
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

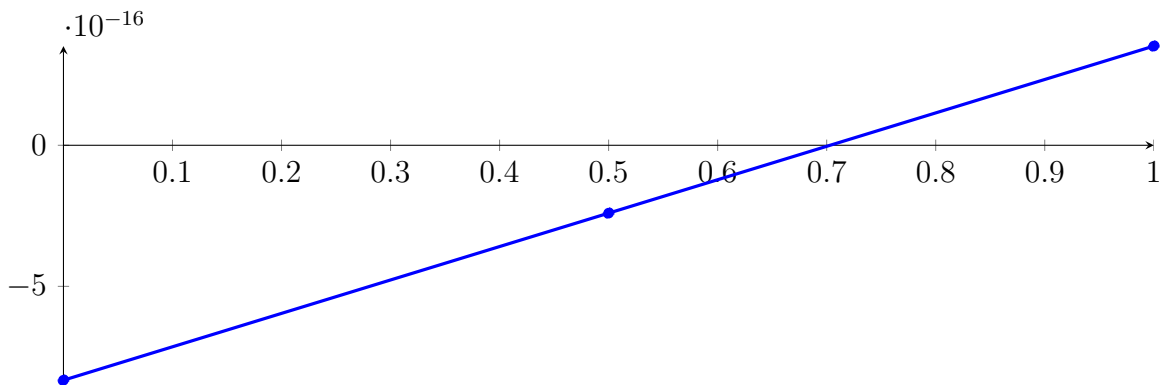
Longest intersection interval: $4.44089 \cdot 10^{-16}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

11.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

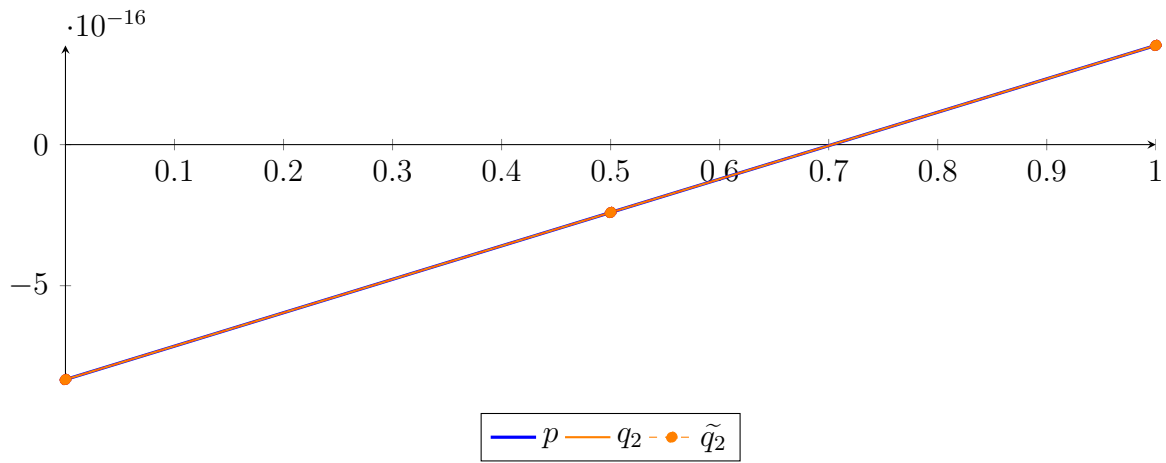
$$\begin{aligned} p &= -1.97215 \cdot 10^{-31} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2}(X) - 2.40548 \cdot 10^{-16} B_{1,2}(X) + 3.51571 \cdot 10^{-16} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.18329 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2} - 2.40548 \cdot 10^{-16} B_{1,2} + 3.51571 \cdot 10^{-16} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 2.5638 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2} - 2.40548 \cdot 10^{-16} B_{1,2} + 3.51571 \cdot 10^{-16} B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.2326 \cdot 10^{-30}$.

Bounding polynomials M and m :

$$M = 1.08468 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$

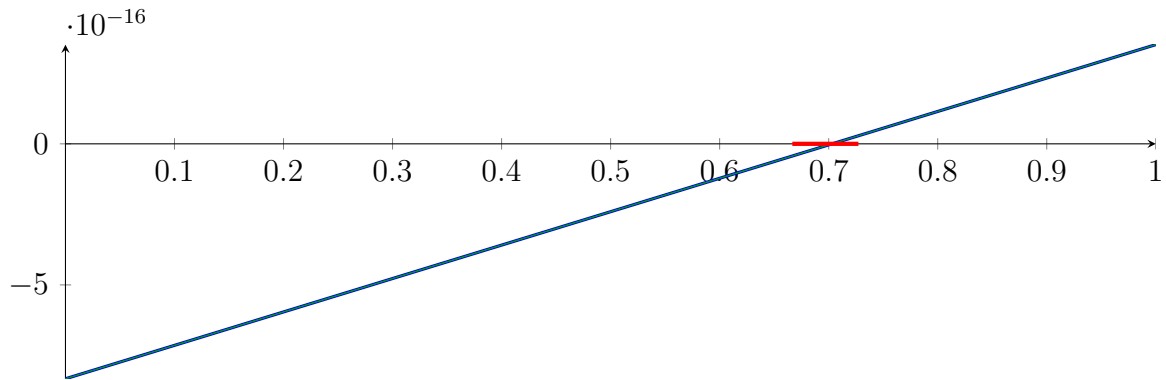
$$m = 1.18329 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$

Root of M and m :

$$N(M) = \{-1.09178 \cdot 10^{15}, 0.727273\}$$

$$N(m) = \{-1.0008 \cdot 10^{15}, 0.666667\}$$

Intersection intervals:



$$[0.666667, 0.727273]$$

Longest intersection interval: 0.0606061

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

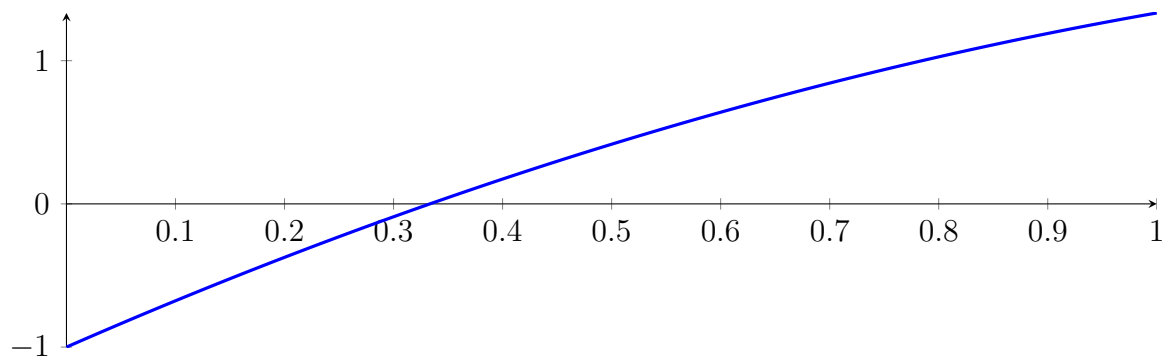
11.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

11.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

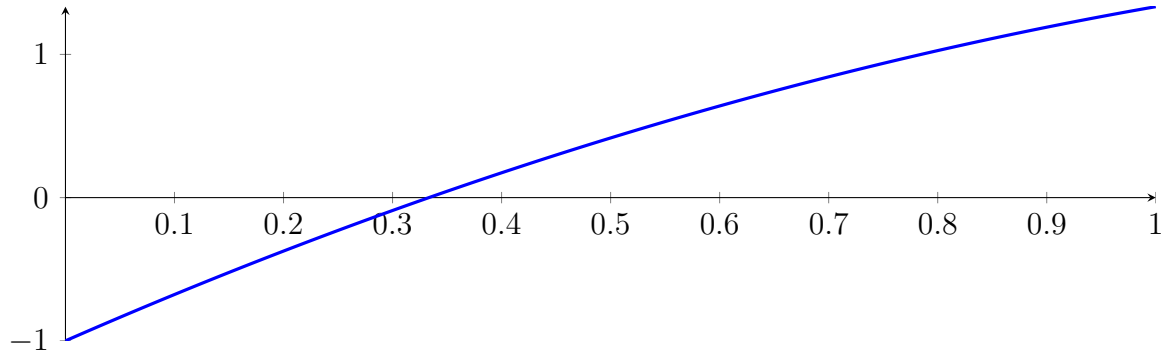
with precision $\varepsilon = 1 \cdot 10^{-16}$.

12 Running CubeClip on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

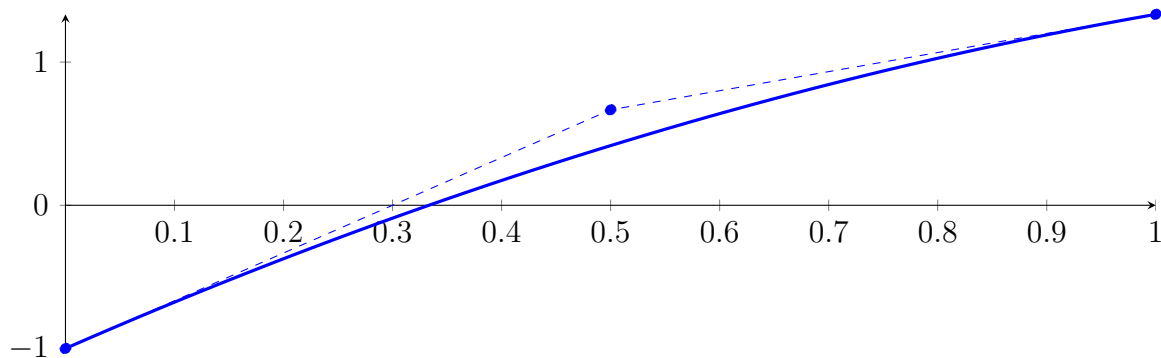
$$p = -1X^2 + 3.33333X - 1$$



12.1 Recursion Branch 1 for Input Interval $[0, 1]$

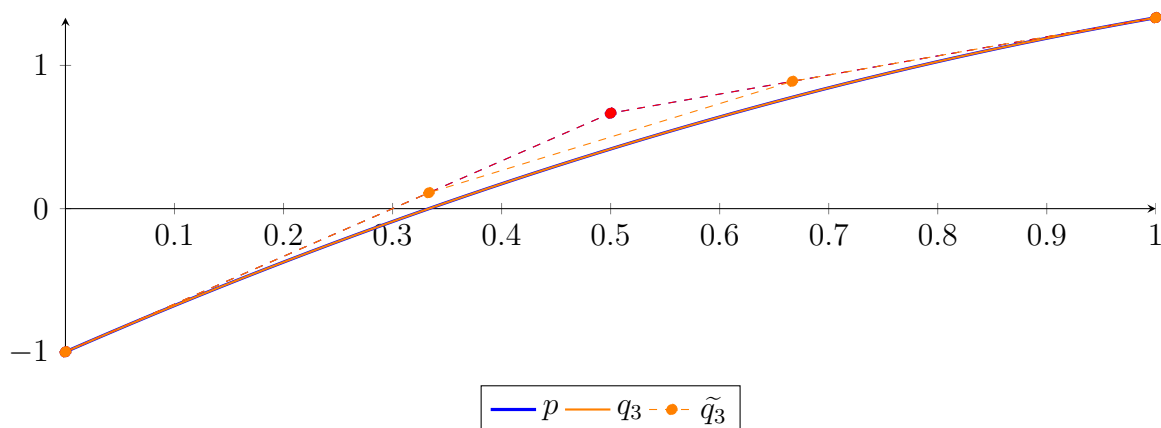
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.90958 \cdot 10^{-14}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.

Bounding polynomials M and m :

$$M = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

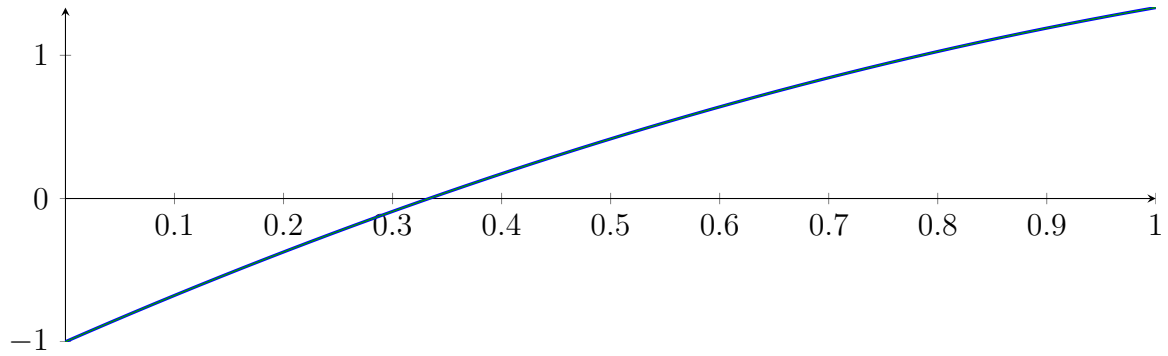
$$m = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-5.17655 \cdot 10^{13}\}$$

$$N(m) = \{-5.17655 \cdot 10^{13}\}$$

Intersection intervals:

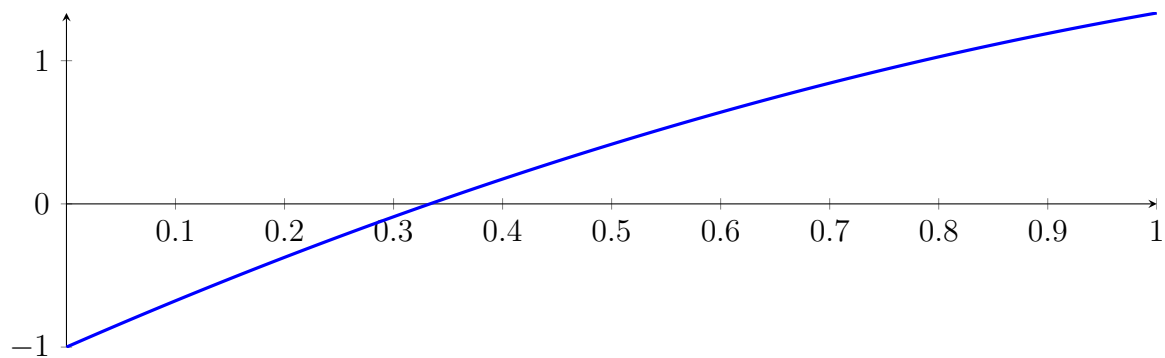


No intersection intervals with the x axis.

12.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

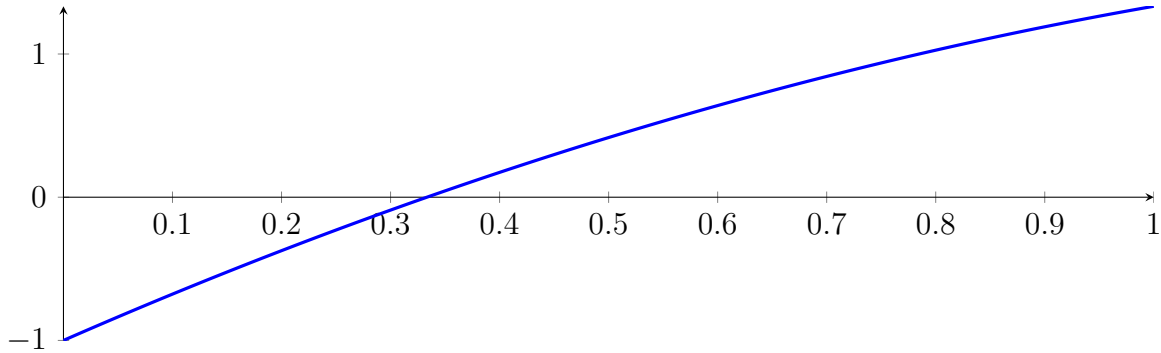
with precision $\varepsilon = 1 \cdot 10^{-16}$.

13 Running BezClip on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

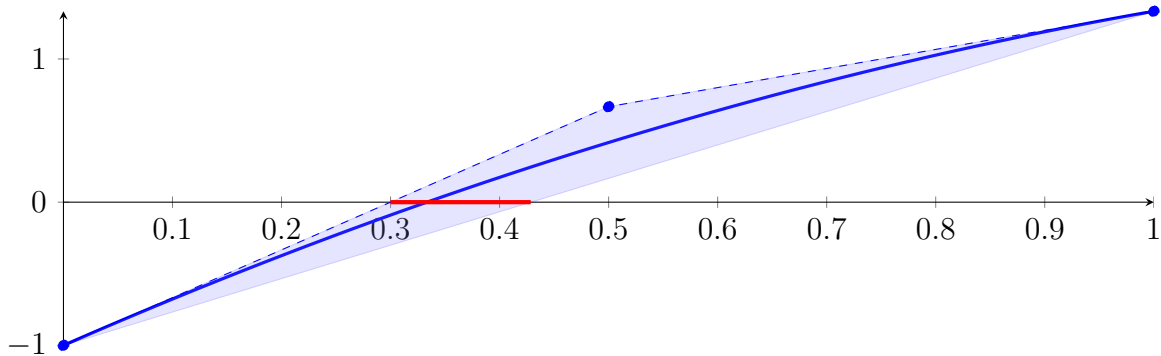
$$p = -1X^2 + 3.33333X - 1$$



13.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

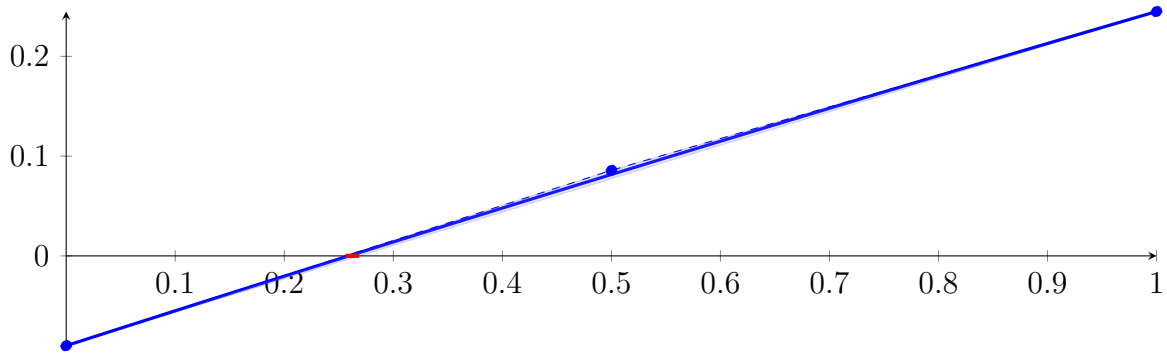
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

13.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

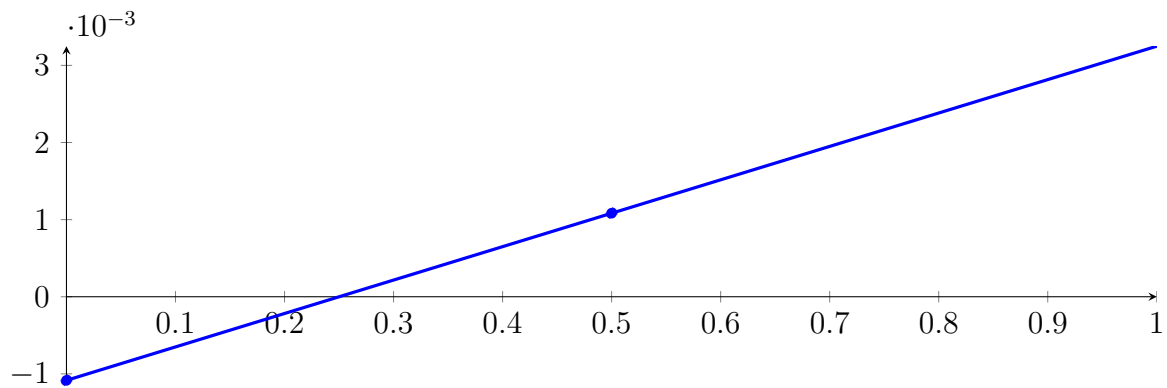
Longest intersection interval: 0.012641

\Rightarrow Selective recursion: interval 1: $[0.332927, 0.334552]$,

13.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

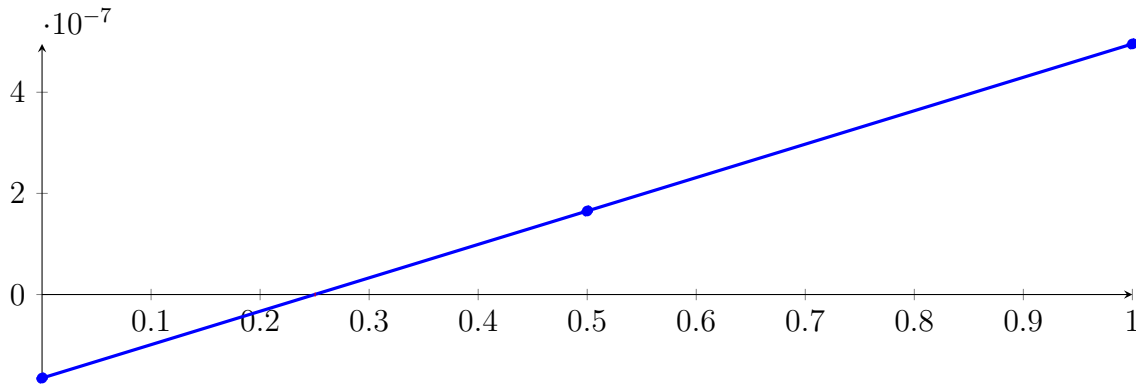
Longest intersection interval: 0.000152462

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333334]$,

13.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07} \\
 &= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

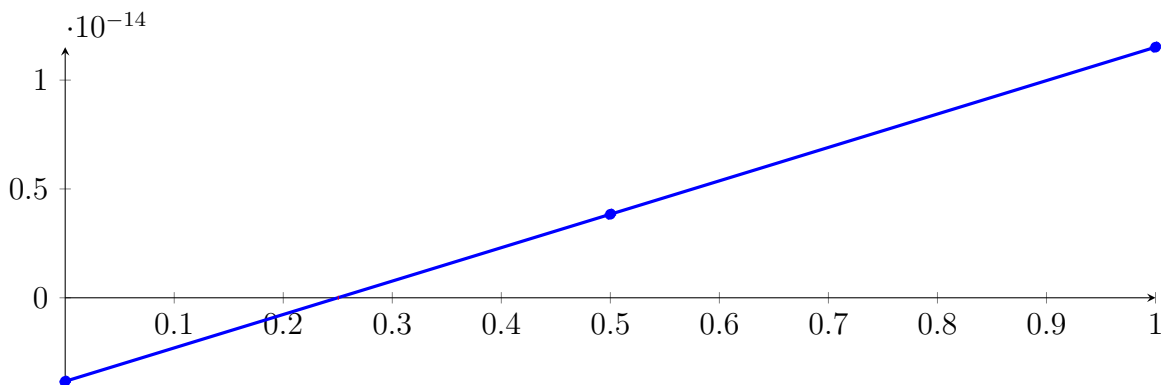
Longest intersection interval: $2.32306 \cdot 10^{-08}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

13.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.31322 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15} \\
 &= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.55112 \cdot 10^{-16}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

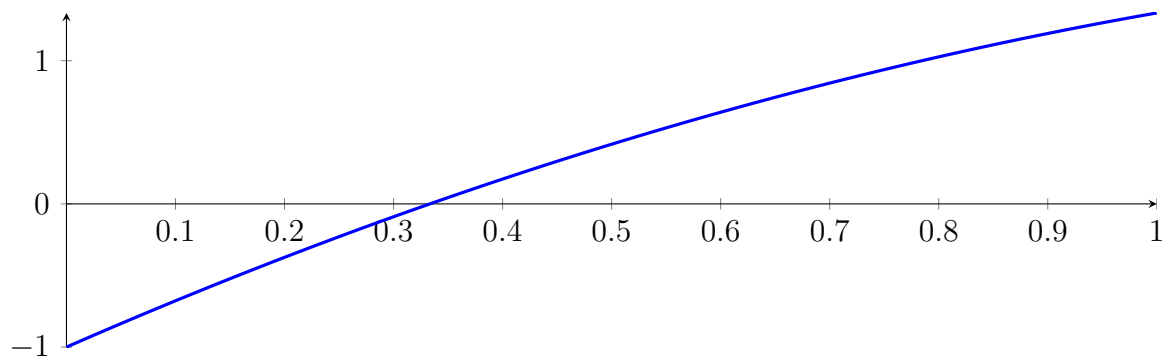
13.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

13.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

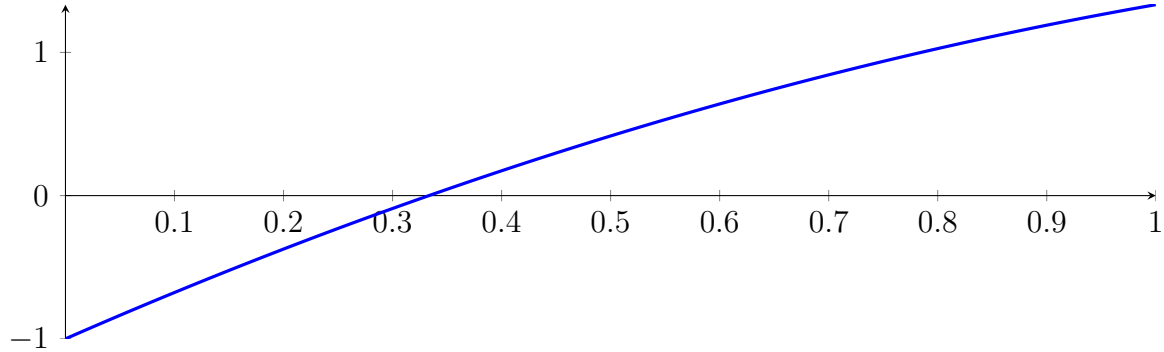
with precision $\varepsilon = 1 \cdot 10^{-32}$.

14 Running QuadClip on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

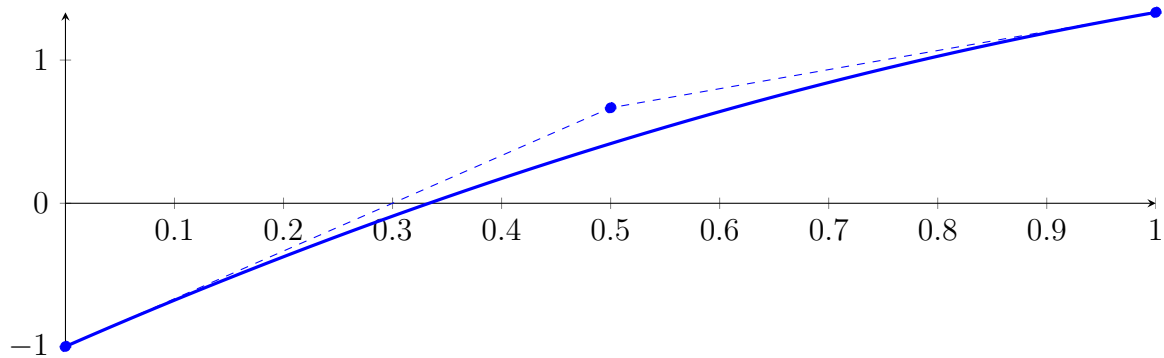
$$p = -1X^2 + 3.33333X - 1$$



14.1 Recursion Branch 1 for Input Interval $[0, 1]$

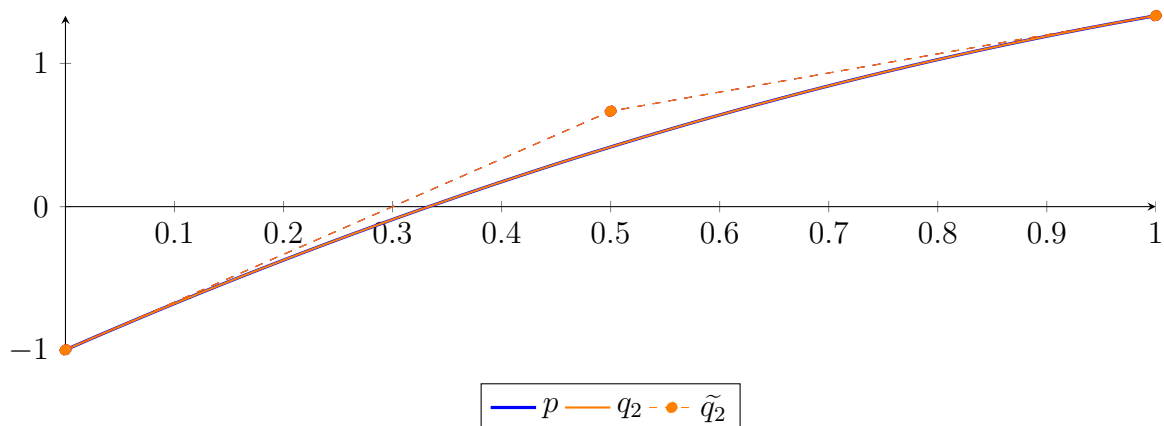
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

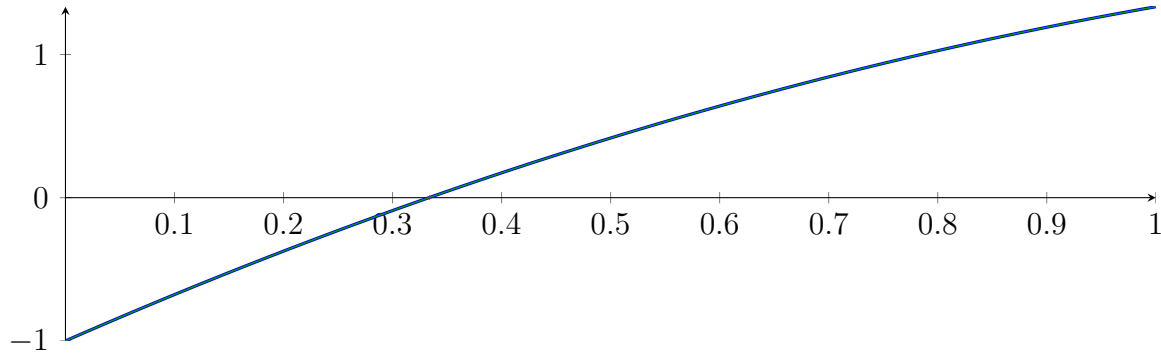
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

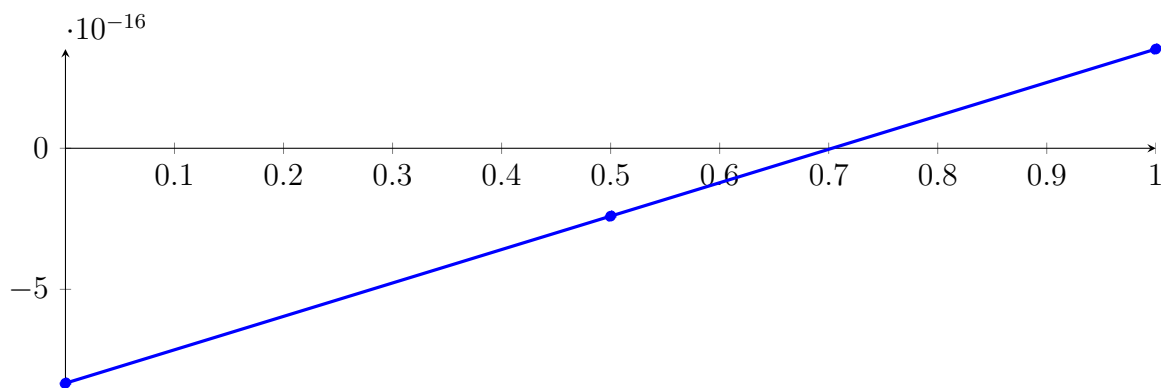
Longest intersection interval: $4.44089 \cdot 10^{-16}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

14.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

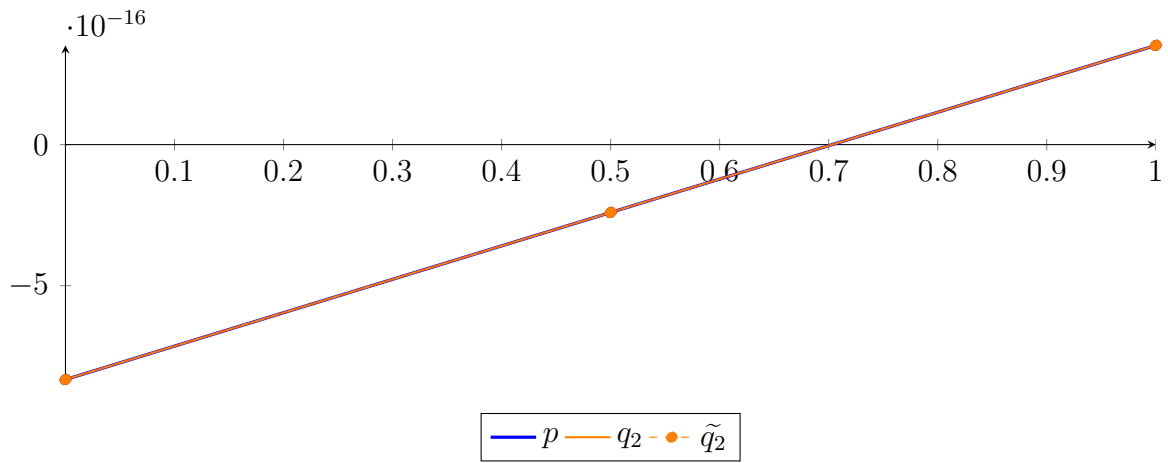
$$\begin{aligned} p &= -1.97215 \cdot 10^{-31} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2}(X) - 2.40548 \cdot 10^{-16} B_{1,2}(X) + 3.51571 \cdot 10^{-16} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.18329 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2} - 2.40548 \cdot 10^{-16} B_{1,2} + 3.51571 \cdot 10^{-16} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 2.5638 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2} - 2.40548 \cdot 10^{-16} B_{1,2} + 3.51571 \cdot 10^{-16} B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.2326 \cdot 10^{-30}$.

Bounding polynomials M and m :

$$M = 1.08468 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$

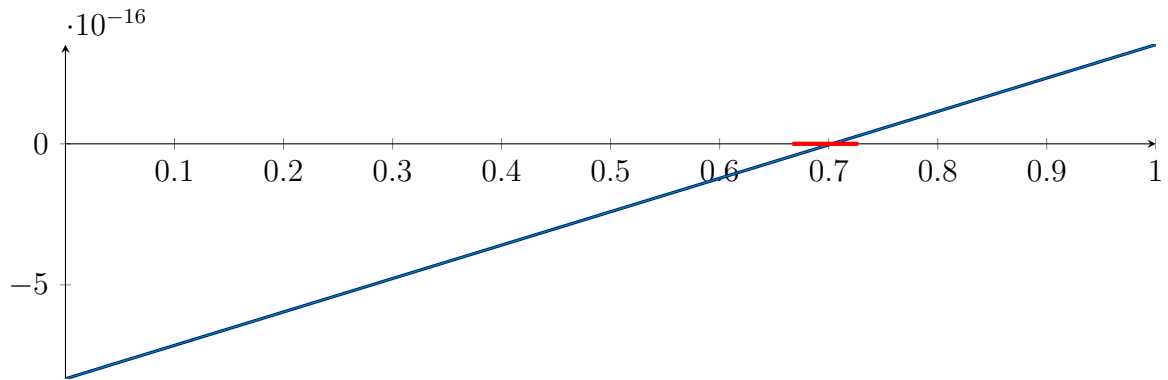
$$m = 1.18329 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$

Root of M and m :

$$N(M) = \{-1.09178 \cdot 10^{15}, 0.727273\}$$

$$N(m) = \{-1.0008 \cdot 10^{15}, 0.666667\}$$

Intersection intervals:



$$[0.666667, 0.727273]$$

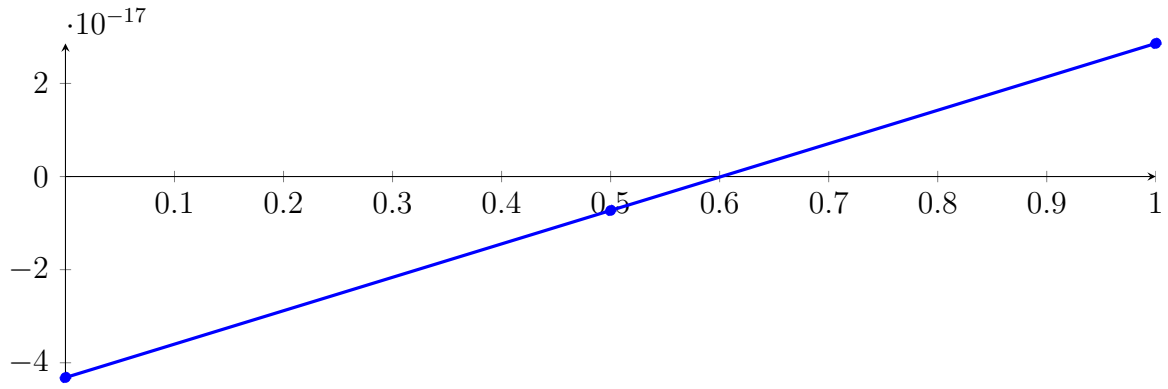
Longest intersection interval: 0.0606061

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

14.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

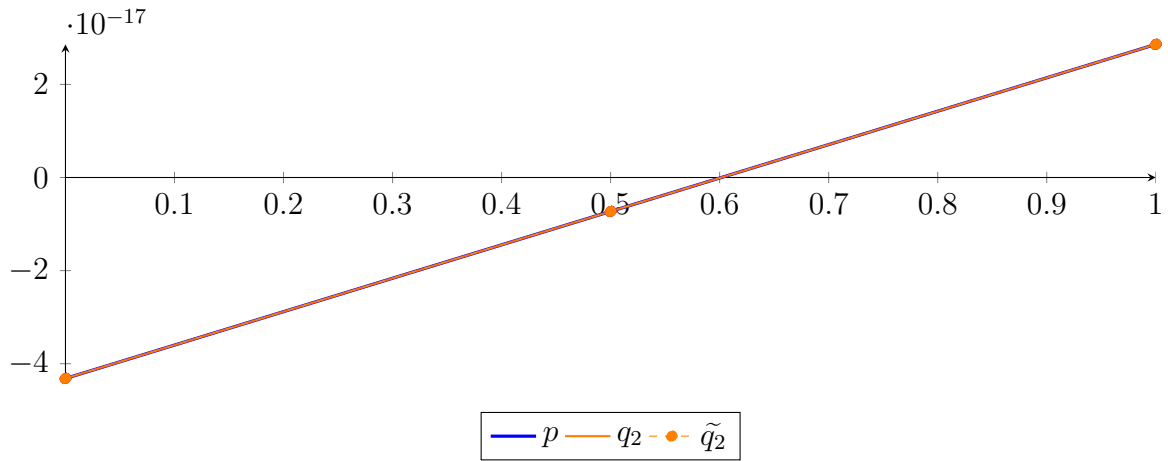
$$\begin{aligned} p &= 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17} \\ &= -4.31753 \cdot 10^{-17} B_{0,2}(X) - 7.28934 \cdot 10^{-18} B_{1,2}(X) + 2.85967 \cdot 10^{-17} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 5.54668 \cdot 10^{-32} X^2 + 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17} \\ &= -4.31753 \cdot 10^{-17} B_{0,2} - 7.28934 \cdot 10^{-18} B_{1,2} + 2.85967 \cdot 10^{-17} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 1.10934 \cdot 10^{-31} X^2 + 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17} \\ &= -4.31753 \cdot 10^{-17} B_{0,2} - 7.28934 \cdot 10^{-18} B_{1,2} + 2.85967 \cdot 10^{-17} B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.93038 \cdot 10^{-32}$.

Bounding polynomials M and m :

$$M = 5.54668 \cdot 10^{-32} X^2 + 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17}$$

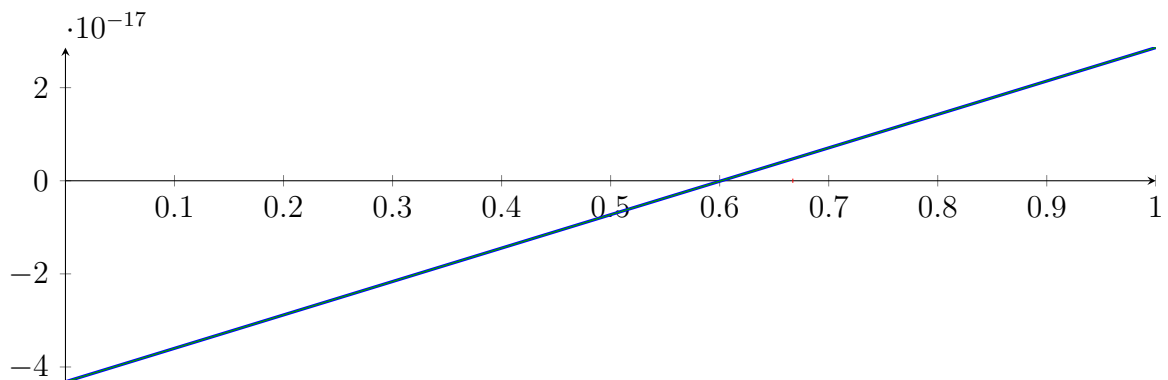
$$m = 5.54668 \cdot 10^{-32} X^2 + 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17}$$

Root of M and m :

$$N(M) = \{-1.29396 \cdot 10^{15}, 0.666667\}$$

$$N(m) = \{-1.29396 \cdot 10^{15}, 0.666667\}$$

Intersection intervals:



$[0.666667, 0.666667]$

Longest intersection interval: 0

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

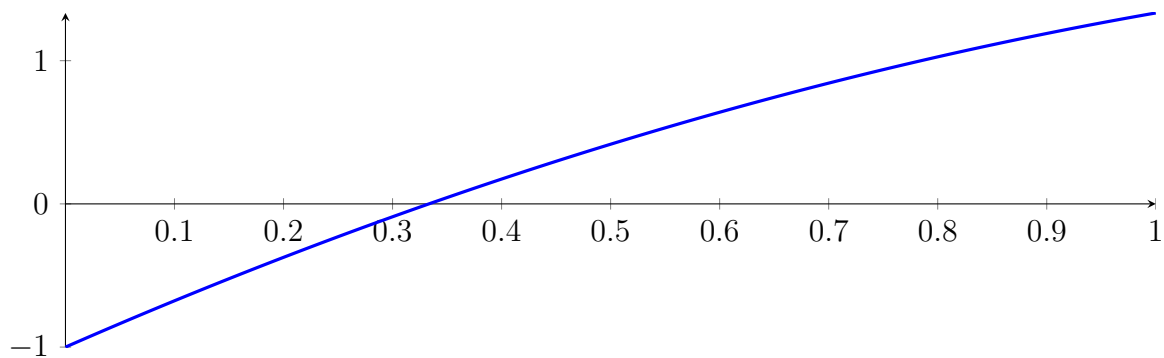
14.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

14.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

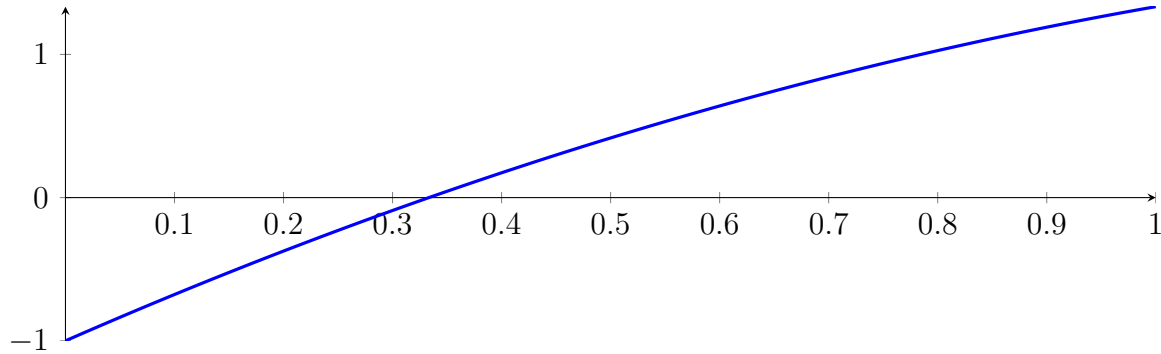
with precision $\varepsilon = 1 \cdot 10^{-32}$.

15 Running CubeClip on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

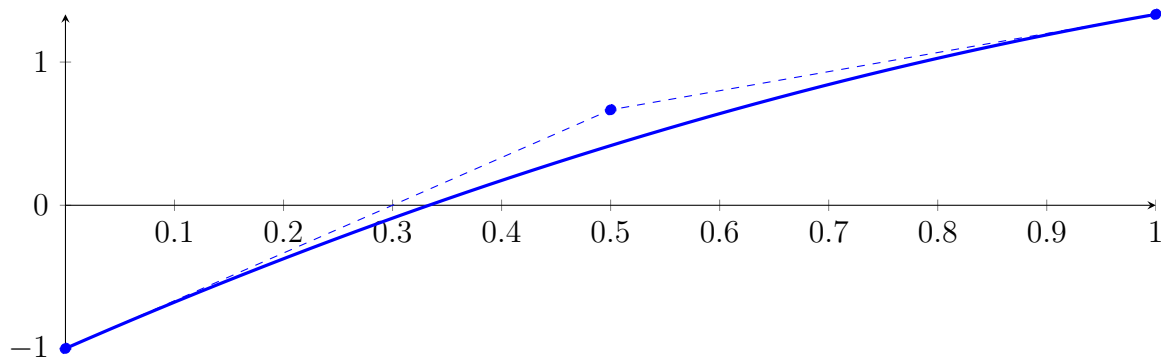
$$p = -1X^2 + 3.33333X - 1$$



15.1 Recursion Branch 1 for Input Interval $[0, 1]$

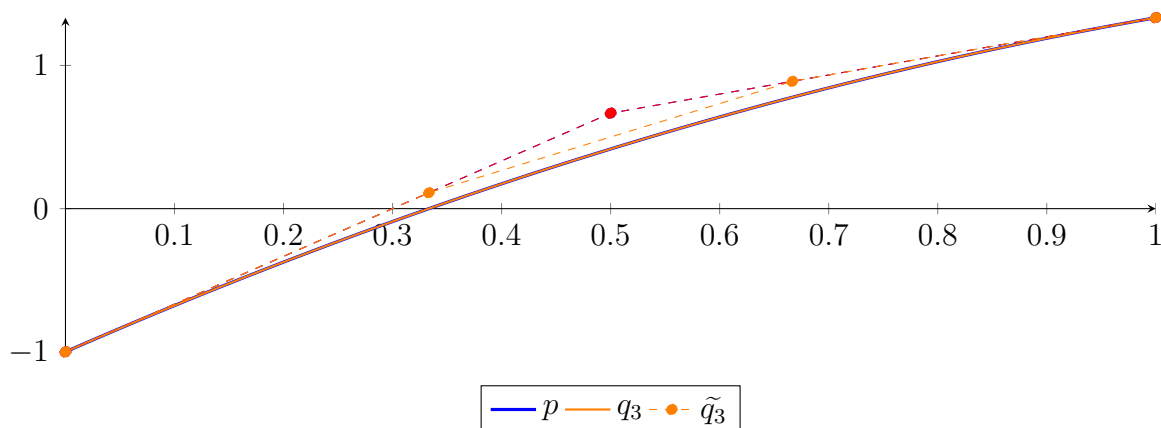
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.90958 \cdot 10^{-14}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.

Bounding polynomials M and m :

$$M = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

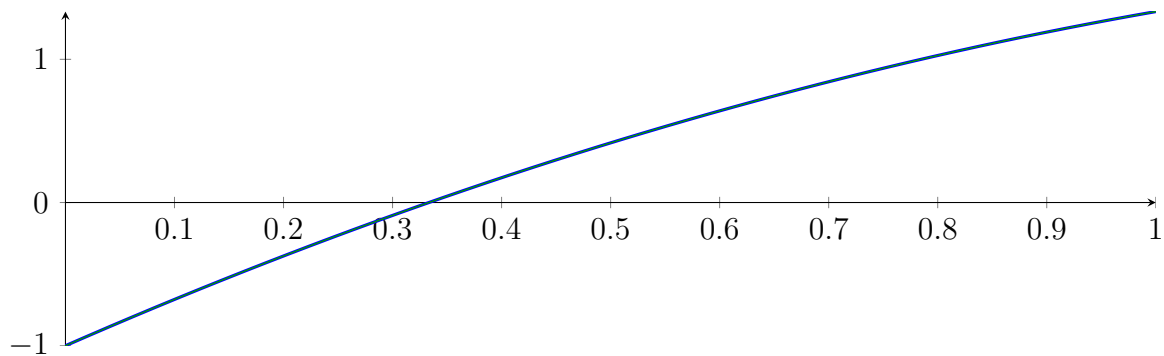
$$m = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-5.17655 \cdot 10^{13}\}$$

$$N(m) = \{-5.17655 \cdot 10^{13}\}$$

Intersection intervals:

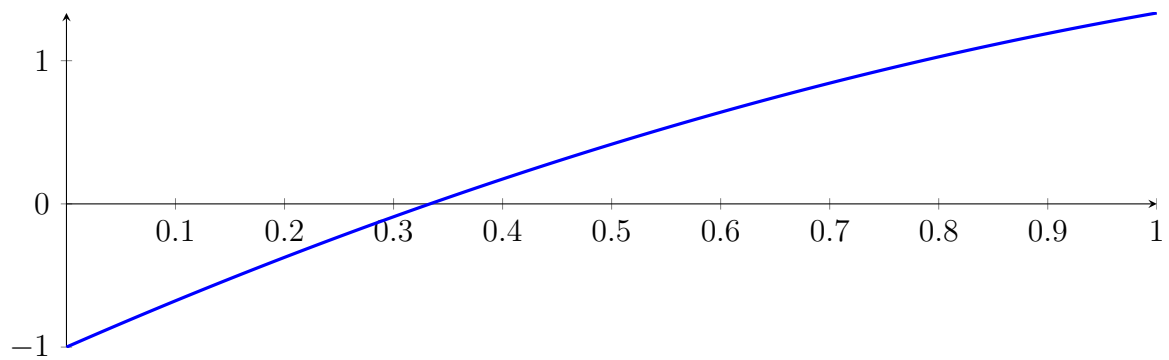


No intersection intervals with the x axis.

15.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

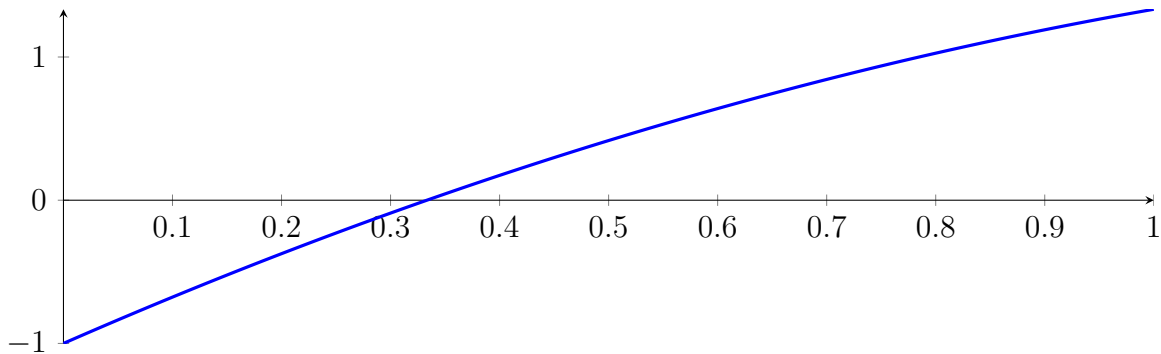
with precision $\varepsilon = 1 \cdot 10^{-32}$.

16 Running BezClip on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

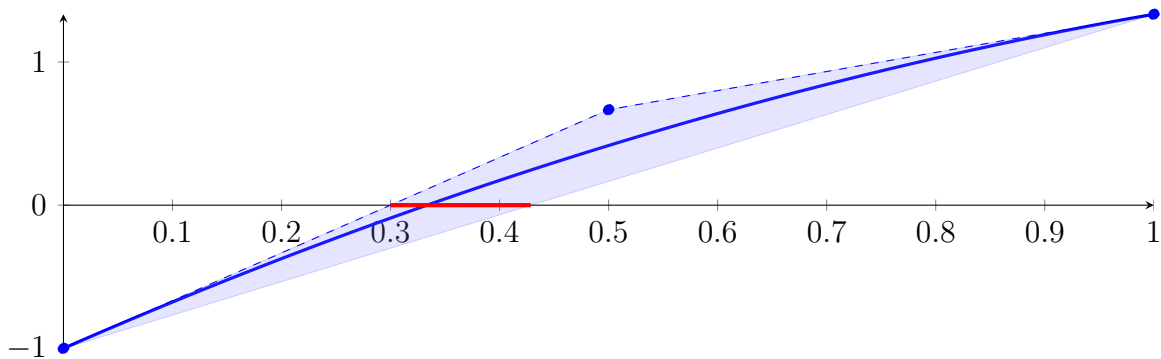
$$p = -1X^2 + 3.33333X - 1$$



16.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

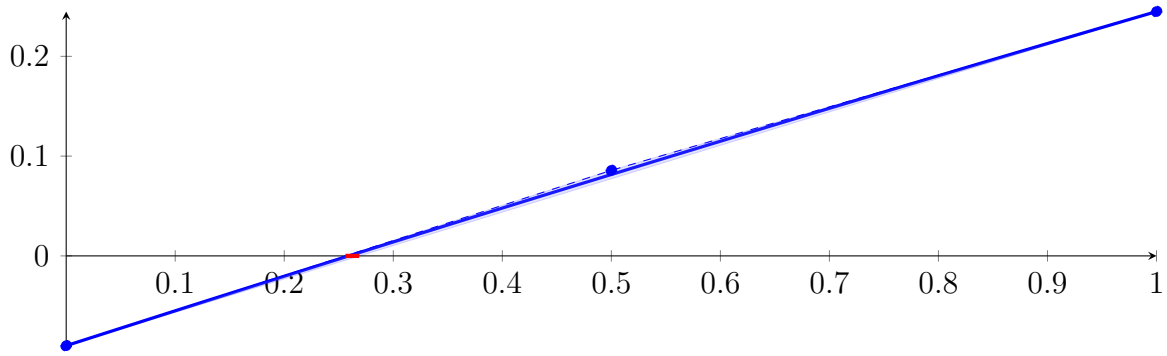
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

16.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

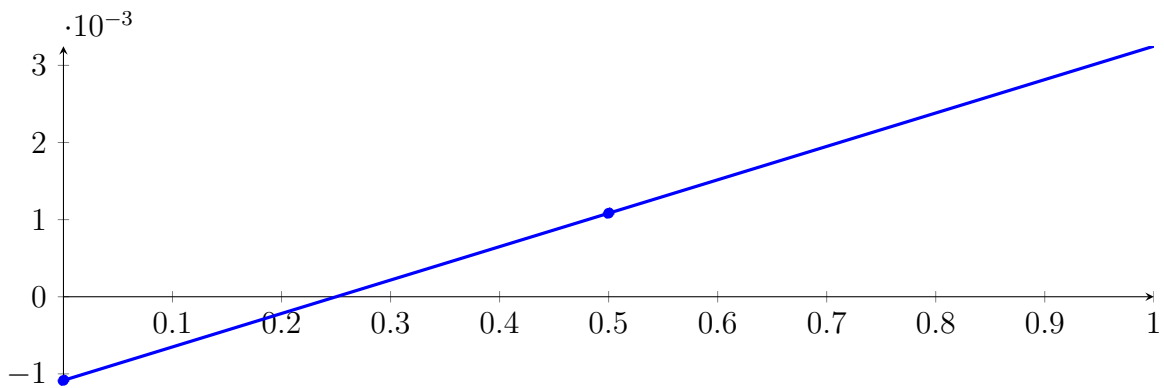
Longest intersection interval: 0.012641

\implies Selective recursion: interval 1: $[0.332927, 0.334552]$,

16.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

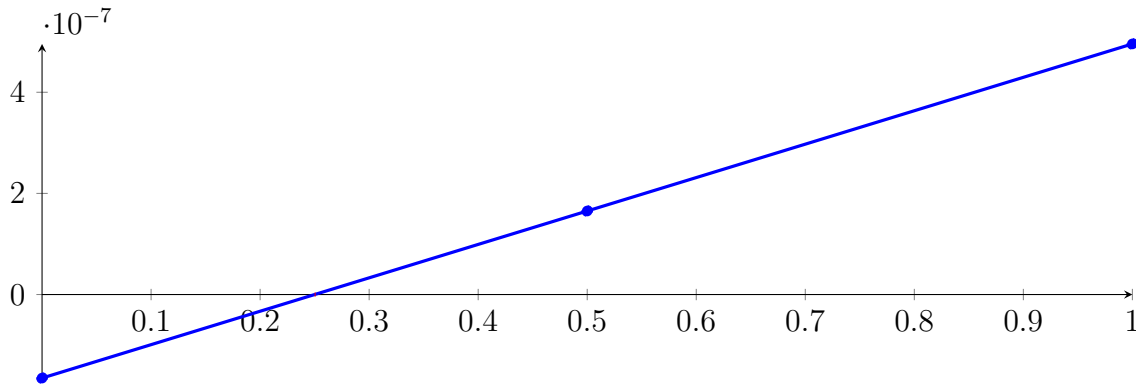
Longest intersection interval: 0.000152462

\implies Selective recursion: interval 1: $[0.333333, 0.333334]$,

16.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07} \\
 &= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

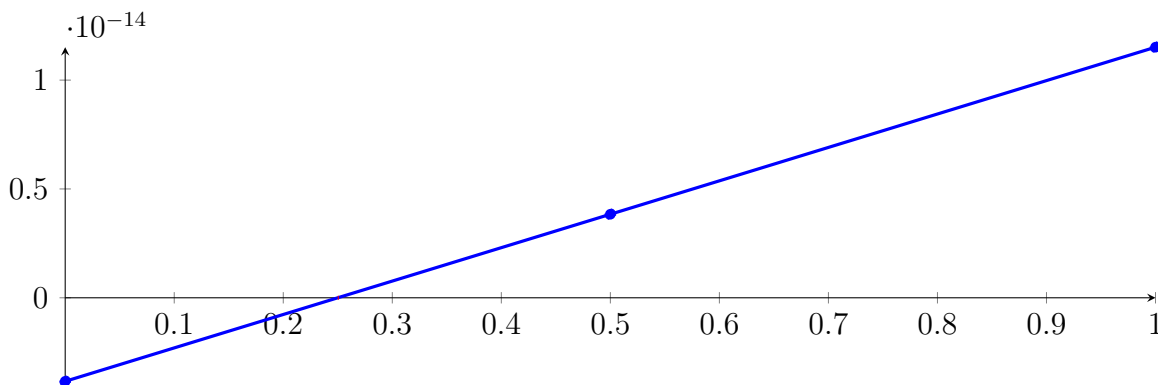
Longest intersection interval: $2.32306 \cdot 10^{-08}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

16.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.31322 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15} \\
 &= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.55112 \cdot 10^{-16}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

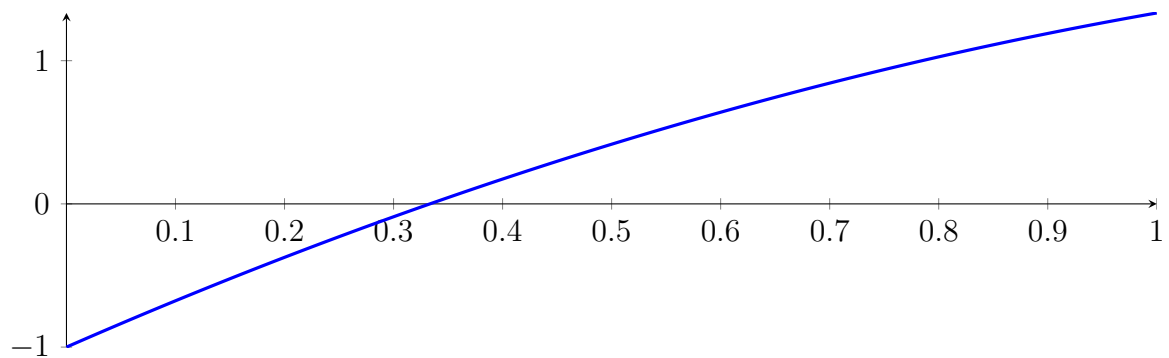
16.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

16.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

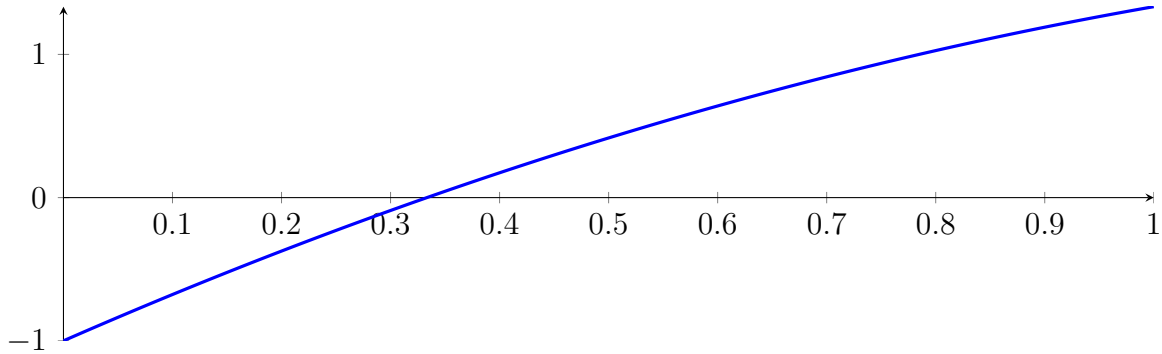
with precision $\varepsilon = 1 \cdot 10^{-64}$.

17 Running QuadClip on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

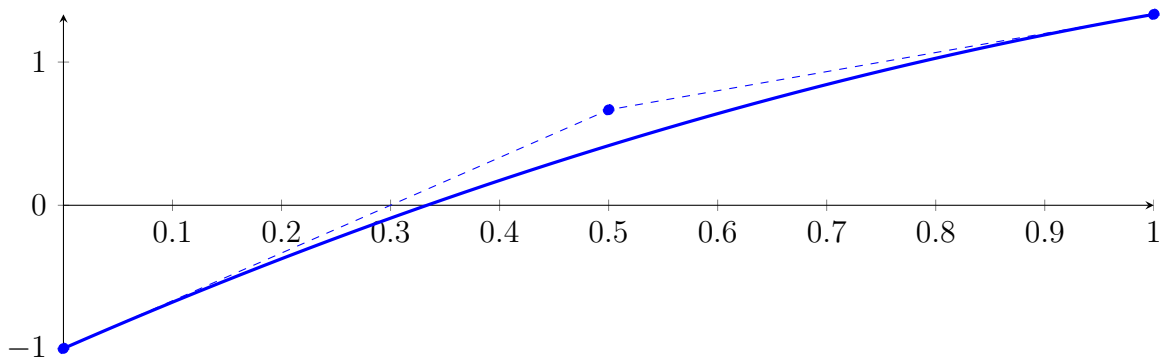
$$p = -1X^2 + 3.33333X - 1$$



17.1 Recursion Branch 1 for Input Interval $[0, 1]$

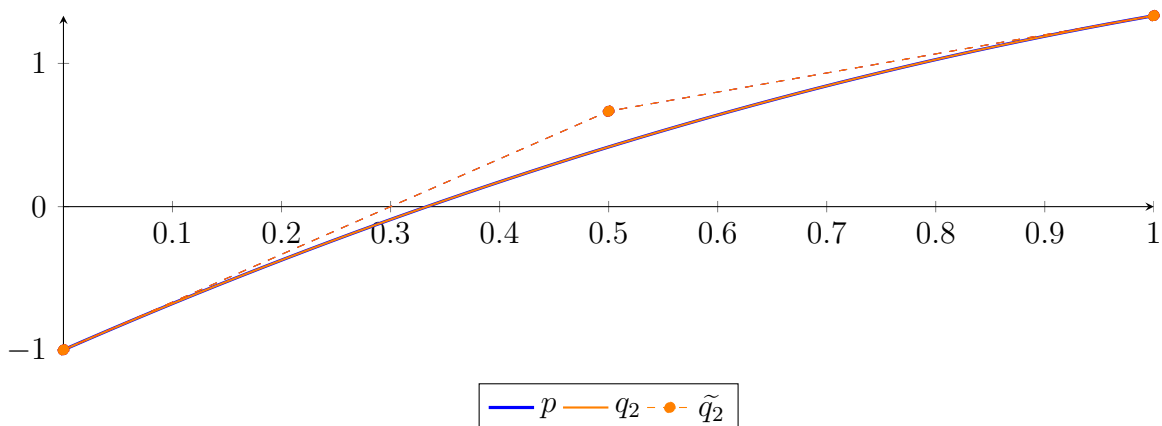
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

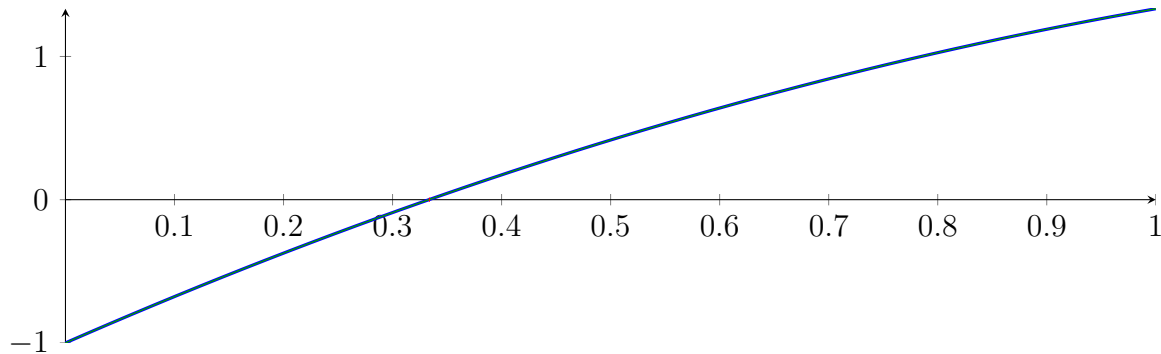
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

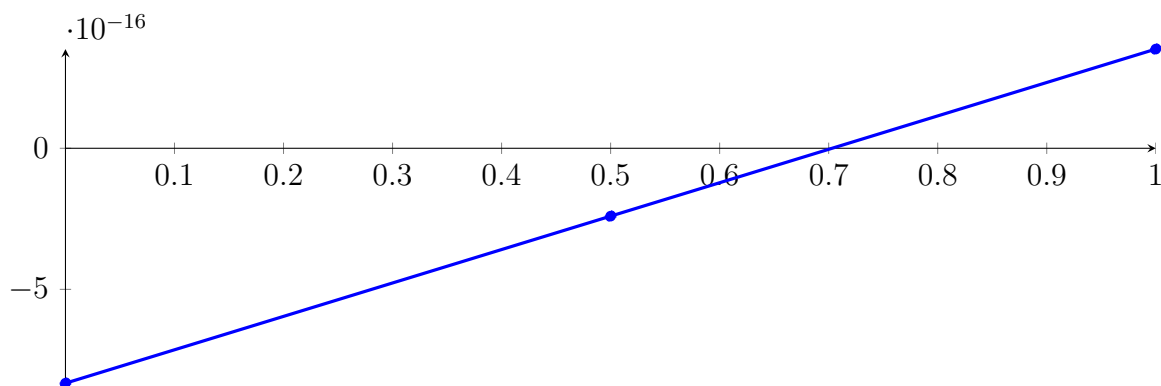
Longest intersection interval: $4.44089 \cdot 10^{-16}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

17.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

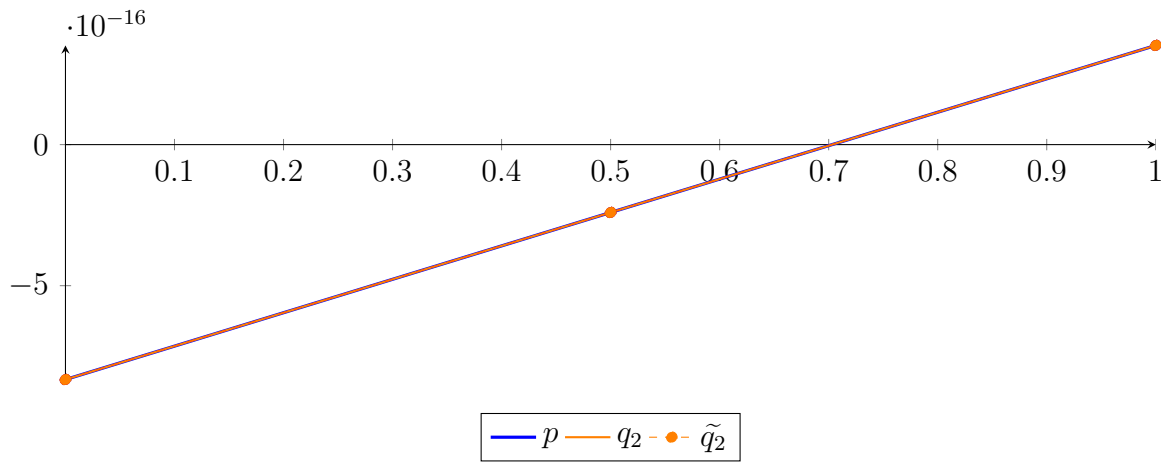
$$\begin{aligned} p &= -1.97215 \cdot 10^{-31} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2}(X) - 2.40548 \cdot 10^{-16} B_{1,2}(X) + 3.51571 \cdot 10^{-16} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.18329 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2} - 2.40548 \cdot 10^{-16} B_{1,2} + 3.51571 \cdot 10^{-16} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 2.5638 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2} - 2.40548 \cdot 10^{-16} B_{1,2} + 3.51571 \cdot 10^{-16} B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.2326 \cdot 10^{-30}$.

Bounding polynomials M and m :

$$M = 1.08468 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$

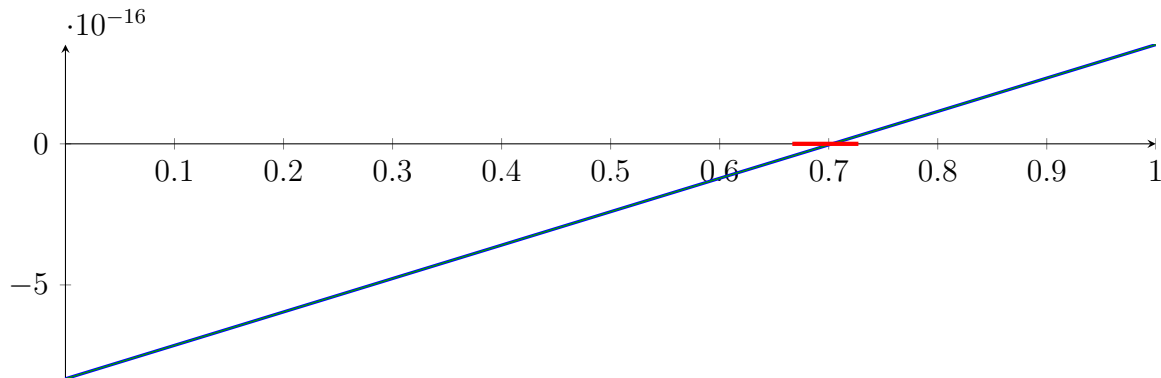
$$m = 1.18329 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$

Root of M and m :

$$N(M) = \{-1.09178 \cdot 10^{15}, 0.727273\}$$

$$N(m) = \{-1.0008 \cdot 10^{15}, 0.666667\}$$

Intersection intervals:



$$[0.666667, 0.727273]$$

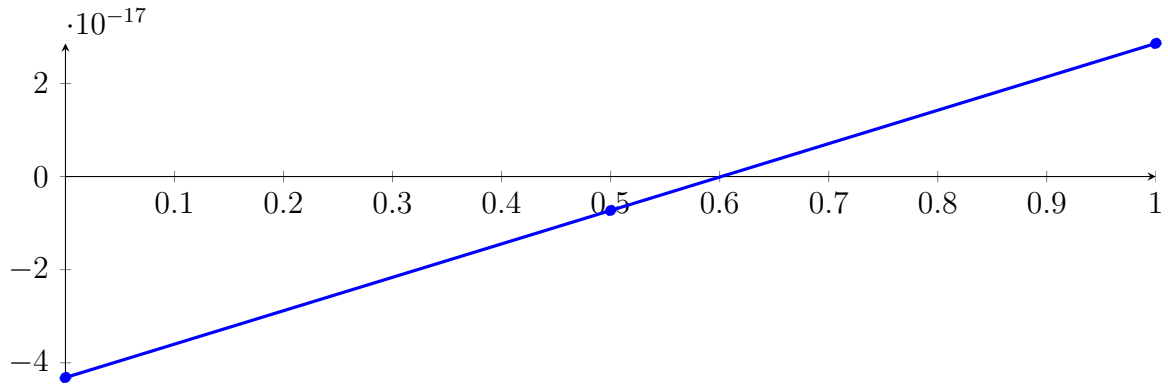
Longest intersection interval: 0.0606061

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

17.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

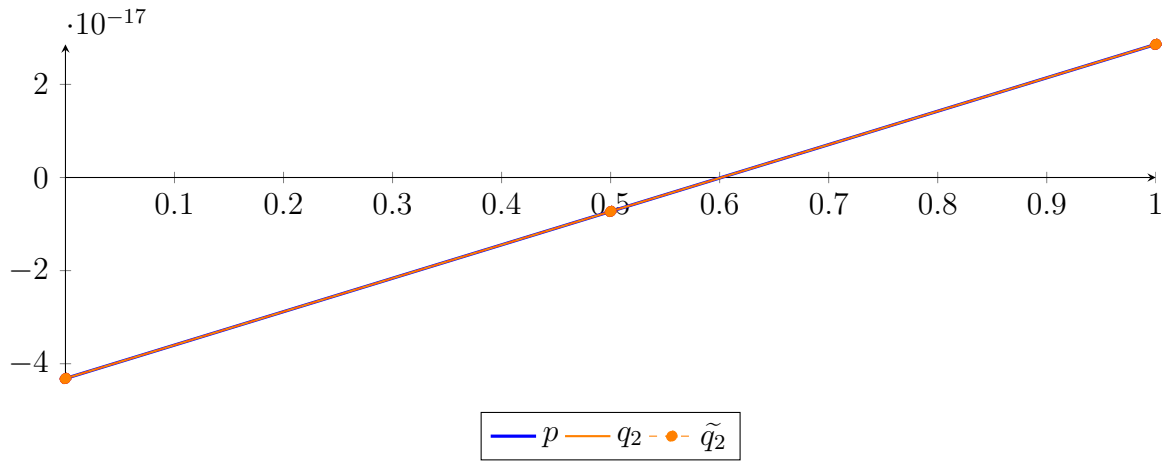
$$\begin{aligned} p &= 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17} \\ &= -4.31753 \cdot 10^{-17} B_{0,2}(X) - 7.28934 \cdot 10^{-18} B_{1,2}(X) + 2.85967 \cdot 10^{-17} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 5.54668 \cdot 10^{-32} X^2 + 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17} \\ &= -4.31753 \cdot 10^{-17} B_{0,2} - 7.28934 \cdot 10^{-18} B_{1,2} + 2.85967 \cdot 10^{-17} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 1.10934 \cdot 10^{-31} X^2 + 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17} \\ &= -4.31753 \cdot 10^{-17} B_{0,2} - 7.28934 \cdot 10^{-18} B_{1,2} + 2.85967 \cdot 10^{-17} B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.93038 \cdot 10^{-32}$.

Bounding polynomials M and m :

$$M = 5.54668 \cdot 10^{-32} X^2 + 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17}$$

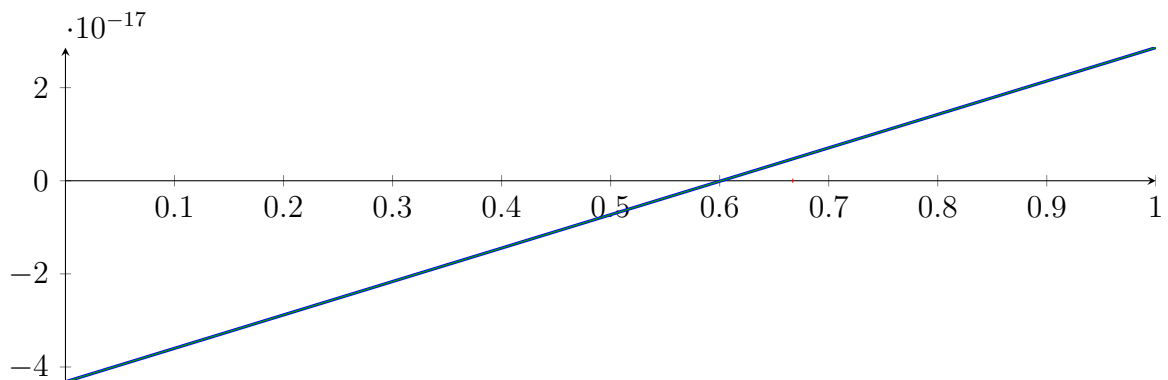
$$m = 5.54668 \cdot 10^{-32} X^2 + 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17}$$

Root of M and m :

$$N(M) = \{-1.29396 \cdot 10^{15}, 0.666667\}$$

$$N(m) = \{-1.29396 \cdot 10^{15}, 0.666667\}$$

Intersection intervals:



$[0.666667, 0.666667]$

Longest intersection interval: 0

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

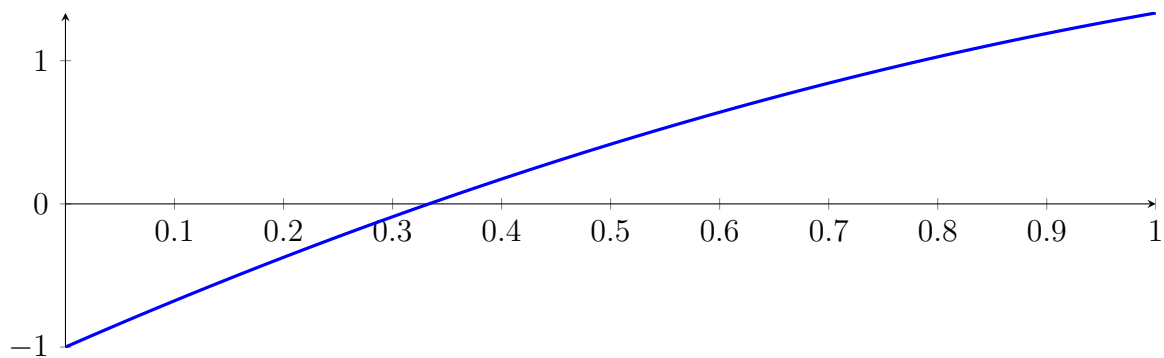
17.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

17.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

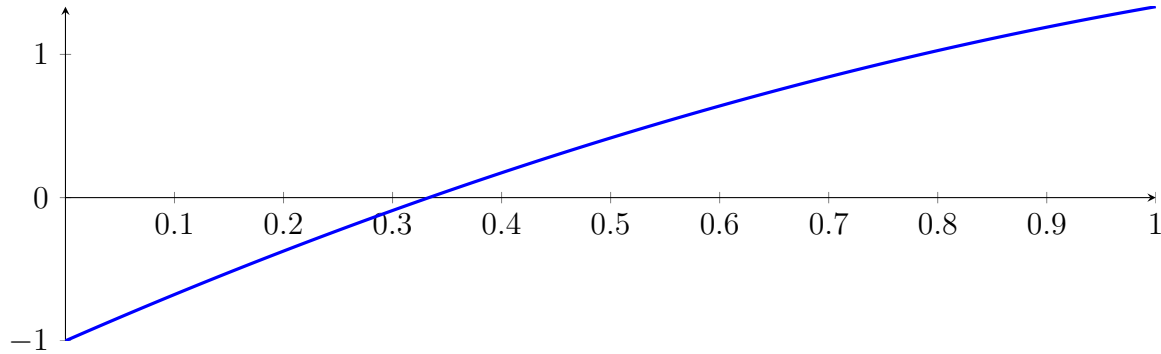
with precision $\varepsilon = 1 \cdot 10^{-64}$.

18 Running CubeClip on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

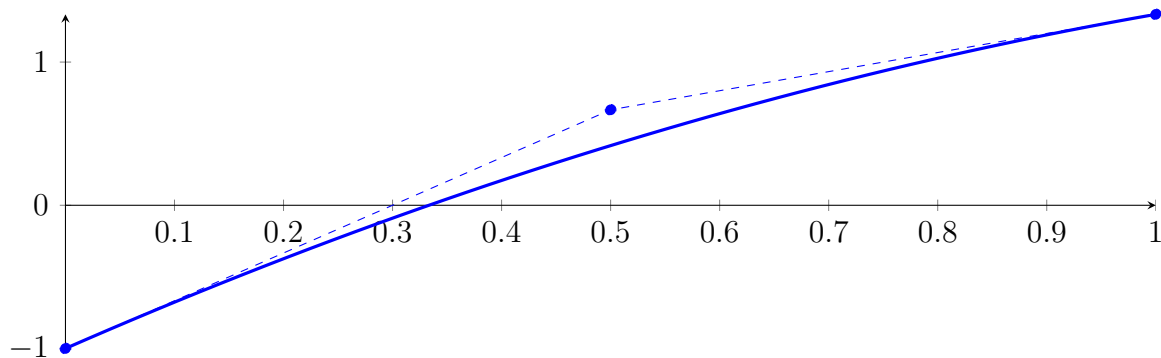
$$p = -1X^2 + 3.33333X - 1$$



18.1 Recursion Branch 1 for Input Interval $[0, 1]$

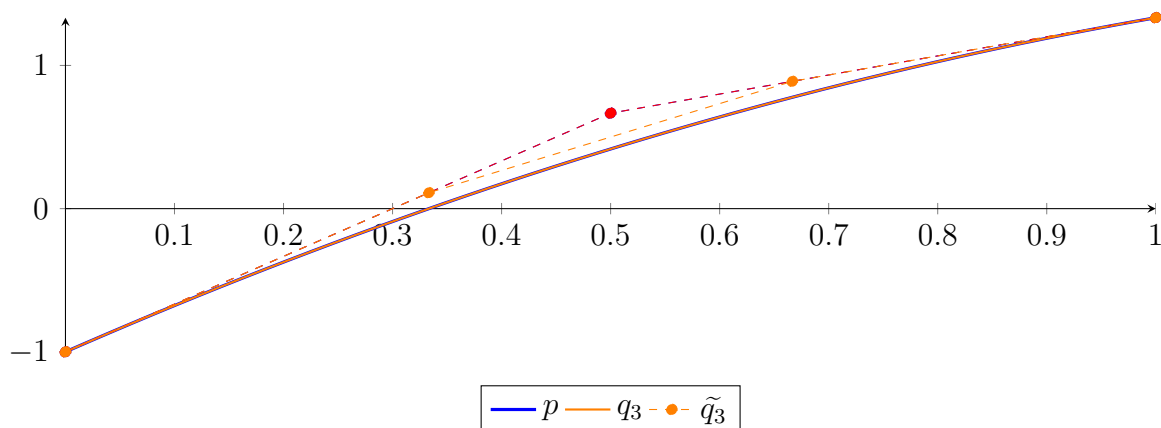
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.90958 \cdot 10^{-14}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.

Bounding polynomials M and m :

$$M = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

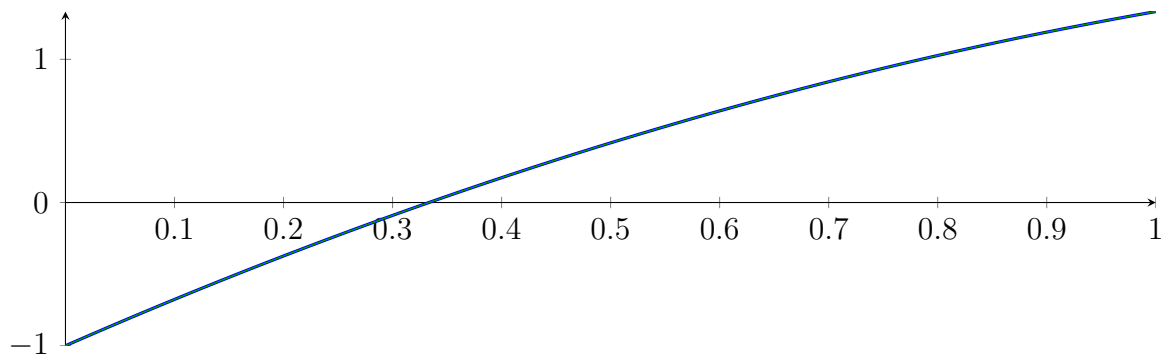
$$m = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-5.17655 \cdot 10^{13}\}$$

$$N(m) = \{-5.17655 \cdot 10^{13}\}$$

Intersection intervals:

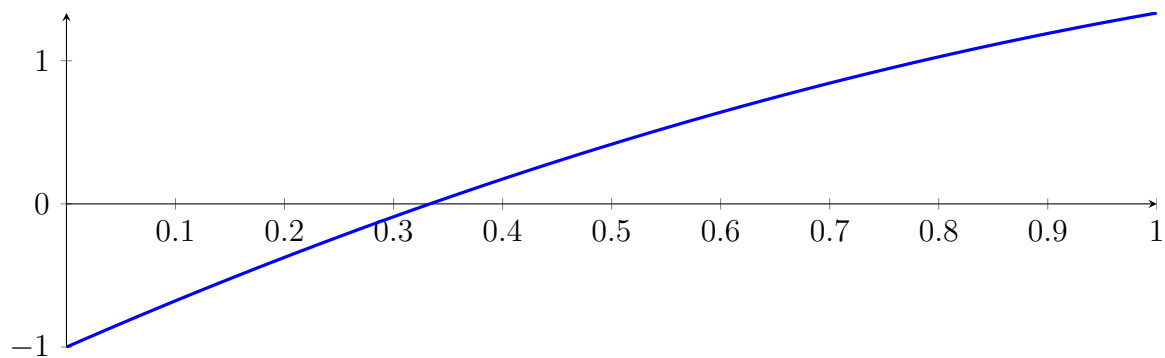


No intersection intervals with the x axis.

18.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

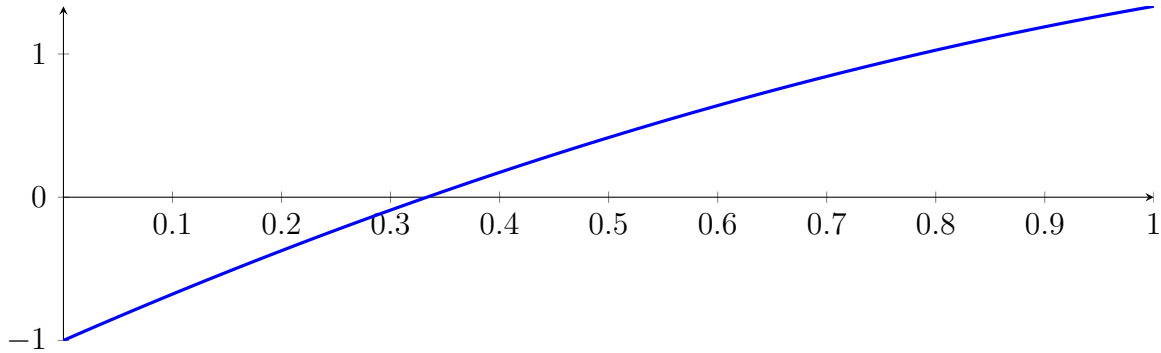
with precision $\varepsilon = 1 \cdot 10^{-64}$.

19 Running BezClip on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

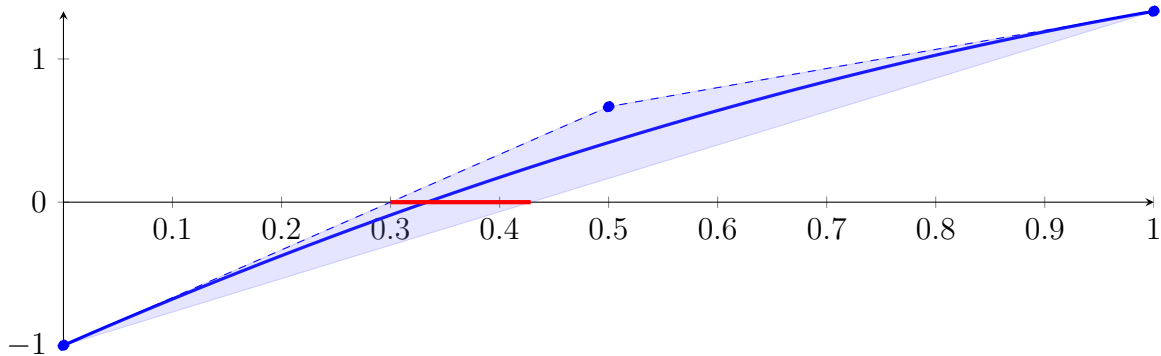
$$p = -1X^2 + 3.33333X - 1$$



19.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

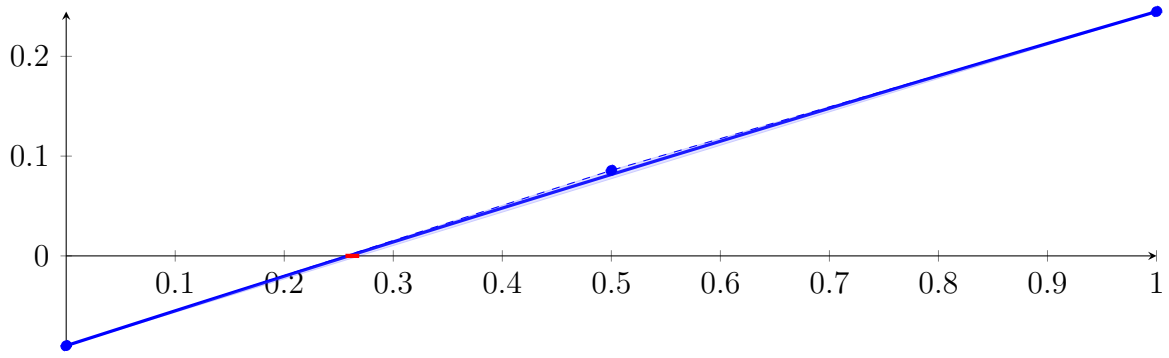
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

19.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

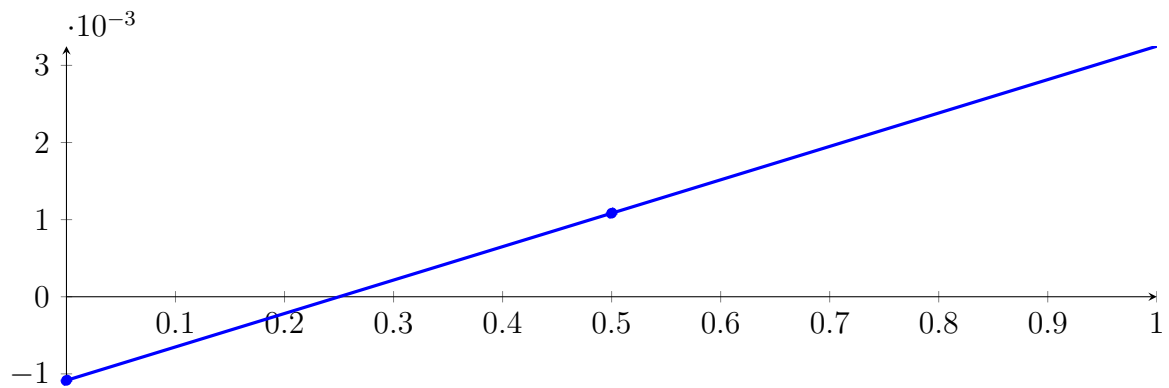
Longest intersection interval: 0.012641

\Rightarrow Selective recursion: interval 1: $[0.332927, 0.334552]$,

19.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

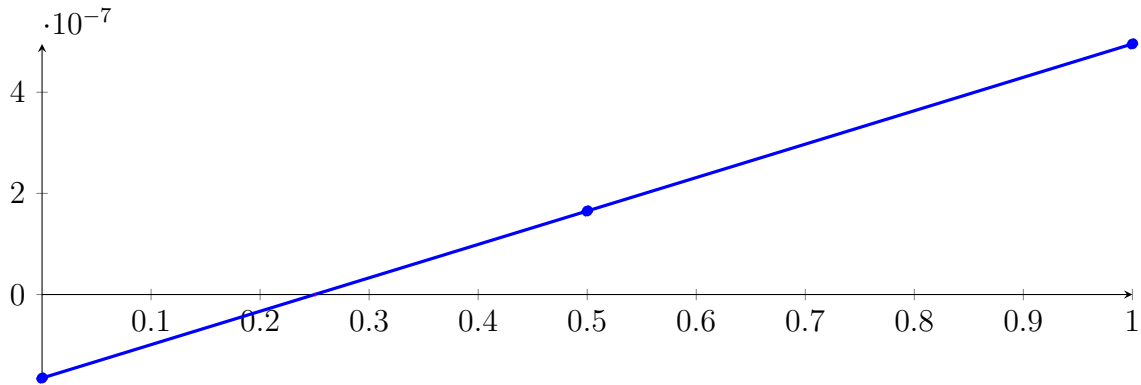
Longest intersection interval: 0.000152462

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333334]$,

19.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07} \\
 &= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

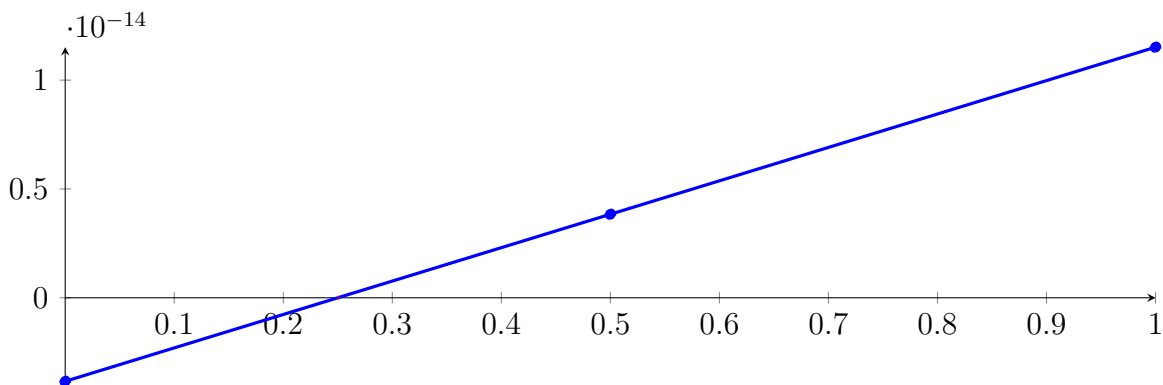
Longest intersection interval: $2.32306 \cdot 10^{-08}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

19.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.31322 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15} \\
 &= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.55112 \cdot 10^{-16}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

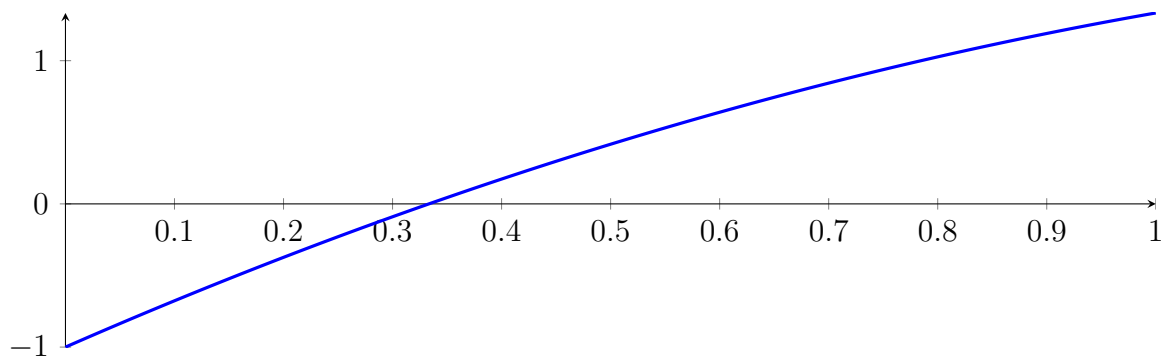
19.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

19.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

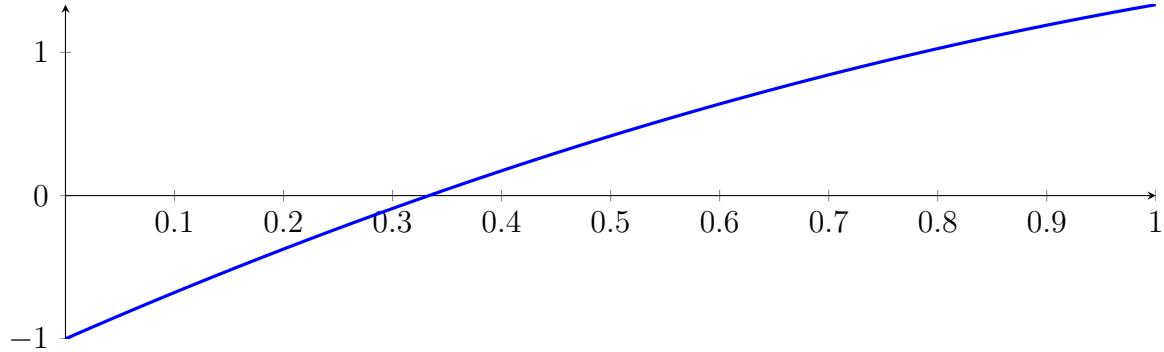
with precision $\varepsilon = 1 \cdot 10^{-128}$.

20 Running QuadClip on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

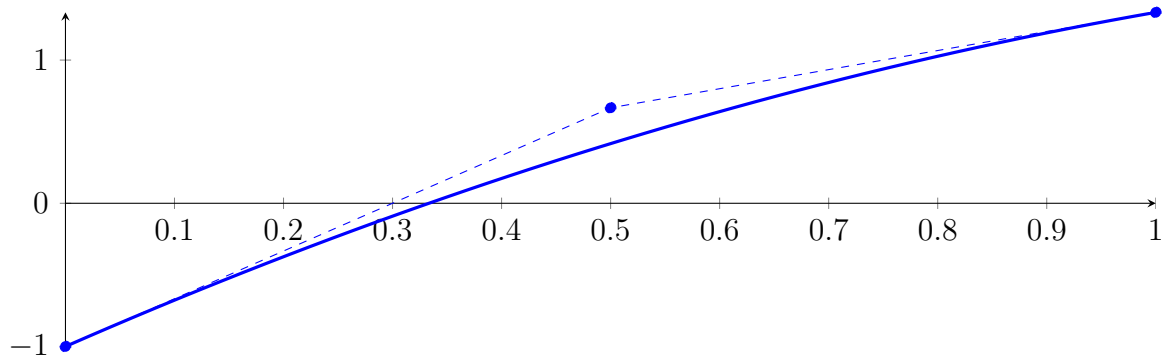
$$p = -1X^2 + 3.33333X - 1$$



20.1 Recursion Branch 1 for Input Interval $[0, 1]$

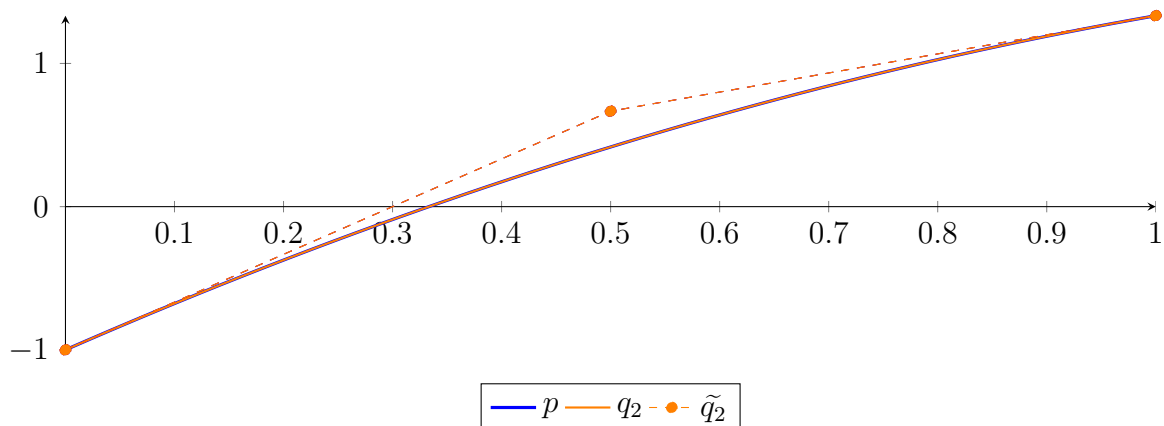
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

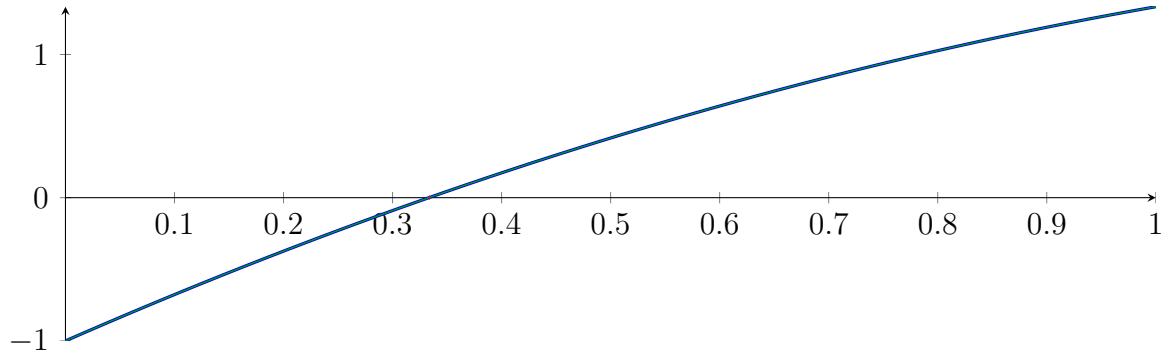
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

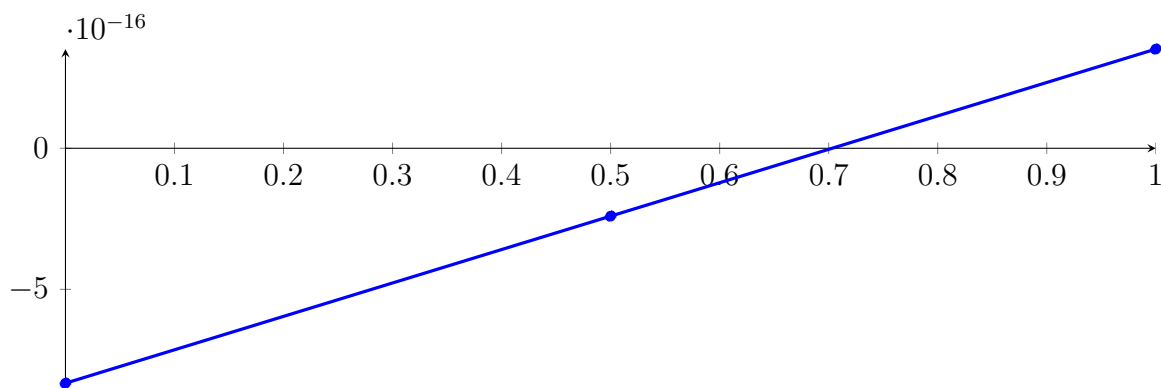
Longest intersection interval: $4.44089 \cdot 10^{-16}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

20.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

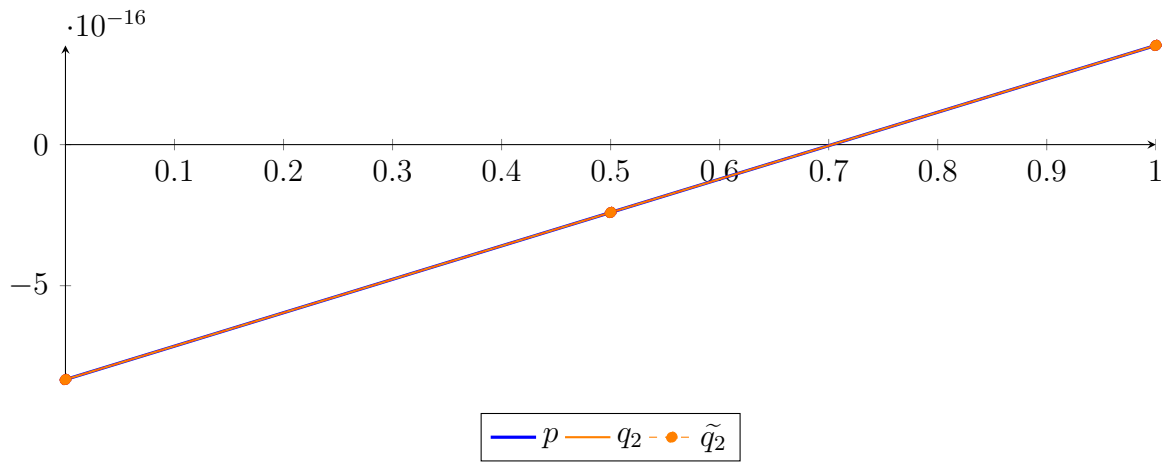
$$\begin{aligned} p &= -1.97215 \cdot 10^{-31} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2}(X) - 2.40548 \cdot 10^{-16} B_{1,2}(X) + 3.51571 \cdot 10^{-16} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 1.18329 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2} - 2.40548 \cdot 10^{-16} B_{1,2} + 3.51571 \cdot 10^{-16} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 2.5638 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16} \\ &= -8.32667 \cdot 10^{-16} B_{0,2} - 2.40548 \cdot 10^{-16} B_{1,2} + 3.51571 \cdot 10^{-16} B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.2326 \cdot 10^{-30}$.

Bounding polynomials M and m :

$$M = 1.08468 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$

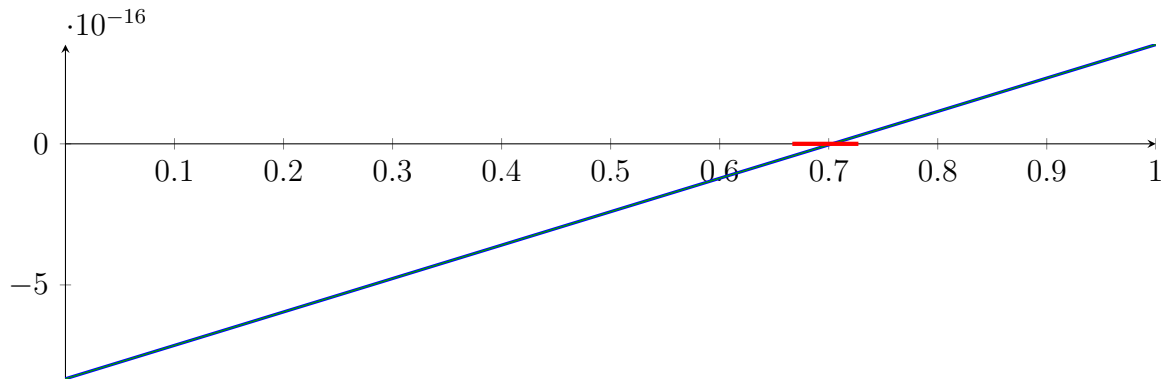
$$m = 1.18329 \cdot 10^{-30} X^2 + 1.18424 \cdot 10^{-15} X - 8.32667 \cdot 10^{-16}$$

Root of M and m :

$$N(M) = \{-1.09178 \cdot 10^{15}, 0.727273\}$$

$$N(m) = \{-1.0008 \cdot 10^{15}, 0.666667\}$$

Intersection intervals:



$$[0.666667, 0.727273]$$

Longest intersection interval: 0.0606061

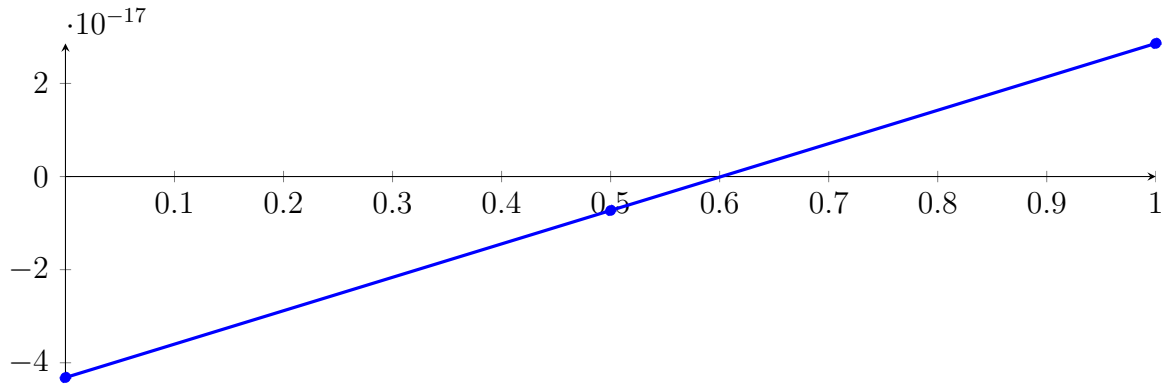
\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

20.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17}$$

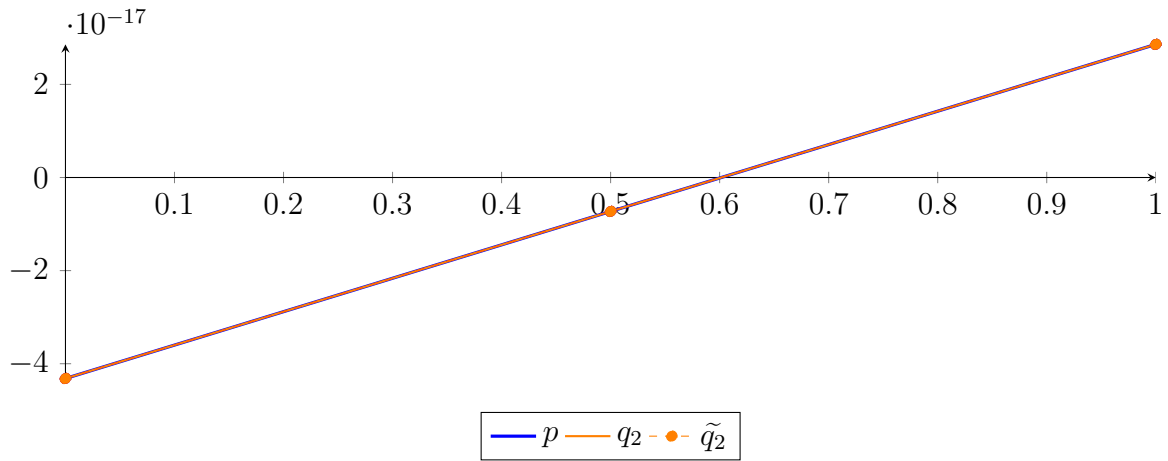
$$= -4.31753 \cdot 10^{-17} B_{0,2}(X) - 7.28934 \cdot 10^{-18} B_{1,2}(X) + 2.85967 \cdot 10^{-17} B_{2,2}(X)$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 5.54668 \cdot 10^{-32} X^2 + 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17} \\ &= -4.31753 \cdot 10^{-17} B_{0,2} - 7.28934 \cdot 10^{-18} B_{1,2} + 2.85967 \cdot 10^{-17} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 1.10934 \cdot 10^{-31} X^2 + 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17} \\ &= -4.31753 \cdot 10^{-17} B_{0,2} - 7.28934 \cdot 10^{-18} B_{1,2} + 2.85967 \cdot 10^{-17} B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.93038 \cdot 10^{-32}$.

Bounding polynomials M and m :

$$M = 5.54668 \cdot 10^{-32} X^2 + 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17}$$

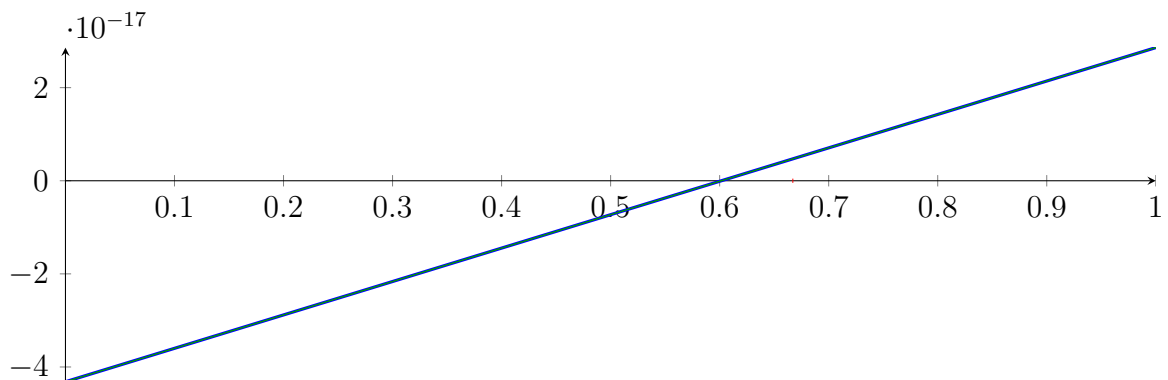
$$m = 5.54668 \cdot 10^{-32} X^2 + 7.1772 \cdot 10^{-17} X - 4.31753 \cdot 10^{-17}$$

Root of M and m :

$$N(M) = \{-1.29396 \cdot 10^{15}, 0.666667\}$$

$$N(m) = \{-1.29396 \cdot 10^{15}, 0.666667\}$$

Intersection intervals:



$[0.666667, 0.666667]$

Longest intersection interval: 0

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

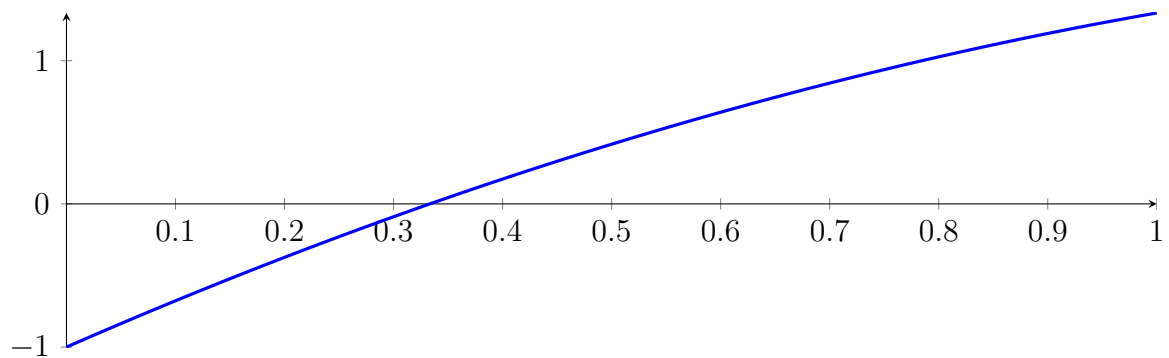
20.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

20.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

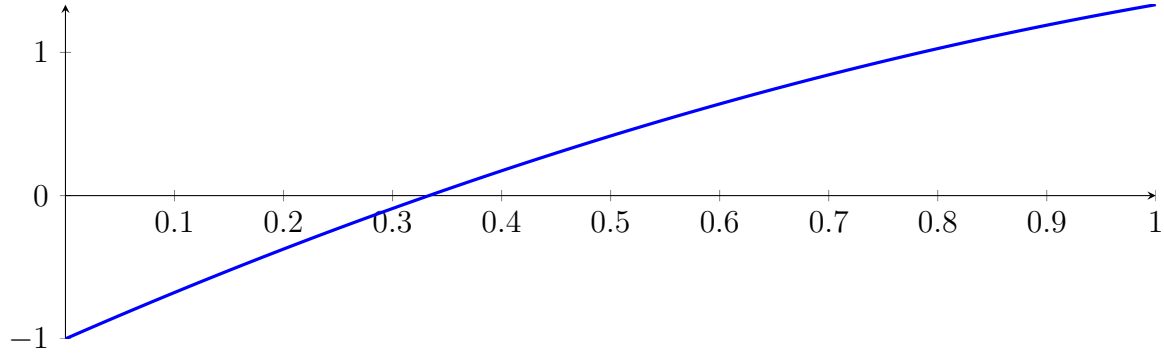
with precision $\varepsilon = 1 \cdot 10^{-128}$.

21 Running CubeClip on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

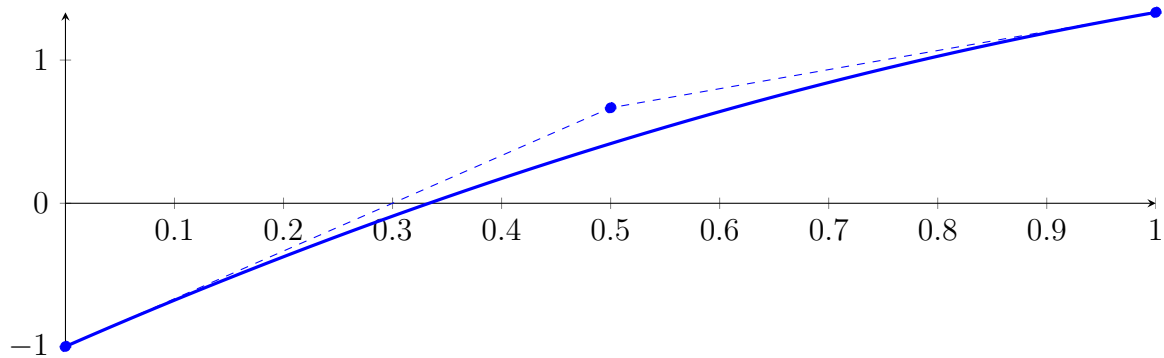
$$p = -1X^2 + 3.33333X - 1$$



21.1 Recursion Branch 1 for Input Interval $[0, 1]$

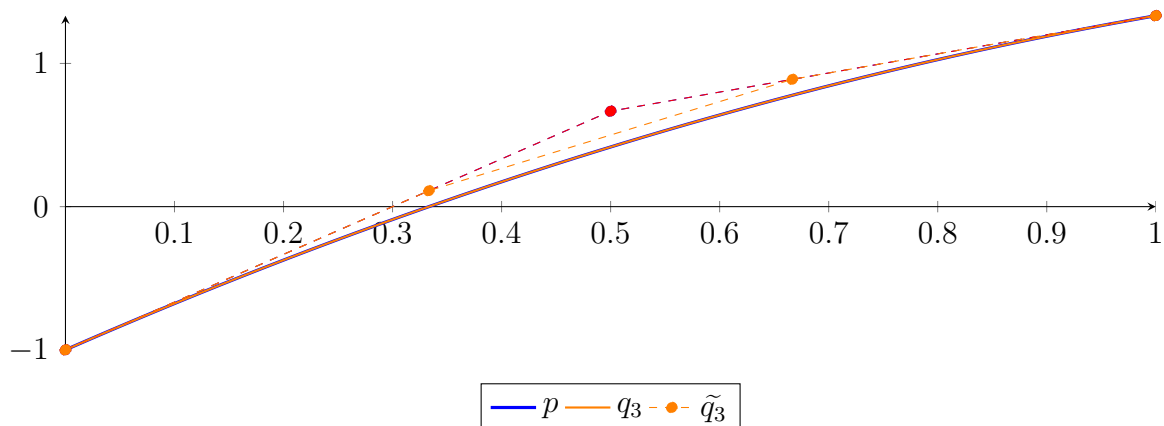
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.90958 \cdot 10^{-14}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.66134 \cdot 10^{-16}$.

Bounding polynomials M and m :

$$M = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

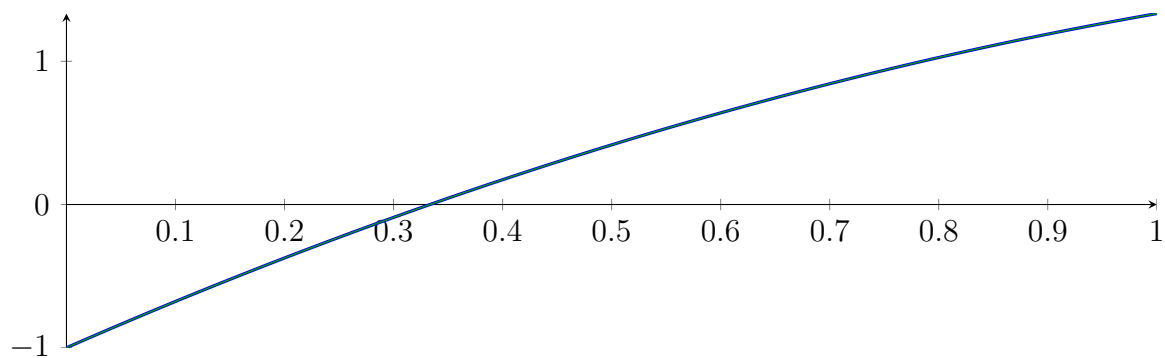
$$m = -1.88738 \cdot 10^{-14} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-5.17655 \cdot 10^{13}\}$$

$$N(m) = \{-5.17655 \cdot 10^{13}\}$$

Intersection intervals:

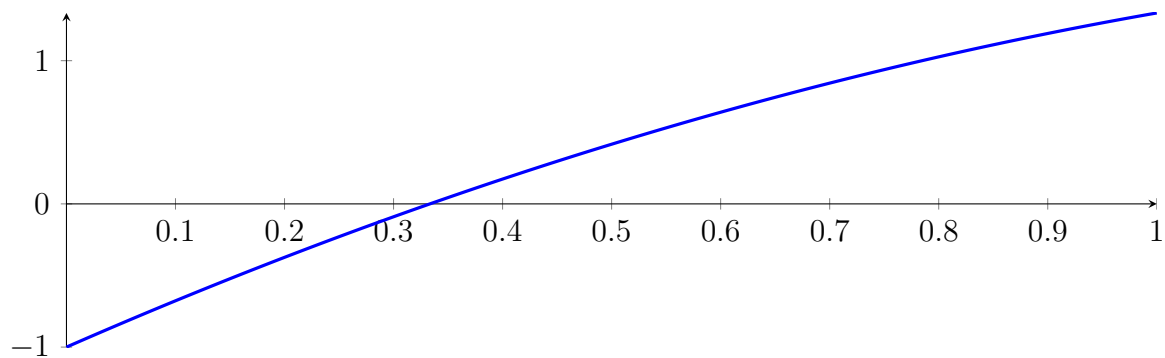


No intersection intervals with the x axis.

21.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

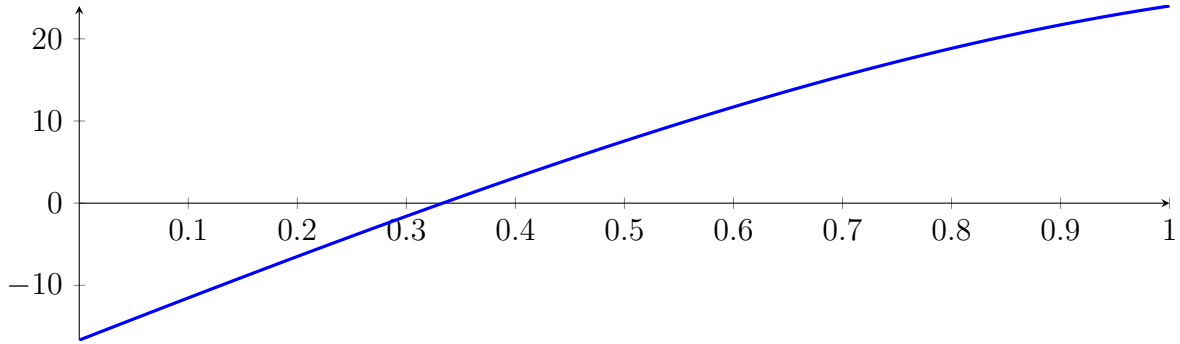
with precision $\varepsilon = 1 \cdot 10^{-128}$.

22 Running BezClip on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

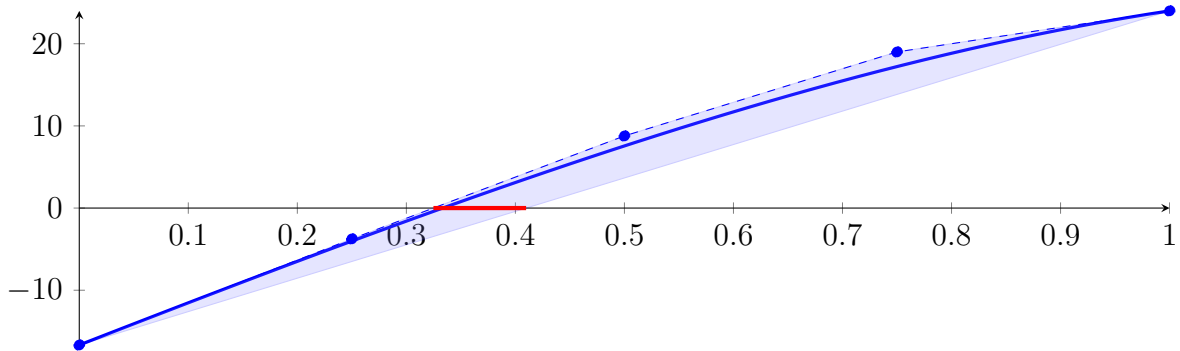
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



22.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

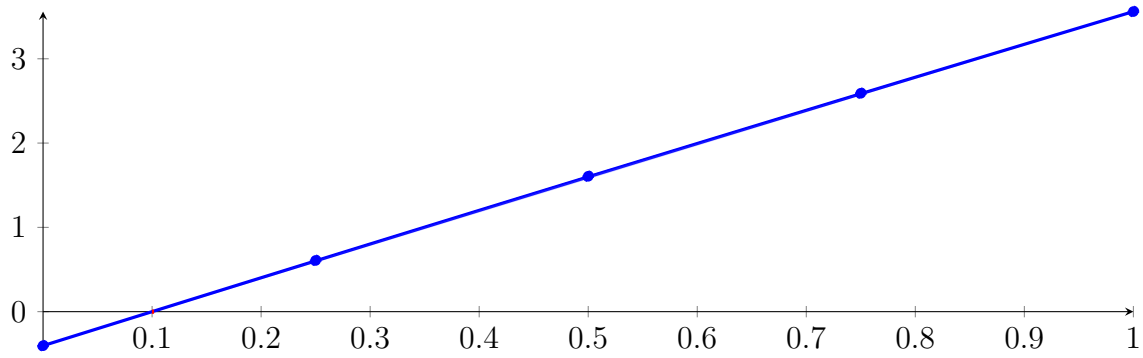
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

22.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877

\implies Selective recursion: [interval 1: \[0.333317, 0.333491\]](#),

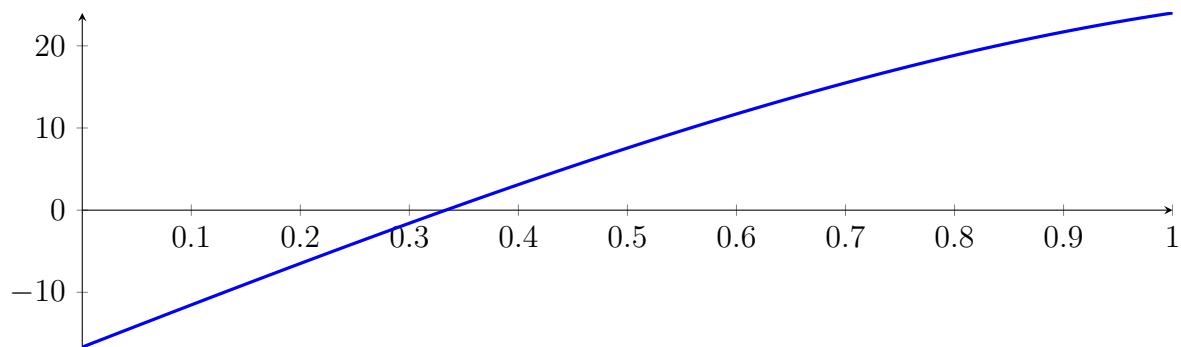
22.3 Recursion Branch 1 1 1 in Interval 1: [0.333317, 0.333491]

Found root in interval [0.333317, 0.333491] at recursion depth 3!

22.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333317, 0.333491]$$

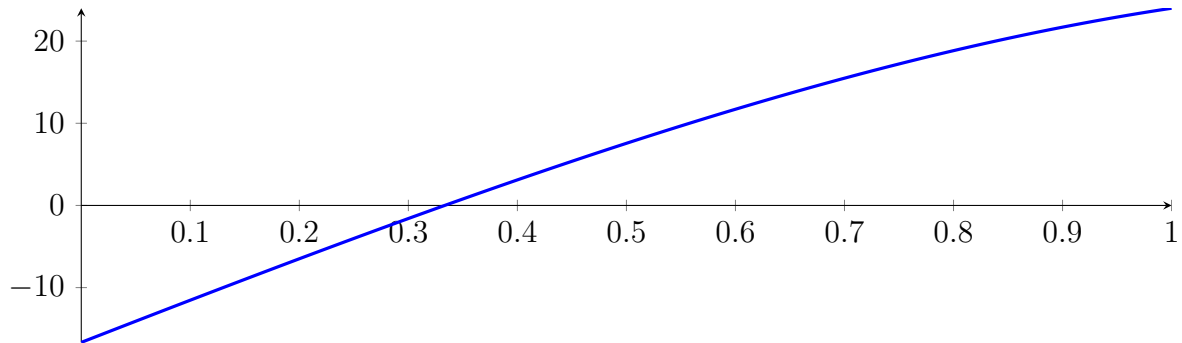
with precision $\varepsilon = 0.01$.

23 Running QuadClip on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

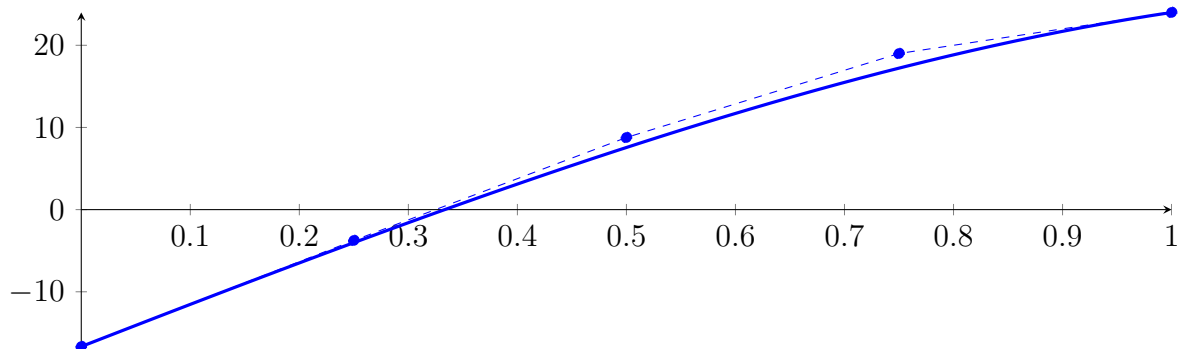
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



23.1 Recursion Branch 1 for Input Interval $[0, 1]$

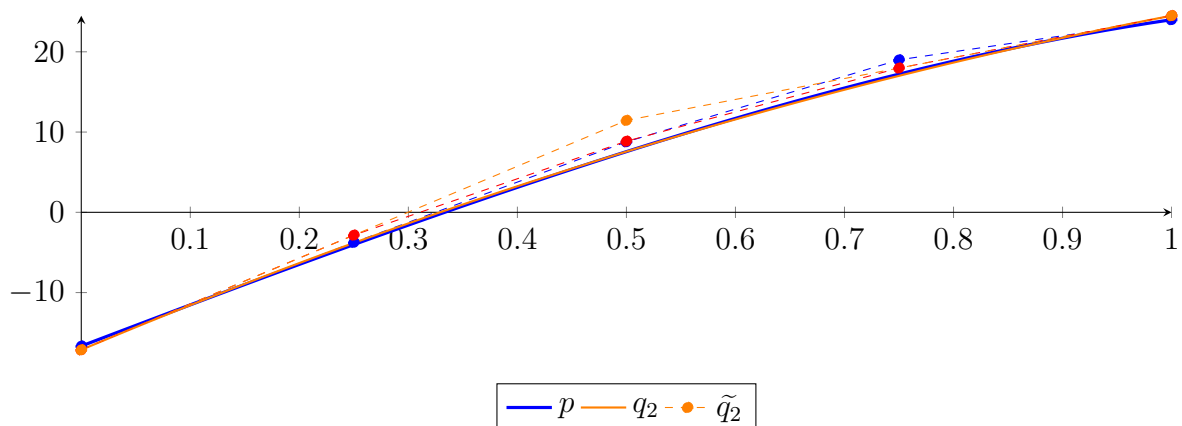
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \\ \tilde{q}_2 &= -2.67519 \cdot 10^{-12}X^4 + 5.32907 \cdot 10^{-12}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

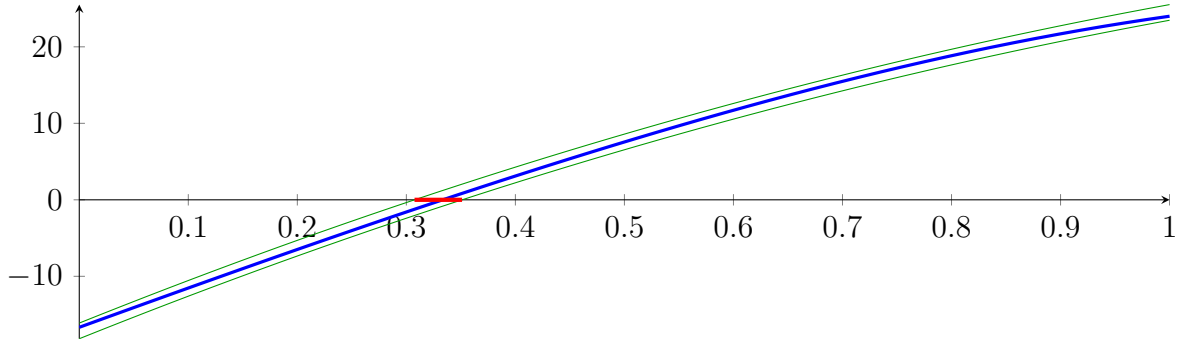
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

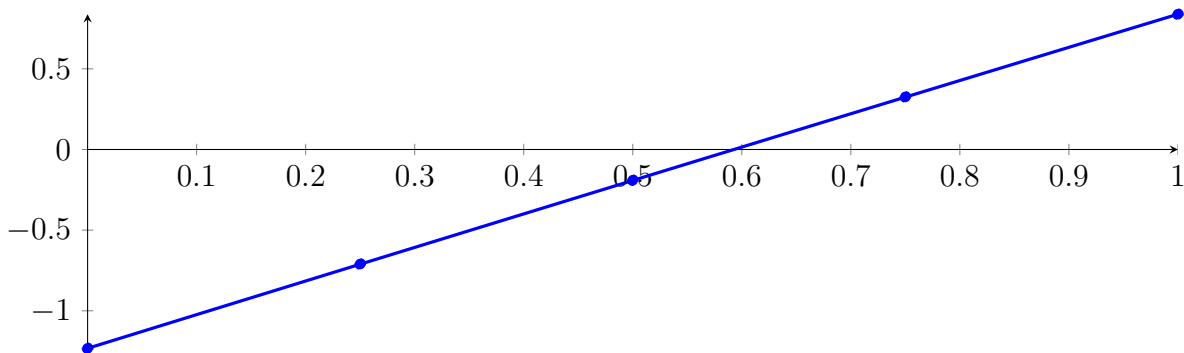
Longest intersection interval: 0.0436205

⇒ Selective recursion: interval 1: $[0.307477, 0.351097]$,

23.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

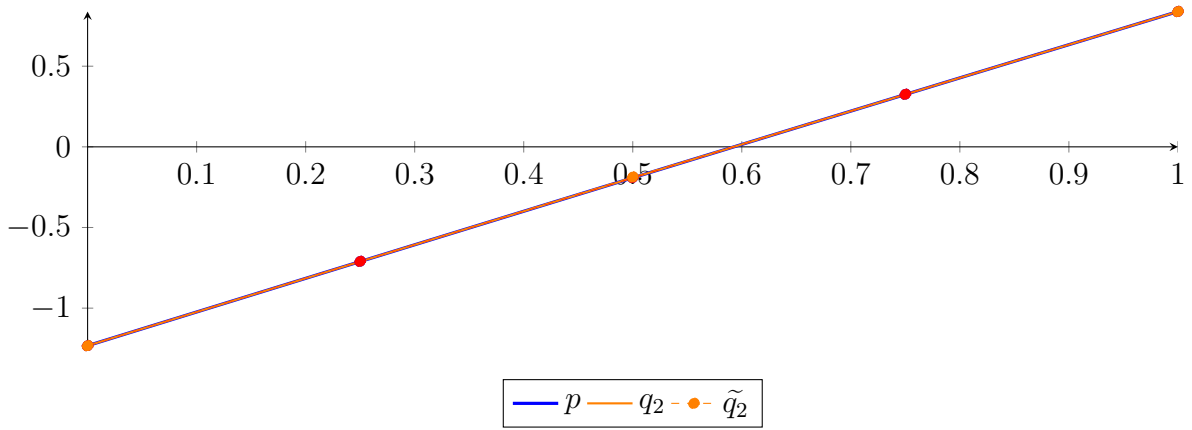
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 2.43583 \cdot 10^{-13} X^4 - 4.3876 \cdot 10^{-13} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

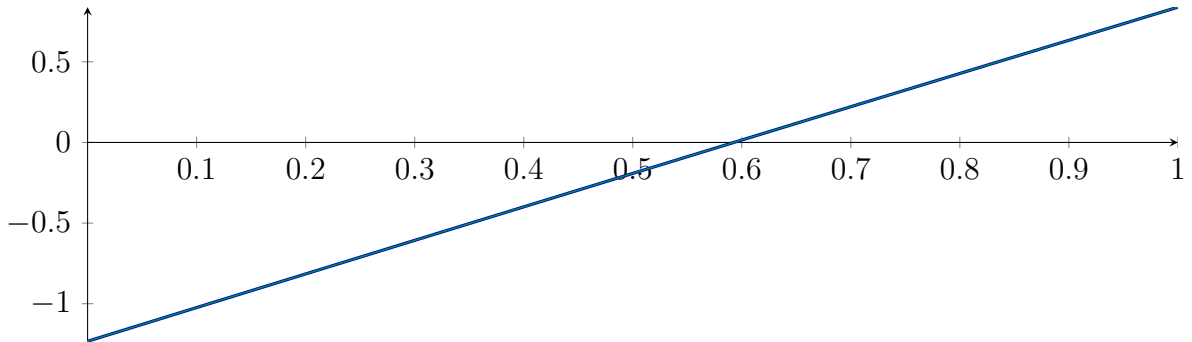
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

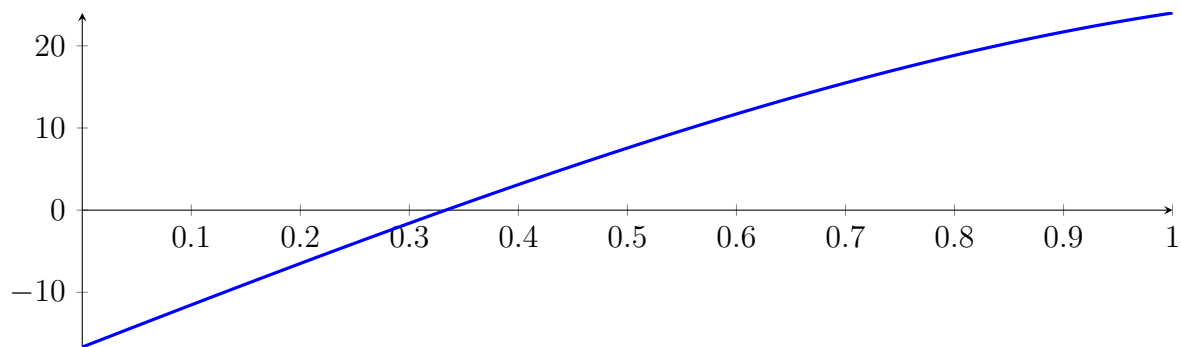
23.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Found root in interval $[0.333332, 0.333335]$ at recursion depth 3!

23.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333332, 0.333335]$$

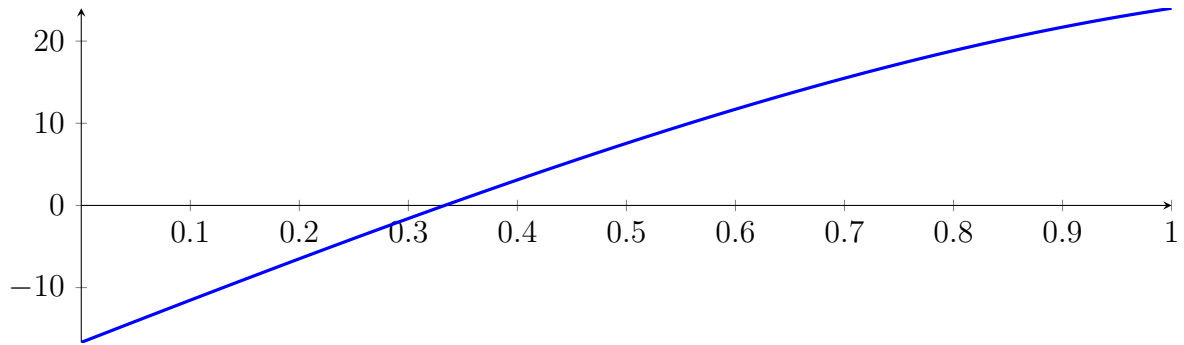
with precision $\varepsilon = 0.01$.

24 Running CubeClip on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

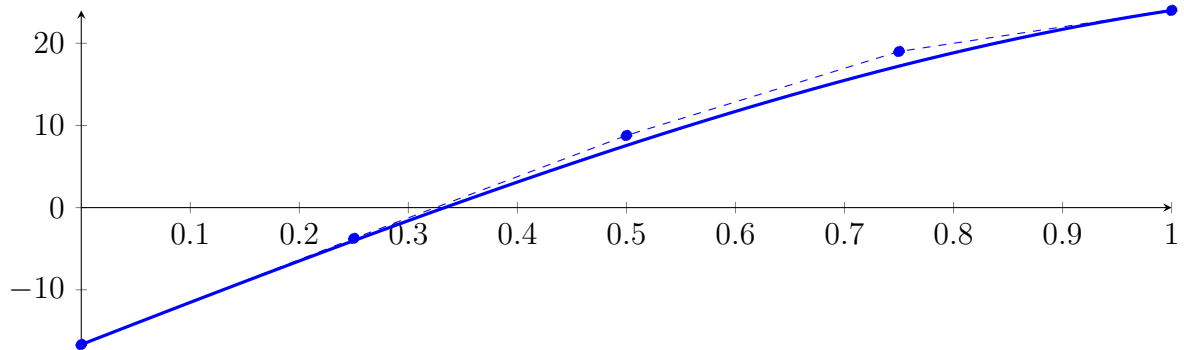
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



24.1 Recursion Branch 1 for Input Interval $[0, 1]$

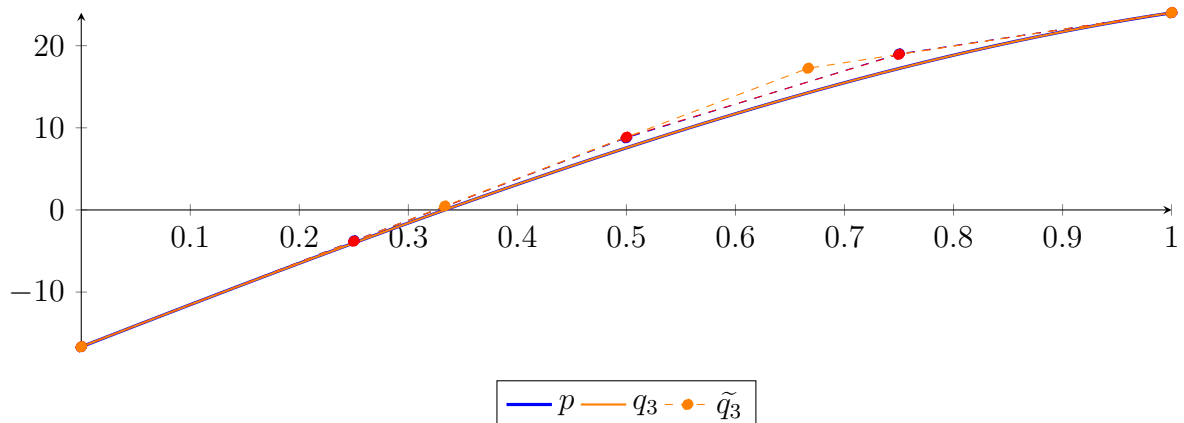
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= -2.28084 \cdot 10^{-12}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

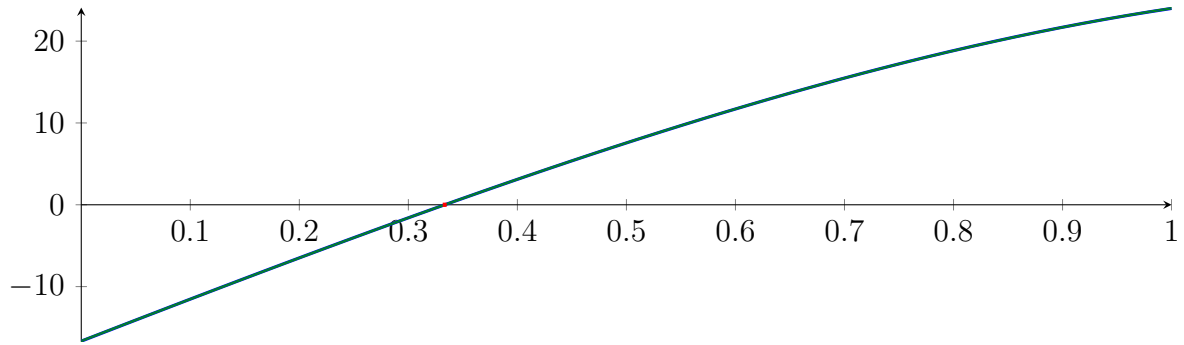
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204

\implies Selective recursion: interval 1: [\[0.331524, 0.335136\]](#),

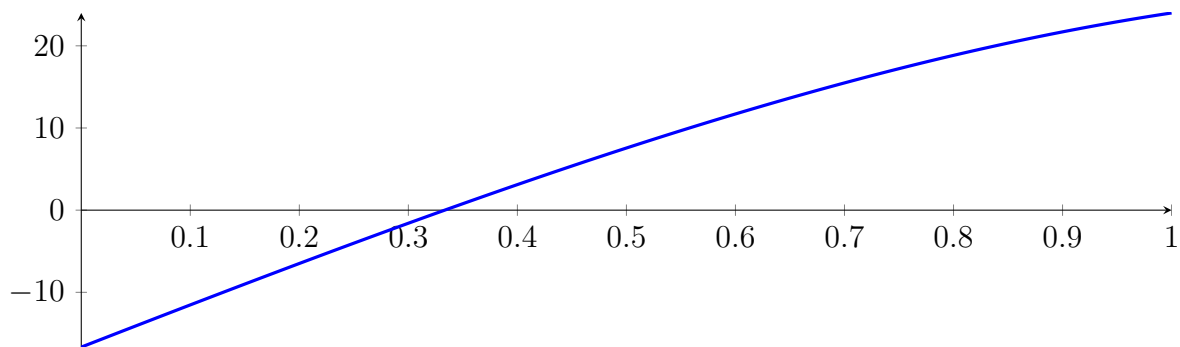
24.2 Recursion Branch 1 1 in Interval 1: [\[0.331524, 0.335136\]](#)

Found root in interval [\[0.331524, 0.335136\]](#) at recursion depth 2!

24.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.331524, 0.335136]$$

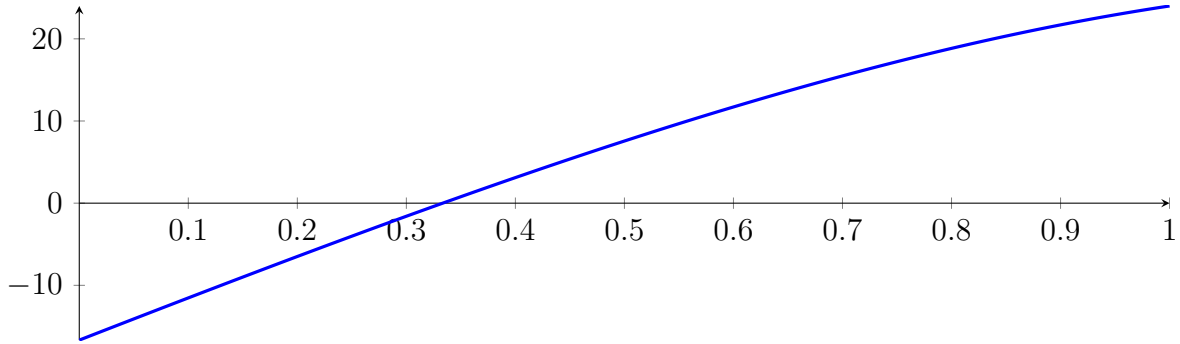
with precision $\varepsilon = 0.01$.

25 Running BezClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

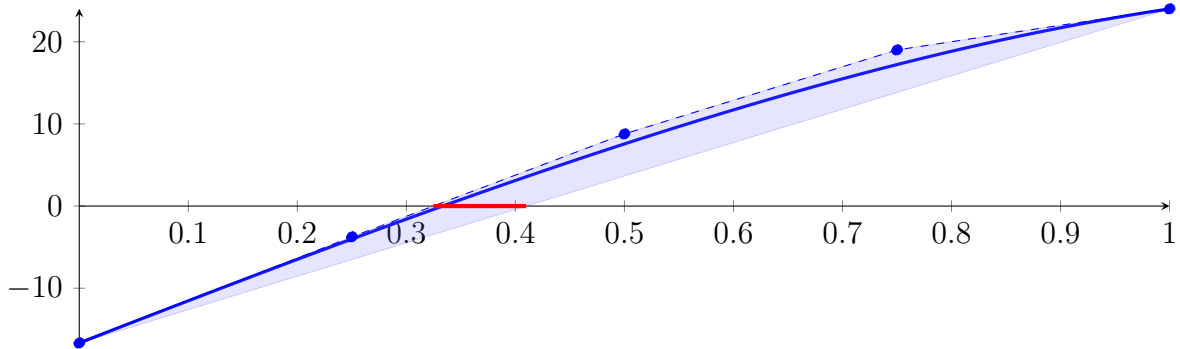
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



25.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

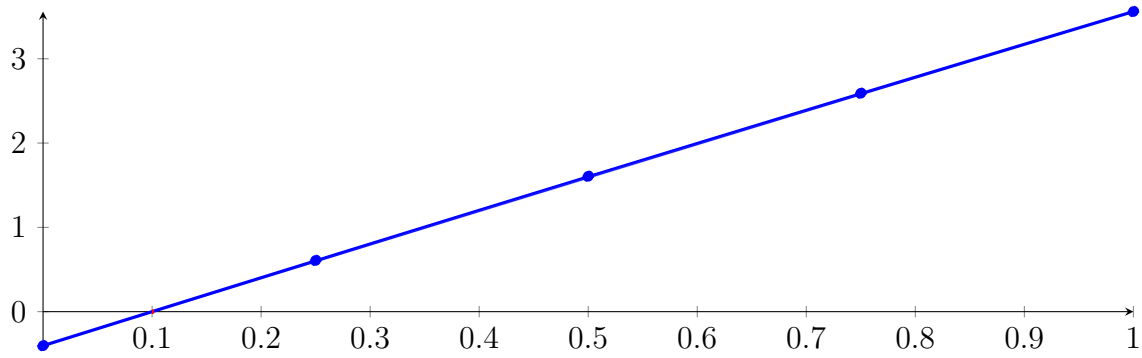
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

25.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

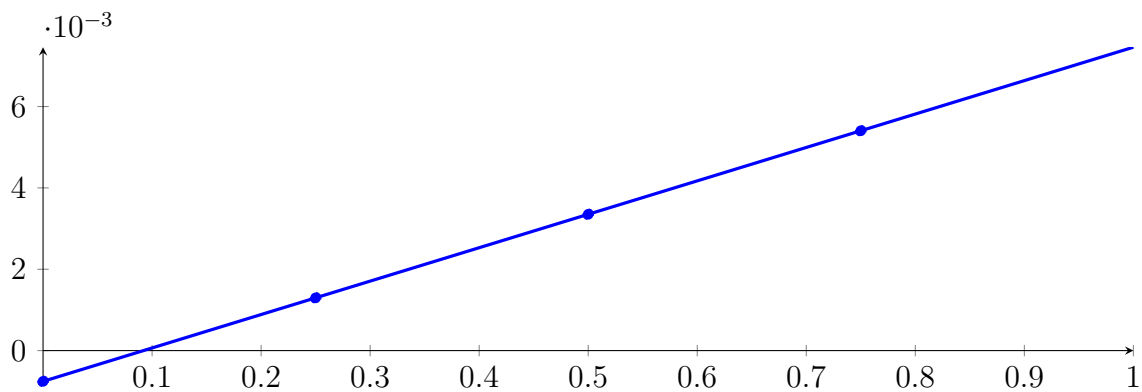
Longest intersection interval: 0.00203877

\implies Selective recursion: interval 1: $[0.333317, 0.333491]$,

25.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.06393 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

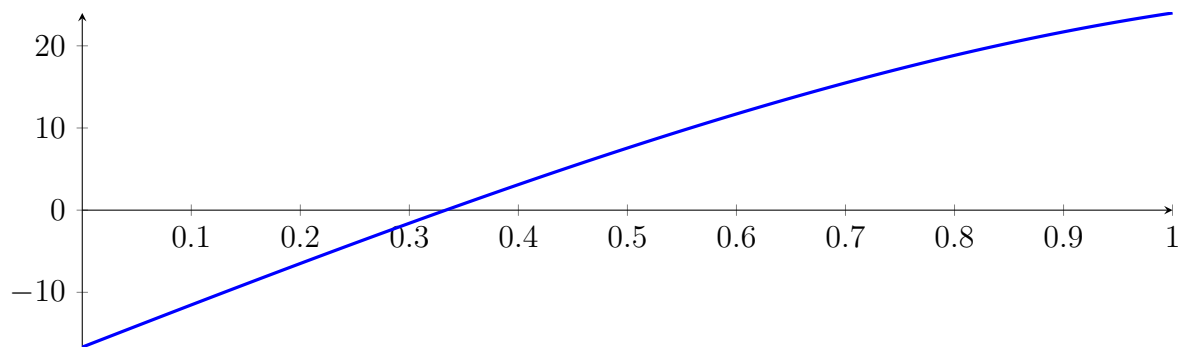
25.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

25.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

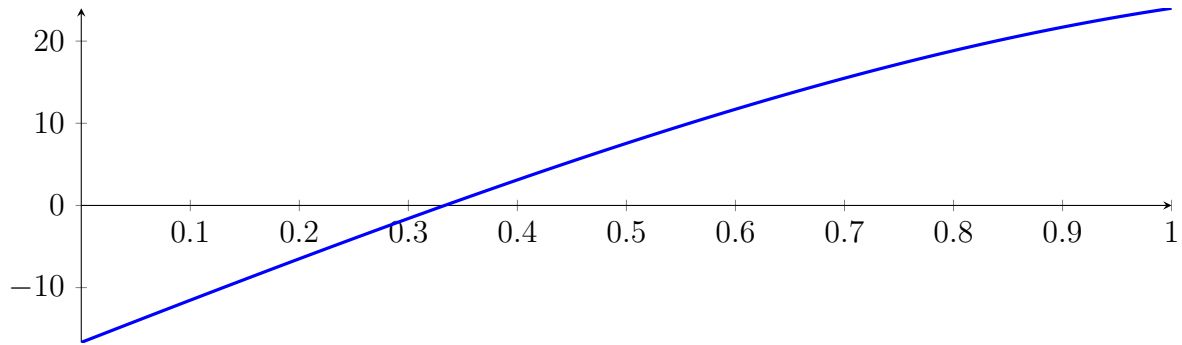
with precision $\varepsilon = 0.0001$.

26 Running QuadClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

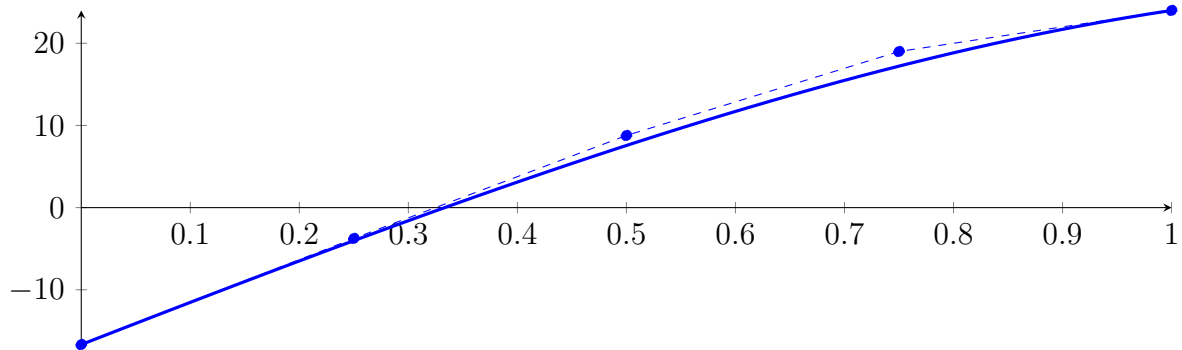
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



26.1 Recursion Branch 1 for Input Interval $[0, 1]$

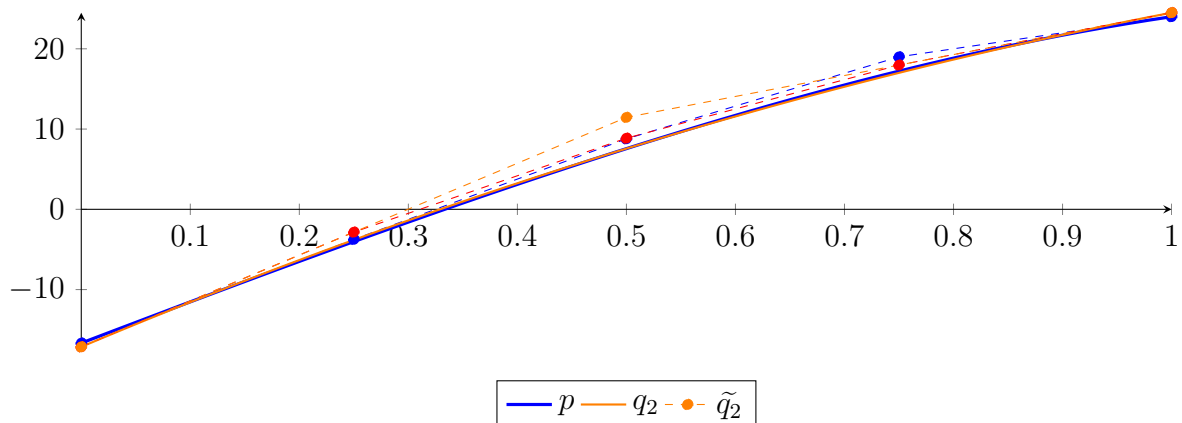
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \\ \tilde{q}_2 &= -2.67519 \cdot 10^{-12}X^4 + 5.32907 \cdot 10^{-12}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

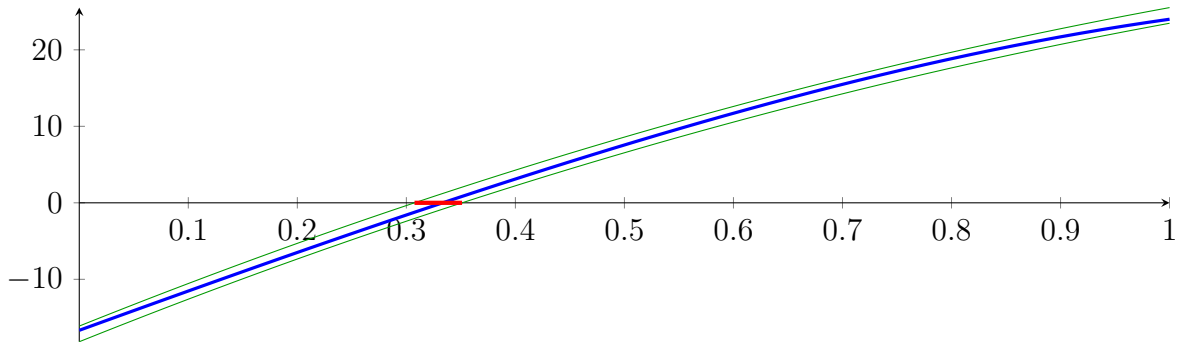
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

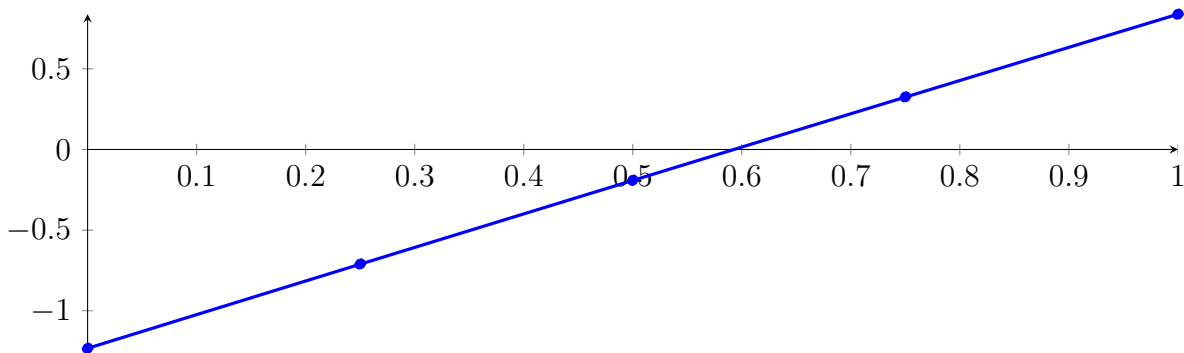
Longest intersection interval: 0.0436205

⇒ Selective recursion: interval 1: $[0.307477, 0.351097]$,

26.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

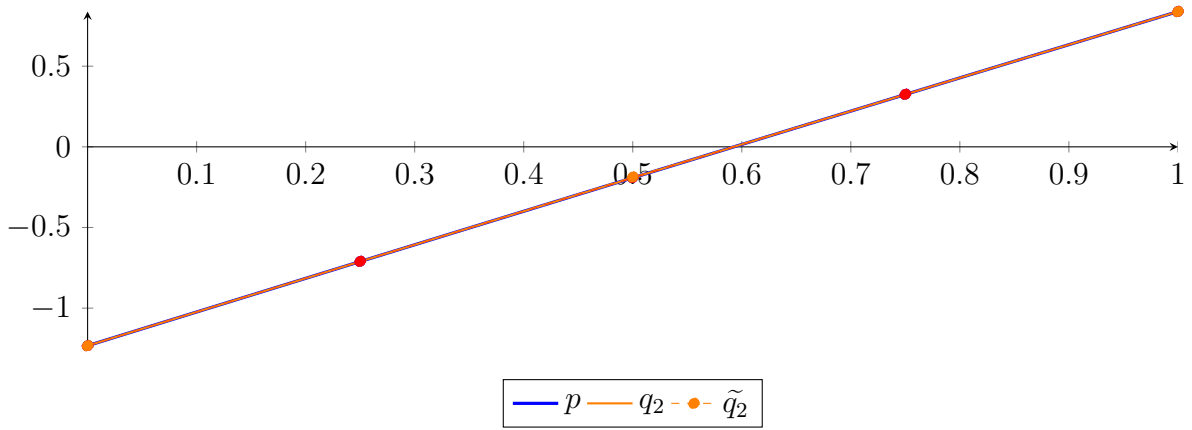
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 2.43583 \cdot 10^{-13} X^4 - 4.3876 \cdot 10^{-13} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

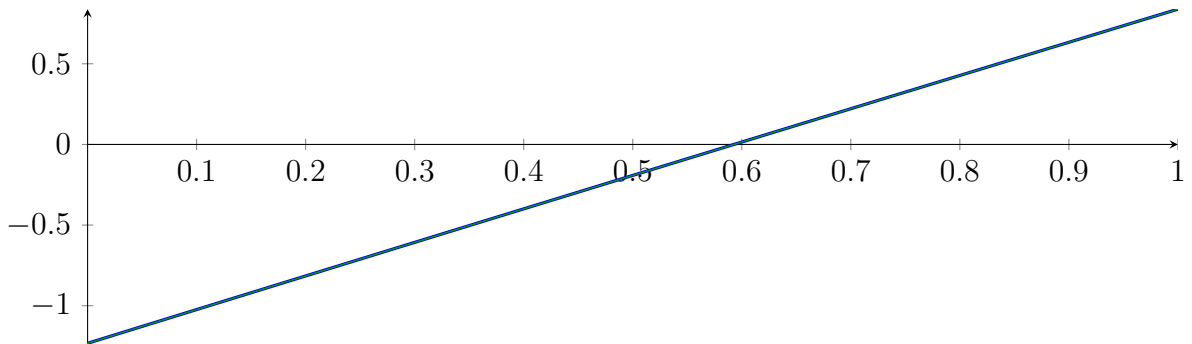
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

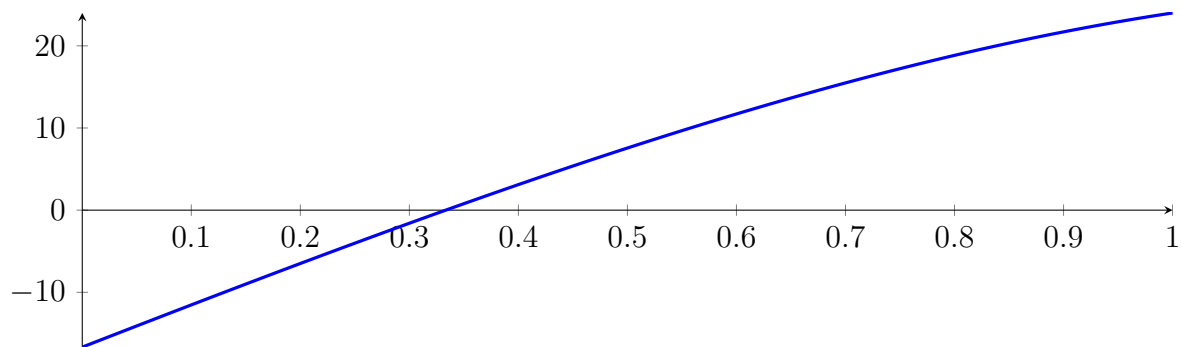
26.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Found root in interval $[0.333332, 0.333335]$ at recursion depth 3!

26.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333332, 0.333335]$$

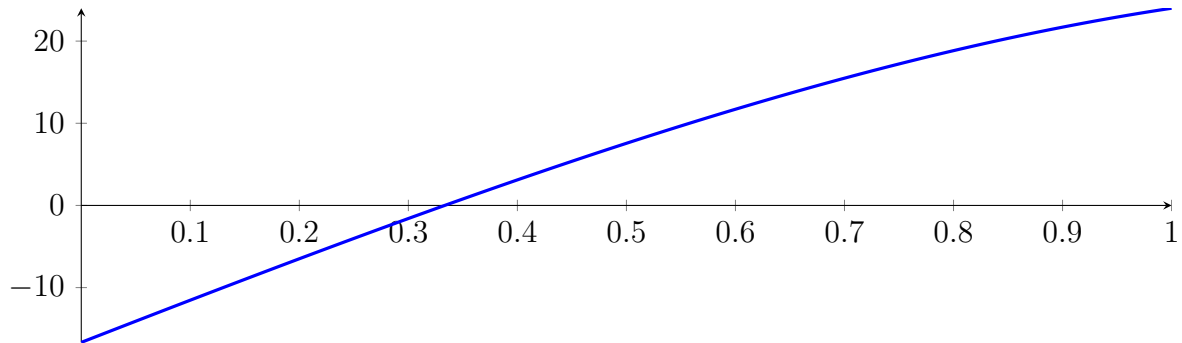
with precision $\varepsilon = 0.0001$.

27 Running CubeClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

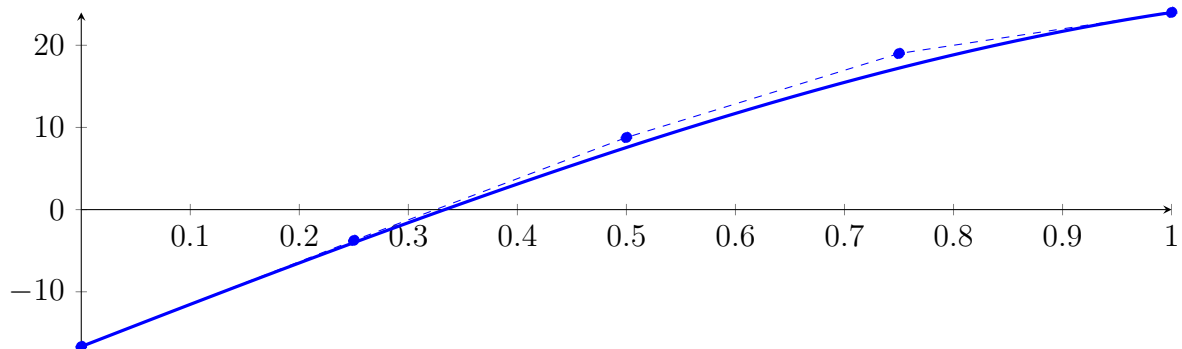
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



27.1 Recursion Branch 1 for Input Interval $[0, 1]$

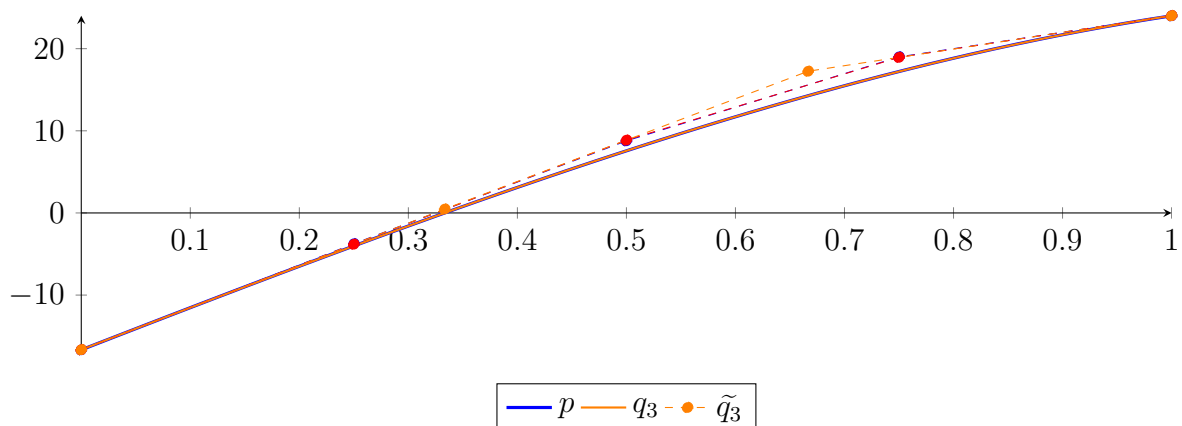
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= -2.28084 \cdot 10^{-12}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

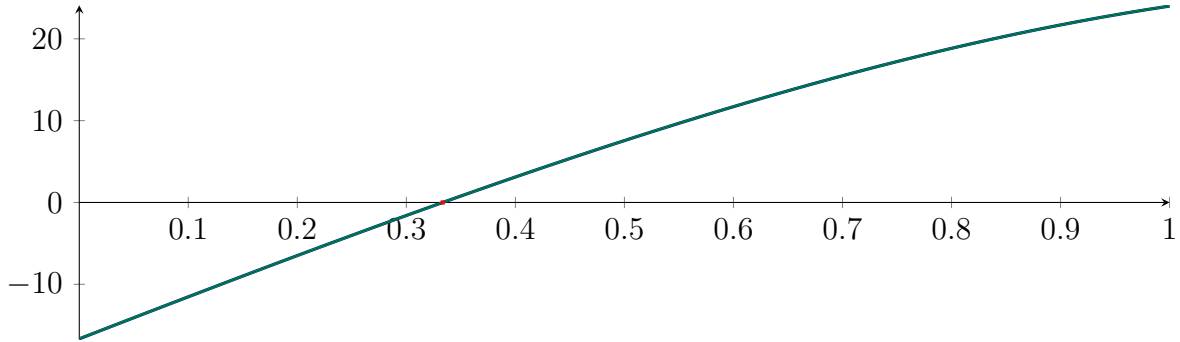
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

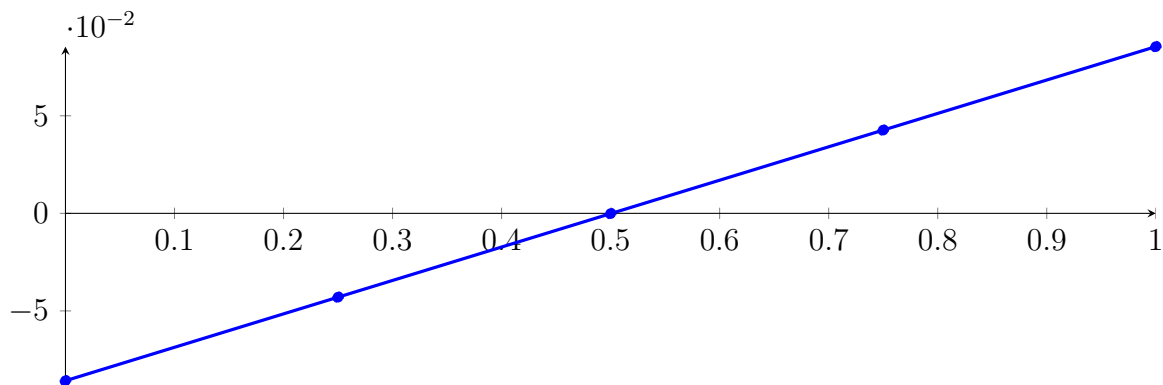
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

27.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

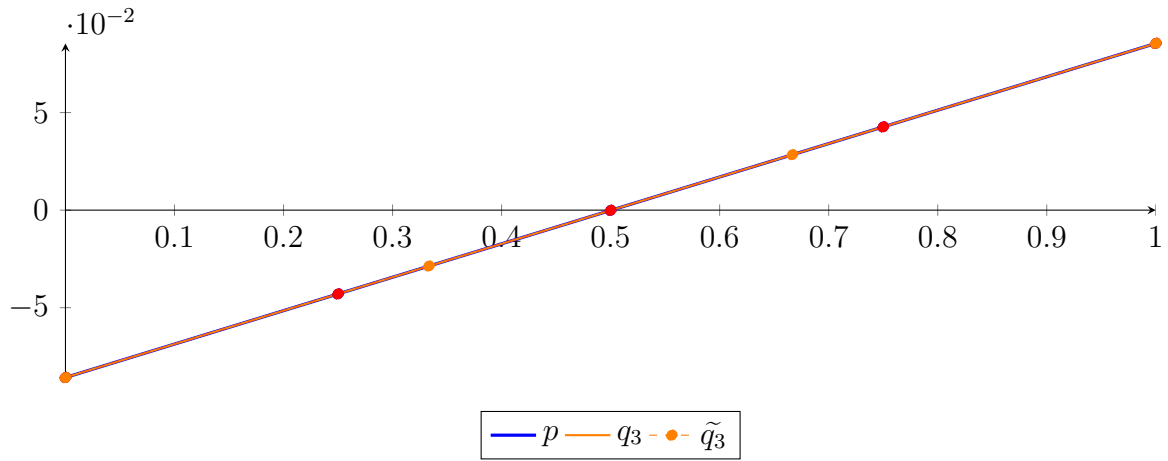
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 1.2032 \cdot 10^{-14} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45913 \cdot 10^{-11}$.

Bounding polynomials M and m :

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

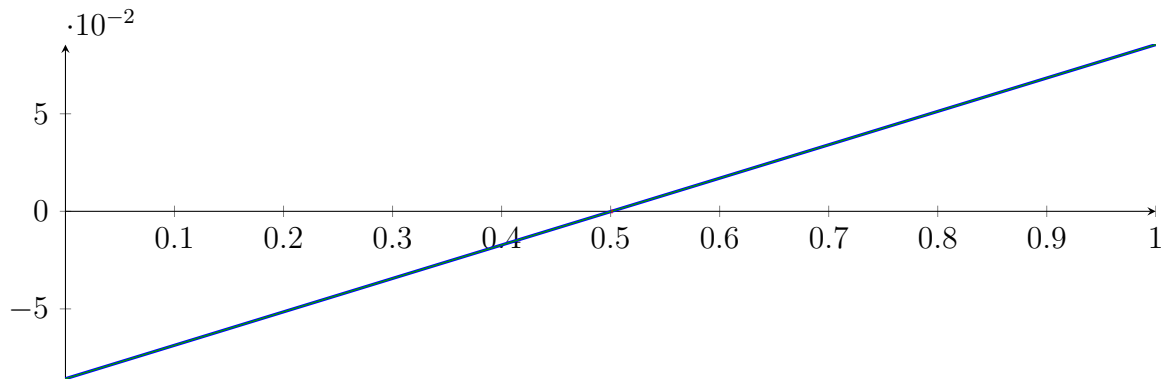
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\}$$

$$N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

Longest intersection interval: $1.70047 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

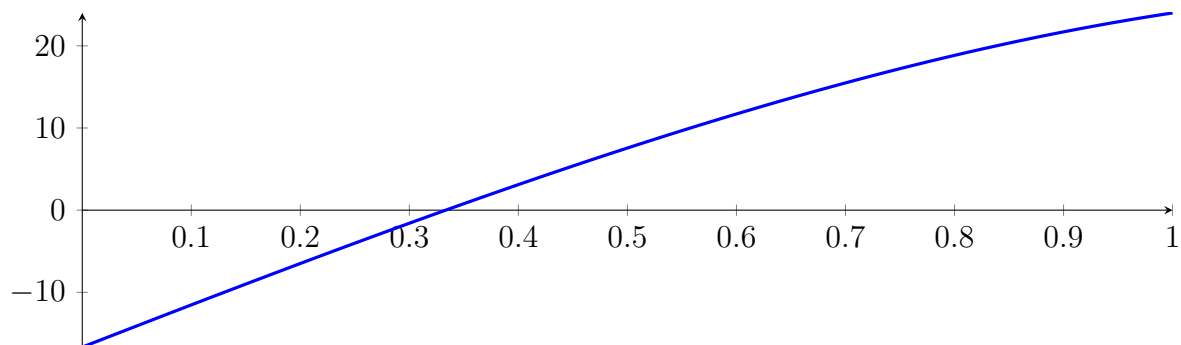
27.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

27.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

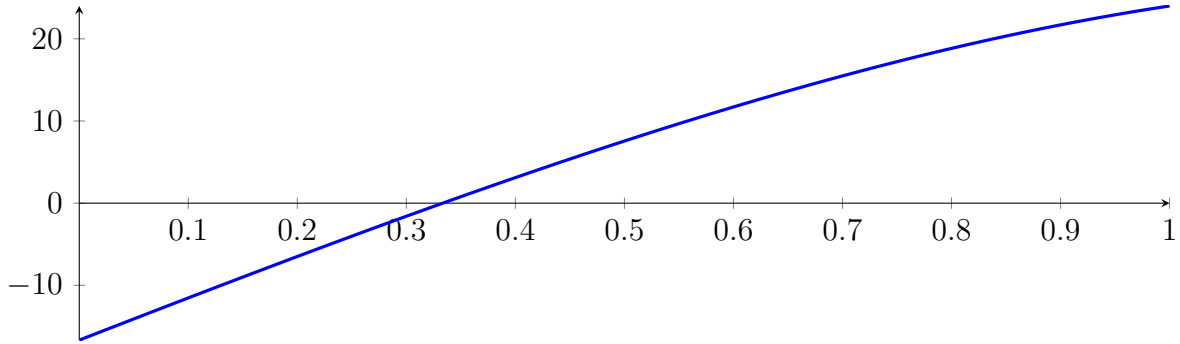
with precision $\varepsilon = 0.0001$.

28 Running BezClip on f_4 with epsilon 8

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

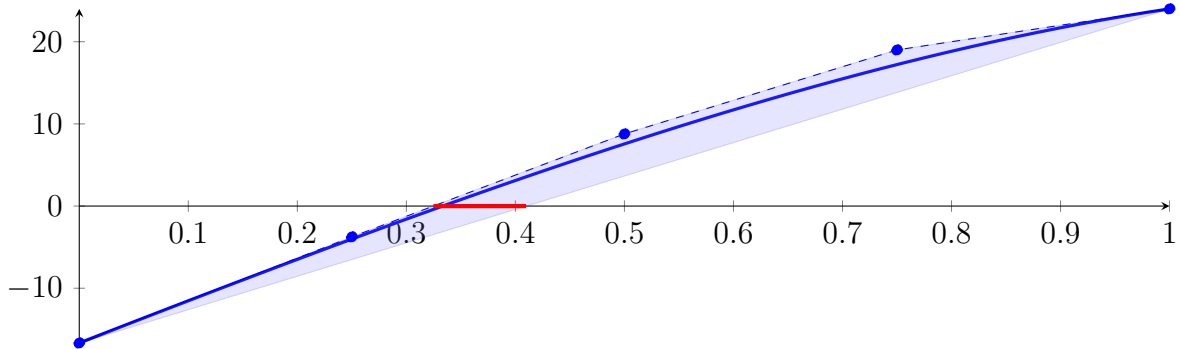
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



28.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

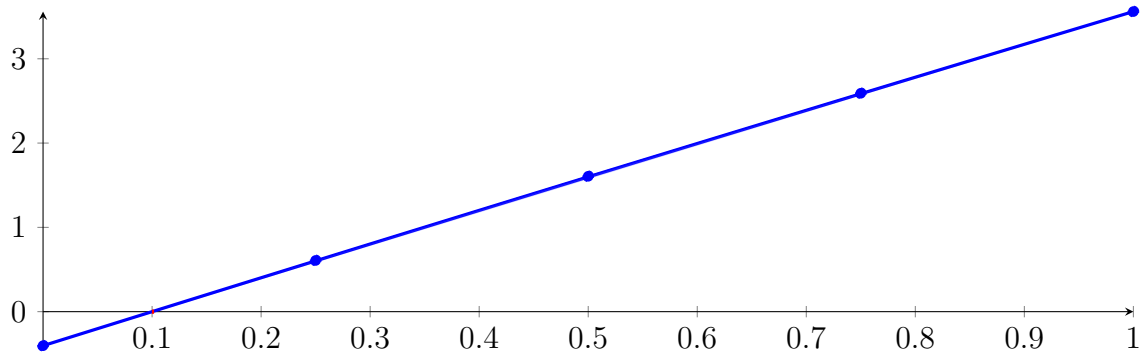
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

28.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

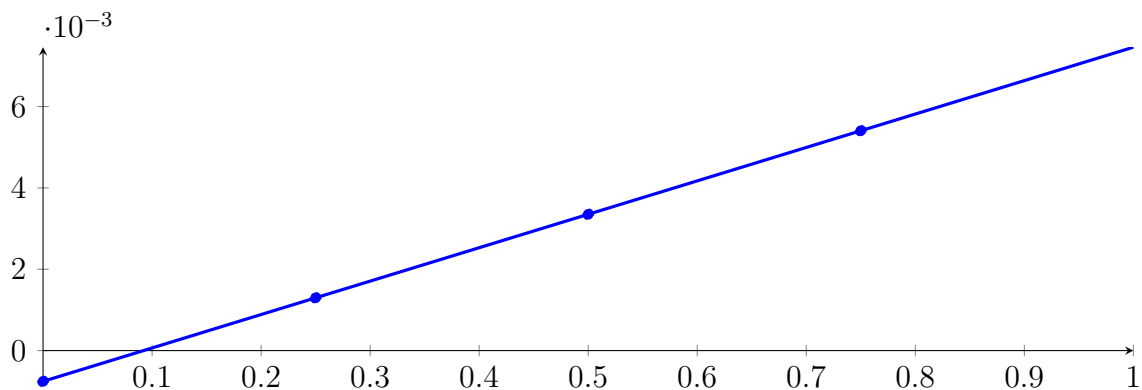
Longest intersection interval: 0.00203877

\implies Selective recursion: interval 1: $[0.333317, 0.333491]$,

28.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.06393 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

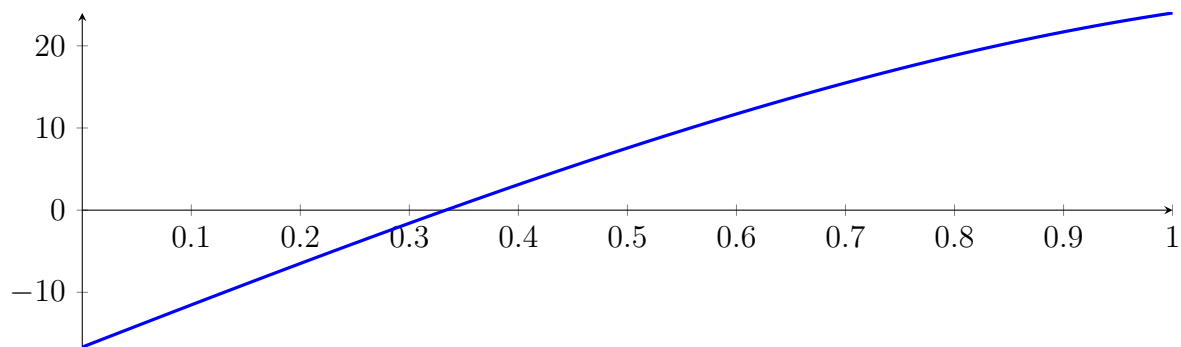
28.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

28.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

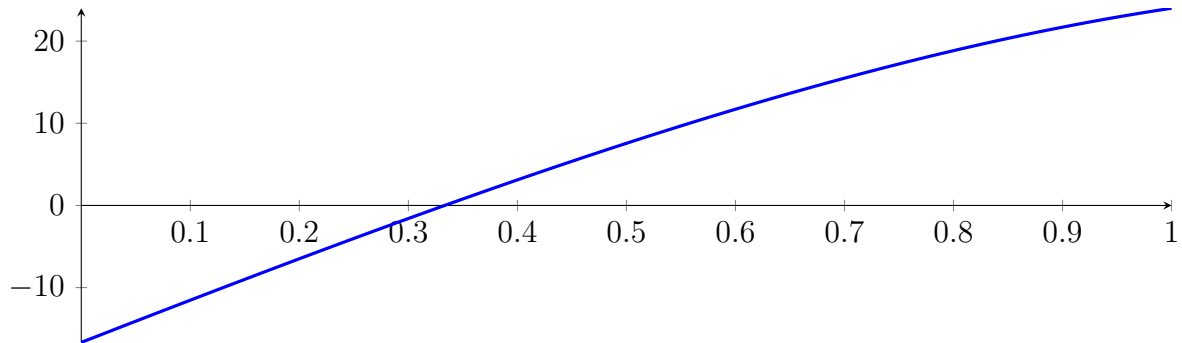
with precision $\varepsilon = 1 \cdot 10^{-08}$.

29 Running QuadClip on f_4 with epsilon 8

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

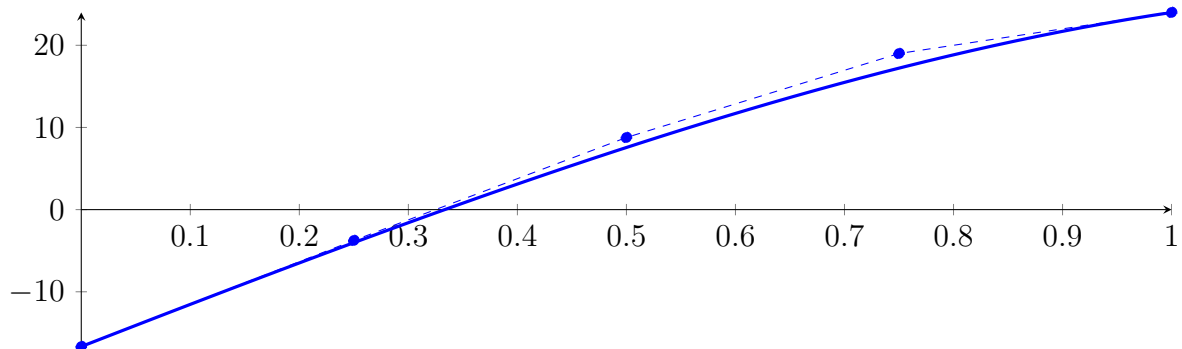
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



29.1 Recursion Branch 1 for Input Interval $[0, 1]$

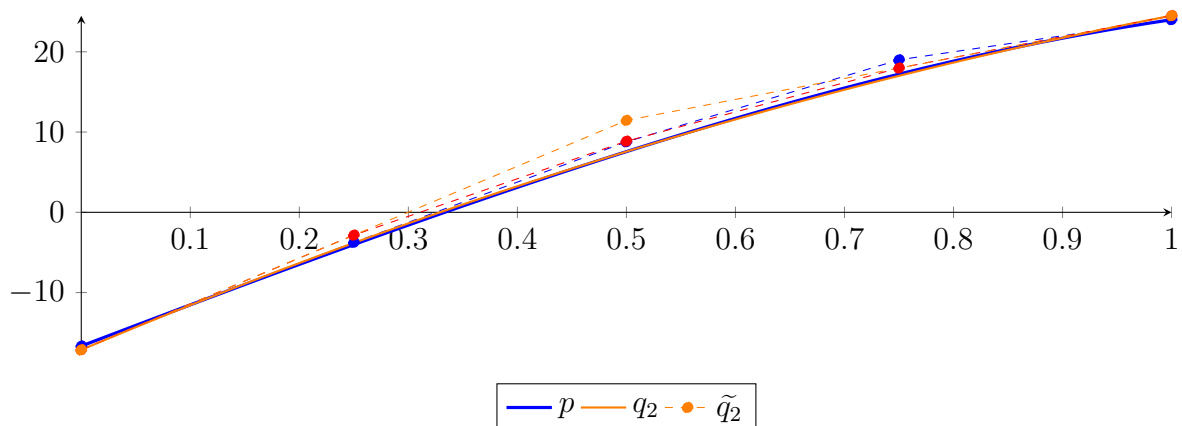
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \\ \tilde{q}_2 &= -2.67519 \cdot 10^{-12}X^4 + 5.32907 \cdot 10^{-12}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

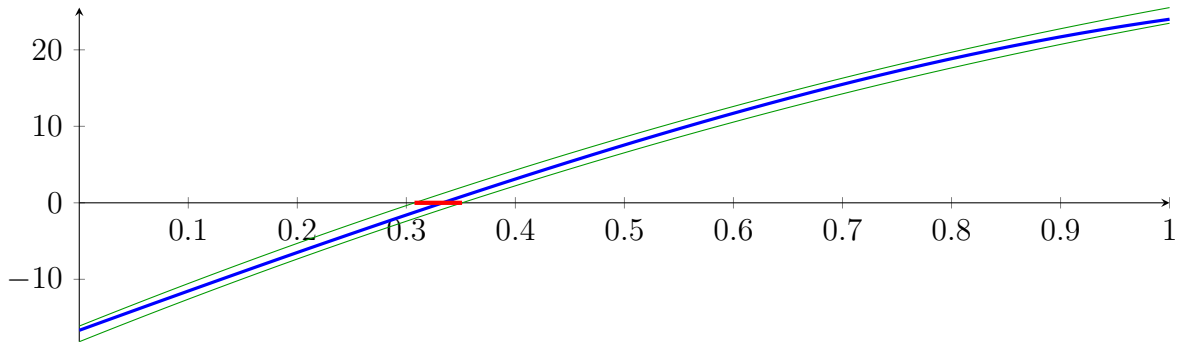
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

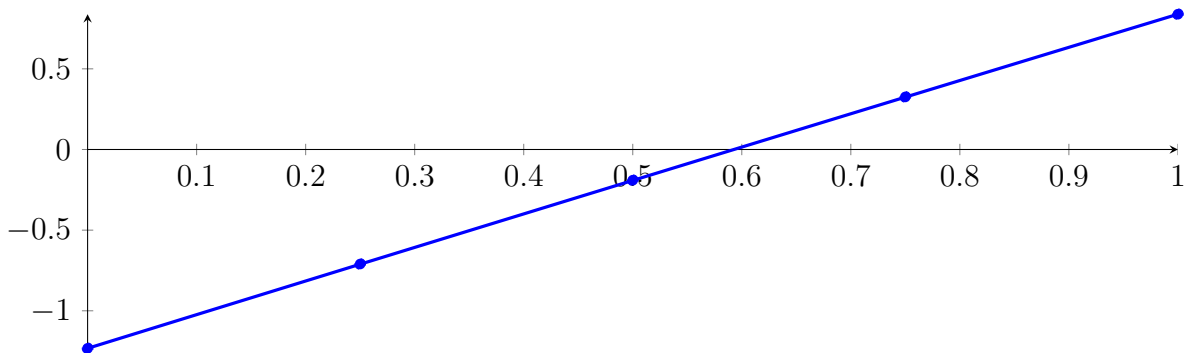
Longest intersection interval: 0.0436205

⇒ Selective recursion: interval 1: $[0.307477, 0.351097]$,

29.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

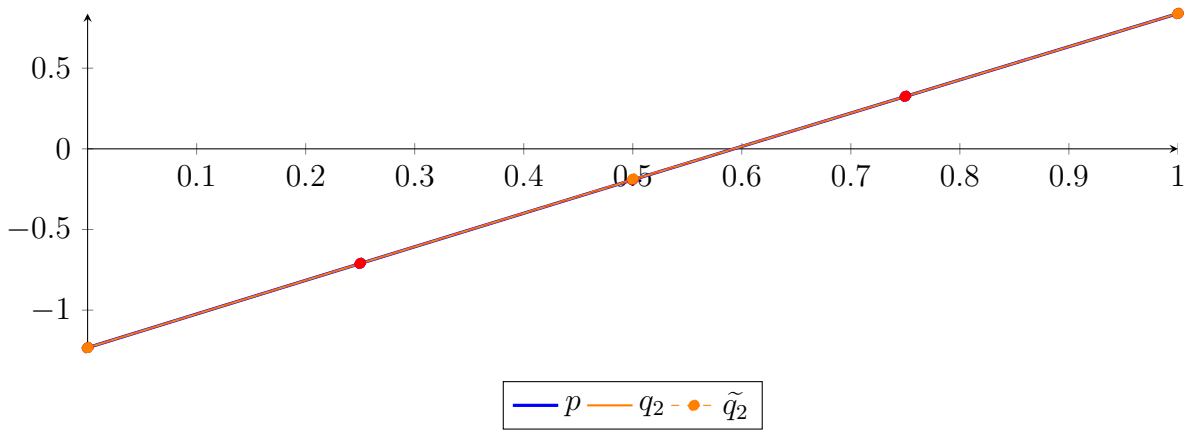
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 2.43583 \cdot 10^{-13} X^4 - 4.3876 \cdot 10^{-13} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

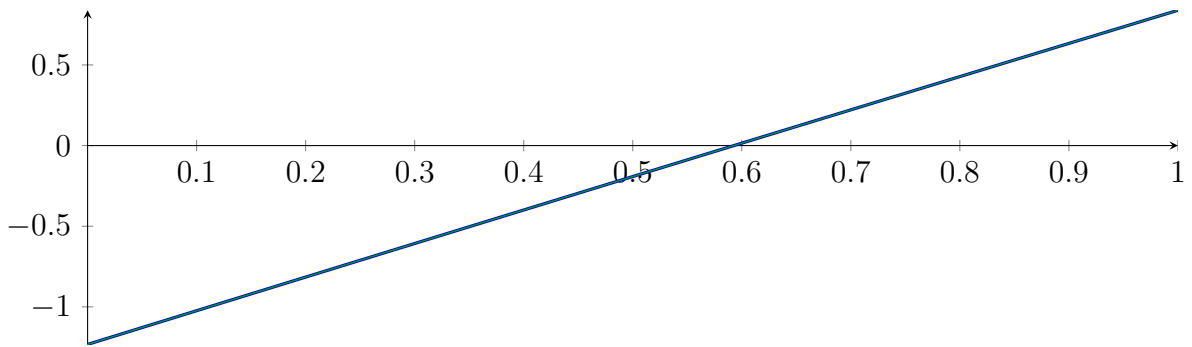
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

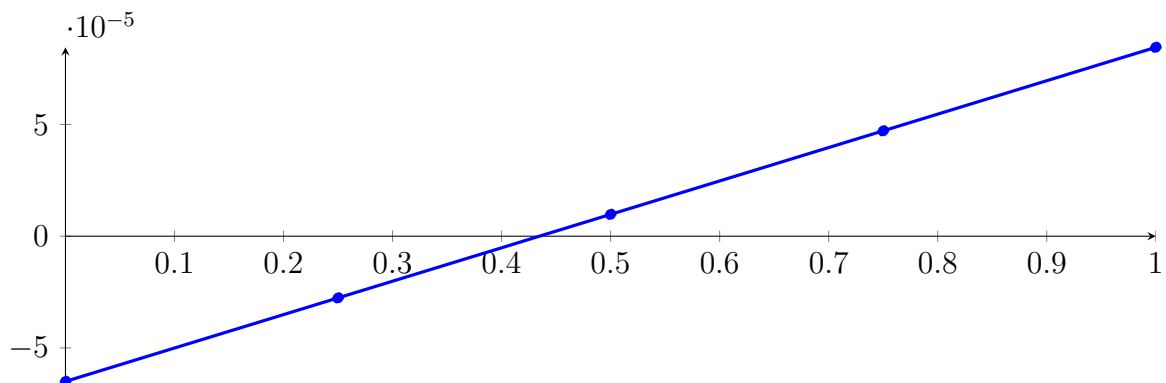
Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

29.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -2.71051 \cdot 10^{-20} X^4 - 2.82489 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05} \\
 &= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \\
 &\quad \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)
 \end{aligned}$$



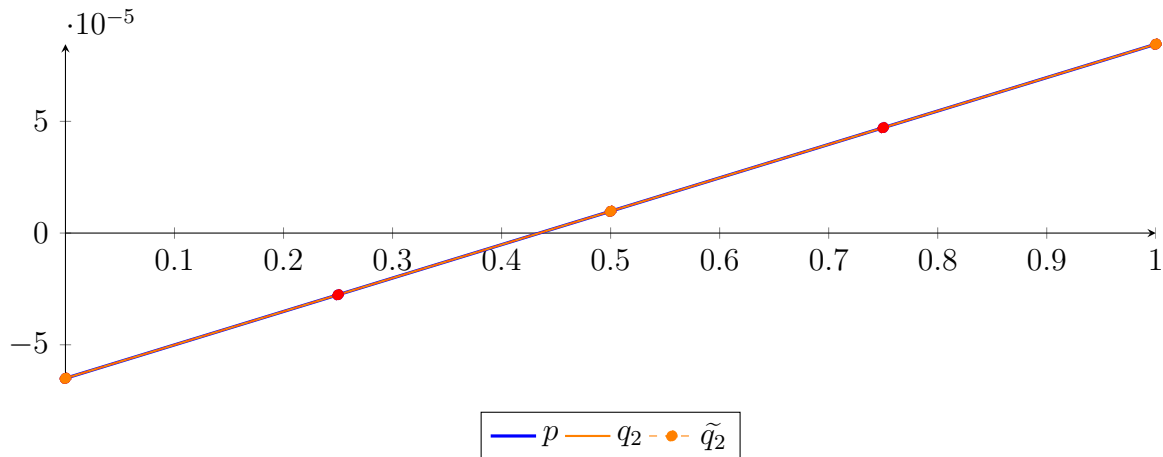
Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = 6.72205 \cdot 10^{-18} X^4 - 1.21431 \cdot 10^{-17} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.88601 \cdot 10^{-17}$.

Bounding polynomials M and m :

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

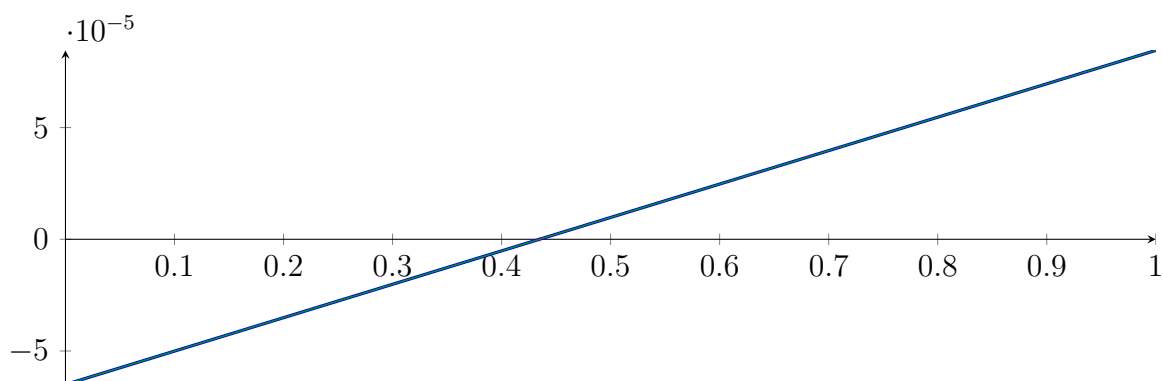
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.434685, 1.4089 \cdot 10^6\}$$

$$N(m) = \{0.434685, 1.4089 \cdot 10^6\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

Longest intersection interval: $1.27678 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

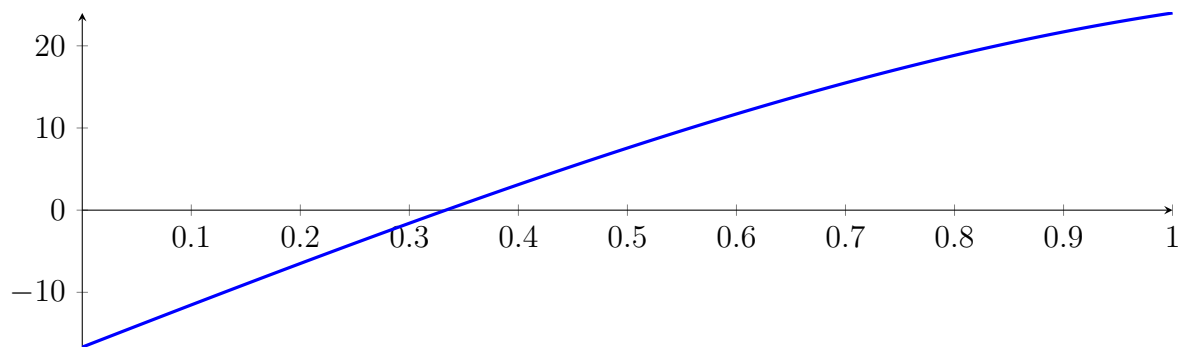
29.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

29.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

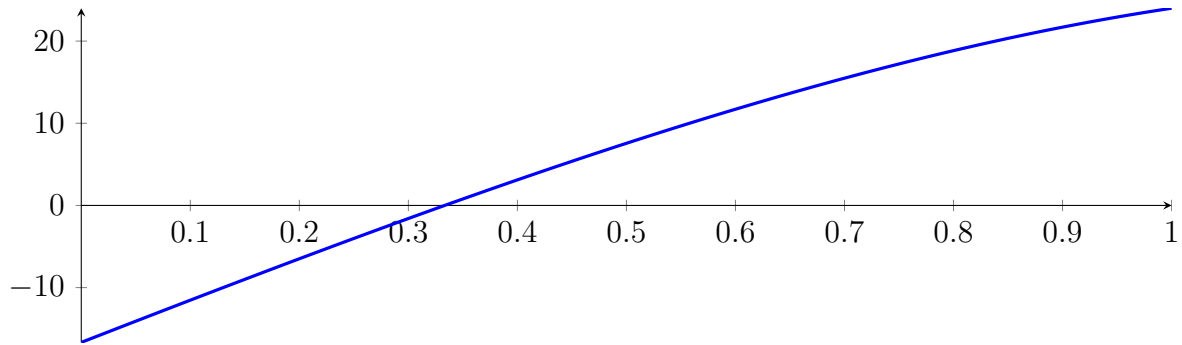
with precision $\varepsilon = 1 \cdot 10^{-08}$.

30 Running CubeClip on f_4 with epsilon 8

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

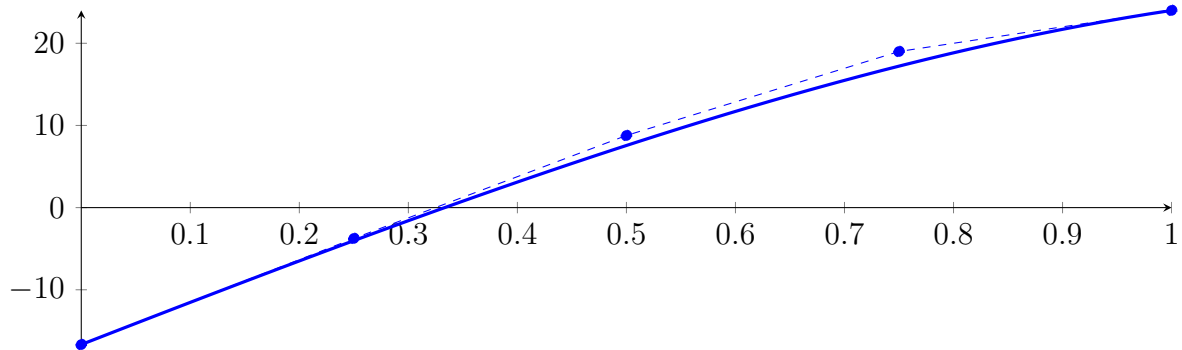
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



30.1 Recursion Branch 1 for Input Interval $[0, 1]$

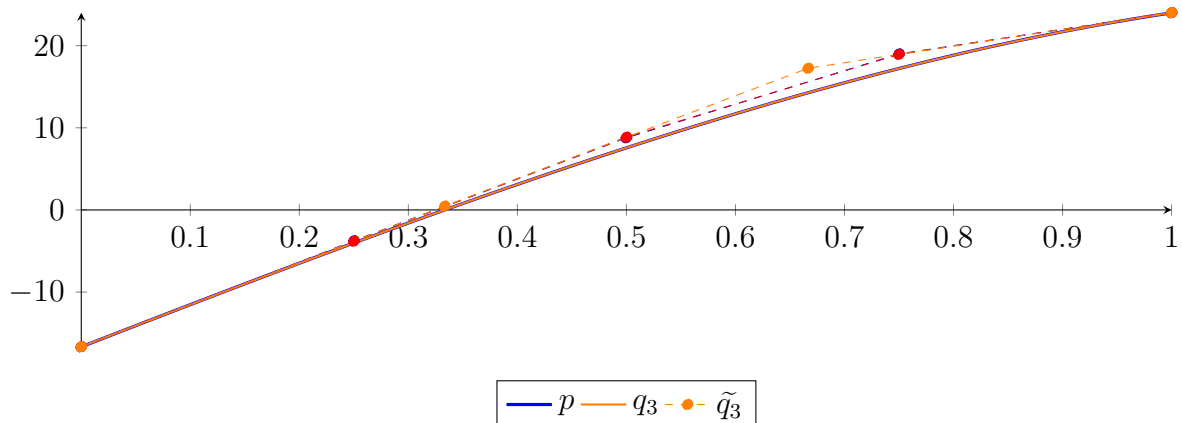
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= -2.28084 \cdot 10^{-12}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

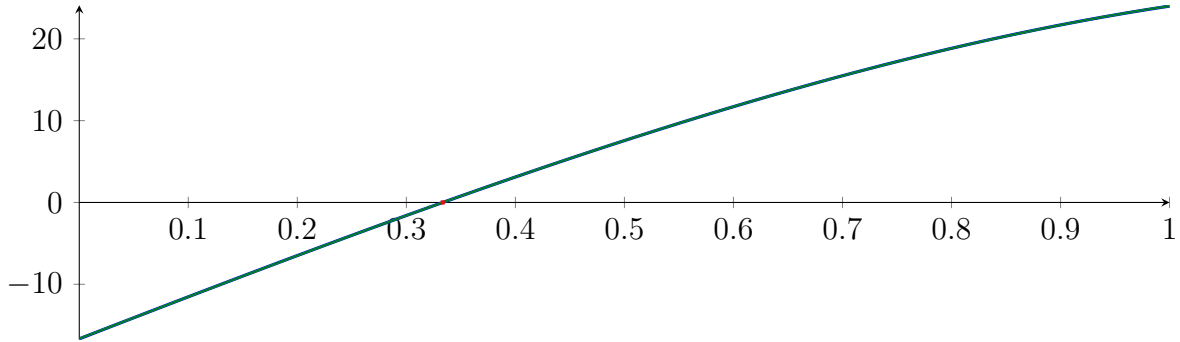
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

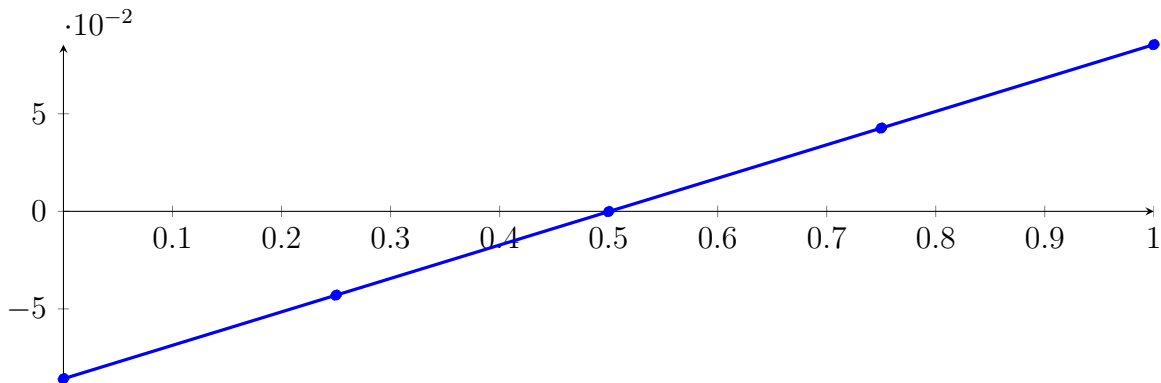
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

30.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

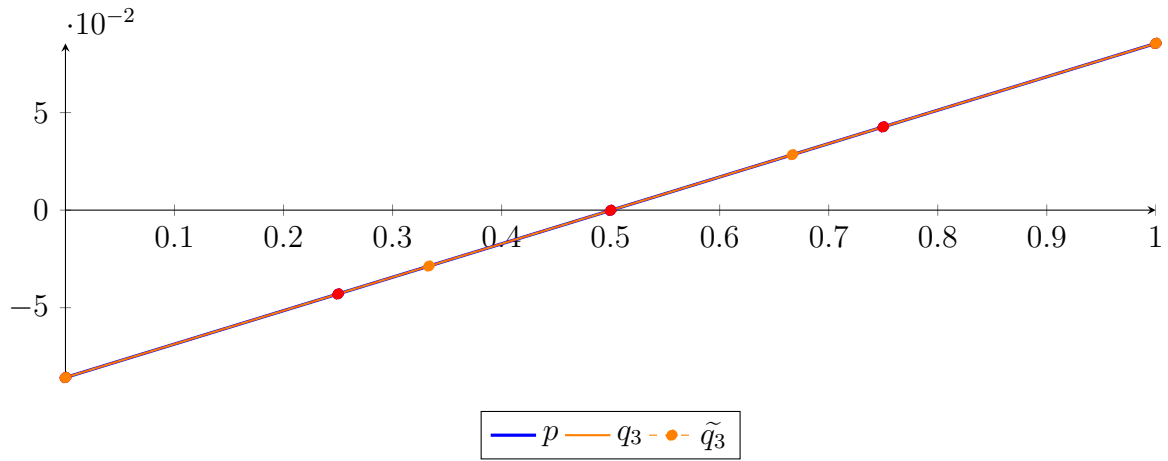
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 1.2032 \cdot 10^{-14} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45913 \cdot 10^{-11}$.

Bounding polynomials M and m :

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

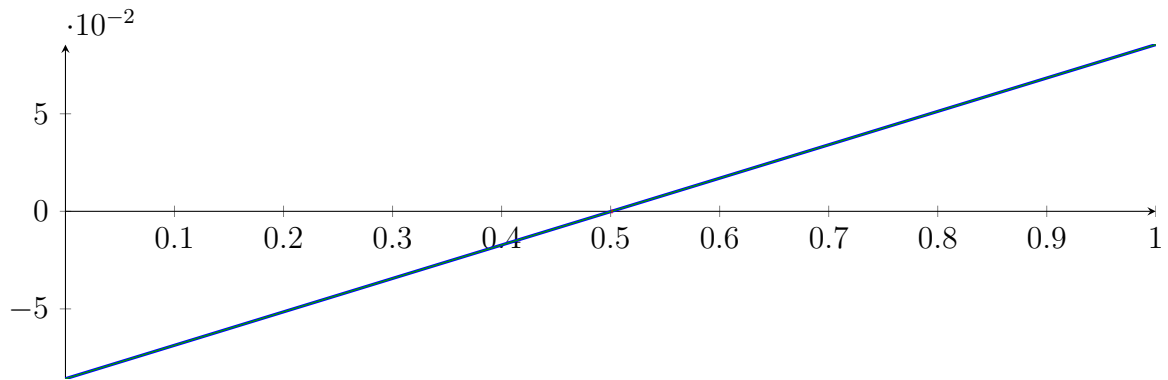
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\}$$

$$N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

Longest intersection interval: $1.70047 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

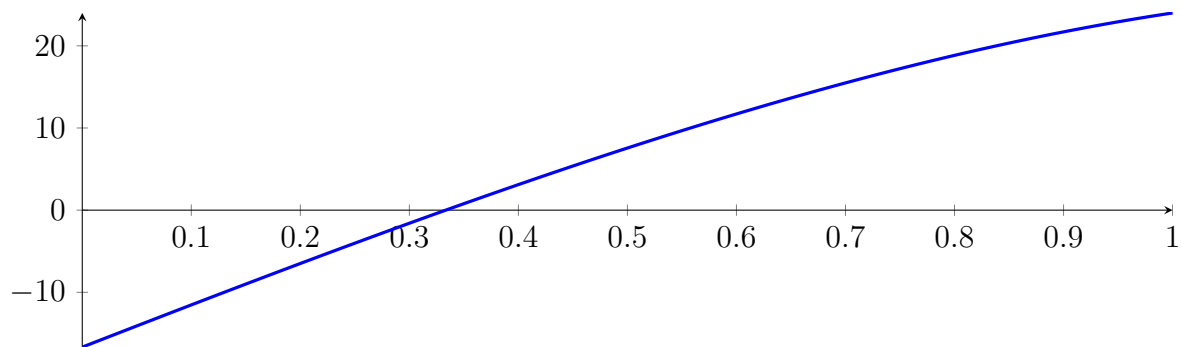
30.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

30.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

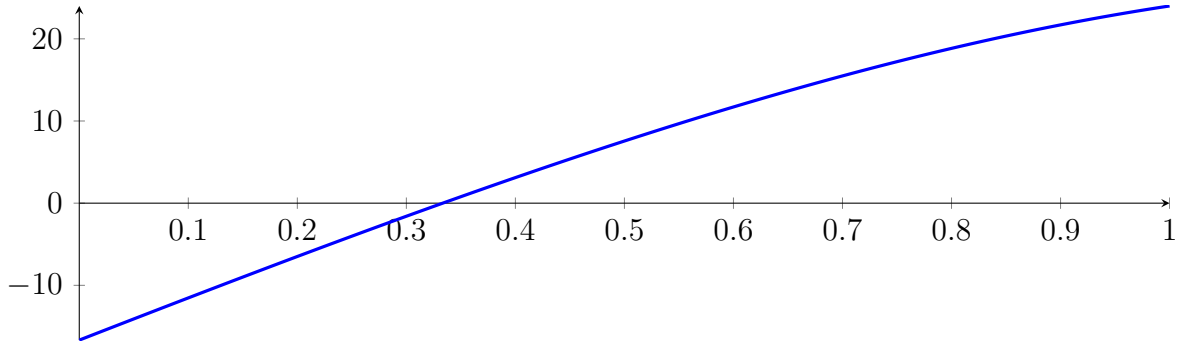
with precision $\varepsilon = 1 \cdot 10^{-08}$.

31 Running BezClip on f_4 with epsilon 16

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

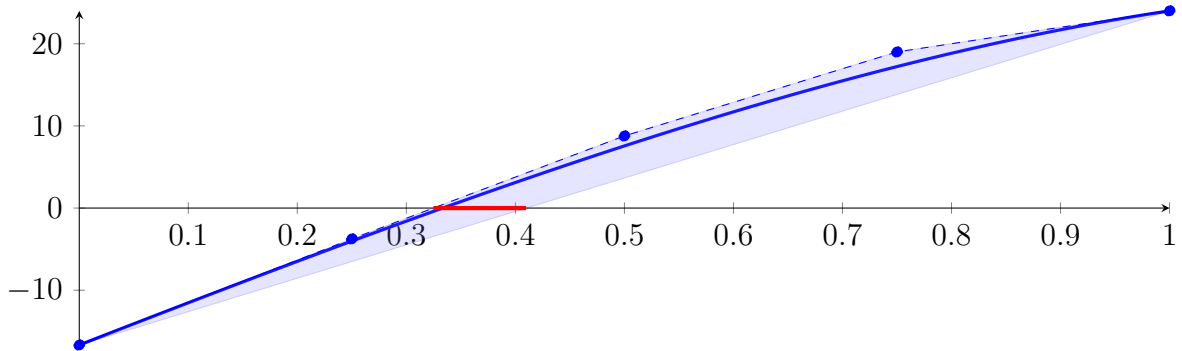
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



31.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

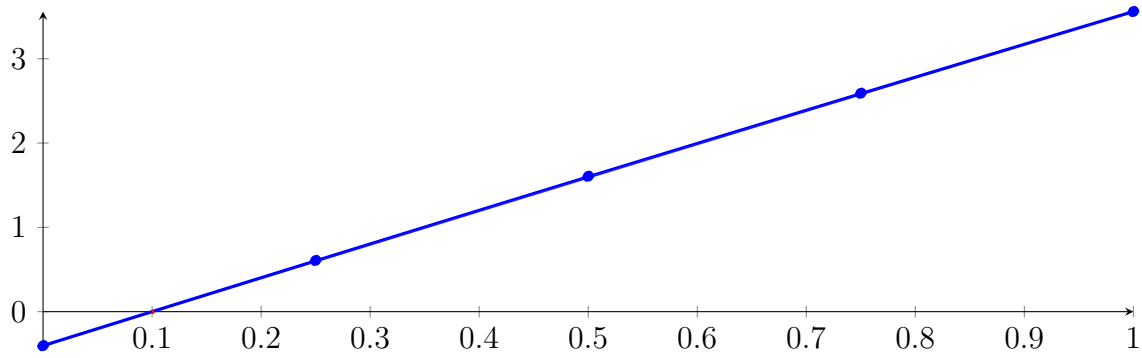
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

31.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

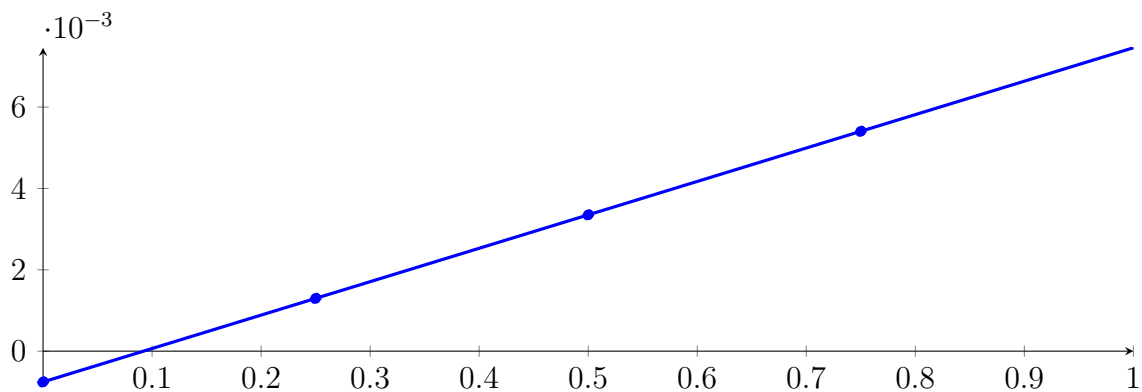
Longest intersection interval: 0.00203877

\implies Selective recursion: interval 1: $[0.333317, 0.333491]$,

31.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.06393 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

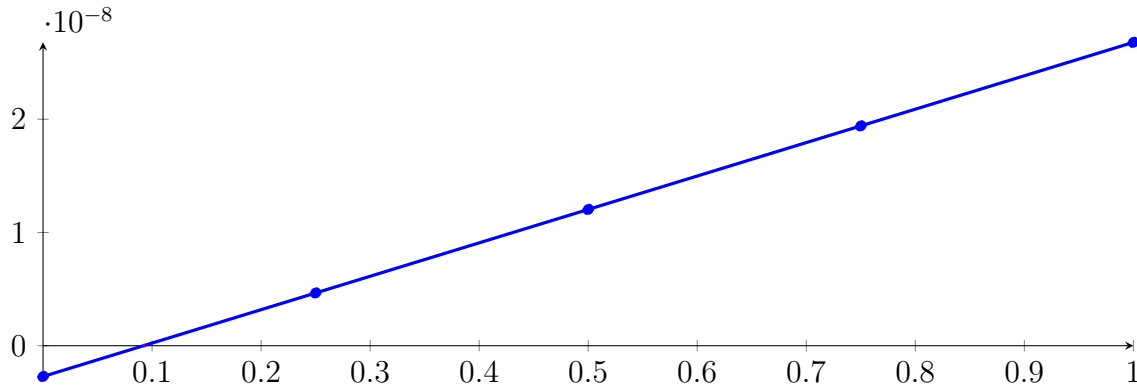
Longest intersection interval: $3.59185 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

31.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.92617 \cdot 10^{-24} X^4 + 6.61744 \cdot 10^{-24} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09} \\ &= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \\ &\quad \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28974 \cdot 10^{-11}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

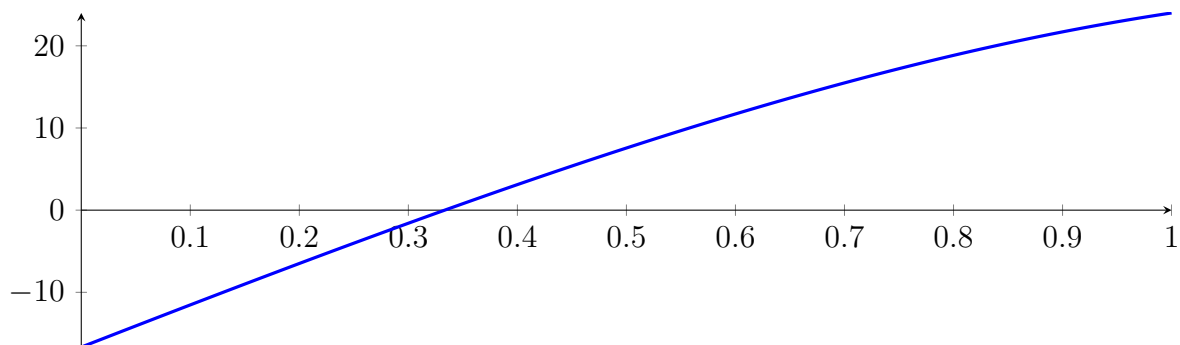
31.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

31.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

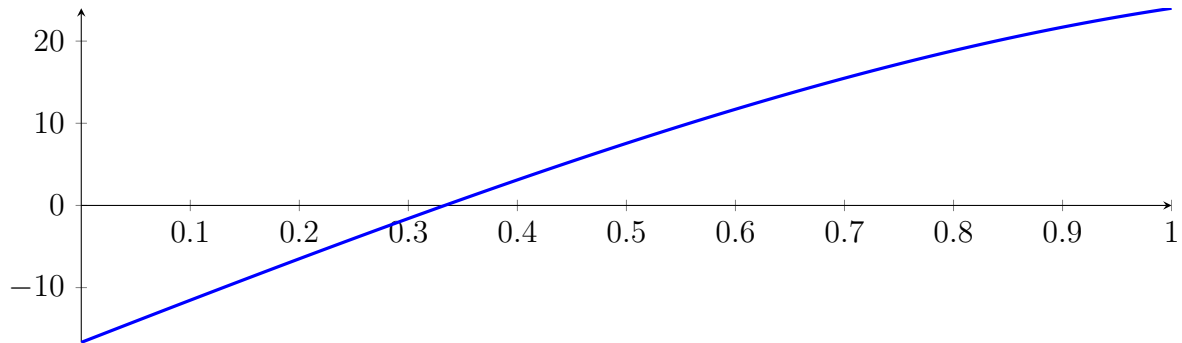
with precision $\varepsilon = 1 \cdot 10^{-16}$.

32 Running QuadClip on f_4 with epsilon 16

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

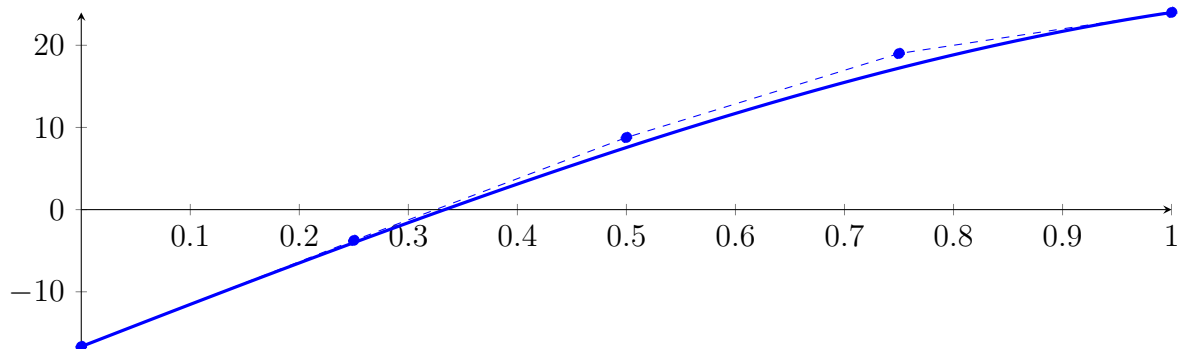
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



32.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

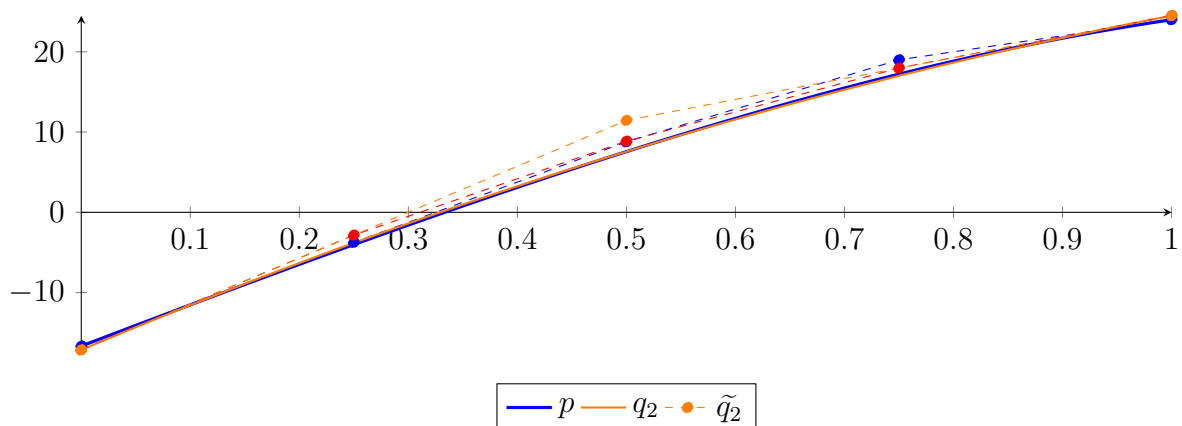
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -2.67519 \cdot 10^{-12}X^4 + 5.32907 \cdot 10^{-12}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

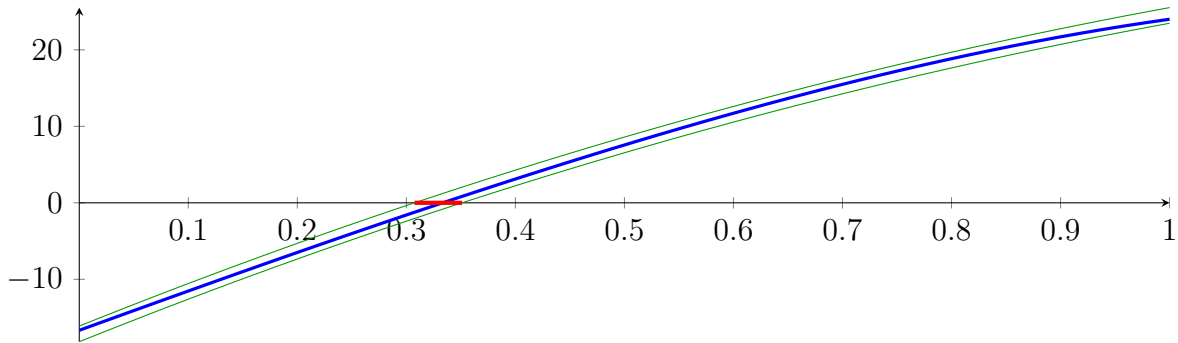
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

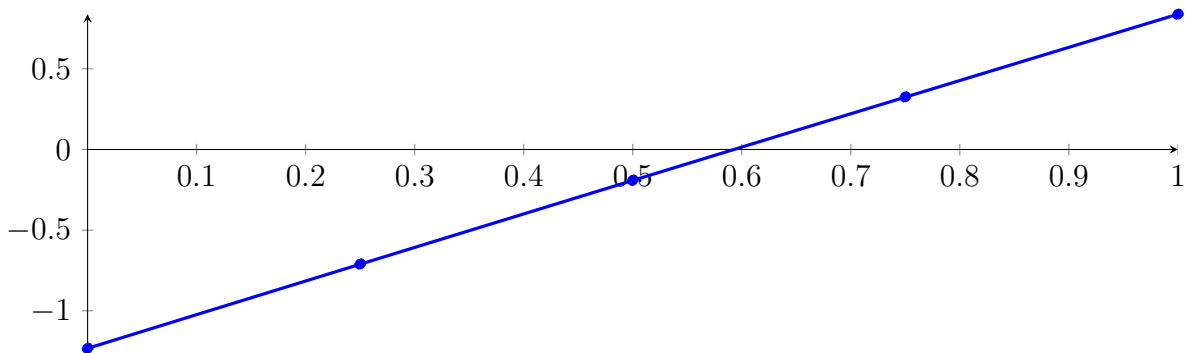
Longest intersection interval: 0.0436205

⇒ Selective recursion: **interval 1:** $[0.307477, 0.351097]$,

32.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

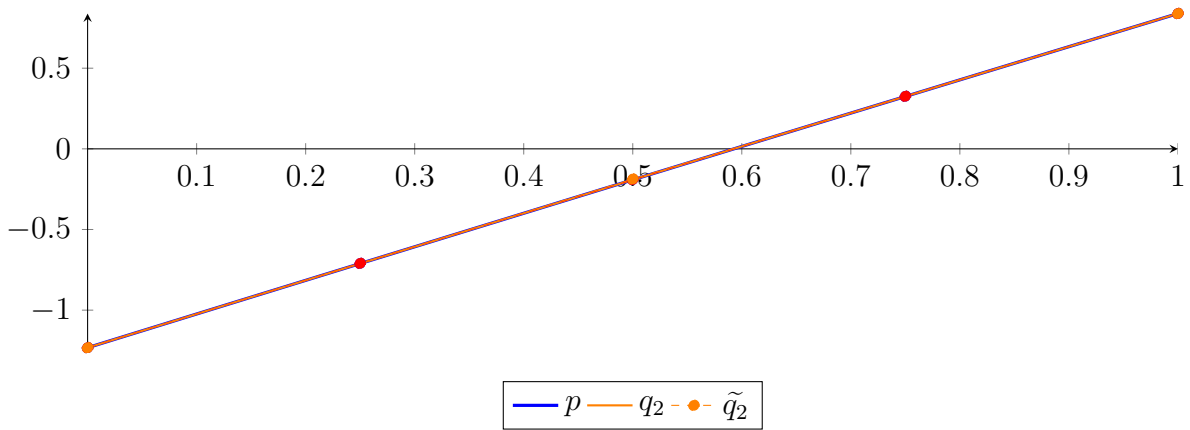
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 2.43583 \cdot 10^{-13} X^4 - 4.3876 \cdot 10^{-13} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

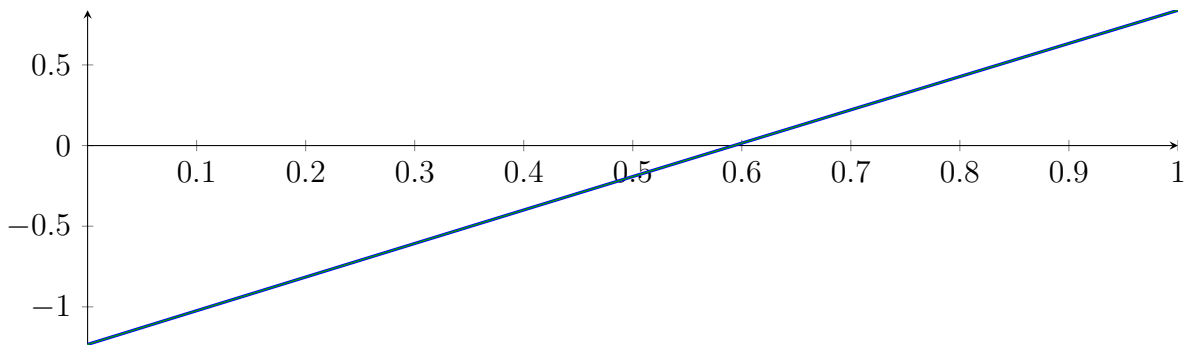
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

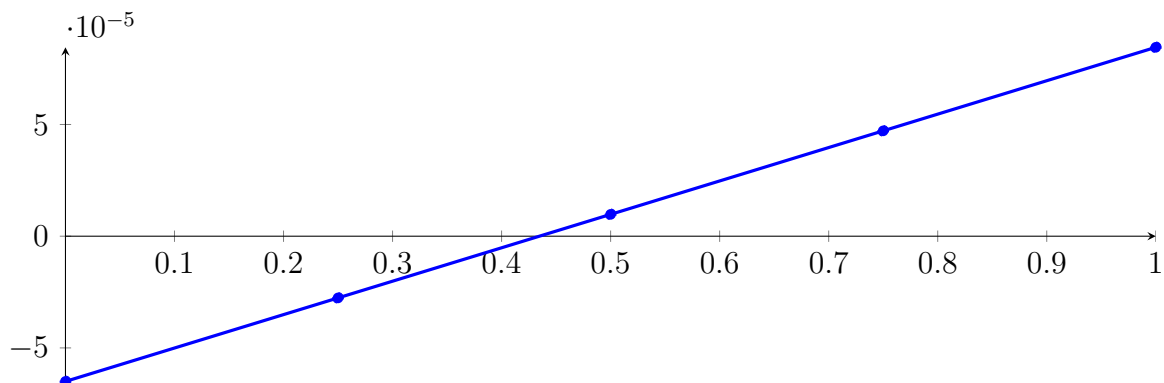
Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

32.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -2.71051 \cdot 10^{-20} X^4 - 2.82489 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05} \\
 &= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \\
 &\quad \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)
 \end{aligned}$$



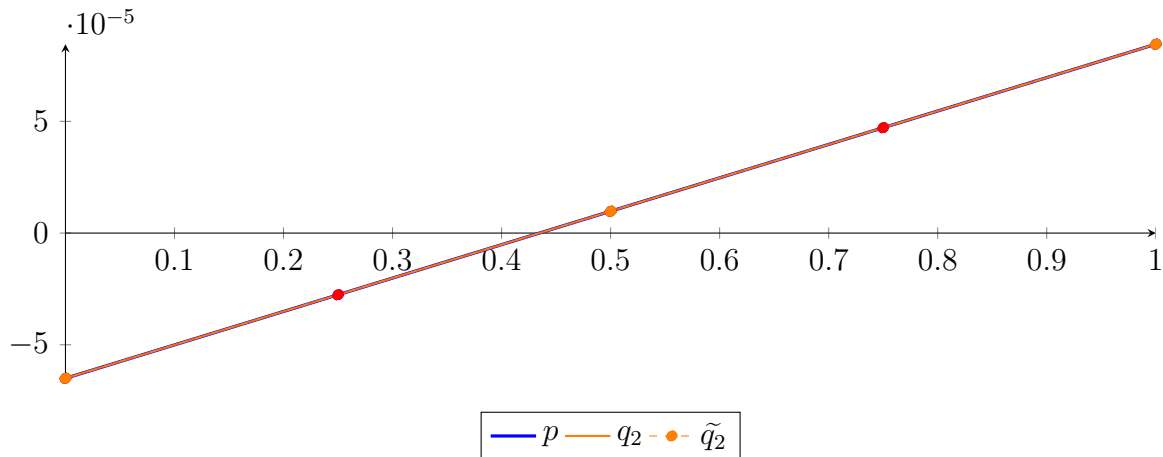
Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = 6.72205 \cdot 10^{-18} X^4 - 1.21431 \cdot 10^{-17} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.88601 \cdot 10^{-17}$.

Bounding polynomials M and m :

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

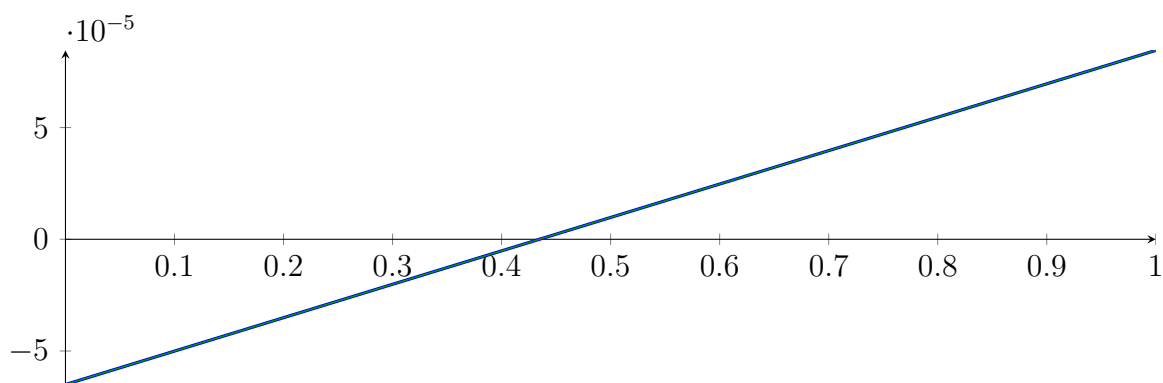
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.434685, 1.4089 \cdot 10^6\}$$

$$N(m) = \{0.434685, 1.4089 \cdot 10^6\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

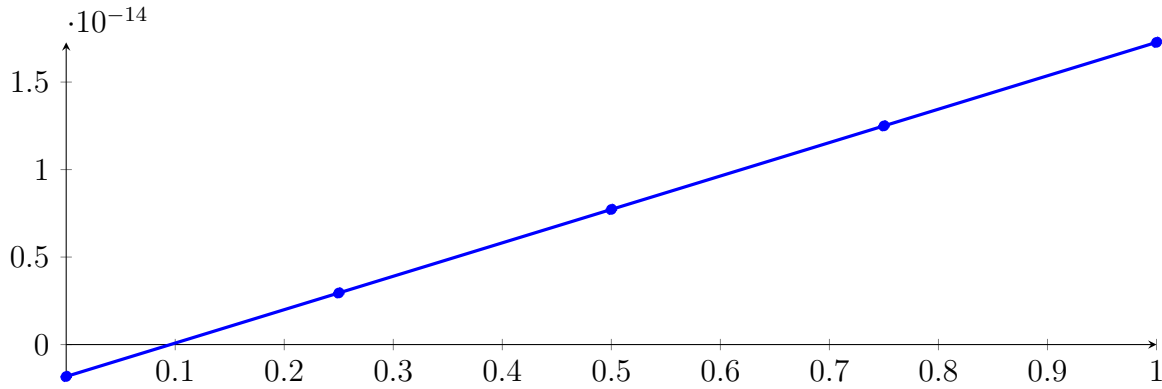
Longest intersection interval: $1.27678 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

32.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

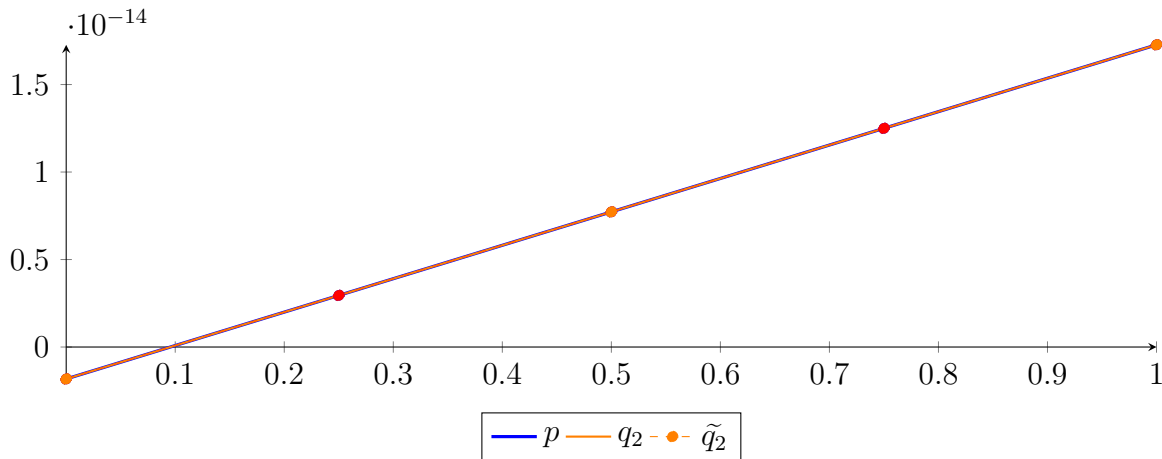
$$\begin{aligned}
 p &= -1.41995 \cdot 10^{-29} X^4 + 6.31089 \cdot 10^{-30} X^3 + 4.73317 \cdot 10^{-30} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15} \\
 &= -1.8241 \cdot 10^{-15} B_{0,4}(X) + 2.94943 \cdot 10^{-15} B_{1,4}(X) + 7.72295 \\
 &\quad \cdot 10^{-15} B_{2,4}(X) + 1.24965 \cdot 10^{-14} B_{3,4}(X) + 1.727 \cdot 10^{-14} B_{4,4}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -2.68213 \cdot 10^{-29} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15} \\
 &= -1.8241 \cdot 10^{-15} B_{0,2} + 7.72295 \cdot 10^{-15} B_{1,2} + 1.727 \cdot 10^{-14} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -2.18041 \cdot 10^{-27} X^4 + 3.9443 \cdot 10^{-27} X^3 - 2.24352 \cdot 10^{-27} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15} \\
 &= -1.8241 \cdot 10^{-15} B_{0,4} + 2.94943 \cdot 10^{-15} B_{1,4} + 7.72295 \cdot 10^{-15} B_{2,4} + 1.24965 \cdot 10^{-14} B_{3,4} + 1.727 \cdot 10^{-14} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.73549 \cdot 10^{-28}$.

Bounding polynomials M and m :

$$M = -2.68213 \cdot 10^{-29} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15}$$

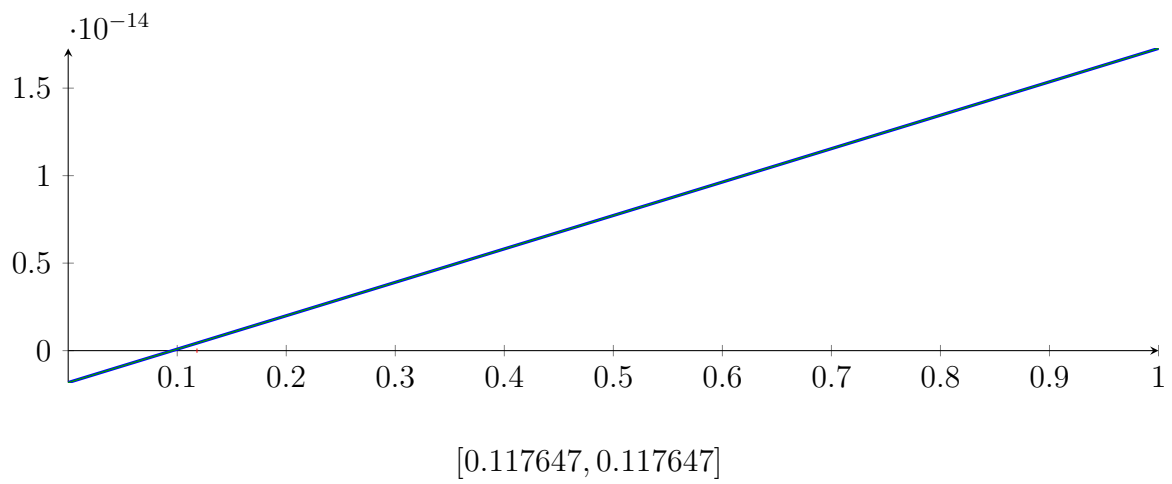
$$m = -2.68213 \cdot 10^{-29} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15}$$

Root of M and m :

$$N(M) = \{0.117647, 7.11901 \cdot 10^{14}\}$$

$$N(m) = \{0.117647, 7.11901 \cdot 10^{14}\}$$

Intersection intervals:



Longest intersection interval: 0

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

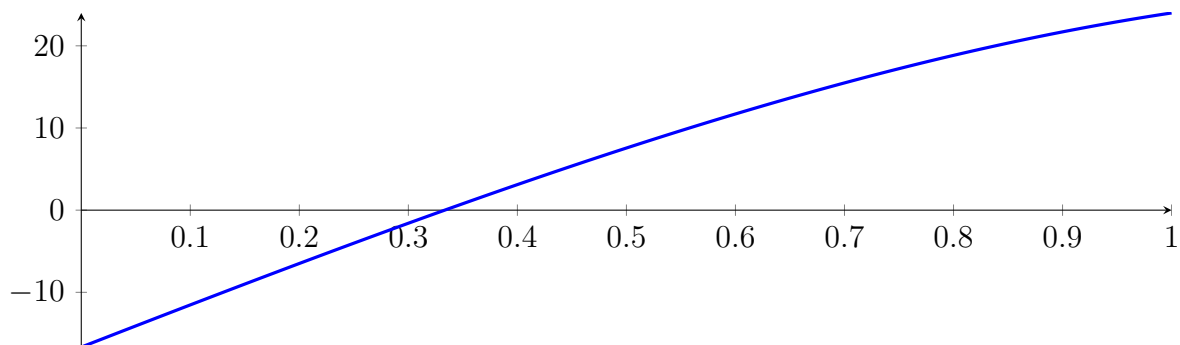
32.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

32.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

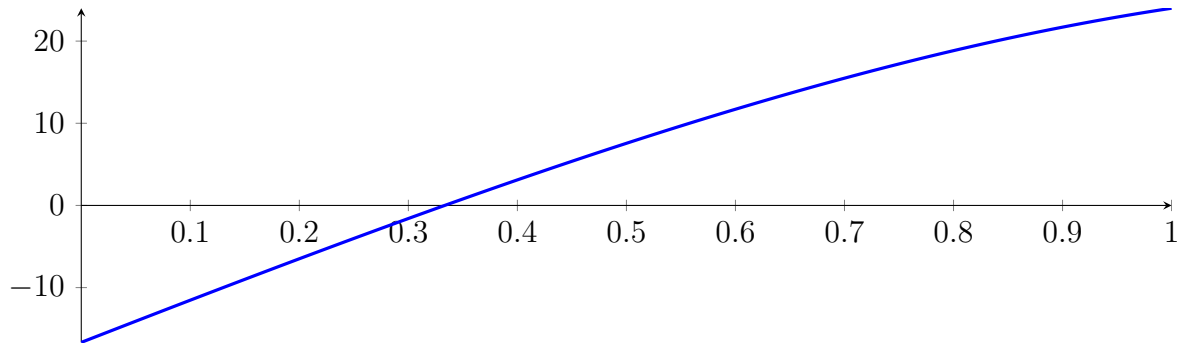
with precision $\varepsilon = 1 \cdot 10^{-16}$.

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$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

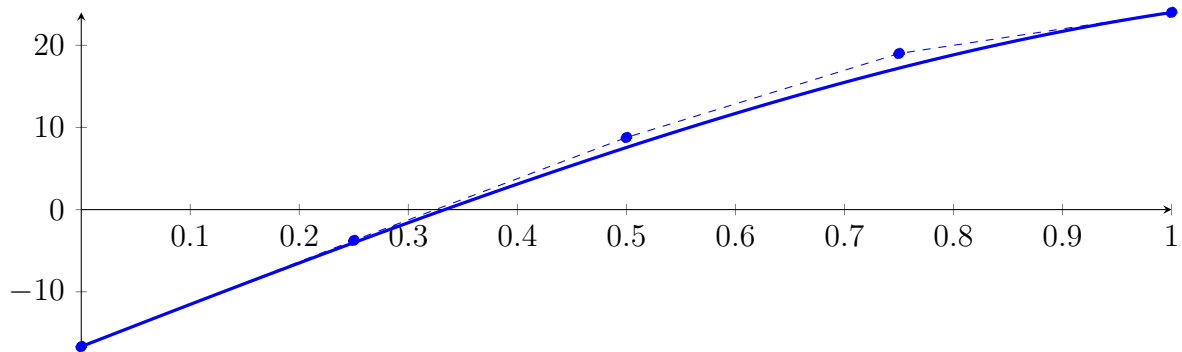
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



33.1 Recursion Branch 1 for Input Interval $[0, 1]$

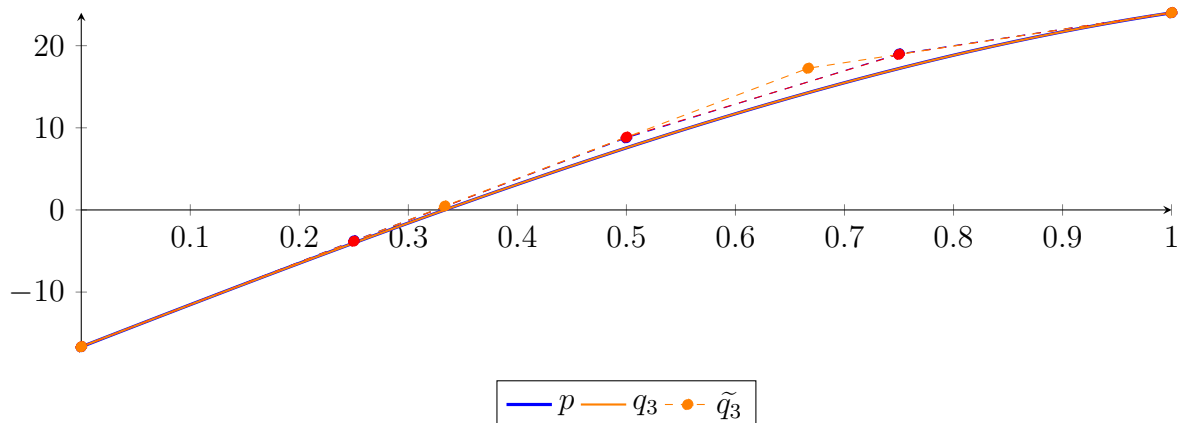
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= -2.28084 \cdot 10^{-12}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

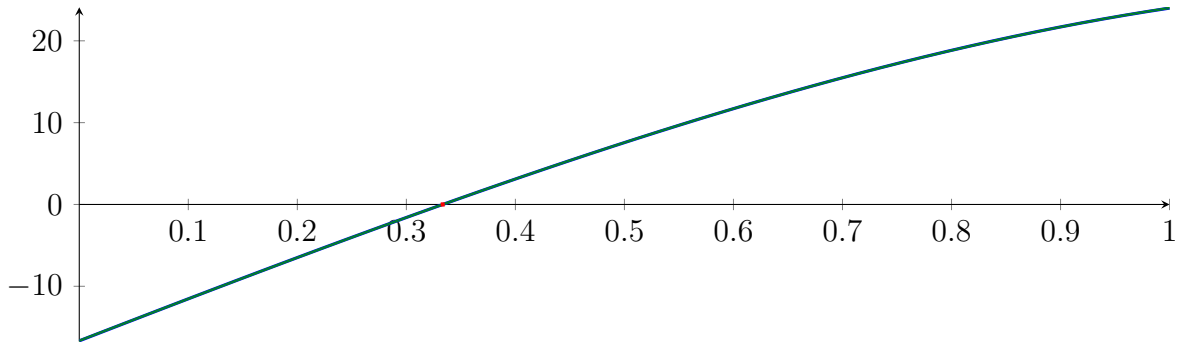
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

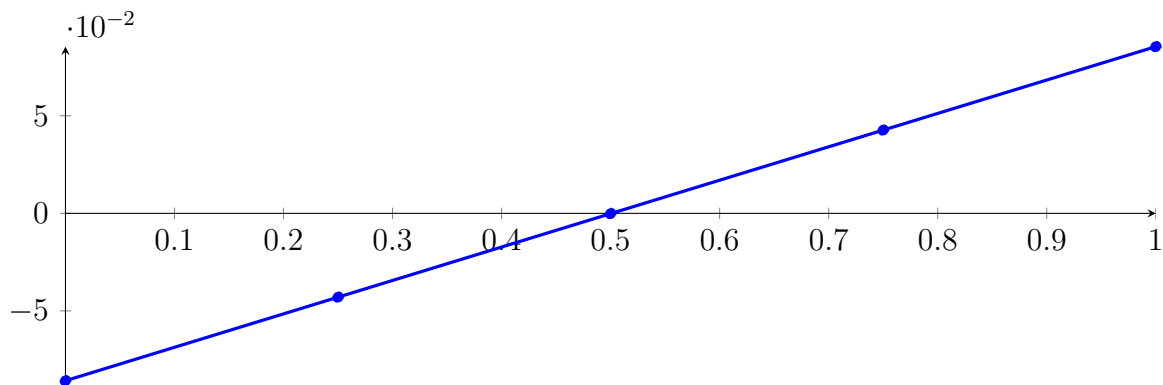
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

33.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

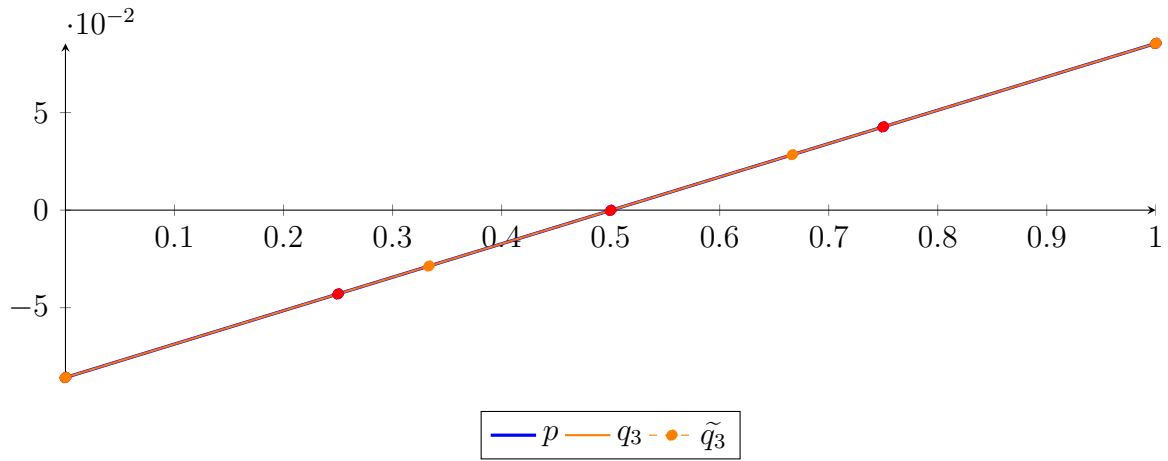
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 1.2032 \cdot 10^{-14} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45913 \cdot 10^{-11}$.

Bounding polynomials M and m :

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

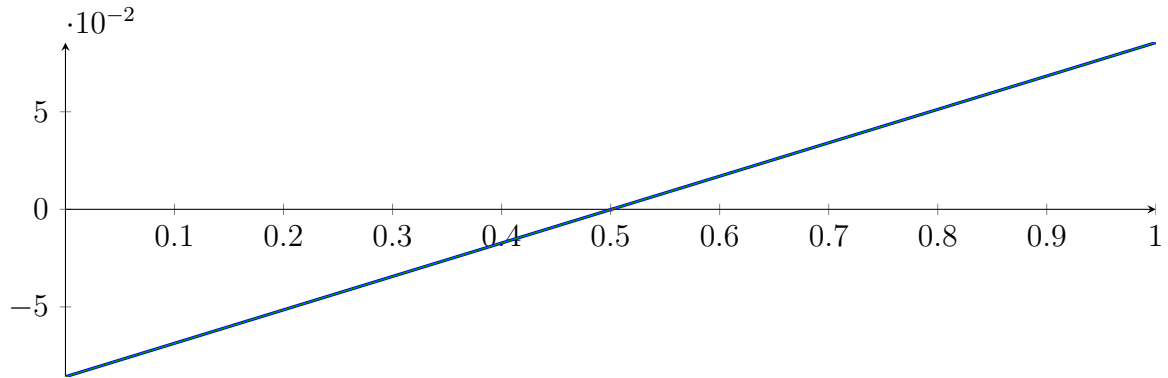
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\}$$

$$N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

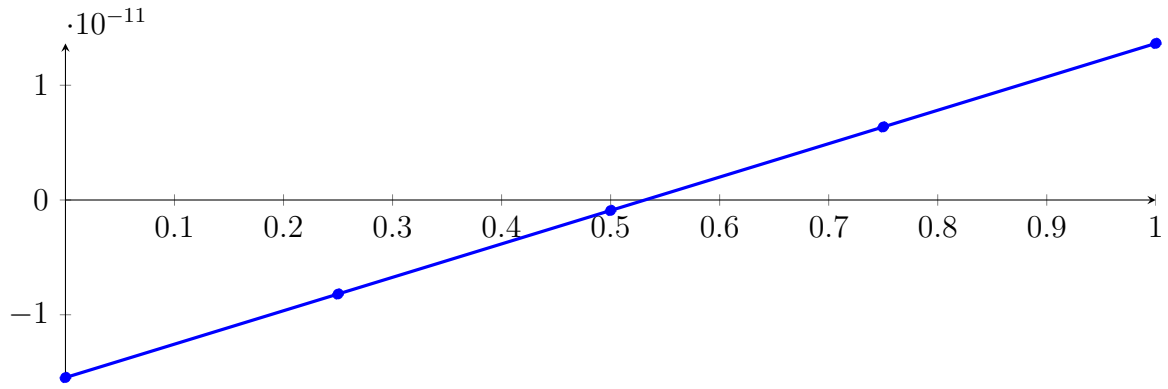
Longest intersection interval: $1.70047 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

33.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

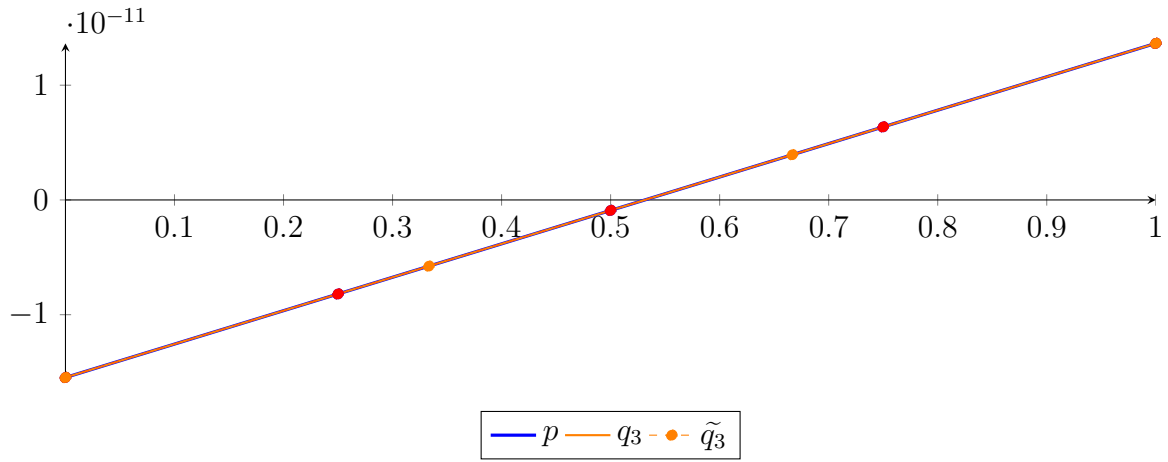
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.01312 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\ &= -1.5473 \cdot 10^{-11} B_{0,4}(X) - 8.19335 \cdot 10^{-12} B_{1,4}(X) - 9.13745 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36586 \cdot 10^{-12} B_{3,4}(X) + 1.36455 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -6.78547 \cdot 10^{-25} X^3 - 2.95652 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\
 &= -1.5473 \cdot 10^{-11} B_{0,3} - 5.76681 \cdot 10^{-12} B_{1,3} + 3.93932 \cdot 10^{-12} B_{2,3} + 1.36455 \cdot 10^{-11} B_{3,3} \\
 \tilde{q}_3 &= 2.83697 \cdot 10^{-24} X^4 - 6.85009 \cdot 10^{-24} X^3 + 1.39587 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\
 &= -1.5473 \cdot 10^{-11} B_{0,4} - 8.19335 \cdot 10^{-12} B_{1,4} - 9.13745 \cdot 10^{-13} B_{2,4} + 6.36586 \cdot 10^{-12} B_{3,4} + 1.36455 \cdot 10^{-11} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.84343 \cdot 10^{-25}$.

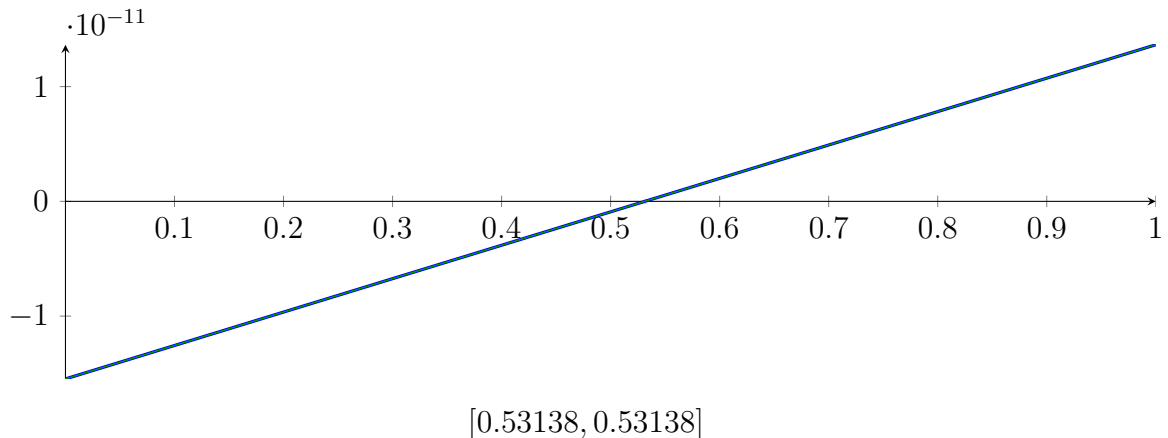
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -6.78547 \cdot 10^{-25} X^3 - 2.95652 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\
 m &= -6.78547 \cdot 10^{-25} X^3 - 2.95652 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-6.51982 \cdot 10^6, 0.53138, 6.51982 \cdot 10^6\} \quad N(m) = \{-6.51982 \cdot 10^6, 0.53138, 6.51982 \cdot 10^6\}$$

Intersection intervals:



Longest intersection interval: 0

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

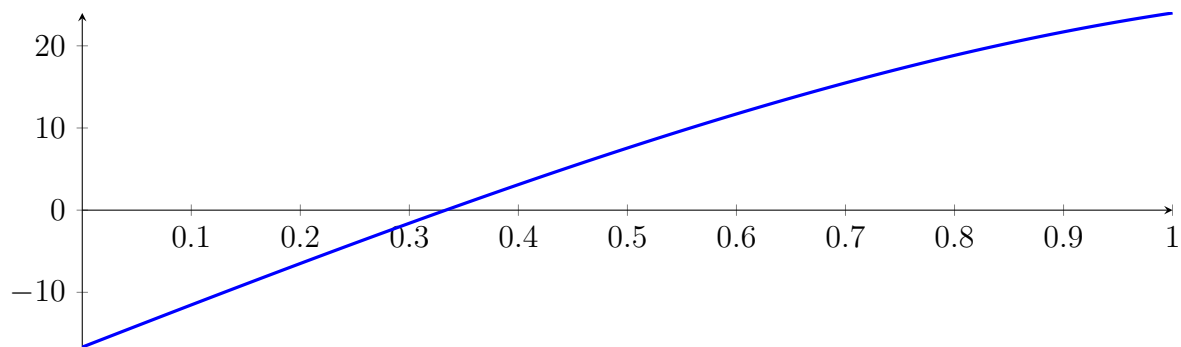
33.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

33.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

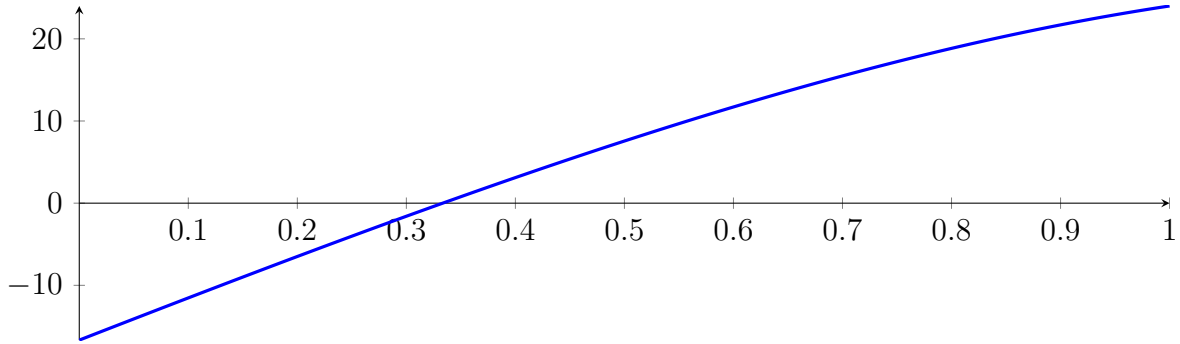
with precision $\varepsilon = 1 \cdot 10^{-16}$.

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$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

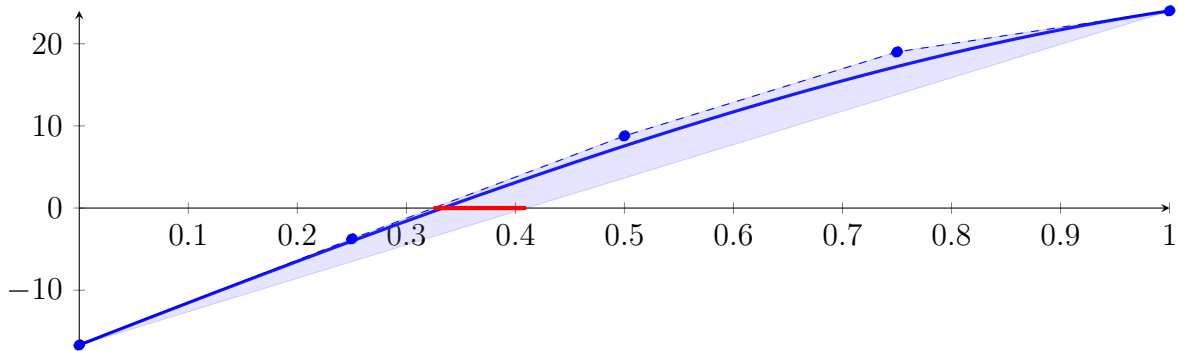
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



34.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

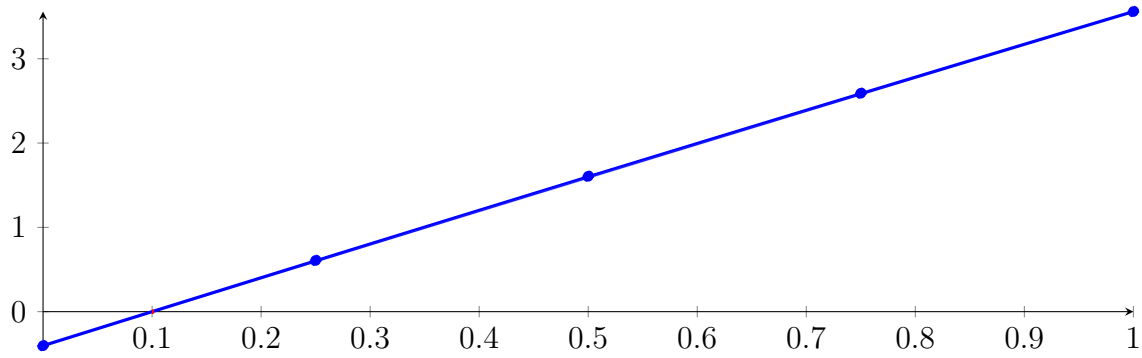
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

34.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

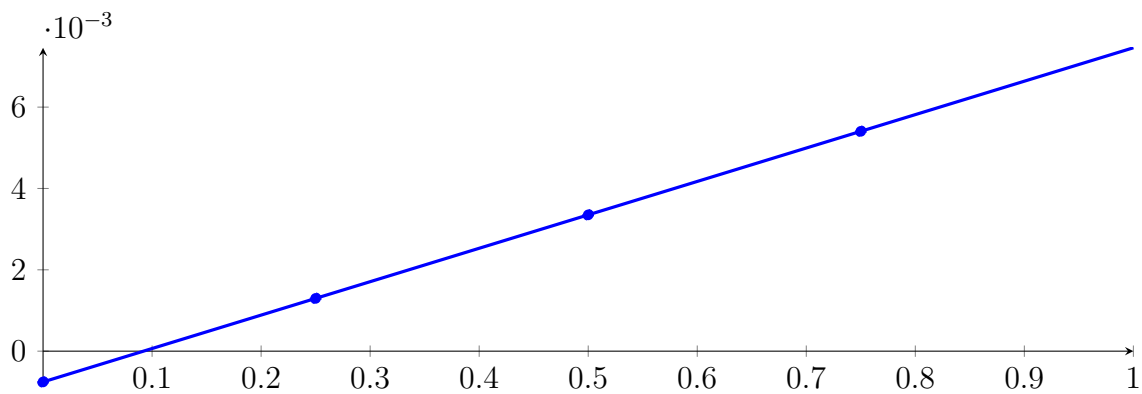
Longest intersection interval: 0.00203877

\implies Selective recursion: interval 1: $[0.333317, 0.333491]$,

34.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.06393 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

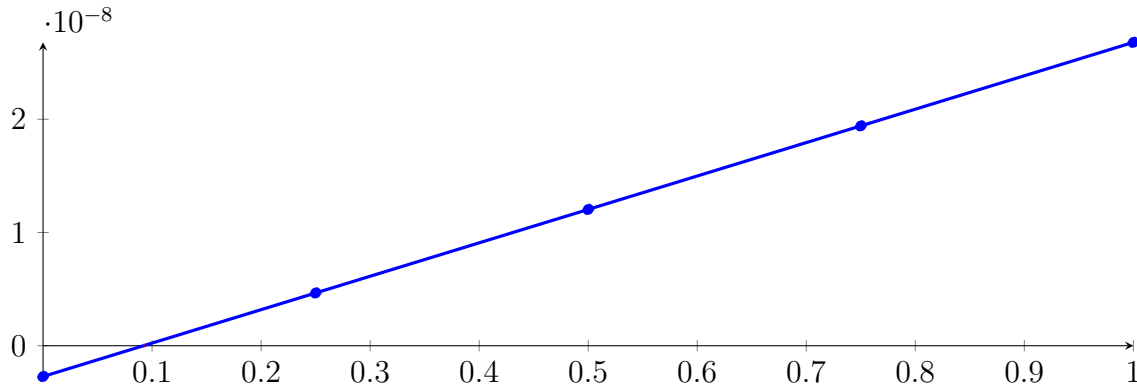
Longest intersection interval: $3.59185 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

34.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.92617 \cdot 10^{-24} X^4 + 6.61744 \cdot 10^{-24} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09} \\ &= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \\ &\quad \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28974 \cdot 10^{-11}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

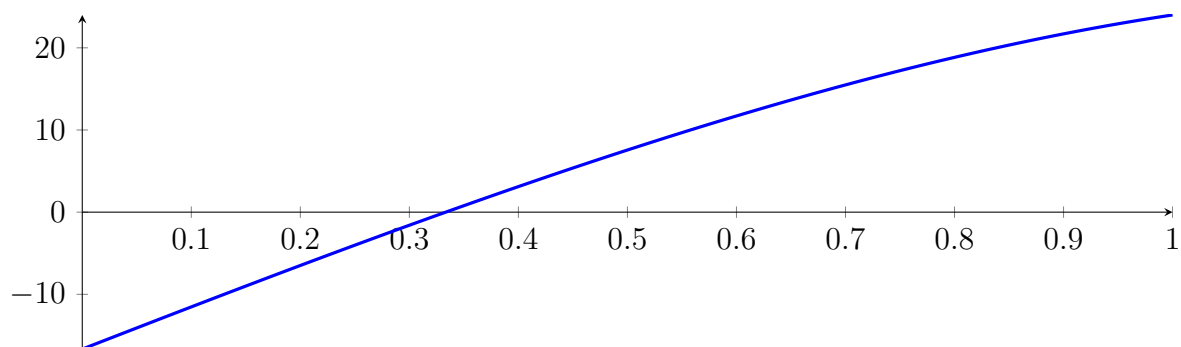
34.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

34.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

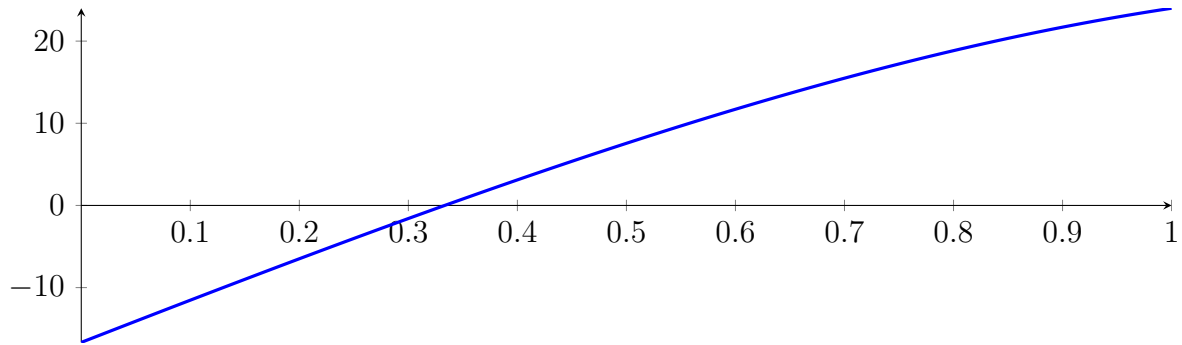
with precision $\varepsilon = 1 \cdot 10^{-32}$.

35 Running QuadClip on f_4 with epsilon 32

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

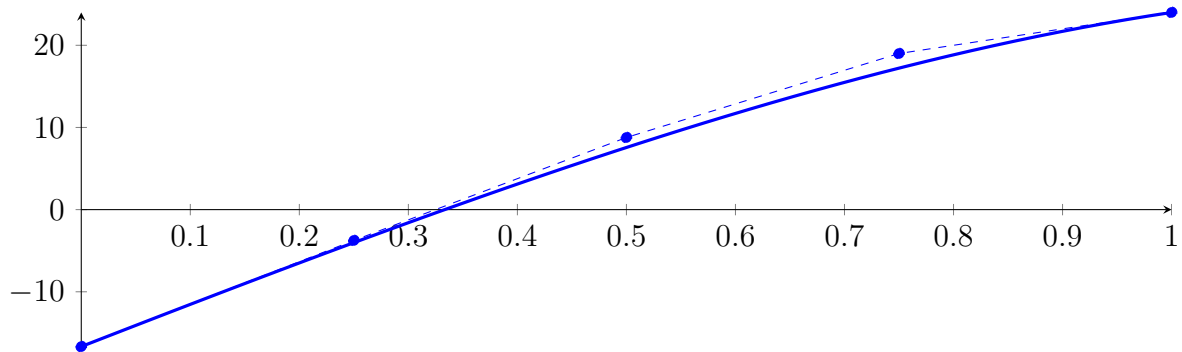
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



35.1 Recursion Branch 1 for Input Interval $[0, 1]$

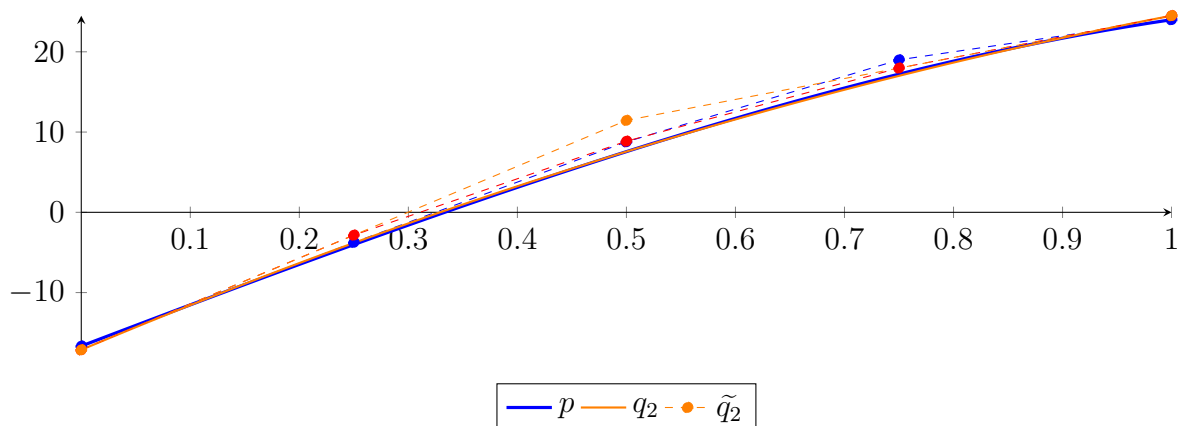
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \\ \tilde{q}_2 &= -2.67519 \cdot 10^{-12}X^4 + 5.32907 \cdot 10^{-12}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

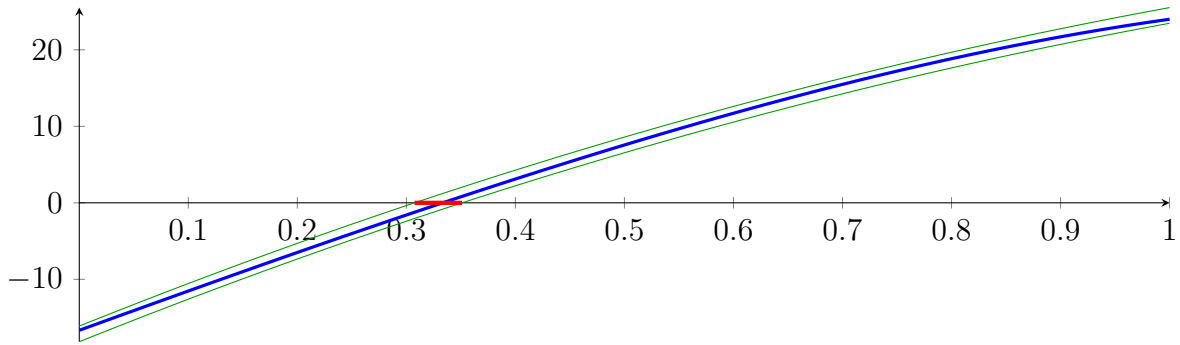
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

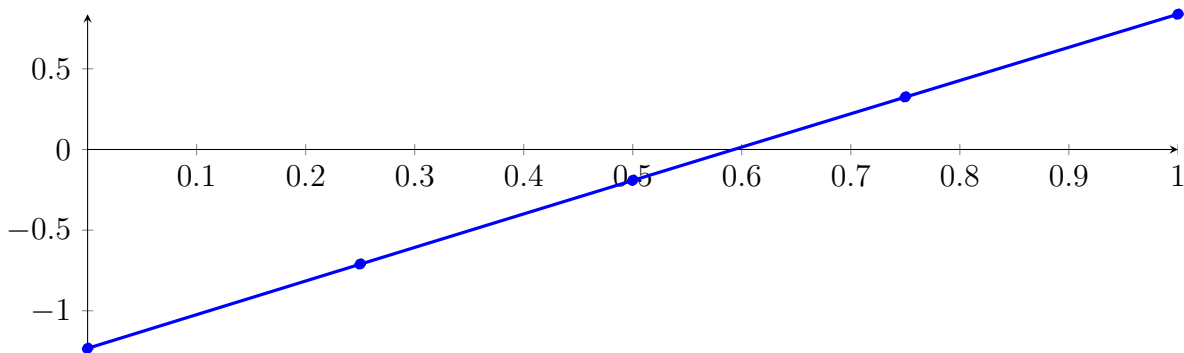
Longest intersection interval: 0.0436205

⇒ Selective recursion: **interval 1:** $[0.307477, 0.351097]$,

35.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

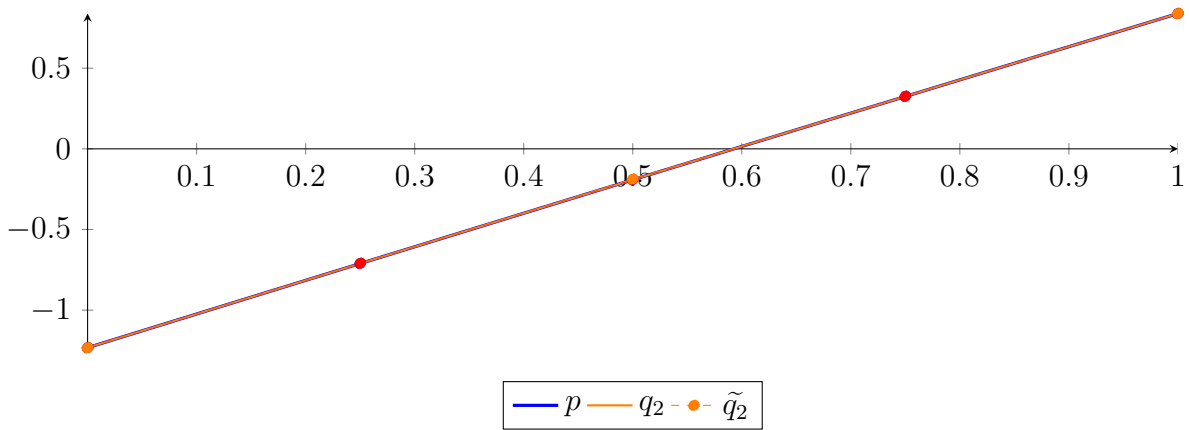
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 2.43583 \cdot 10^{-13} X^4 - 4.3876 \cdot 10^{-13} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

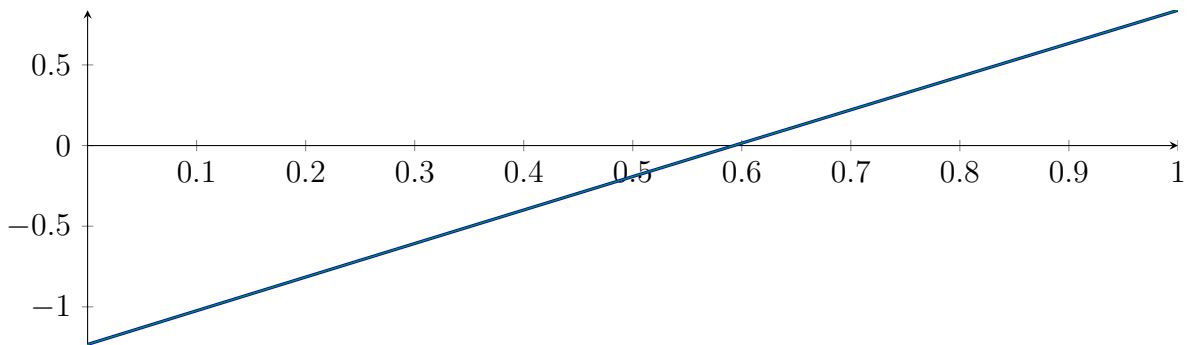
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

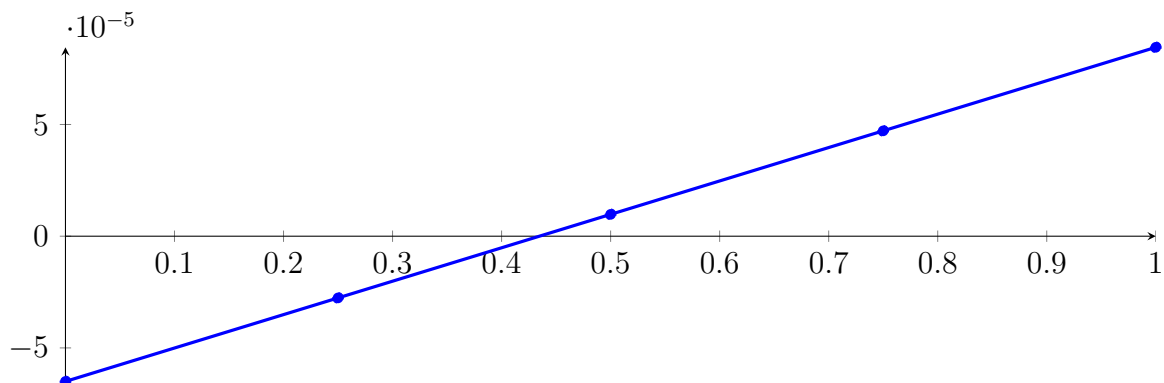
Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

35.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -2.71051 \cdot 10^{-20} X^4 - 2.82489 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05} \\
 &= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \\
 &\quad \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)
 \end{aligned}$$



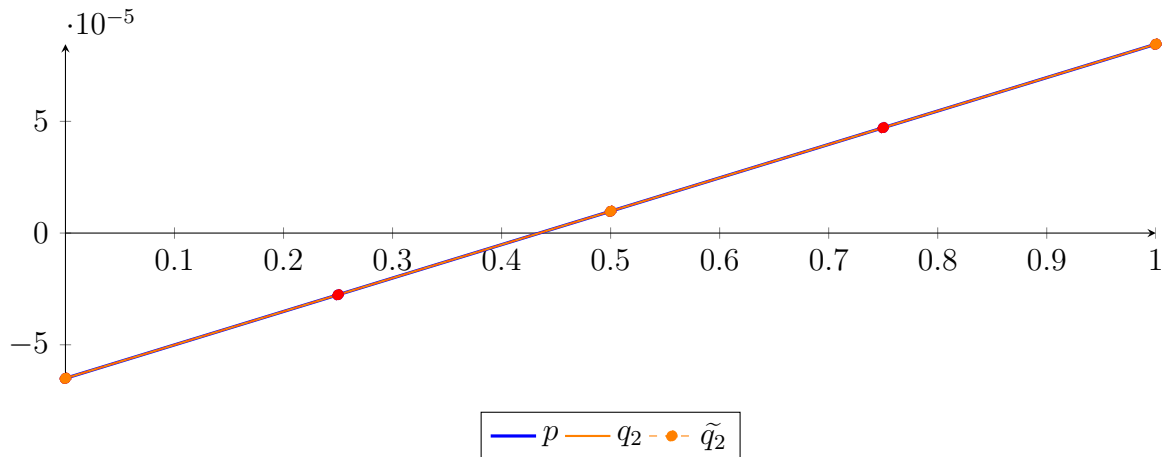
Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = 6.72205 \cdot 10^{-18} X^4 - 1.21431 \cdot 10^{-17} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.88601 \cdot 10^{-17}$.

Bounding polynomials M and m :

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

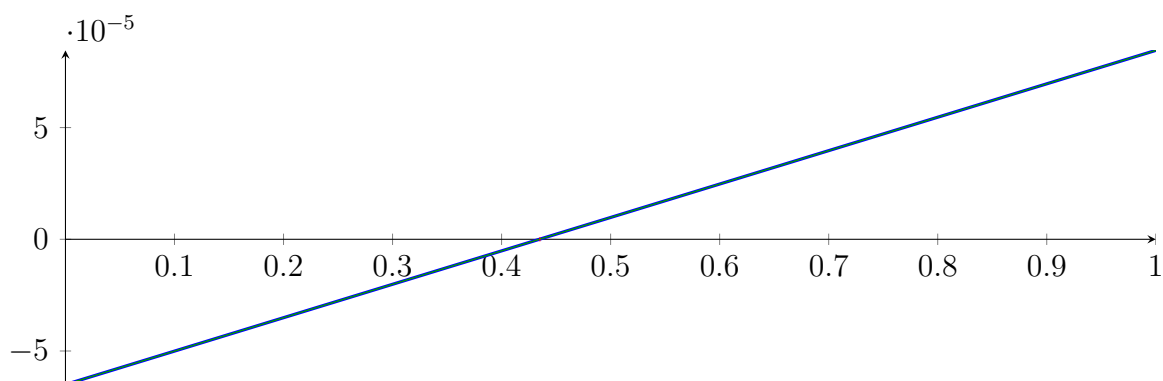
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.434685, 1.4089 \cdot 10^6\}$$

$$N(m) = \{0.434685, 1.4089 \cdot 10^6\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

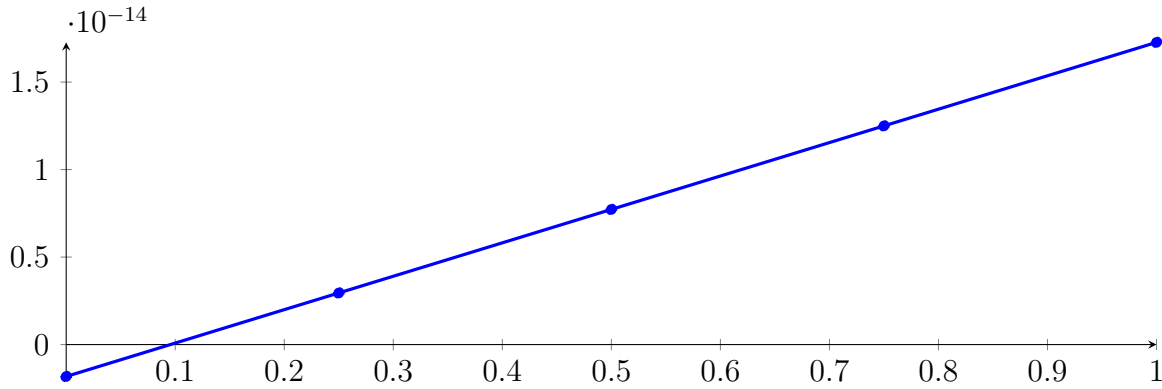
Longest intersection interval: $1.27678 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

35.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

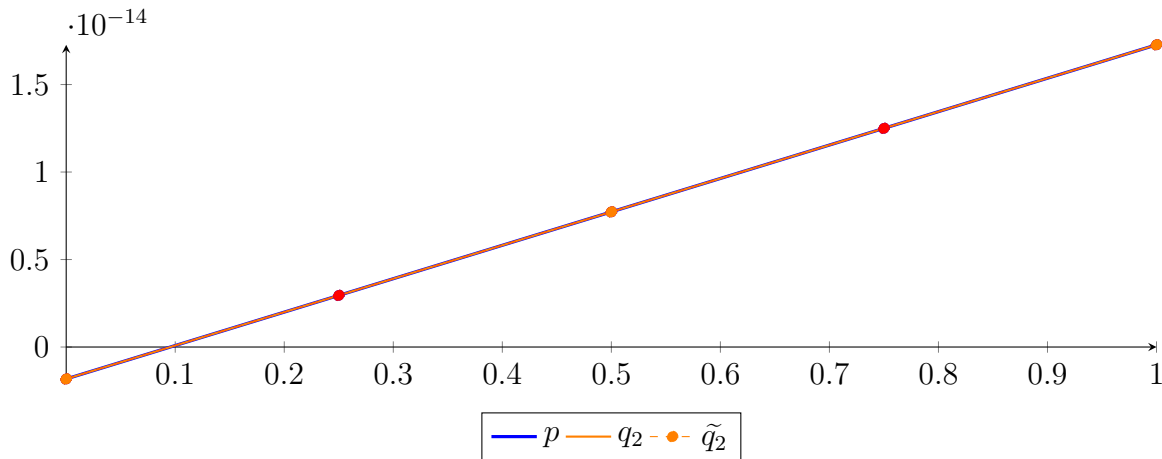
$$\begin{aligned}
 p &= -1.41995 \cdot 10^{-29} X^4 + 6.31089 \cdot 10^{-30} X^3 + 4.73317 \cdot 10^{-30} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15} \\
 &= -1.8241 \cdot 10^{-15} B_{0,4}(X) + 2.94943 \cdot 10^{-15} B_{1,4}(X) + 7.72295 \\
 &\quad \cdot 10^{-15} B_{2,4}(X) + 1.24965 \cdot 10^{-14} B_{3,4}(X) + 1.727 \cdot 10^{-14} B_{4,4}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -2.68213 \cdot 10^{-29} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15} \\
 &= -1.8241 \cdot 10^{-15} B_{0,2} + 7.72295 \cdot 10^{-15} B_{1,2} + 1.727 \cdot 10^{-14} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -2.18041 \cdot 10^{-27} X^4 + 3.9443 \cdot 10^{-27} X^3 - 2.24352 \cdot 10^{-27} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15} \\
 &= -1.8241 \cdot 10^{-15} B_{0,4} + 2.94943 \cdot 10^{-15} B_{1,4} + 7.72295 \cdot 10^{-15} B_{2,4} + 1.24965 \cdot 10^{-14} B_{3,4} + 1.727 \cdot 10^{-14} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.73549 \cdot 10^{-28}$.

Bounding polynomials M and m :

$$M = -2.68213 \cdot 10^{-29} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15}$$

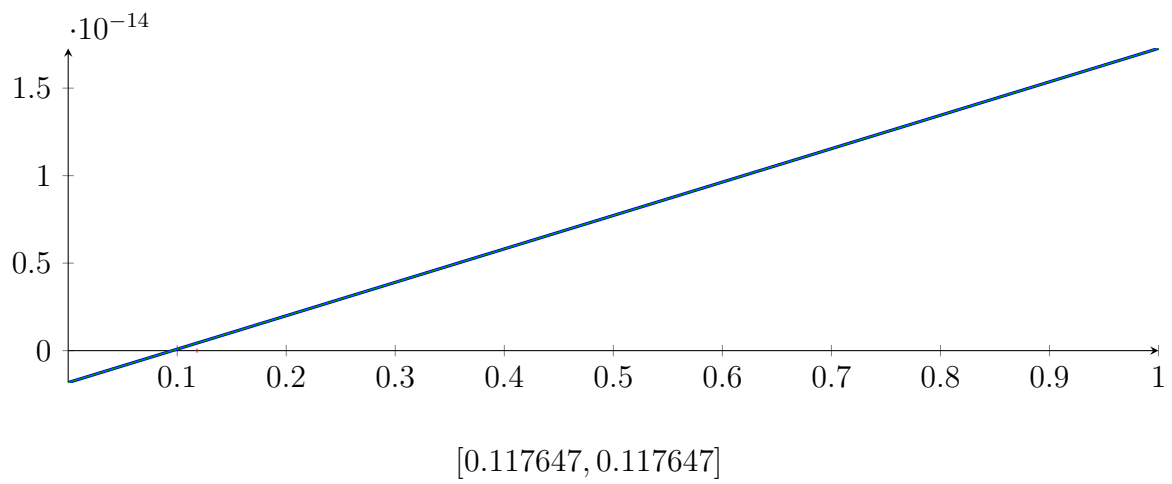
$$m = -2.68213 \cdot 10^{-29} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15}$$

Root of M and m :

$$N(M) = \{0.117647, 7.11901 \cdot 10^{14}\}$$

$$N(m) = \{0.117647, 7.11901 \cdot 10^{14}\}$$

Intersection intervals:



Longest intersection interval: 0

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

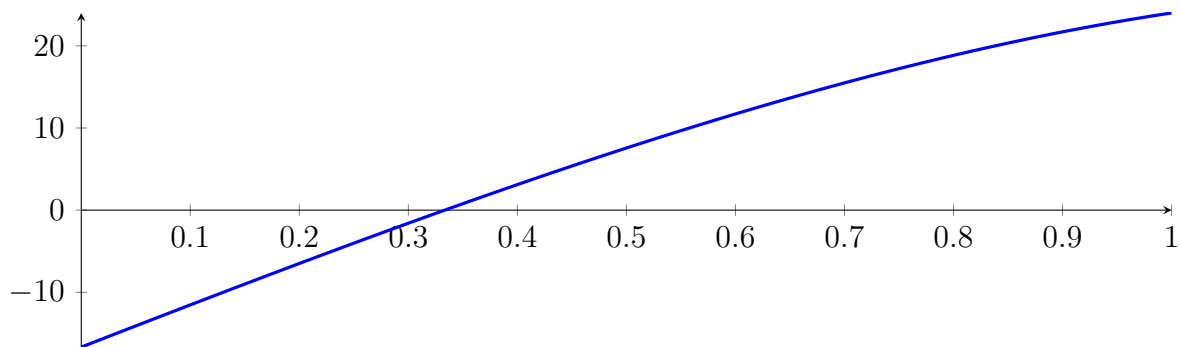
35.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

35.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

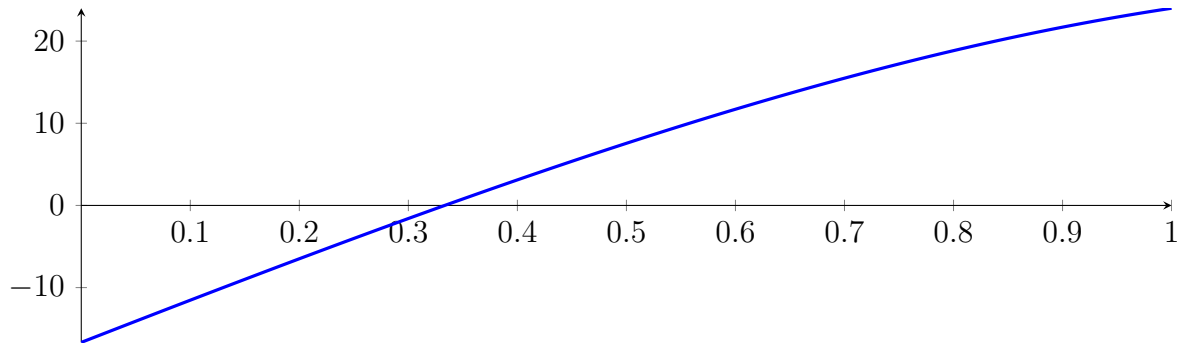
with precision $\varepsilon = 1 \cdot 10^{-32}$.

36 Running CubeClip on f_4 with epsilon 32

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

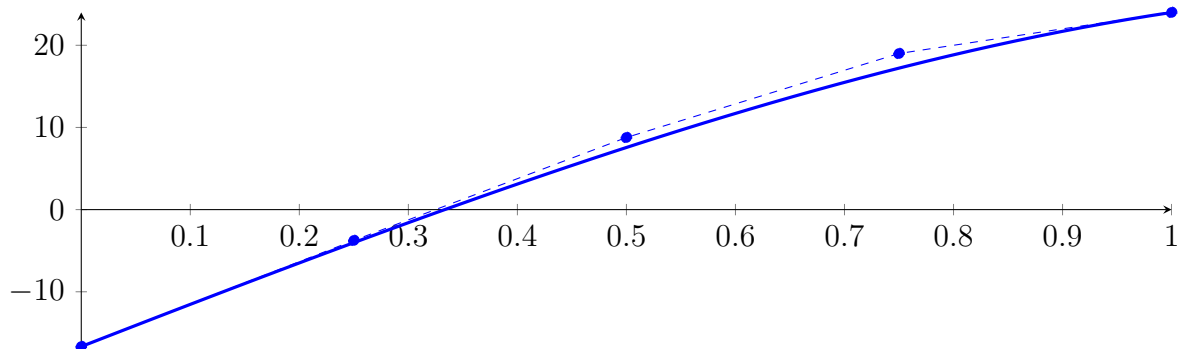
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



36.1 Recursion Branch 1 for Input Interval $[0, 1]$

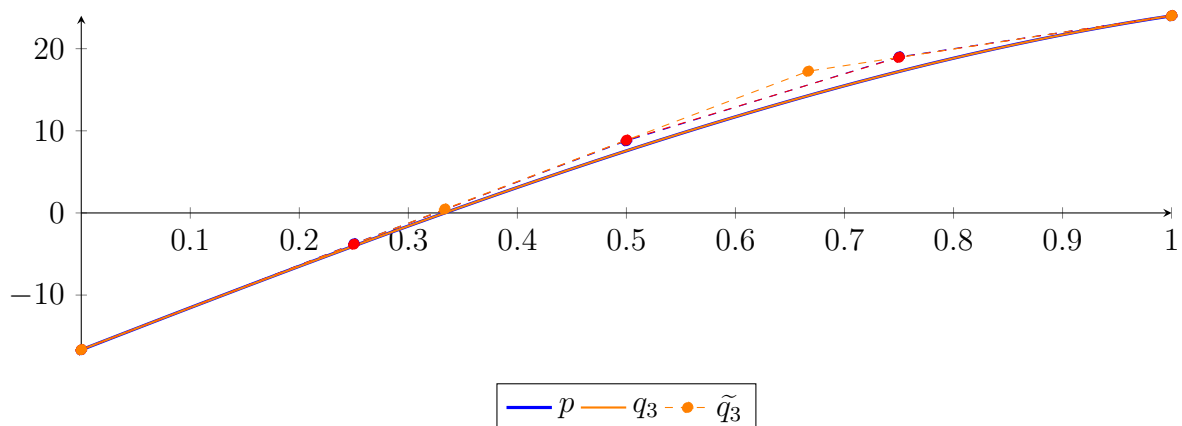
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= -2.28084 \cdot 10^{-12}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

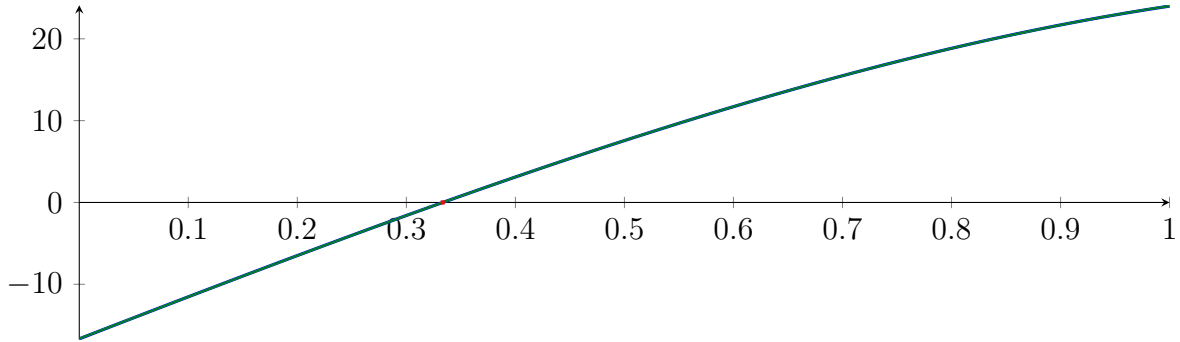
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

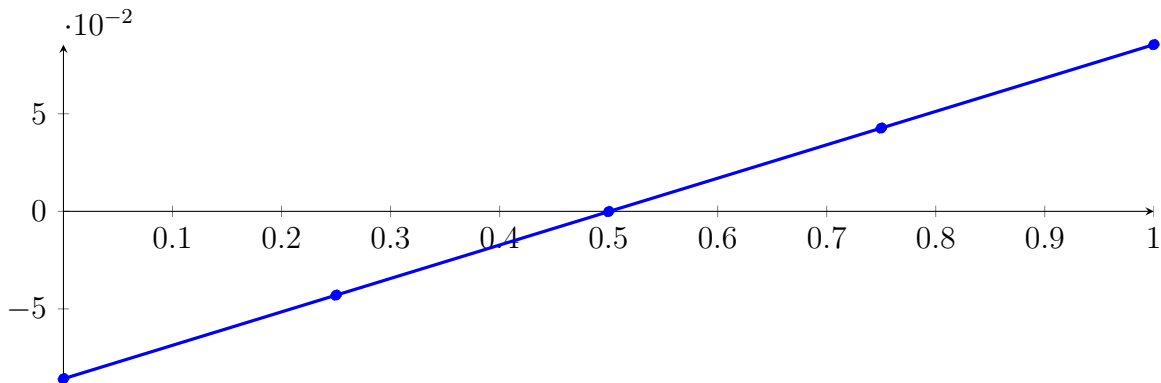
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

36.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

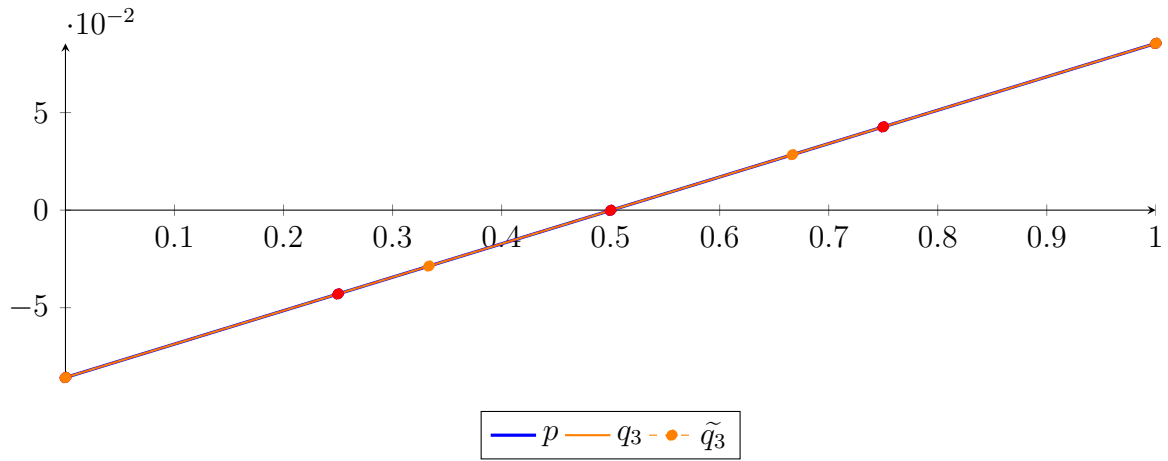
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 1.2032 \cdot 10^{-14} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45913 \cdot 10^{-11}$.

Bounding polynomials M and m :

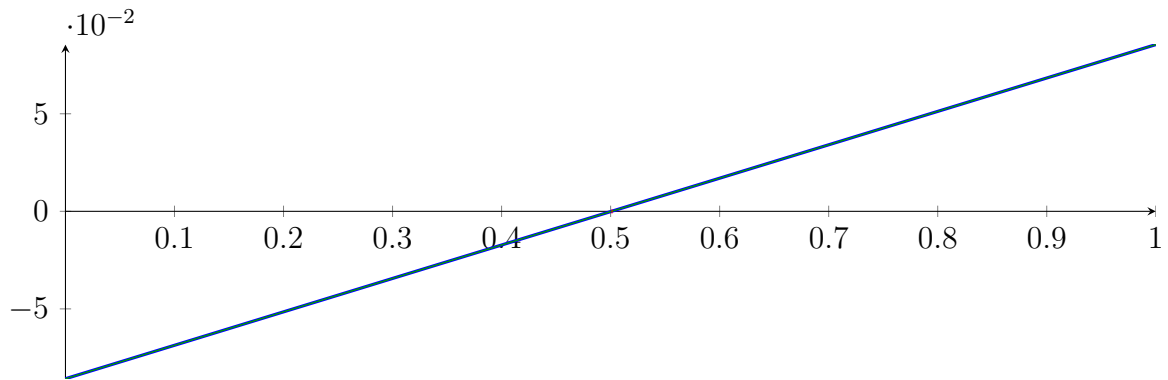
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\} \quad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

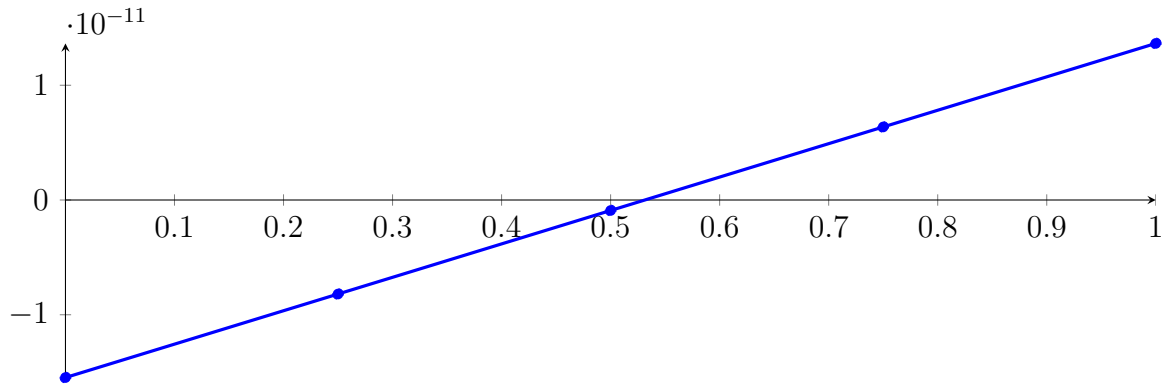
Longest intersection interval: $1.70047 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

36.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

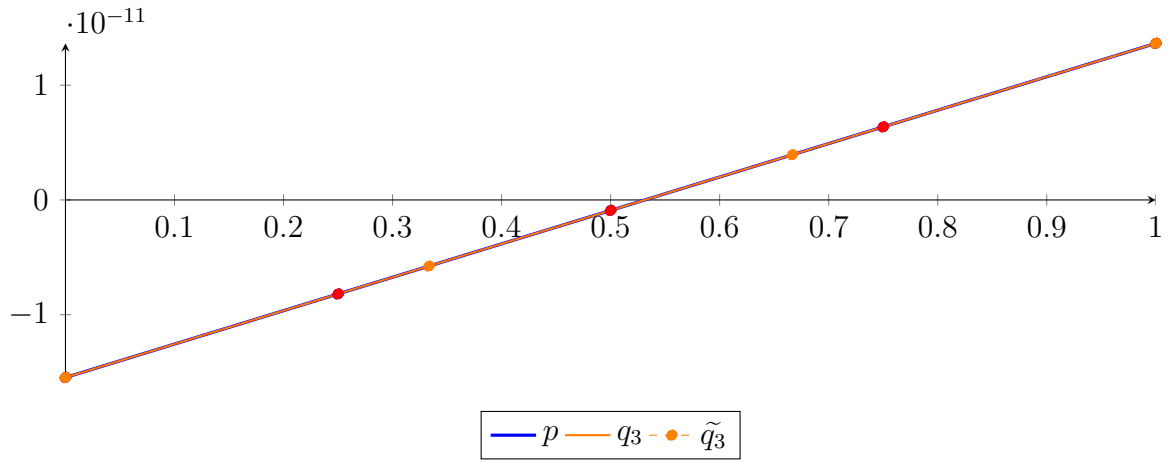
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.01312 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\ &= -1.5473 \cdot 10^{-11} B_{0,4}(X) - 8.19335 \cdot 10^{-12} B_{1,4}(X) - 9.13745 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36586 \cdot 10^{-12} B_{3,4}(X) + 1.36455 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -6.78547 \cdot 10^{-25} X^3 - 2.95652 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\
 &= -1.5473 \cdot 10^{-11} B_{0,3} - 5.76681 \cdot 10^{-12} B_{1,3} + 3.93932 \cdot 10^{-12} B_{2,3} + 1.36455 \cdot 10^{-11} B_{3,3} \\
 \tilde{q}_3 &= 2.83697 \cdot 10^{-24} X^4 - 6.85009 \cdot 10^{-24} X^3 + 1.39587 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\
 &= -1.5473 \cdot 10^{-11} B_{0,4} - 8.19335 \cdot 10^{-12} B_{1,4} - 9.13745 \cdot 10^{-13} B_{2,4} + 6.36586 \cdot 10^{-12} B_{3,4} + 1.36455 \cdot 10^{-11} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.84343 \cdot 10^{-25}$.

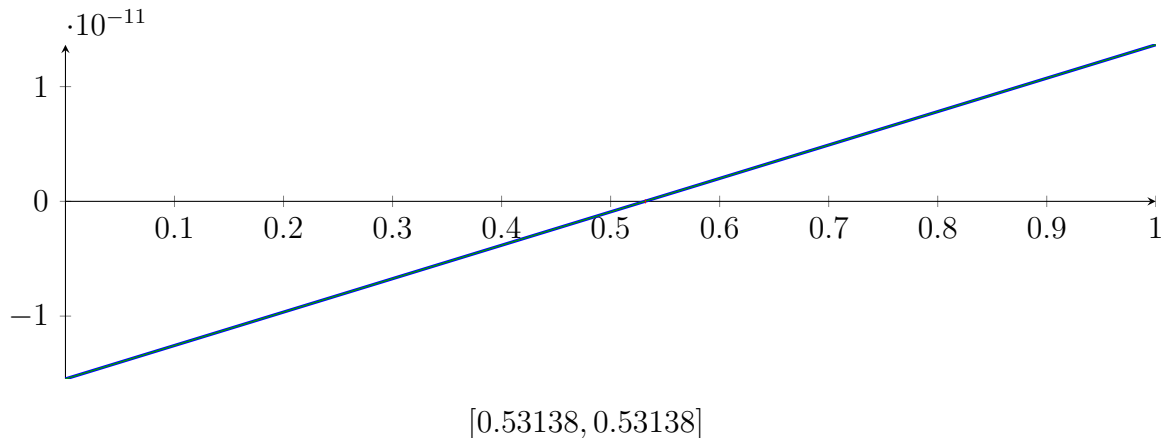
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -6.78547 \cdot 10^{-25} X^3 - 2.95652 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\
 m &= -6.78547 \cdot 10^{-25} X^3 - 2.95652 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-6.51982 \cdot 10^6, 0.53138, 6.51982 \cdot 10^6\} \quad N(m) = \{-6.51982 \cdot 10^6, 0.53138, 6.51982 \cdot 10^6\}$$

Intersection intervals:



Longest intersection interval: 0

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333333]$,

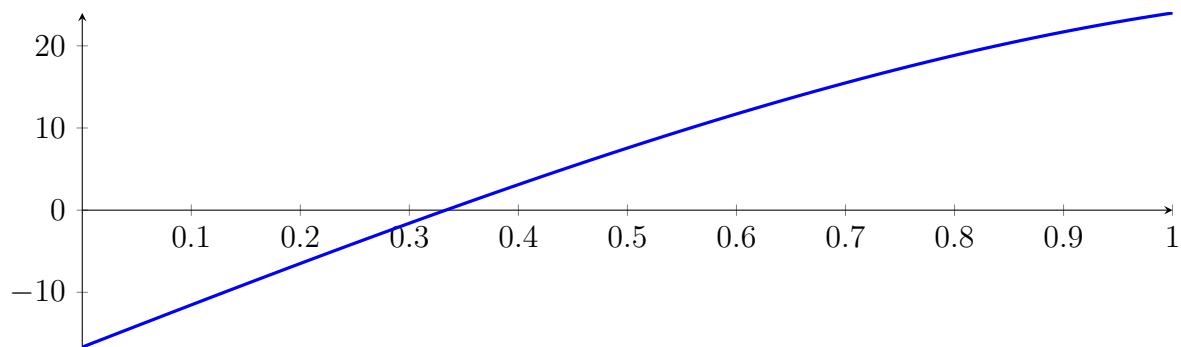
36.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

36.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

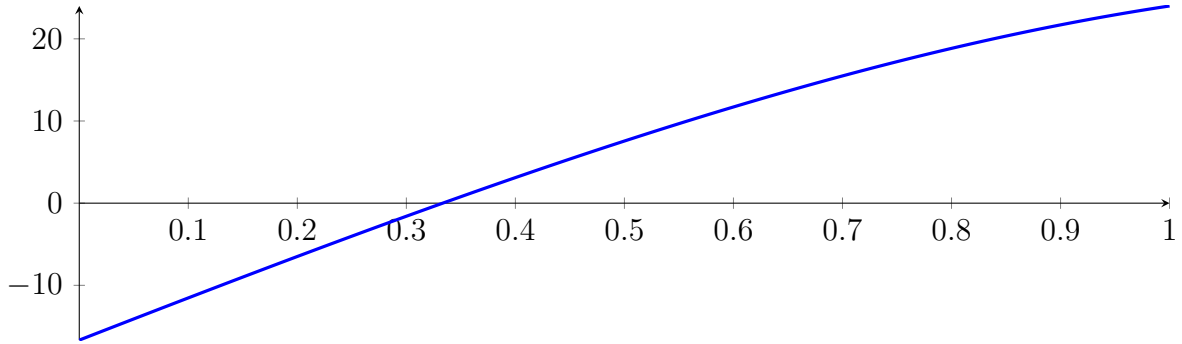
with precision $\varepsilon = 1 \cdot 10^{-32}$.

37 Running BezClip on f_4 with epsilon 64

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

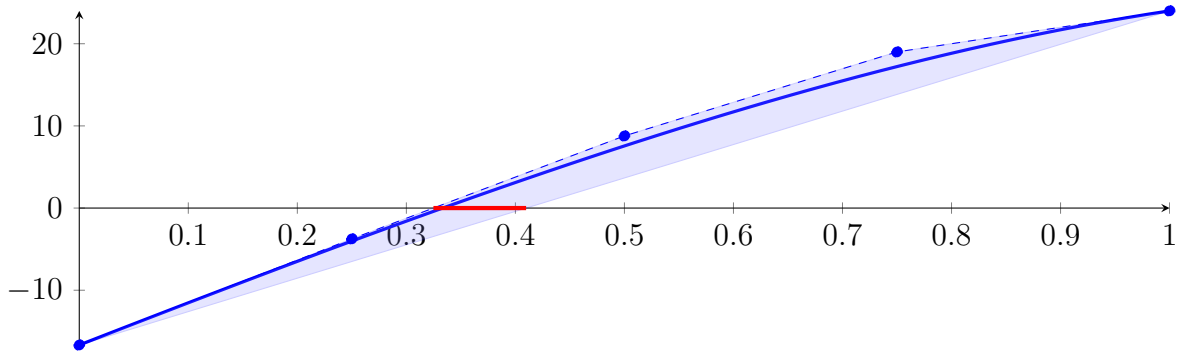
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



37.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

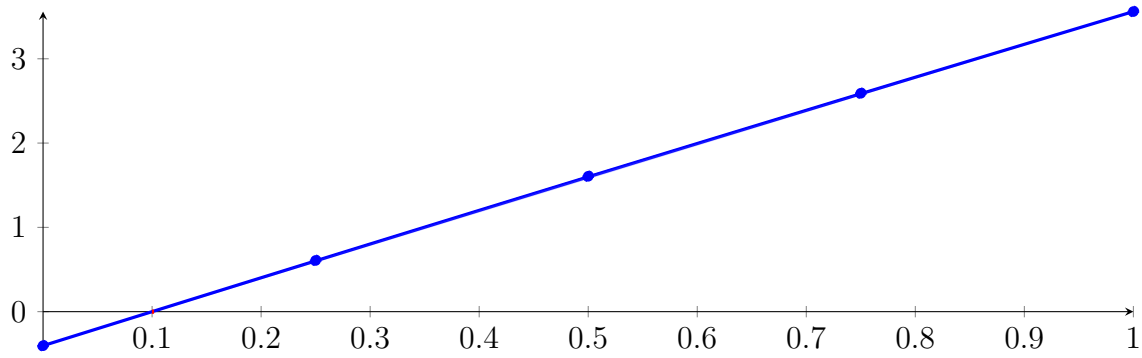
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

37.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

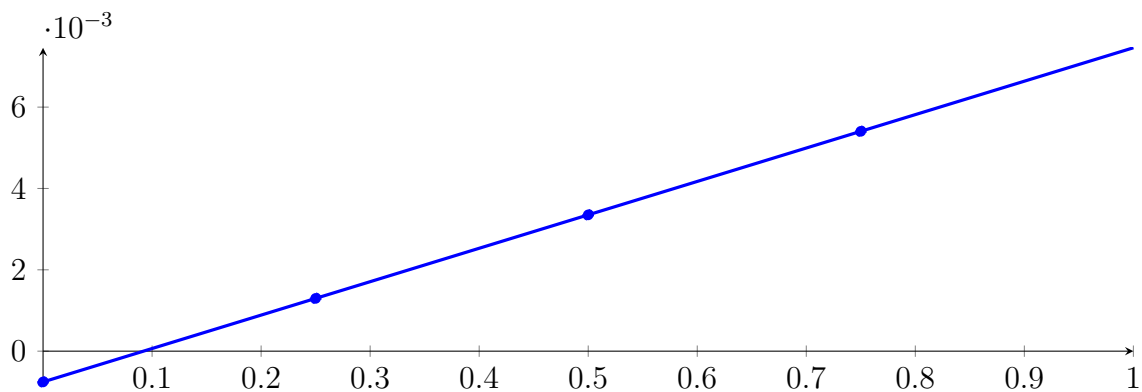
Longest intersection interval: 0.00203877

\implies Selective recursion: interval 1: $[0.333317, 0.333491]$,

37.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.06393 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

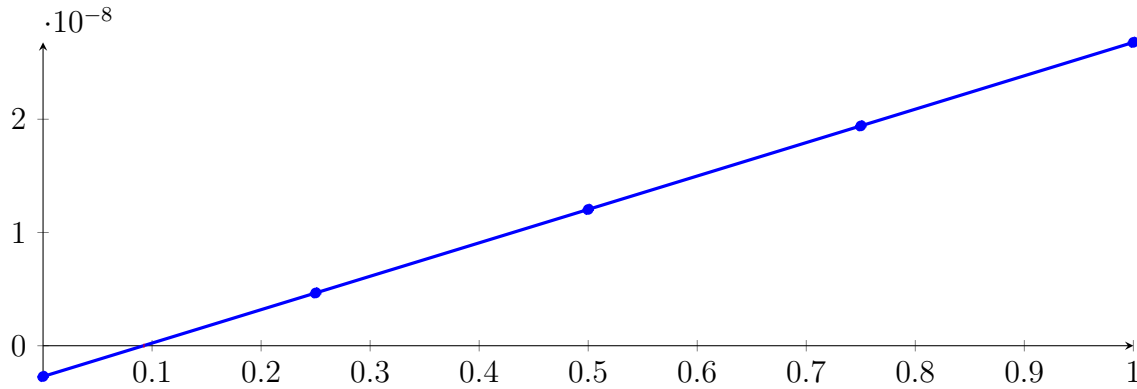
Longest intersection interval: $3.59185 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

37.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.92617 \cdot 10^{-24} X^4 + 6.61744 \cdot 10^{-24} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09} \\ &= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \\ &\quad \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28974 \cdot 10^{-11}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

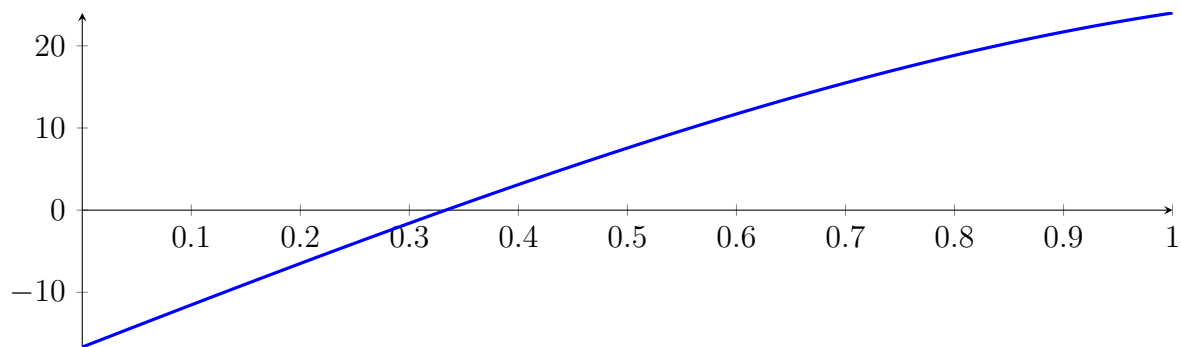
37.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

37.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

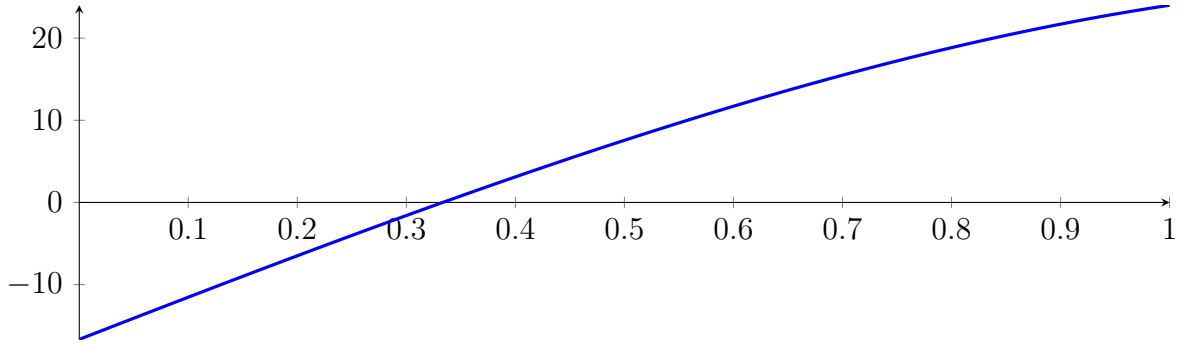
with precision $\varepsilon = 1 \cdot 10^{-64}$.

38 Running QuadClip on f_4 with epsilon 64

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

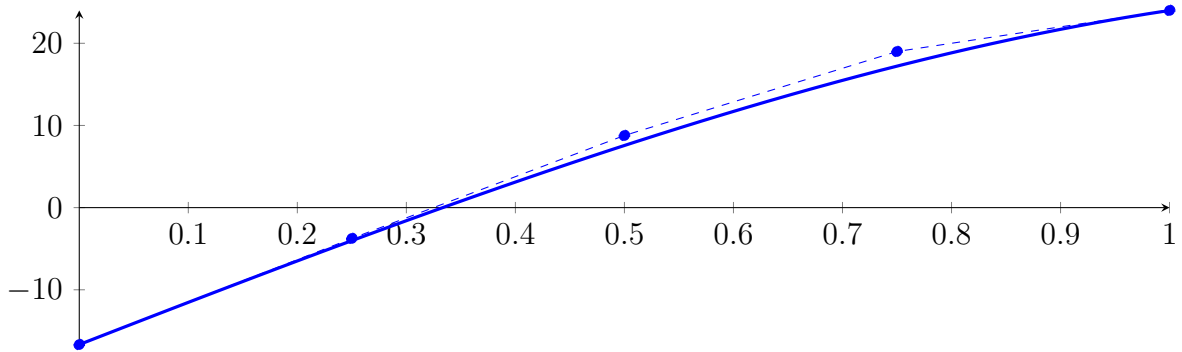
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



38.1 Recursion Branch 1 for Input Interval $[0, 1]$

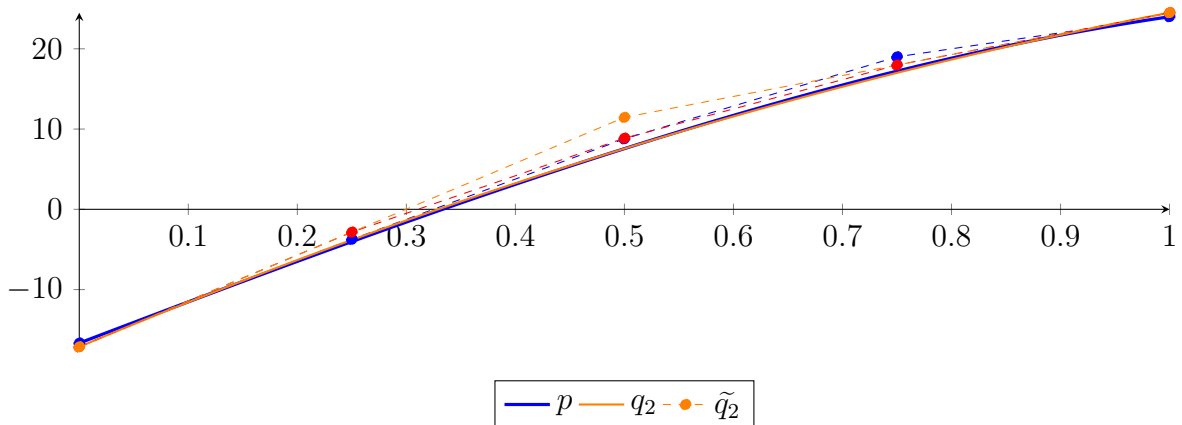
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \\ \tilde{q}_2 &= -2.67519 \cdot 10^{-12}X^4 + 5.32907 \cdot 10^{-12}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

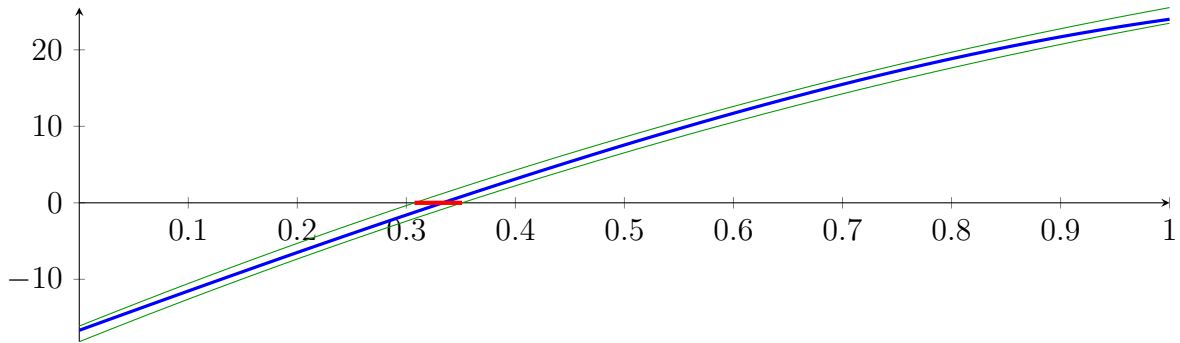
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

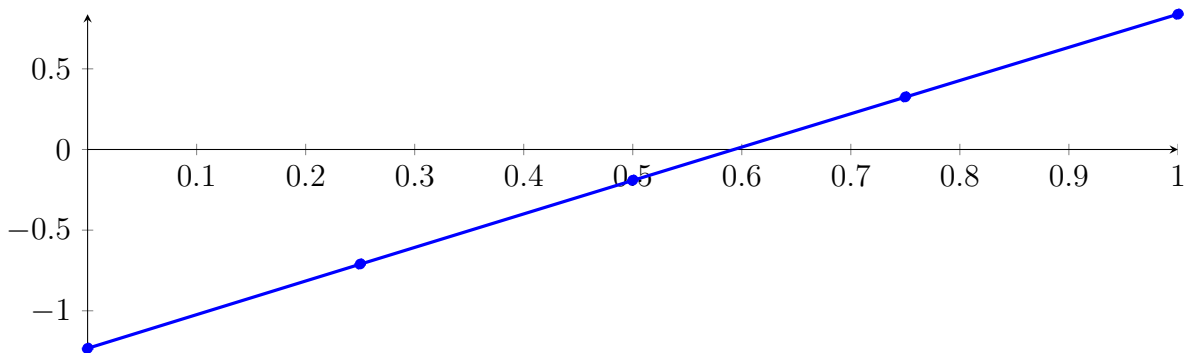
Longest intersection interval: 0.0436205

⇒ Selective recursion: interval 1: $[0.307477, 0.351097]$,

38.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

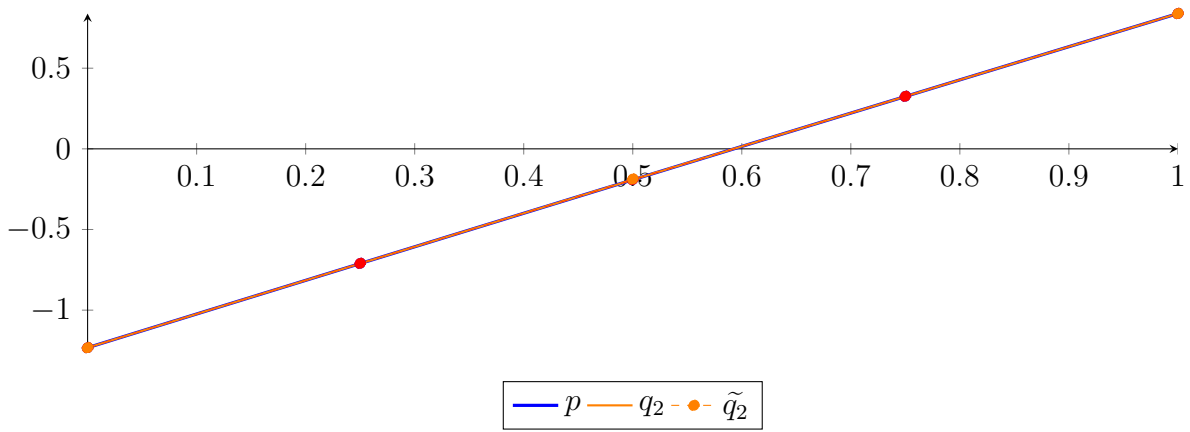
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 2.43583 \cdot 10^{-13} X^4 - 4.3876 \cdot 10^{-13} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

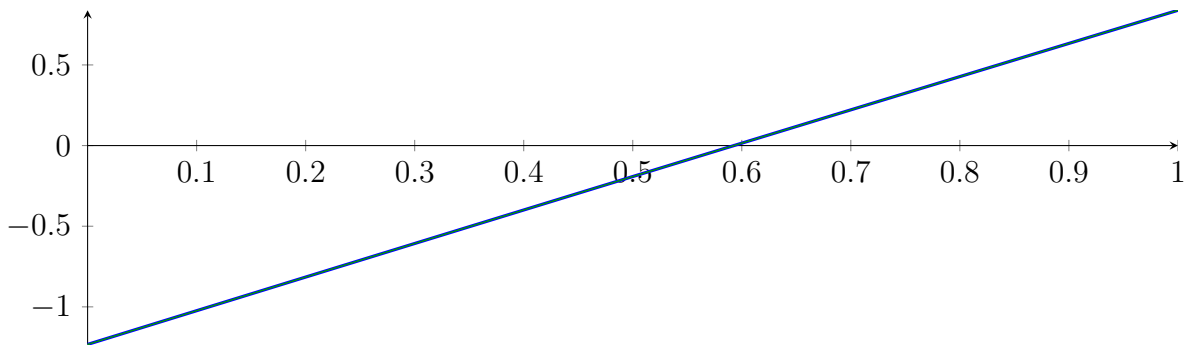
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

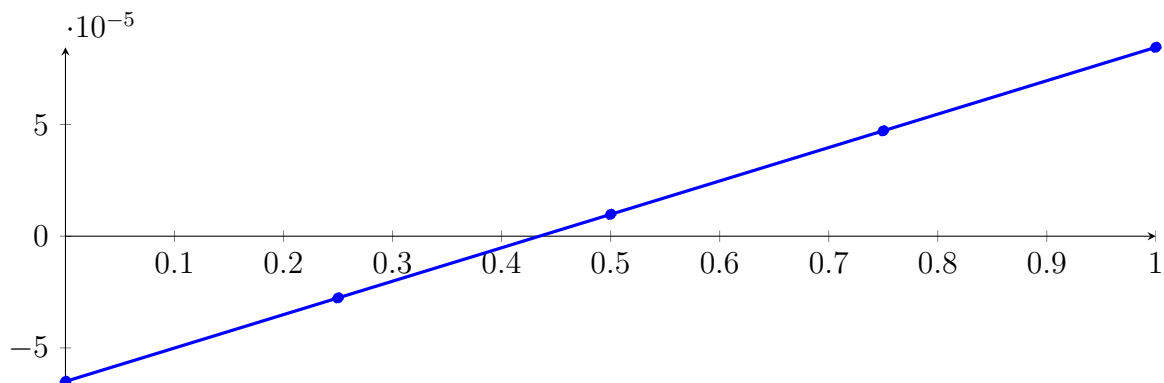
Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

38.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -2.71051 \cdot 10^{-20} X^4 - 2.82489 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05} \\
 &= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \\
 &\quad \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)
 \end{aligned}$$



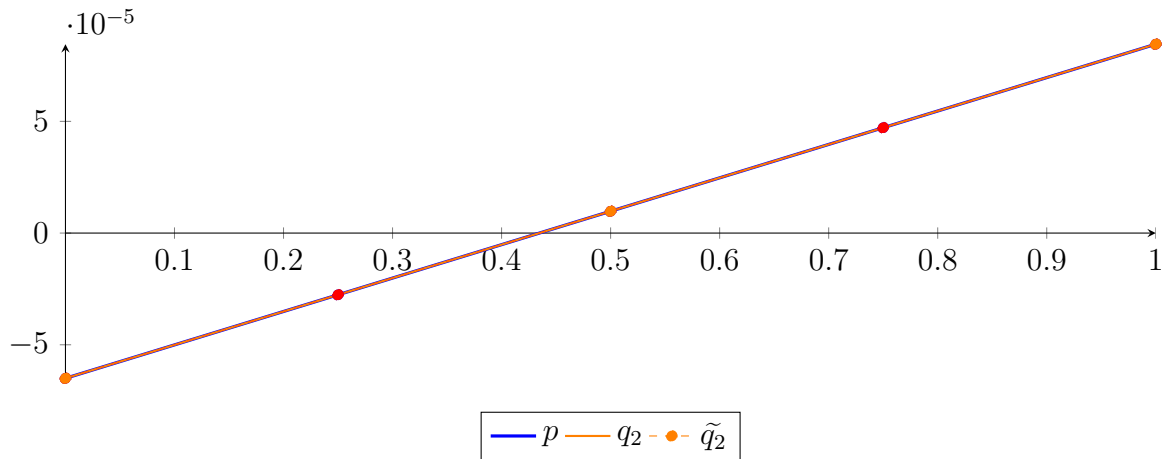
Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = 6.72205 \cdot 10^{-18} X^4 - 1.21431 \cdot 10^{-17} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.88601 \cdot 10^{-17}$.

Bounding polynomials M and m :

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

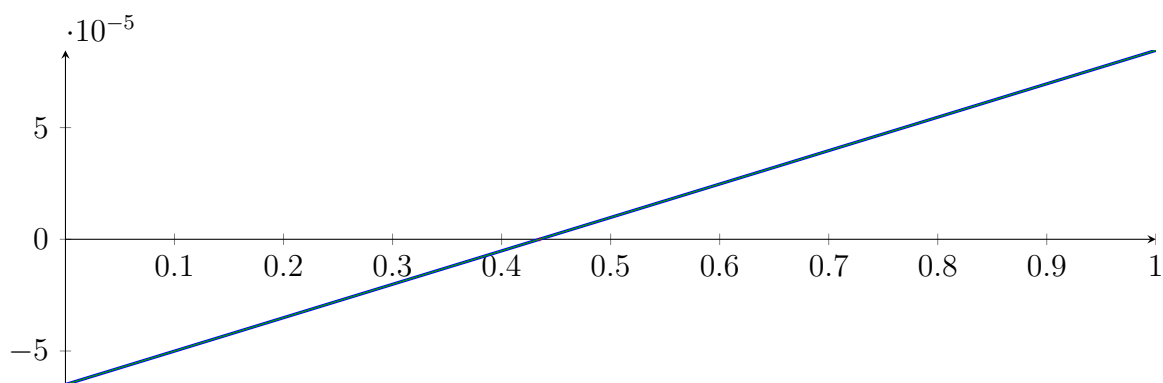
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.434685, 1.4089 \cdot 10^6\}$$

$$N(m) = \{0.434685, 1.4089 \cdot 10^6\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

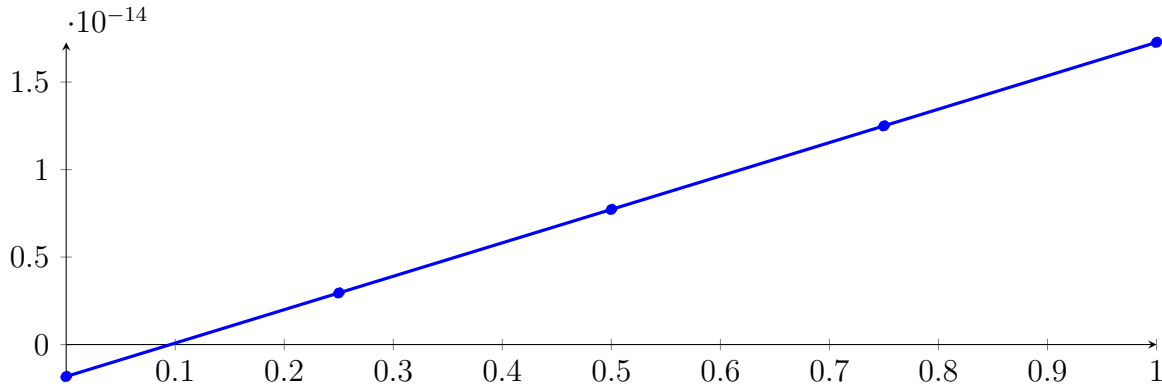
Longest intersection interval: $1.27678 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

38.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

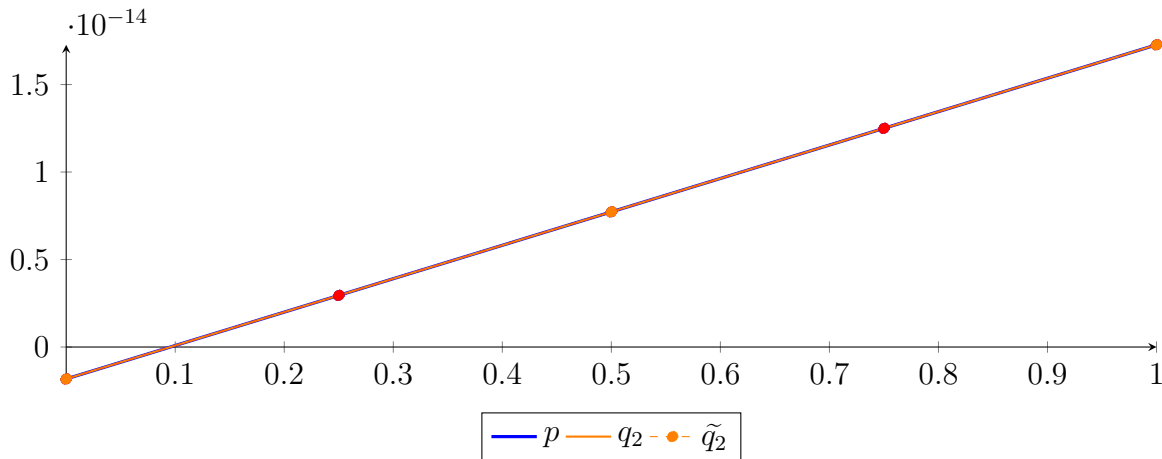
$$\begin{aligned}
 p &= -1.41995 \cdot 10^{-29} X^4 + 6.31089 \cdot 10^{-30} X^3 + 4.73317 \cdot 10^{-30} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15} \\
 &= -1.8241 \cdot 10^{-15} B_{0,4}(X) + 2.94943 \cdot 10^{-15} B_{1,4}(X) + 7.72295 \\
 &\quad \cdot 10^{-15} B_{2,4}(X) + 1.24965 \cdot 10^{-14} B_{3,4}(X) + 1.727 \cdot 10^{-14} B_{4,4}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -2.68213 \cdot 10^{-29} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15} \\
 &= -1.8241 \cdot 10^{-15} B_{0,2} + 7.72295 \cdot 10^{-15} B_{1,2} + 1.727 \cdot 10^{-14} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -2.18041 \cdot 10^{-27} X^4 + 3.9443 \cdot 10^{-27} X^3 - 2.24352 \cdot 10^{-27} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15} \\
 &= -1.8241 \cdot 10^{-15} B_{0,4} + 2.94943 \cdot 10^{-15} B_{1,4} + 7.72295 \cdot 10^{-15} B_{2,4} + 1.24965 \cdot 10^{-14} B_{3,4} + 1.727 \cdot 10^{-14} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.73549 \cdot 10^{-28}$.

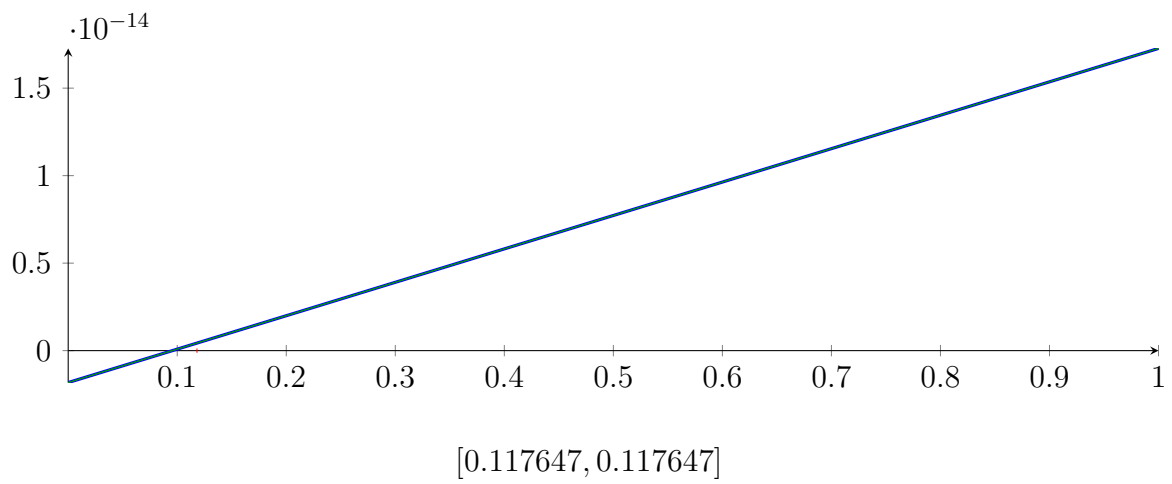
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -2.68213 \cdot 10^{-29} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15} \\
 m &= -2.68213 \cdot 10^{-29} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.117647, 7.11901 \cdot 10^{14}\} \quad N(m) = \{0.117647, 7.11901 \cdot 10^{14}\}$$

Intersection intervals:



Longest intersection interval: 0

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

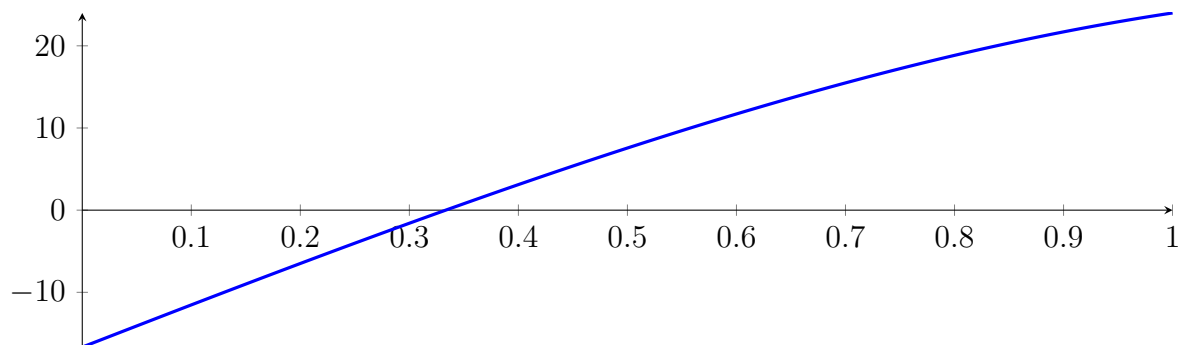
38.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

38.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

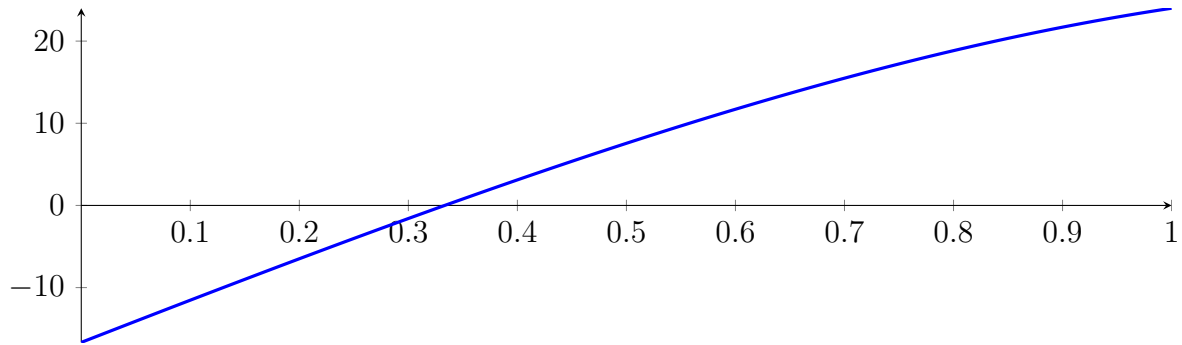
with precision $\varepsilon = 1 \cdot 10^{-64}$.

39 Running CubeClip on f_4 with epsilon 64

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

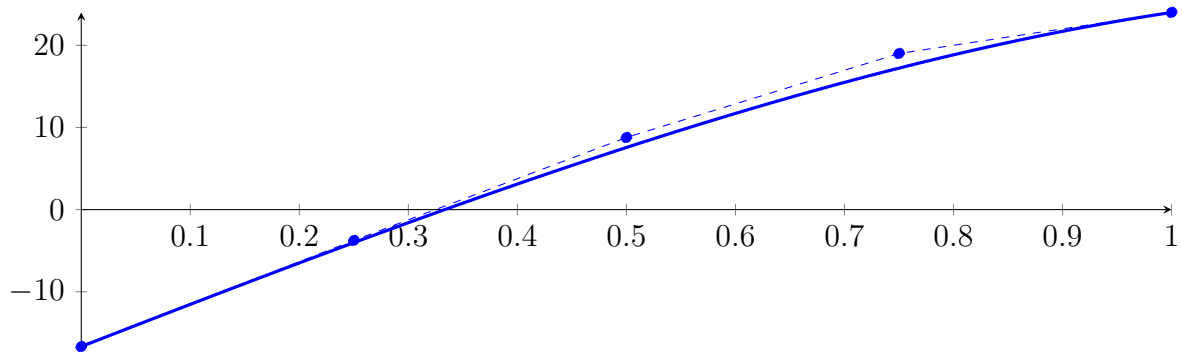
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



39.1 Recursion Branch 1 for Input Interval $[0, 1]$

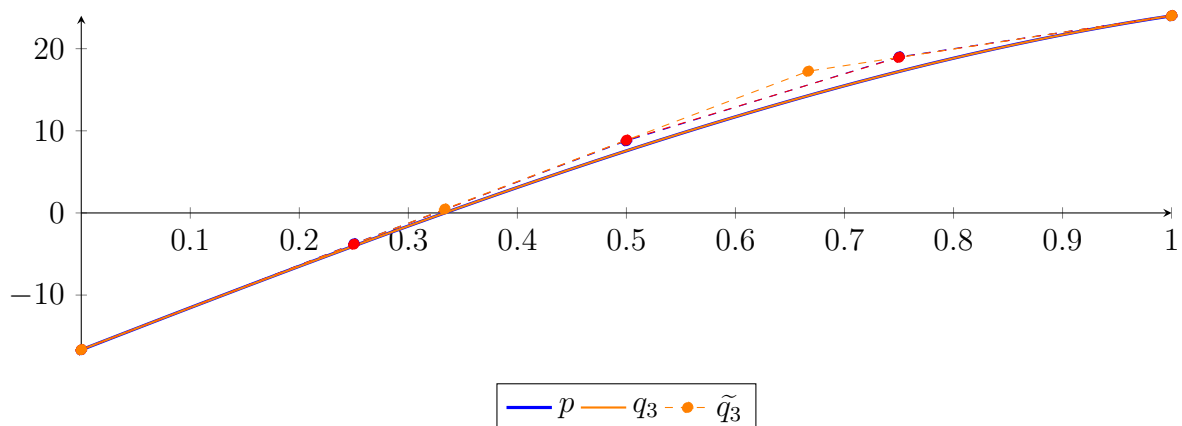
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= -2.28084 \cdot 10^{-12}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

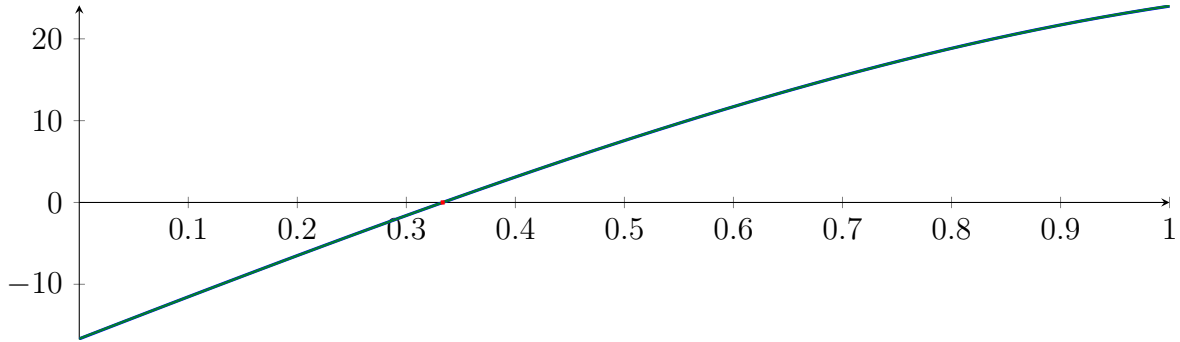
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

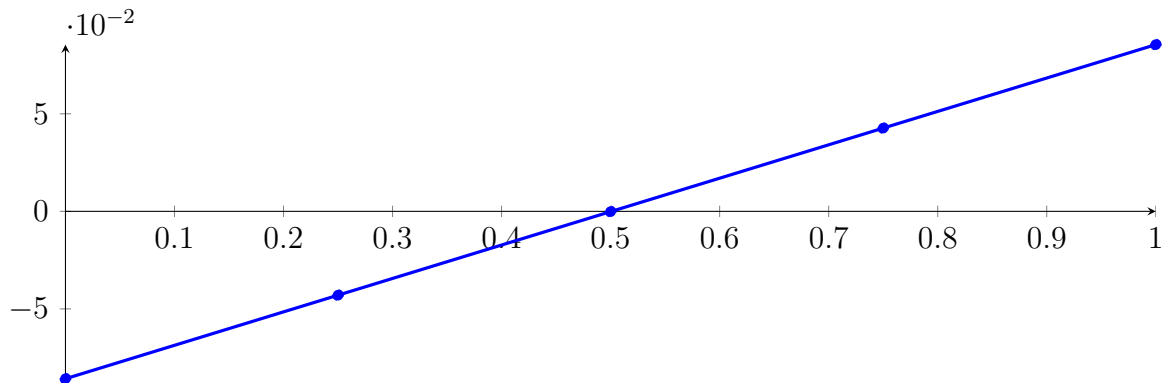
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

39.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

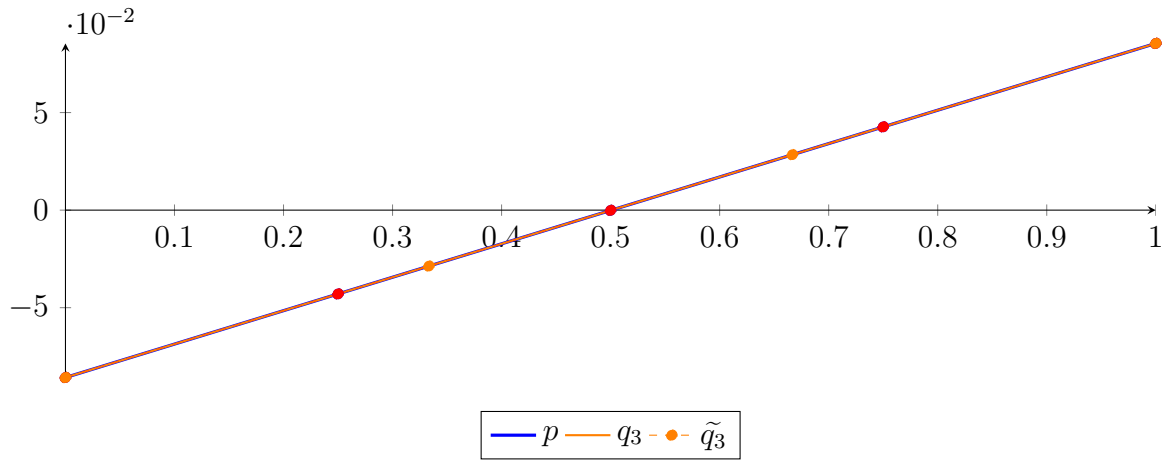
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 1.2032 \cdot 10^{-14} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45913 \cdot 10^{-11}$.

Bounding polynomials M and m :

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

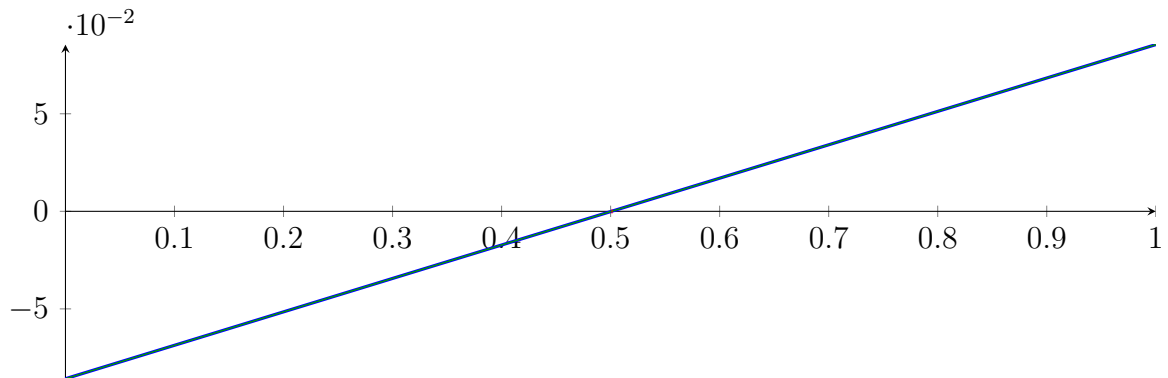
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\}$$

$$N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

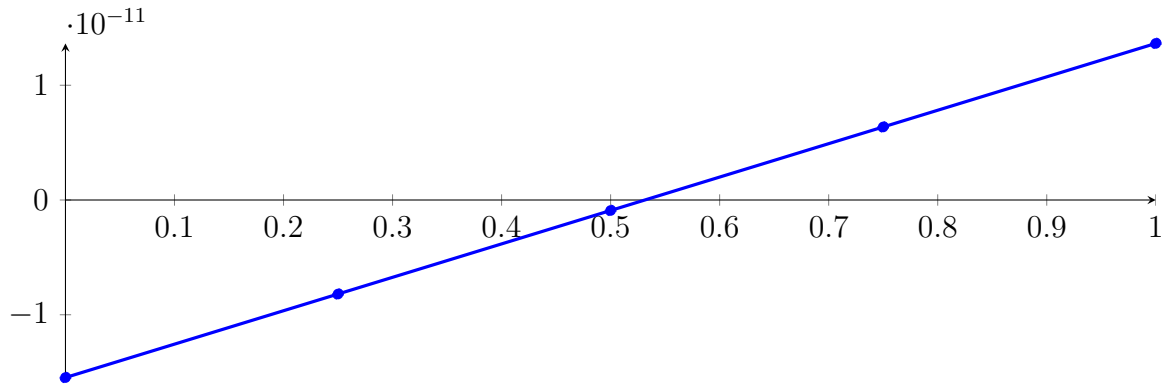
Longest intersection interval: $1.70047 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

39.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

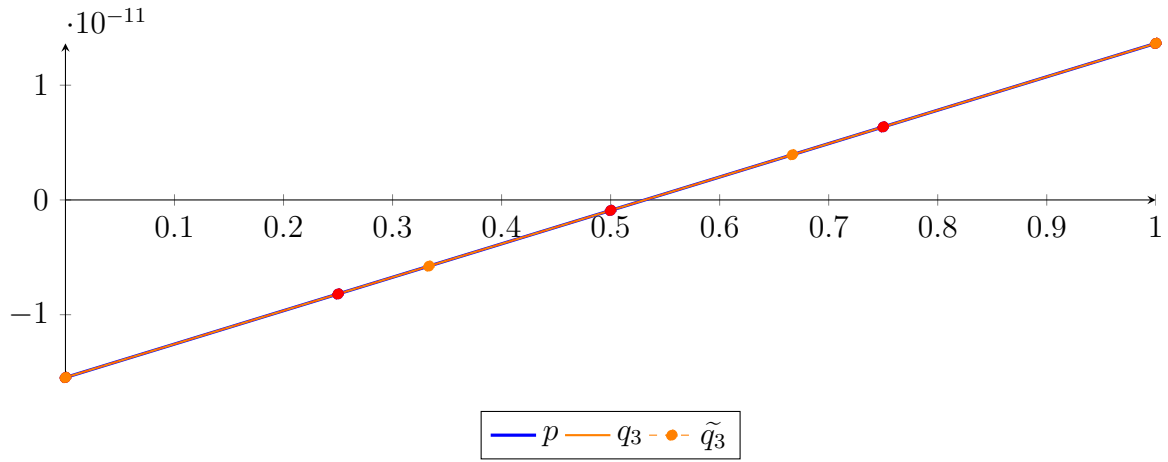
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.01312 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\ &= -1.5473 \cdot 10^{-11} B_{0,4}(X) - 8.19335 \cdot 10^{-12} B_{1,4}(X) - 9.13745 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36586 \cdot 10^{-12} B_{3,4}(X) + 1.36455 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -6.78547 \cdot 10^{-25} X^3 - 2.95652 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\
 &= -1.5473 \cdot 10^{-11} B_{0,3} - 5.76681 \cdot 10^{-12} B_{1,3} + 3.93932 \cdot 10^{-12} B_{2,3} + 1.36455 \cdot 10^{-11} B_{3,3} \\
 \tilde{q}_3 &= 2.83697 \cdot 10^{-24} X^4 - 6.85009 \cdot 10^{-24} X^3 + 1.39587 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\
 &= -1.5473 \cdot 10^{-11} B_{0,4} - 8.19335 \cdot 10^{-12} B_{1,4} - 9.13745 \cdot 10^{-13} B_{2,4} + 6.36586 \cdot 10^{-12} B_{3,4} + 1.36455 \cdot 10^{-11} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.84343 \cdot 10^{-25}$.

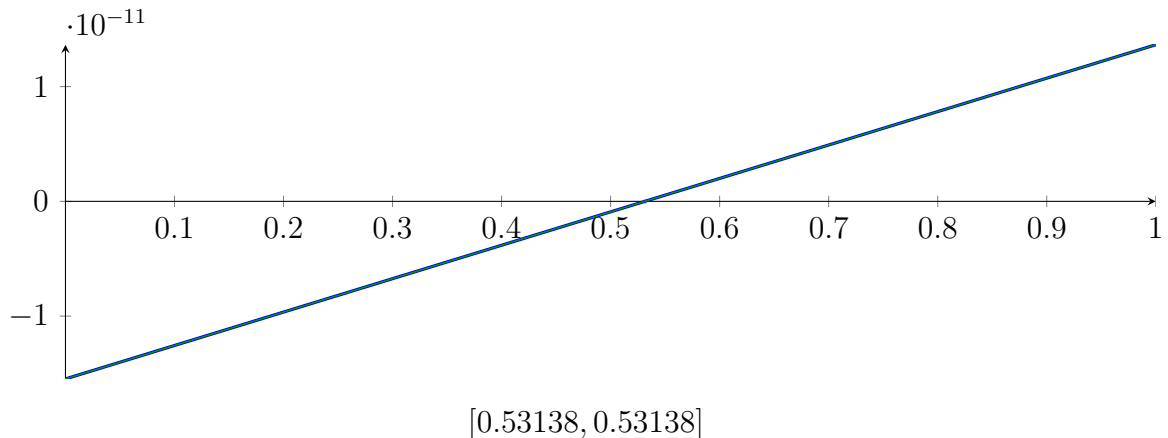
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -6.78547 \cdot 10^{-25} X^3 - 2.95652 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\
 m &= -6.78547 \cdot 10^{-25} X^3 - 2.95652 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-6.51982 \cdot 10^6, 0.53138, 6.51982 \cdot 10^6\} \quad N(m) = \{-6.51982 \cdot 10^6, 0.53138, 6.51982 \cdot 10^6\}$$

Intersection intervals:



Longest intersection interval: 0

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

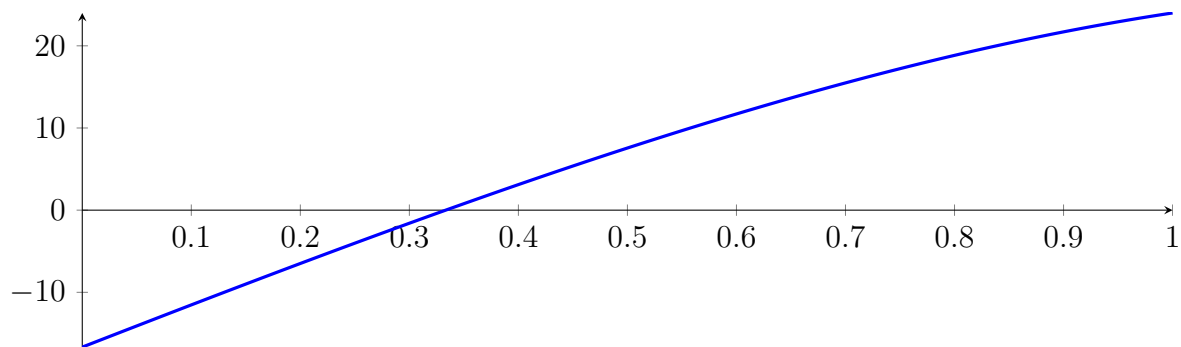
39.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

39.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

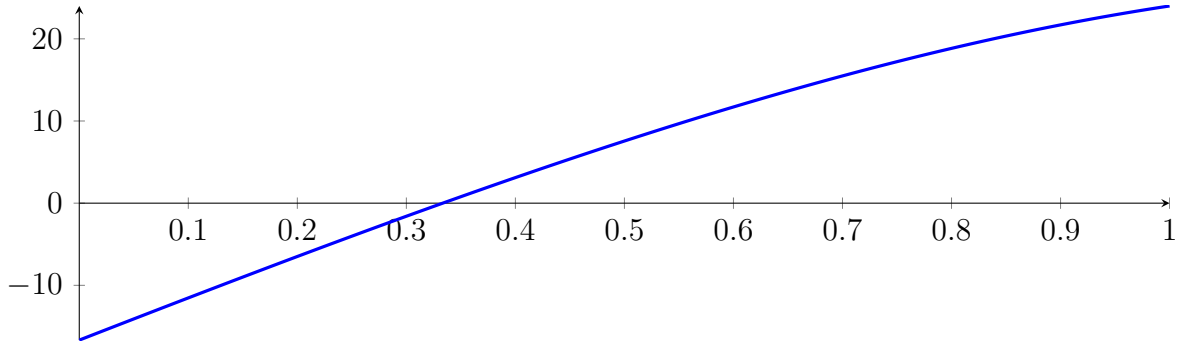
with precision $\varepsilon = 1 \cdot 10^{-64}$.

40 Running BezClip on f_4 with epsilon 128

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

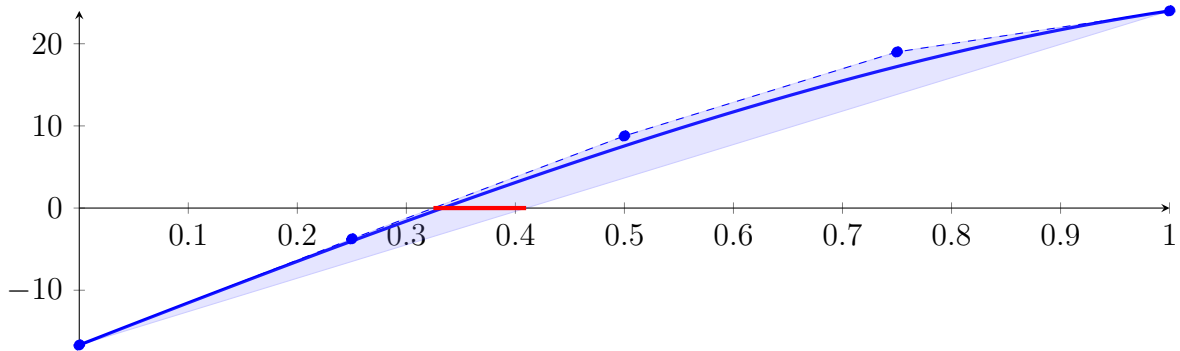
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



40.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

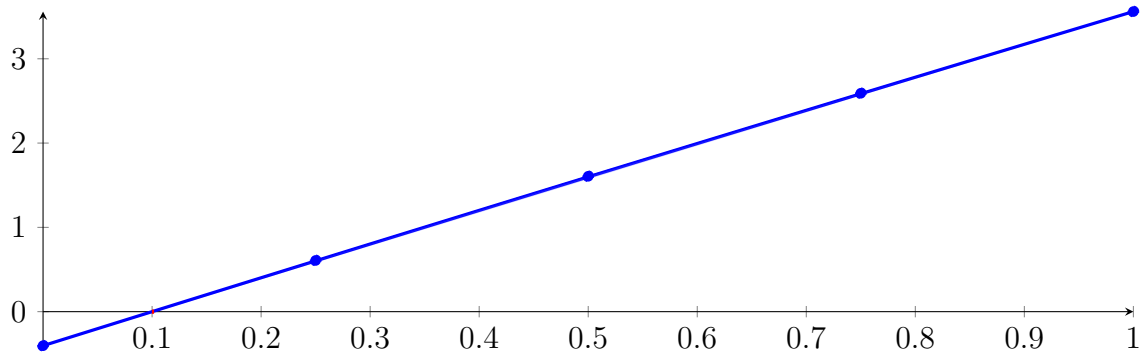
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

40.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

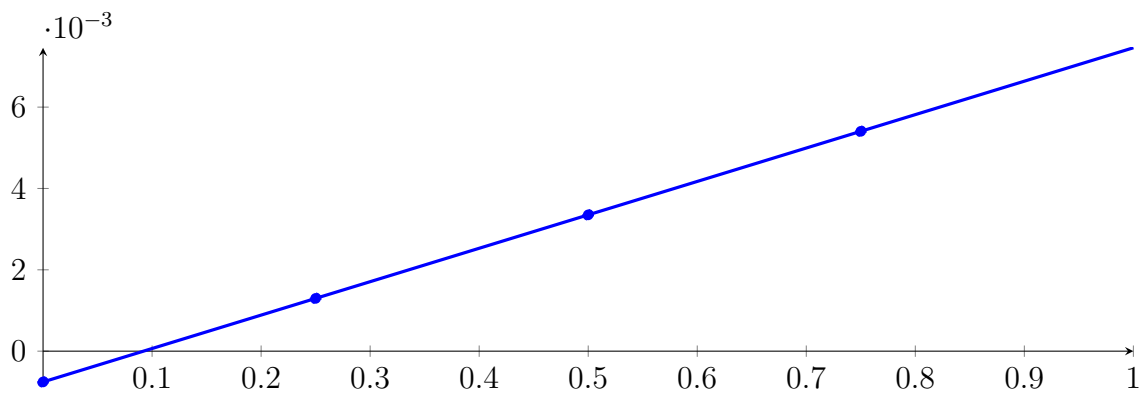
Longest intersection interval: 0.00203877

\implies Selective recursion: interval 1: $[0.333317, 0.333491]$,

40.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.06393 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

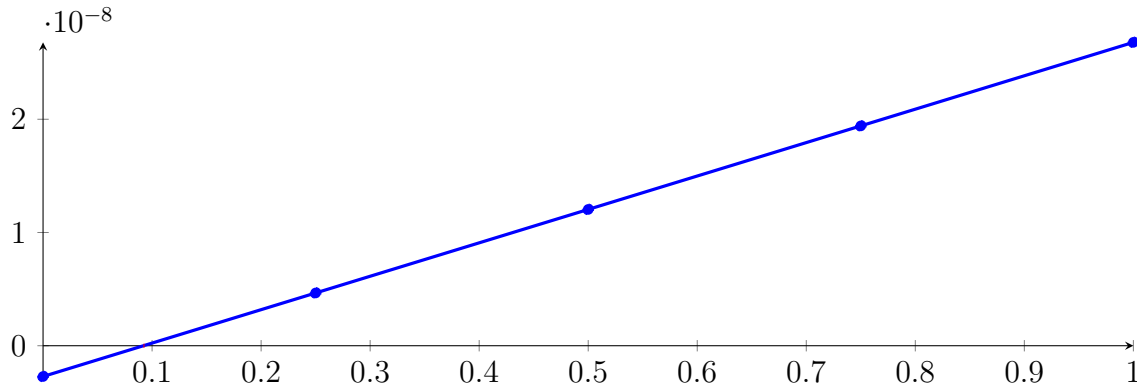
Longest intersection interval: $3.59185 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

40.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.92617 \cdot 10^{-24} X^4 + 6.61744 \cdot 10^{-24} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09} \\ &= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \\ &\quad \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28974 \cdot 10^{-11}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

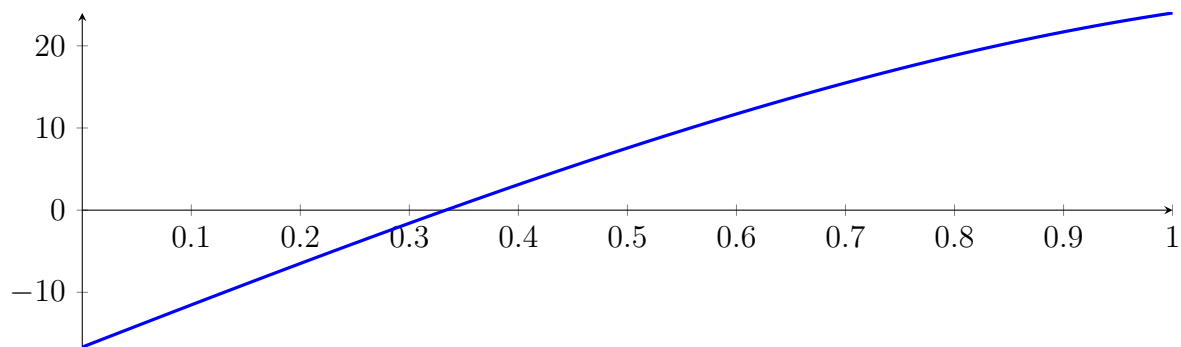
40.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

40.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

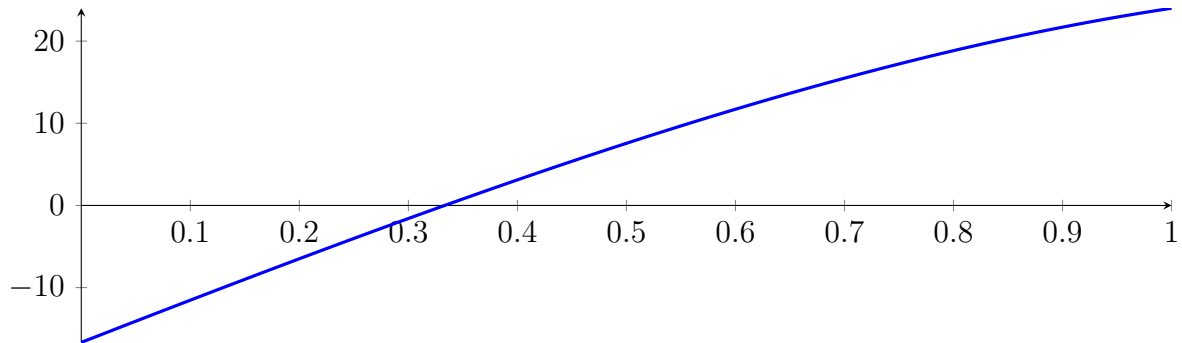
with precision $\varepsilon = 1 \cdot 10^{-128}$.

41 Running QuadClip on f_4 with epsilon 128

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

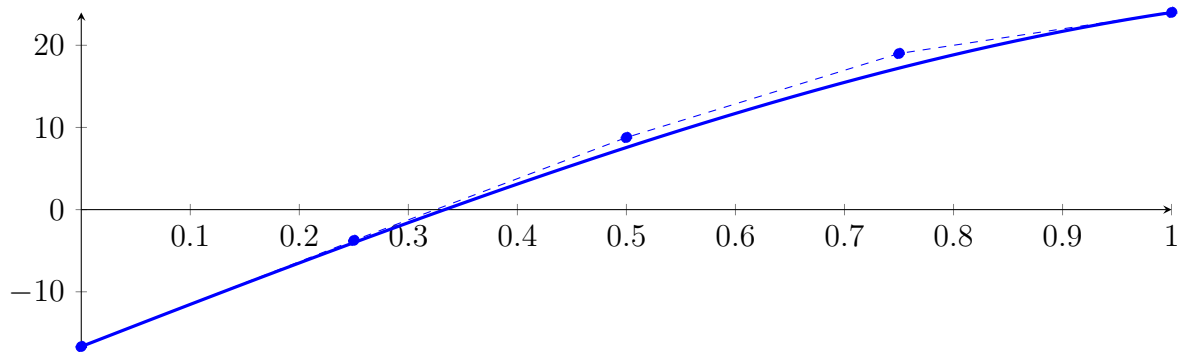
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



41.1 Recursion Branch 1 for Input Interval $[0, 1]$

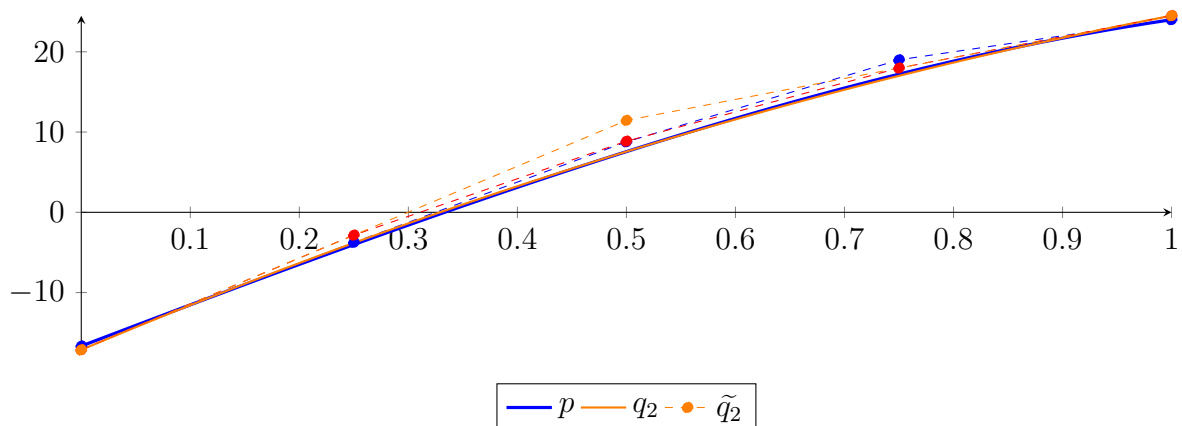
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \\ \tilde{q}_2 &= -2.67519 \cdot 10^{-12}X^4 + 5.32907 \cdot 10^{-12}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

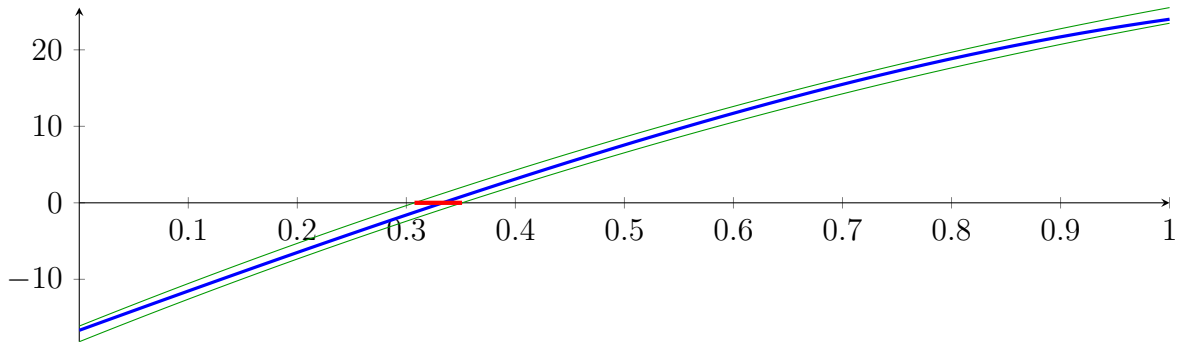
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

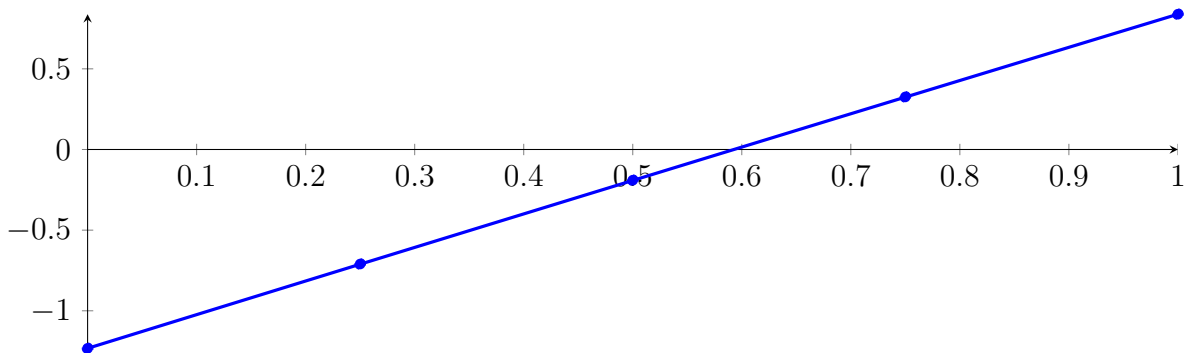
Longest intersection interval: 0.0436205

⇒ Selective recursion: interval 1: $[0.307477, 0.351097]$,

41.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

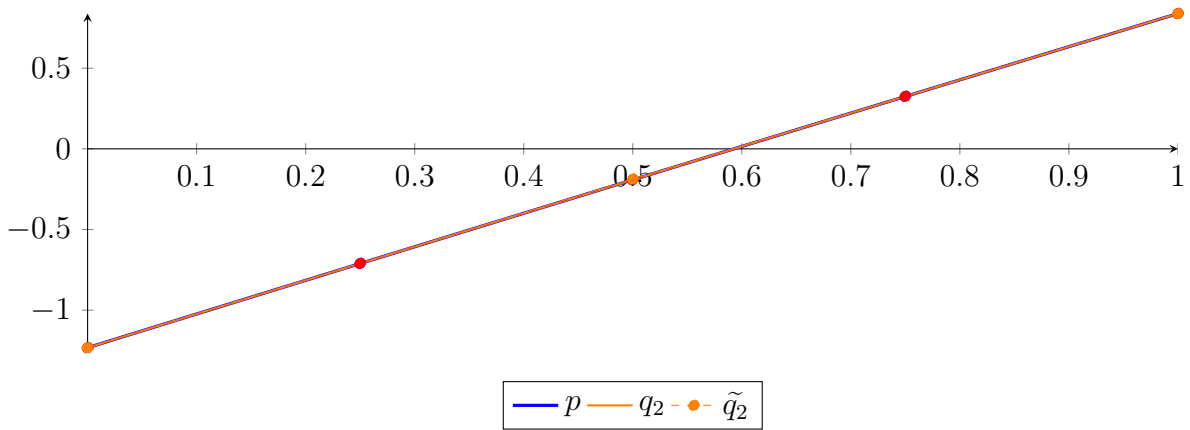
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 2.43583 \cdot 10^{-13} X^4 - 4.3876 \cdot 10^{-13} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

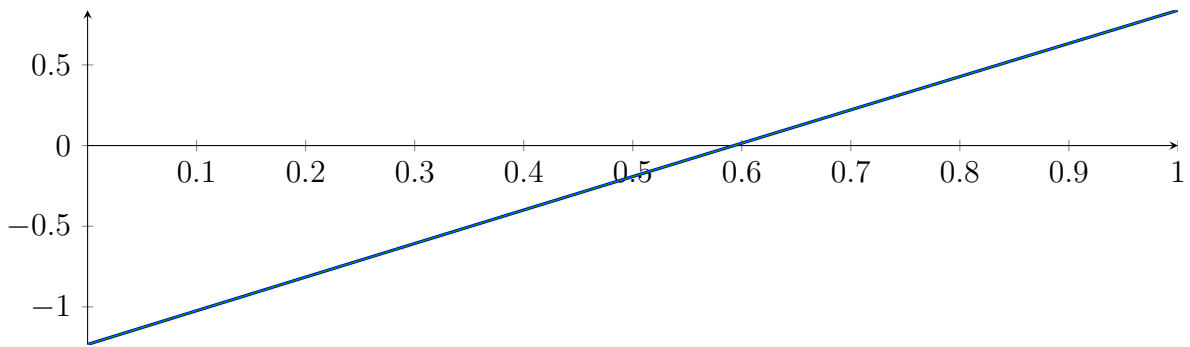
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

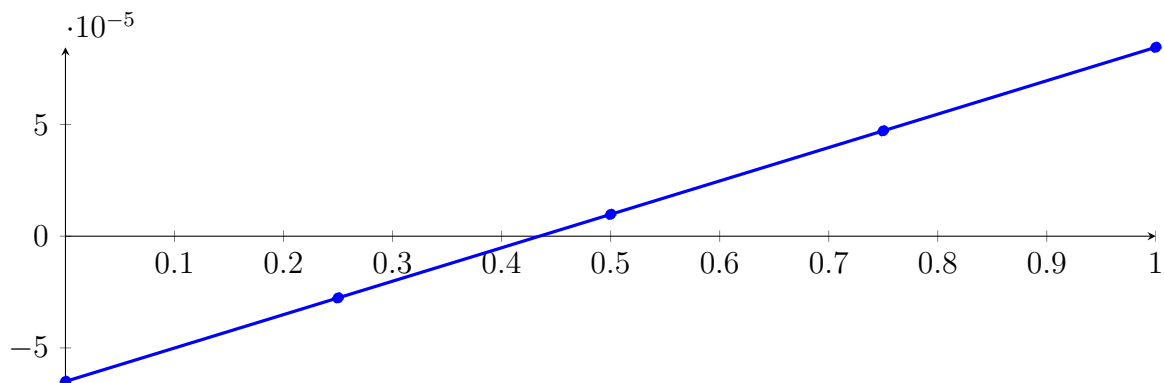
Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

41.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -2.71051 \cdot 10^{-20} X^4 - 2.82489 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05} \\
 &= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \\
 &\quad \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)
 \end{aligned}$$



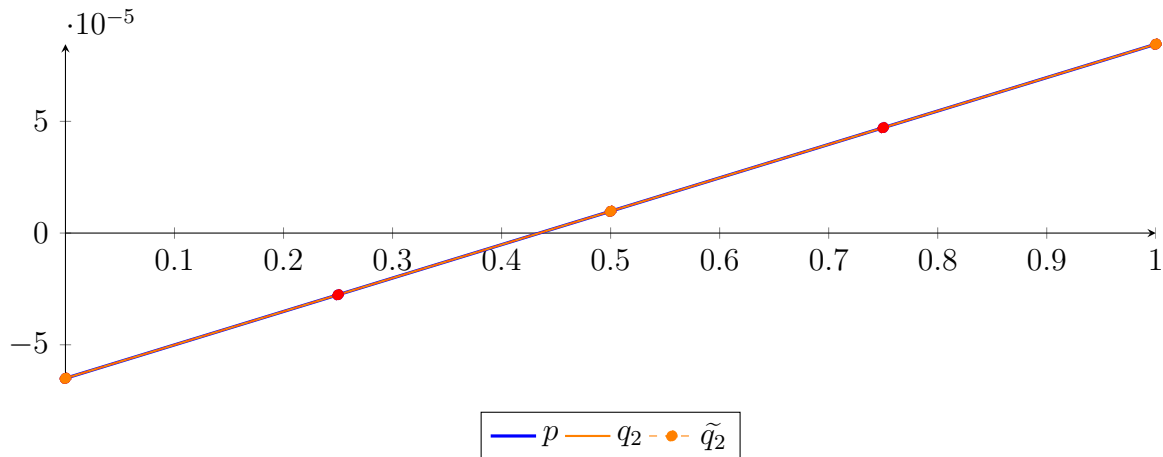
Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = 6.72205 \cdot 10^{-18} X^4 - 1.21431 \cdot 10^{-17} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.88601 \cdot 10^{-17}$.

Bounding polynomials M and m :

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

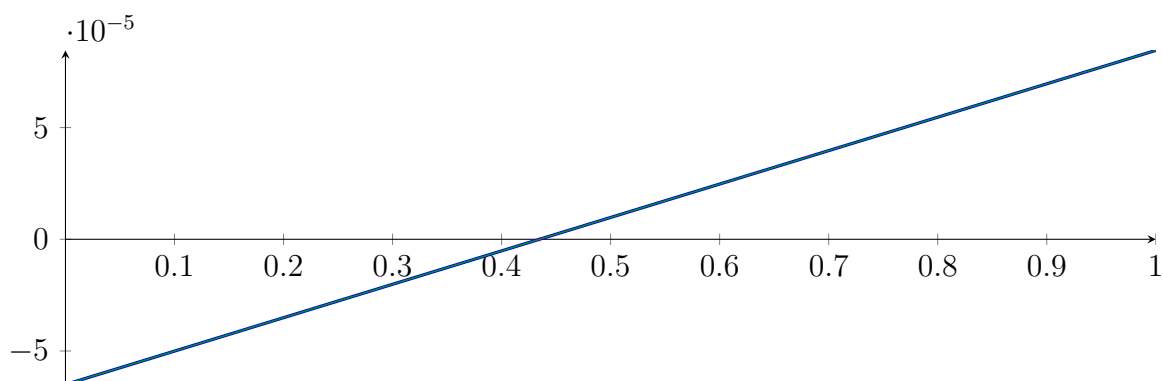
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.434685, 1.4089 \cdot 10^6\}$$

$$N(m) = \{0.434685, 1.4089 \cdot 10^6\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

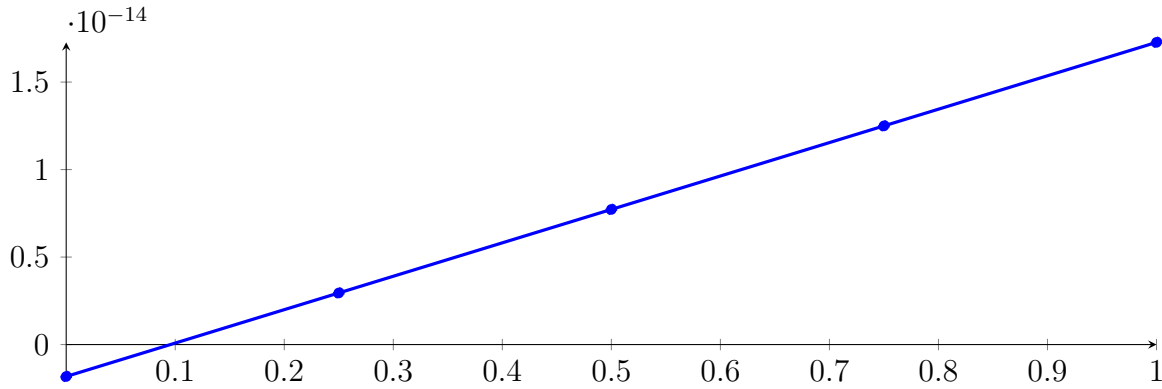
Longest intersection interval: $1.27678 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

41.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

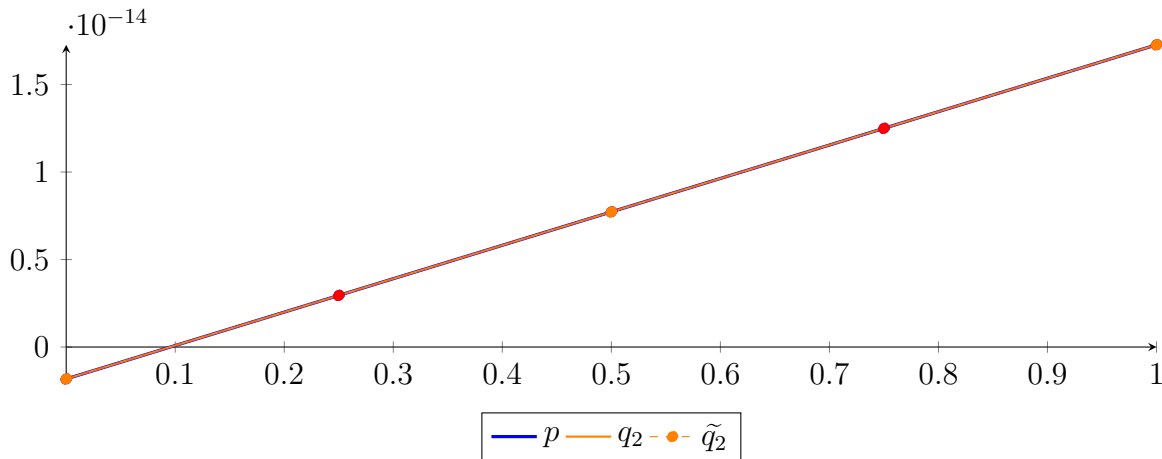
$$\begin{aligned}
 p &= -1.41995 \cdot 10^{-29} X^4 + 6.31089 \cdot 10^{-30} X^3 + 4.73317 \cdot 10^{-30} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15} \\
 &= -1.8241 \cdot 10^{-15} B_{0,4}(X) + 2.94943 \cdot 10^{-15} B_{1,4}(X) + 7.72295 \\
 &\quad \cdot 10^{-15} B_{2,4}(X) + 1.24965 \cdot 10^{-14} B_{3,4}(X) + 1.727 \cdot 10^{-14} B_{4,4}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -2.68213 \cdot 10^{-29} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15} \\
 &= -1.8241 \cdot 10^{-15} B_{0,2} + 7.72295 \cdot 10^{-15} B_{1,2} + 1.727 \cdot 10^{-14} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -2.18041 \cdot 10^{-27} X^4 + 3.9443 \cdot 10^{-27} X^3 - 2.24352 \cdot 10^{-27} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15} \\
 &= -1.8241 \cdot 10^{-15} B_{0,4} + 2.94943 \cdot 10^{-15} B_{1,4} + 7.72295 \cdot 10^{-15} B_{2,4} + 1.24965 \cdot 10^{-14} B_{3,4} + 1.727 \cdot 10^{-14} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.73549 \cdot 10^{-28}$.

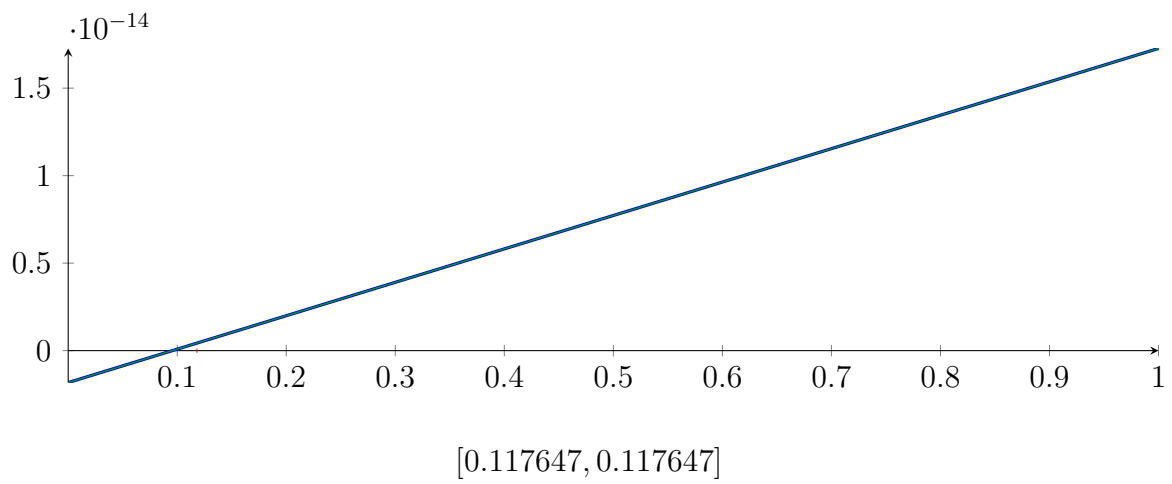
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -2.68213 \cdot 10^{-29} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15} \\
 m &= -2.68213 \cdot 10^{-29} X^2 + 1.90941 \cdot 10^{-14} X - 1.8241 \cdot 10^{-15}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.117647, 7.11901 \cdot 10^{14}\} \quad N(m) = \{0.117647, 7.11901 \cdot 10^{14}\}$$

Intersection intervals:



Longest intersection interval: 0

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

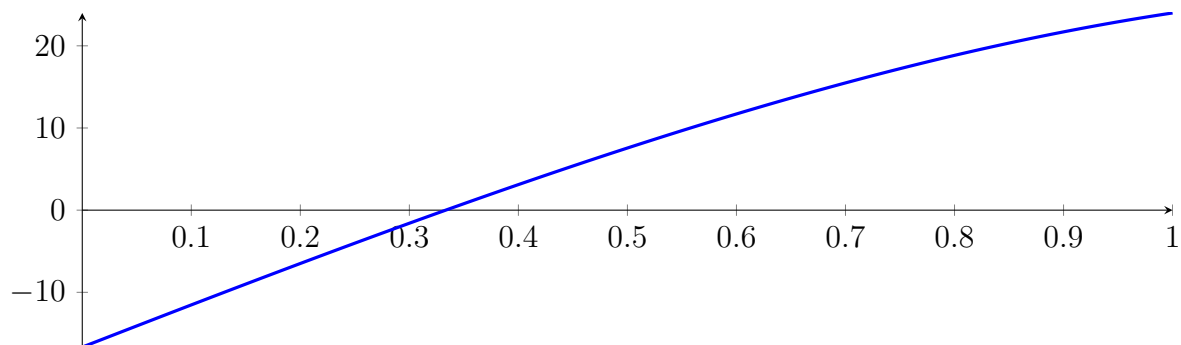
41.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

41.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

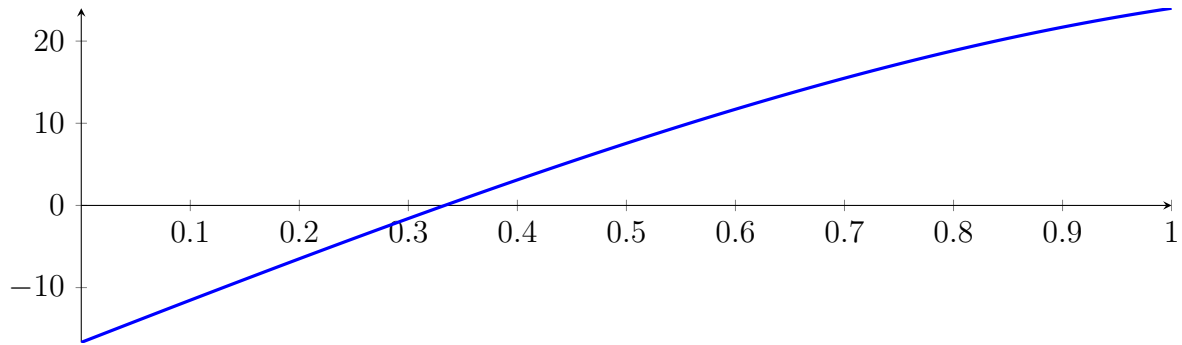
with precision $\varepsilon = 1 \cdot 10^{-128}$.

42 Running CubeClip on f_4 with epsilon 128

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

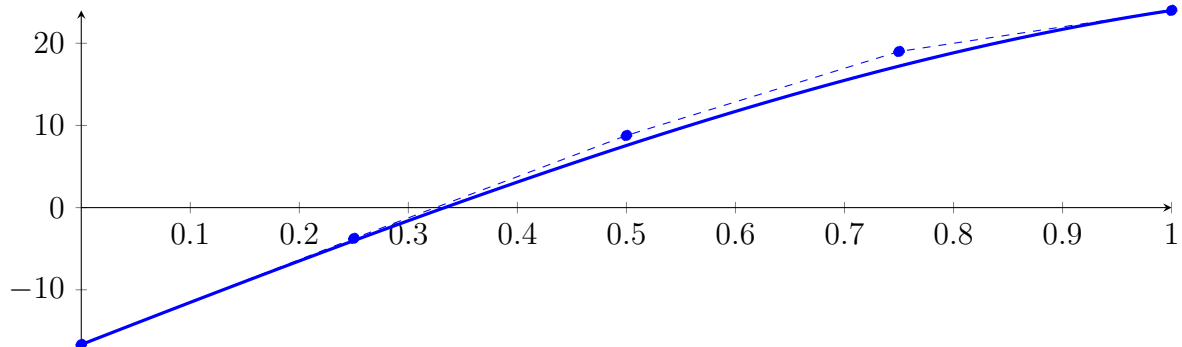
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



42.1 Recursion Branch 1 for Input Interval $[0, 1]$

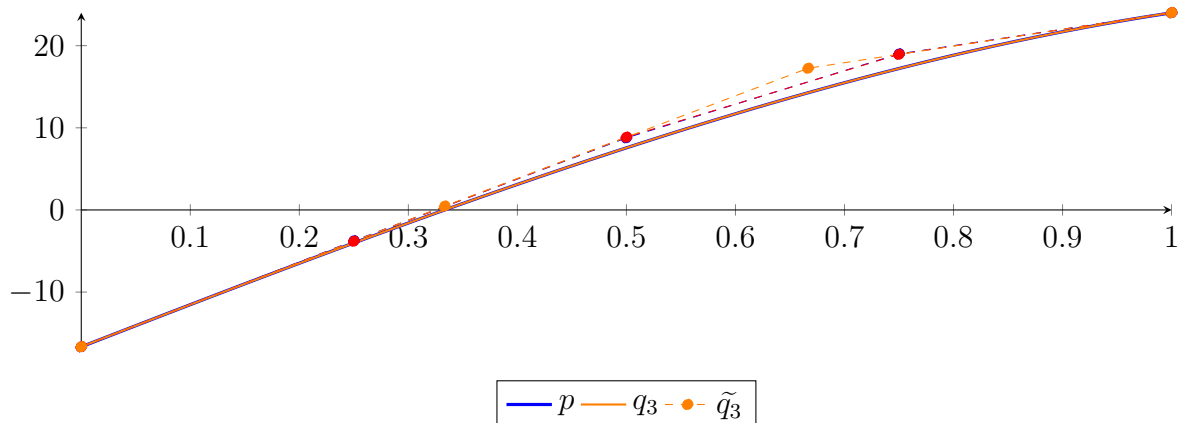
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= -2.28084 \cdot 10^{-12}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

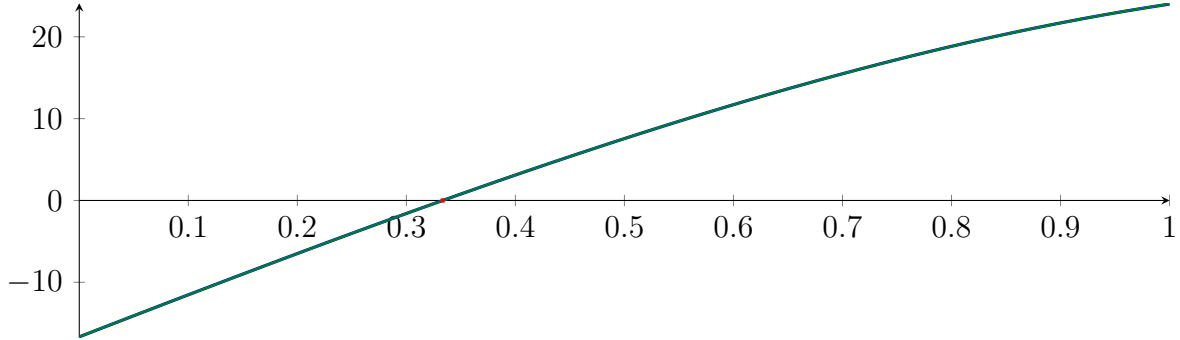
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

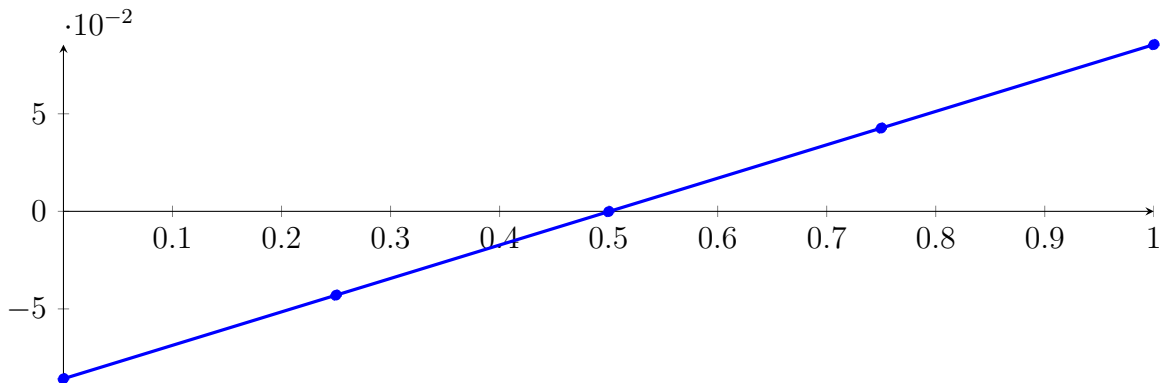
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

42.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

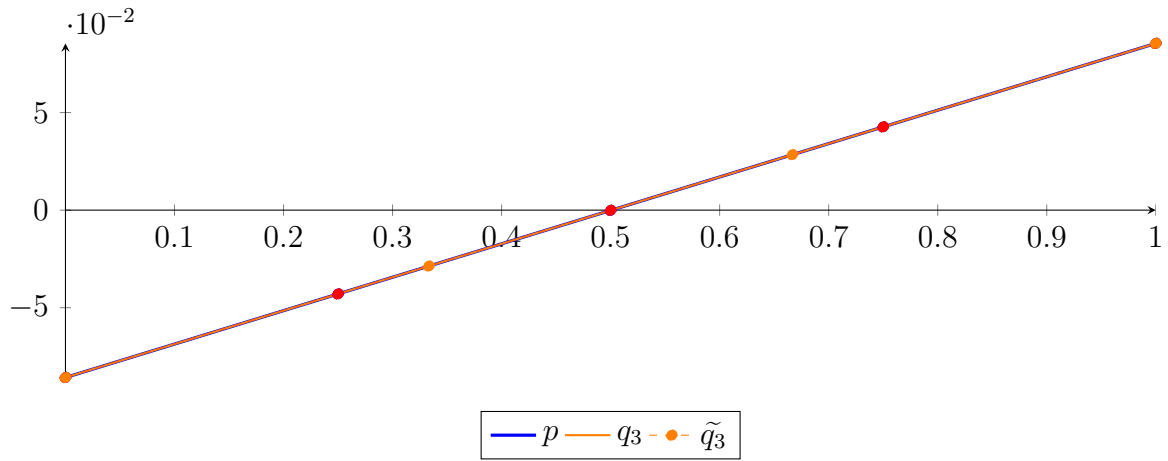
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 1.2032 \cdot 10^{-14} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45913 \cdot 10^{-11}$.

Bounding polynomials M and m :

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

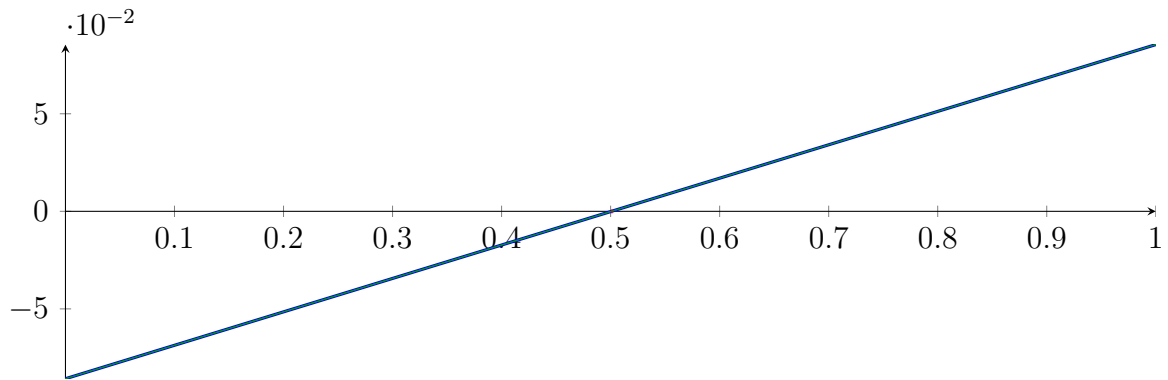
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\}$$

$$N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

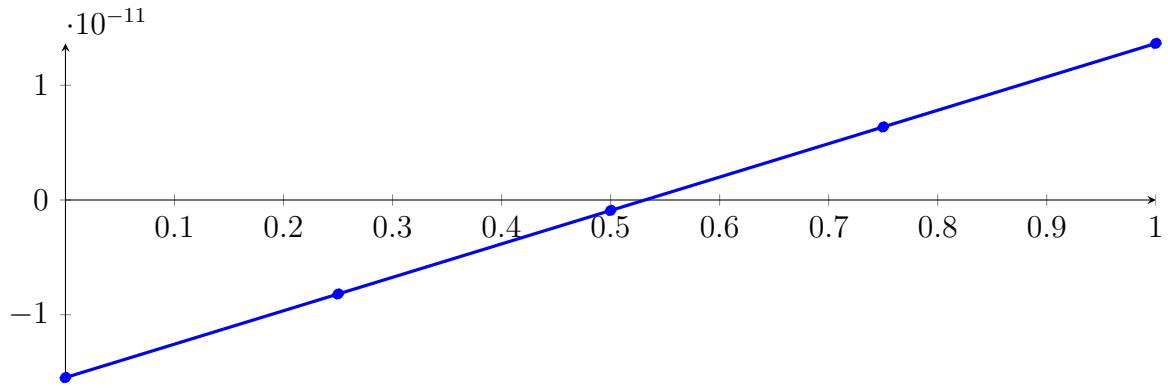
Longest intersection interval: $1.70047 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

42.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

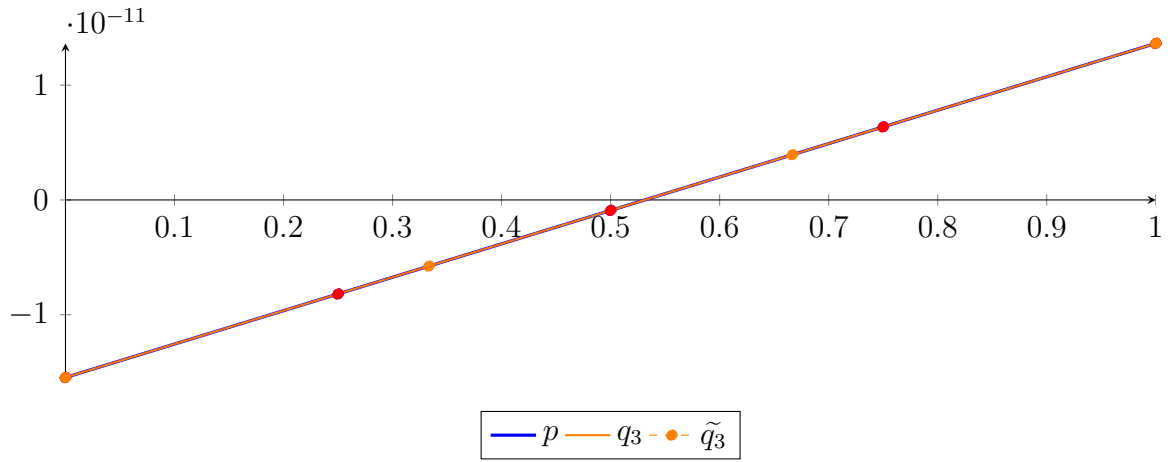
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.01312 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\ &= -1.5473 \cdot 10^{-11} B_{0,4}(X) - 8.19335 \cdot 10^{-12} B_{1,4}(X) - 9.13745 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36586 \cdot 10^{-12} B_{3,4}(X) + 1.36455 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -6.78547 \cdot 10^{-25} X^3 - 2.95652 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\
 &= -1.5473 \cdot 10^{-11} B_{0,3} - 5.76681 \cdot 10^{-12} B_{1,3} + 3.93932 \cdot 10^{-12} B_{2,3} + 1.36455 \cdot 10^{-11} B_{3,3} \\
 \tilde{q}_3 &= 2.83697 \cdot 10^{-24} X^4 - 6.85009 \cdot 10^{-24} X^3 + 1.39587 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\
 &= -1.5473 \cdot 10^{-11} B_{0,4} - 8.19335 \cdot 10^{-12} B_{1,4} - 9.13745 \cdot 10^{-13} B_{2,4} + 6.36586 \cdot 10^{-12} B_{3,4} + 1.36455 \cdot 10^{-11} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.84343 \cdot 10^{-25}$.

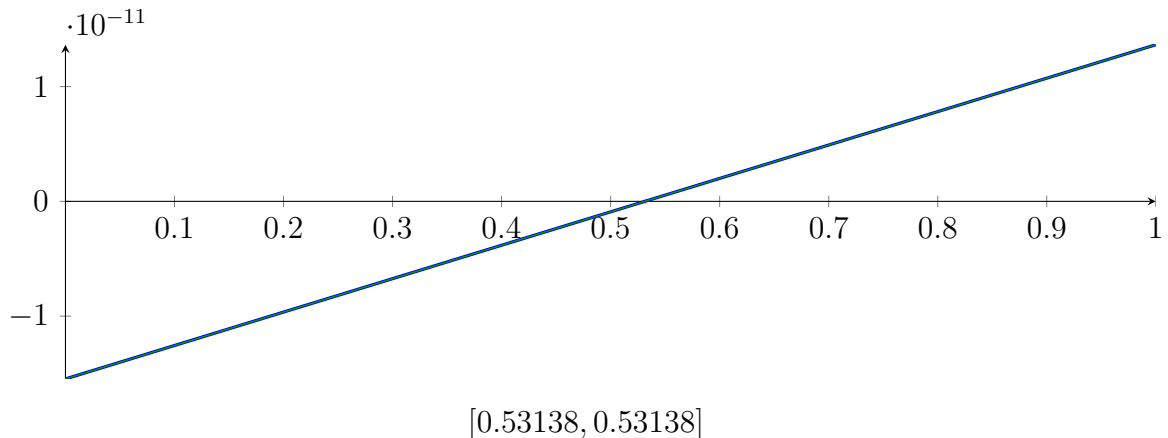
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -6.78547 \cdot 10^{-25} X^3 - 2.95652 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11} \\
 m &= -6.78547 \cdot 10^{-25} X^3 - 2.95652 \cdot 10^{-24} X^2 + 2.91184 \cdot 10^{-11} X - 1.5473 \cdot 10^{-11}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-6.51982 \cdot 10^6, 0.53138, 6.51982 \cdot 10^6\} \quad N(m) = \{-6.51982 \cdot 10^6, 0.53138, 6.51982 \cdot 10^6\}$$

Intersection intervals:



Longest intersection interval: 0

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

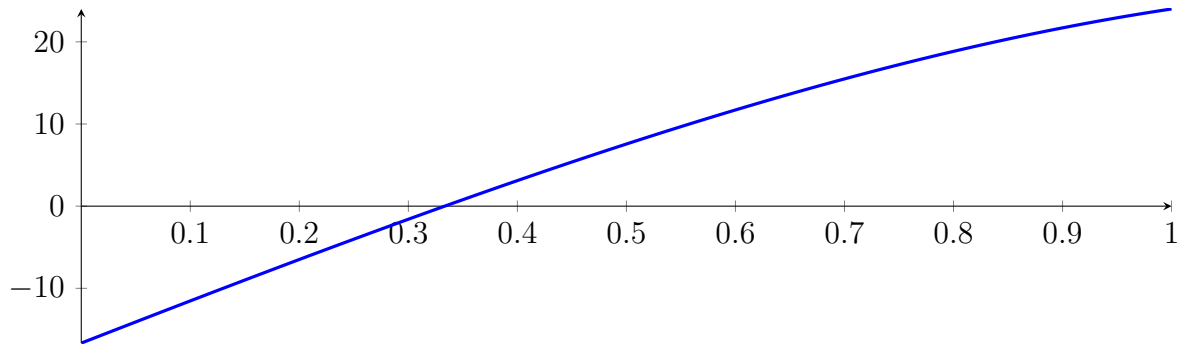
42.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

42.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

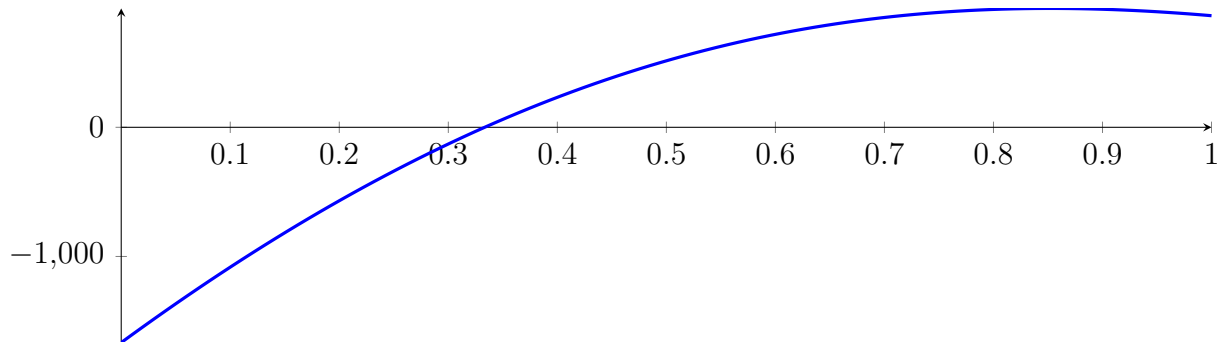
with precision $\varepsilon = 1 \cdot 10^{-128}$.

43 Running BezClip on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

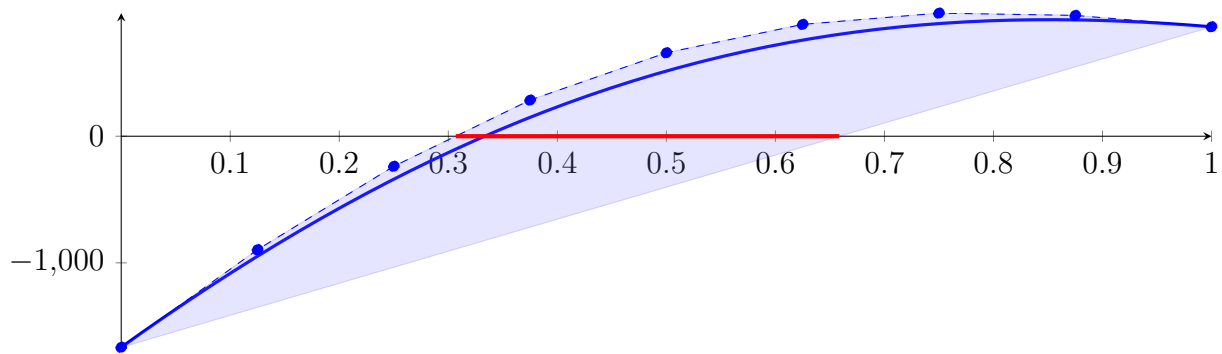
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



43.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

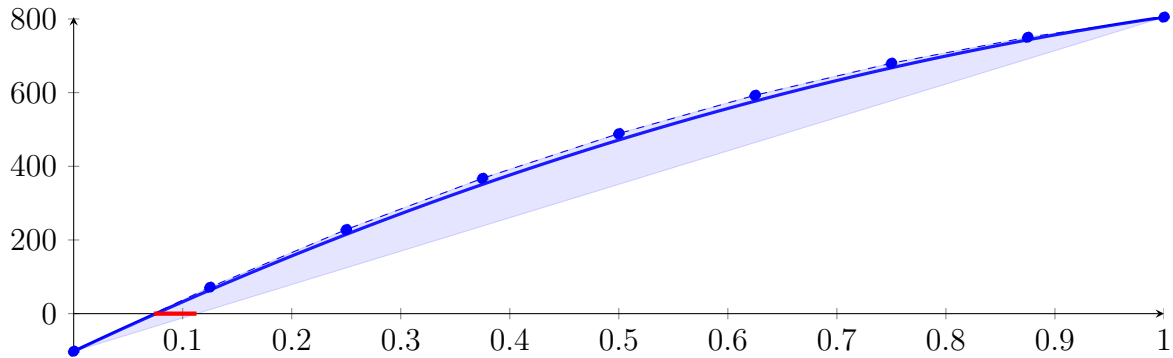
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

43.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

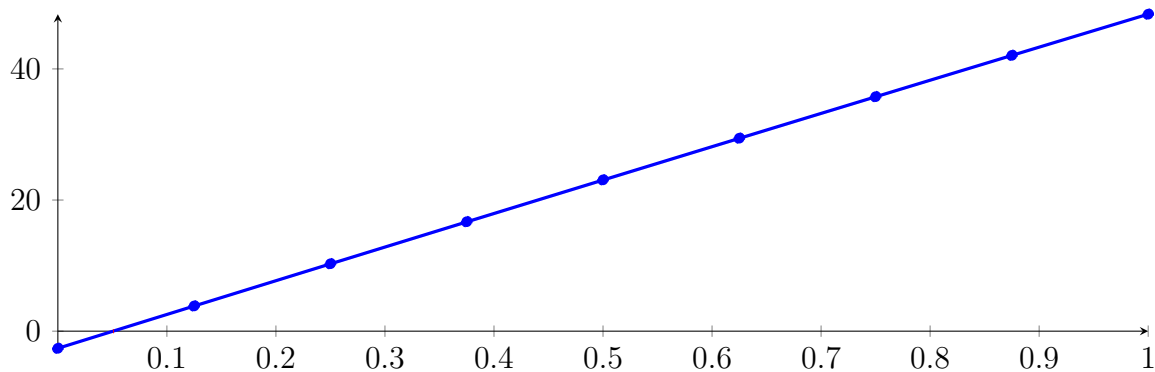
Longest intersection interval: 0.0391855

\implies Selective recursion: interval 1: [0.332635, 0.34642],

43.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -5.41789 \cdot 10^{-13}X^8 - 1.26477 \cdot 10^{-12}X^7 - 5.05786 \cdot 10^{-10}X^6 + 3.66765 \cdot 10^{-08}X^5 \\
 &\quad + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589

\implies Selective recursion: interval 1: [0.333333, 0.333343],

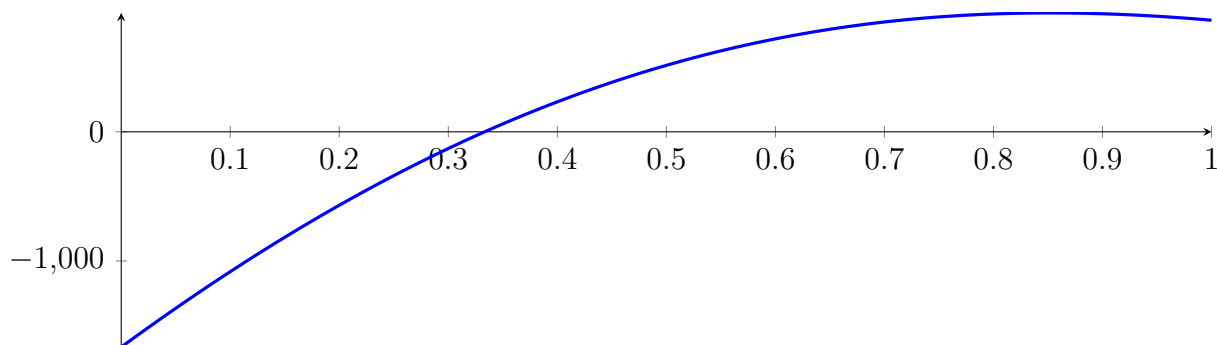
43.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Found root in interval [0.333333, 0.333343] at recursion depth 4!

43.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333343]$$

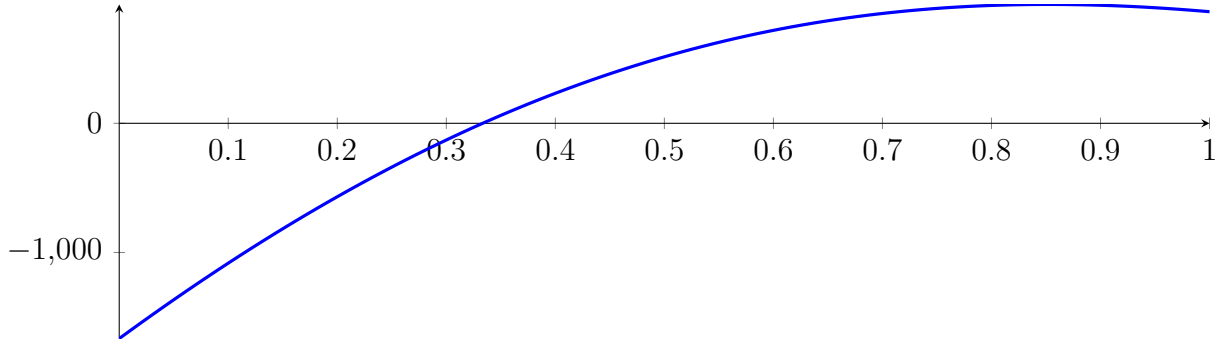
with precision $\varepsilon = 0.01$.

44 Running QuadClip on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

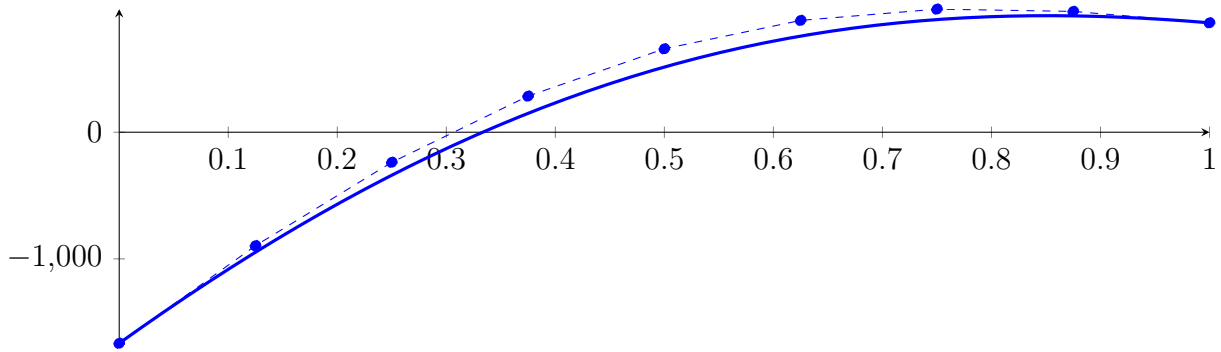
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



44.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

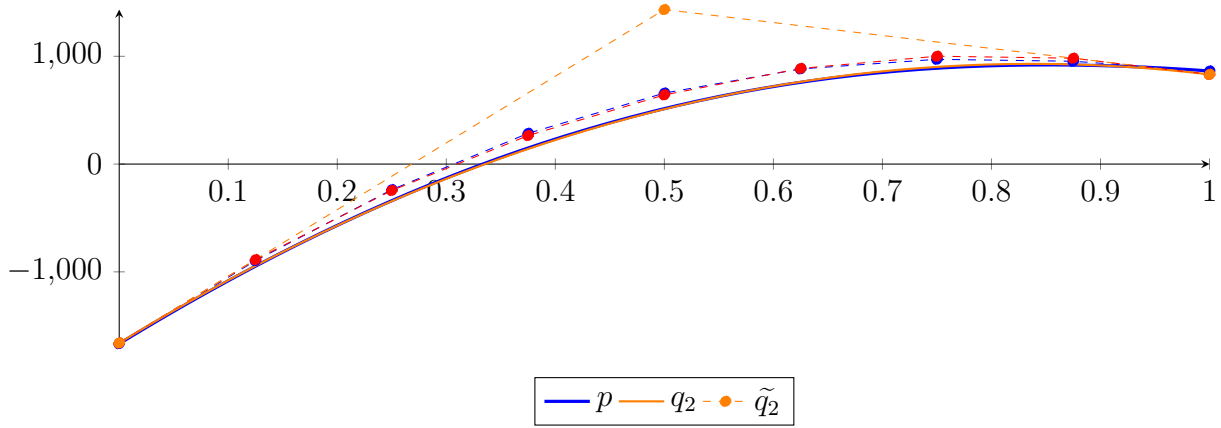
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -3.91297 \cdot 10^{-06}X^8 + 1.58774 \cdot 10^{-05}X^7 - 2.63335 \cdot 10^{-05}X^6 + 2.29285 \cdot 10^{-05}X^5 \\ &\quad - 1.11931 \cdot 10^{-05}X^4 + 3.0313 \cdot 10^{-06}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

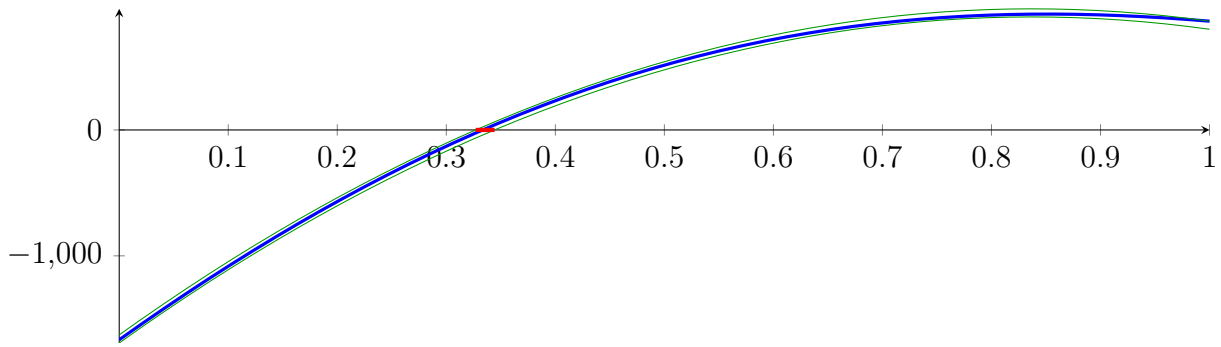
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

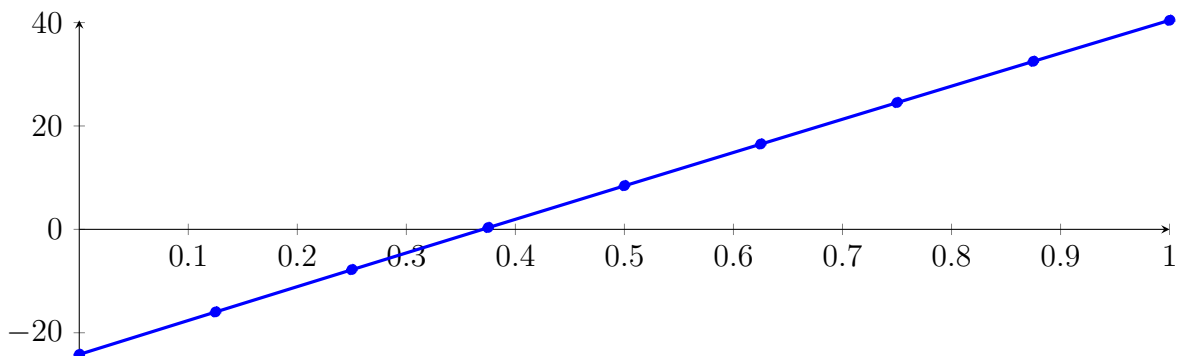
Longest intersection interval: 0.0173372

\implies Selective recursion: interval 1: $[0.326917, 0.344255]$,

44.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

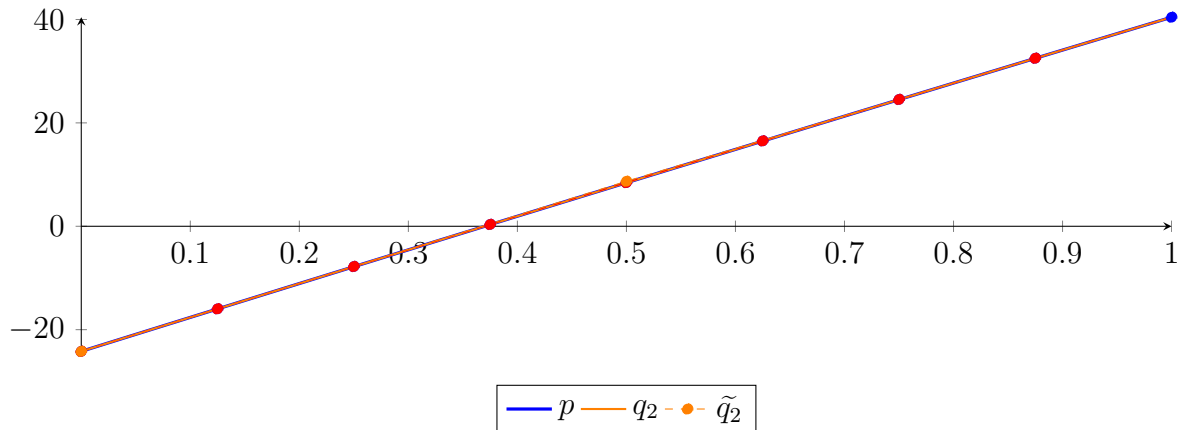
$$\begin{aligned} p &= -5.40012 \cdot 10^{-13} X^8 - 7.38964 \cdot 10^{-12} X^7 - 1.94416 \cdot 10^{-09} X^6 + 1.19265 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 1.00159 \cdot 10^{-08} X^8 - 3.3372 \cdot 10^{-08} X^7 + 4.23875 \cdot 10^{-08} X^6 - 2.49721 \cdot 10^{-08} X^5 \\ &\quad + 6.08793 \cdot 10^{-09} X^4 + 1.46429 \cdot 10^{-10} X^3 - 1.18261X^2 + 65.8162X - 24.1946 \\ &= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8} \\ &\quad + 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

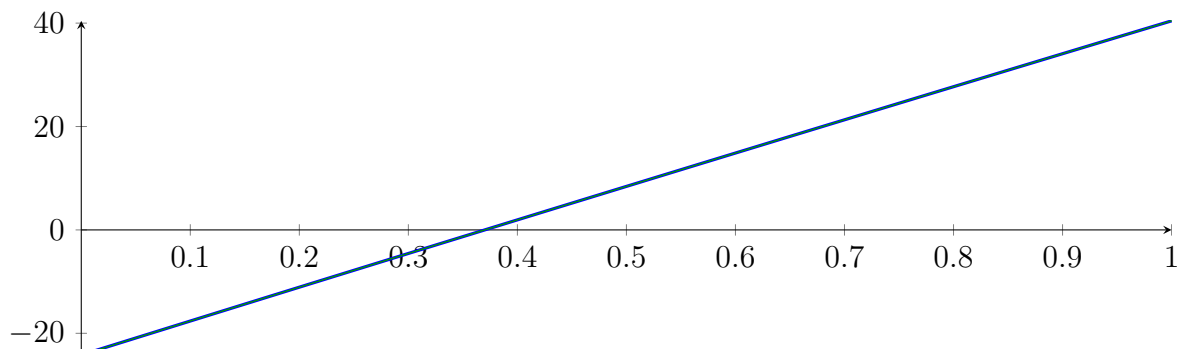
Bounding polynomials M and m :

$$\begin{aligned} M &= -1.18261X^2 + 65.8162X - 24.1945 \\ m &= -1.18261X^2 + 65.8162X - 24.1946 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\} \quad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

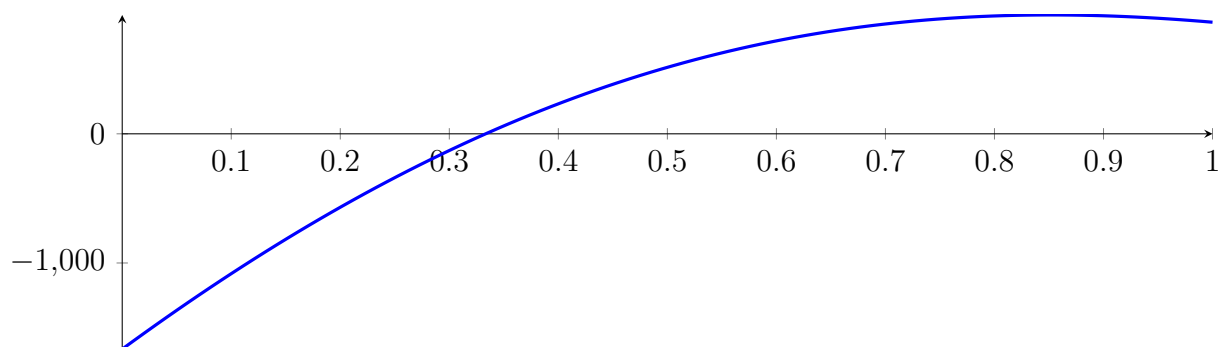
44.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

44.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

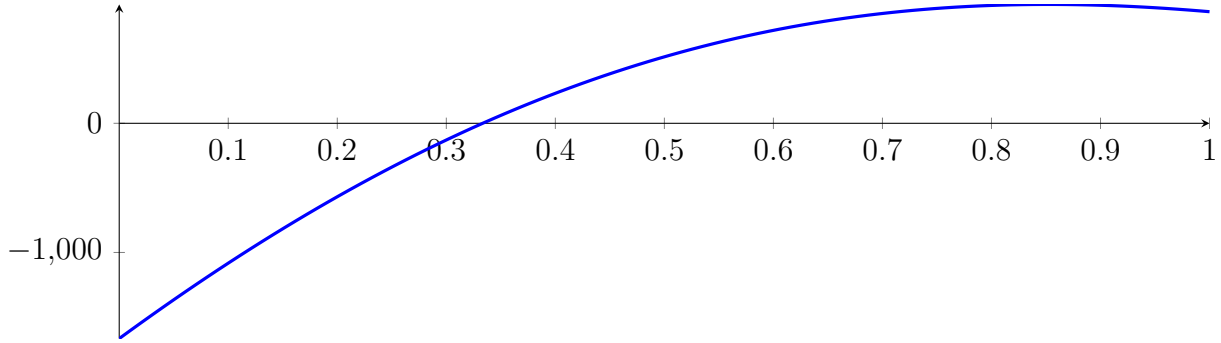
with precision $\varepsilon = 0.01$.

45 Running CubeClip on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

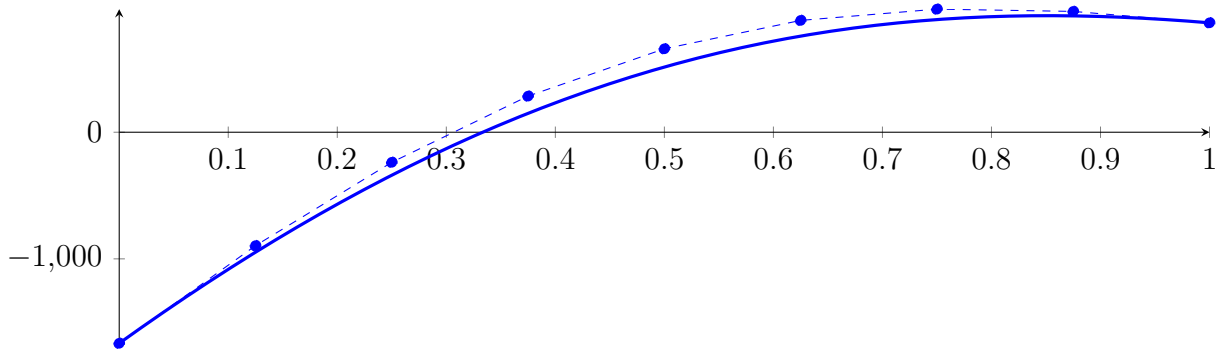
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



45.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

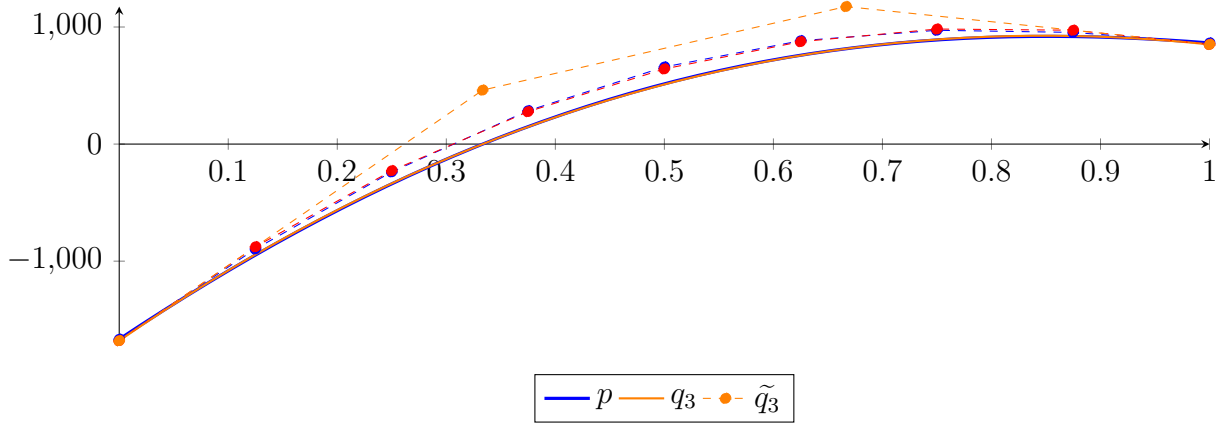
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 2.38117 \cdot 10^{-06}X^8 - 9.49957 \cdot 10^{-06}X^7 + 1.54757 \cdot 10^{-05}X^6 - 1.32609 \cdot 10^{-05}X^5 \\ &\quad + 6.41819 \cdot 10^{-06}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

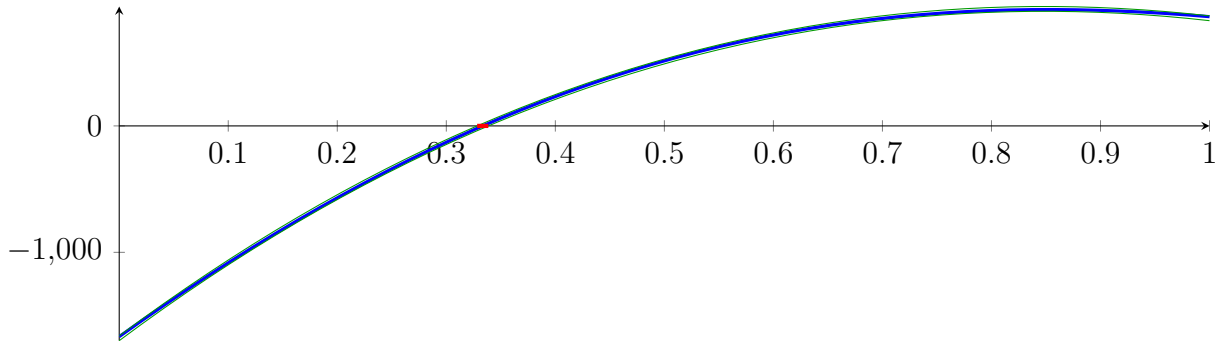
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

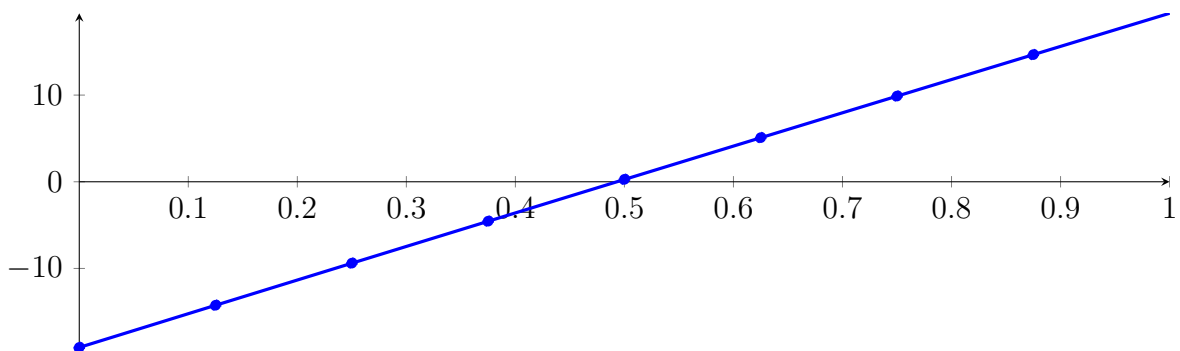
Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

45.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

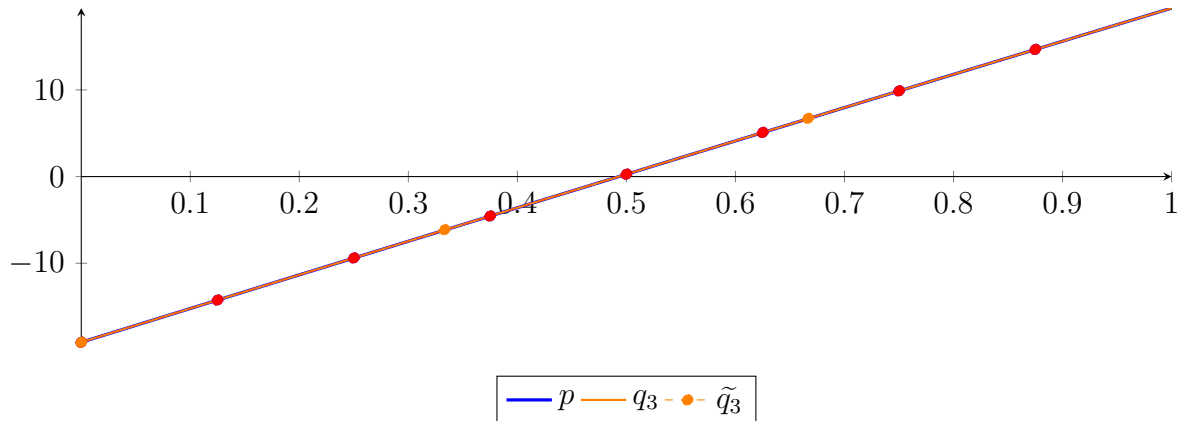
$$\begin{aligned} p &= -1.27898 \cdot 10^{-13} X^8 - 1.13687 \cdot 10^{-13} X^7 - 8.23661 \cdot 10^{-11} X^6 + 8.72882 \cdot 10^{-09} X^5 \\ &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 2.3353 \cdot 10^{-08}X^8 - 9.31856 \cdot 10^{-08}X^7 + 1.51861 \cdot 10^{-07}X^6 - 1.30228 \cdot 10^{-07}X^5 \\ &\quad + 6.31618 \cdot 10^{-08}X^4 - 0.000240962X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16887 \cdot 10^{-07}$.

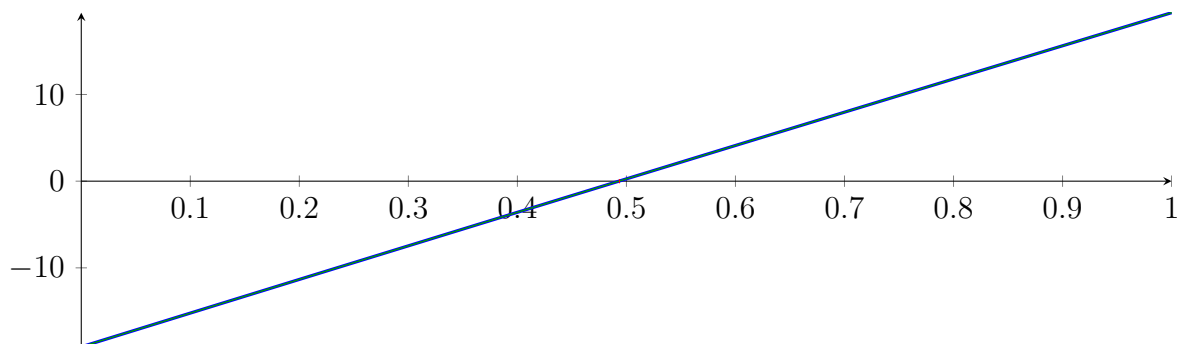
Bounding polynomials M and m :

$$\begin{aligned} M &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ m &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \end{aligned}$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \quad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.12517 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

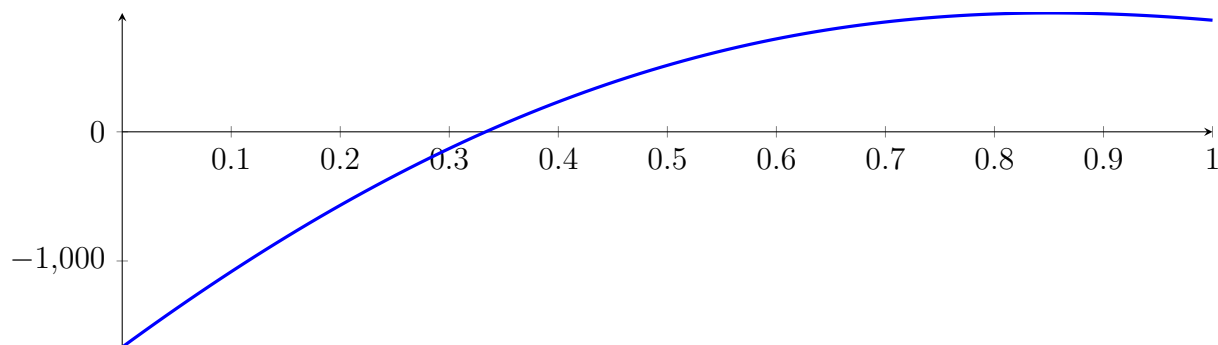
45.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

45.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

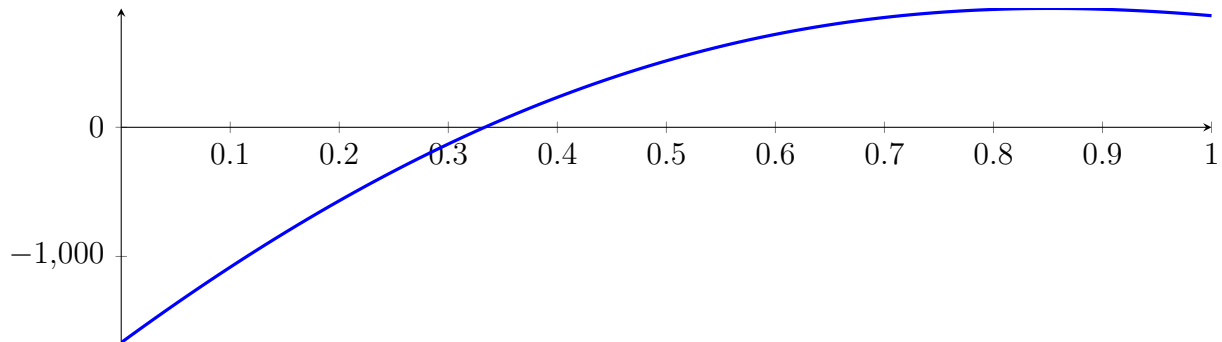
with precision $\varepsilon = 0.01$.

46 Running BezClip on f_8 with epsilon 4

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

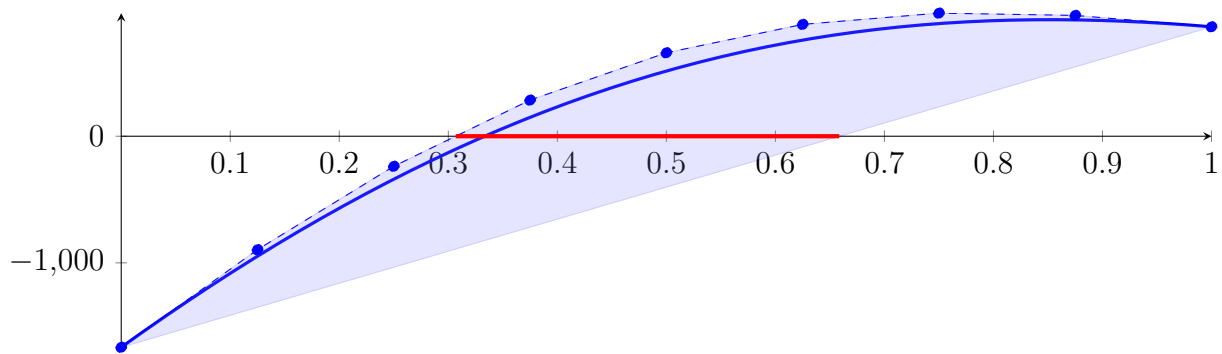
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



46.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

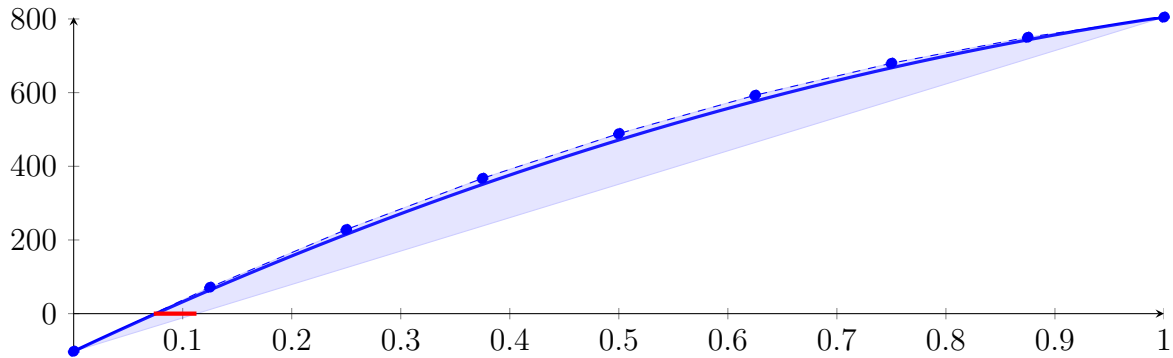
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

46.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

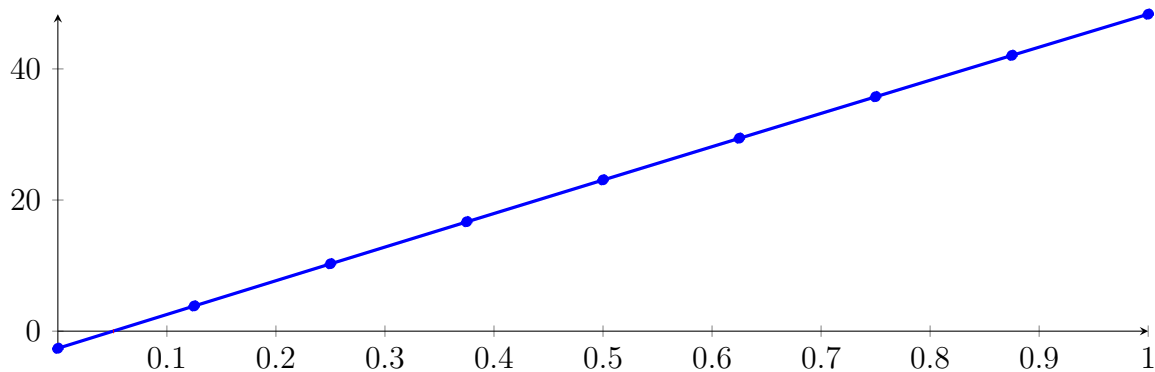
Longest intersection interval: 0.0391855

\Rightarrow Selective recursion: interval 1: [0.332635, 0.34642],

46.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -5.41789 \cdot 10^{-13}X^8 - 1.26477 \cdot 10^{-12}X^7 - 5.05786 \cdot 10^{-10}X^6 + 3.66765 \cdot 10^{-08}X^5 \\
 &\quad + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589

\Rightarrow Selective recursion: interval 1: [0.333333, 0.333343],

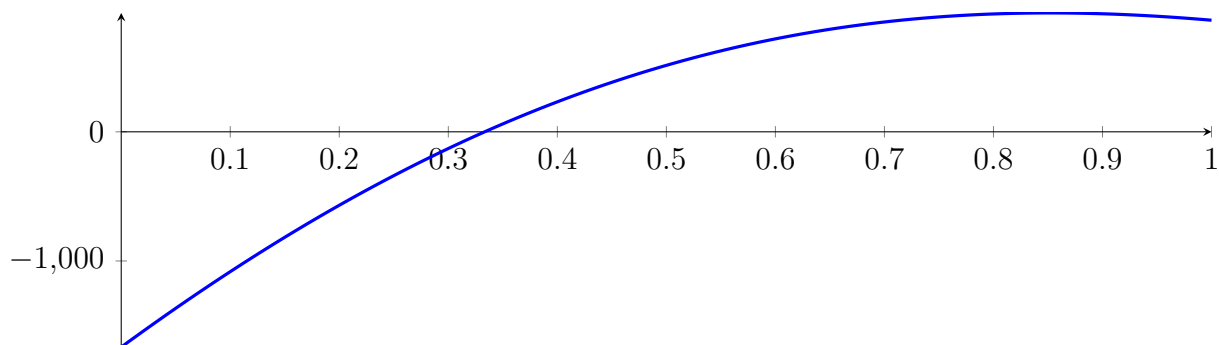
46.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Found root in interval [0.333333, 0.333343] at recursion depth 4!

46.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333343]$$

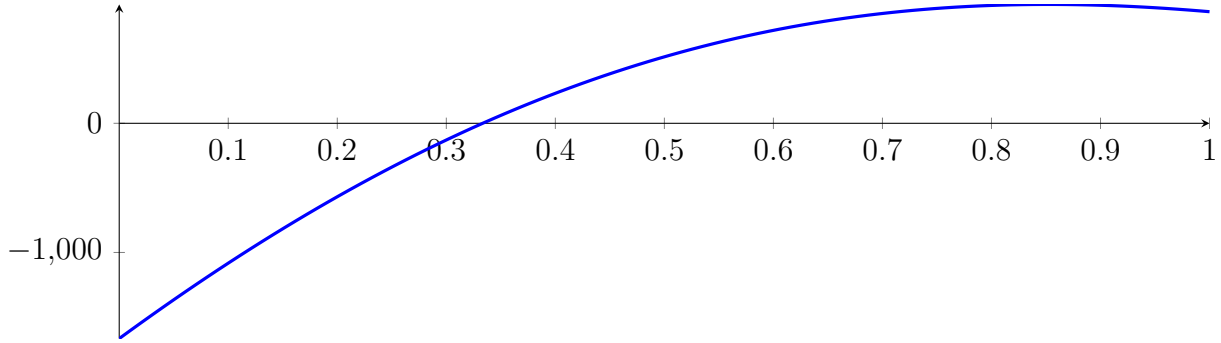
with precision $\varepsilon = 0.0001$.

47 Running QuadClip on f_8 with epsilon 4

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

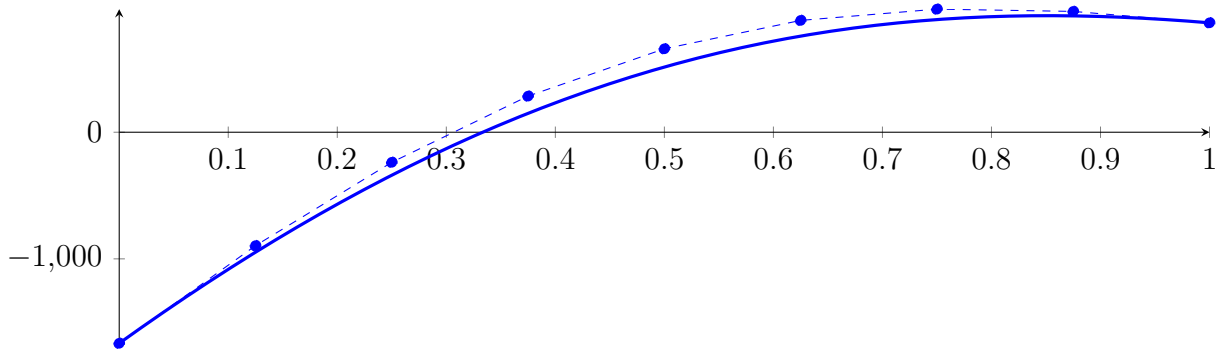
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



47.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

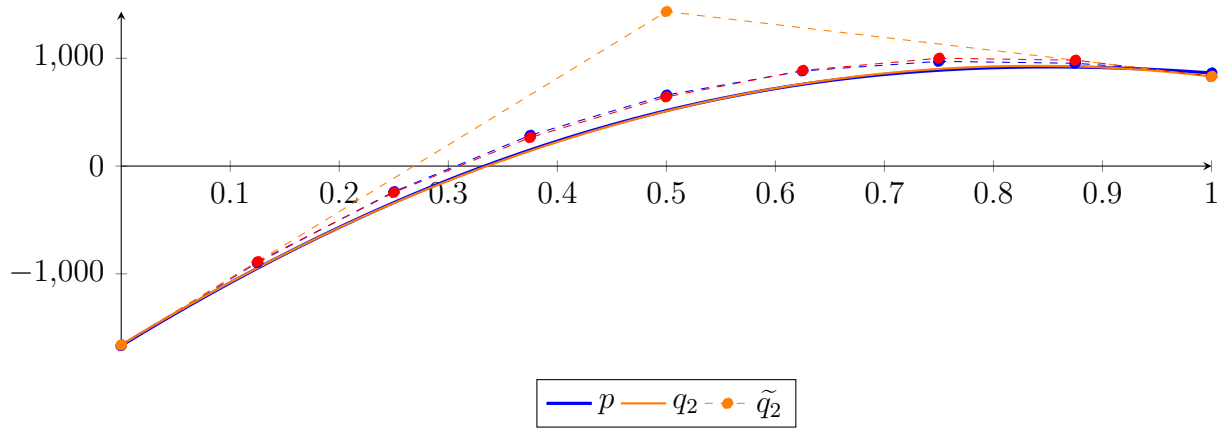
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -3.91297 \cdot 10^{-06}X^8 + 1.58774 \cdot 10^{-05}X^7 - 2.63335 \cdot 10^{-05}X^6 + 2.29285 \cdot 10^{-05}X^5 \\ &\quad - 1.11931 \cdot 10^{-05}X^4 + 3.0313 \cdot 10^{-06}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

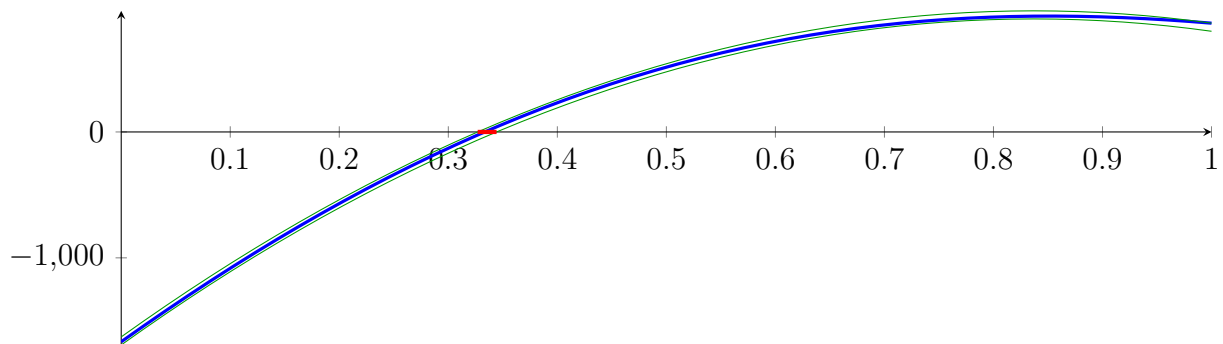
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

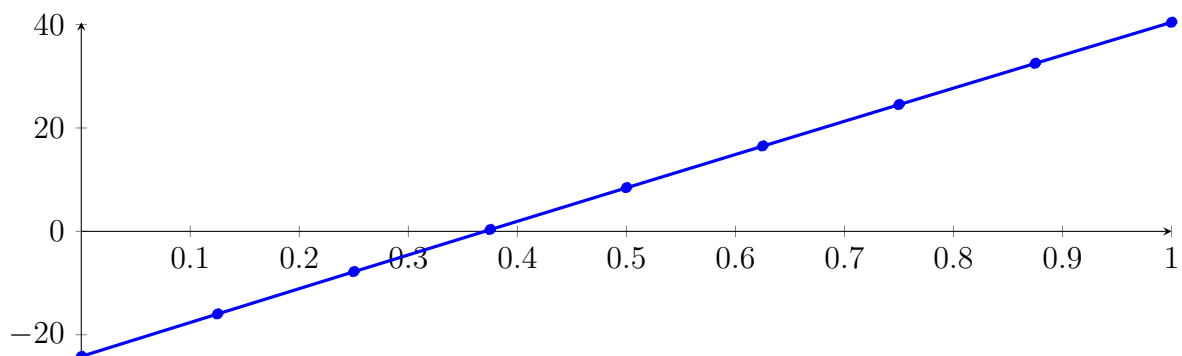
Longest intersection interval: 0.0173372

\implies Selective recursion: interval 1: $[0.326917, 0.344255]$,

47.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.40012 \cdot 10^{-13} X^8 - 7.38964 \cdot 10^{-12} X^7 - 1.94416 \cdot 10^{-09} X^6 + 1.19265 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

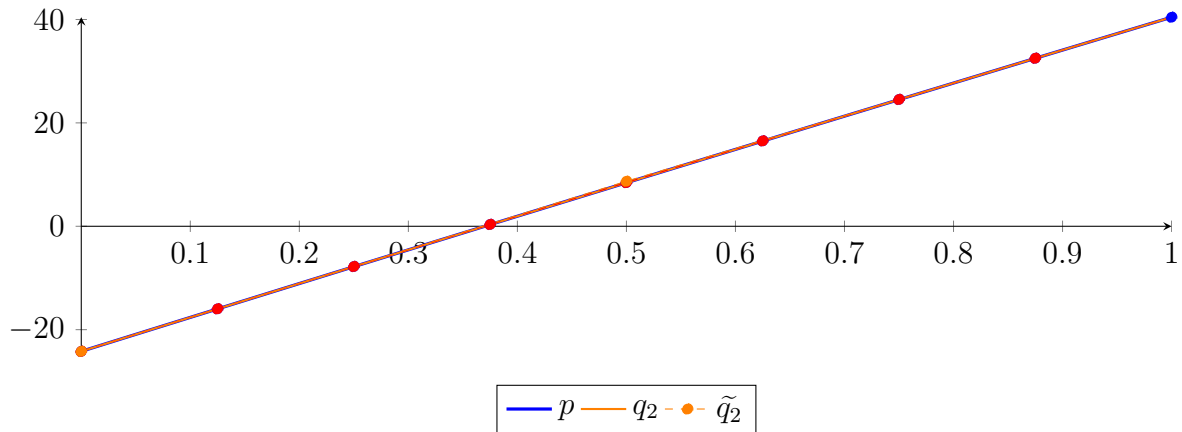
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = 1.00159 \cdot 10^{-08} X^8 - 3.3372 \cdot 10^{-08} X^7 + 4.23875 \cdot 10^{-08} X^6 - 2.49721 \cdot 10^{-08} X^5$$

$$+ 6.08793 \cdot 10^{-09} X^4 + 1.46429 \cdot 10^{-10} X^3 - 1.18261X^2 + 65.8162X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

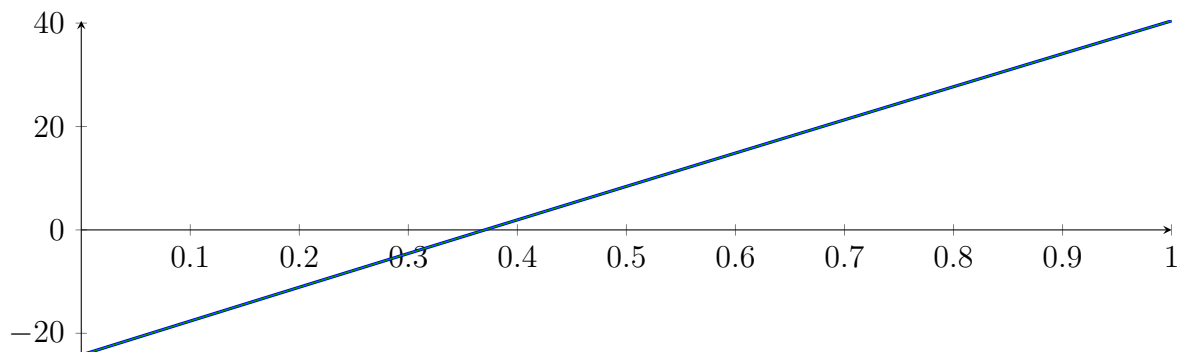
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

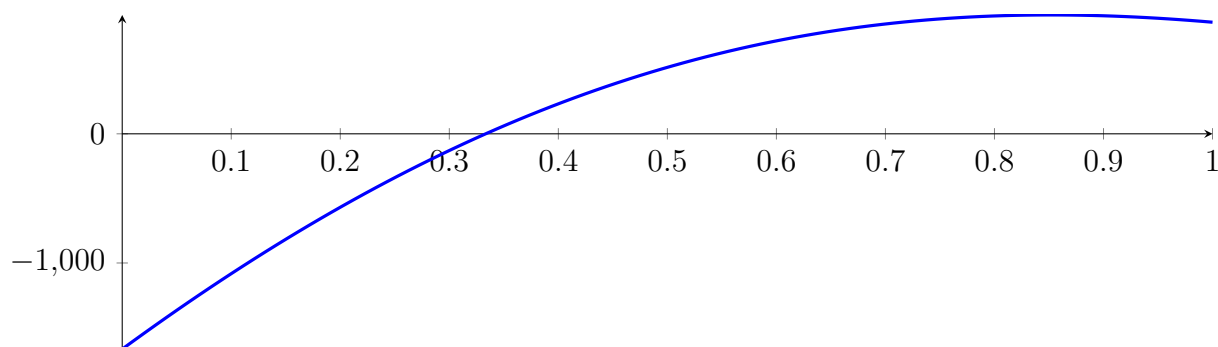
47.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

47.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

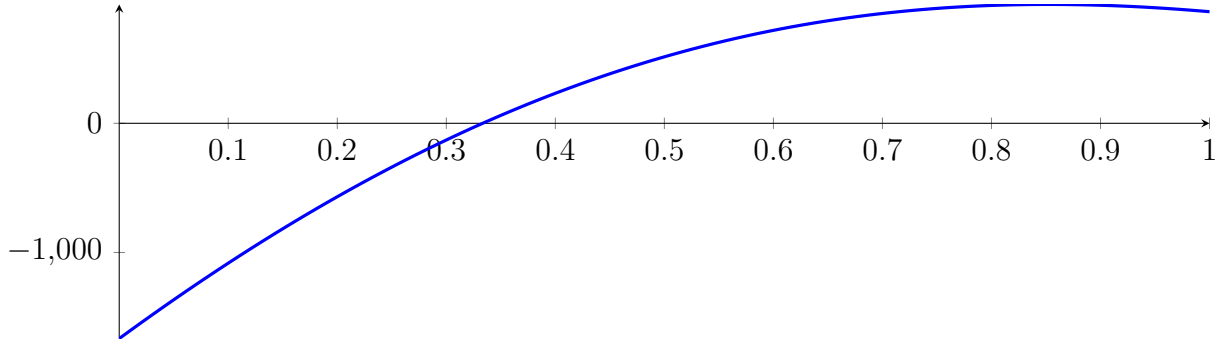
with precision $\varepsilon = 0.0001$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

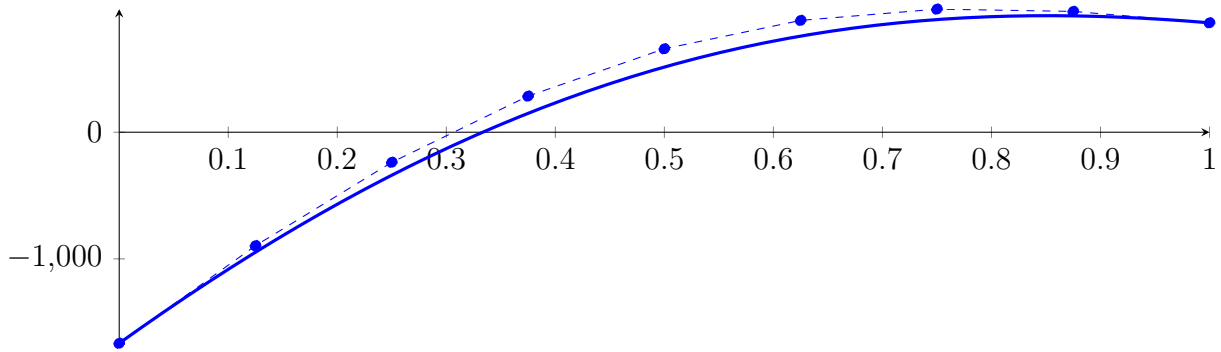
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



48.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

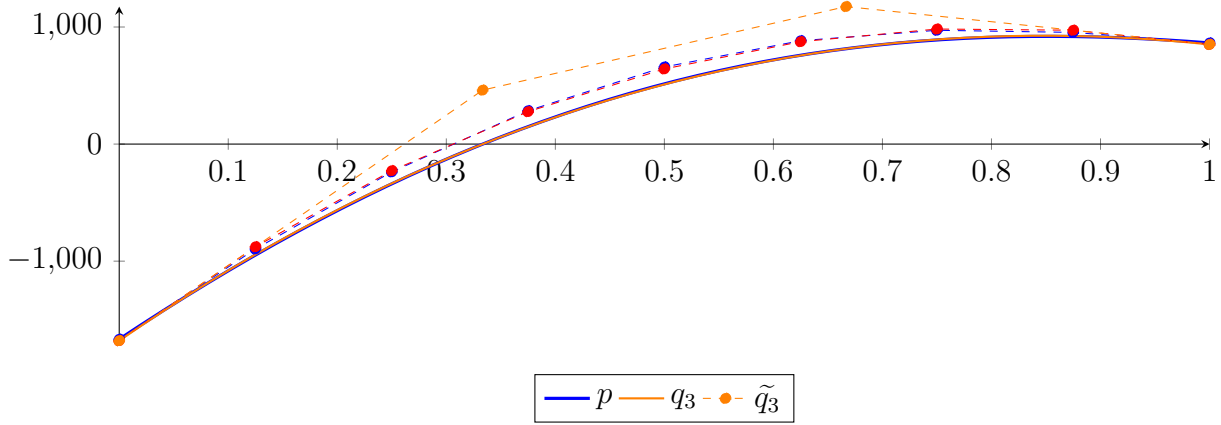
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 2.38117 \cdot 10^{-06}X^8 - 9.49957 \cdot 10^{-06}X^7 + 1.54757 \cdot 10^{-05}X^6 - 1.32609 \cdot 10^{-05}X^5 \\ &\quad + 6.41819 \cdot 10^{-06}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

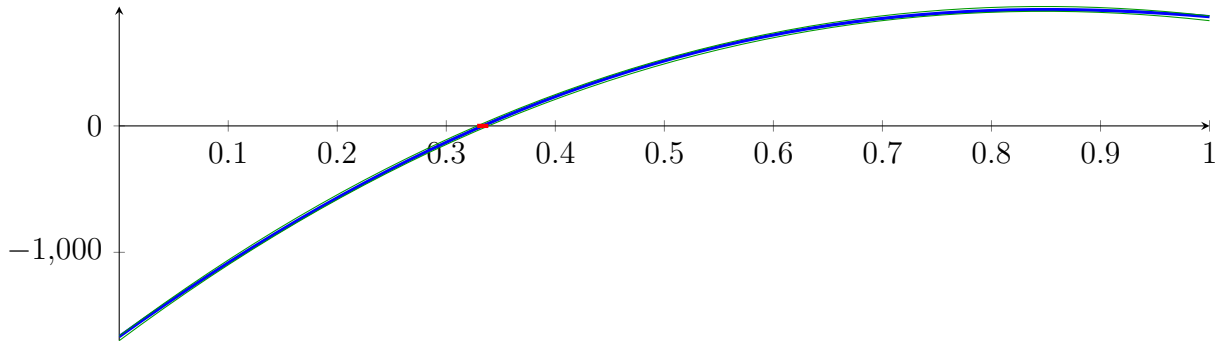
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

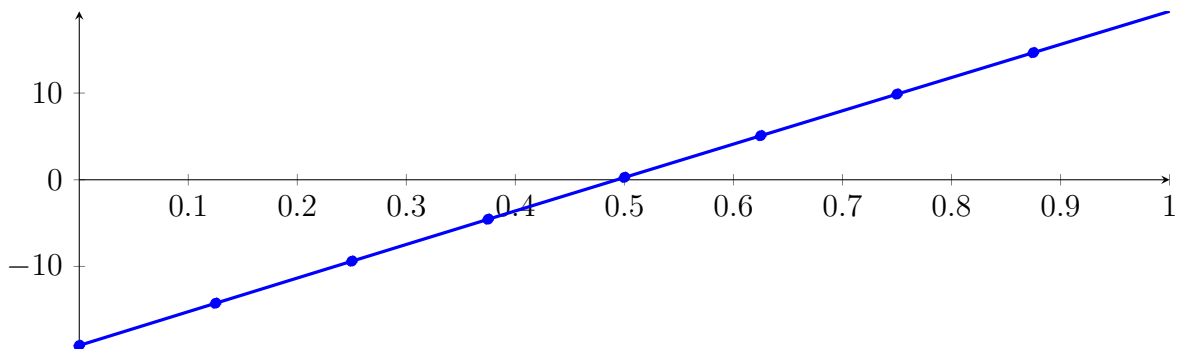
Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

48.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

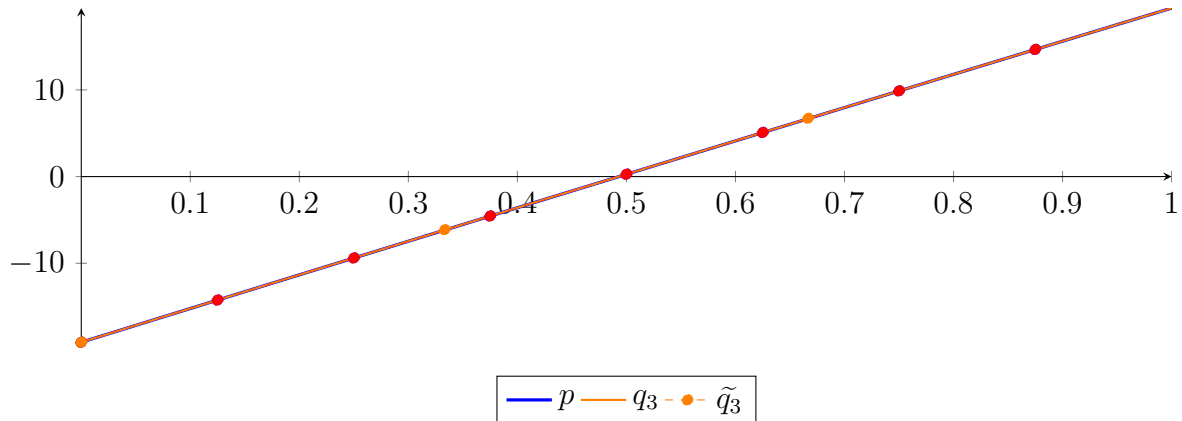
$$\begin{aligned}
 p &= -1.27898 \cdot 10^{-13} X^8 - 1.13687 \cdot 10^{-13} X^7 - 8.23661 \cdot 10^{-11} X^6 + 8.72882 \cdot 10^{-09} X^5 \\
 &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\
 &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\
 &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 2.3353 \cdot 10^{-08}X^8 - 9.31856 \cdot 10^{-08}X^7 + 1.51861 \cdot 10^{-07}X^6 - 1.30228 \cdot 10^{-07}X^5 \\ &\quad + 6.31618 \cdot 10^{-08}X^4 - 0.000240962X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ &= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8} \\ &\quad + 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16887 \cdot 10^{-07}$.

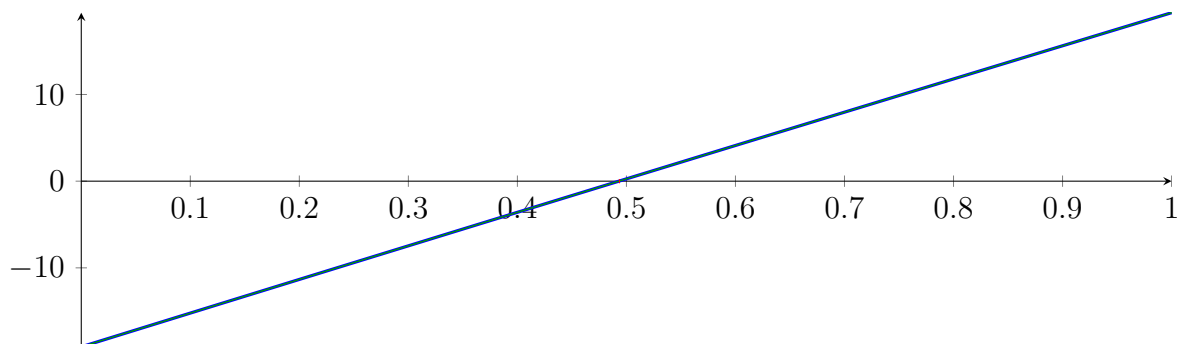
Bounding polynomials M and m :

$$\begin{aligned} M &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \\ m &= -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124 \end{aligned}$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \quad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.12517 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

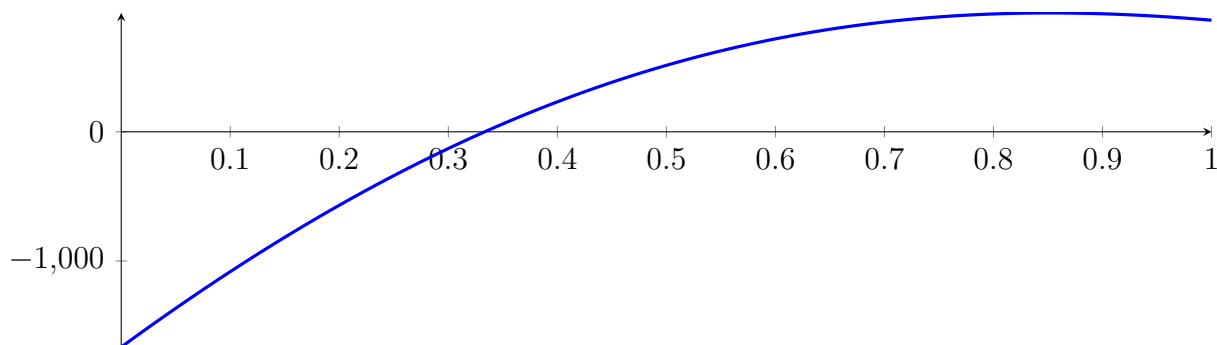
48.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

48.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

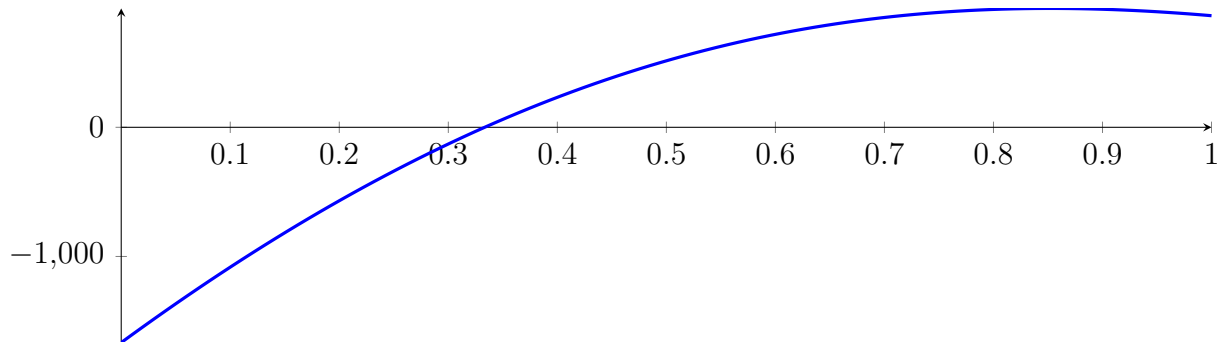
with precision $\varepsilon = 0.0001$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

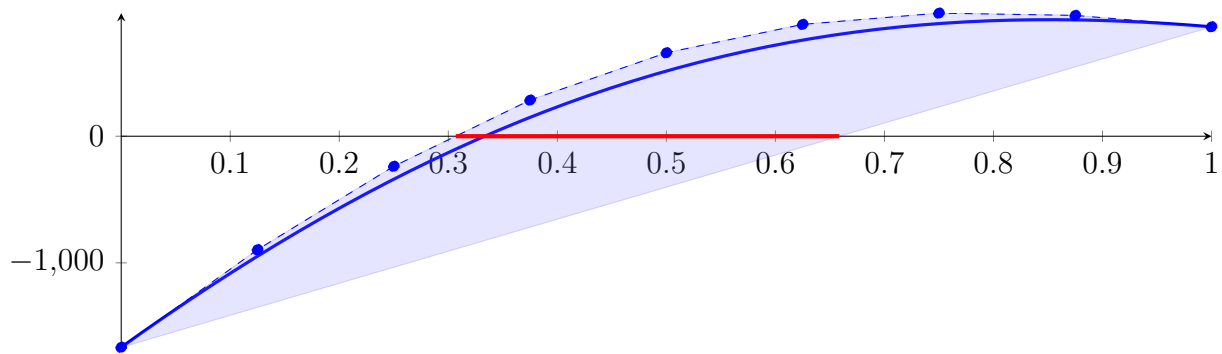
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



49.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

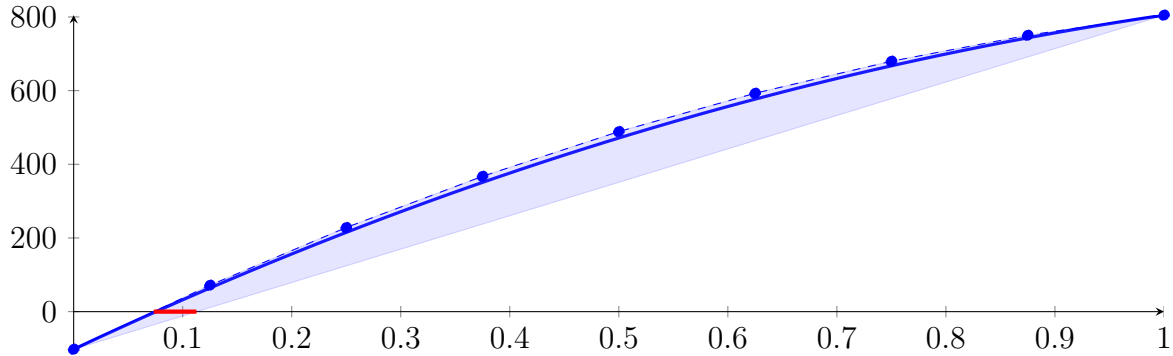
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

49.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

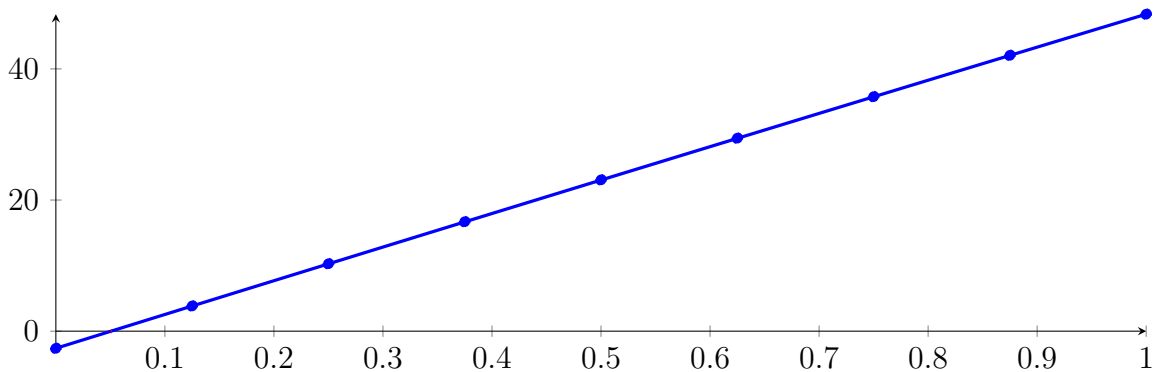
Longest intersection interval: 0.0391855

\implies Selective recursion: interval 1: [0.332635, 0.34642],

49.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -5.41789 \cdot 10^{-13}X^8 - 1.26477 \cdot 10^{-12}X^7 - 5.05786 \cdot 10^{-10}X^6 + 3.66765 \cdot 10^{-08}X^5 \\
 &\quad + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

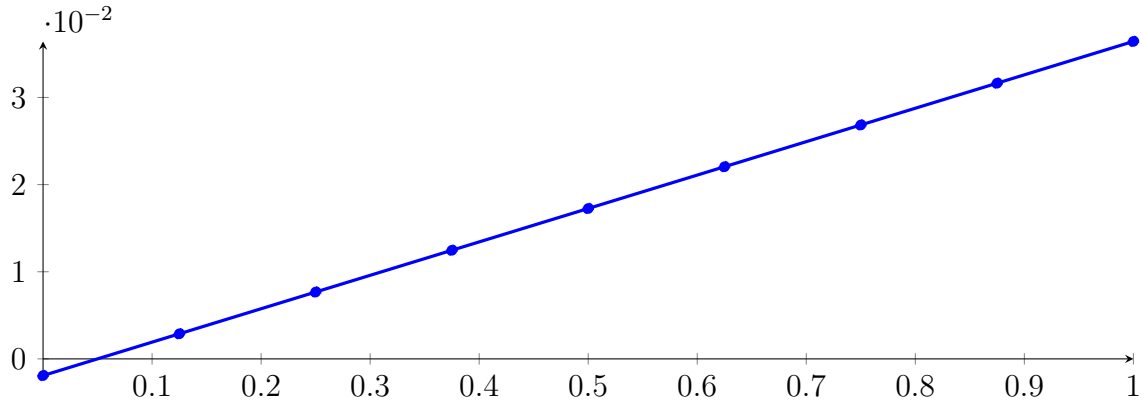
Longest intersection interval: 0.000742589

\implies Selective recursion: interval 1: [0.333333, 0.333343],

49.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -7.11237 \cdot 10^{-16} X^8 + 5.27356 \cdot 10^{-16} X^7 - 7.38298 \cdot 10^{-15} X^6 + 1.06859 \cdot 10^{-15} X^5 \\
 &\quad - 1.09288 \cdot 10^{-15} X^4 - 2.37227 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\
 &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\
 &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\
 &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

Longest intersection interval: $5.36469 \cdot 10^{-07}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

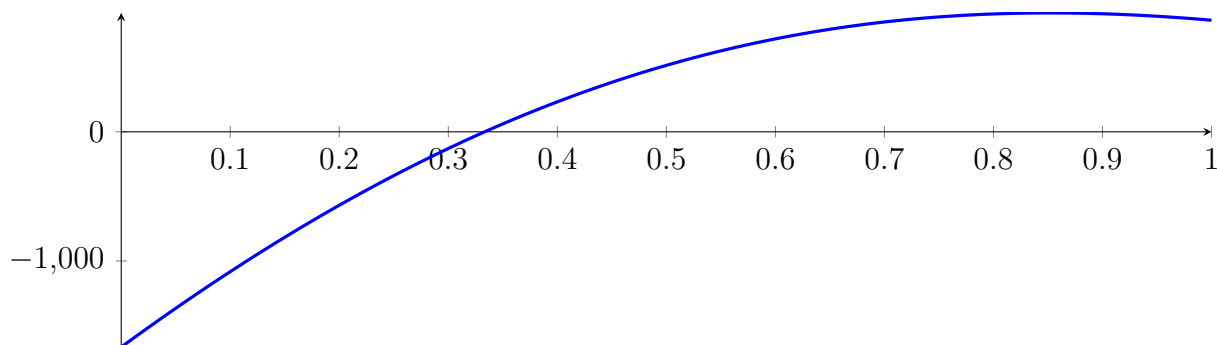
49.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

49.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

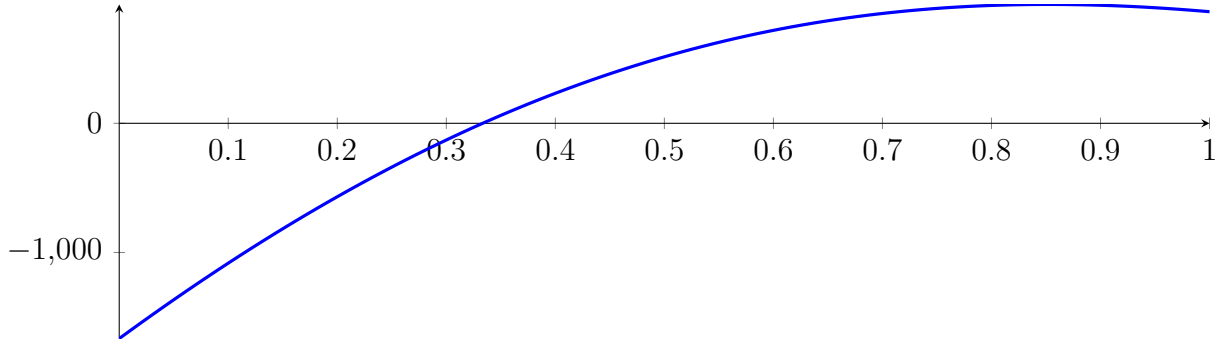
with precision $\varepsilon = 1 \cdot 10^{-08}$.

50 Running QuadClip on f_8 with epsilon 8

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

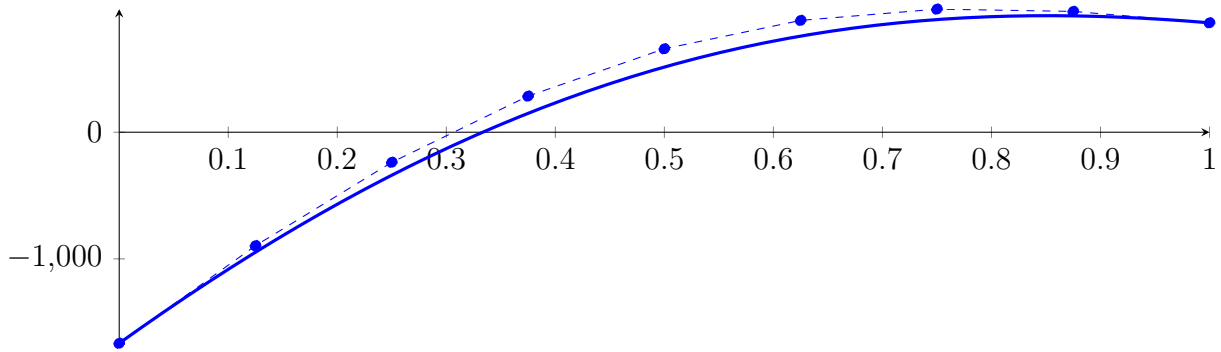
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



50.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

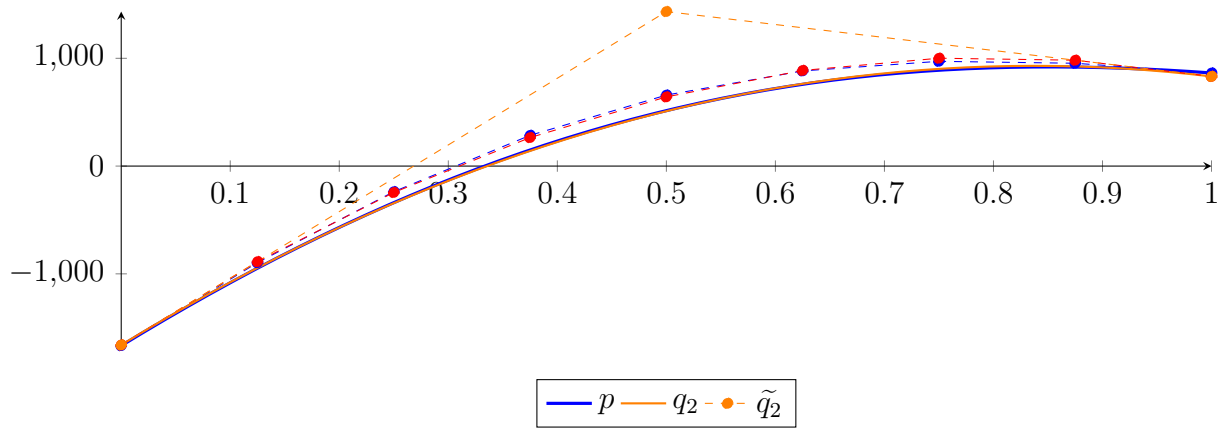
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -3.91297 \cdot 10^{-06}X^8 + 1.58774 \cdot 10^{-05}X^7 - 2.63335 \cdot 10^{-05}X^6 + 2.29285 \cdot 10^{-05}X^5 \\ &\quad - 1.11931 \cdot 10^{-05}X^4 + 3.0313 \cdot 10^{-06}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

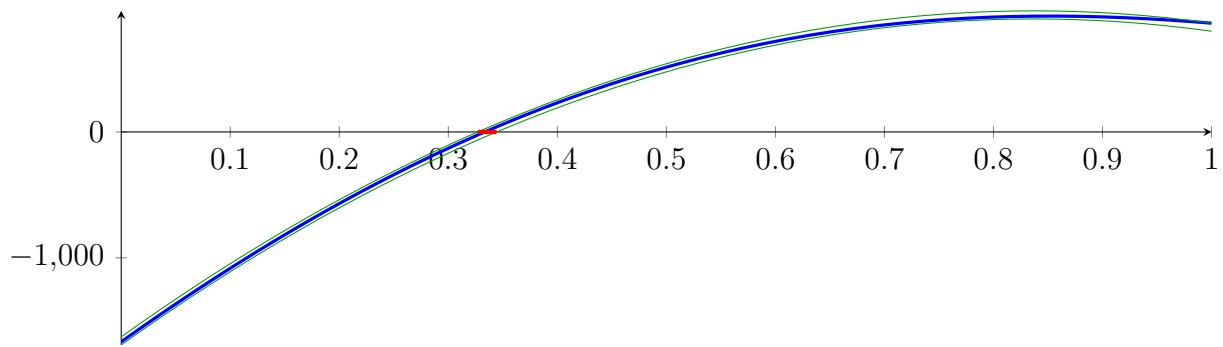
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

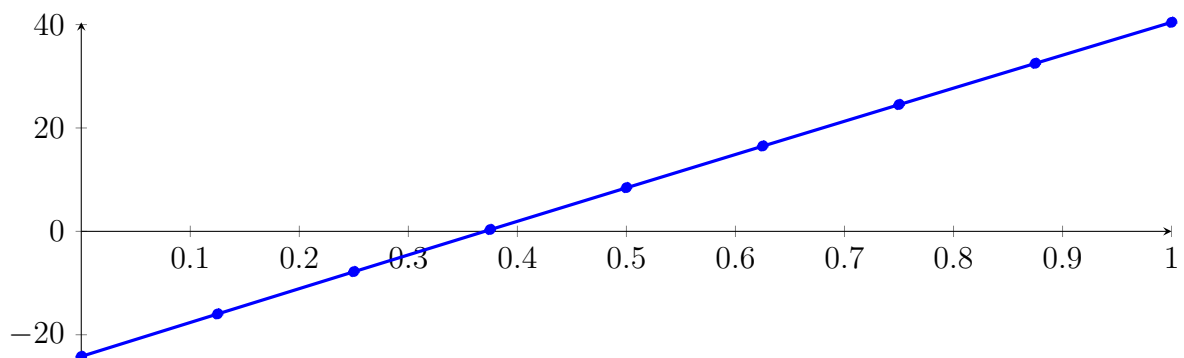
Longest intersection interval: 0.0173372

\implies Selective recursion: interval 1: $[0.326917, 0.344255]$,

50.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.40012 \cdot 10^{-13} X^8 - 7.38964 \cdot 10^{-12} X^7 - 1.94416 \cdot 10^{-09} X^6 + 1.19265 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

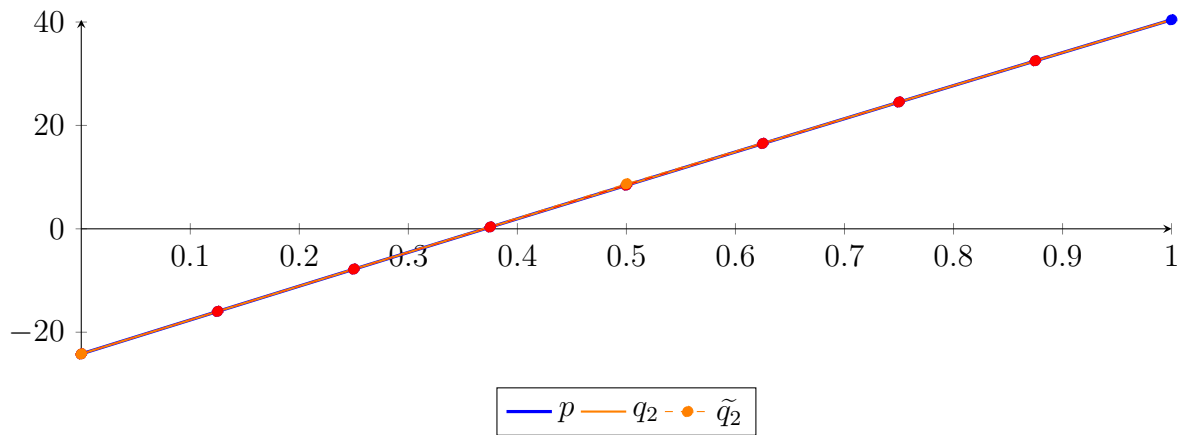
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = 1.00159 \cdot 10^{-08} X^8 - 3.3372 \cdot 10^{-08} X^7 + 4.23875 \cdot 10^{-08} X^6 - 2.49721 \cdot 10^{-08} X^5$$

$$+ 6.08793 \cdot 10^{-09} X^4 + 1.46429 \cdot 10^{-10} X^3 - 1.18261 X^2 + 65.8162 X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

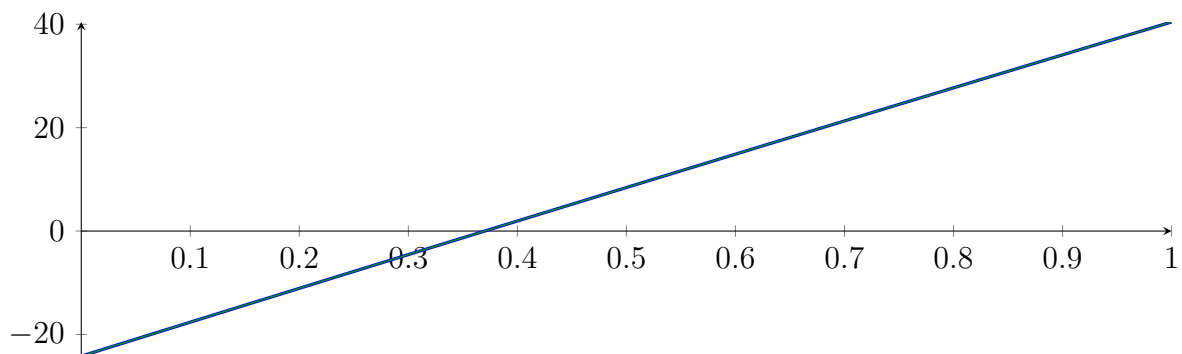
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

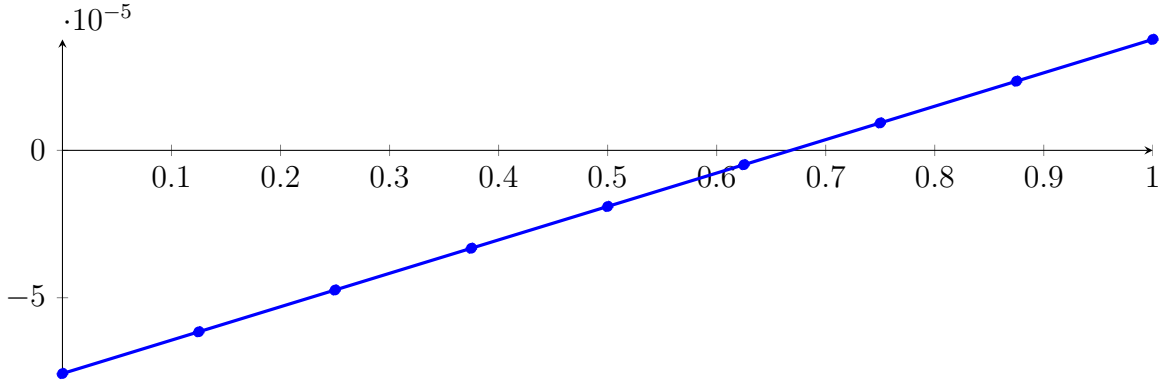
Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

50.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

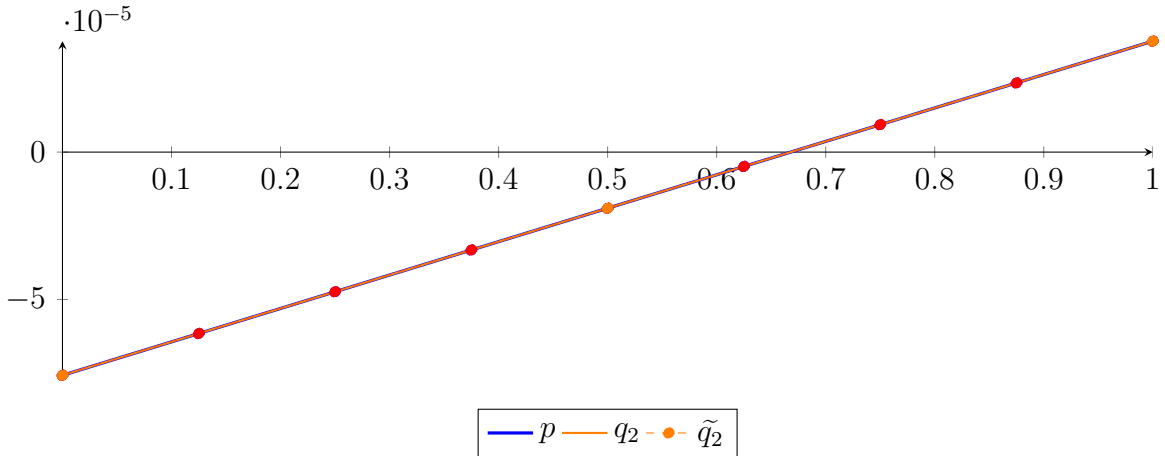
$$\begin{aligned}
 p &= -2.1684 \cdot 10^{-19} X^8 - 4.33681 \cdot 10^{-19} X^7 + 2.12504 \cdot 10^{-17} X^6 - 1.51788 \cdot 10^{-18} X^5 \\
 &\quad + 7.58942 \cdot 10^{-18} X^4 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) \\
 &\quad - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) \\
 &\quad + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 5.62292 \cdot 10^{-14} X^8 - 2.22643 \cdot 10^{-13} X^7 + 3.60043 \cdot 10^{-13} X^6 - 3.05846 \cdot 10^{-13} X^5 + 1.46182 \\
 &\quad \cdot 10^{-13} X^4 - 3.90612 \cdot 10^{-14} X^3 - 3.59793 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \\
 &\quad \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.98887 \cdot 10^{-16}$.

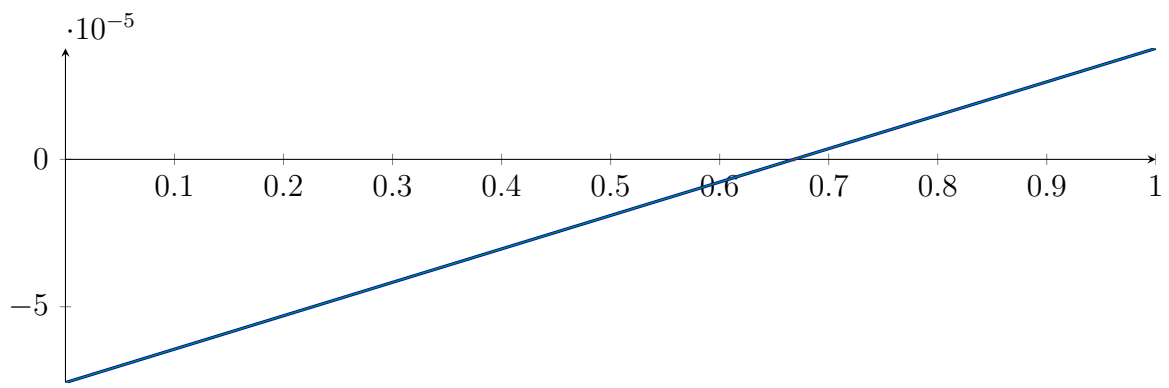
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 m &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.667955, 3.14644 \cdot 10^7\} \qquad N(m) = \{0.667955, 3.14644 \cdot 10^7\}$$

Intersection intervals:



[0.667955, 0.667955]

Longest intersection interval: $1.88052 \cdot 10^{-09}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

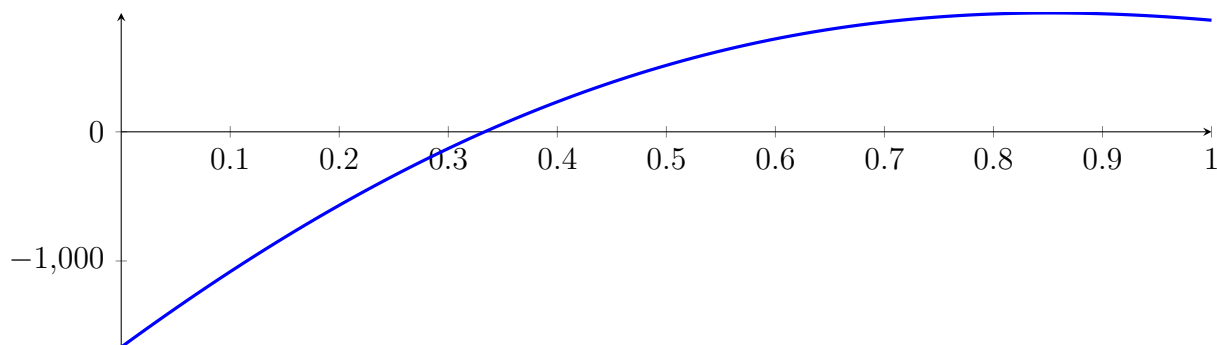
50.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

50.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

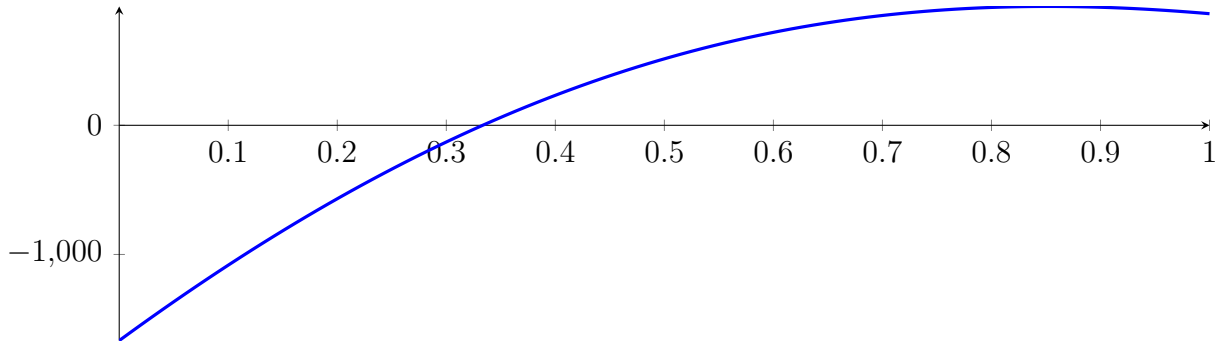
with precision $\varepsilon = 1 \cdot 10^{-08}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

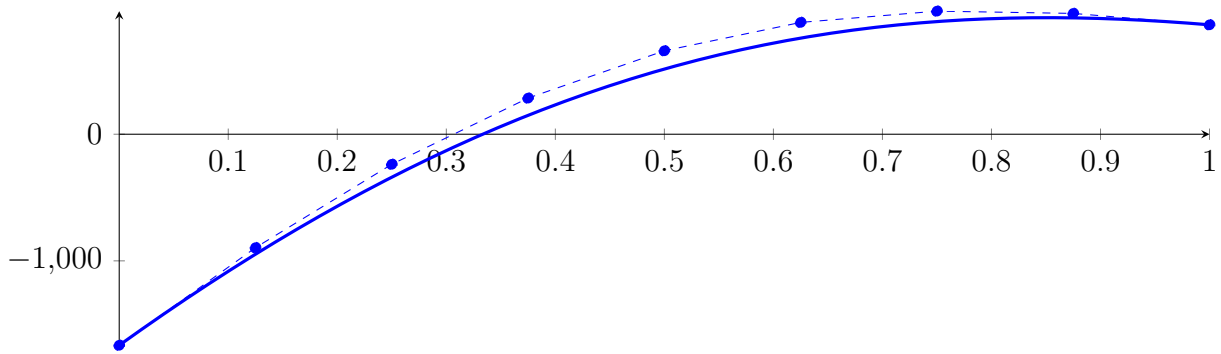
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



51.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

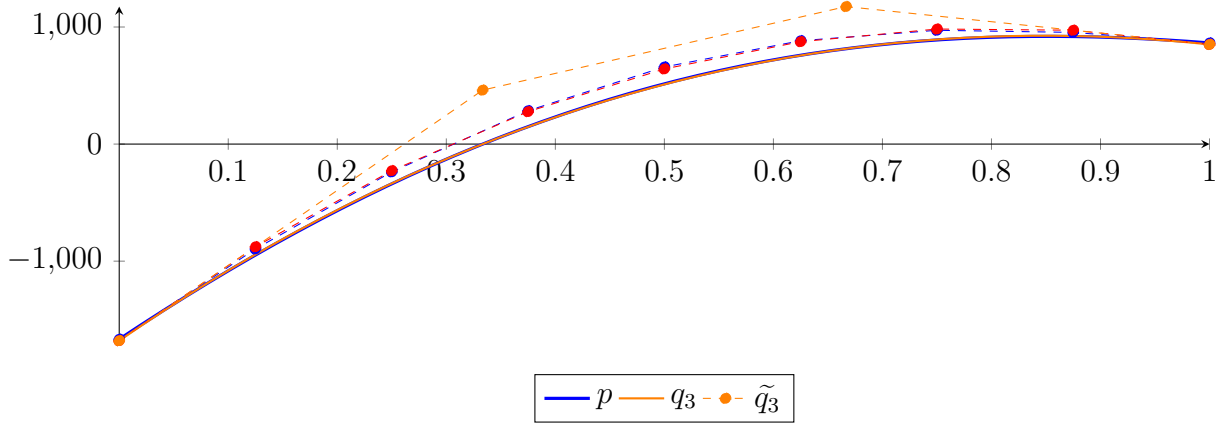
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 2.38117 \cdot 10^{-06}X^8 - 9.49957 \cdot 10^{-06}X^7 + 1.54757 \cdot 10^{-05}X^6 - 1.32609 \cdot 10^{-05}X^5 \\ &\quad + 6.41819 \cdot 10^{-06}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

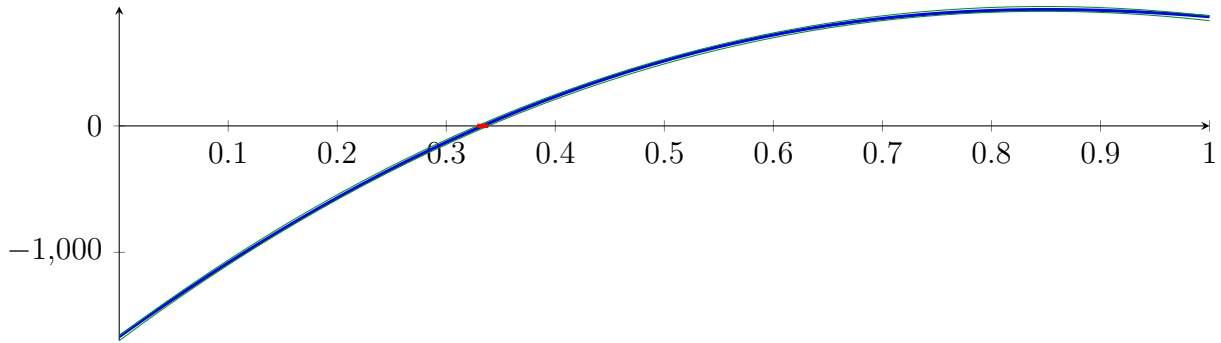
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

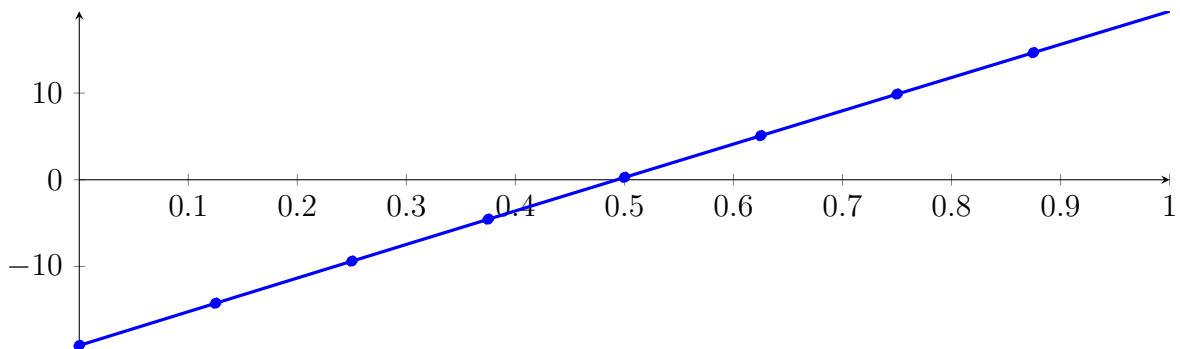
Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

51.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.27898 \cdot 10^{-13} X^8 - 1.13687 \cdot 10^{-13} X^7 - 8.23661 \cdot 10^{-11} X^6 + 8.72882 \cdot 10^{-09} X^5 \\ &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

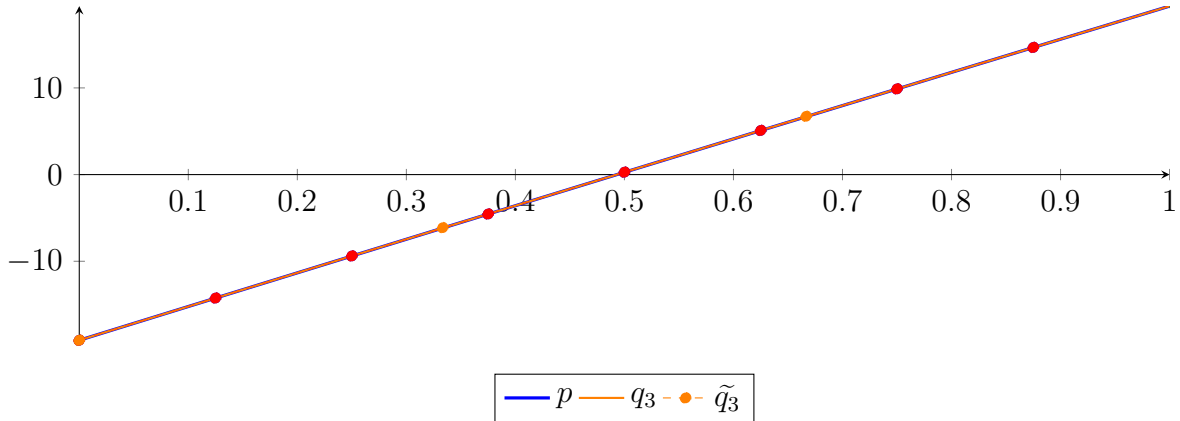
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = 2.3353 \cdot 10^{-08}X^8 - 9.31856 \cdot 10^{-08}X^7 + 1.51861 \cdot 10^{-07}X^6 - 1.30228 \cdot 10^{-07}X^5$$

$$+ 6.31618 \cdot 10^{-08}X^4 - 0.000240962X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16887 \cdot 10^{-07}$.

Bounding polynomials M and m :

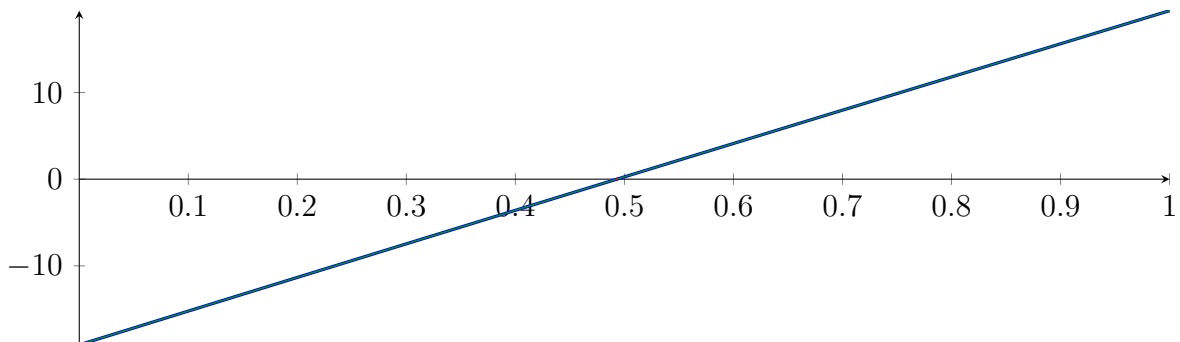
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \quad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.12517 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

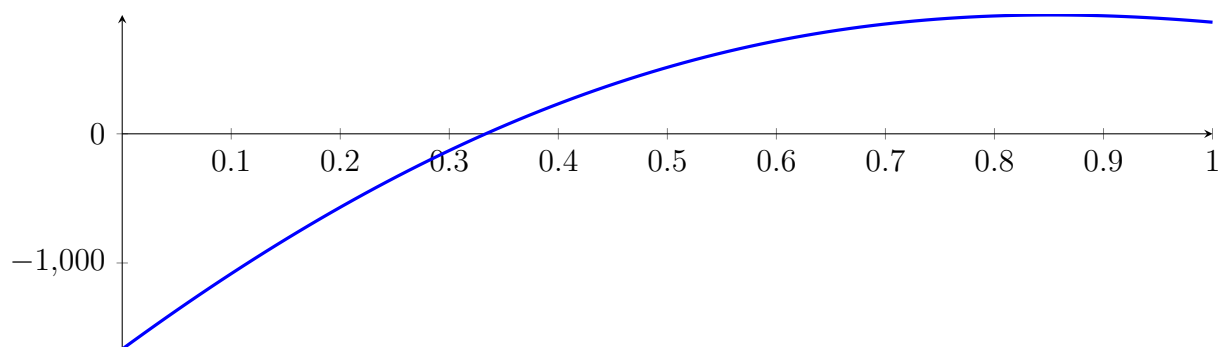
51.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

51.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

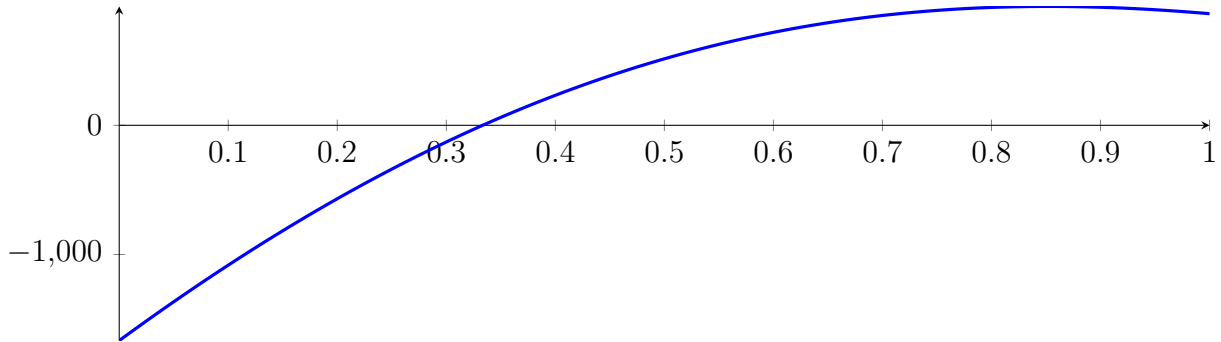
with precision $\varepsilon = 1 \cdot 10^{-08}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

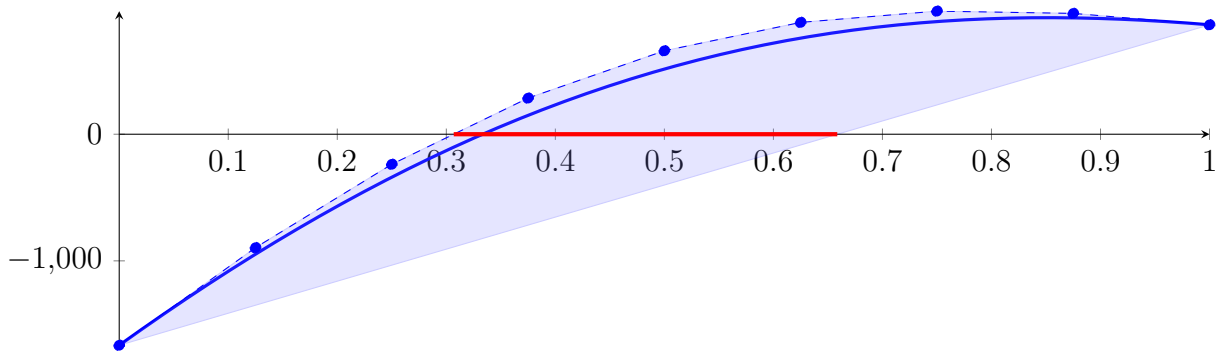
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



52.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

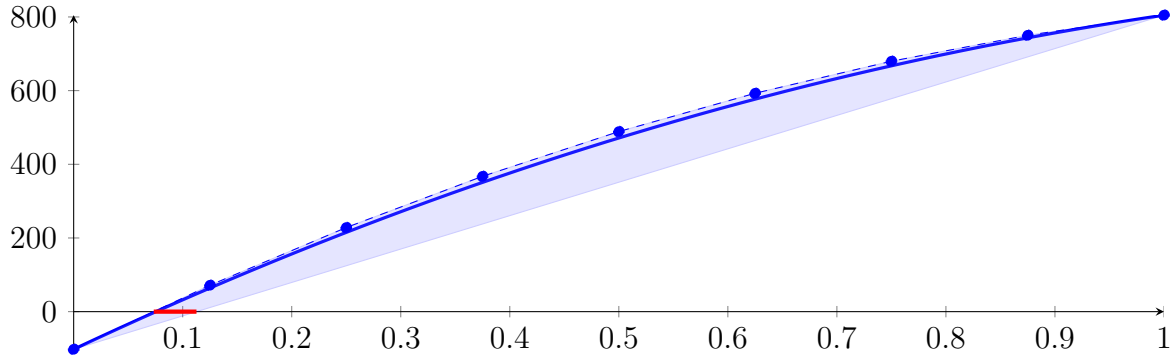
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

52.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

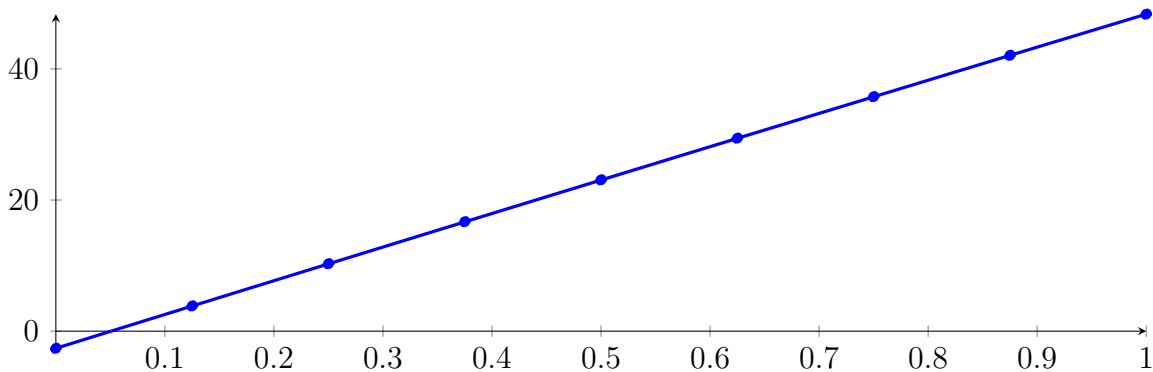
Longest intersection interval: 0.0391855

\implies Selective recursion: interval 1: [0.332635, 0.34642],

52.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -5.41789 \cdot 10^{-13}X^8 - 1.26477 \cdot 10^{-12}X^7 - 5.05786 \cdot 10^{-10}X^6 + 3.66765 \cdot 10^{-08}X^5 \\
 &\quad + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

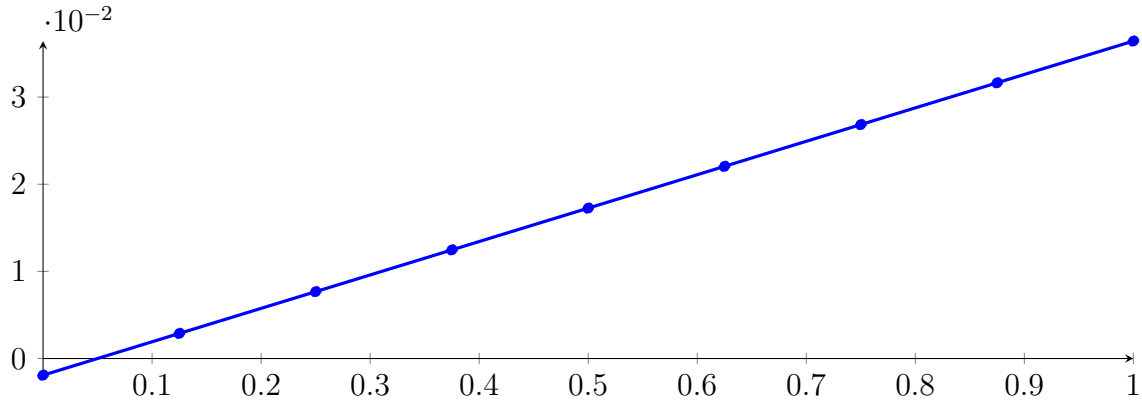
Longest intersection interval: 0.000742589

\implies Selective recursion: interval 1: [0.333333, 0.333343],

52.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -7.11237 \cdot 10^{-16} X^8 + 5.27356 \cdot 10^{-16} X^7 - 7.38298 \cdot 10^{-15} X^6 + 1.06859 \cdot 10^{-15} X^5 \\
 &\quad - 1.09288 \cdot 10^{-15} X^4 - 2.37227 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\
 &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\
 &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\
 &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

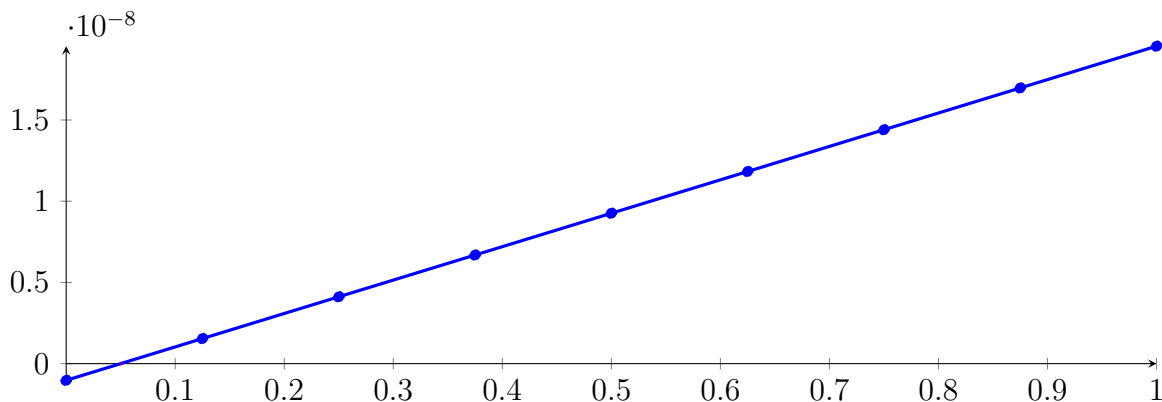
Longest intersection interval: $5.36469 \cdot 10^{-07}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

52.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.09366 \cdot 10^{-22} X^8 + 4.49986 \cdot 10^{-22} X^7 - 4.03002 \cdot 10^{-21} X^6 + 3.70577 \cdot 10^{-22} X^5 - 3.47416 \\
 &\quad \cdot 10^{-22} X^4 + 9.26442 \cdot 10^{-23} X^3 - 1.18608 \cdot 10^{-19} X^2 + 2.05706 \cdot 10^{-08} X - 1.02667 \cdot 10^{-09} \\
 &= -1.02667 \cdot 10^{-09} B_{0,8}(X) + 1.54465 \cdot 10^{-09} B_{1,8}(X) + 4.11597 \cdot 10^{-09} B_{2,8}(X) \\
 &\quad + 6.68729 \cdot 10^{-09} B_{3,8}(X) + 9.25862 \cdot 10^{-09} B_{4,8}(X) + 1.18299 \cdot 10^{-08} B_{5,8}(X) \\
 &\quad + 1.44013 \cdot 10^{-08} B_{6,8}(X) + 1.69726 \cdot 10^{-08} B_{7,8}(X) + 1.95439 \cdot 10^{-08} B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87728 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

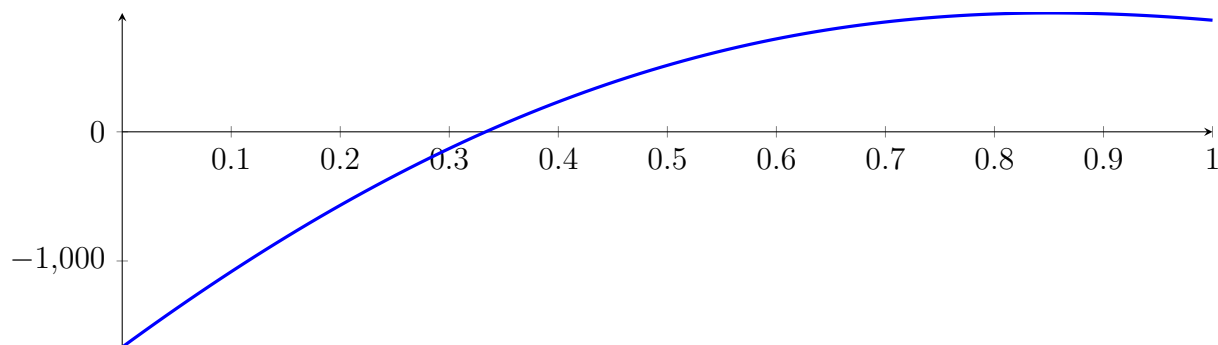
52.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

52.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

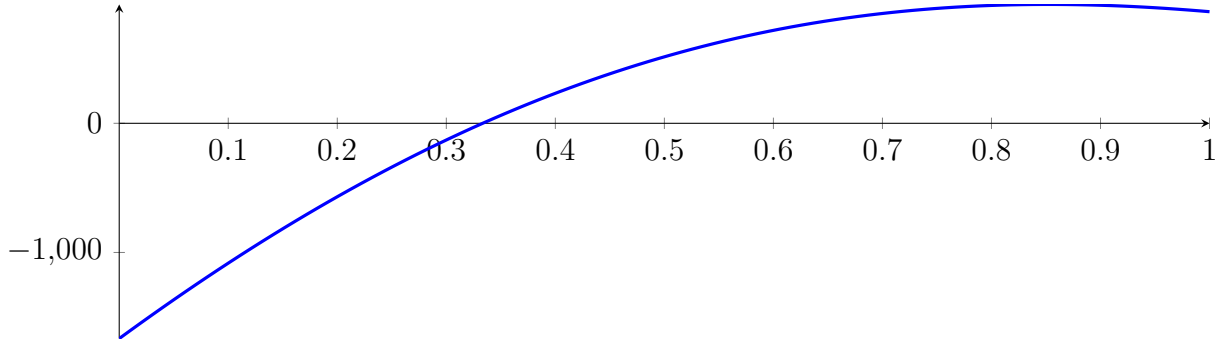
with precision $\varepsilon = 1 \cdot 10^{-16}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

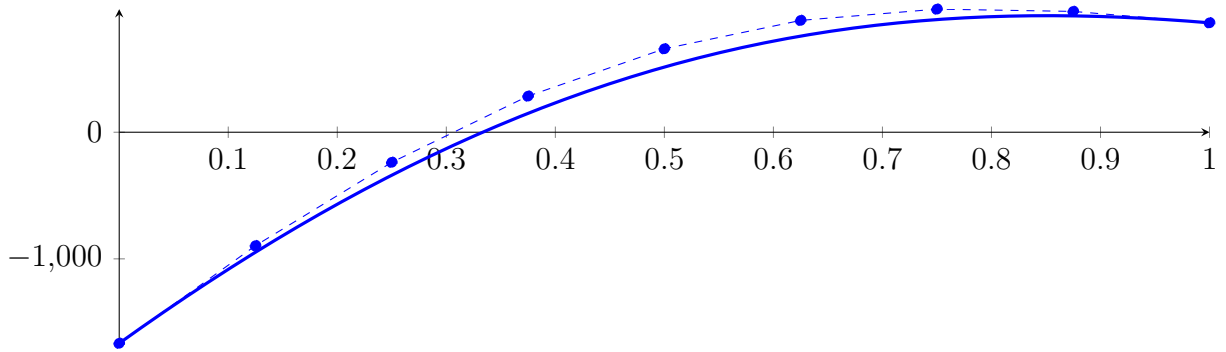
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



53.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

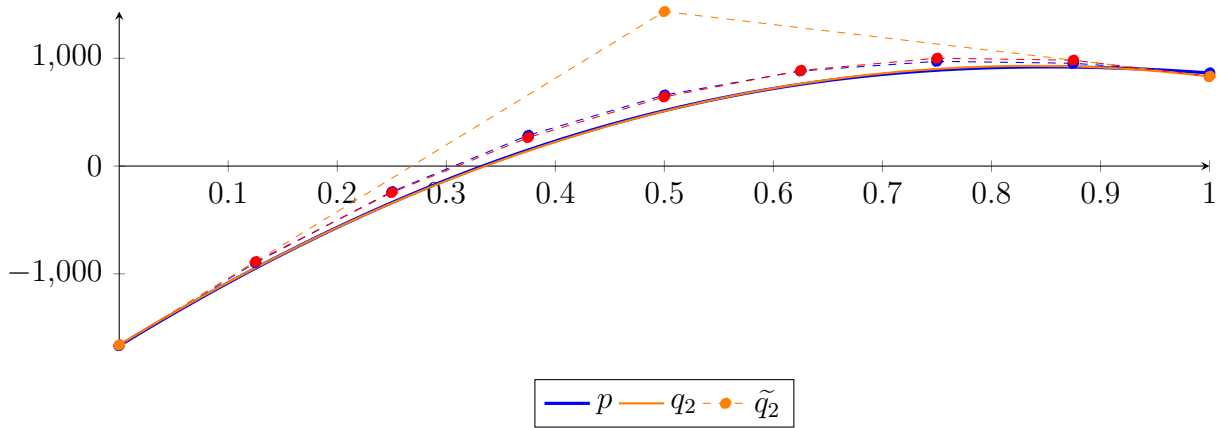
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -3.91297 \cdot 10^{-06}X^8 + 1.58774 \cdot 10^{-05}X^7 - 2.63335 \cdot 10^{-05}X^6 + 2.29285 \cdot 10^{-05}X^5 \\ &\quad - 1.11931 \cdot 10^{-05}X^4 + 3.0313 \cdot 10^{-06}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

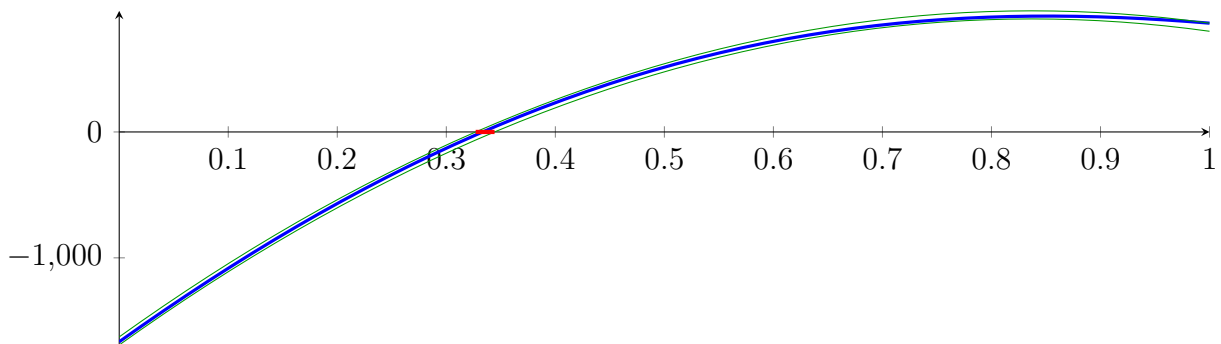
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

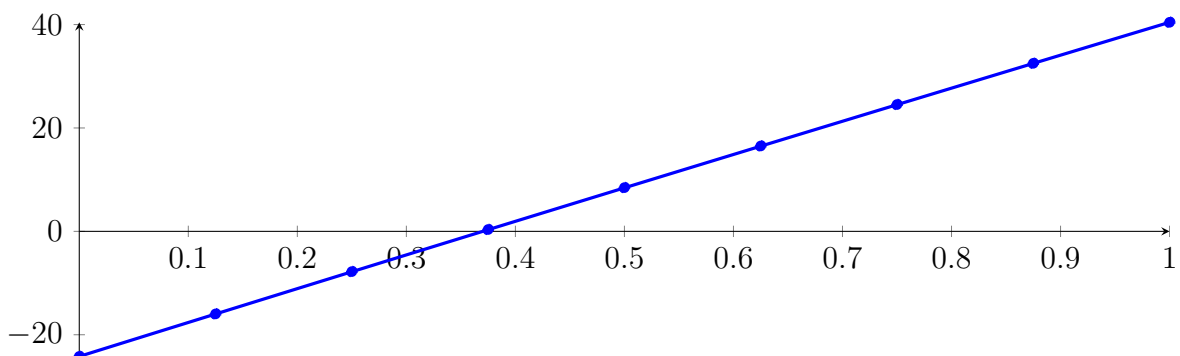
Longest intersection interval: 0.0173372

\implies Selective recursion: interval 1: $[0.326917, 0.344255]$,

53.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.40012 \cdot 10^{-13} X^8 - 7.38964 \cdot 10^{-12} X^7 - 1.94416 \cdot 10^{-09} X^6 + 1.19265 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

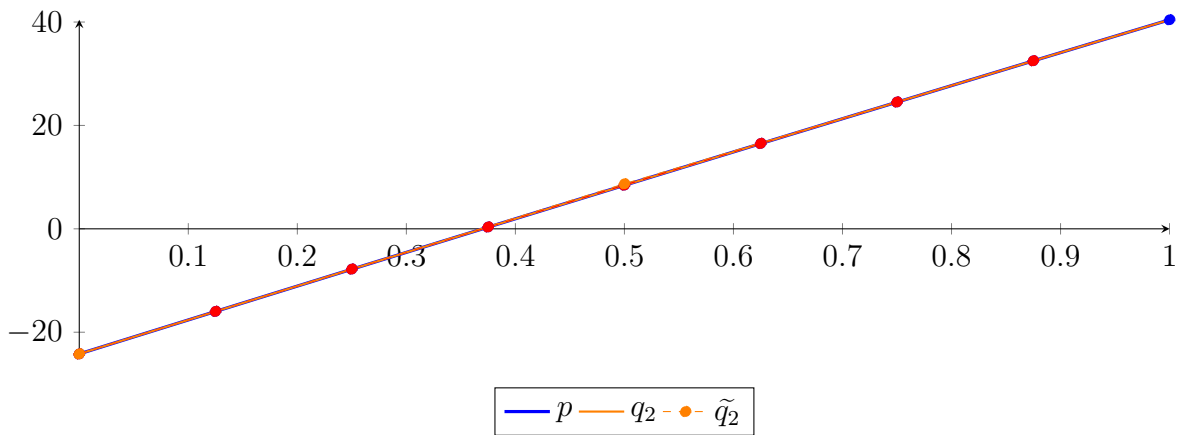
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = 1.00159 \cdot 10^{-08} X^8 - 3.3372 \cdot 10^{-08} X^7 + 4.23875 \cdot 10^{-08} X^6 - 2.49721 \cdot 10^{-08} X^5$$

$$+ 6.08793 \cdot 10^{-09} X^4 + 1.46429 \cdot 10^{-10} X^3 - 1.18261 X^2 + 65.8162 X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

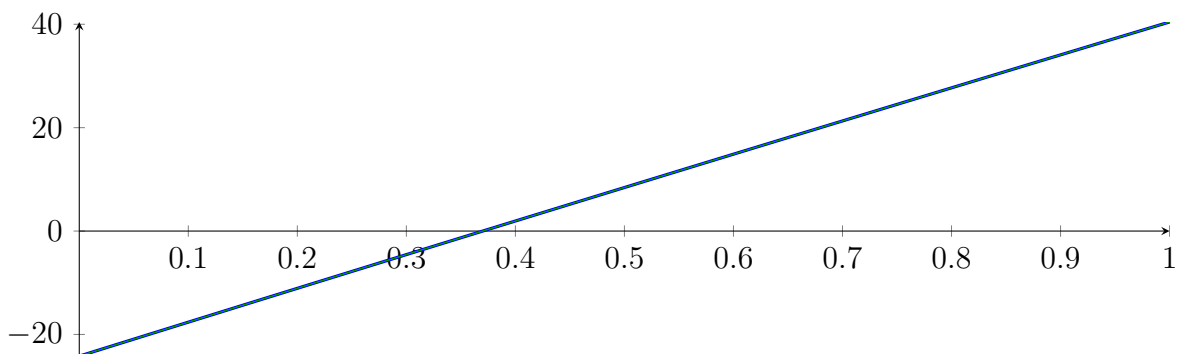
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

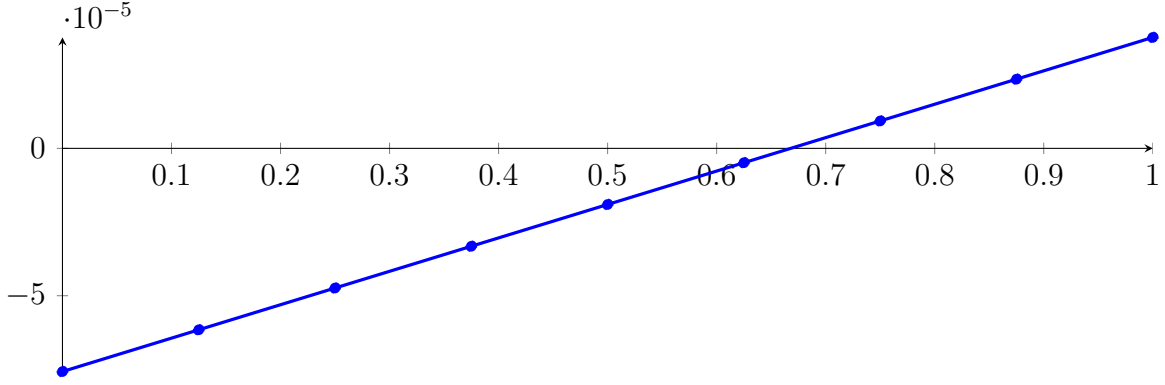
Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

53.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

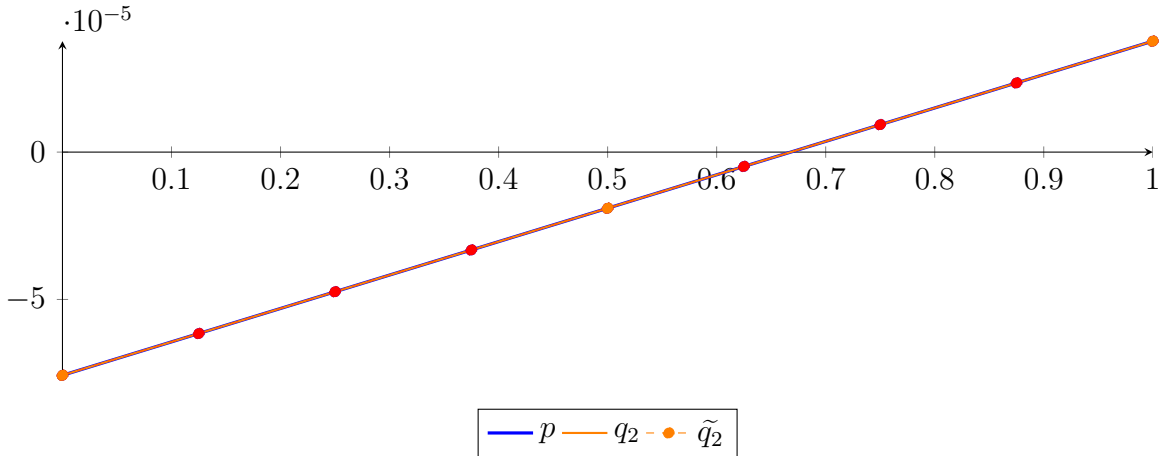
$$\begin{aligned}
 p &= -2.1684 \cdot 10^{-19} X^8 - 4.33681 \cdot 10^{-19} X^7 + 2.12504 \cdot 10^{-17} X^6 - 1.51788 \cdot 10^{-18} X^5 \\
 &\quad + 7.58942 \cdot 10^{-18} X^4 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) \\
 &\quad - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) \\
 &\quad + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 5.62292 \cdot 10^{-14} X^8 - 2.22643 \cdot 10^{-13} X^7 + 3.60043 \cdot 10^{-13} X^6 - 3.05846 \cdot 10^{-13} X^5 + 1.46182 \\
 &\quad \cdot 10^{-13} X^4 - 3.90612 \cdot 10^{-14} X^3 - 3.59793 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \\
 &\quad \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.98887 \cdot 10^{-16}$.

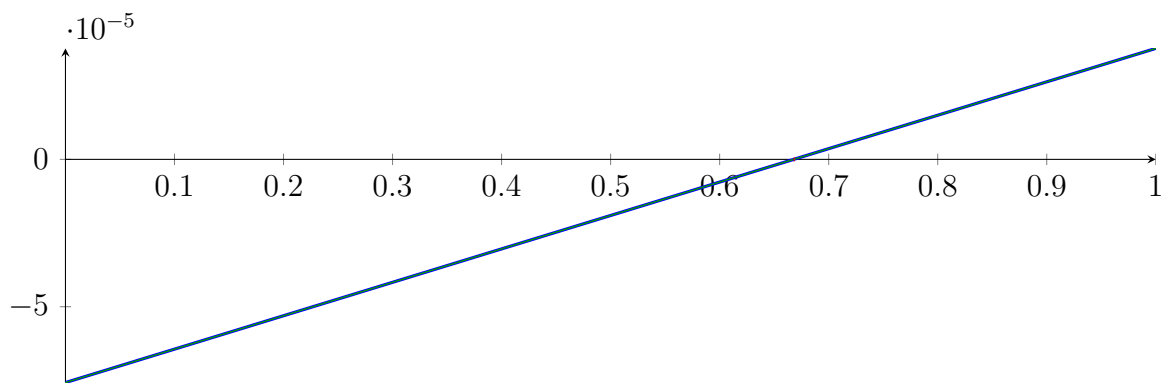
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 m &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.667955, 3.14644 \cdot 10^7\} \qquad N(m) = \{0.667955, 3.14644 \cdot 10^7\}$$

Intersection intervals:



[0.667955, 0.667955]

Longest intersection interval: $1.88052 \cdot 10^{-09}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

53.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

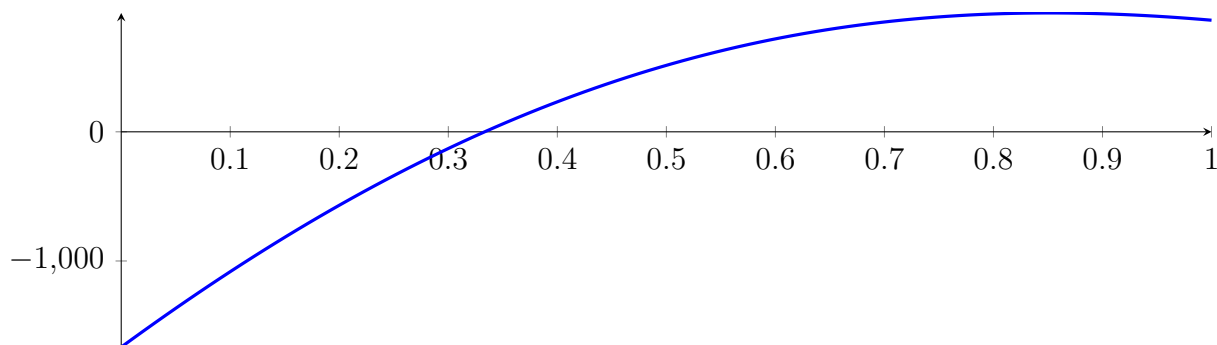
Reached interval [0.333333, 0.333333] **without sign change** at depth 4!

$p(0) = 4.52469e-14$ - $p(1) 2.58458e-13$

53.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

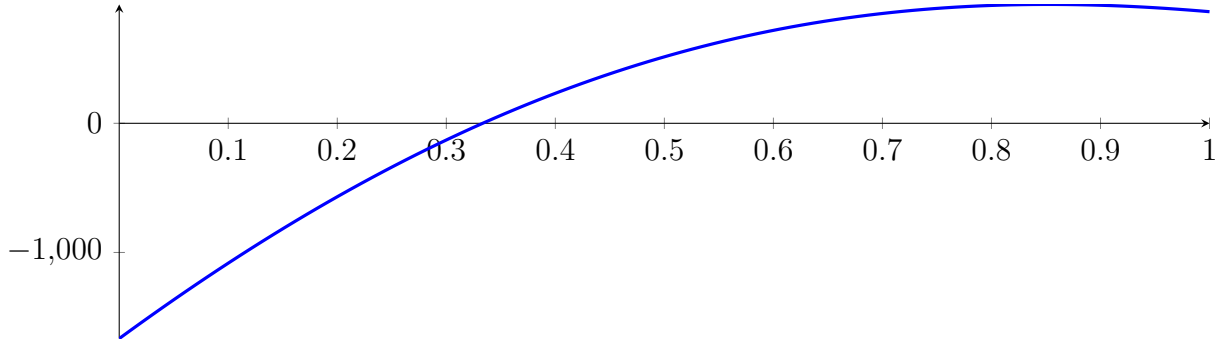
with precision $\varepsilon = 1 \cdot 10^{-16}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

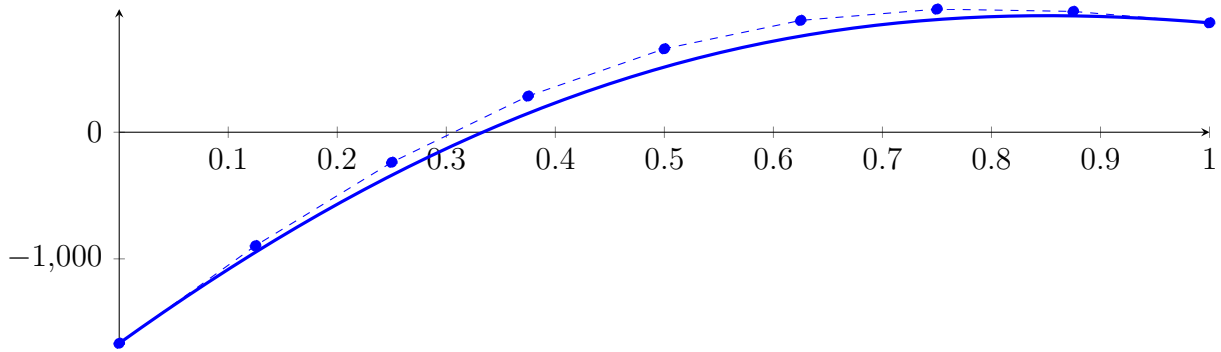
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



54.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

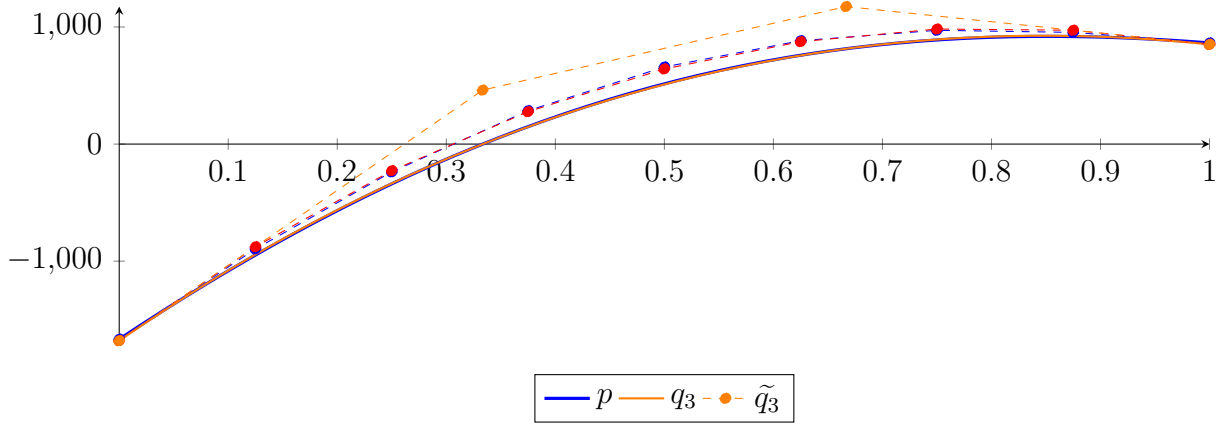
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 2.38117 \cdot 10^{-06}X^8 - 9.49957 \cdot 10^{-06}X^7 + 1.54757 \cdot 10^{-05}X^6 - 1.32609 \cdot 10^{-05}X^5 \\ &\quad + 6.41819 \cdot 10^{-06}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

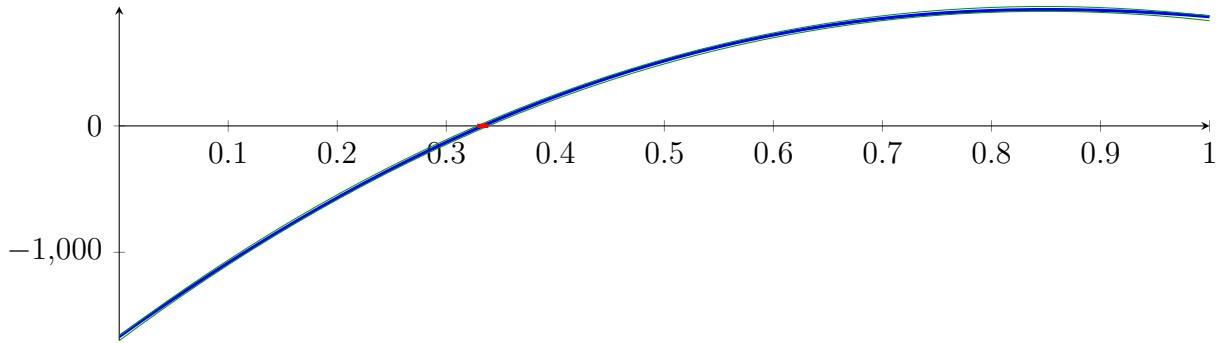
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

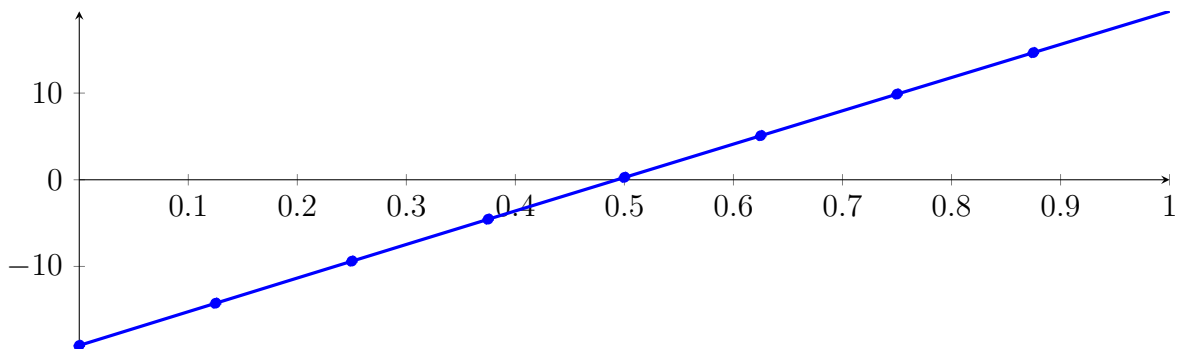
Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

54.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.27898 \cdot 10^{-13} X^8 - 1.13687 \cdot 10^{-13} X^7 - 8.23661 \cdot 10^{-11} X^6 + 8.72882 \cdot 10^{-09} X^5 \\ &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

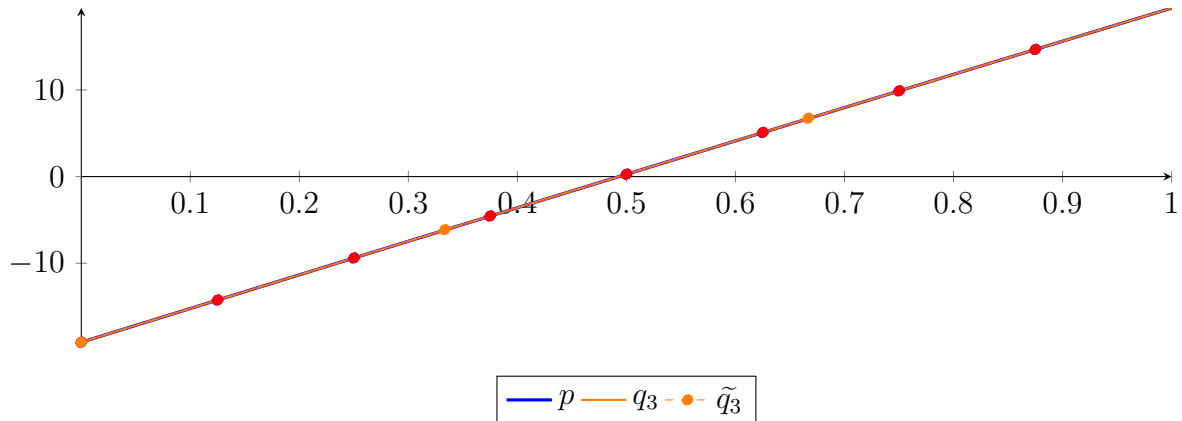
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = 2.3353 \cdot 10^{-08}X^8 - 9.31856 \cdot 10^{-08}X^7 + 1.51861 \cdot 10^{-07}X^6 - 1.30228 \cdot 10^{-07}X^5$$

$$+ 6.31618 \cdot 10^{-08}X^4 - 0.000240962X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16887 \cdot 10^{-07}$.

Bounding polynomials M and m :

$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

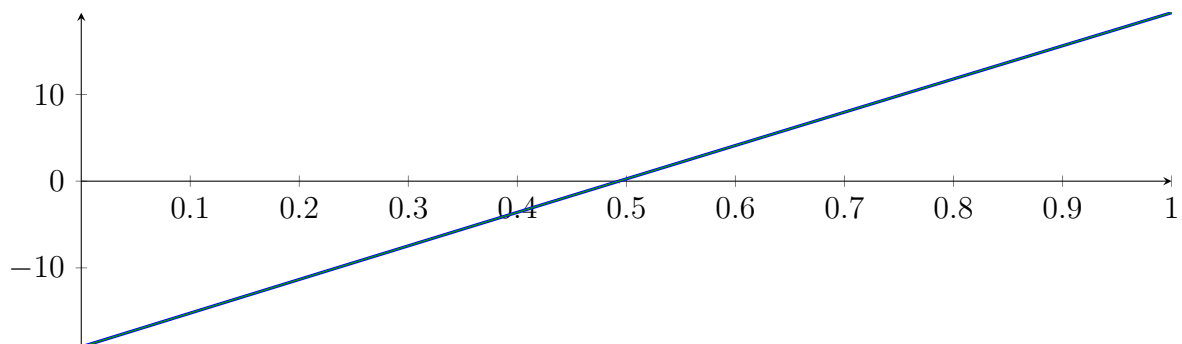
$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\}$$

$$N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

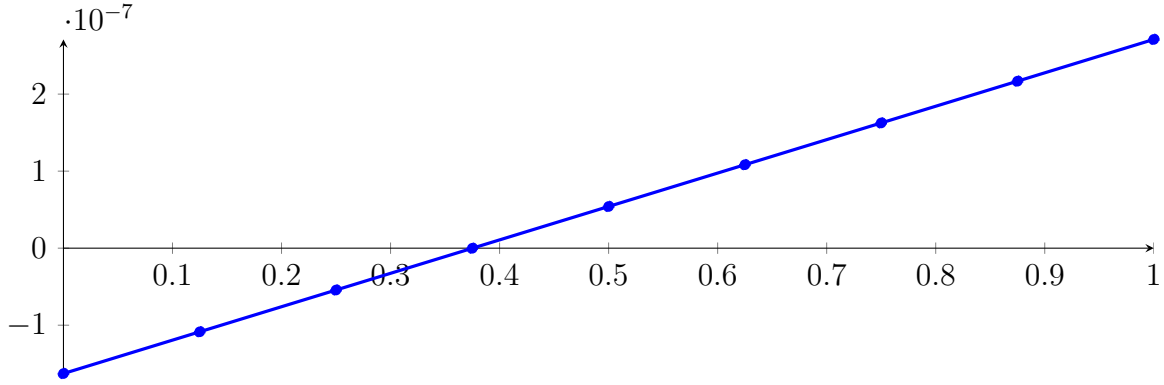
Longest intersection interval: $1.12517 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

54.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

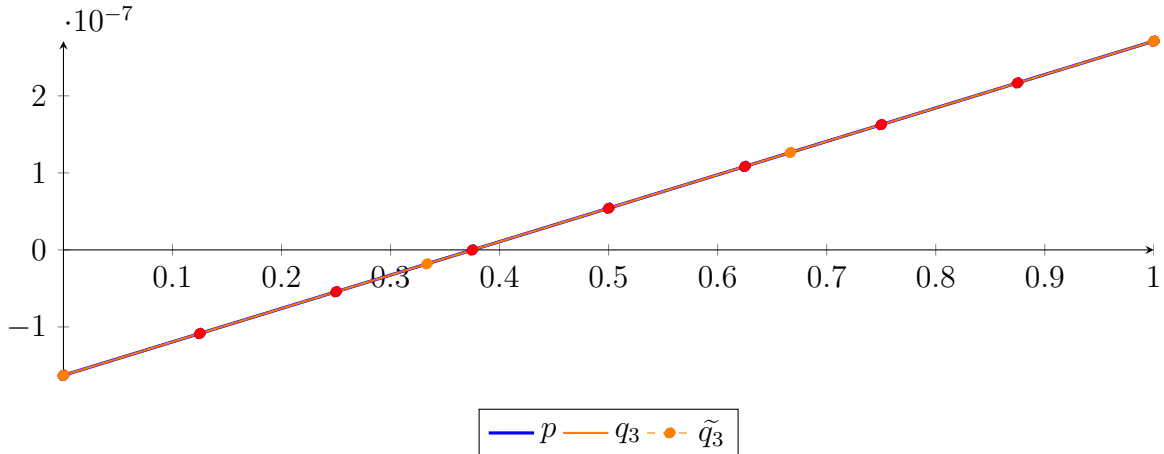
$$\begin{aligned}
 p &= -3.81165 \cdot 10^{-21} X^8 + 6.77626 \cdot 10^{-21} X^7 + 2.96462 \cdot 10^{-21} X^6 + 5.92923 \cdot 10^{-21} X^5 + 1.11173 \\
 &\quad \cdot 10^{-20} X^4 + 1.48231 \cdot 10^{-21} X^3 - 5.27494 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07} \\
 &= -1.6278 \cdot 10^{-07} B_{0,8}(X) - 1.08555 \cdot 10^{-07} B_{1,8}(X) - 5.43313 \cdot 10^{-08} B_{2,8}(X) \\
 &\quad - 1.07093 \cdot 10^{-10} B_{3,8}(X) + 5.41171 \cdot 10^{-08} B_{4,8}(X) + 1.08341 \cdot 10^{-07} B_{5,8}(X) \\
 &\quad + 1.62565 \cdot 10^{-07} B_{6,8}(X) + 2.1679 \cdot 10^{-07} B_{7,8}(X) + 2.71014 \cdot 10^{-07} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -1.90582 \cdot 10^{-21} X^3 - 5.2746 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07} \\
 &= -1.6278 \cdot 10^{-07} B_{0,3} - 1.81818 \cdot 10^{-08} B_{1,3} + 1.26416 \cdot 10^{-07} B_{2,3} + 2.71014 \cdot 10^{-07} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 1.50585 \cdot 10^{-16} X^8 - 5.82707 \cdot 10^{-16} X^7 + 9.15943 \cdot 10^{-16} X^6 - 7.54824 \cdot 10^{-16} X^5 + 3.52096 \\
 &\quad \cdot 10^{-16} X^4 - 9.31289 \cdot 10^{-17} X^3 - 3.98474 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07} \\
 &= -1.6278 \cdot 10^{-07} B_{0,8} - 1.08555 \cdot 10^{-07} B_{1,8} - 5.43313 \cdot 10^{-08} B_{2,8} - 1.07093 \cdot 10^{-10} B_{3,8} + 5.41171 \\
 &\quad \cdot 10^{-08} B_{4,8} + 1.08341 \cdot 10^{-07} B_{5,8} + 1.62565 \cdot 10^{-07} B_{6,8} + 2.1679 \cdot 10^{-07} B_{7,8} + 2.71014 \cdot 10^{-07} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 8.66435 \cdot 10^{-19}$.

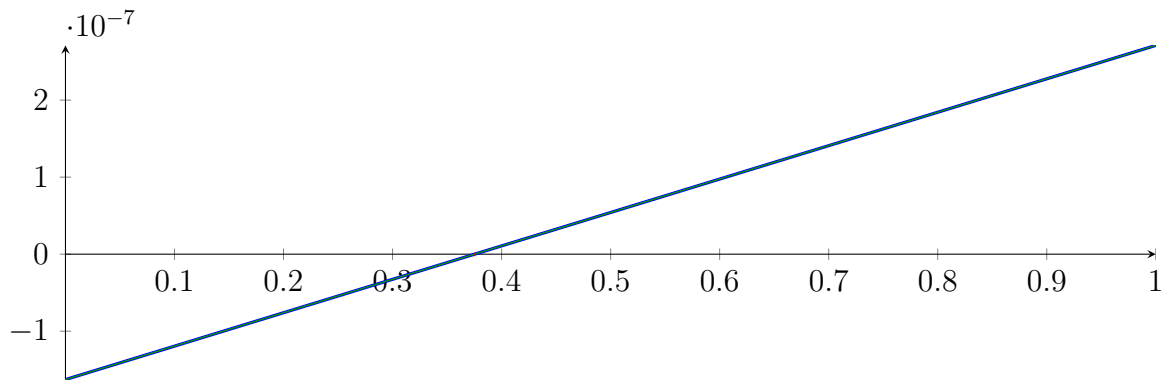
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -1.85288 \cdot 10^{-21} X^3 - 5.2746 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07} \\
 m &= -1.90582 \cdot 10^{-21} X^3 - 5.2746 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-1.49969 \cdot 10^7, 0.375247, 1.49696 \cdot 10^7\} \quad N(m) = \{-1.46018 \cdot 10^7, 0.375247, 1.45759 \cdot 10^7\}$$

Intersection intervals:



[0.375247, 0.375247]

Longest intersection interval: $7.69251 \cdot 10^{-09}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

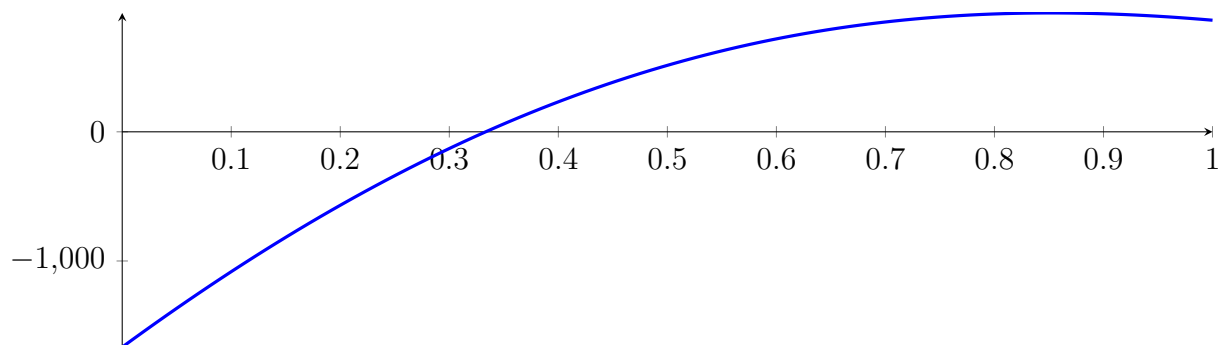
54.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

54.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

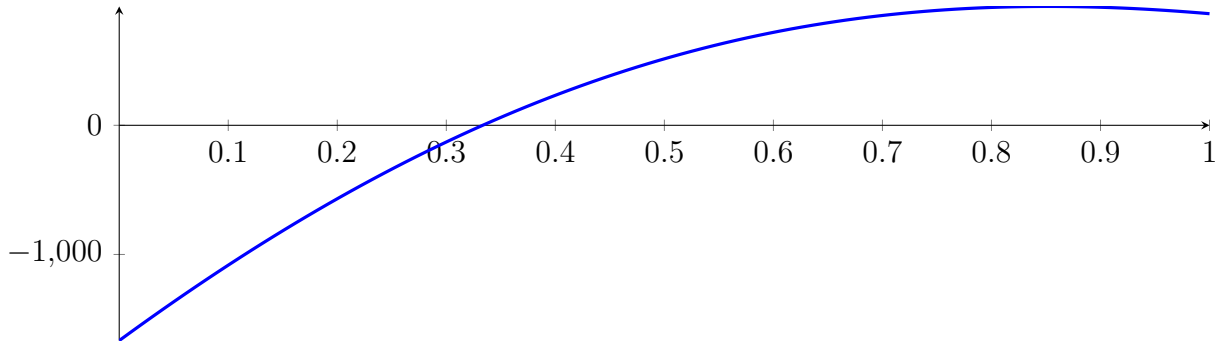
with precision $\varepsilon = 1 \cdot 10^{-16}$.

55 Running BezClip on f_8 with epsilon 32

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

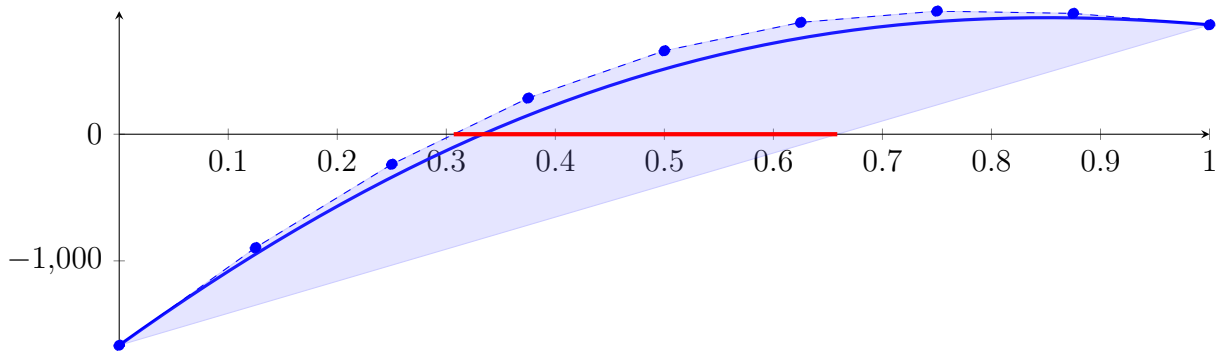
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



55.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

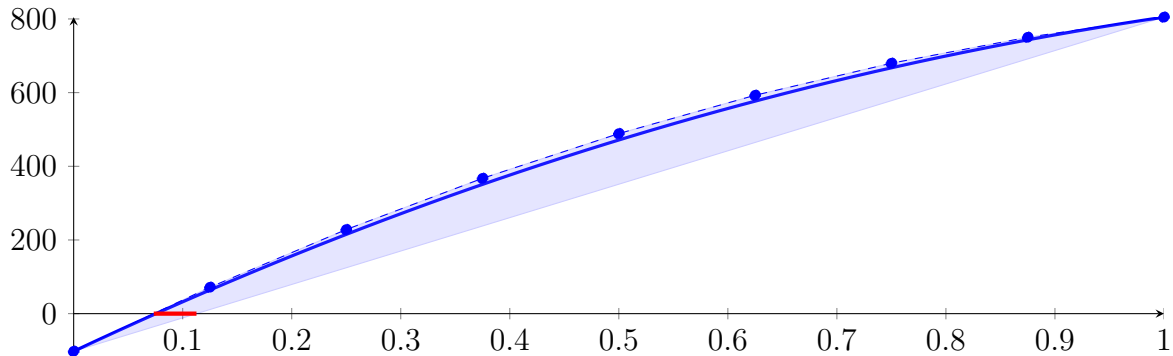
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

55.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

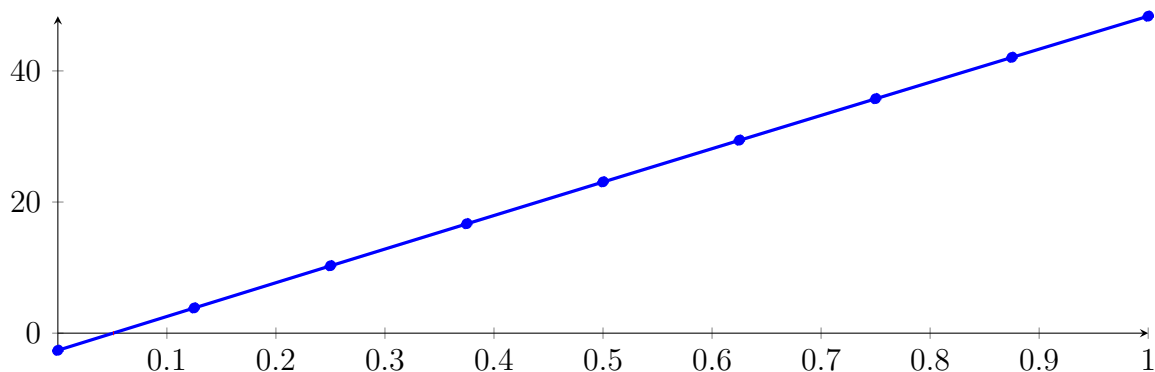
Longest intersection interval: 0.0391855

⇒ Selective recursion: interval 1: [0.332635, 0.34642],

55.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -5.41789 \cdot 10^{-13}X^8 - 1.26477 \cdot 10^{-12}X^7 - 5.05786 \cdot 10^{-10}X^6 + 3.66765 \cdot 10^{-08}X^5 \\
 &\quad + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

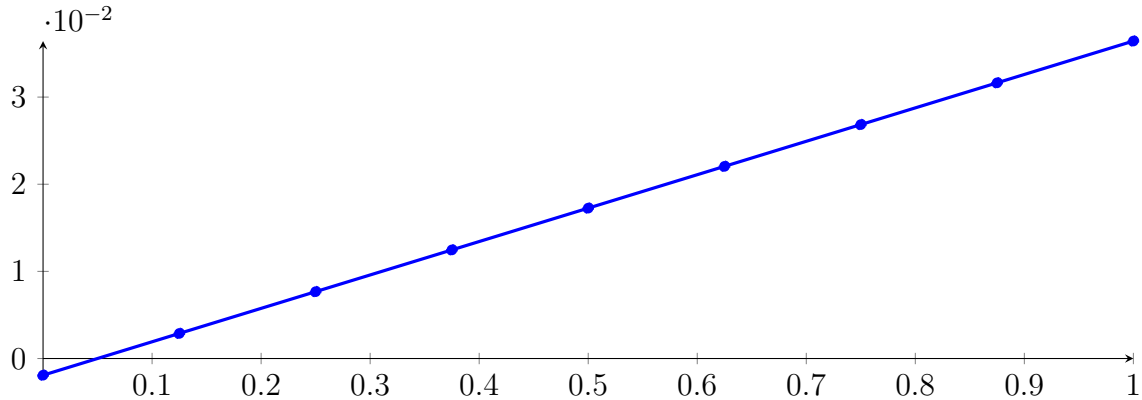
Longest intersection interval: 0.000742589

⇒ Selective recursion: interval 1: [0.333333, 0.333343],

55.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -7.11237 \cdot 10^{-16} X^8 + 5.27356 \cdot 10^{-16} X^7 - 7.38298 \cdot 10^{-15} X^6 + 1.06859 \cdot 10^{-15} X^5 \\
 &\quad - 1.09288 \cdot 10^{-15} X^4 - 2.37227 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\
 &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\
 &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\
 &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

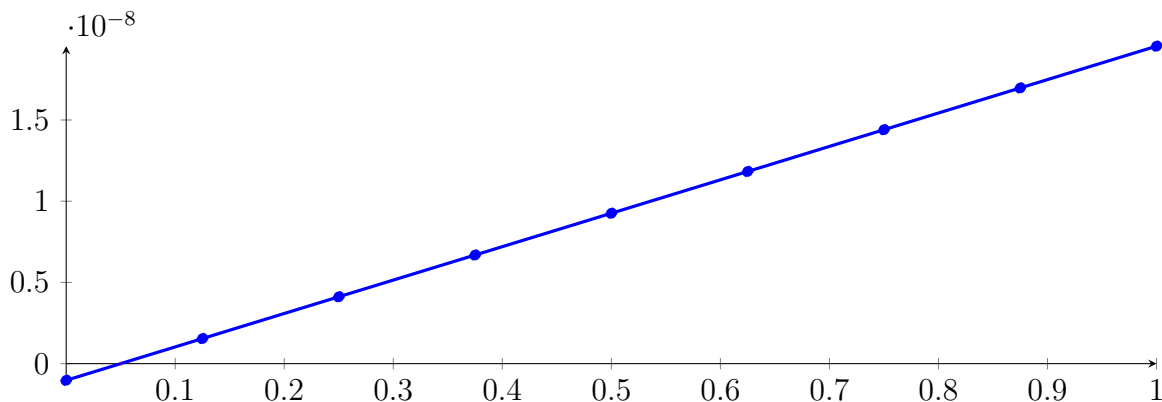
Longest intersection interval: $5.36469 \cdot 10^{-07}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

55.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.09366 \cdot 10^{-22} X^8 + 4.49986 \cdot 10^{-22} X^7 - 4.03002 \cdot 10^{-21} X^6 + 3.70577 \cdot 10^{-22} X^5 - 3.47416 \\
 &\quad \cdot 10^{-22} X^4 + 9.26442 \cdot 10^{-23} X^3 - 1.18608 \cdot 10^{-19} X^2 + 2.05706 \cdot 10^{-08} X - 1.02667 \cdot 10^{-09} \\
 &= -1.02667 \cdot 10^{-09} B_{0,8}(X) + 1.54465 \cdot 10^{-09} B_{1,8}(X) + 4.11597 \cdot 10^{-09} B_{2,8}(X) \\
 &\quad + 6.68729 \cdot 10^{-09} B_{3,8}(X) + 9.25862 \cdot 10^{-09} B_{4,8}(X) + 1.18299 \cdot 10^{-08} B_{5,8}(X) \\
 &\quad + 1.44013 \cdot 10^{-08} B_{6,8}(X) + 1.69726 \cdot 10^{-08} B_{7,8}(X) + 1.95439 \cdot 10^{-08} B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87728 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

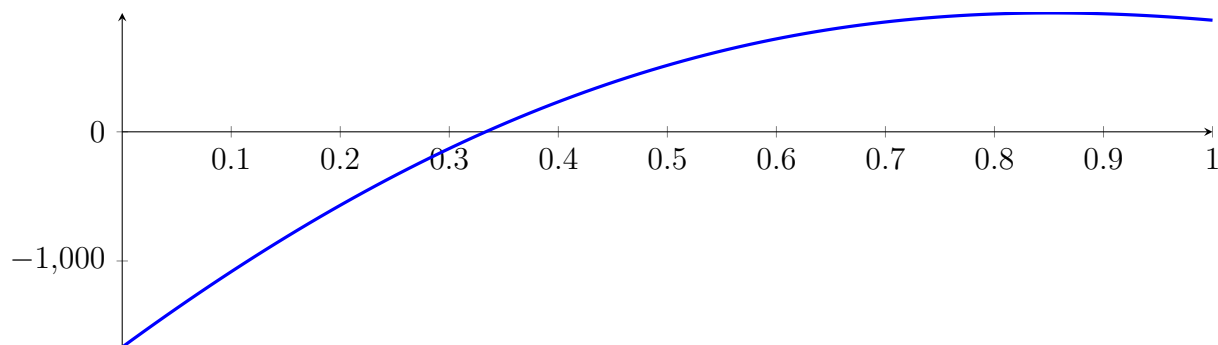
55.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

55.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

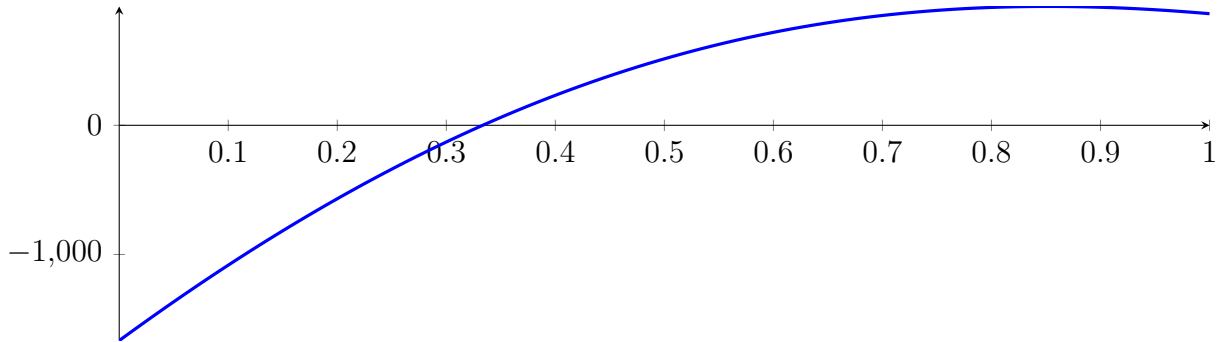
with precision $\varepsilon = 1 \cdot 10^{-32}$.

56 Running QuadClip on f_8 with epsilon 32

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

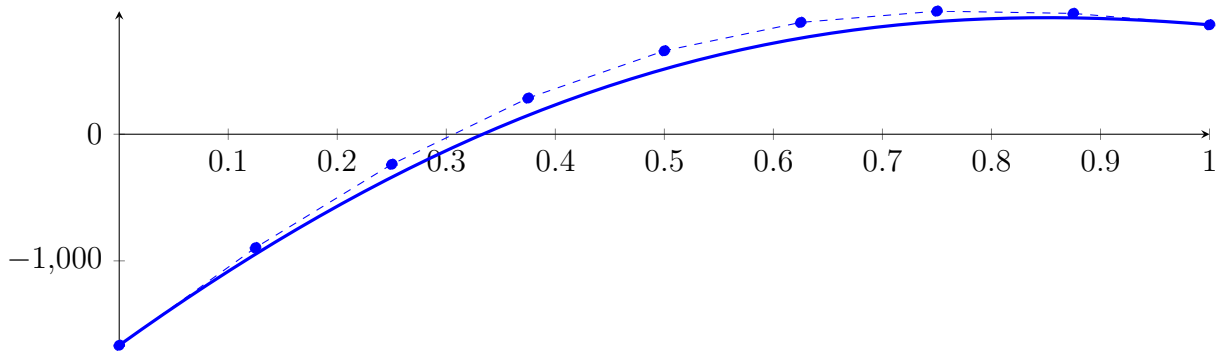
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



56.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

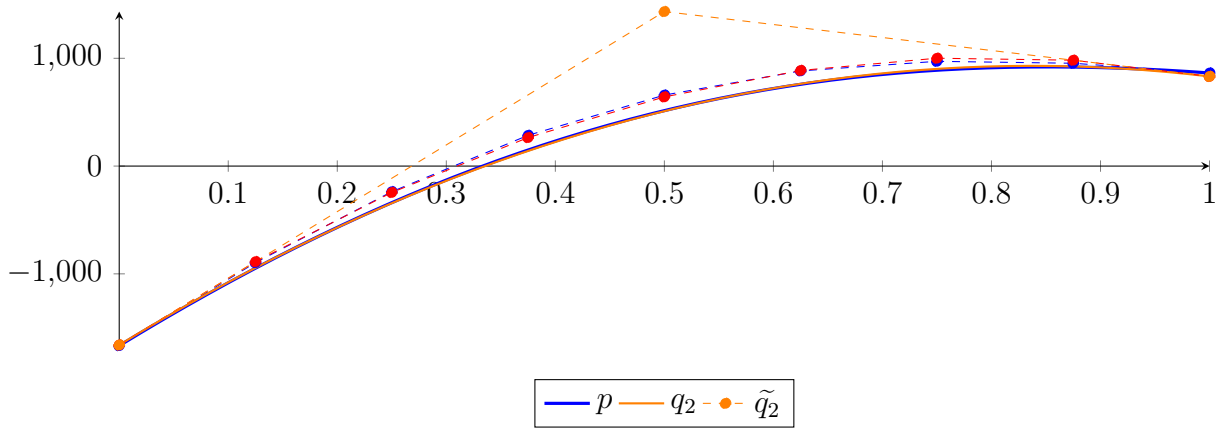
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -3.91297 \cdot 10^{-06}X^8 + 1.58774 \cdot 10^{-05}X^7 - 2.63335 \cdot 10^{-05}X^6 + 2.29285 \cdot 10^{-05}X^5 \\ &\quad - 1.11931 \cdot 10^{-05}X^4 + 3.0313 \cdot 10^{-06}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

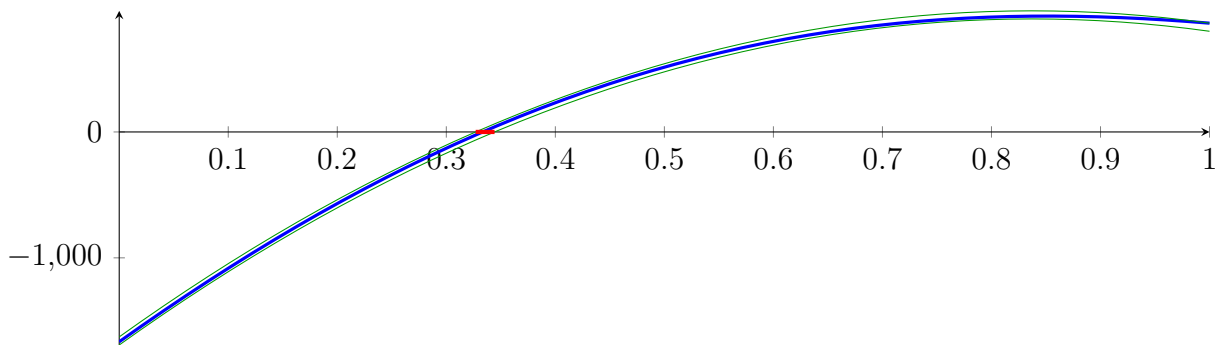
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

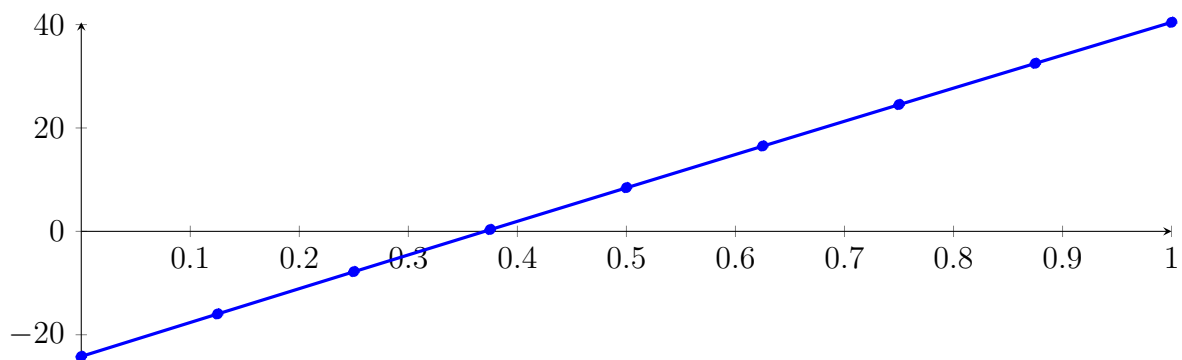
Longest intersection interval: 0.0173372

\implies Selective recursion: interval 1: $[0.326917, 0.344255]$,

56.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.40012 \cdot 10^{-13} X^8 - 7.38964 \cdot 10^{-12} X^7 - 1.94416 \cdot 10^{-09} X^6 + 1.19265 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

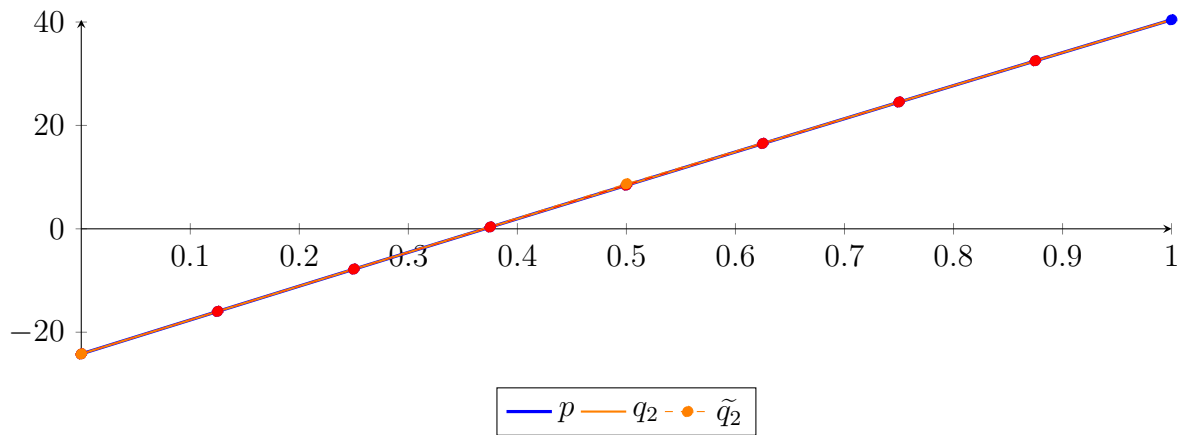
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = 1.00159 \cdot 10^{-08} X^8 - 3.3372 \cdot 10^{-08} X^7 + 4.23875 \cdot 10^{-08} X^6 - 2.49721 \cdot 10^{-08} X^5$$

$$+ 6.08793 \cdot 10^{-09} X^4 + 1.46429 \cdot 10^{-10} X^3 - 1.18261 X^2 + 65.8162 X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

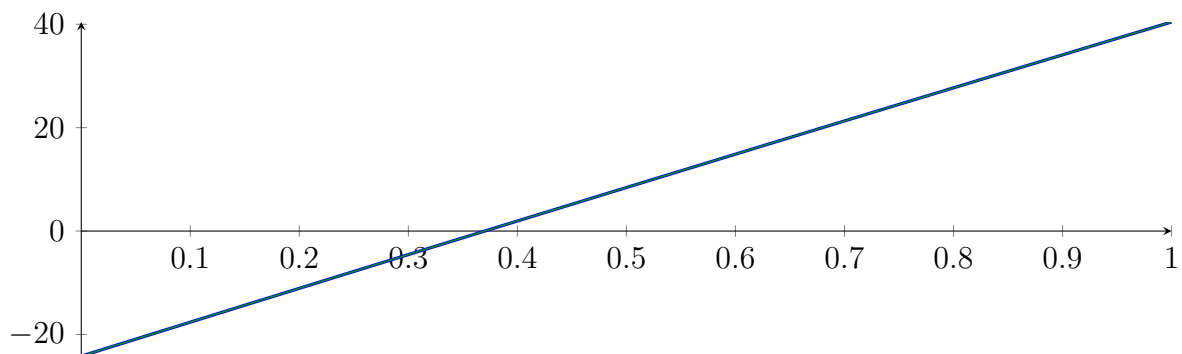
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

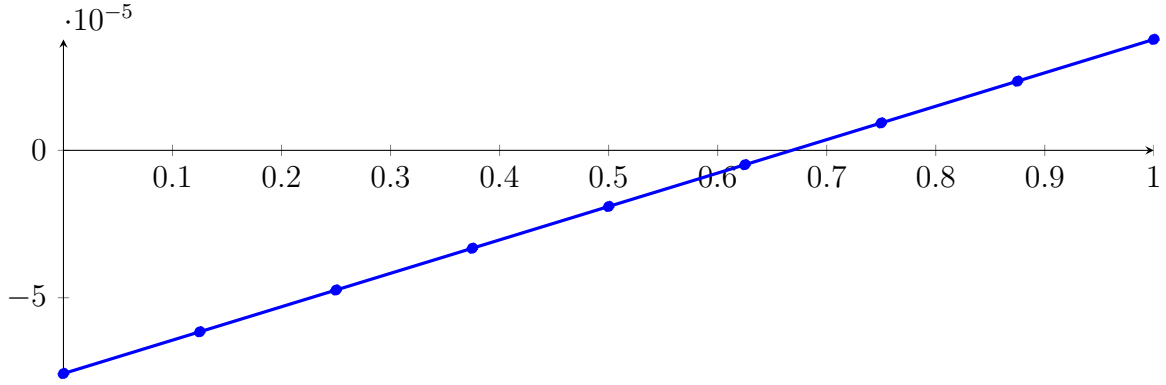
Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

56.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

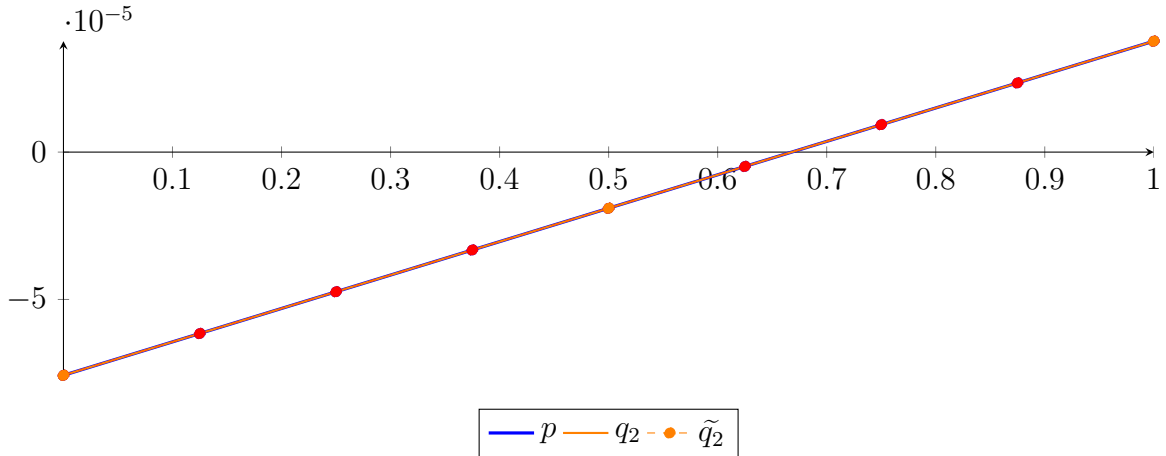
$$\begin{aligned}
 p &= -2.1684 \cdot 10^{-19} X^8 - 4.33681 \cdot 10^{-19} X^7 + 2.12504 \cdot 10^{-17} X^6 - 1.51788 \cdot 10^{-18} X^5 \\
 &\quad + 7.58942 \cdot 10^{-18} X^4 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) \\
 &\quad - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) \\
 &\quad + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 5.62292 \cdot 10^{-14} X^8 - 2.22643 \cdot 10^{-13} X^7 + 3.60043 \cdot 10^{-13} X^6 - 3.05846 \cdot 10^{-13} X^5 + 1.46182 \\
 &\quad \cdot 10^{-13} X^4 - 3.90612 \cdot 10^{-14} X^3 - 3.59793 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \\
 &\quad \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.98887 \cdot 10^{-16}$.

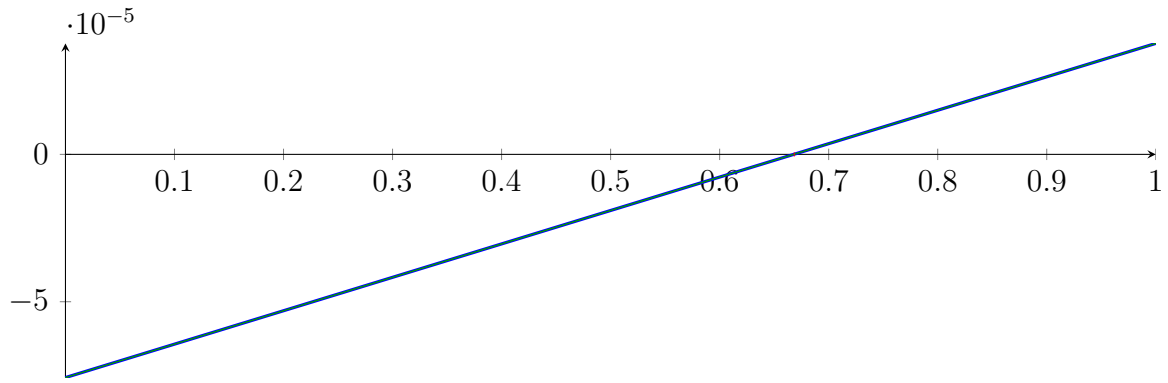
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 m &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.667955, 3.14644 \cdot 10^7\} \qquad N(m) = \{0.667955, 3.14644 \cdot 10^7\}$$

Intersection intervals:



[0.667955, 0.667955]

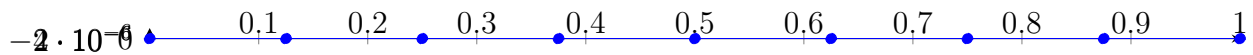
Longest intersection interval: $1.88052 \cdot 10^{-09}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

56.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

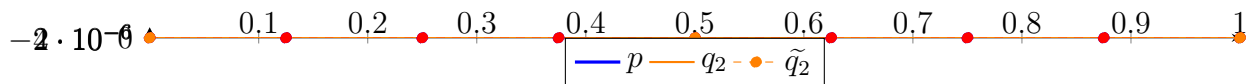
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.31266 \cdot 10^{-27} X^8 - 6.92683 \cdot 10^{-26} X^6 - 8.48183 \cdot 10^{-27} X^5 - 1.06023 \\
 &\quad \cdot 10^{-26} X^4 + 1.41364 \cdot 10^{-27} X^3 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\
 &= 4.52469 \cdot 10^{-14} B_{0,8}(X) + 7.18983 \cdot 10^{-14} B_{1,8}(X) + 9.85497 \cdot 10^{-14} B_{2,8}(X) \\
 &\quad + 1.25201 \cdot 10^{-13} B_{3,8}(X) + 1.51852 \cdot 10^{-13} B_{4,8}(X) + 1.78504 \cdot 10^{-13} B_{5,8}(X) \\
 &\quad + 2.05155 \cdot 10^{-13} B_{6,8}(X) + 2.31807 \cdot 10^{-13} B_{7,8}(X) + 2.58458 \cdot 10^{-13} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= 2.44862 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\
 &= 4.52469 \cdot 10^{-14} B_{0,2} + 1.51852 \cdot 10^{-13} B_{1,2} + 2.58458 \cdot 10^{-13} B_{2,2} \\
 \tilde{q}_2 &= -9.59093 \cdot 10^{-23} X^8 + 4.40717 \cdot 10^{-22} X^7 - 8.25115 \cdot 10^{-22} X^6 + 8.08254 \cdot 10^{-22} X^5 - 4.43405 \\
 &\quad \cdot 10^{-22} X^4 + 1.3589 \cdot 10^{-22} X^3 - 2.18259 \cdot 10^{-23} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\
 &= 4.52469 \cdot 10^{-14} B_{0,8} + 7.18983 \cdot 10^{-14} B_{1,8} + 9.85497 \cdot 10^{-14} B_{2,8} + 1.25201 \cdot 10^{-13} B_{3,8} + 1.51852 \\
 &\quad \cdot 10^{-13} B_{4,8} + 1.78504 \cdot 10^{-13} B_{5,8} + 2.05155 \cdot 10^{-13} B_{6,8} + 2.31807 \cdot 10^{-13} B_{7,8} + 2.58458 \cdot 10^{-13} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.40606 \cdot 10^{-25}$.

Bounding polynomials M and m :

$$\begin{aligned}
 M &= 2.42338 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\
 m &= 2.47387 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-8.79809 \cdot 10^{13}, -0.213542\} \quad N(m) = \{-8.61853 \cdot 10^{13}, -0.214286\}$$

Intersection intervals:

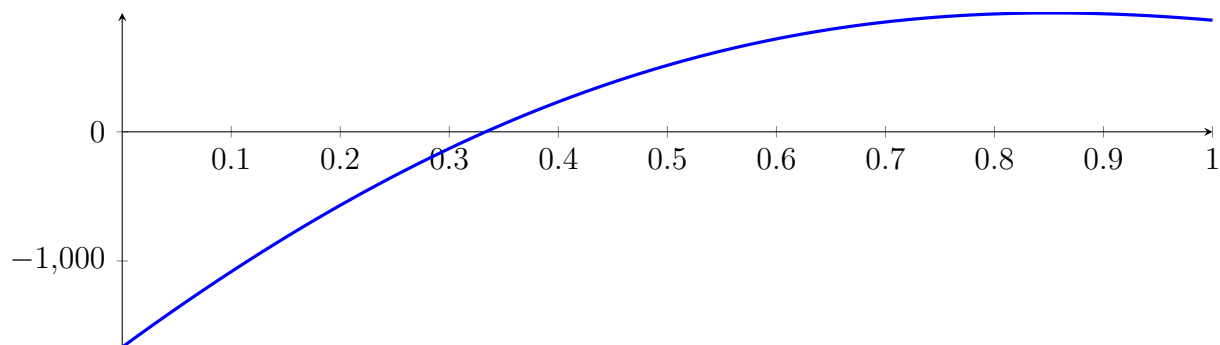


No intersection intervals with the x axis.

56.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

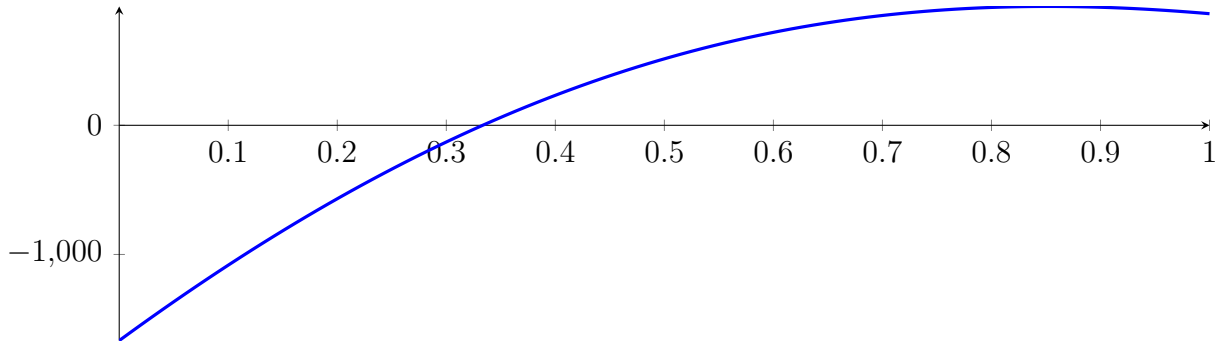
with precision $\varepsilon = 1 \cdot 10^{-32}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

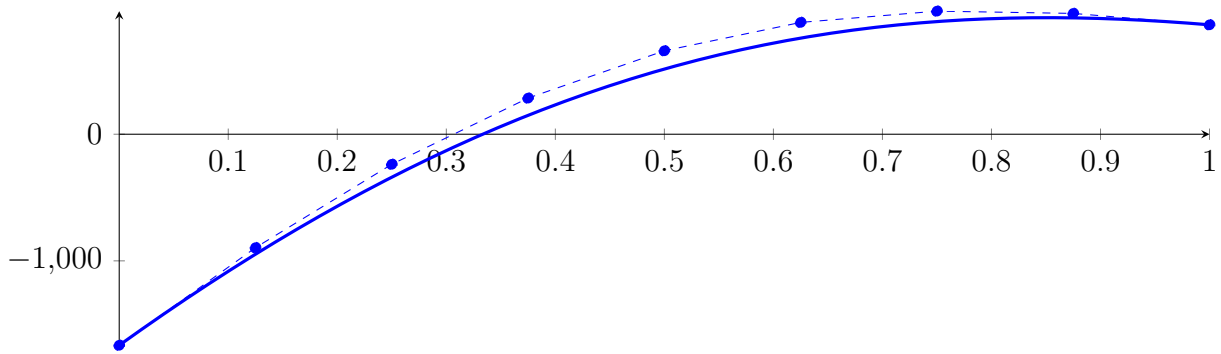
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



57.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

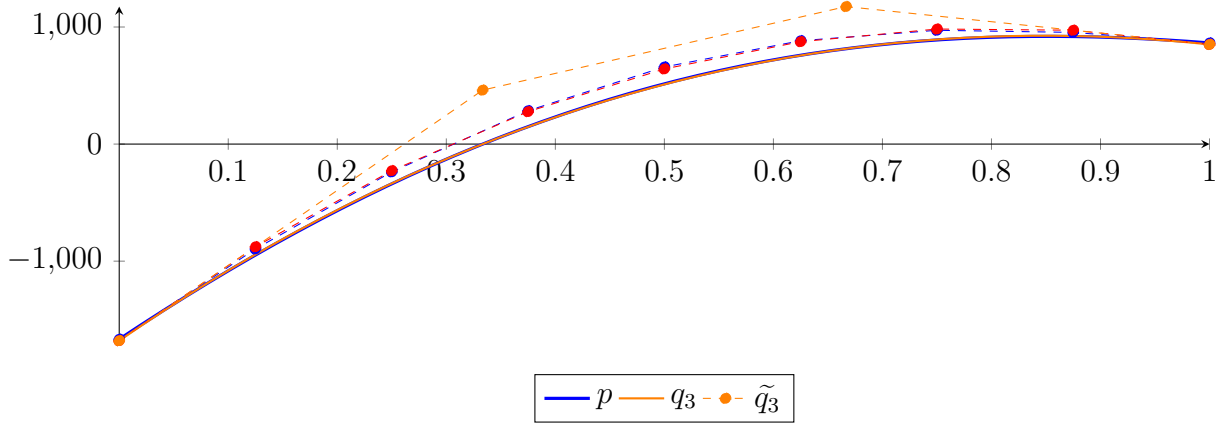
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 2.38117 \cdot 10^{-06}X^8 - 9.49957 \cdot 10^{-06}X^7 + 1.54757 \cdot 10^{-05}X^6 - 1.32609 \cdot 10^{-05}X^5 \\ &\quad + 6.41819 \cdot 10^{-06}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

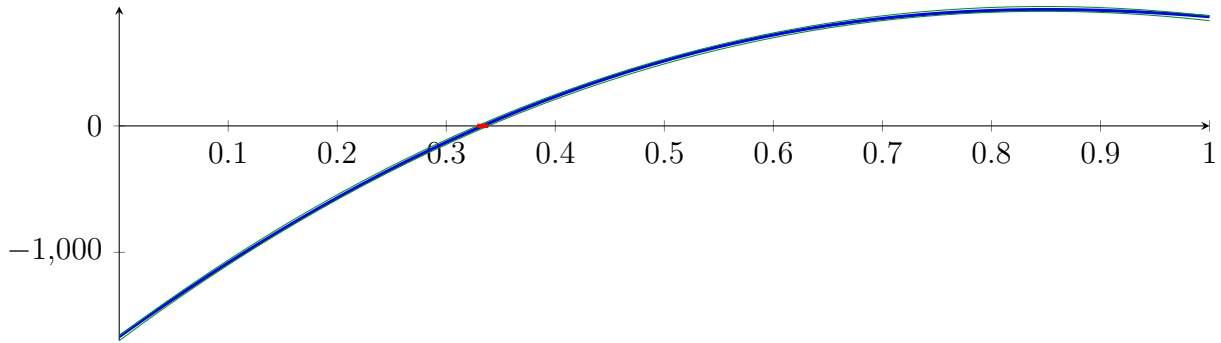
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

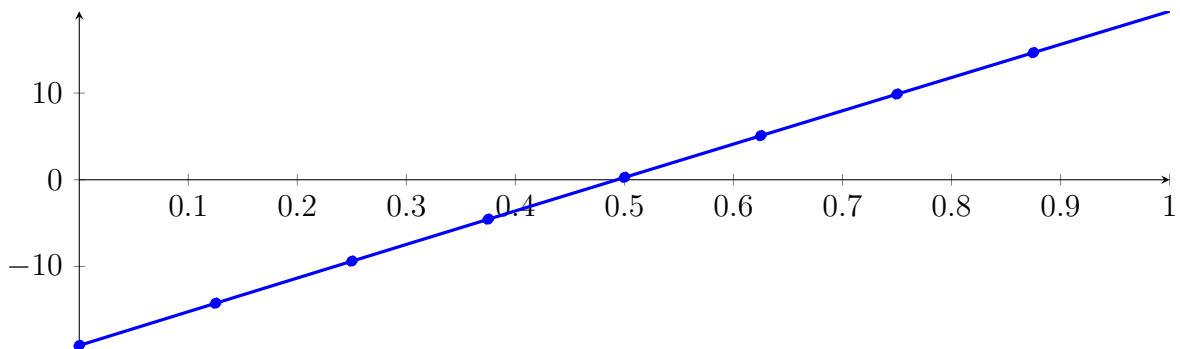
Longest intersection interval: 0.0102926

⇒ Selective recursion: interval 1: $[0.328258, 0.338551]$,

57.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.27898 \cdot 10^{-13} X^8 - 1.13687 \cdot 10^{-13} X^7 - 8.23661 \cdot 10^{-11} X^6 + 8.72882 \cdot 10^{-09} X^5 \\ &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

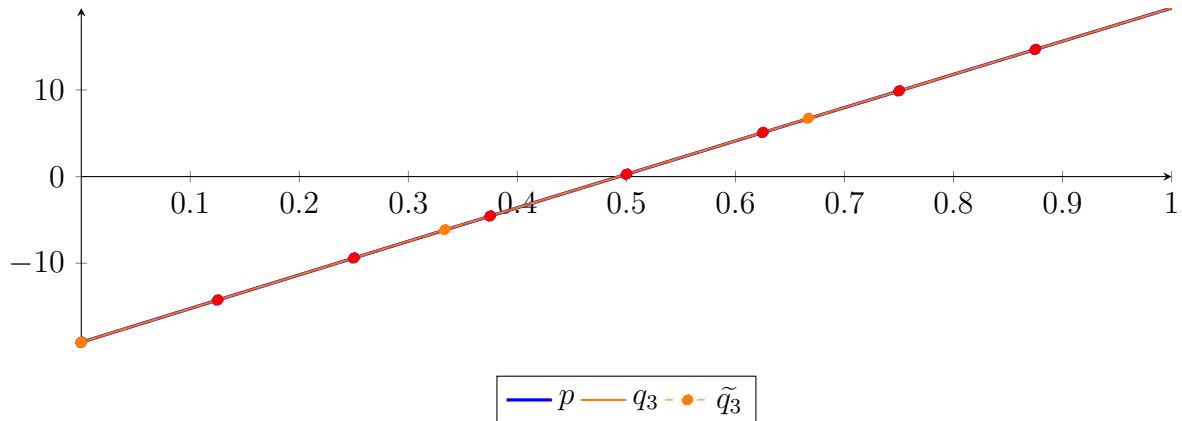
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = 2.3353 \cdot 10^{-08}X^8 - 9.31856 \cdot 10^{-08}X^7 + 1.51861 \cdot 10^{-07}X^6 - 1.30228 \cdot 10^{-07}X^5$$

$$+ 6.31618 \cdot 10^{-08}X^4 - 0.000240962X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16887 \cdot 10^{-07}$.

Bounding polynomials M and m :

$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

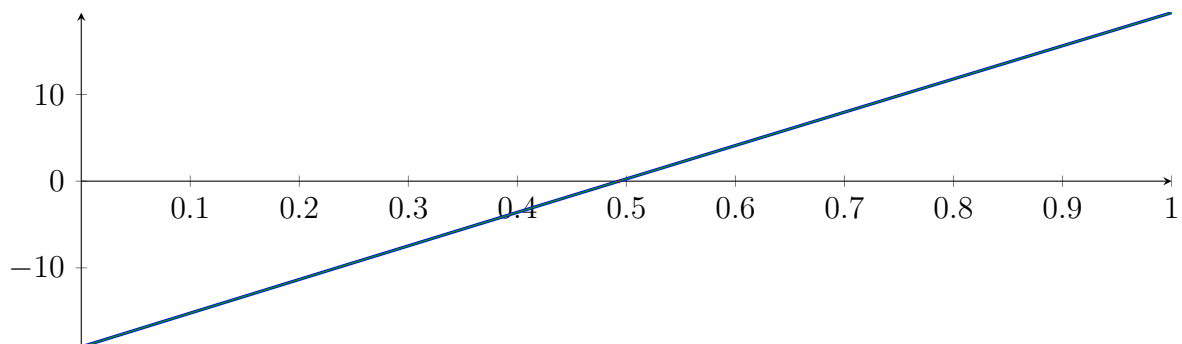
$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\}$$

$$N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

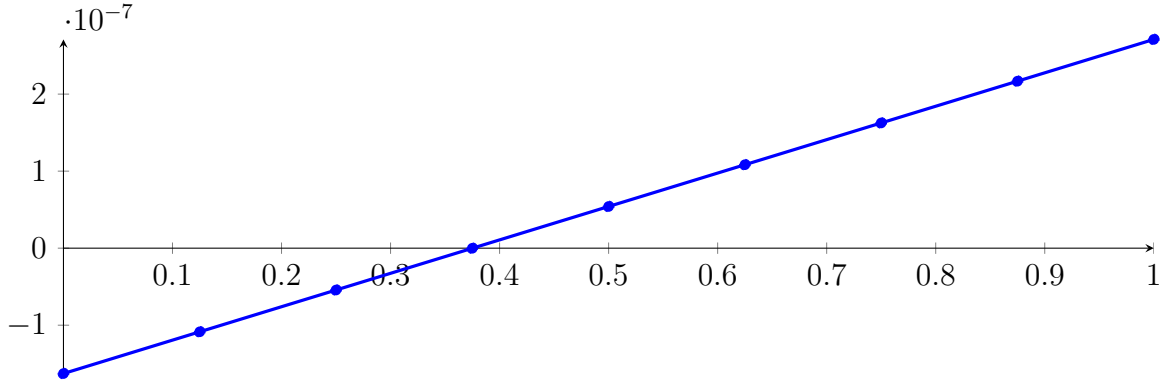
Longest intersection interval: $1.12517 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

57.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

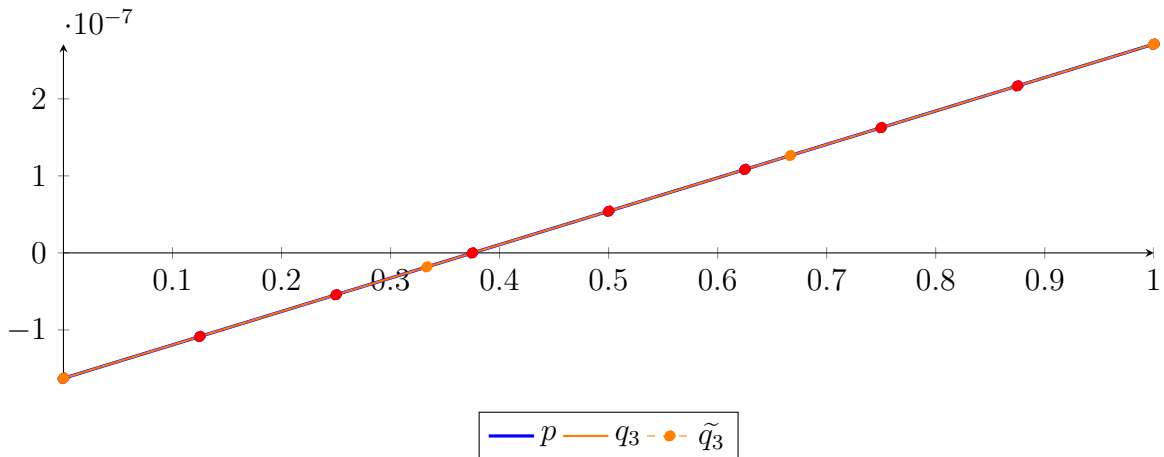
$$\begin{aligned}
 p &= -3.81165 \cdot 10^{-21} X^8 + 6.77626 \cdot 10^{-21} X^7 + 2.96462 \cdot 10^{-21} X^6 + 5.92923 \cdot 10^{-21} X^5 + 1.11173 \\
 &\quad \cdot 10^{-20} X^4 + 1.48231 \cdot 10^{-21} X^3 - 5.27494 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07} \\
 &= -1.6278 \cdot 10^{-07} B_{0,8}(X) - 1.08555 \cdot 10^{-07} B_{1,8}(X) - 5.43313 \cdot 10^{-08} B_{2,8}(X) \\
 &\quad - 1.07093 \cdot 10^{-10} B_{3,8}(X) + 5.41171 \cdot 10^{-08} B_{4,8}(X) + 1.08341 \cdot 10^{-07} B_{5,8}(X) \\
 &\quad + 1.62565 \cdot 10^{-07} B_{6,8}(X) + 2.1679 \cdot 10^{-07} B_{7,8}(X) + 2.71014 \cdot 10^{-07} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -1.90582 \cdot 10^{-21} X^3 - 5.2746 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07} \\
 &= -1.6278 \cdot 10^{-07} B_{0,3} - 1.81818 \cdot 10^{-08} B_{1,3} + 1.26416 \cdot 10^{-07} B_{2,3} + 2.71014 \cdot 10^{-07} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 1.50585 \cdot 10^{-16} X^8 - 5.82707 \cdot 10^{-16} X^7 + 9.15943 \cdot 10^{-16} X^6 - 7.54824 \cdot 10^{-16} X^5 + 3.52096 \\
 &\quad \cdot 10^{-16} X^4 - 9.31289 \cdot 10^{-17} X^3 - 3.98474 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07} \\
 &= -1.6278 \cdot 10^{-07} B_{0,8} - 1.08555 \cdot 10^{-07} B_{1,8} - 5.43313 \cdot 10^{-08} B_{2,8} - 1.07093 \cdot 10^{-10} B_{3,8} + 5.41171 \\
 &\quad \cdot 10^{-08} B_{4,8} + 1.08341 \cdot 10^{-07} B_{5,8} + 1.62565 \cdot 10^{-07} B_{6,8} + 2.1679 \cdot 10^{-07} B_{7,8} + 2.71014 \cdot 10^{-07} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 8.66435 \cdot 10^{-19}$.

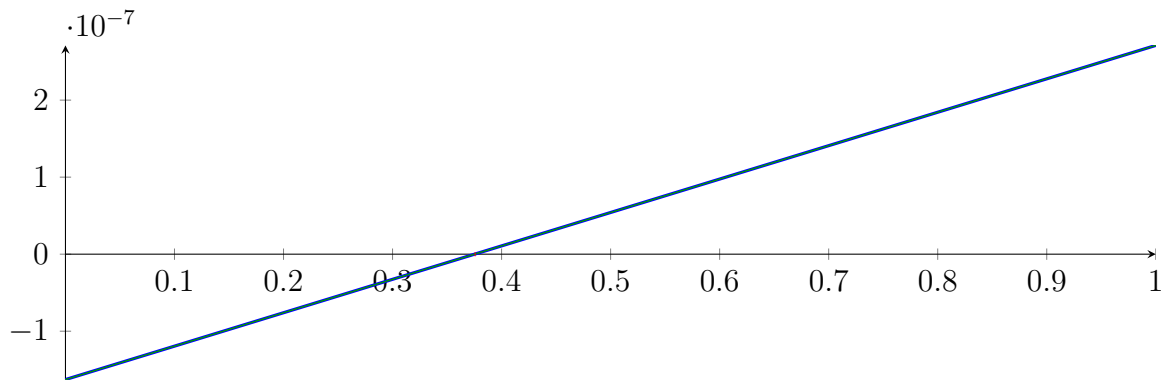
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -1.85288 \cdot 10^{-21} X^3 - 5.2746 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07} \\
 m &= -1.90582 \cdot 10^{-21} X^3 - 5.2746 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-1.49969 \cdot 10^7, 0.375247, 1.49696 \cdot 10^7\} \quad N(m) = \{-1.46018 \cdot 10^7, 0.375247, 1.45759 \cdot 10^7\}$$

Intersection intervals:



[0.375247, 0.375247]

Longest intersection interval: $7.69251 \cdot 10^{-09}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

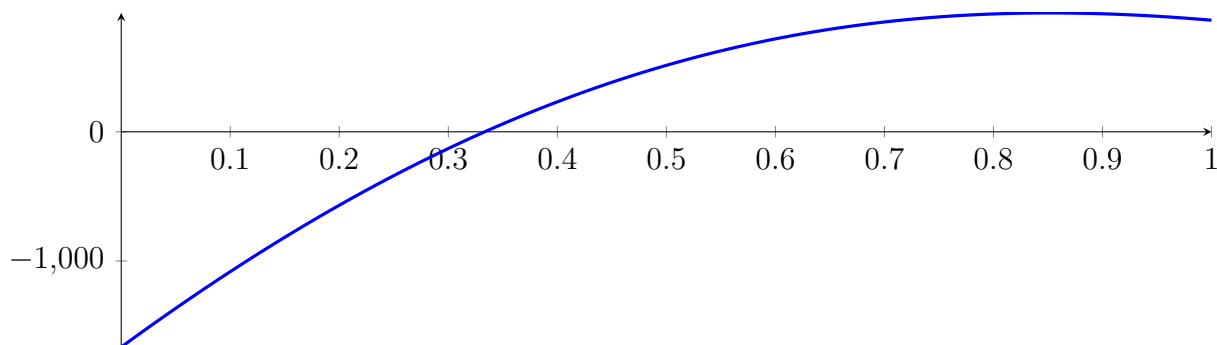
57.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

57.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

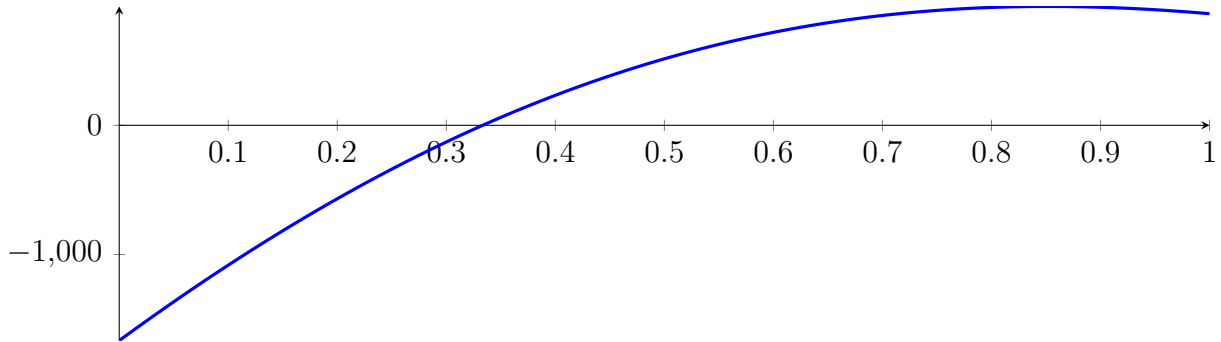
with precision $\varepsilon = 1 \cdot 10^{-32}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

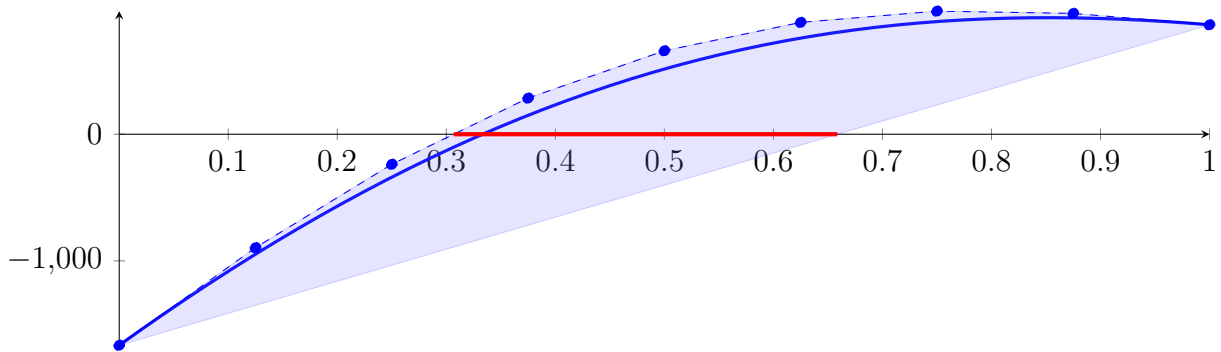
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



58.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

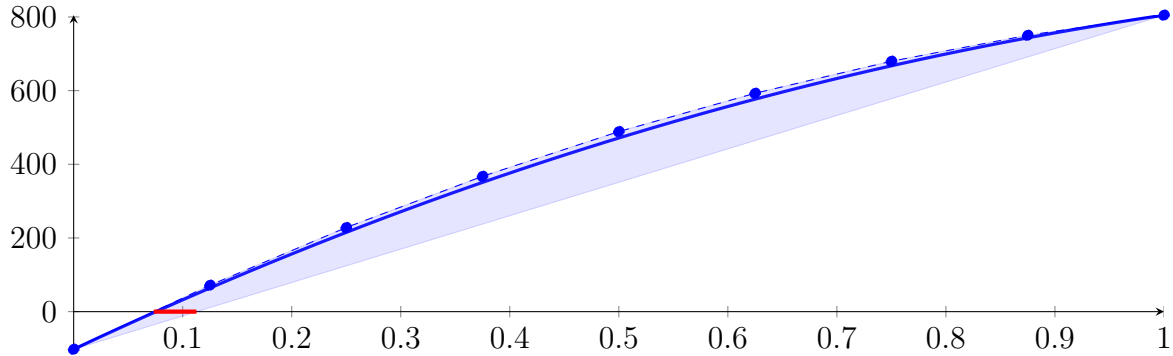
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

58.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

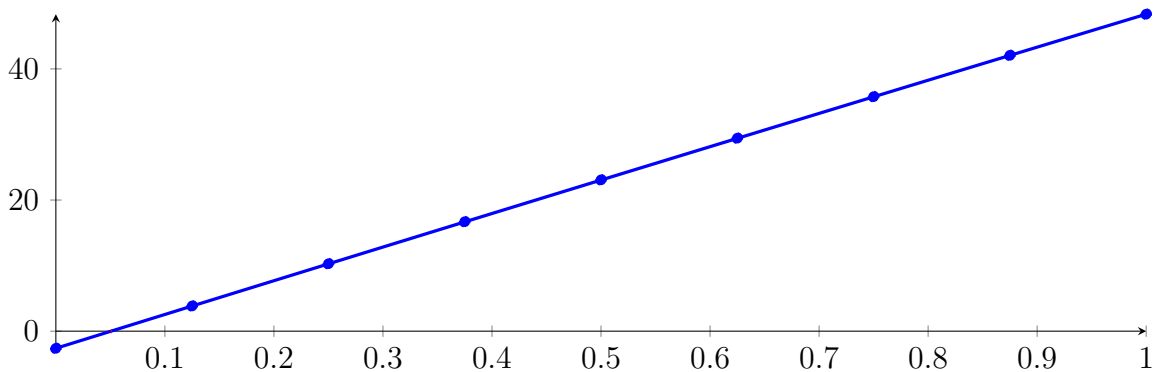
Longest intersection interval: 0.0391855

⇒ Selective recursion: interval 1: [0.332635, 0.34642],

58.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -5.41789 \cdot 10^{-13}X^8 - 1.26477 \cdot 10^{-12}X^7 - 5.05786 \cdot 10^{-10}X^6 + 3.66765 \cdot 10^{-08}X^5 \\
 &\quad + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

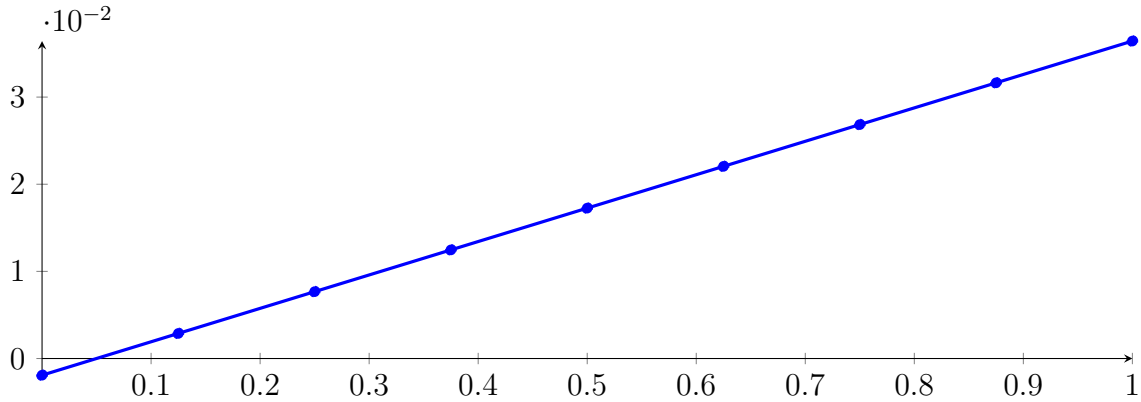
Longest intersection interval: 0.000742589

⇒ Selective recursion: interval 1: [0.333333, 0.333343],

58.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -7.11237 \cdot 10^{-16} X^8 + 5.27356 \cdot 10^{-16} X^7 - 7.38298 \cdot 10^{-15} X^6 + 1.06859 \cdot 10^{-15} X^5 \\
 &\quad - 1.09288 \cdot 10^{-15} X^4 - 2.37227 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\
 &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\
 &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\
 &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

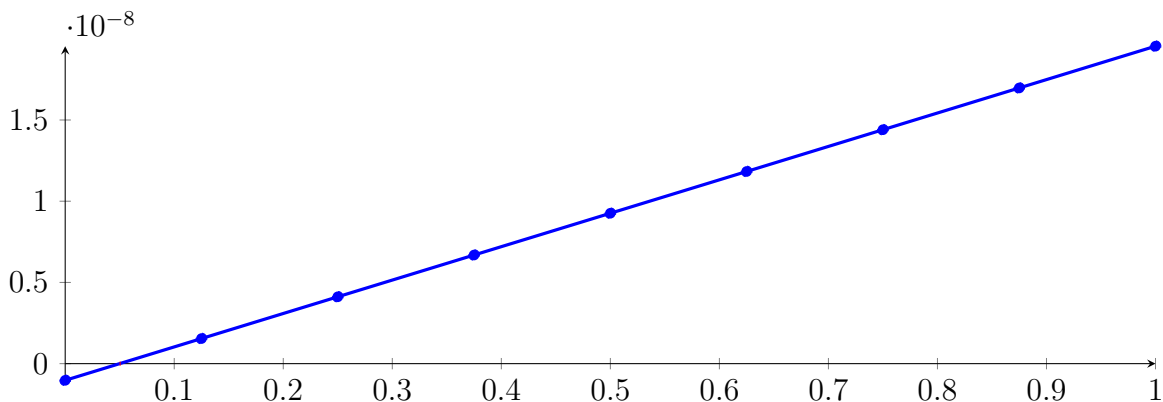
Longest intersection interval: $5.36469 \cdot 10^{-07}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

58.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.09366 \cdot 10^{-22} X^8 + 4.49986 \cdot 10^{-22} X^7 - 4.03002 \cdot 10^{-21} X^6 + 3.70577 \cdot 10^{-22} X^5 - 3.47416 \\
 &\quad \cdot 10^{-22} X^4 + 9.26442 \cdot 10^{-23} X^3 - 1.18608 \cdot 10^{-19} X^2 + 2.05706 \cdot 10^{-08} X - 1.02667 \cdot 10^{-09} \\
 &= -1.02667 \cdot 10^{-09} B_{0,8}(X) + 1.54465 \cdot 10^{-09} B_{1,8}(X) + 4.11597 \cdot 10^{-09} B_{2,8}(X) \\
 &\quad + 6.68729 \cdot 10^{-09} B_{3,8}(X) + 9.25862 \cdot 10^{-09} B_{4,8}(X) + 1.18299 \cdot 10^{-08} B_{5,8}(X) \\
 &\quad + 1.44013 \cdot 10^{-08} B_{6,8}(X) + 1.69726 \cdot 10^{-08} B_{7,8}(X) + 1.95439 \cdot 10^{-08} B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87728 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

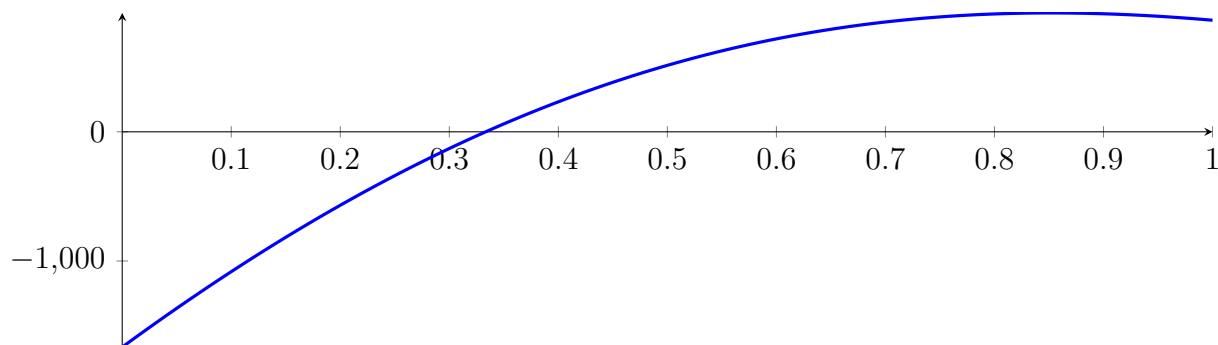
58.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

58.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

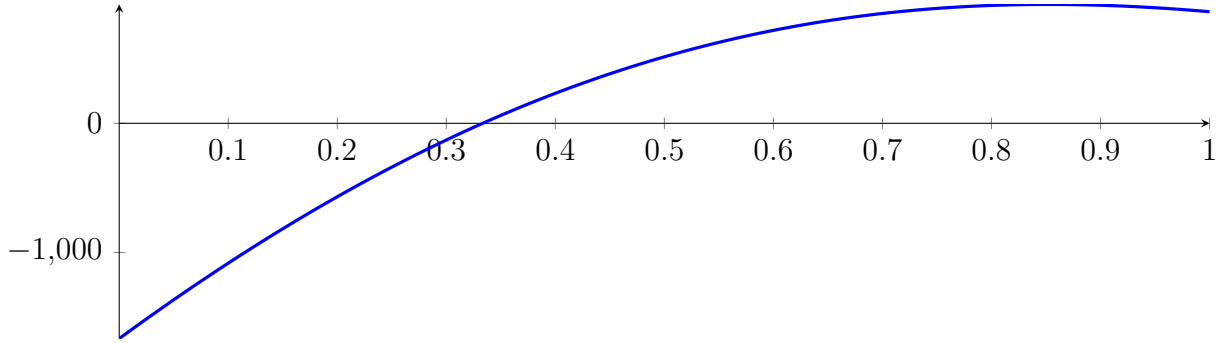
with precision $\varepsilon = 1 \cdot 10^{-64}$.

59 Running QuadClip on f_8 with epsilon 64

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

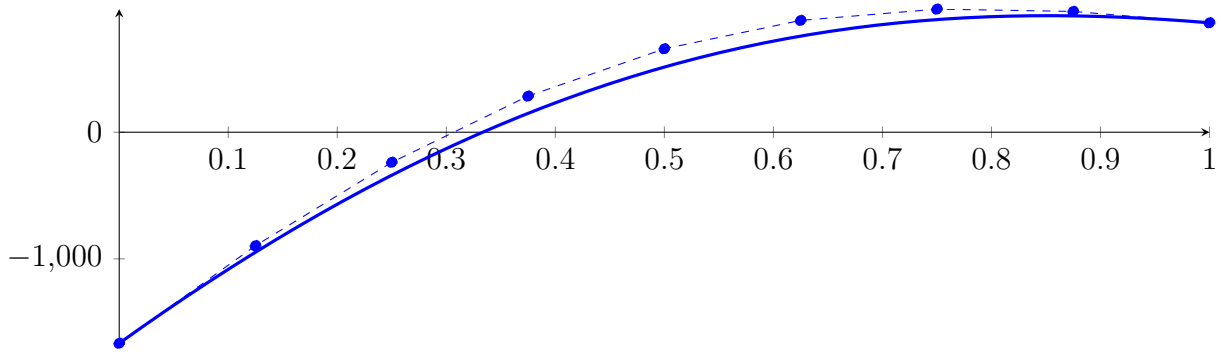
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



59.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

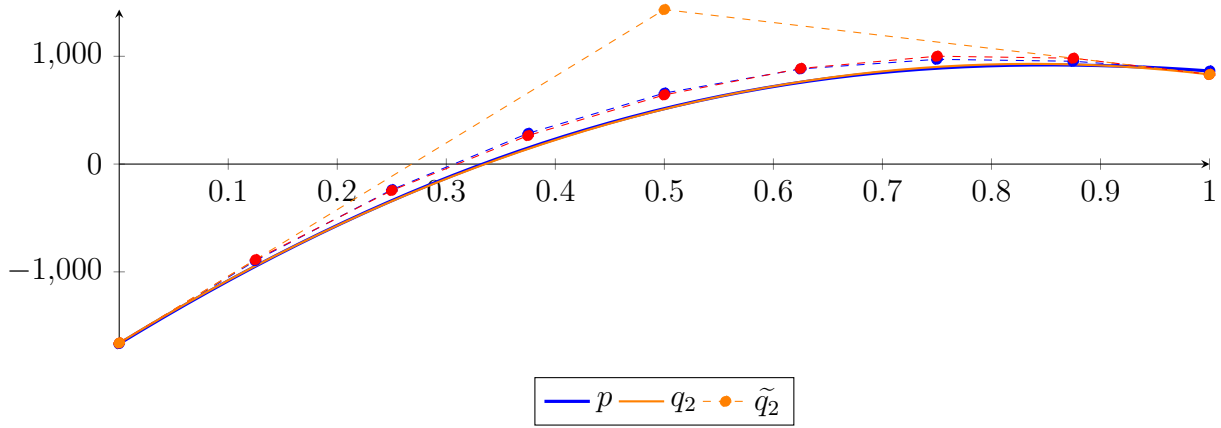
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -3.91297 \cdot 10^{-06}X^8 + 1.58774 \cdot 10^{-05}X^7 - 2.63335 \cdot 10^{-05}X^6 + 2.29285 \cdot 10^{-05}X^5 \\ &\quad - 1.11931 \cdot 10^{-05}X^4 + 3.0313 \cdot 10^{-06}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

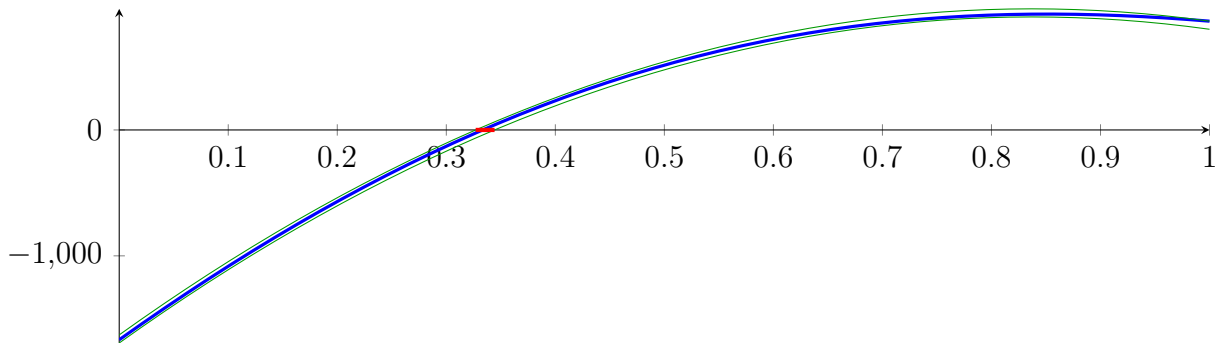
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

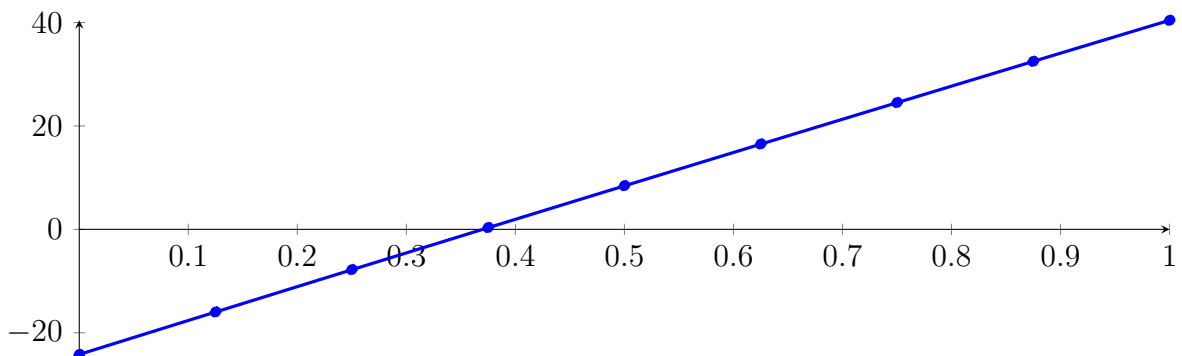
Longest intersection interval: 0.0173372

\implies Selective recursion: **interval 1: $[0.326917, 0.344255]$** ,

59.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.40012 \cdot 10^{-13} X^8 - 7.38964 \cdot 10^{-12} X^7 - 1.94416 \cdot 10^{-09} X^6 + 1.19265 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

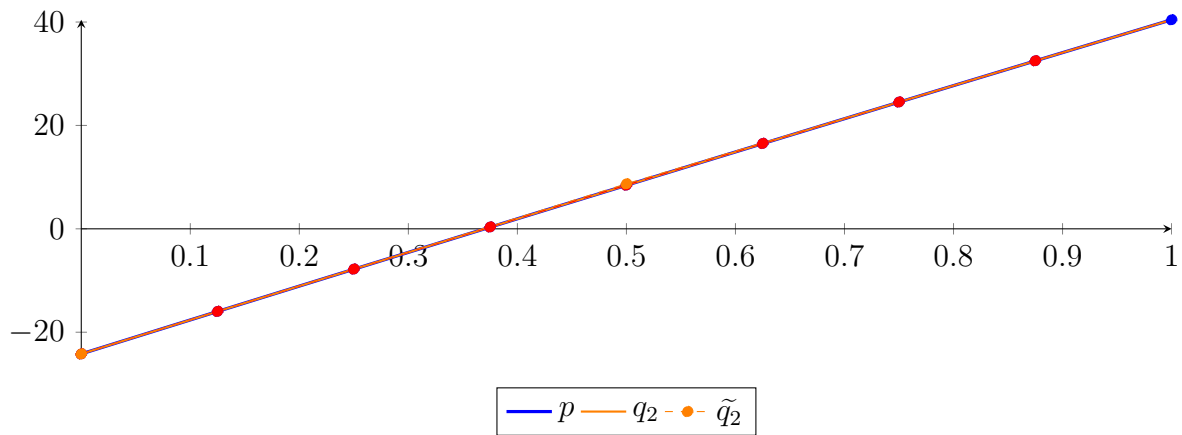
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = 1.00159 \cdot 10^{-08} X^8 - 3.3372 \cdot 10^{-08} X^7 + 4.23875 \cdot 10^{-08} X^6 - 2.49721 \cdot 10^{-08} X^5$$

$$+ 6.08793 \cdot 10^{-09} X^4 + 1.46429 \cdot 10^{-10} X^3 - 1.18261 X^2 + 65.8162 X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

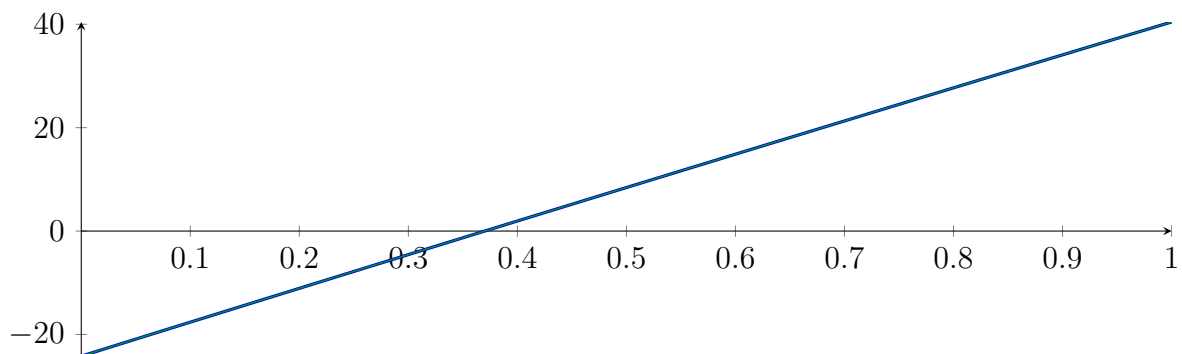
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

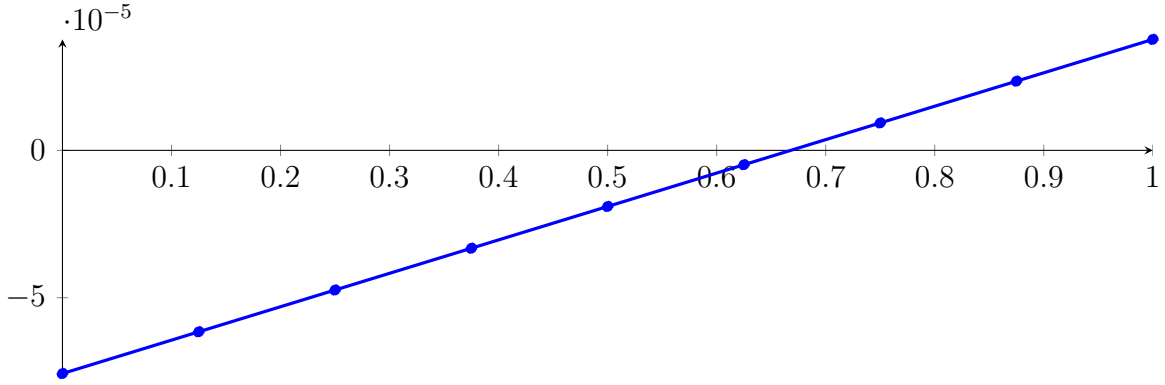
Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

59.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

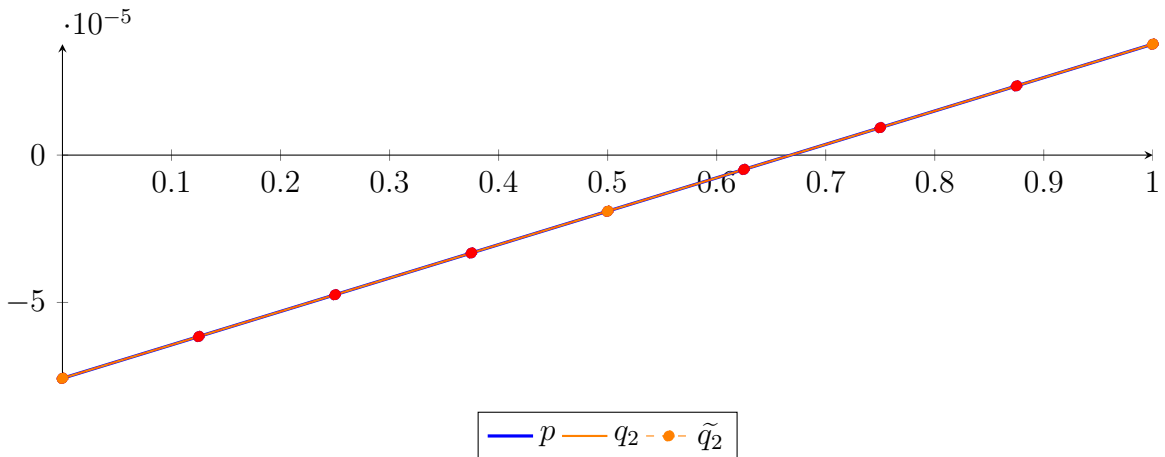
$$\begin{aligned}
 p &= -2.1684 \cdot 10^{-19} X^8 - 4.33681 \cdot 10^{-19} X^7 + 2.12504 \cdot 10^{-17} X^6 - 1.51788 \cdot 10^{-18} X^5 \\
 &\quad + 7.58942 \cdot 10^{-18} X^4 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) \\
 &\quad - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) \\
 &\quad + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 5.62292 \cdot 10^{-14} X^8 - 2.22643 \cdot 10^{-13} X^7 + 3.60043 \cdot 10^{-13} X^6 - 3.05846 \cdot 10^{-13} X^5 + 1.46182 \\
 &\quad \cdot 10^{-13} X^4 - 3.90612 \cdot 10^{-14} X^3 - 3.59793 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \\
 &\quad \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.98887 \cdot 10^{-16}$.

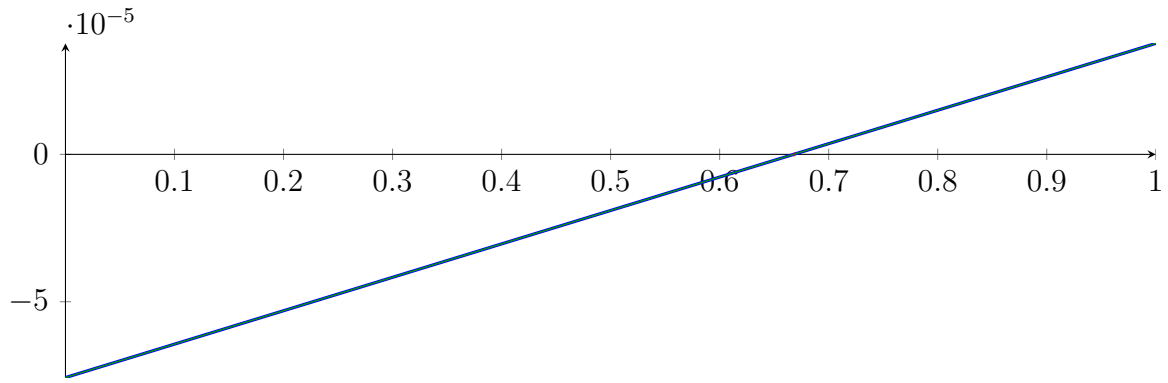
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 m &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.667955, 3.14644 \cdot 10^7\} \qquad N(m) = \{0.667955, 3.14644 \cdot 10^7\}$$

Intersection intervals:



[0.667955, 0.667955]

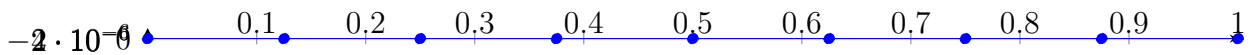
Longest intersection interval: $1.88052 \cdot 10^{-09}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

59.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

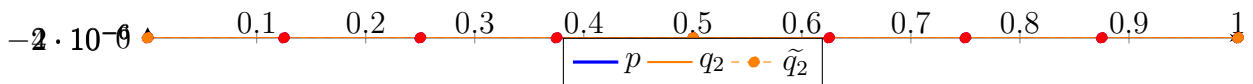
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.31266 \cdot 10^{-27} X^8 - 6.92683 \cdot 10^{-26} X^6 - 8.48183 \cdot 10^{-27} X^5 - 1.06023 \\
 &\quad \cdot 10^{-26} X^4 + 1.41364 \cdot 10^{-27} X^3 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\
 &= 4.52469 \cdot 10^{-14} B_{0,8}(X) + 7.18983 \cdot 10^{-14} B_{1,8}(X) + 9.85497 \cdot 10^{-14} B_{2,8}(X) \\
 &\quad + 1.25201 \cdot 10^{-13} B_{3,8}(X) + 1.51852 \cdot 10^{-13} B_{4,8}(X) + 1.78504 \cdot 10^{-13} B_{5,8}(X) \\
 &\quad + 2.05155 \cdot 10^{-13} B_{6,8}(X) + 2.31807 \cdot 10^{-13} B_{7,8}(X) + 2.58458 \cdot 10^{-13} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= 2.44862 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\
 &= 4.52469 \cdot 10^{-14} B_{0,2} + 1.51852 \cdot 10^{-13} B_{1,2} + 2.58458 \cdot 10^{-13} B_{2,2} \\
 \tilde{q}_2 &= -9.59093 \cdot 10^{-23} X^8 + 4.40717 \cdot 10^{-22} X^7 - 8.25115 \cdot 10^{-22} X^6 + 8.08254 \cdot 10^{-22} X^5 - 4.43405 \\
 &\quad \cdot 10^{-22} X^4 + 1.3589 \cdot 10^{-22} X^3 - 2.18259 \cdot 10^{-23} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\
 &= 4.52469 \cdot 10^{-14} B_{0,8} + 7.18983 \cdot 10^{-14} B_{1,8} + 9.85497 \cdot 10^{-14} B_{2,8} + 1.25201 \cdot 10^{-13} B_{3,8} + 1.51852 \\
 &\quad \cdot 10^{-13} B_{4,8} + 1.78504 \cdot 10^{-13} B_{5,8} + 2.05155 \cdot 10^{-13} B_{6,8} + 2.31807 \cdot 10^{-13} B_{7,8} + 2.58458 \cdot 10^{-13} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.40606 \cdot 10^{-25}$.

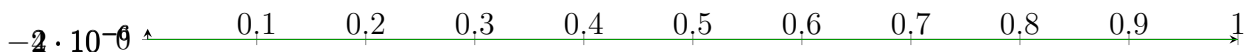
Bounding polynomials M and m :

$$\begin{aligned}
 M &= 2.42338 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\
 m &= 2.47387 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-8.79809 \cdot 10^{13}, -0.213542\} \quad N(m) = \{-8.61853 \cdot 10^{13}, -0.214286\}$$

Intersection intervals:

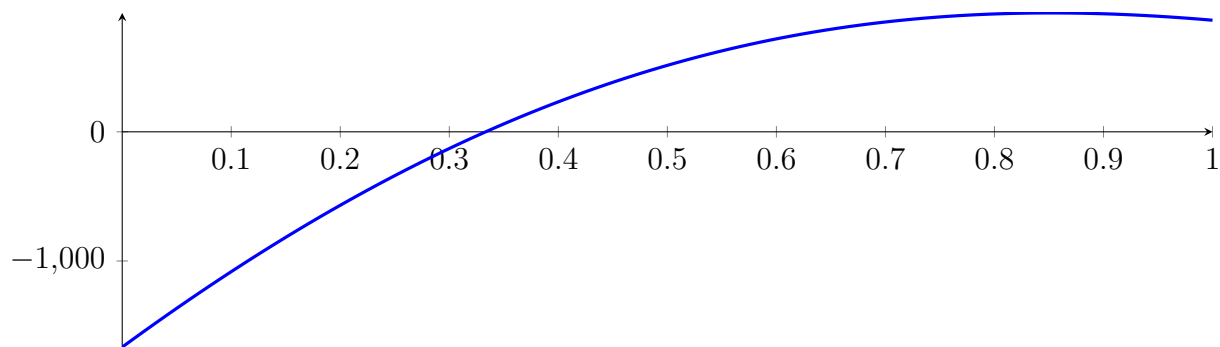


No intersection intervals with the x axis.

59.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

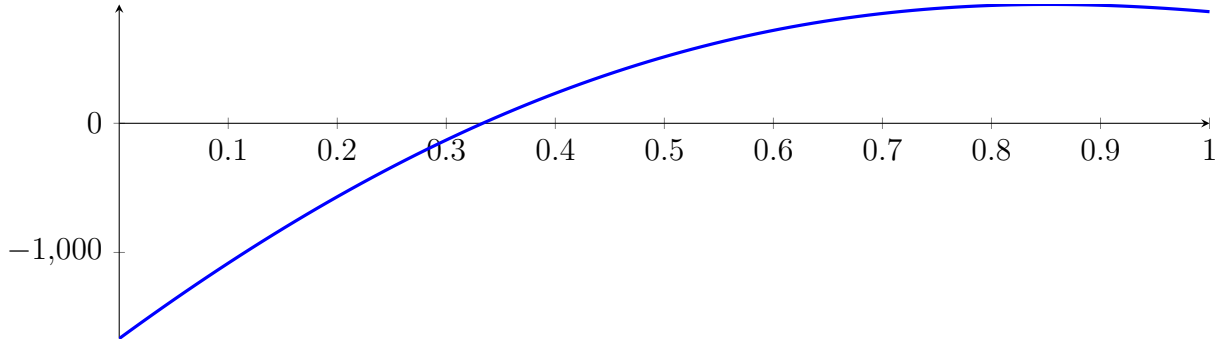
with precision $\varepsilon = 1 \cdot 10^{-64}$.

60 Running CubeClip on f_8 with epsilon 64

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

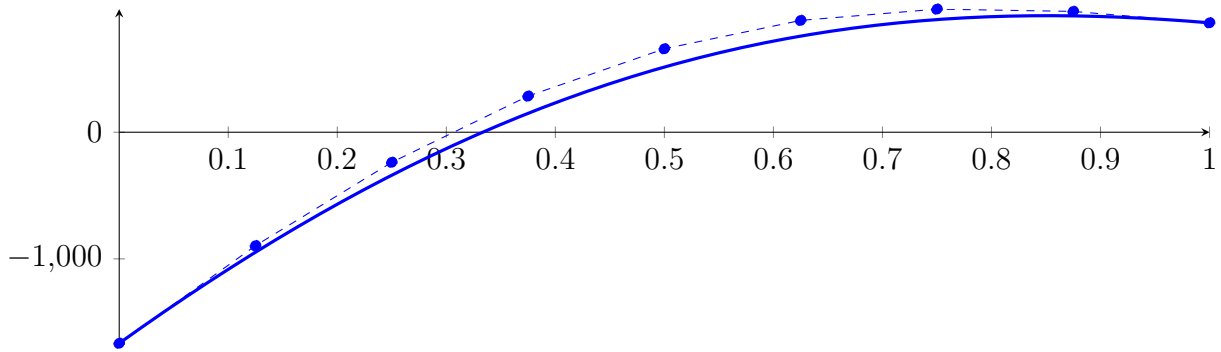
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



60.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

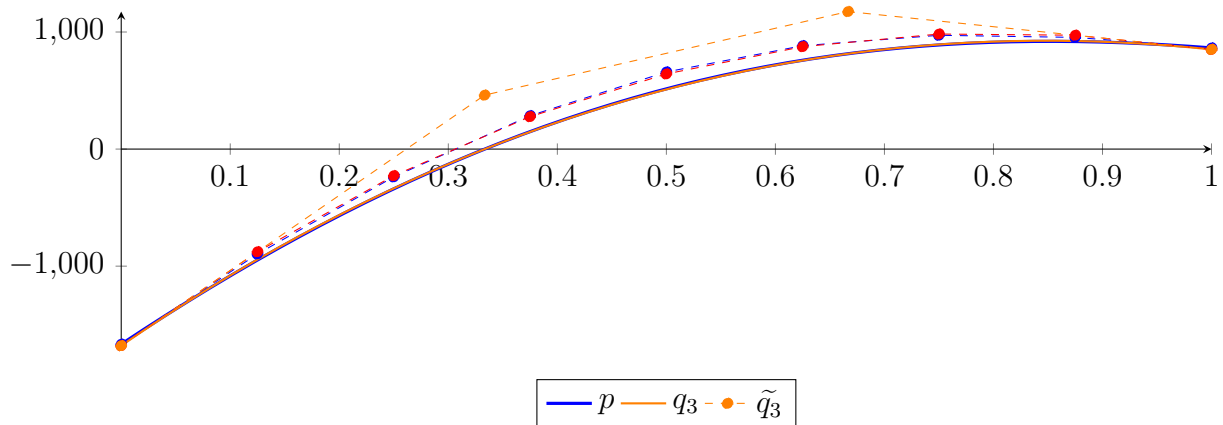
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 2.38117 \cdot 10^{-06}X^8 - 9.49957 \cdot 10^{-06}X^7 + 1.54757 \cdot 10^{-05}X^6 - 1.32609 \cdot 10^{-05}X^5 \\ &\quad + 6.41819 \cdot 10^{-06}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

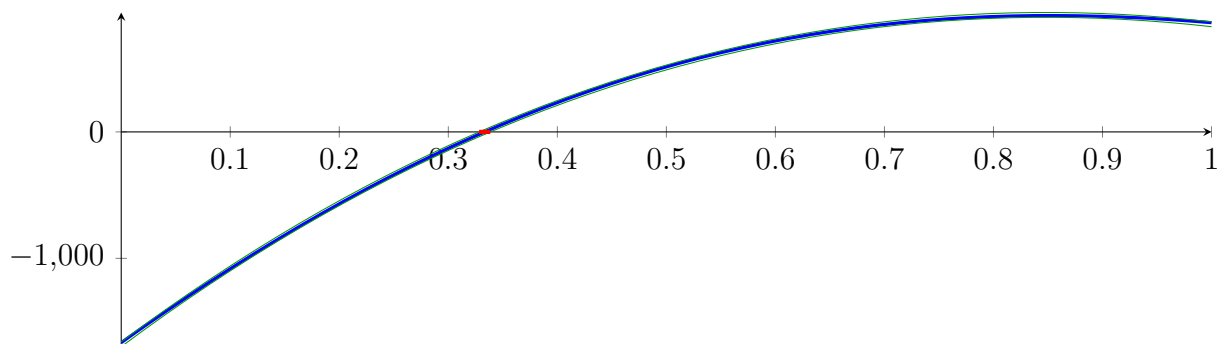
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

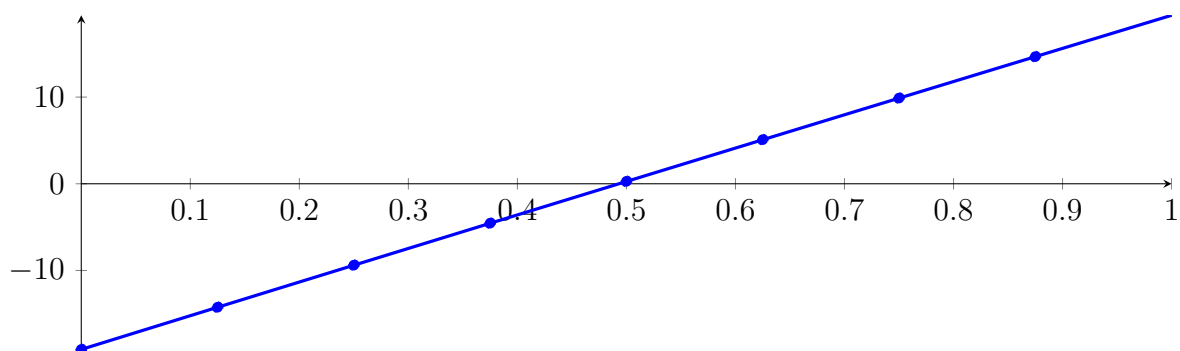
Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

60.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.27898 \cdot 10^{-13} X^8 - 1.13687 \cdot 10^{-13} X^7 - 8.23661 \cdot 10^{-11} X^6 + 8.72882 \cdot 10^{-09} X^5 \\ &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

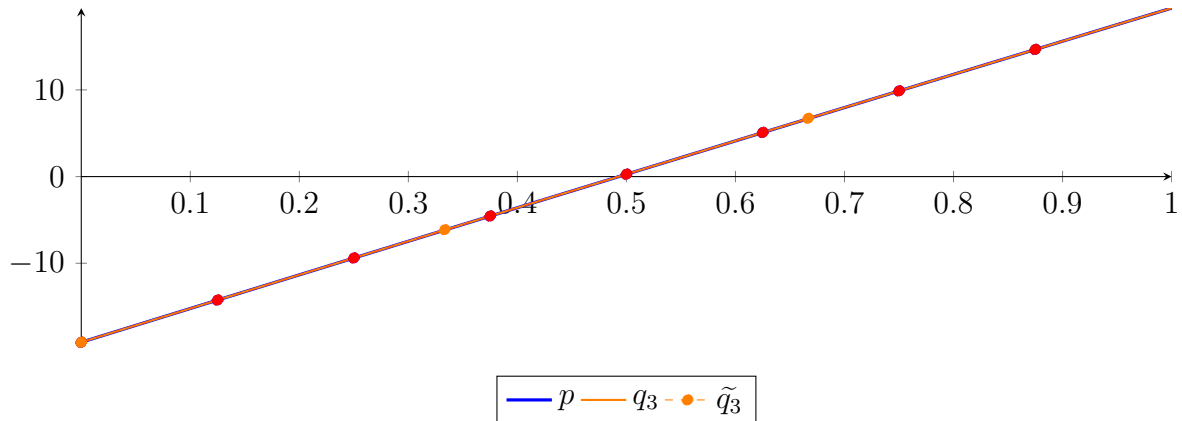
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = 2.3353 \cdot 10^{-08}X^8 - 9.31856 \cdot 10^{-08}X^7 + 1.51861 \cdot 10^{-07}X^6 - 1.30228 \cdot 10^{-07}X^5$$

$$+ 6.31618 \cdot 10^{-08}X^4 - 0.000240962X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16887 \cdot 10^{-07}$.

Bounding polynomials M and m :

$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

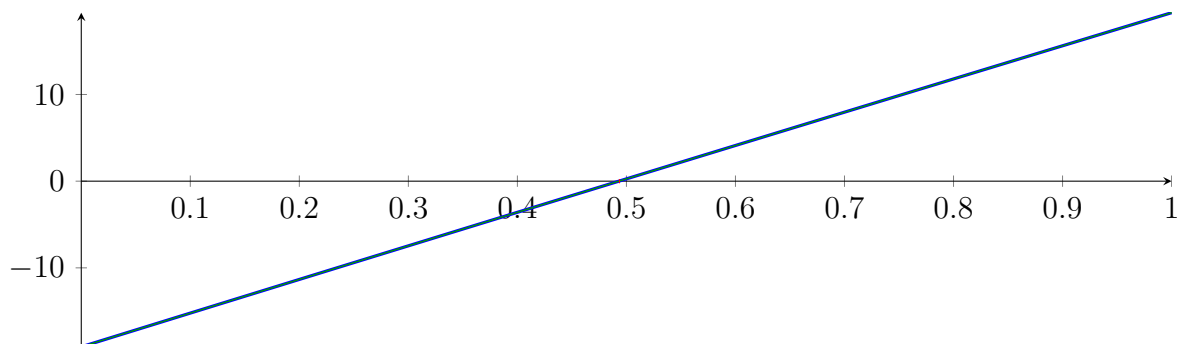
$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\}$$

$$N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

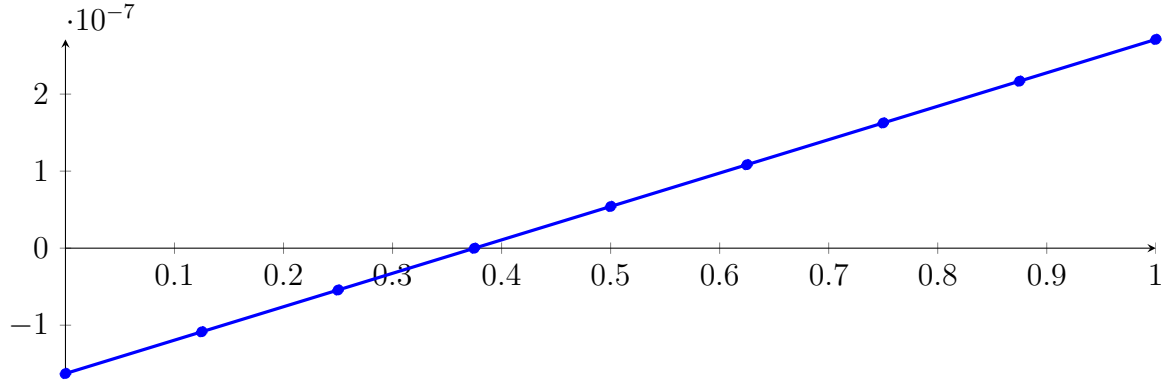
Longest intersection interval: $1.12517 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

60.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

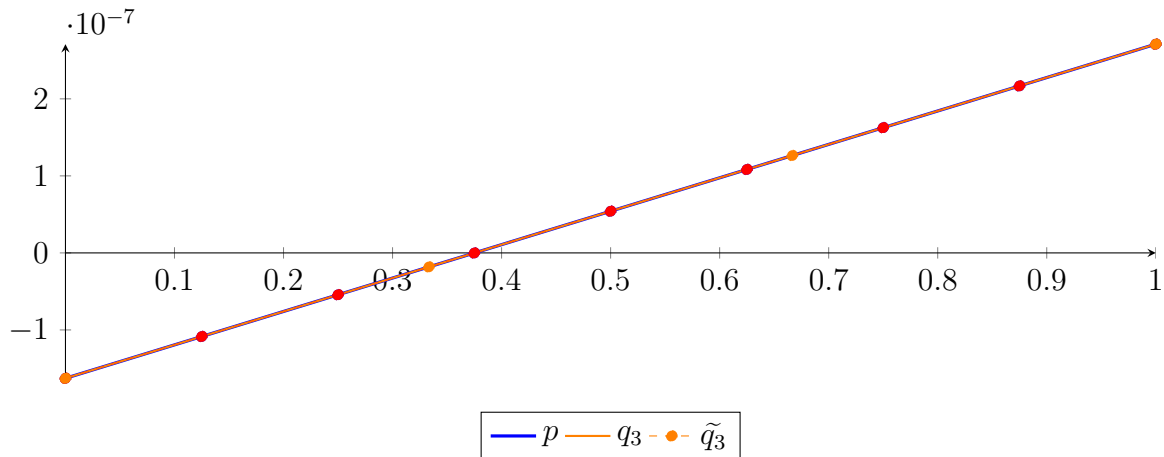
$$\begin{aligned}
 p &= -3.81165 \cdot 10^{-21} X^8 + 6.77626 \cdot 10^{-21} X^7 + 2.96462 \cdot 10^{-21} X^6 + 5.92923 \cdot 10^{-21} X^5 + 1.11173 \\
 &\quad \cdot 10^{-20} X^4 + 1.48231 \cdot 10^{-21} X^3 - 5.27494 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07} \\
 &= -1.6278 \cdot 10^{-07} B_{0,8}(X) - 1.08555 \cdot 10^{-07} B_{1,8}(X) - 5.43313 \cdot 10^{-08} B_{2,8}(X) \\
 &\quad - 1.07093 \cdot 10^{-10} B_{3,8}(X) + 5.41171 \cdot 10^{-08} B_{4,8}(X) + 1.08341 \cdot 10^{-07} B_{5,8}(X) \\
 &\quad + 1.62565 \cdot 10^{-07} B_{6,8}(X) + 2.1679 \cdot 10^{-07} B_{7,8}(X) + 2.71014 \cdot 10^{-07} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -1.90582 \cdot 10^{-21} X^3 - 5.2746 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07} \\
 &= -1.6278 \cdot 10^{-07} B_{0,3} - 1.81818 \cdot 10^{-08} B_{1,3} + 1.26416 \cdot 10^{-07} B_{2,3} + 2.71014 \cdot 10^{-07} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 1.50585 \cdot 10^{-16} X^8 - 5.82707 \cdot 10^{-16} X^7 + 9.15943 \cdot 10^{-16} X^6 - 7.54824 \cdot 10^{-16} X^5 + 3.52096 \\
 &\quad \cdot 10^{-16} X^4 - 9.31289 \cdot 10^{-17} X^3 - 3.98474 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07} \\
 &= -1.6278 \cdot 10^{-07} B_{0,8} - 1.08555 \cdot 10^{-07} B_{1,8} - 5.43313 \cdot 10^{-08} B_{2,8} - 1.07093 \cdot 10^{-10} B_{3,8} + 5.41171 \\
 &\quad \cdot 10^{-08} B_{4,8} + 1.08341 \cdot 10^{-07} B_{5,8} + 1.62565 \cdot 10^{-07} B_{6,8} + 2.1679 \cdot 10^{-07} B_{7,8} + 2.71014 \cdot 10^{-07} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 8.66435 \cdot 10^{-19}$.

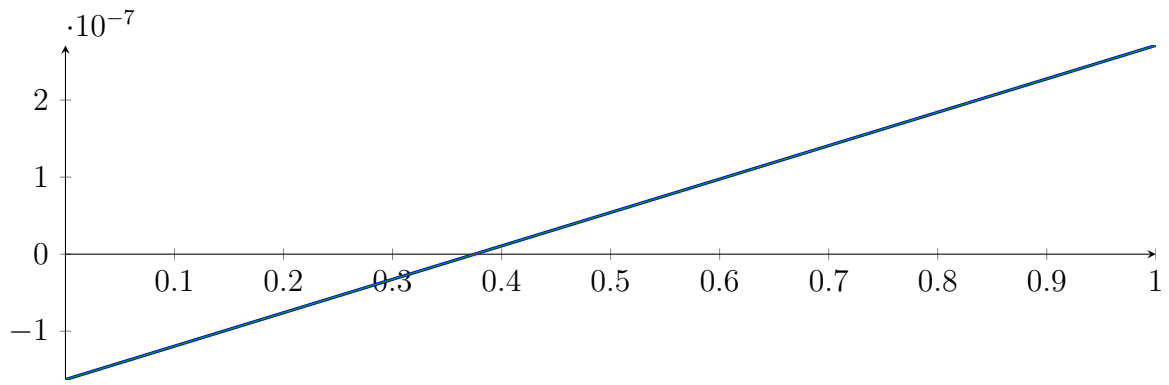
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -1.85288 \cdot 10^{-21} X^3 - 5.2746 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07} \\
 m &= -1.90582 \cdot 10^{-21} X^3 - 5.2746 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-1.49969 \cdot 10^7, 0.375247, 1.49696 \cdot 10^7\} \quad N(m) = \{-1.46018 \cdot 10^7, 0.375247, 1.45759 \cdot 10^7\}$$

Intersection intervals:



[0.375247, 0.375247]

Longest intersection interval: $7.69251 \cdot 10^{-09}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

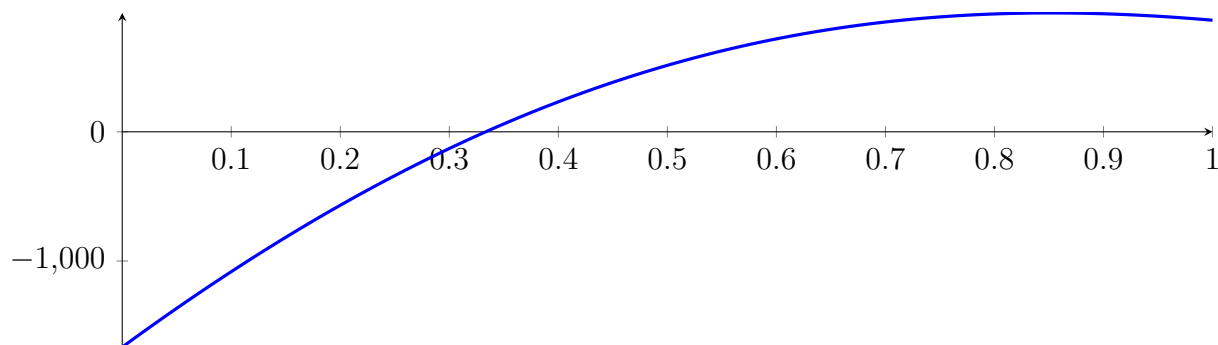
60.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

60.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

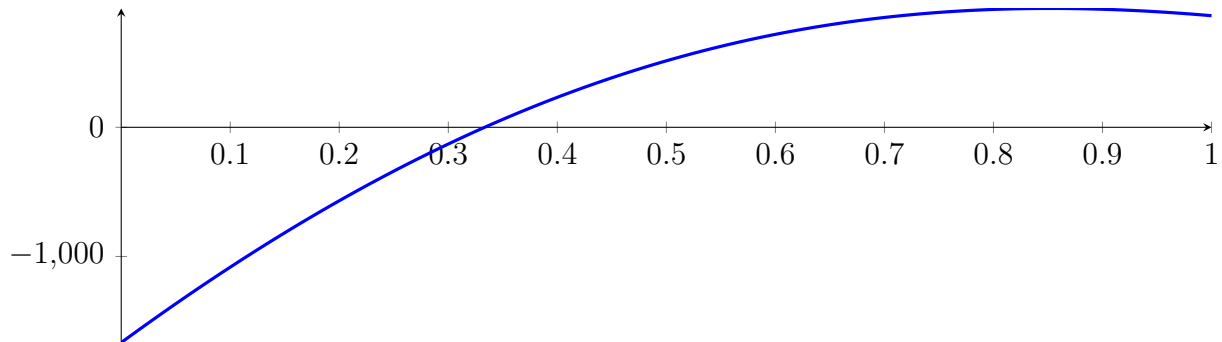
with precision $\varepsilon = 1 \cdot 10^{-64}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

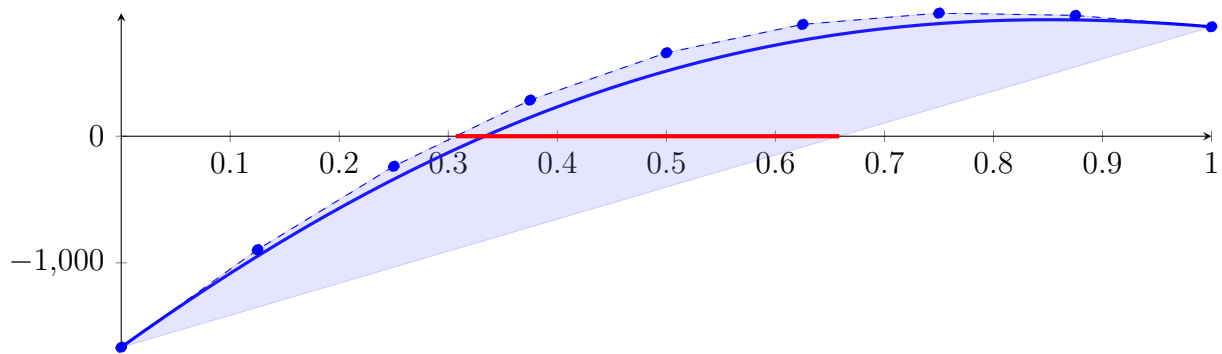
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



61.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

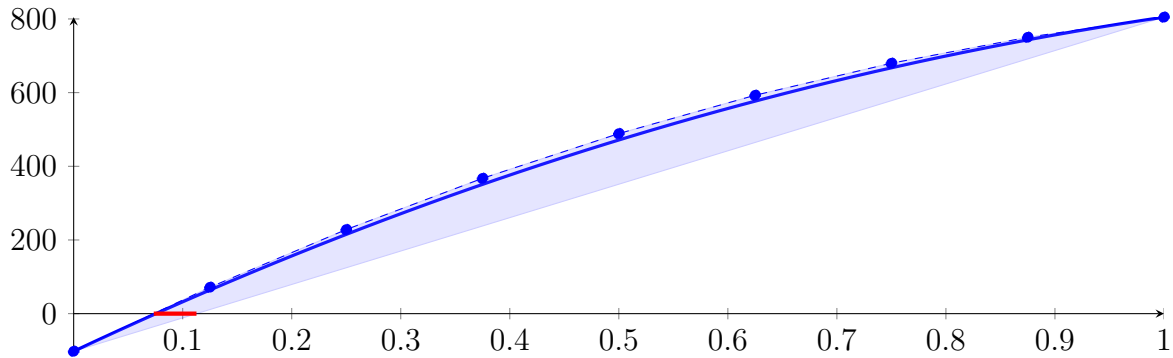
Longest intersection interval: 0.351792

\Rightarrow Selective recursion: interval 1: $[0.306796, 0.658588]$,

61.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

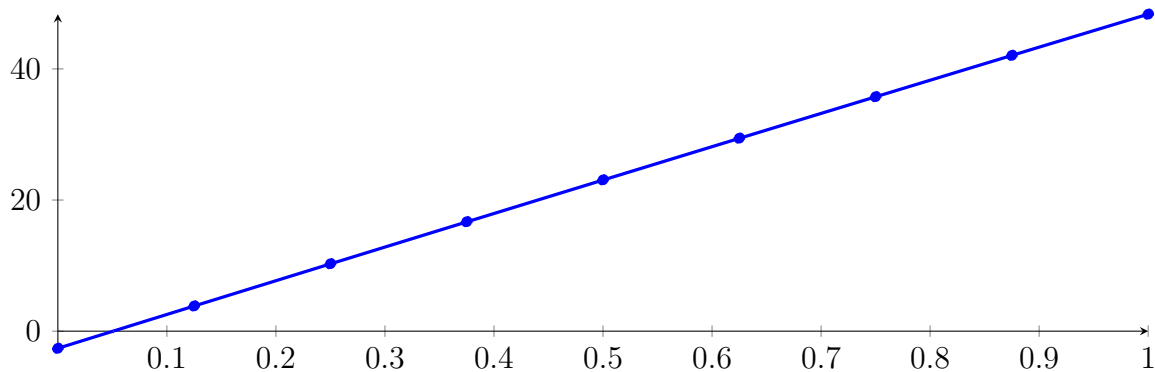
Longest intersection interval: 0.0391855

\implies Selective recursion: interval 1: [0.332635, 0.34642],

61.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -5.41789 \cdot 10^{-13}X^8 - 1.26477 \cdot 10^{-12}X^7 - 5.05786 \cdot 10^{-10}X^6 + 3.66765 \cdot 10^{-08}X^5 \\
 &\quad + 3.25466 \cdot 10^{-05}X^4 - 0.000586142X^3 - 0.747315X^2 + 51.7118X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

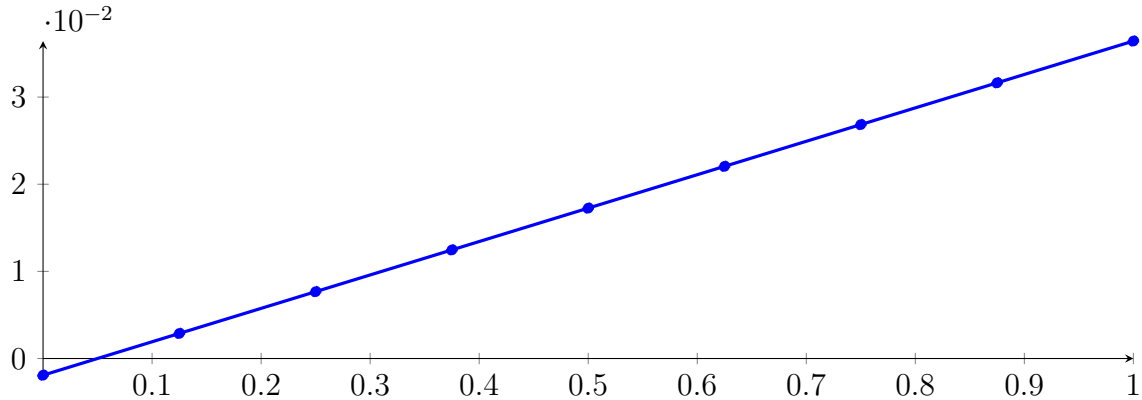
Longest intersection interval: 0.000742589

\implies Selective recursion: interval 1: [0.333333, 0.333343],

61.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -7.11237 \cdot 10^{-16} X^8 + 5.27356 \cdot 10^{-16} X^7 - 7.38298 \cdot 10^{-15} X^6 + 1.06859 \cdot 10^{-15} X^5 \\
 &\quad - 1.09288 \cdot 10^{-15} X^4 - 2.37227 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\
 &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\
 &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\
 &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

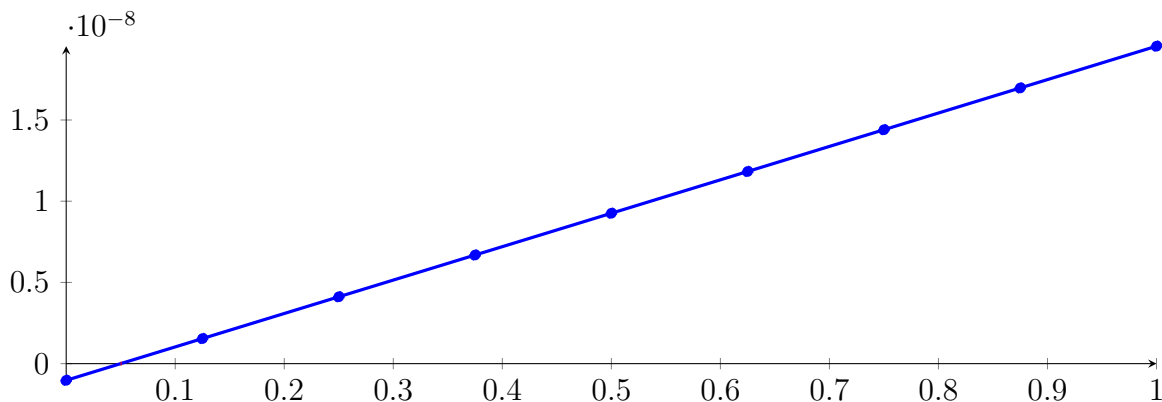
Longest intersection interval: $5.36469 \cdot 10^{-07}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

61.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.09366 \cdot 10^{-22} X^8 + 4.49986 \cdot 10^{-22} X^7 - 4.03002 \cdot 10^{-21} X^6 + 3.70577 \cdot 10^{-22} X^5 - 3.47416 \\
 &\quad \cdot 10^{-22} X^4 + 9.26442 \cdot 10^{-23} X^3 - 1.18608 \cdot 10^{-19} X^2 + 2.05706 \cdot 10^{-08} X - 1.02667 \cdot 10^{-09} \\
 &= -1.02667 \cdot 10^{-09} B_{0,8}(X) + 1.54465 \cdot 10^{-09} B_{1,8}(X) + 4.11597 \cdot 10^{-09} B_{2,8}(X) \\
 &\quad + 6.68729 \cdot 10^{-09} B_{3,8}(X) + 9.25862 \cdot 10^{-09} B_{4,8}(X) + 1.18299 \cdot 10^{-08} B_{5,8}(X) \\
 &\quad + 1.44013 \cdot 10^{-08} B_{6,8}(X) + 1.69726 \cdot 10^{-08} B_{7,8}(X) + 1.95439 \cdot 10^{-08} B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87728 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

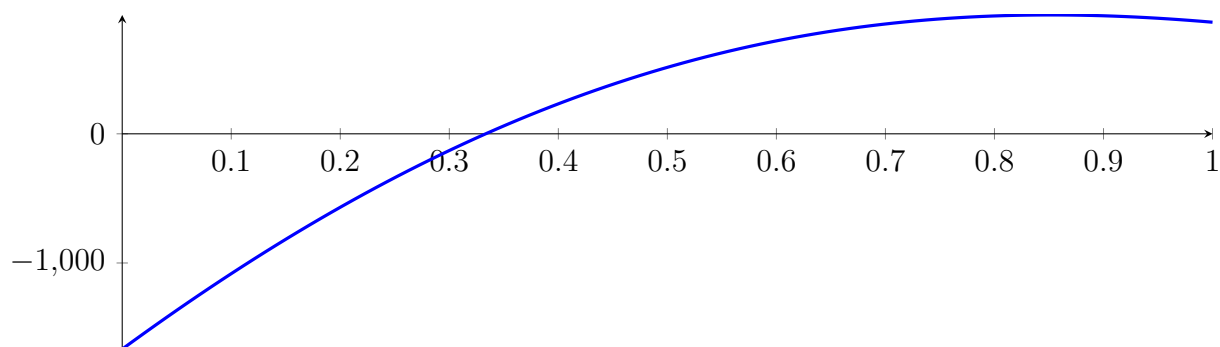
61.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

61.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

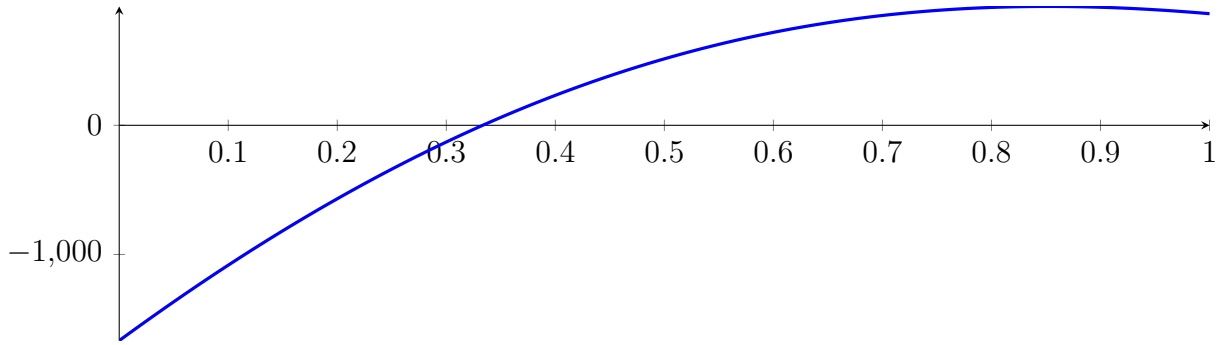
with precision $\varepsilon = 1 \cdot 10^{-128}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

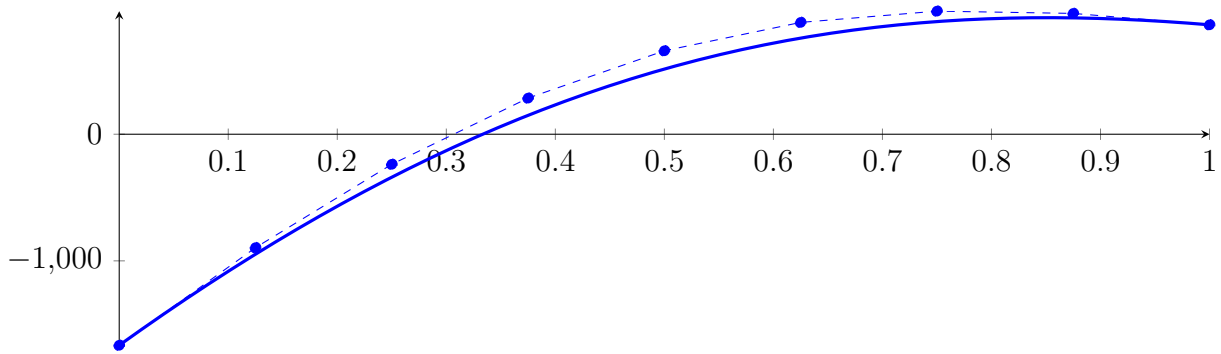
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



62.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

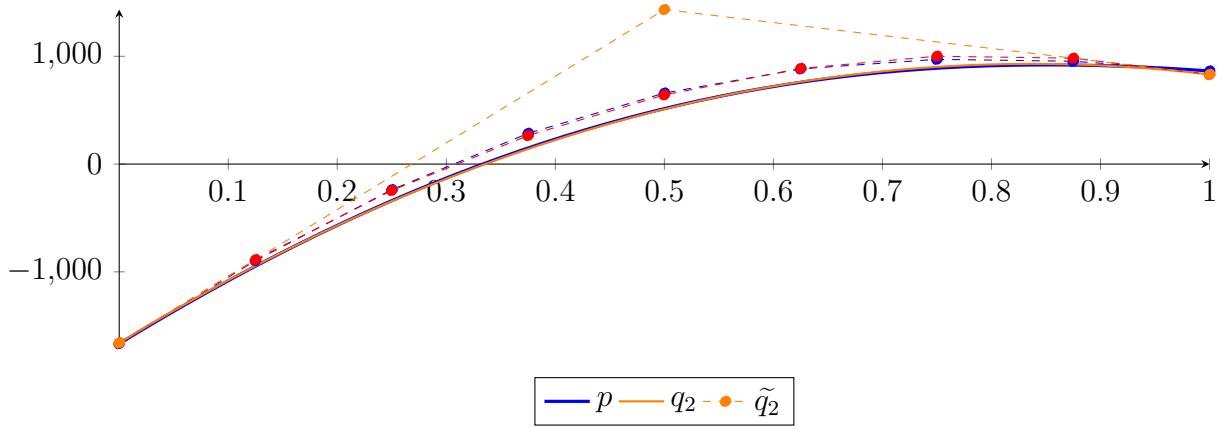
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -3.91297 \cdot 10^{-06}X^8 + 1.58774 \cdot 10^{-05}X^7 - 2.63335 \cdot 10^{-05}X^6 + 2.29285 \cdot 10^{-05}X^5 \\ &\quad - 1.11931 \cdot 10^{-05}X^4 + 3.0313 \cdot 10^{-06}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

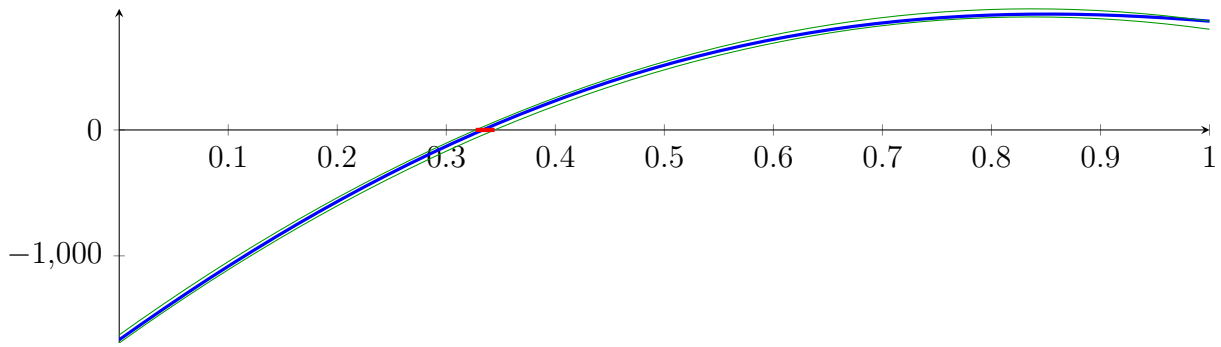
$$M = -3695.78X^2 + 6187.64X - 1627.86$$

$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\} \quad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

Longest intersection interval: 0.0173372

\implies Selective recursion: interval 1: $[0.326917, 0.344255]$,

62.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

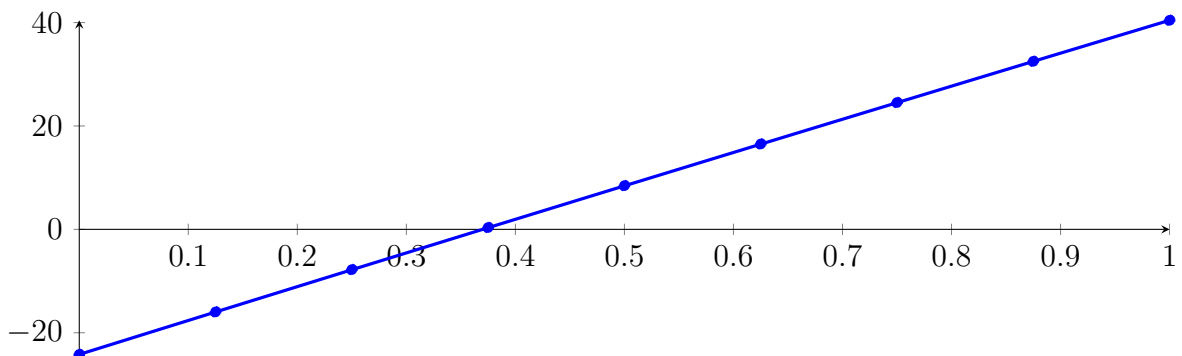
Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -5.40012 \cdot 10^{-13} X^8 - 7.38964 \cdot 10^{-12} X^7 - 1.94416 \cdot 10^{-09} X^6 + 1.19265 \cdot 10^{-07} X^5$$

$$+ 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945$$

$$= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X)$$

$$+ 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X)$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

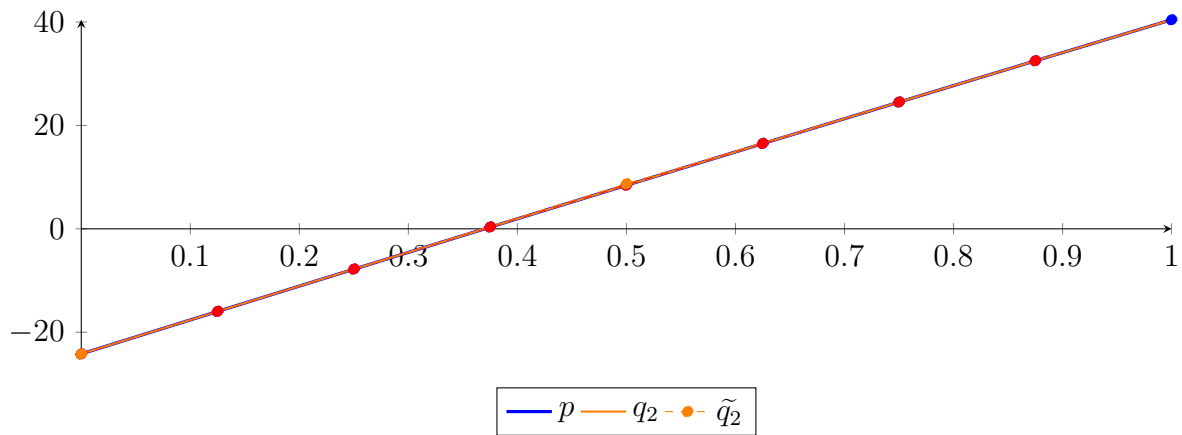
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = 1.00159 \cdot 10^{-08} X^8 - 3.3372 \cdot 10^{-08} X^7 + 4.23875 \cdot 10^{-08} X^6 - 2.49721 \cdot 10^{-08} X^5$$

$$+ 6.08793 \cdot 10^{-09} X^4 + 1.46429 \cdot 10^{-10} X^3 - 1.18261 X^2 + 65.8162 X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

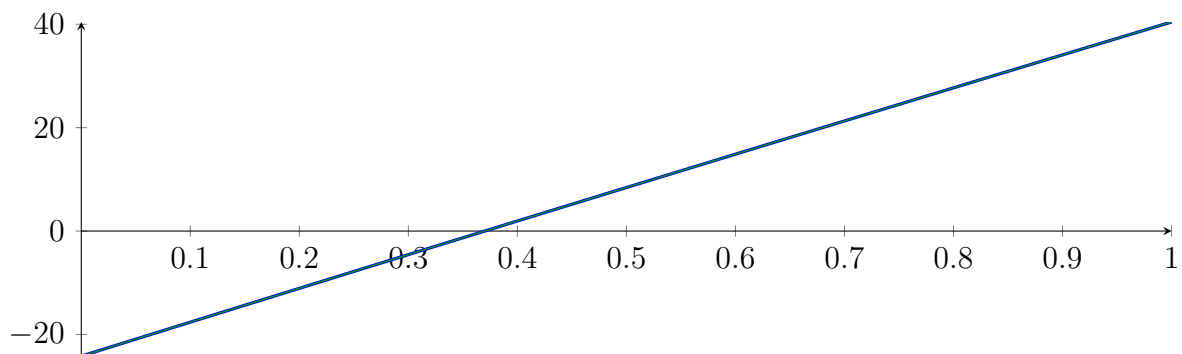
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

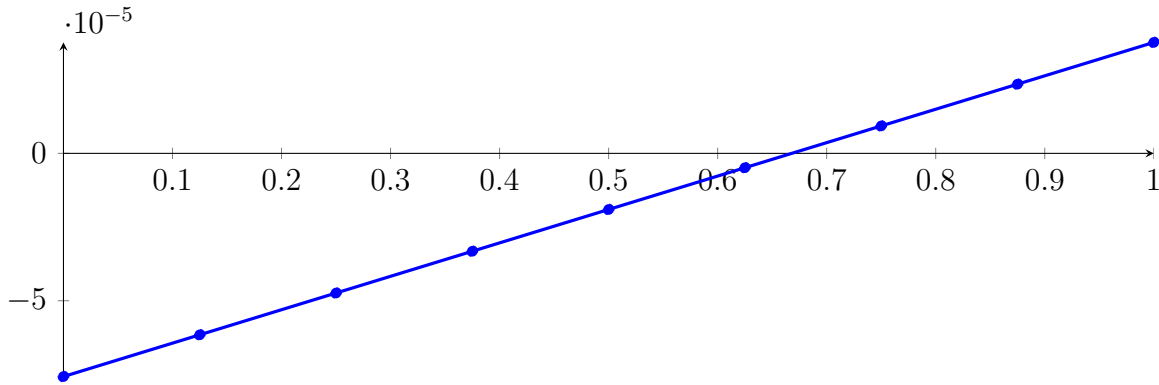
Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

62.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

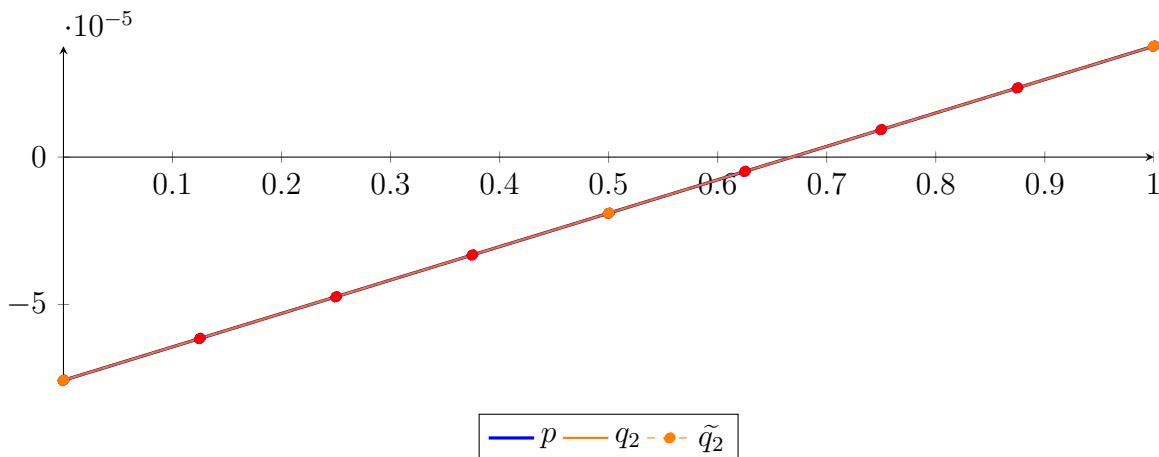
$$\begin{aligned}
 p &= -2.1684 \cdot 10^{-19} X^8 - 4.33681 \cdot 10^{-19} X^7 + 2.12504 \cdot 10^{-17} X^6 - 1.51788 \cdot 10^{-18} X^5 \\
 &\quad + 7.58942 \cdot 10^{-18} X^4 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) \\
 &\quad - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) \\
 &\quad + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 5.62292 \cdot 10^{-14} X^8 - 2.22643 \cdot 10^{-13} X^7 + 3.60043 \cdot 10^{-13} X^6 - 3.05846 \cdot 10^{-13} X^5 + 1.46182 \\
 &\quad \cdot 10^{-13} X^4 - 3.90612 \cdot 10^{-14} X^3 - 3.59793 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \\
 &\quad \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.98887 \cdot 10^{-16}$.

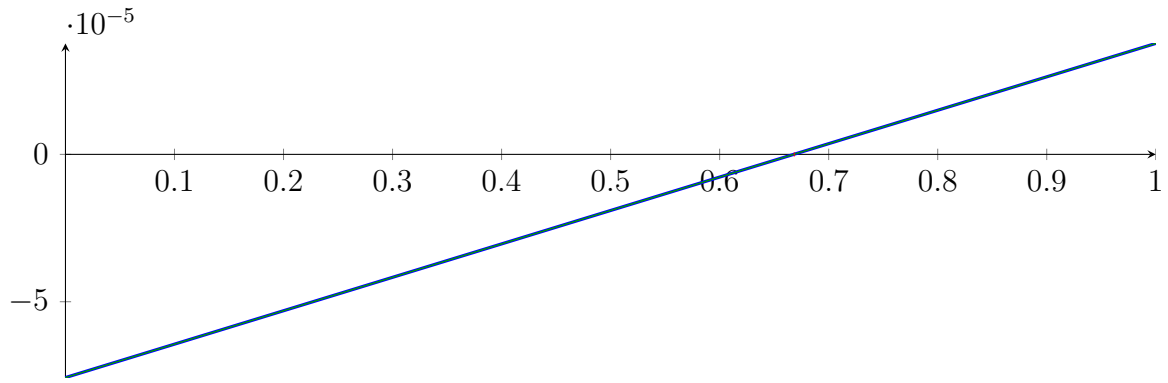
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 m &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.667955, 3.14644 \cdot 10^7\} \quad N(m) = \{0.667955, 3.14644 \cdot 10^7\}$$

Intersection intervals:



[0.667955, 0.667955]

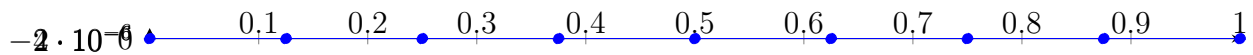
Longest intersection interval: $1.88052 \cdot 10^{-09}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

62.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

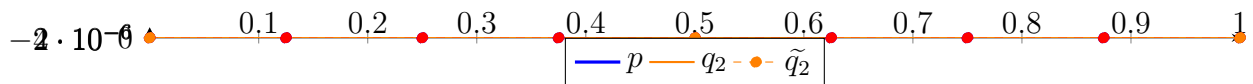
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.31266 \cdot 10^{-27} X^8 - 6.92683 \cdot 10^{-26} X^6 - 8.48183 \cdot 10^{-27} X^5 - 1.06023 \\
 &\quad \cdot 10^{-26} X^4 + 1.41364 \cdot 10^{-27} X^3 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\
 &= 4.52469 \cdot 10^{-14} B_{0,8}(X) + 7.18983 \cdot 10^{-14} B_{1,8}(X) + 9.85497 \cdot 10^{-14} B_{2,8}(X) \\
 &\quad + 1.25201 \cdot 10^{-13} B_{3,8}(X) + 1.51852 \cdot 10^{-13} B_{4,8}(X) + 1.78504 \cdot 10^{-13} B_{5,8}(X) \\
 &\quad + 2.05155 \cdot 10^{-13} B_{6,8}(X) + 2.31807 \cdot 10^{-13} B_{7,8}(X) + 2.58458 \cdot 10^{-13} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= 2.44862 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\
 &= 4.52469 \cdot 10^{-14} B_{0,2} + 1.51852 \cdot 10^{-13} B_{1,2} + 2.58458 \cdot 10^{-13} B_{2,2} \\
 \tilde{q}_2 &= -9.59093 \cdot 10^{-23} X^8 + 4.40717 \cdot 10^{-22} X^7 - 8.25115 \cdot 10^{-22} X^6 + 8.08254 \cdot 10^{-22} X^5 - 4.43405 \\
 &\quad \cdot 10^{-22} X^4 + 1.3589 \cdot 10^{-22} X^3 - 2.18259 \cdot 10^{-23} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\
 &= 4.52469 \cdot 10^{-14} B_{0,8} + 7.18983 \cdot 10^{-14} B_{1,8} + 9.85497 \cdot 10^{-14} B_{2,8} + 1.25201 \cdot 10^{-13} B_{3,8} + 1.51852 \\
 &\quad \cdot 10^{-13} B_{4,8} + 1.78504 \cdot 10^{-13} B_{5,8} + 2.05155 \cdot 10^{-13} B_{6,8} + 2.31807 \cdot 10^{-13} B_{7,8} + 2.58458 \cdot 10^{-13} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.40606 \cdot 10^{-25}$.

Bounding polynomials M and m :

$$\begin{aligned}
 M &= 2.42338 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14} \\
 m &= 2.47387 \cdot 10^{-27} X^2 + 2.13211 \cdot 10^{-13} X + 4.52469 \cdot 10^{-14}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-8.79809 \cdot 10^{13}, -0.213542\} \quad N(m) = \{-8.61853 \cdot 10^{13}, -0.214286\}$$

Intersection intervals:

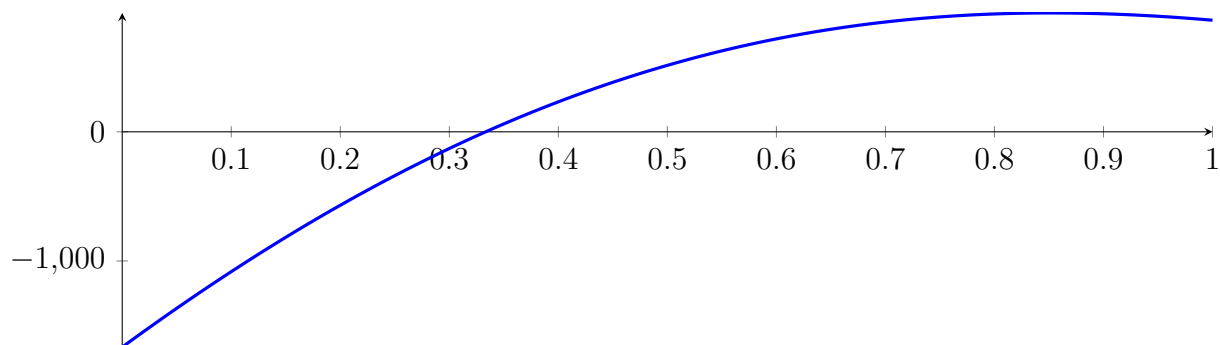


No intersection intervals with the x axis.

62.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

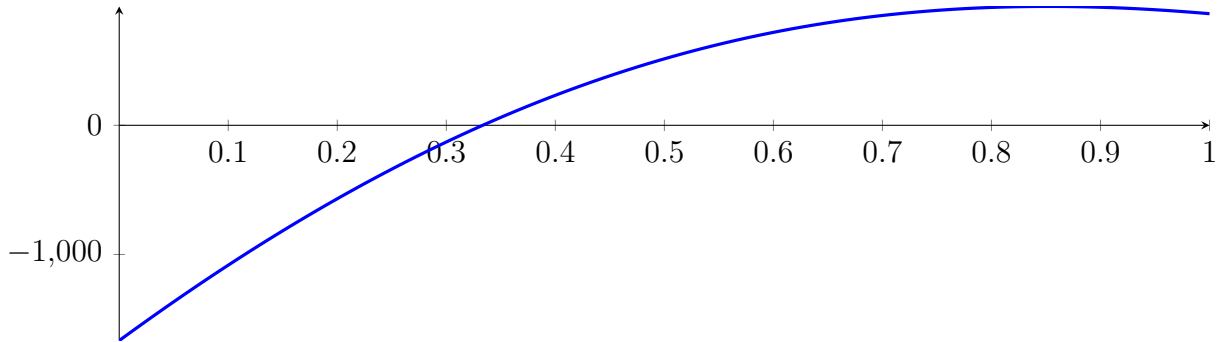
with precision $\varepsilon = 1 \cdot 10^{-128}$.

63 Running CubeClip on f_8 with epsilon 128

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

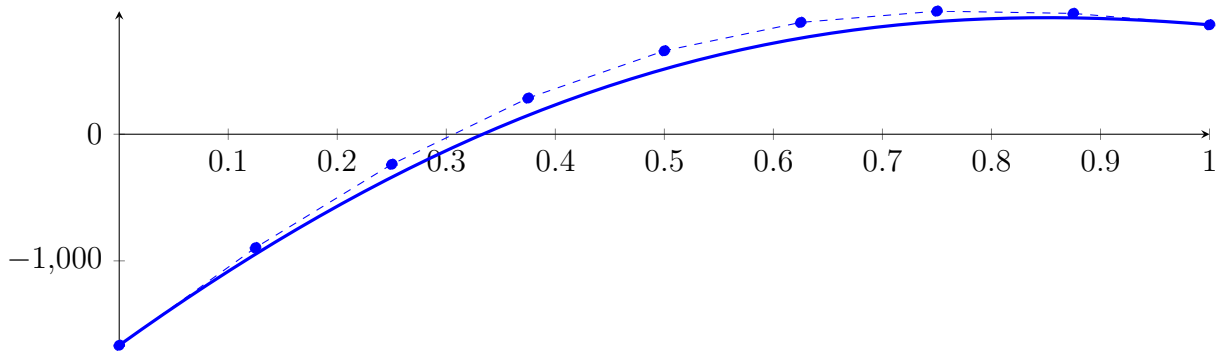
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



63.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

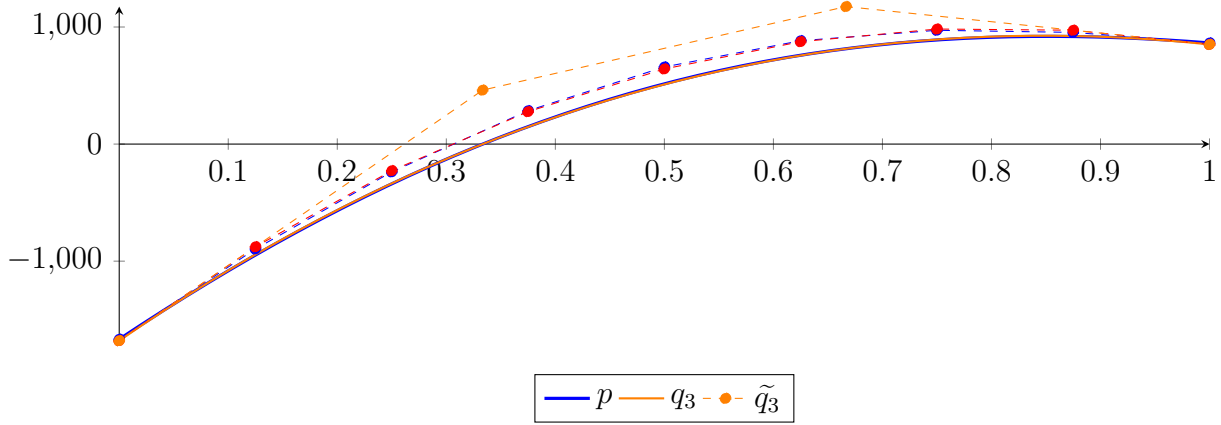
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 2.38117 \cdot 10^{-06}X^8 - 9.49957 \cdot 10^{-06}X^7 + 1.54757 \cdot 10^{-05}X^6 - 1.32609 \cdot 10^{-05}X^5 \\ &\quad + 6.41819 \cdot 10^{-06}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

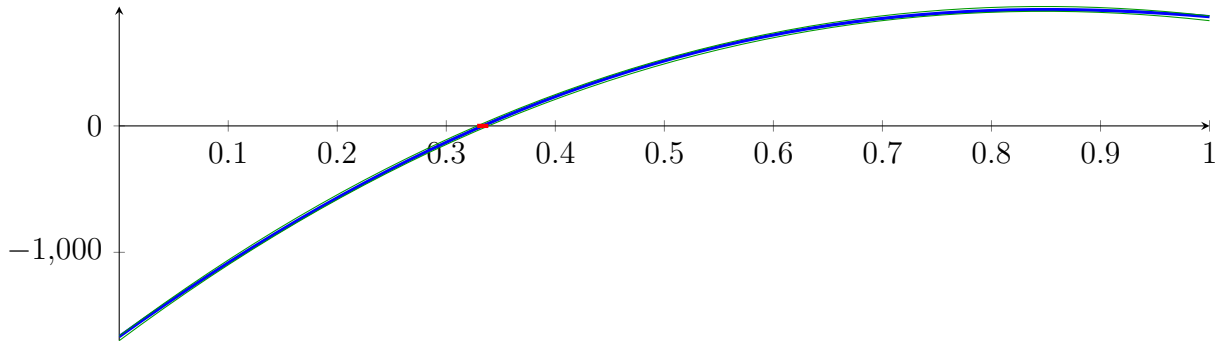
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

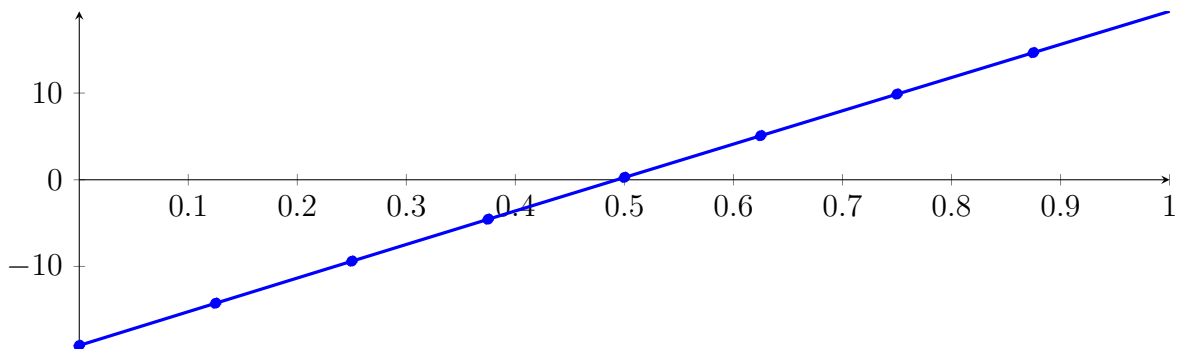
Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

63.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.27898 \cdot 10^{-13} X^8 - 1.13687 \cdot 10^{-13} X^7 - 8.23661 \cdot 10^{-11} X^6 + 8.72882 \cdot 10^{-09} X^5 \\ &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

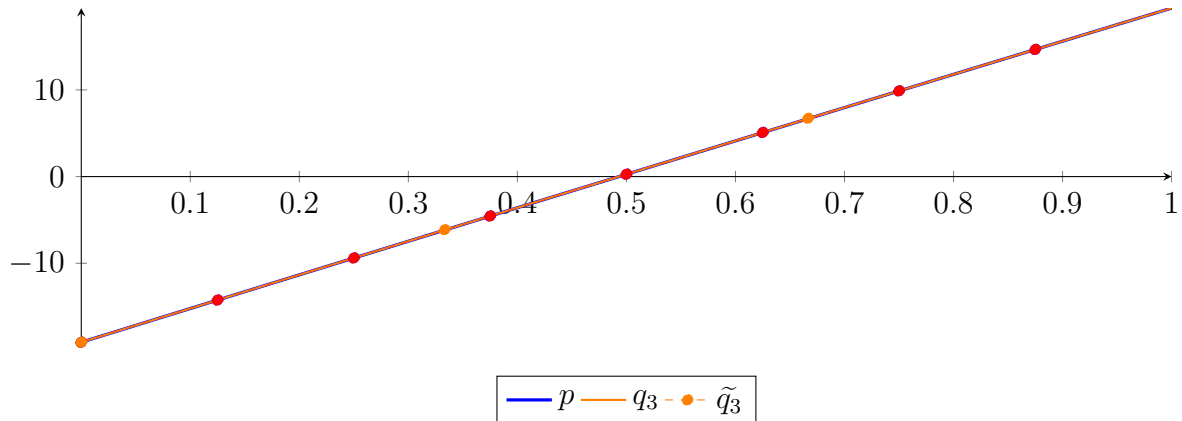
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = 2.3353 \cdot 10^{-08}X^8 - 9.31856 \cdot 10^{-08}X^7 + 1.51861 \cdot 10^{-07}X^6 - 1.30228 \cdot 10^{-07}X^5$$

$$+ 6.31618 \cdot 10^{-08}X^4 - 0.000240962X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16887 \cdot 10^{-07}$.

Bounding polynomials M and m :

$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

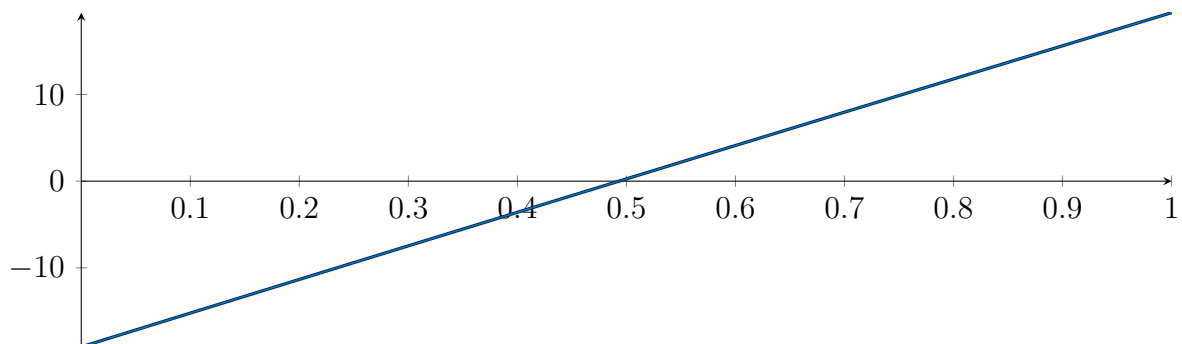
$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\}$$

$$N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

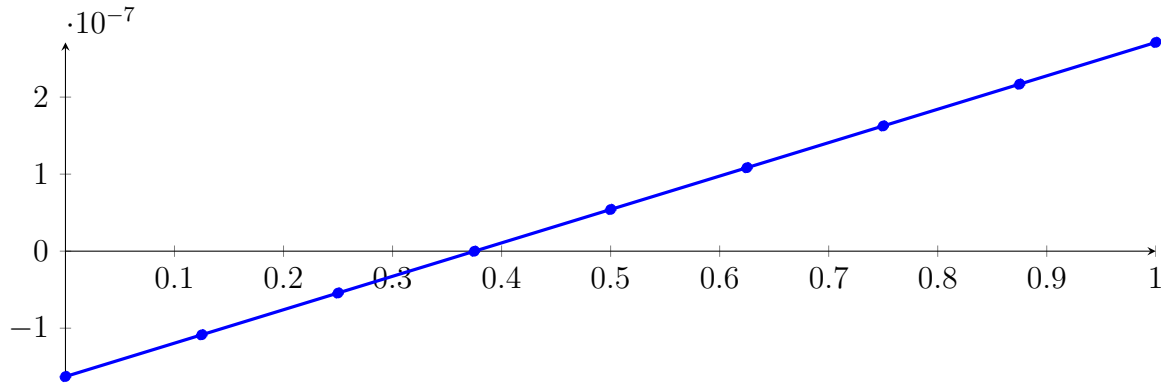
Longest intersection interval: $1.12517 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

63.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

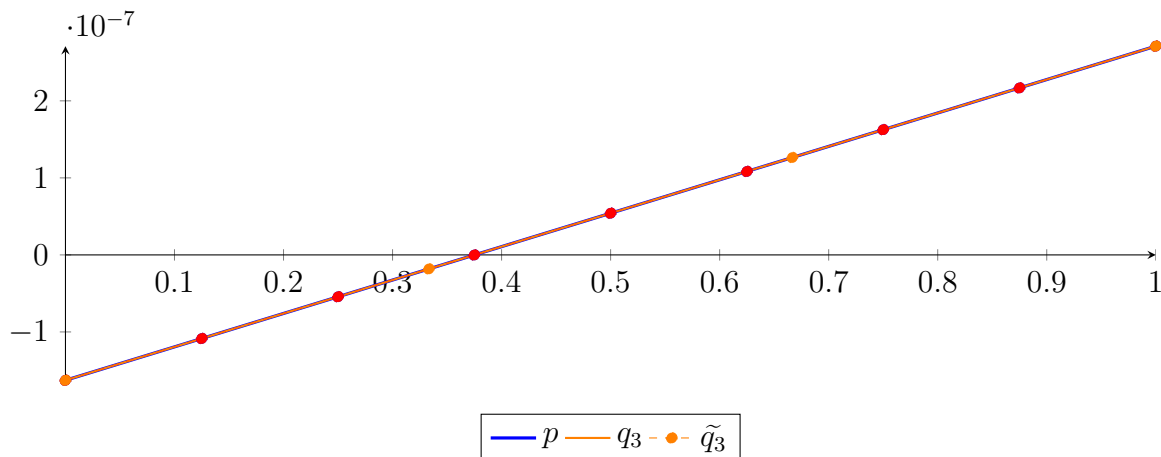
$$\begin{aligned}
 p &= -3.81165 \cdot 10^{-21} X^8 + 6.77626 \cdot 10^{-21} X^7 + 2.96462 \cdot 10^{-21} X^6 + 5.92923 \cdot 10^{-21} X^5 + 1.11173 \\
 &\quad \cdot 10^{-20} X^4 + 1.48231 \cdot 10^{-21} X^3 - 5.27494 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07} \\
 &= -1.6278 \cdot 10^{-07} B_{0,8}(X) - 1.08555 \cdot 10^{-07} B_{1,8}(X) - 5.43313 \cdot 10^{-08} B_{2,8}(X) \\
 &\quad - 1.07093 \cdot 10^{-10} B_{3,8}(X) + 5.41171 \cdot 10^{-08} B_{4,8}(X) + 1.08341 \cdot 10^{-07} B_{5,8}(X) \\
 &\quad + 1.62565 \cdot 10^{-07} B_{6,8}(X) + 2.1679 \cdot 10^{-07} B_{7,8}(X) + 2.71014 \cdot 10^{-07} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -1.90582 \cdot 10^{-21} X^3 - 5.2746 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07} \\
 &= -1.6278 \cdot 10^{-07} B_{0,3} - 1.81818 \cdot 10^{-08} B_{1,3} + 1.26416 \cdot 10^{-07} B_{2,3} + 2.71014 \cdot 10^{-07} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 1.50585 \cdot 10^{-16} X^8 - 5.82707 \cdot 10^{-16} X^7 + 9.15943 \cdot 10^{-16} X^6 - 7.54824 \cdot 10^{-16} X^5 + 3.52096 \\
 &\quad \cdot 10^{-16} X^4 - 9.31289 \cdot 10^{-17} X^3 - 3.98474 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07} \\
 &= -1.6278 \cdot 10^{-07} B_{0,8} - 1.08555 \cdot 10^{-07} B_{1,8} - 5.43313 \cdot 10^{-08} B_{2,8} - 1.07093 \cdot 10^{-10} B_{3,8} + 5.41171 \\
 &\quad \cdot 10^{-08} B_{4,8} + 1.08341 \cdot 10^{-07} B_{5,8} + 1.62565 \cdot 10^{-07} B_{6,8} + 2.1679 \cdot 10^{-07} B_{7,8} + 2.71014 \cdot 10^{-07} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 8.66435 \cdot 10^{-19}$.

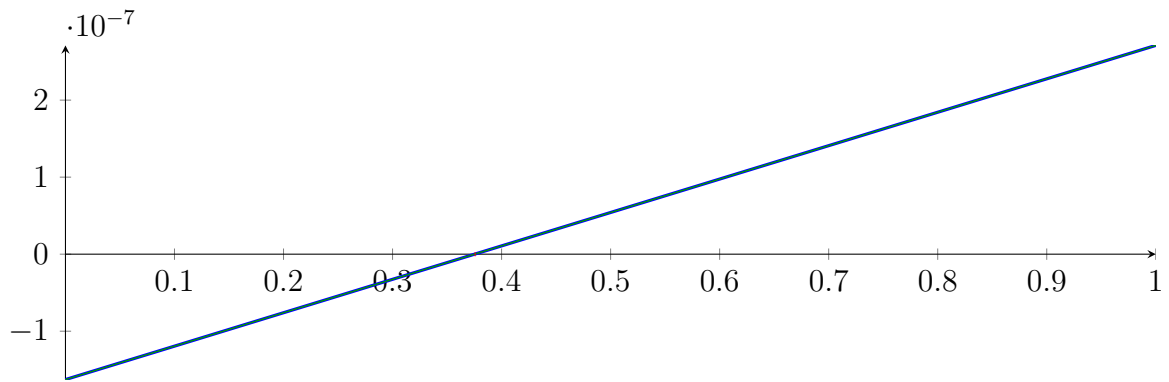
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -1.85288 \cdot 10^{-21} X^3 - 5.2746 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07} \\
 m &= -1.90582 \cdot 10^{-21} X^3 - 5.2746 \cdot 10^{-17} X^2 + 4.33793 \cdot 10^{-07} X - 1.6278 \cdot 10^{-07}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-1.49969 \cdot 10^7, 0.375247, 1.49696 \cdot 10^7\} \quad N(m) = \{-1.46018 \cdot 10^7, 0.375247, 1.45759 \cdot 10^7\}$$

Intersection intervals:



[0.375247, 0.375247]

Longest intersection interval: $7.69251 \cdot 10^{-09}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

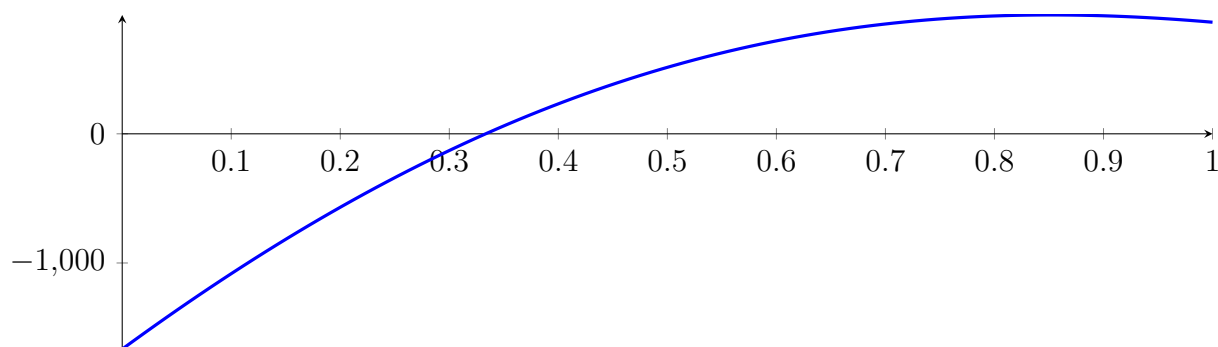
63.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

63.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

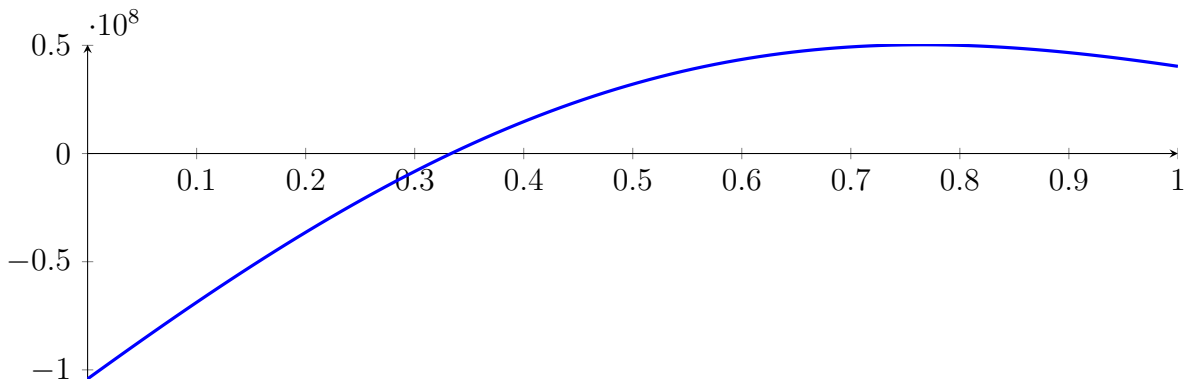
with precision $\varepsilon = 1 \cdot 10^{-128}$.

64 Running BezClip on f_{16} with epsilon 2

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

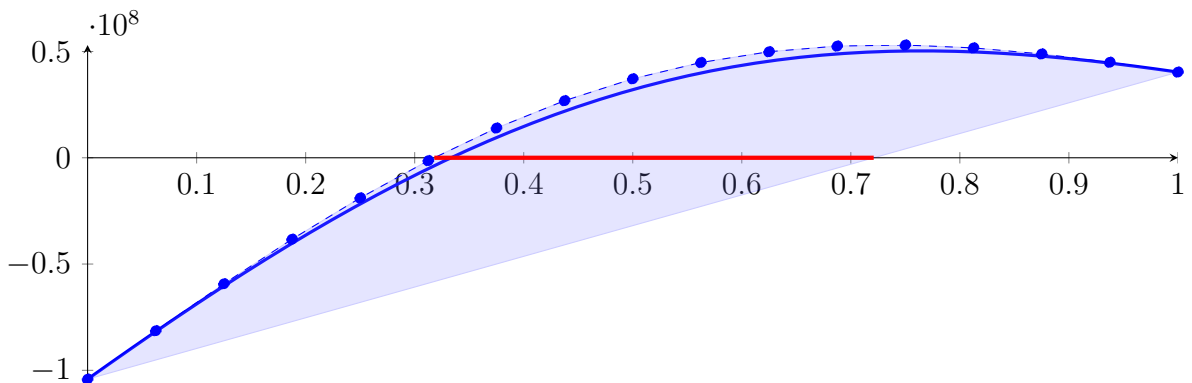
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



64.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

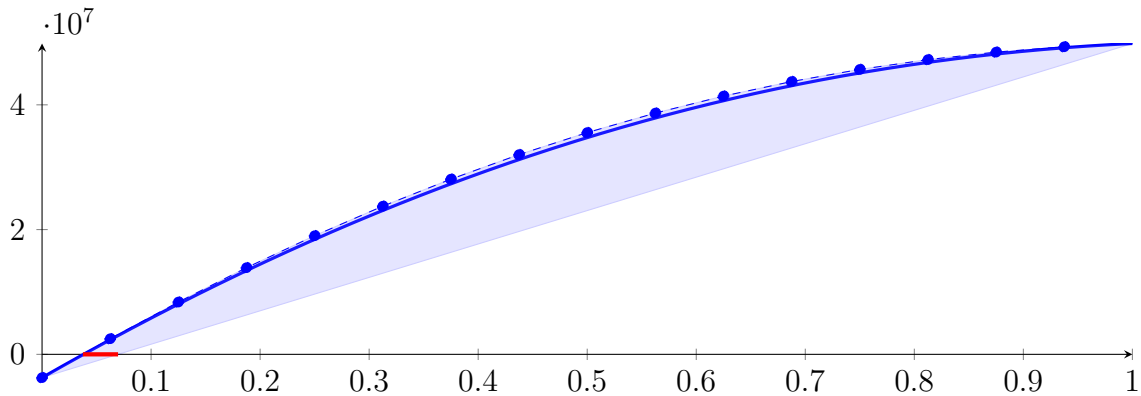
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

64.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.00483322X^{16} + 0.0186809X^{15} - 0.0194312X^{14} - 0.0738695X^{13} - 1.11673X^{12} \\ &\quad - 5.0471X^{11} + 36.3082X^{10} + 692.914X^9 + 1886.96X^8 - 25792X^7 - 149671X^6 + 492605X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

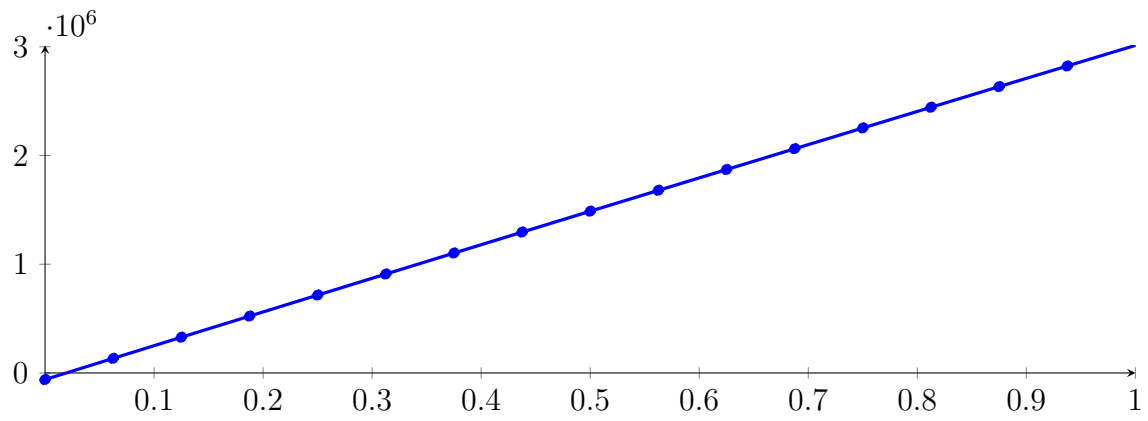
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

64.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.000205056X^{16} + 0.000776167X^{15} - 0.000863333X^{14} - 0.000217499X^{13} - 0.00809276X^{12} \\ &\quad - 5.09773 \cdot 10^{-05} X^{11} - 0.00564923X^{10} - 0.000162811X^9 - 0.000215376X^8 - 3.32948 \cdot 10^{-07} X^7 \\ &\quad - 0.000259866X^6 + 0.0161077X^5 + 4.36155X^4 - 234.216X^3 - 45622.2X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554

⇒ Selective recursion: interval 1: $[0.333333, 0.333337]$,

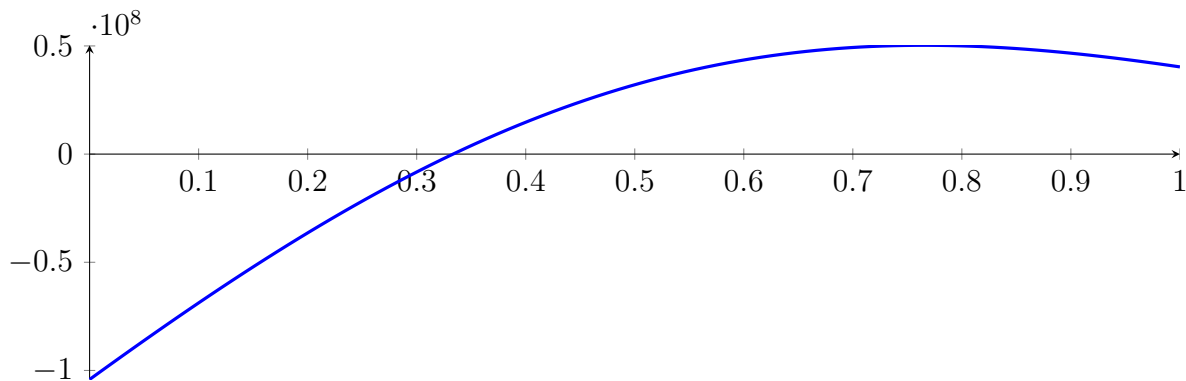
64.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Found root in interval $[0.333333, 0.333337]$ at recursion depth 4!

64.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333337]$$

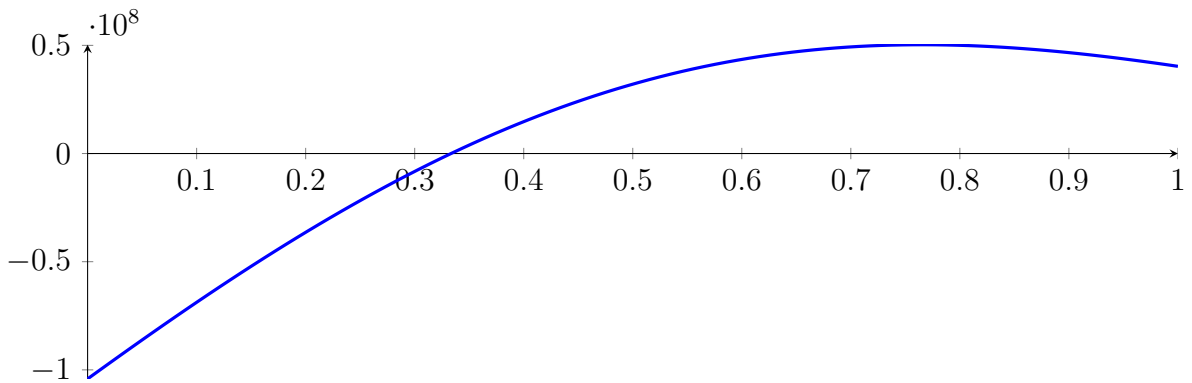
with precision $\varepsilon = 0.01$.

65 Running QuadClip on f_{16} with epsilon 2

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

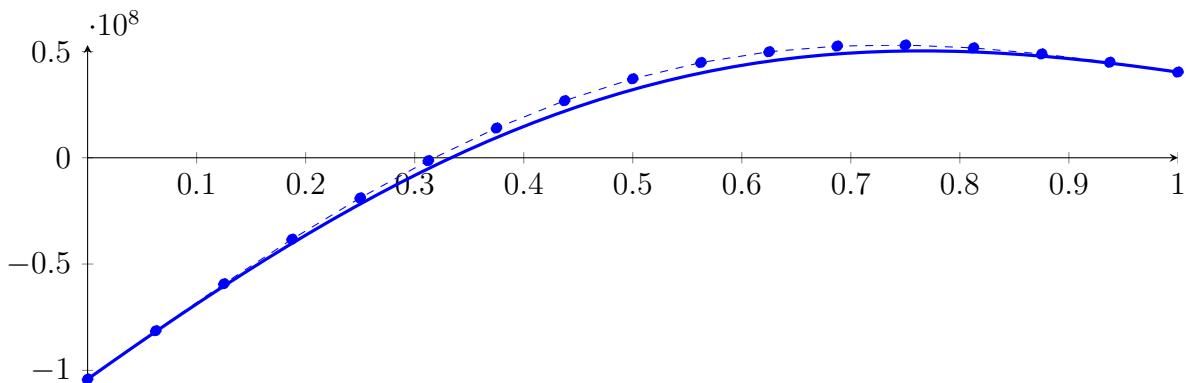
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



65.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = -1.41787 \cdot 10^6 X^{16} + 1.11761 \cdot 10^7 X^{15} - 3.98898 \cdot 10^7 X^{14} + 8.52437 \cdot 10^7 X^{13} - 1.21528 \cdot 10^8 X^{12}$$

$$+ 1.21946 \cdot 10^8 X^{11} - 8.86062 \cdot 10^7 X^{10} + 4.72904 \cdot 10^7 X^9 - 1.86355 \cdot 10^7 X^8 + 5.41059 \cdot 10^6 X^7 - 1.14487$$

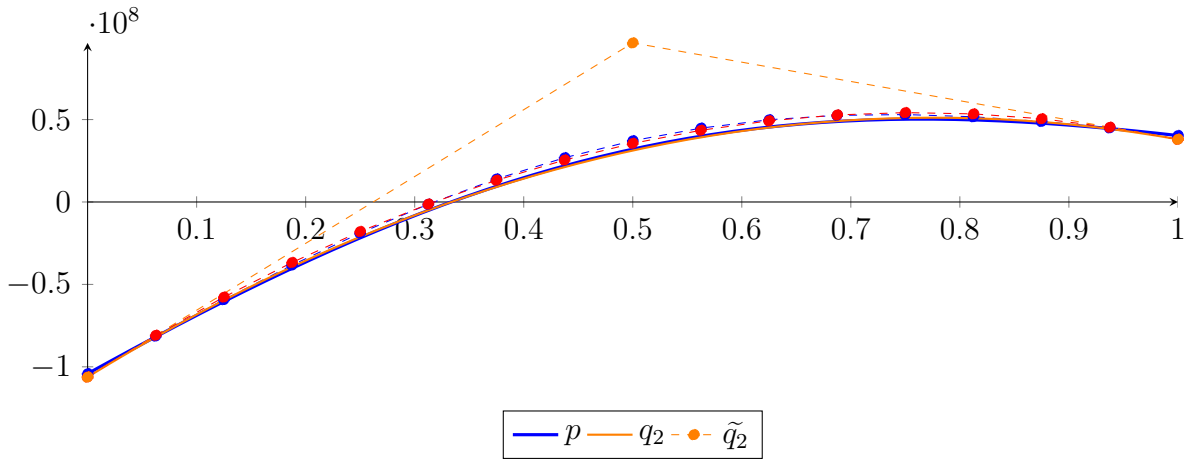
$$\cdot 10^6 X^6 + 172240 X^5 - 17636.6 X^4 + 1159.74 X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26923 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55691 \cdot 10^7 B_{8,16}$$

$$+ 4.34957 \cdot 10^7 B_{9,16} + 4.92455 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

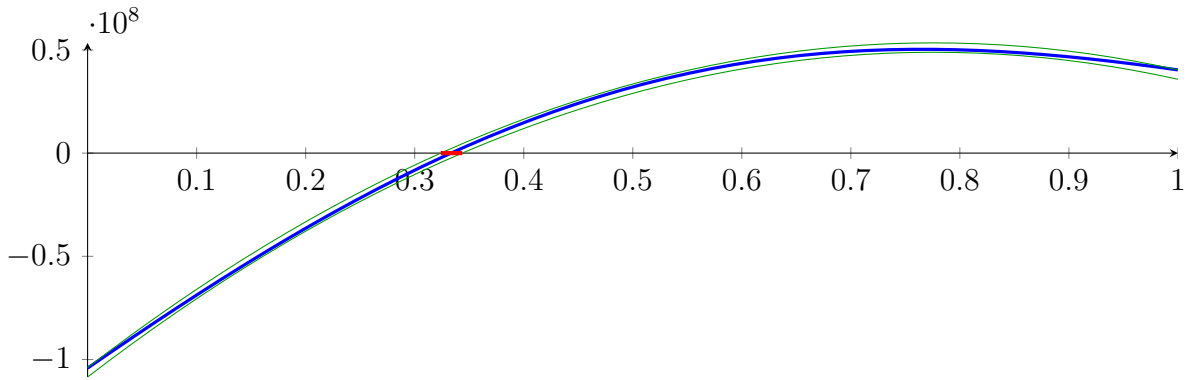
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

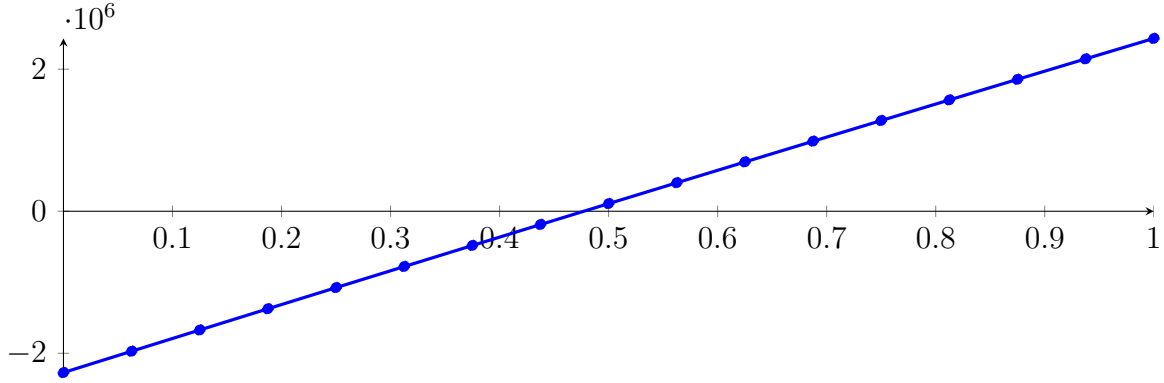
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

65.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

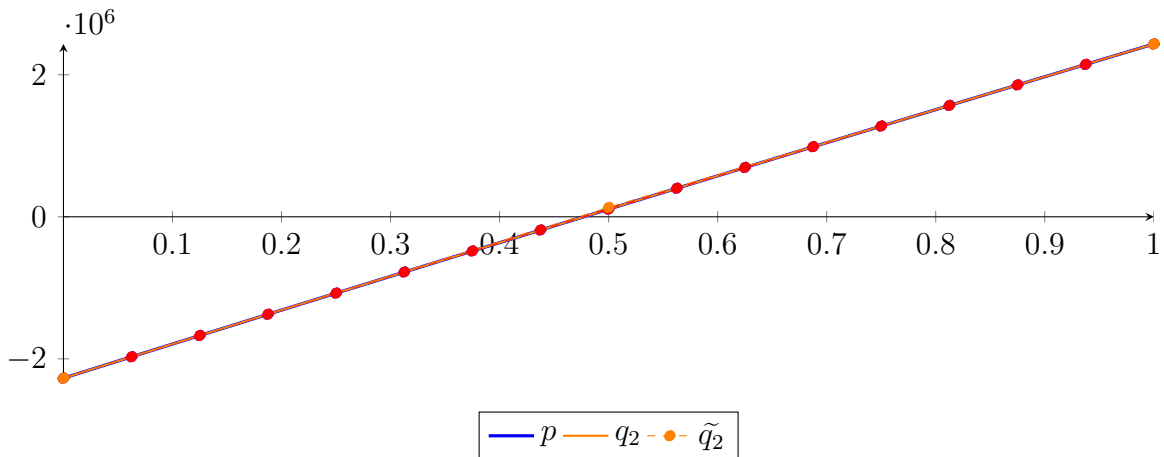
$$\begin{aligned}
 p &= -2.82438 \cdot 10^{-05} X^{16} + 2.90051 \cdot 10^{-05} X^{15} + 0.000231285 X^{14} + 0.000648014 X^{13} + 0.00318916 X^{12} \\
 &\quad + 0.00323204 X^{11} + 0.00460533 X^{10} + 0.00220012 X^9 - 0.000101882 X^8 + 5.32717 \cdot 10^{-06} X^7 \\
 &\quad - 0.00186824 X^6 + 0.132741 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 43556.7 X^{16} - 347904 X^{15} + 1.2616 \cdot 10^6 X^{14} - 2.74835 \cdot 10^6 X^{13} + 4.01042 \cdot 10^6 X^{12} - 4.13709 \\
 &\quad \cdot 10^6 X^{11} + 3.10251 \cdot 10^6 X^{10} - 1.71209 \cdot 10^6 X^9 + 695531 X^8 - 205965 X^7 + 43591.2 X^6 \\
 &\quad - 6402.94 X^5 + 625.678 X^4 - 37.485 X^3 - 104264 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481117 B_{6,16} - 185755 B_{7,16} + 108741 B_{8,16} \\
 &\quad + 402365 B_{9,16} + 695123 B_{10,16} + 987010 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

$$M = -104265 X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

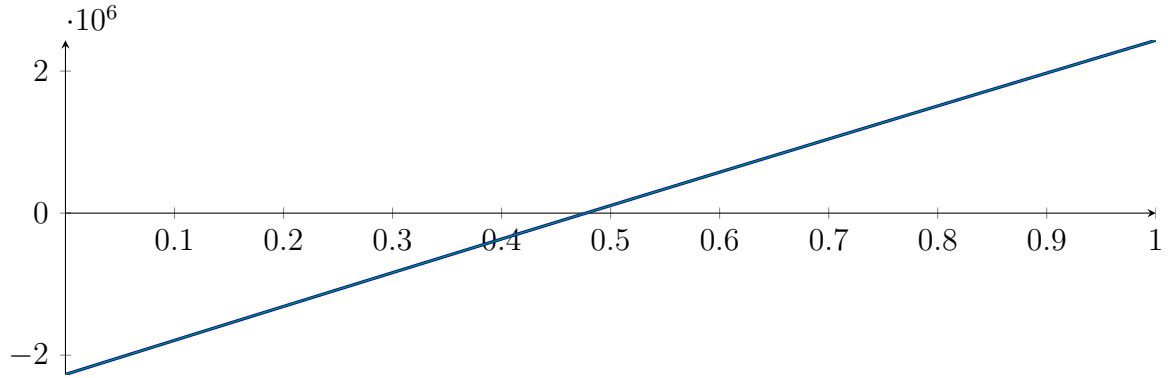
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\}$$

$$N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

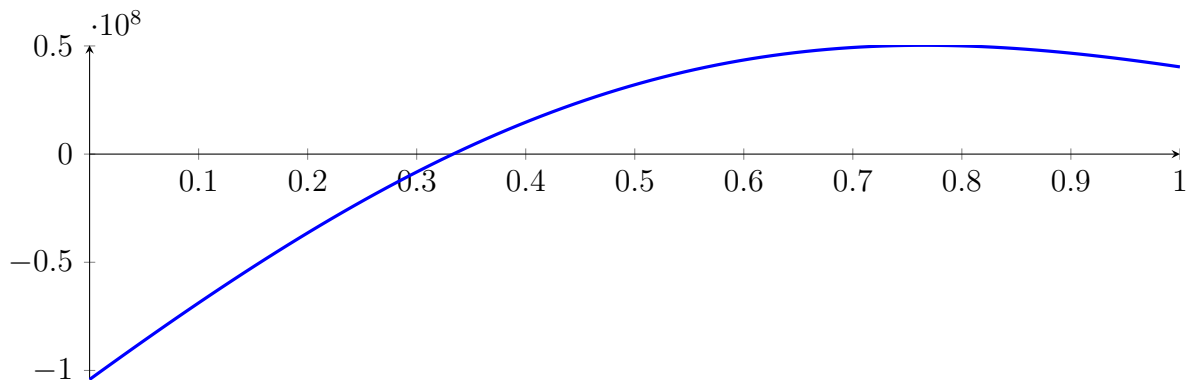
65.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

65.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

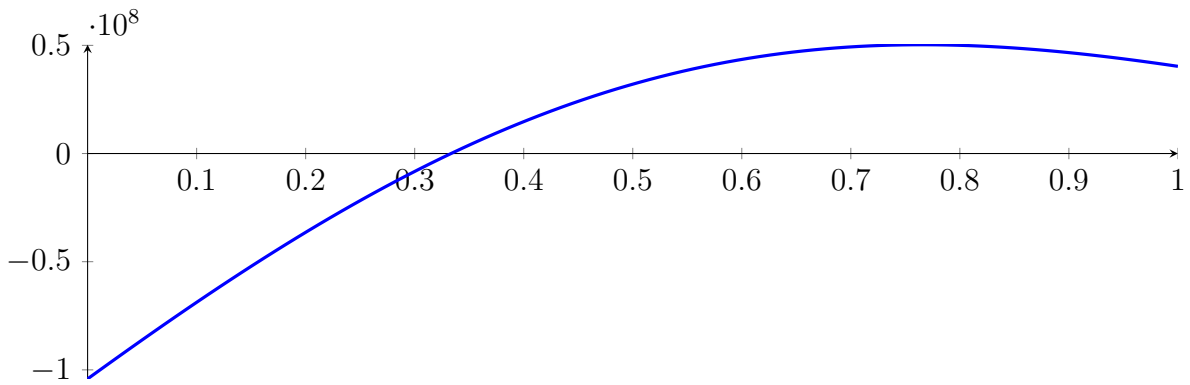
with precision $\varepsilon = 0.01$.

66 Running CubeClip on f_{16} with epsilon 2

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

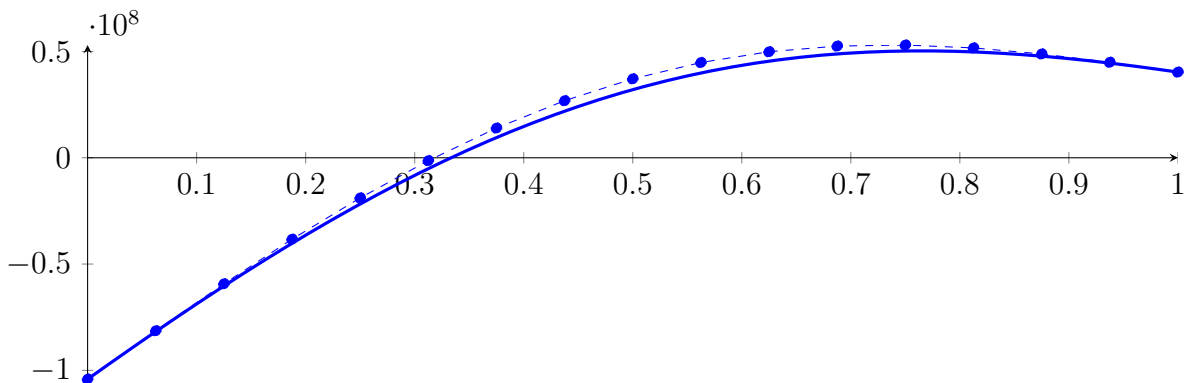
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



66.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 2.11088 \cdot 10^6 X^{16} - 1.70537 \cdot 10^7 X^{15} + 6.25913 \cdot 10^7 X^{14} - 1.38111 \cdot 10^8 X^{13} + 2.043 \cdot 10^8 X^{12} - 2.13799$$

$$\cdot 10^8 X^{11} + 1.62714 \cdot 10^8 X^{10} - 9.11092 \cdot 10^7 X^9 + 3.75254 \cdot 10^7 X^8 - 1.12529 \cdot 10^7 X^7 + 2.40899 \cdot 10^6 X^6$$

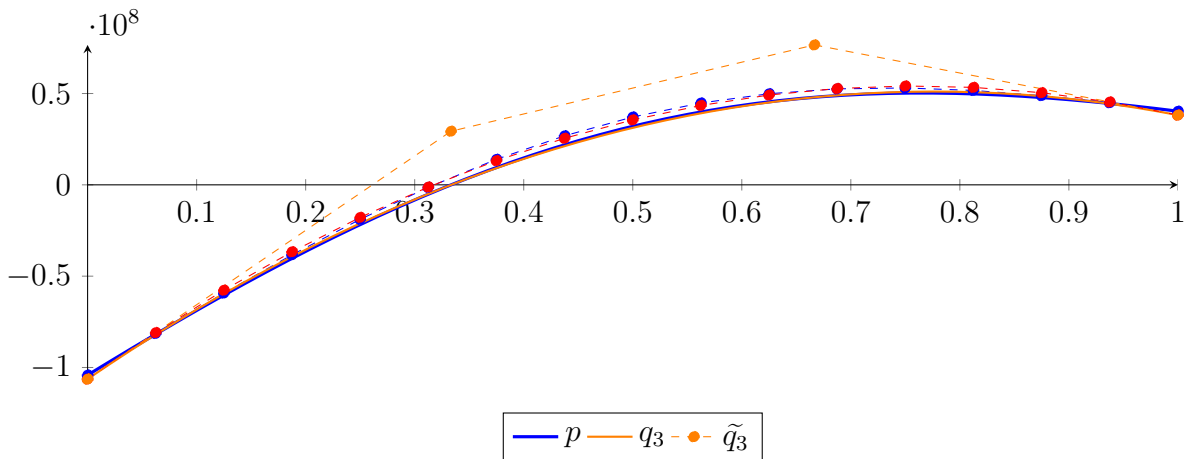
$$- 357156 X^5 + 34986.1 X^4 + 2.75602 \cdot 10^6 X^3 - 2.65318 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18553 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91816 \cdot 10^7 B_{10,16} + 5.27352 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

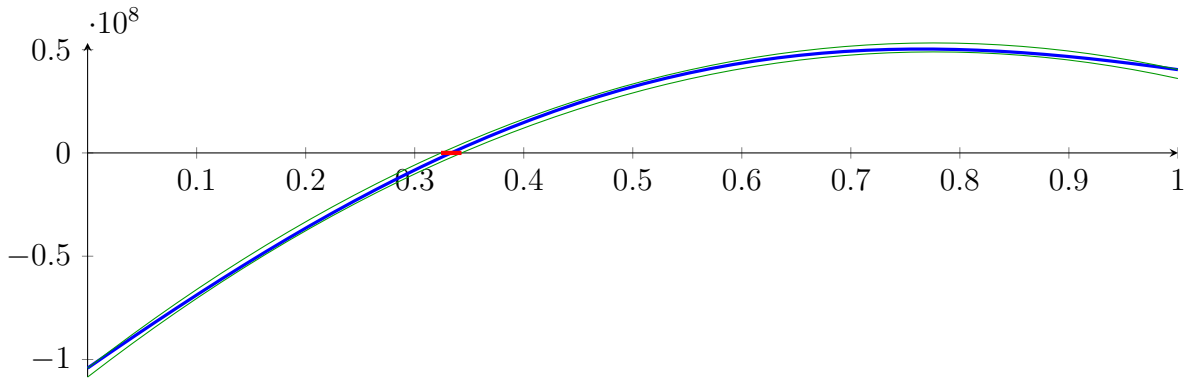
$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\} \quad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

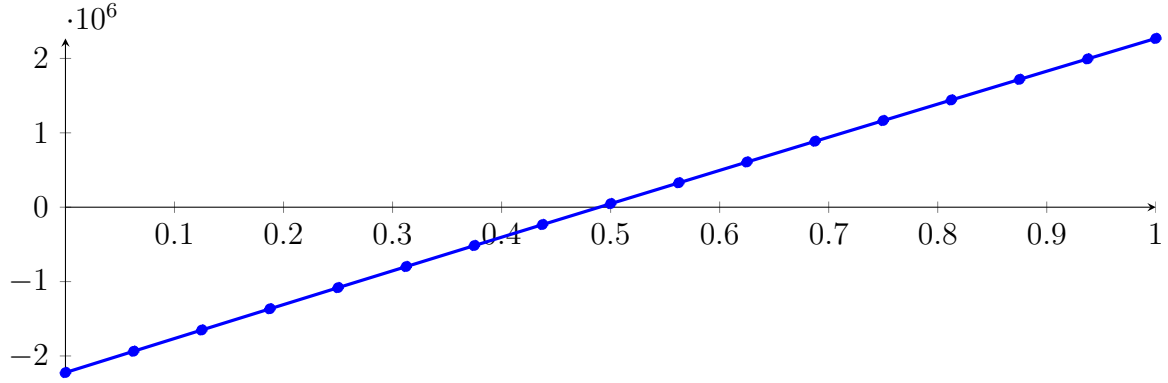
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

66.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

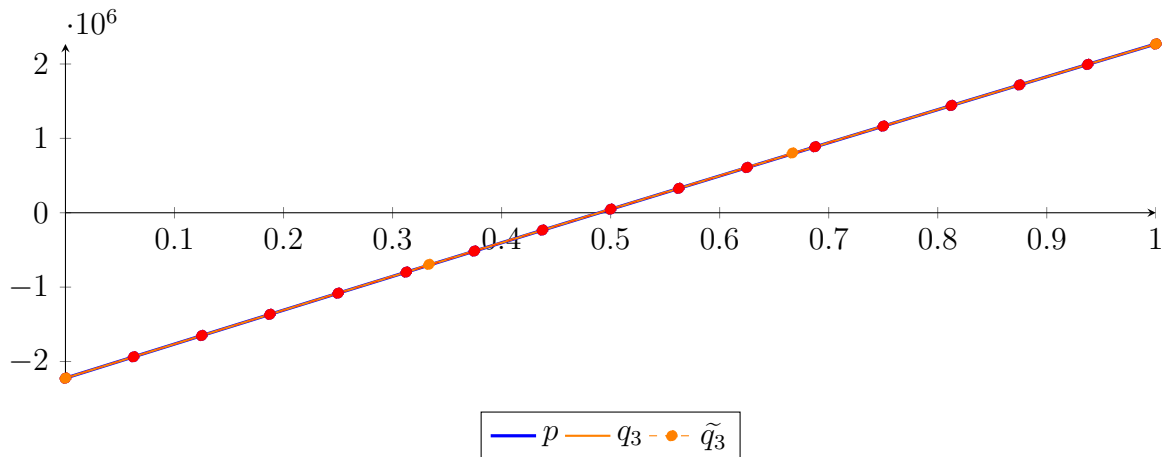
$$\begin{aligned}
 p &= -1.93035 \cdot 10^{-05} X^{16} + 1.08927 \cdot 10^{-05} X^{15} + 0.000255816 X^{14} + 0.000616983 X^{13} + 0.00371715 X^{12} \\
 &\quad + 0.00325035 X^{11} + 0.00510875 X^{10} + 0.00190713 X^9 + 0.000179792 X^8 + 1.06543 \cdot 10^{-05} X^7 \\
 &\quad - 0.00136482 X^6 + 0.104959 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 15290.6 X^{16} - 125412 X^{15} + 468045 X^{14} - 1.05198 \cdot 10^6 X^{13} + 1.588 \cdot 10^6 X^{12} - 1.69893 \\
 &\quad \cdot 10^6 X^{11} + 1.32382 \cdot 10^6 X^{10} - 759402 X^9 + 320067 X^8 - 97826.5 X^7 + 21197.8 X^6 \\
 &\quad - 3162.8 X^5 + 313.931 X^4 - 720.165 X^3 - 93879.2 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.7 B_{8,16} \\
 &\quad + 328649 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.457751$.

Bounding polynomials M and m :

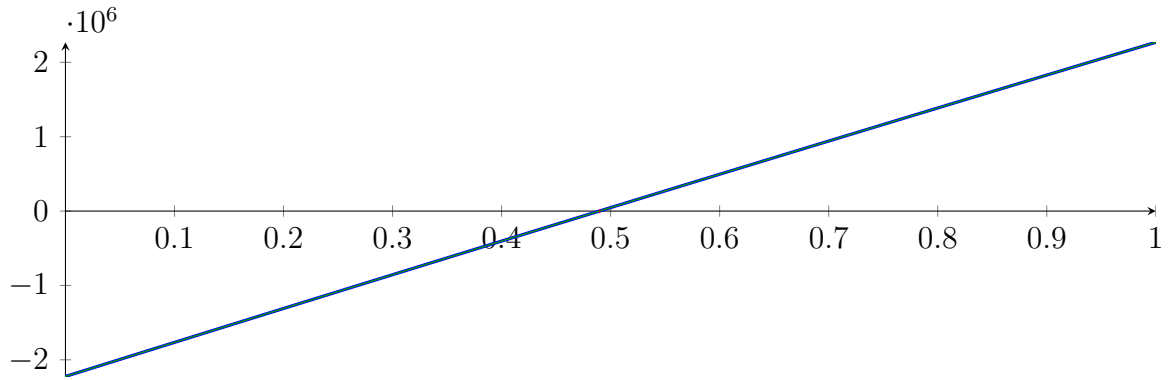
$$M = -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\} \quad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $2.03684 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

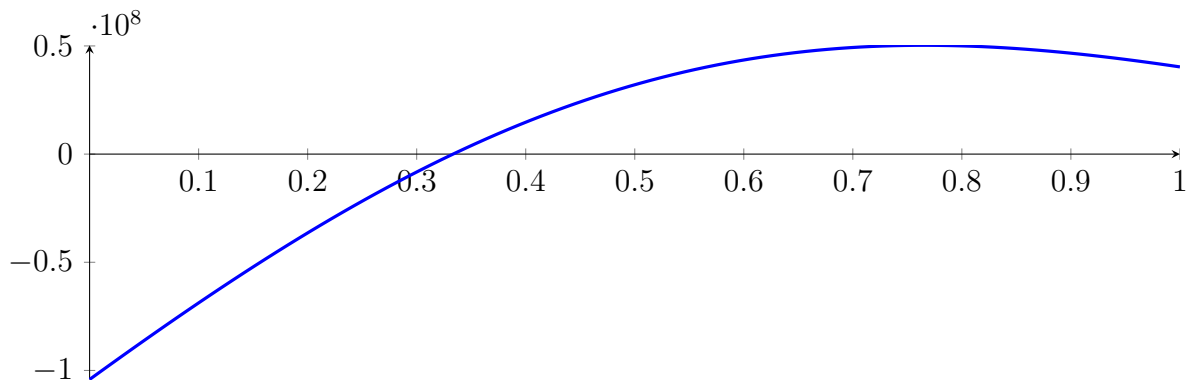
66.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

66.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

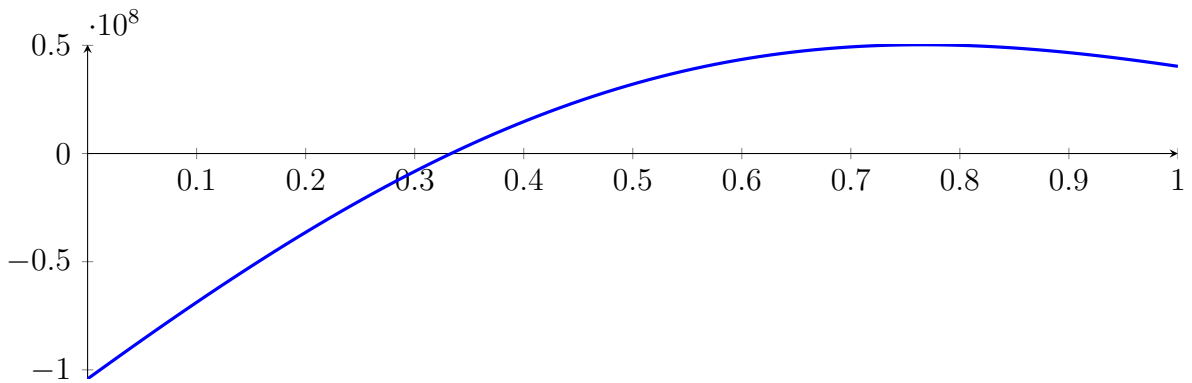
with precision $\varepsilon = 0.01$.

67 Running BezClip on f_{16} with epsilon 4

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

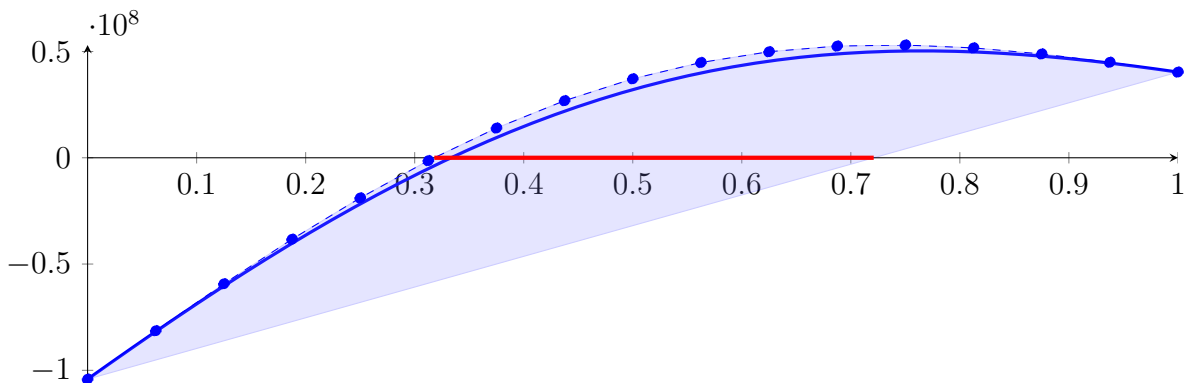
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



67.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

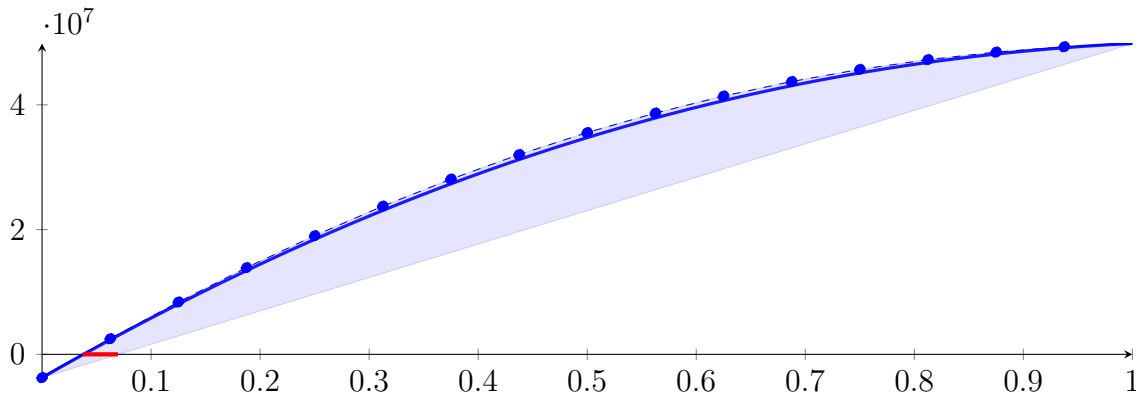
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

67.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.00483322X^{16} + 0.0186809X^{15} - 0.0194312X^{14} - 0.0738695X^{13} - 1.11673X^{12} \\ &\quad - 5.0471X^{11} + 36.3082X^{10} + 692.914X^9 + 1886.96X^8 - 25792X^7 - 149671X^6 + 492605X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

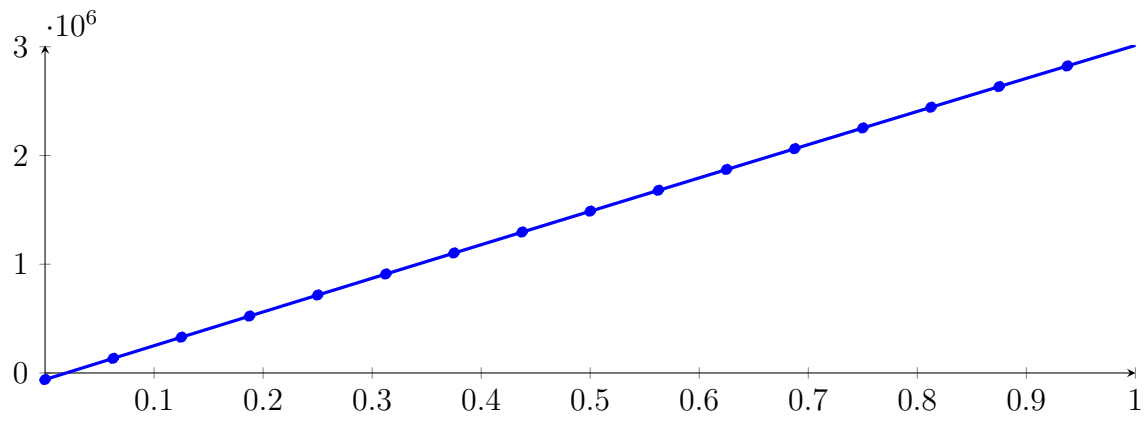
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

67.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.000205056X^{16} + 0.000776167X^{15} - 0.000863333X^{14} - 0.000217499X^{13} - 0.00809276X^{12} \\ &\quad - 5.09773 \cdot 10^{-05} X^{11} - 0.00564923X^{10} - 0.000162811X^9 - 0.000215376X^8 - 3.32948 \cdot 10^{-07} X^7 \\ &\quad - 0.000259866X^6 + 0.0161077X^5 + 4.36155X^4 - 234.216X^3 - 45622.2X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554

\implies Selective recursion: interval 1: $[0.333333, 0.333337]$,

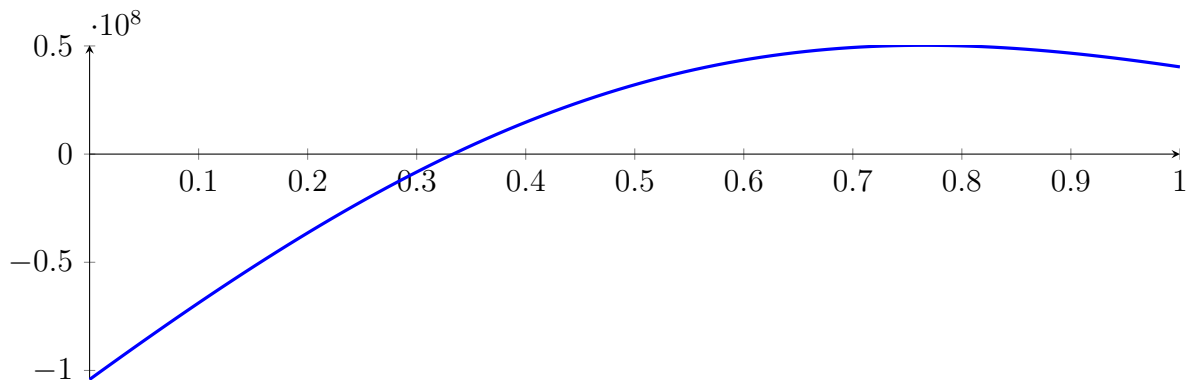
67.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Found root in interval $[0.333333, 0.333337]$ at recursion depth 4!

67.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333337]$$

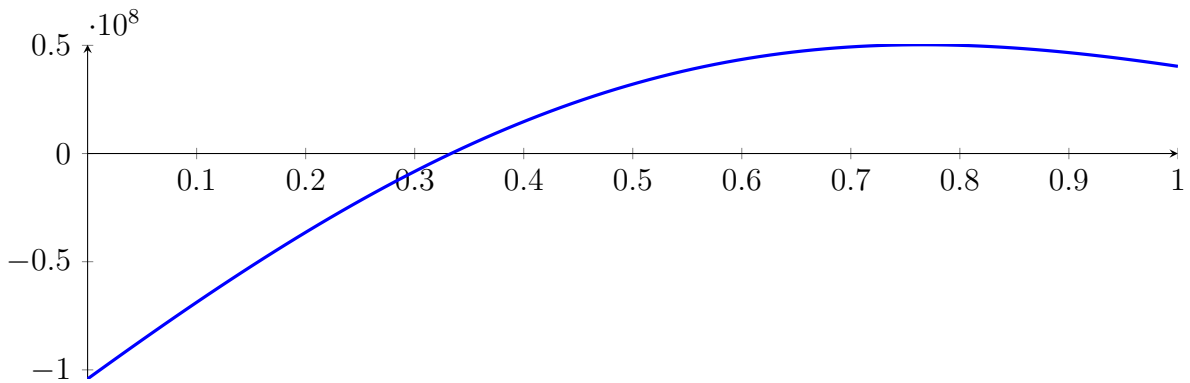
with precision $\varepsilon = 0.0001$.

68 Running QuadClip on f_{16} with epsilon 4

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

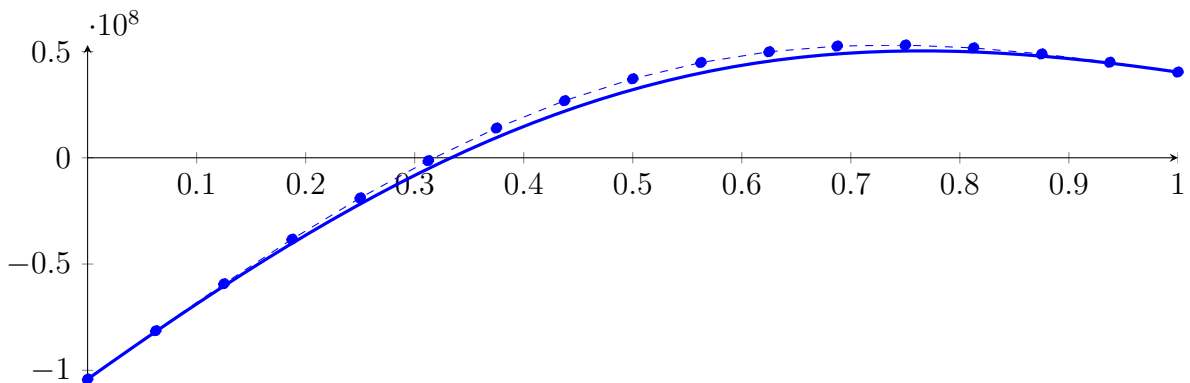
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



68.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = -1.41787 \cdot 10^6 X^{16} + 1.11761 \cdot 10^7 X^{15} - 3.98898 \cdot 10^7 X^{14} + 8.52437 \cdot 10^7 X^{13} - 1.21528 \cdot 10^8 X^{12}$$

$$+ 1.21946 \cdot 10^8 X^{11} - 8.86062 \cdot 10^7 X^{10} + 4.72904 \cdot 10^7 X^9 - 1.86355 \cdot 10^7 X^8 + 5.41059 \cdot 10^6 X^7 - 1.14487$$

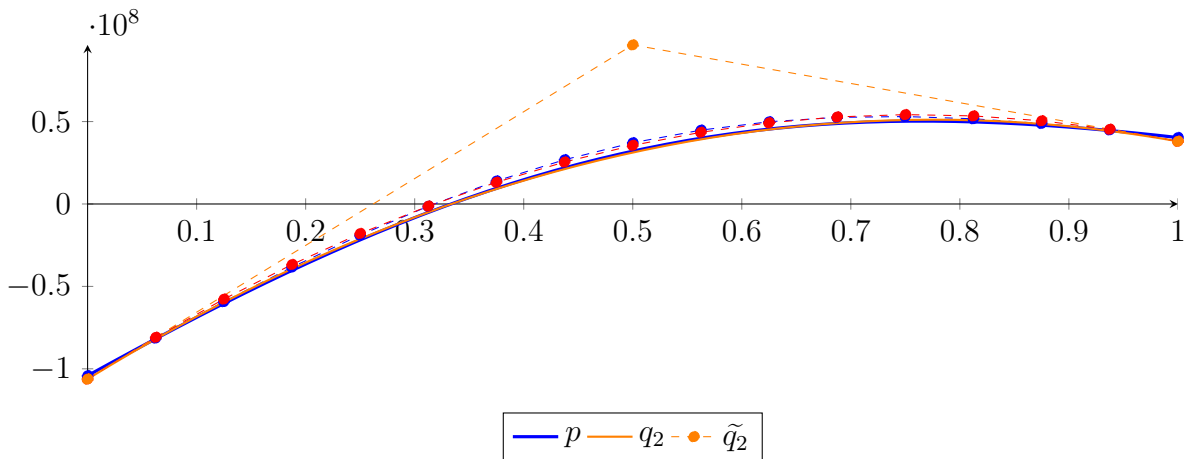
$$\cdot 10^6 X^6 + 172240 X^5 - 17636.6 X^4 + 1159.74 X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26923 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55691 \cdot 10^7 B_{8,16}$$

$$+ 4.34957 \cdot 10^7 B_{9,16} + 4.92455 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

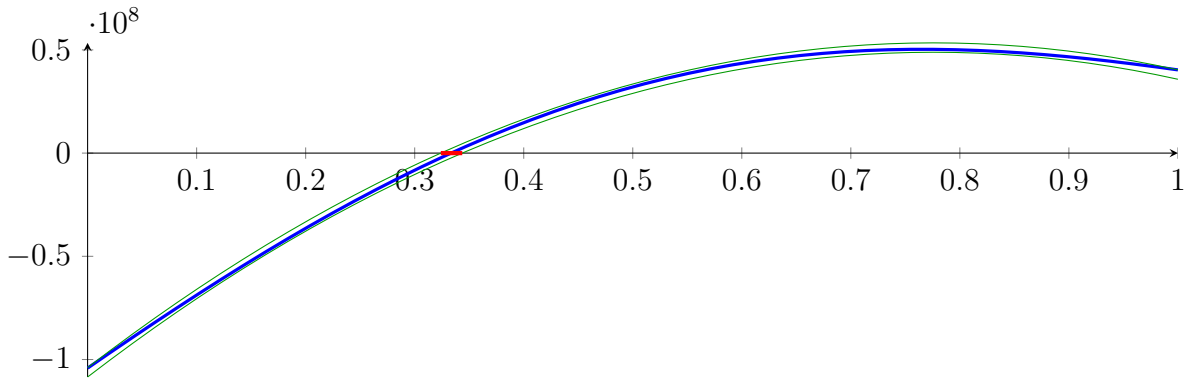
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

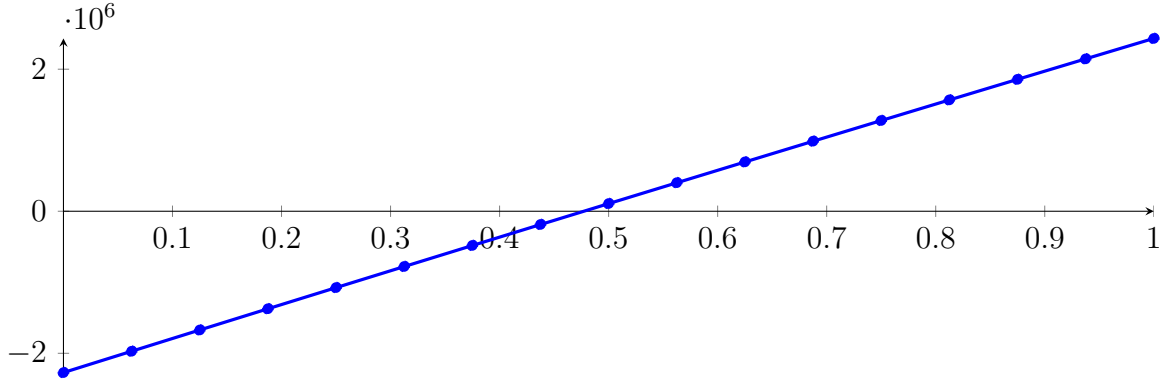
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

68.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

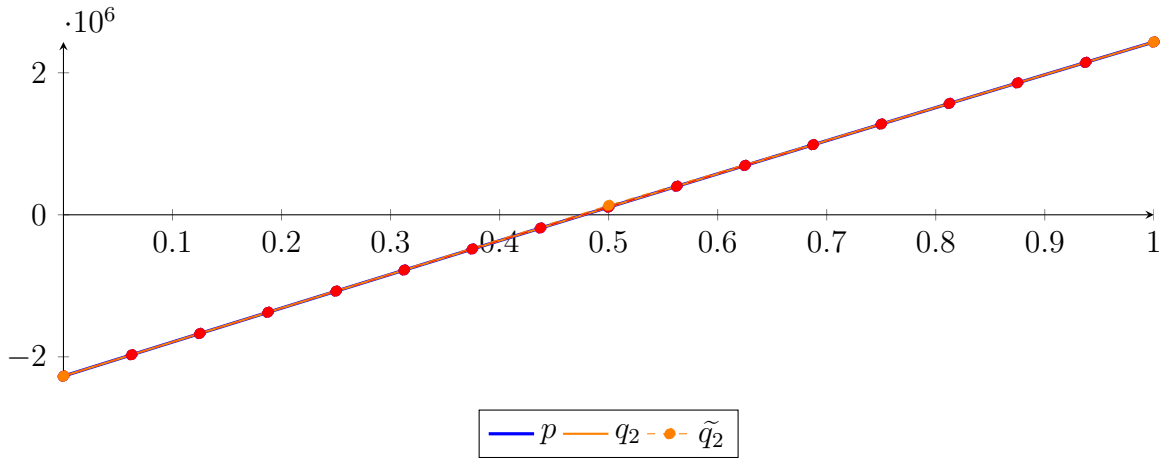
$$\begin{aligned}
 p &= -2.82438 \cdot 10^{-05} X^{16} + 2.90051 \cdot 10^{-05} X^{15} + 0.000231285 X^{14} + 0.000648014 X^{13} + 0.00318916 X^{12} \\
 &\quad + 0.00323204 X^{11} + 0.00460533 X^{10} + 0.00220012 X^9 - 0.000101882 X^8 + 5.32717 \cdot 10^{-06} X^7 \\
 &\quad - 0.00186824 X^6 + 0.132741 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 43556.7 X^{16} - 347904 X^{15} + 1.2616 \cdot 10^6 X^{14} - 2.74835 \cdot 10^6 X^{13} + 4.01042 \cdot 10^6 X^{12} - 4.13709 \\
 &\quad \cdot 10^6 X^{11} + 3.10251 \cdot 10^6 X^{10} - 1.71209 \cdot 10^6 X^9 + 695531 X^8 - 205965 X^7 + 43591.2 X^6 \\
 &\quad - 6402.94 X^5 + 625.678 X^4 - 37.485 X^3 - 104264 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481117 B_{6,16} - 185755 B_{7,16} + 108741 B_{8,16} \\
 &\quad + 402365 B_{9,16} + 695123 B_{10,16} + 987010 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

$$M = -104265 X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

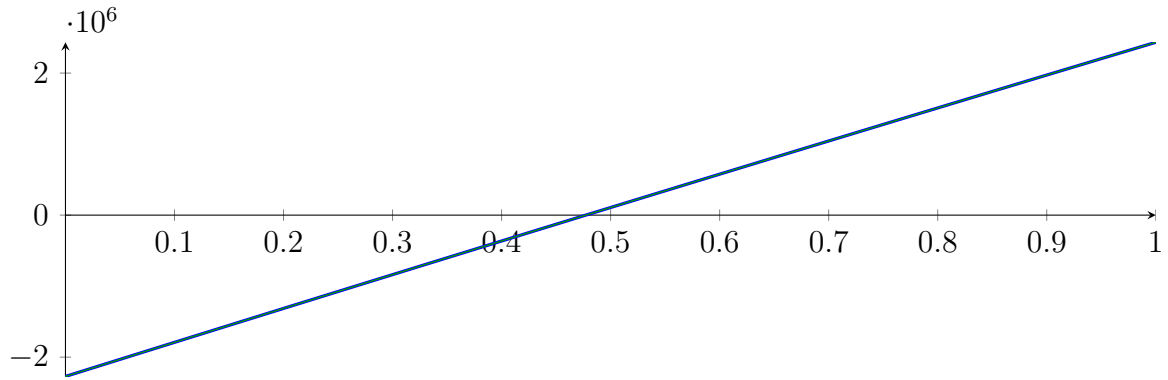
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\}$$

$$N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

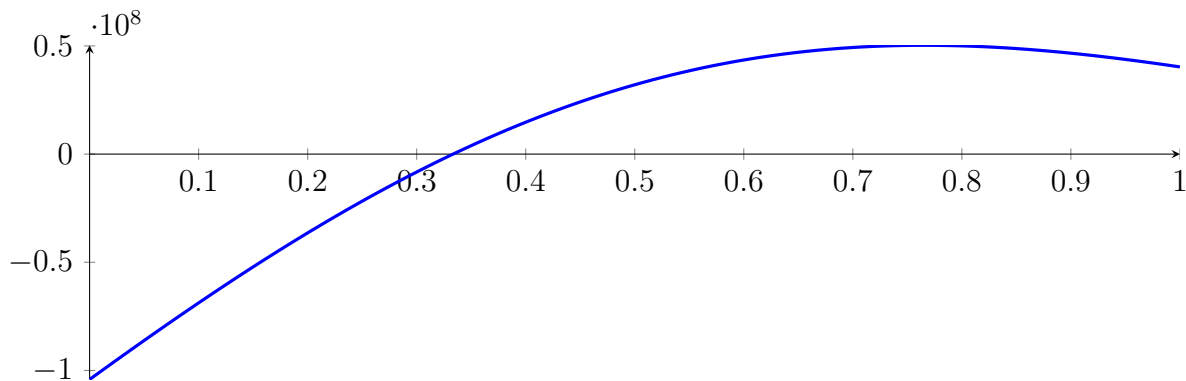
68.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

68.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

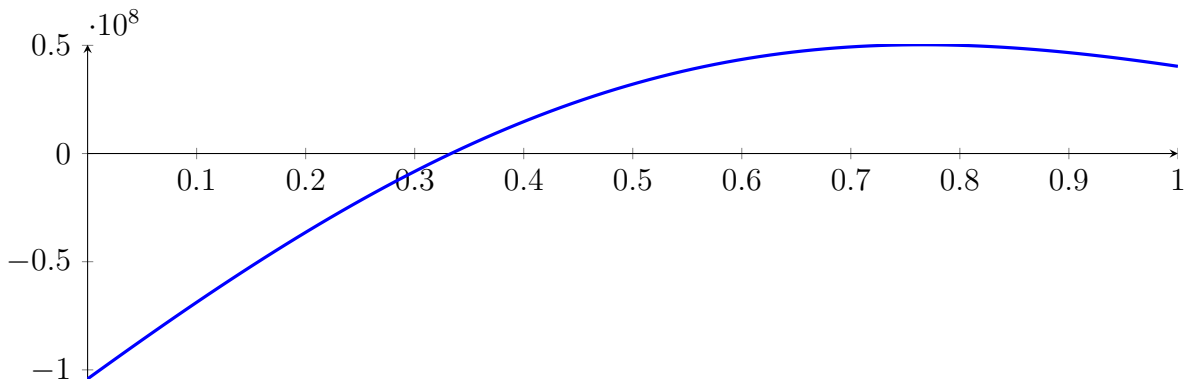
with precision $\varepsilon = 0.0001$.

69 Running CubeClip on f_{16} with epsilon 4

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

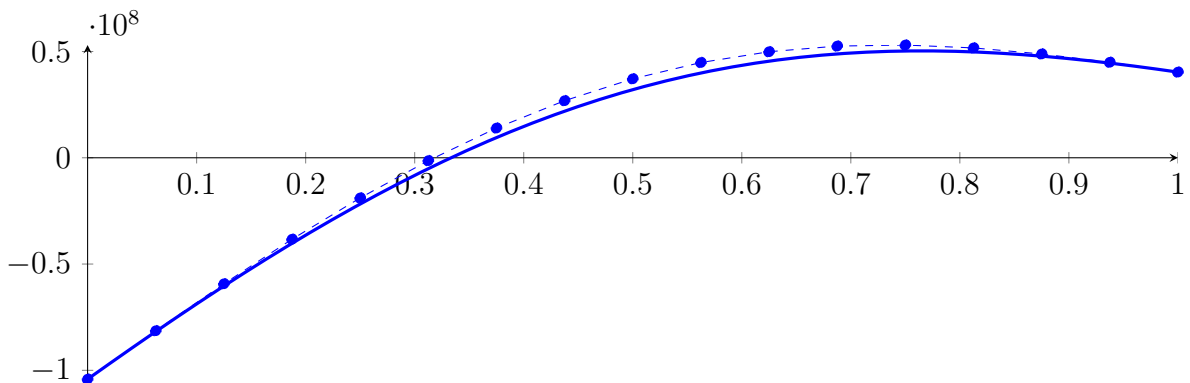
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



69.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 2.11088 \cdot 10^6 X^{16} - 1.70537 \cdot 10^7 X^{15} + 6.25913 \cdot 10^7 X^{14} - 1.38111 \cdot 10^8 X^{13} + 2.043 \cdot 10^8 X^{12} - 2.13799$$

$$\cdot 10^8 X^{11} + 1.62714 \cdot 10^8 X^{10} - 9.11092 \cdot 10^7 X^9 + 3.75254 \cdot 10^7 X^8 - 1.12529 \cdot 10^7 X^7 + 2.40899 \cdot 10^6 X^6$$

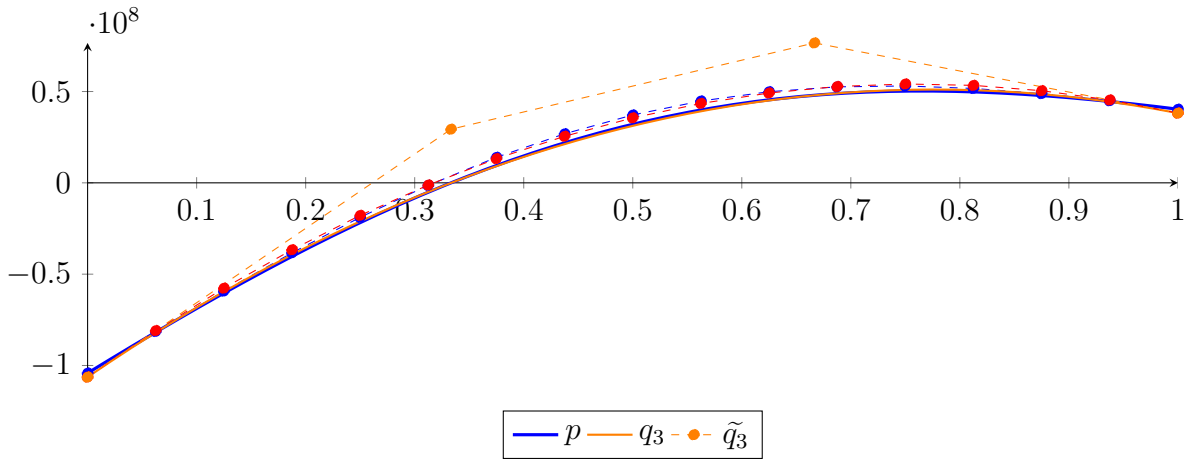
$$- 357156 X^5 + 34986.1 X^4 + 2.75602 \cdot 10^6 X^3 - 2.65318 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18553 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91816 \cdot 10^7 B_{10,16} + 5.27352 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

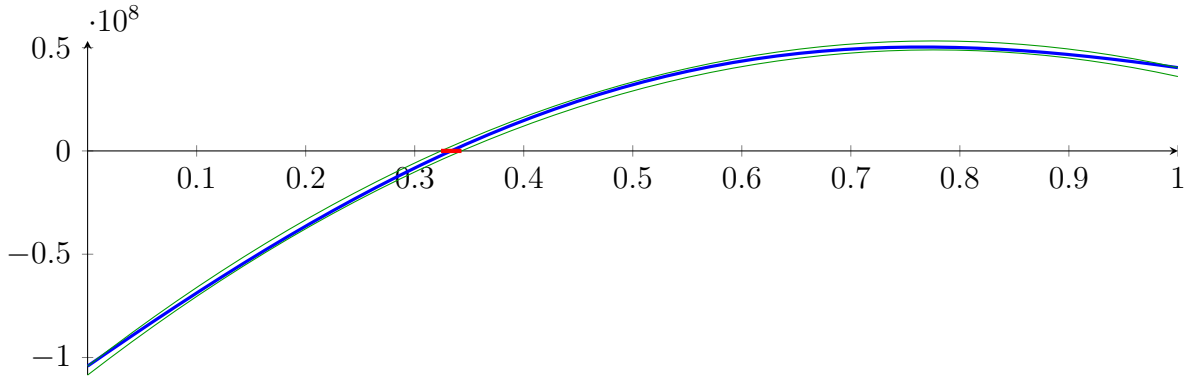
$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\}$$

$$N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

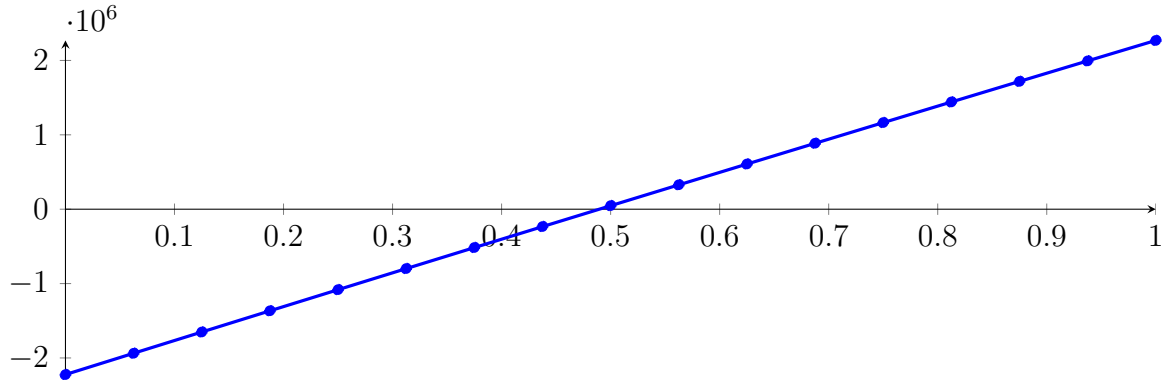
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

69.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

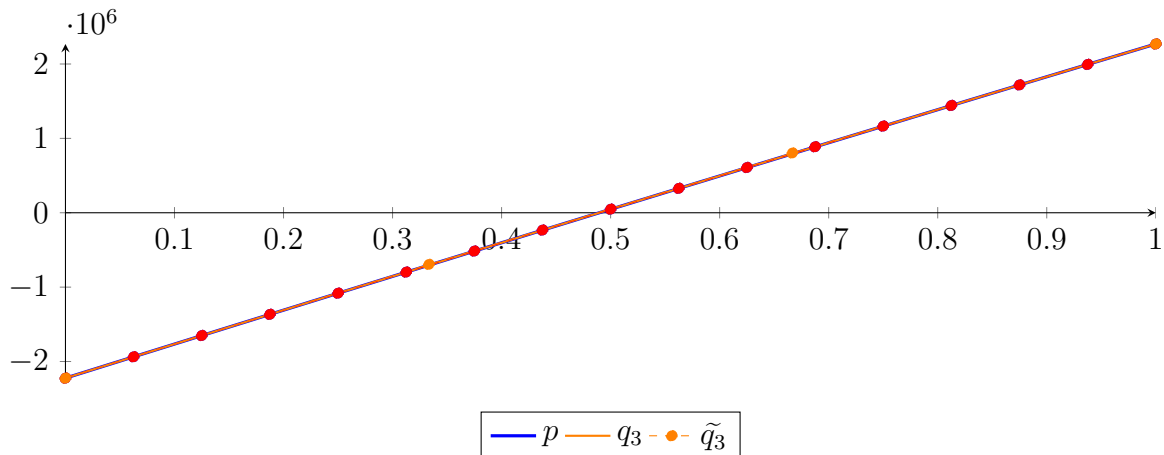
$$\begin{aligned}
 p &= -1.93035 \cdot 10^{-05} X^{16} + 1.08927 \cdot 10^{-05} X^{15} + 0.000255816 X^{14} + 0.000616983 X^{13} + 0.00371715 X^{12} \\
 &\quad + 0.00325035 X^{11} + 0.00510875 X^{10} + 0.00190713 X^9 + 0.000179792 X^8 + 1.06543 \cdot 10^{-05} X^7 \\
 &\quad - 0.00136482 X^6 + 0.104959 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 15290.6 X^{16} - 125412 X^{15} + 468045 X^{14} - 1.05198 \cdot 10^6 X^{13} + 1.588 \cdot 10^6 X^{12} - 1.69893 \\
 &\quad \cdot 10^6 X^{11} + 1.32382 \cdot 10^6 X^{10} - 759402 X^9 + 320067 X^8 - 97826.5 X^7 + 21197.8 X^6 \\
 &\quad - 3162.8 X^5 + 313.931 X^4 - 720.165 X^3 - 93879.2 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.7 B_{8,16} \\
 &\quad + 328649 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.457751$.

Bounding polynomials M and m :

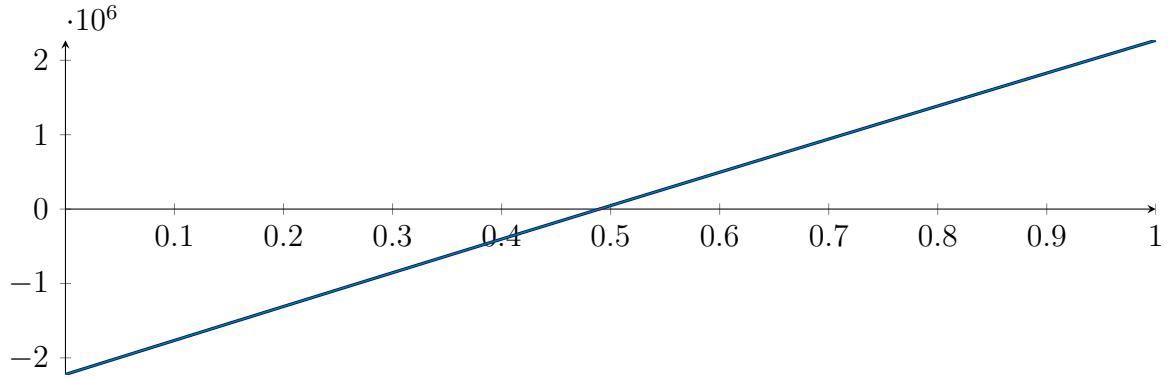
$$M = -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\} \quad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $2.03684 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

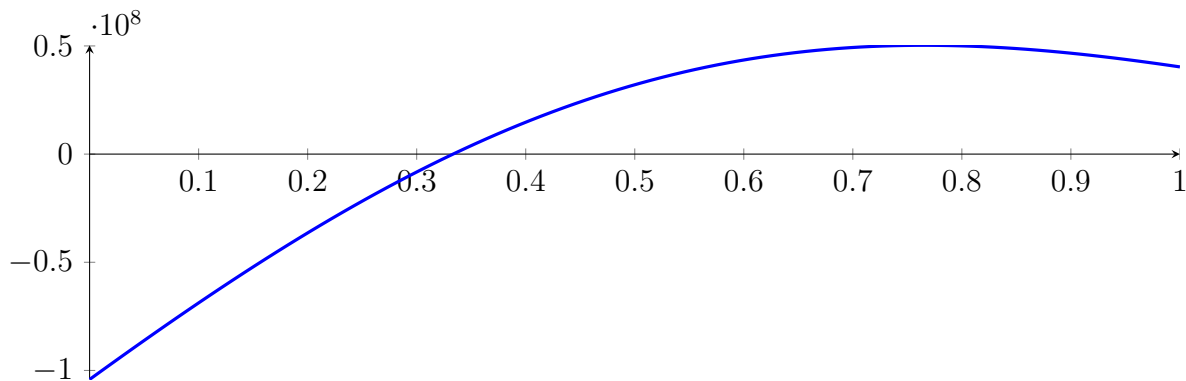
69.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

69.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

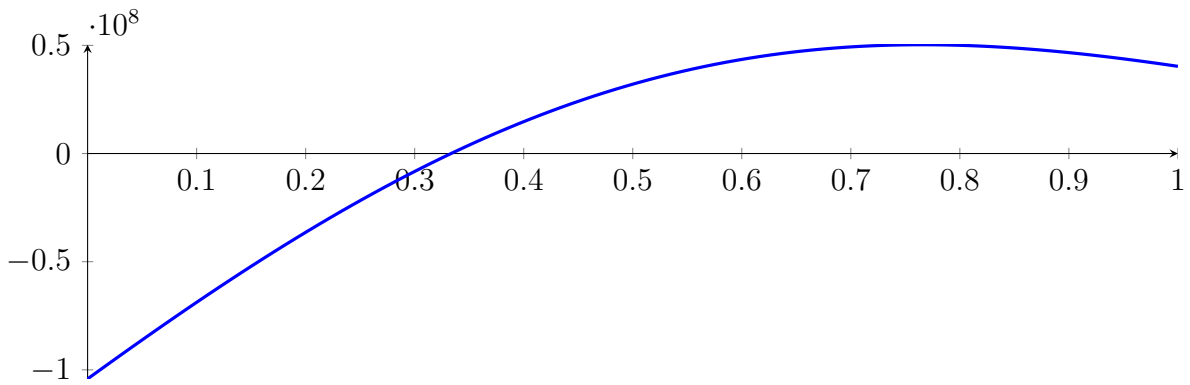
with precision $\varepsilon = 0.0001$.

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$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

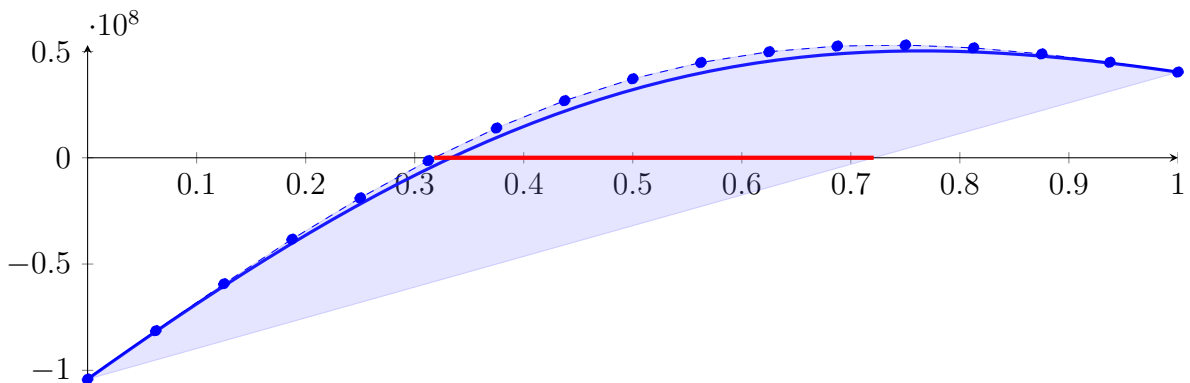
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



70.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

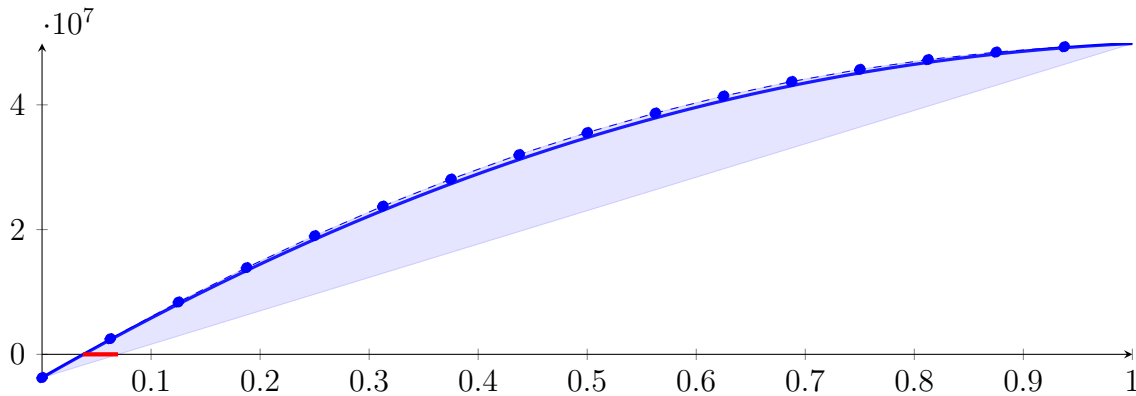
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

70.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.00483322X^{16} + 0.0186809X^{15} - 0.0194312X^{14} - 0.0738695X^{13} - 1.11673X^{12} \\ &\quad - 5.0471X^{11} + 36.3082X^{10} + 692.914X^9 + 1886.96X^8 - 25792X^7 - 149671X^6 + 492605X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

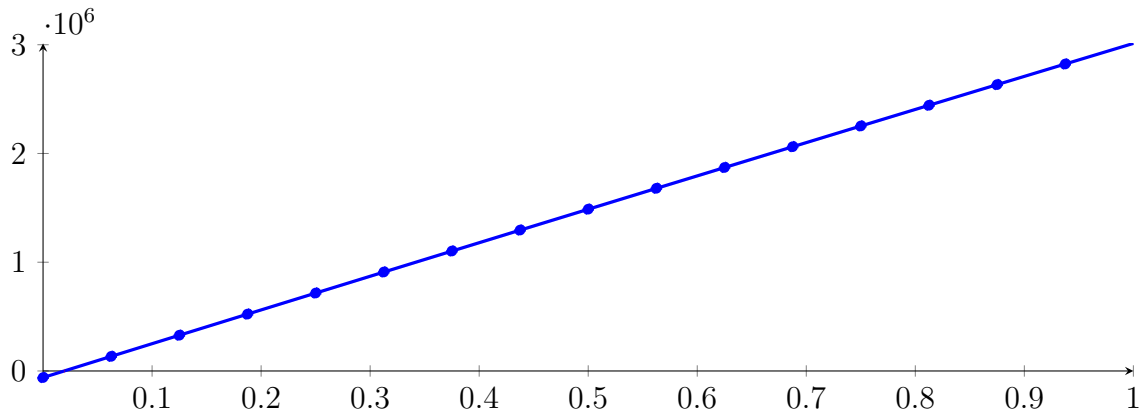
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

70.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.000205056X^{16} + 0.000776167X^{15} - 0.000863333X^{14} - 0.000217499X^{13} - 0.00809276X^{12} \\ &\quad - 5.09773 \cdot 10^{-05} X^{11} - 0.00564923X^{10} - 0.000162811X^9 - 0.000215376X^8 - 3.32948 \cdot 10^{-07} X^7 \\ &\quad - 0.000259866X^6 + 0.0161077X^5 + 4.36155X^4 - 234.216X^3 - 45622.2X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

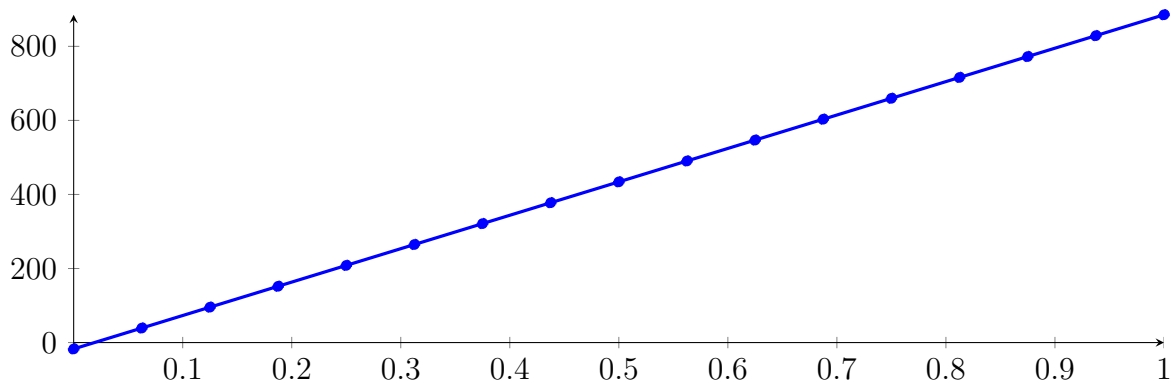
Longest intersection interval: 0.000289554

⇒ Selective recursion: interval 1: [\[0.333333, 0.333337\]](#),

70.4 Recursion Branch 1 1 1 1 in Interval 1: [\[0.333333, 0.333337\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -5.9692 \cdot 10^{-08} X^{16} + 2.16103 \cdot 10^{-07} X^{15} - 2.28456 \cdot 10^{-07} X^{14} - 1.17238 \cdot 10^{-07} X^{13} \\
 &\quad - 2.29525 \cdot 10^{-06} X^{12} - 8.31778 \cdot 10^{-08} X^{11} - 1.74251 \cdot 10^{-06} X^{10} - 9.42919 \cdot 10^{-08} X^9 \\
 &\quad - 7.38891 \cdot 10^{-08} X^8 + 3.25144 \cdot 10^{-09} X^7 - 2.61741 \cdot 10^{-08} X^6 + 7.44876 \cdot 10^{-10} X^5 \\
 &\quad - 2.58638 \cdot 10^{-10} X^4 - 5.65024 \cdot 10^{-09} X^3 - 0.00382618 X^2 + 902.448 X - 17.178 \\
 &= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
 &\quad + 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
 &\quad + 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
 &\quad + 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045 \cdot 10^{-08}$

⇒ Selective recursion: interval 1: [\[0.333333, 0.333333\]](#),

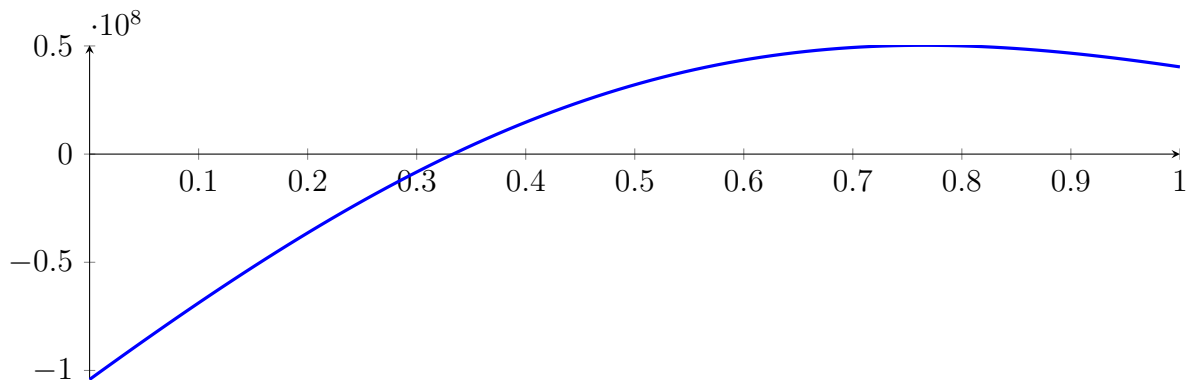
70.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

70.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

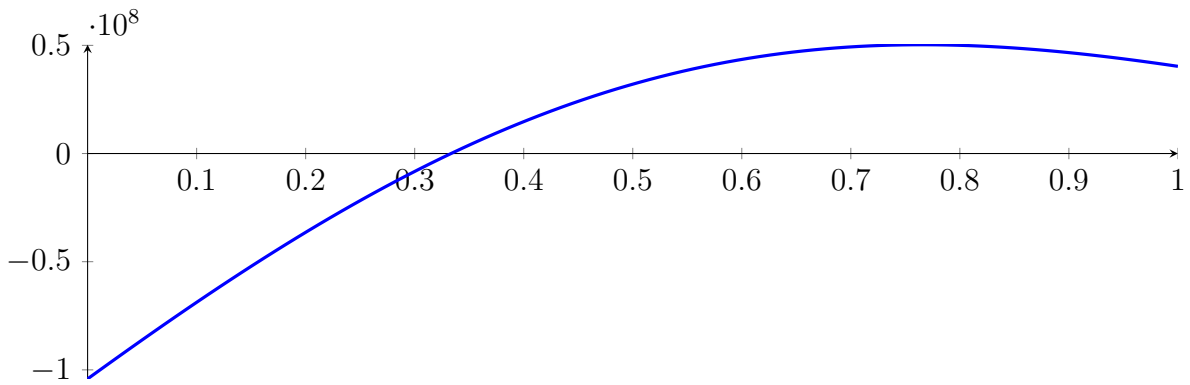
with precision $\varepsilon = 1 \cdot 10^{-08}$.

71 Running QuadClip on f_{16} with epsilon 8

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

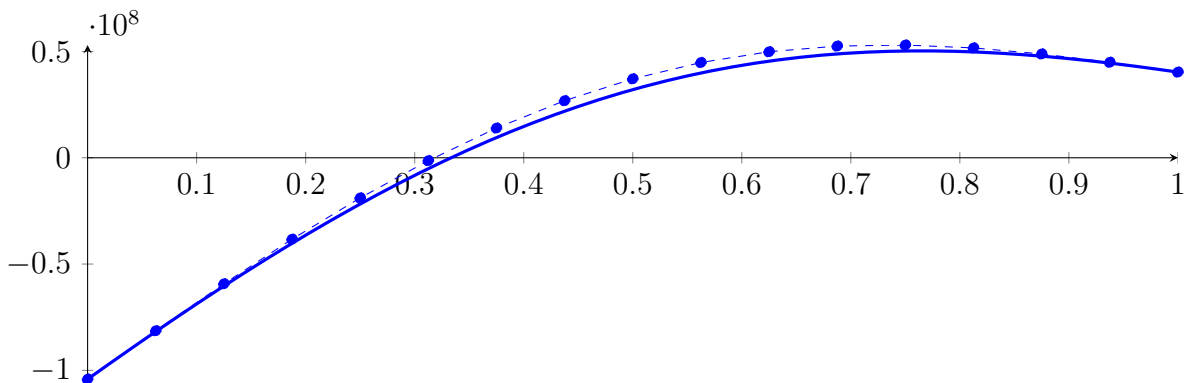
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



71.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = -1.41787 \cdot 10^6 X^{16} + 1.11761 \cdot 10^7 X^{15} - 3.98898 \cdot 10^7 X^{14} + 8.52437 \cdot 10^7 X^{13} - 1.21528 \cdot 10^8 X^{12}$$

$$+ 1.21946 \cdot 10^8 X^{11} - 8.86062 \cdot 10^7 X^{10} + 4.72904 \cdot 10^7 X^9 - 1.86355 \cdot 10^7 X^8 + 5.41059 \cdot 10^6 X^7 - 1.14487$$

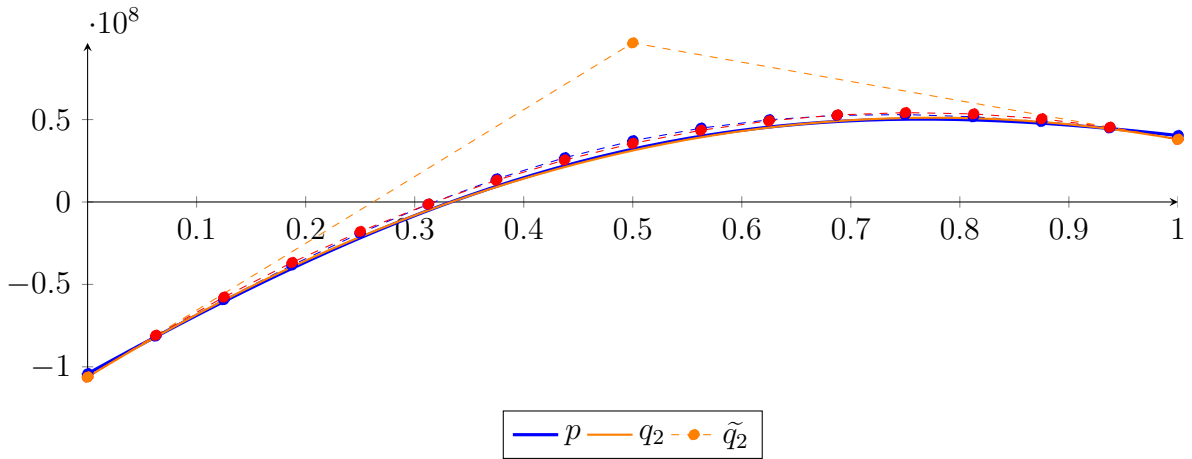
$$\cdot 10^6 X^6 + 172240 X^5 - 17636.6 X^4 + 1159.74 X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26923 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55691 \cdot 10^7 B_{8,16}$$

$$+ 4.34957 \cdot 10^7 B_{9,16} + 4.92455 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

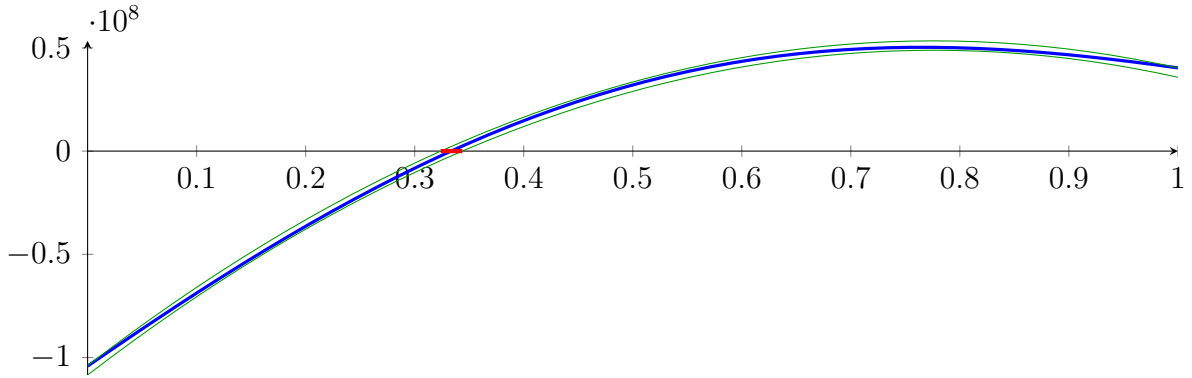
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\} \quad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

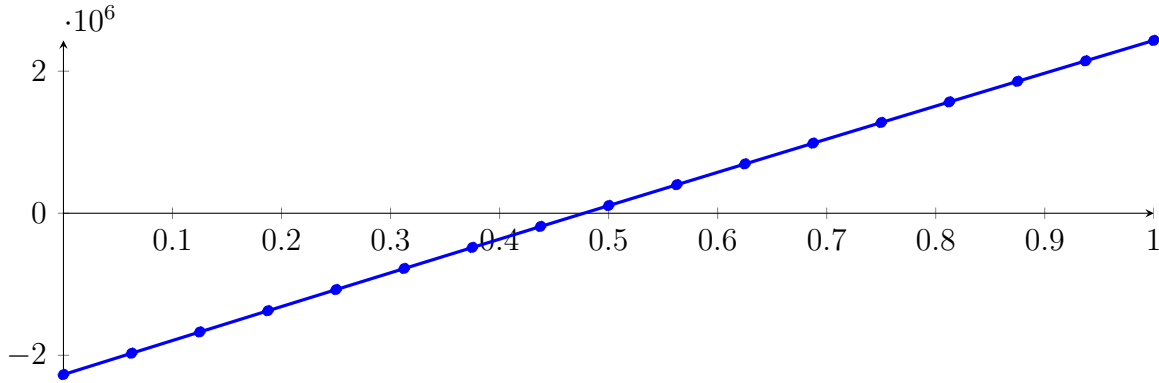
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

71.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

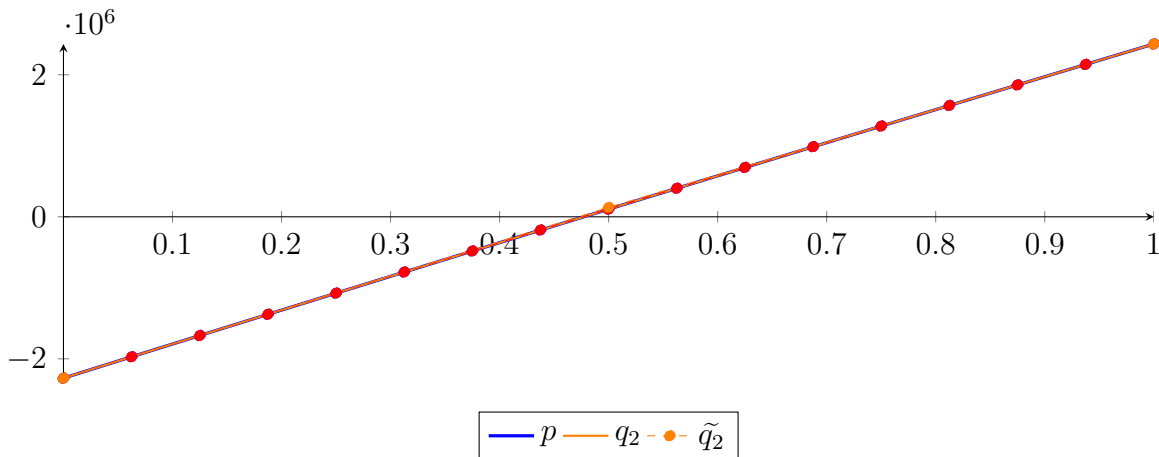
$$\begin{aligned}
 p &= -2.82438 \cdot 10^{-05} X^{16} + 2.90051 \cdot 10^{-05} X^{15} + 0.000231285 X^{14} + 0.000648014 X^{13} + 0.00318916 X^{12} \\
 &\quad + 0.00323204 X^{11} + 0.00460533 X^{10} + 0.00220012 X^9 - 0.000101882 X^8 + 5.32717 \cdot 10^{-06} X^7 \\
 &\quad - 0.00186824 X^6 + 0.132741 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 43556.7 X^{16} - 347904 X^{15} + 1.2616 \cdot 10^6 X^{14} - 2.74835 \cdot 10^6 X^{13} + 4.01042 \cdot 10^6 X^{12} - 4.13709 \\
 &\quad \cdot 10^6 X^{11} + 3.10251 \cdot 10^6 X^{10} - 1.71209 \cdot 10^6 X^9 + 695531 X^8 - 205965 X^7 + 43591.2 X^6 \\
 &\quad - 6402.94 X^5 + 625.678 X^4 - 37.485 X^3 - 104264 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481117 B_{6,16} - 185755 B_{7,16} + 108741 B_{8,16} \\
 &\quad + 402365 B_{9,16} + 695123 B_{10,16} + 987010 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

$$M = -104265 X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

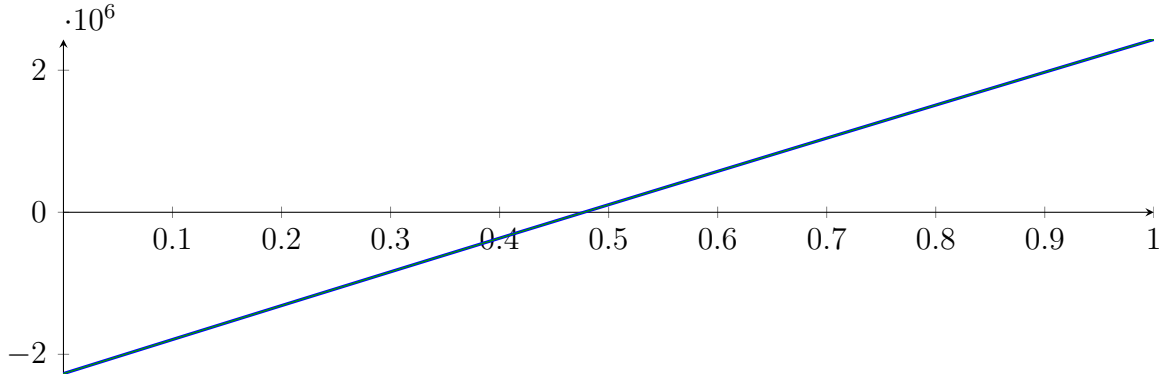
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\}$$

$$N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

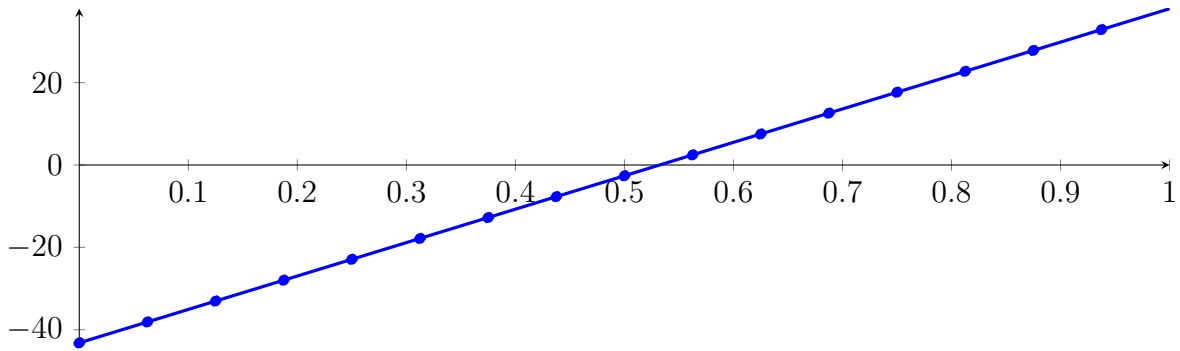
Longest intersection interval: $1.72301 \cdot 10^{-05}$

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333333]$,

71.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

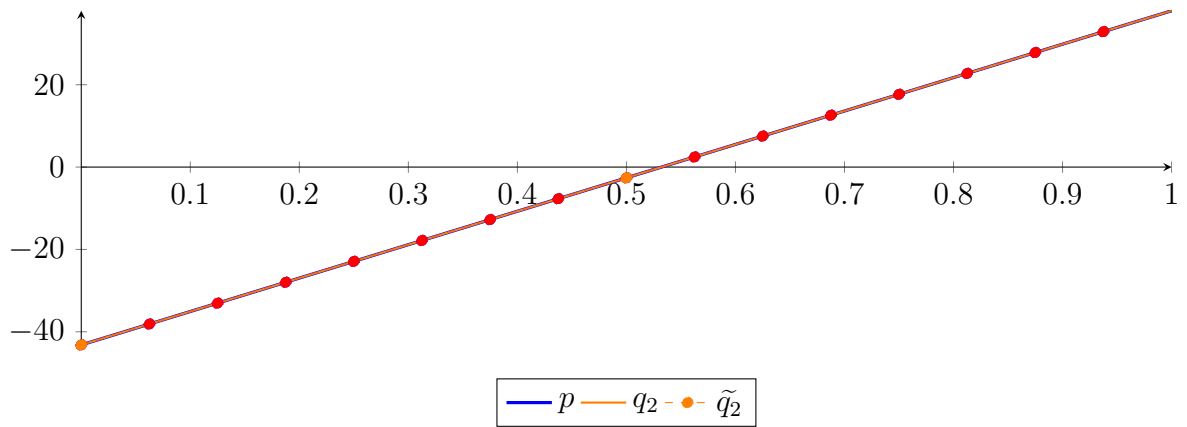
$$\begin{aligned} p &= 8.74252 \cdot 10^{-11} X^{16} - 1.56979 \cdot 10^{-09} X^{15} + 6.68479 \cdot 10^{-09} X^{14} + 1.20008 \cdot 10^{-08} X^{13} + 9.07301 \cdot 10^{-08} X^{12} \\ &+ 5.58657 \cdot 10^{-08} X^{11} + 1.13801 \cdot 10^{-07} X^{10} + 3.70665 \cdot 10^{-08} X^9 + 7.31575 \cdot 10^{-10} X^8 + 1.30058 \cdot 10^{-09} X^7 \\ &+ 5.00722 \cdot 10^{-09} X^6 + 1.24146 \cdot 10^{-10} X^5 + 1.03455 \cdot 10^{-10} X^4 - 3.09388 \cdot 10^{-05} X^2 + 81.1505X - 43.1911 \\ &= -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\ &- 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68777B_{7,16}(X) - 2.61587B_{8,16}(X) \\ &+ 2.45604B_{9,16}(X) + 7.52794B_{10,16}(X) + 12.5998B_{11,16}(X) + 17.6718B_{12,16}(X) \\ &+ 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09388 \cdot 10^{-05} X^2 + 81.1505X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 0.721495X^{16} - 5.74915X^{15} + 20.7933X^{14} - 45.1627X^{13} + 65.6806X^{12} - 67.5044X^{11} \\ &+ 50.4286X^{10} - 27.728X^9 + 11.2318X^8 - 3.32011X^7 + 0.702408X^6 - 0.103415X^5 \\ &+ 0.0102099X^4 - 0.000624725X^3 - 1.10834 \cdot 10^{-05} X^2 + 81.1505X - 43.1911 \\ &= -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} \\ &- 12.7597B_{6,16} - 7.68779B_{7,16} - 2.61585B_{8,16} + 2.45602B_{9,16} + 7.52795B_{10,16} + 12.5998B_{11,16} \\ &+ 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.57956 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -3.09388 \cdot 10^{-05} X^2 + 81.1505X - 43.1911$$

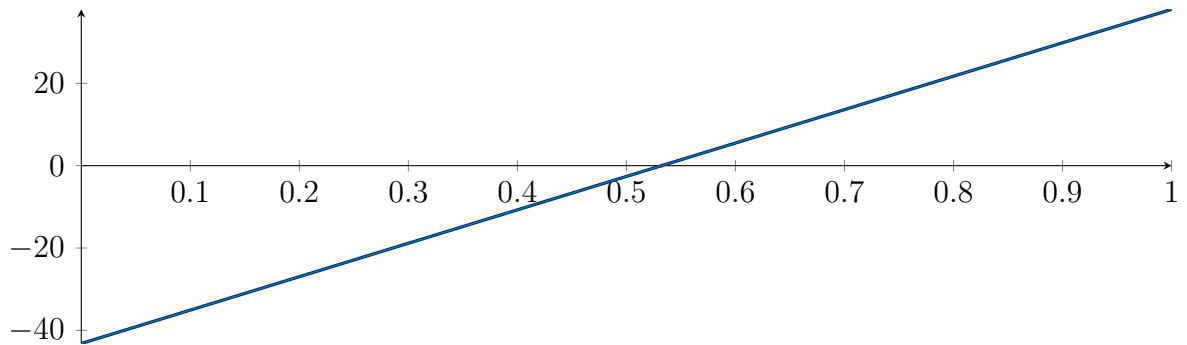
$$m = -3.09388 \cdot 10^{-05} X^2 + 81.1505X - 43.1911$$

Root of M and m :

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\}$$

$$N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

Longest intersection interval: $3.8903 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

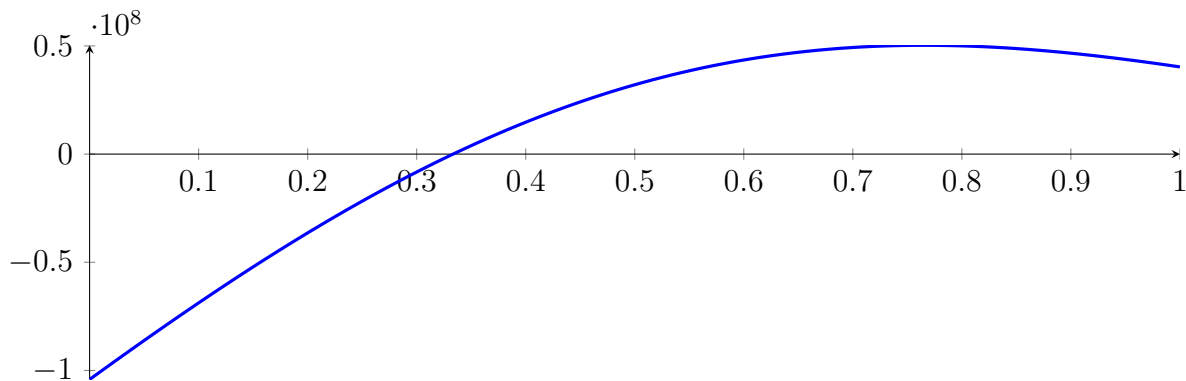
71.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

71.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

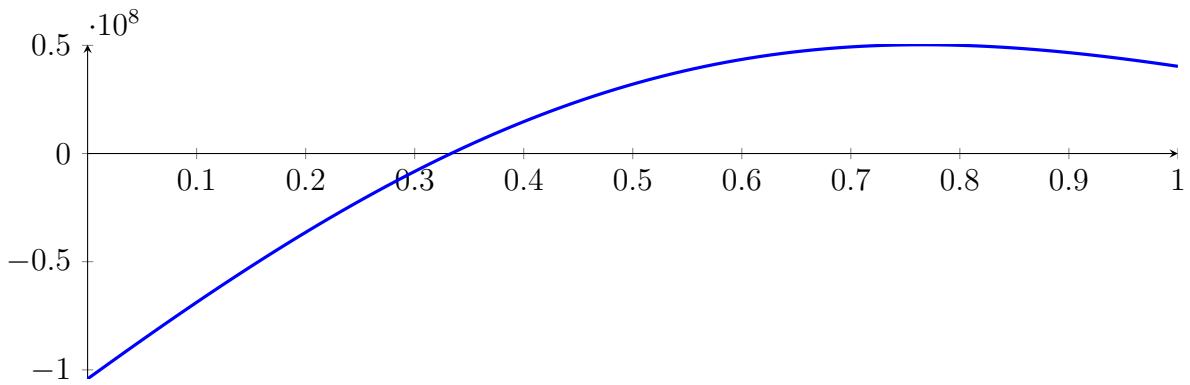
with precision $\varepsilon = 1 \cdot 10^{-08}$.

72 Running CubeClip on f_{16} with epsilon 8

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

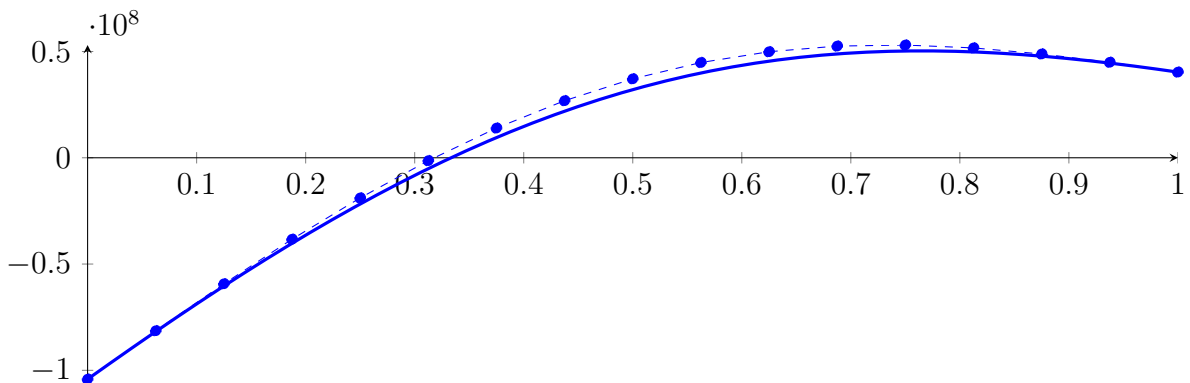
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



72.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 2.11088 \cdot 10^6 X^{16} - 1.70537 \cdot 10^7 X^{15} + 6.25913 \cdot 10^7 X^{14} - 1.38111 \cdot 10^8 X^{13} + 2.043 \cdot 10^8 X^{12} - 2.13799$$

$$\cdot 10^8 X^{11} + 1.62714 \cdot 10^8 X^{10} - 9.11092 \cdot 10^7 X^9 + 3.75254 \cdot 10^7 X^8 - 1.12529 \cdot 10^7 X^7 + 2.40899 \cdot 10^6 X^6$$

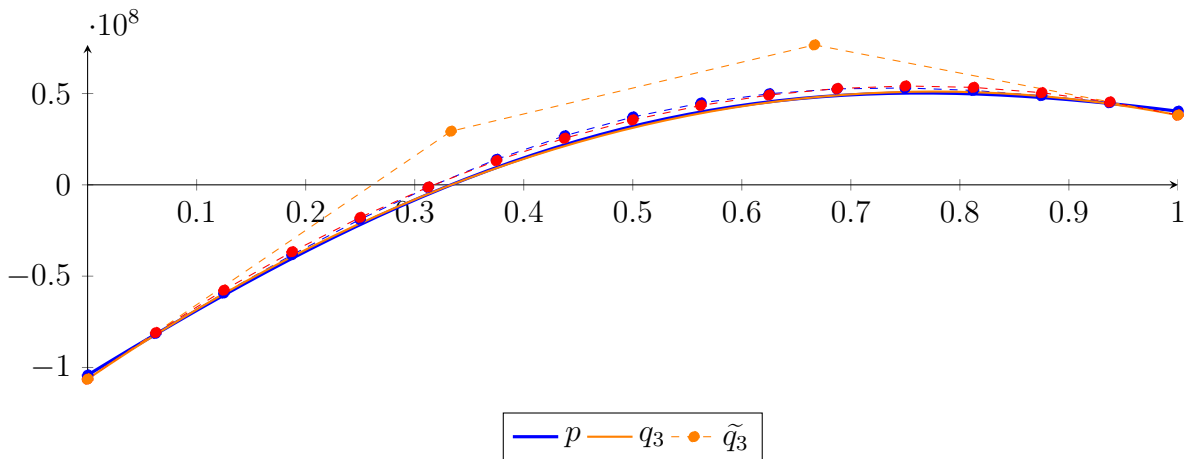
$$- 357156 X^5 + 34986.1 X^4 + 2.75602 \cdot 10^6 X^3 - 2.65318 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18553 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91816 \cdot 10^7 B_{10,16} + 5.27352 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

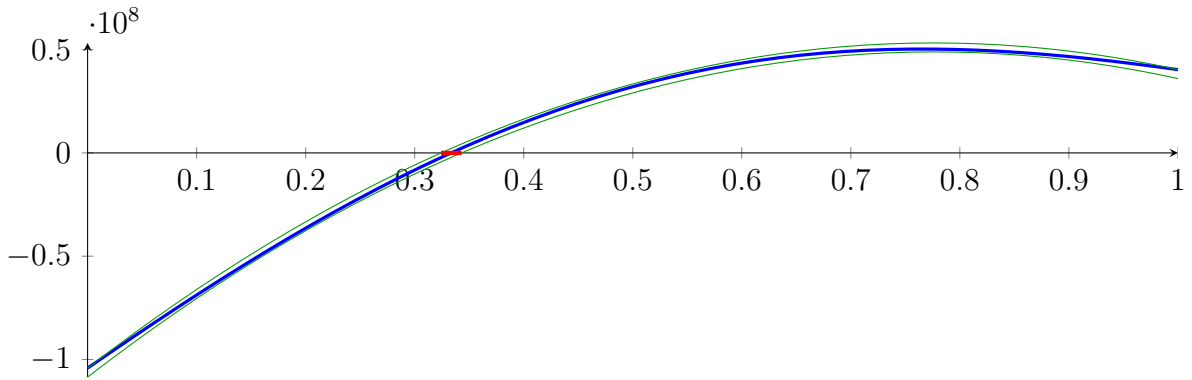
$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\} \quad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

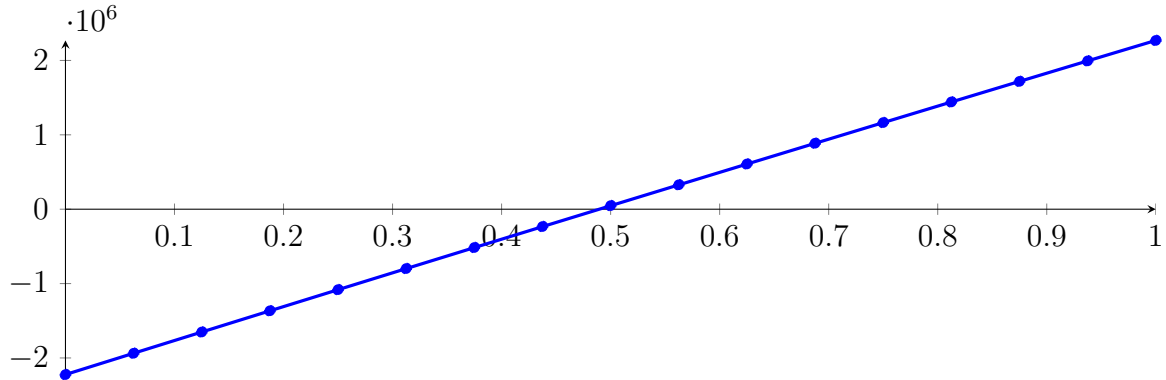
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

72.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

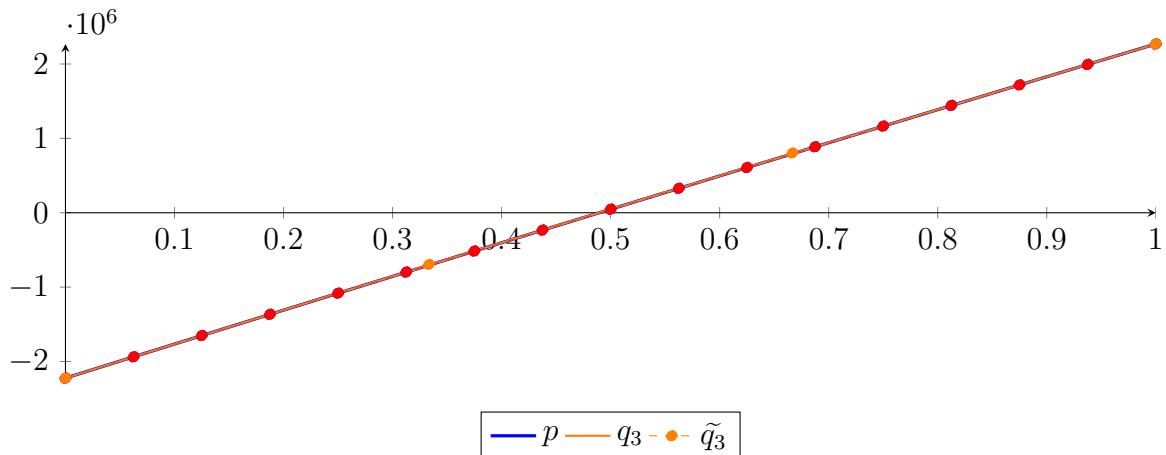
$$\begin{aligned}
 p &= -1.93035 \cdot 10^{-05} X^{16} + 1.08927 \cdot 10^{-05} X^{15} + 0.000255816 X^{14} + 0.000616983 X^{13} + 0.00371715 X^{12} \\
 &+ 0.00325035 X^{11} + 0.00510875 X^{10} + 0.00190713 X^9 + 0.000179792 X^8 + 1.06543 \cdot 10^{-05} X^7 \\
 &- 0.00136482 X^6 + 0.104959 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &- 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &+ 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 15290.6 X^{16} - 125412 X^{15} + 468045 X^{14} - 1.05198 \cdot 10^6 X^{13} + 1.588 \cdot 10^6 X^{12} - 1.69893 \\
 &\cdot 10^6 X^{11} + 1.32382 \cdot 10^6 X^{10} - 759402 X^9 + 320067 X^8 - 97826.5 X^7 + 21197.8 X^6 \\
 &- 3162.8 X^5 + 313.931 X^4 - 720.165 X^3 - 93879.2 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &- 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.7 B_{8,16} \\
 &+ 328649 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.457751$.

Bounding polynomials M and m :

$$M = -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

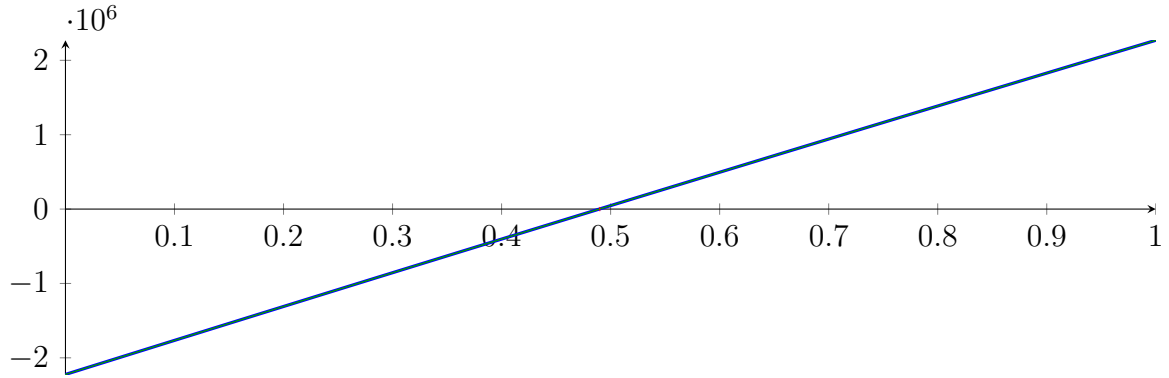
$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\}$$

$$N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $2.03684 \cdot 10^{-07}$

\implies Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

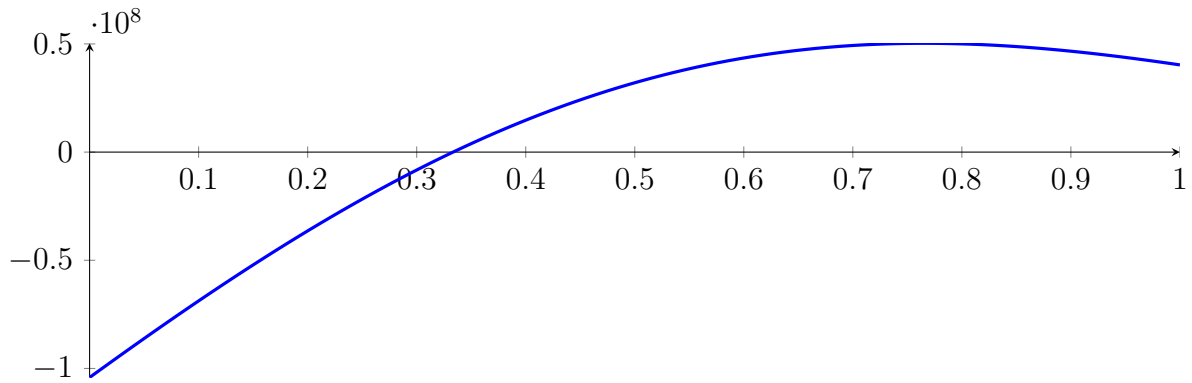
72.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 3!

72.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

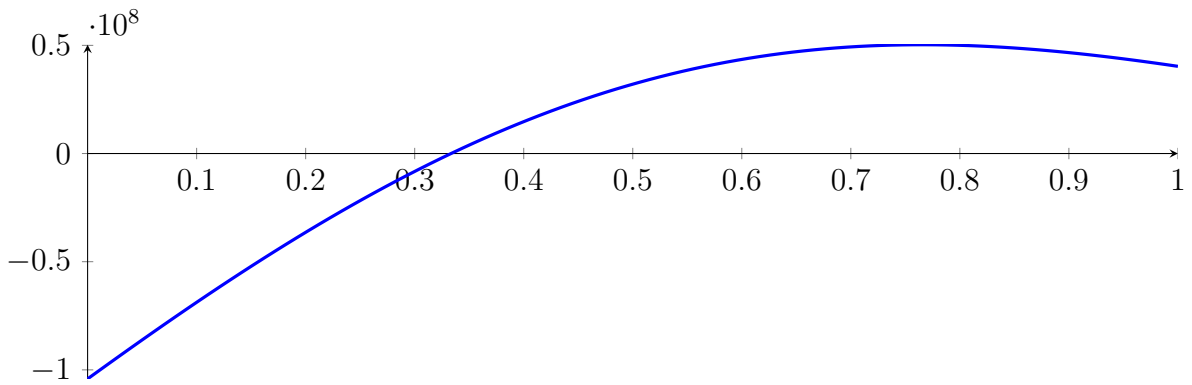
with precision $\varepsilon = 1 \cdot 10^{-08}$.

73 Running BezClip on f_{16} with epsilon 16

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

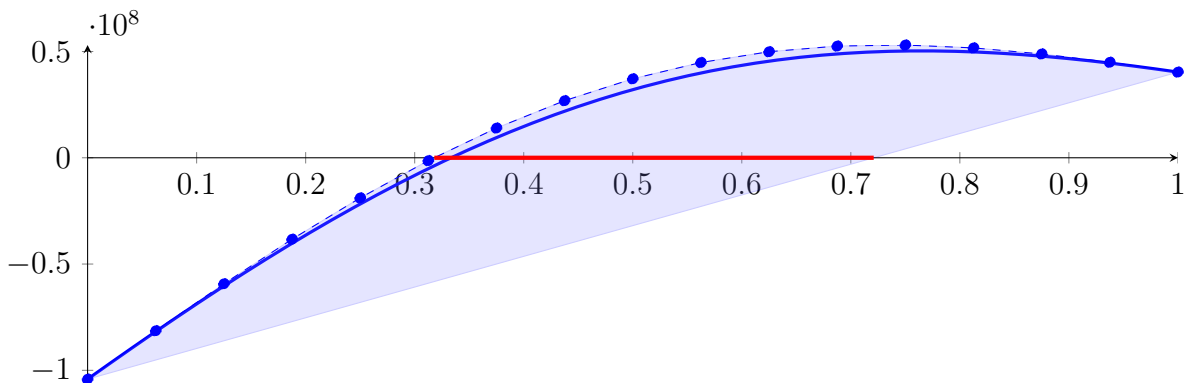
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



73.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

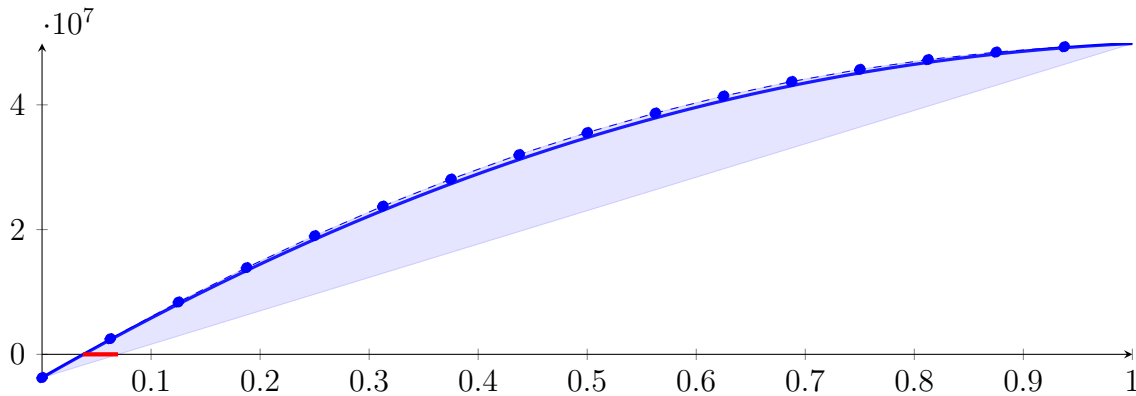
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

73.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.00483322X^{16} + 0.0186809X^{15} - 0.0194312X^{14} - 0.0738695X^{13} - 1.11673X^{12} \\ &\quad - 5.0471X^{11} + 36.3082X^{10} + 692.914X^9 + 1886.96X^8 - 25792X^7 - 149671X^6 + 492605X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

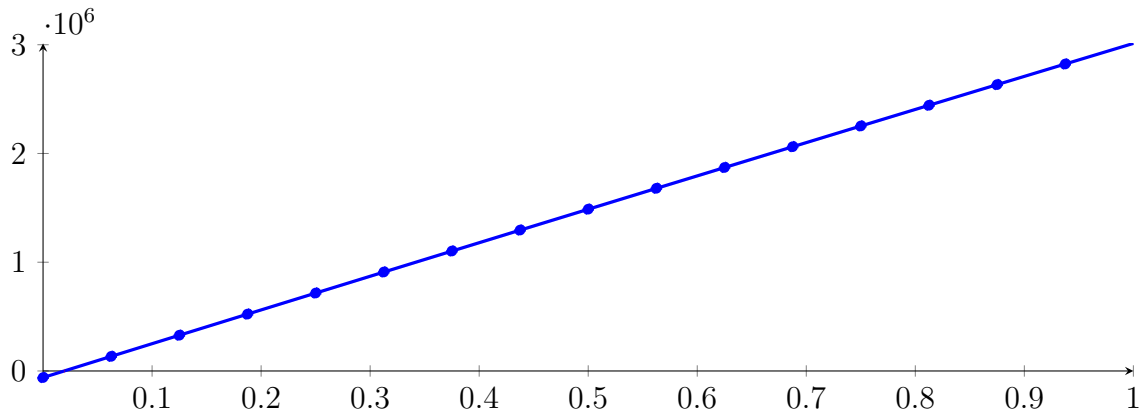
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

73.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.000205056X^{16} + 0.000776167X^{15} - 0.000863333X^{14} - 0.000217499X^{13} - 0.00809276X^{12} \\ &\quad - 5.09773 \cdot 10^{-05} X^{11} - 0.00564923X^{10} - 0.000162811X^9 - 0.000215376X^8 - 3.32948 \cdot 10^{-07} X^7 \\ &\quad - 0.000259866X^6 + 0.0161077X^5 + 4.36155X^4 - 234.216X^3 - 45622.2X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

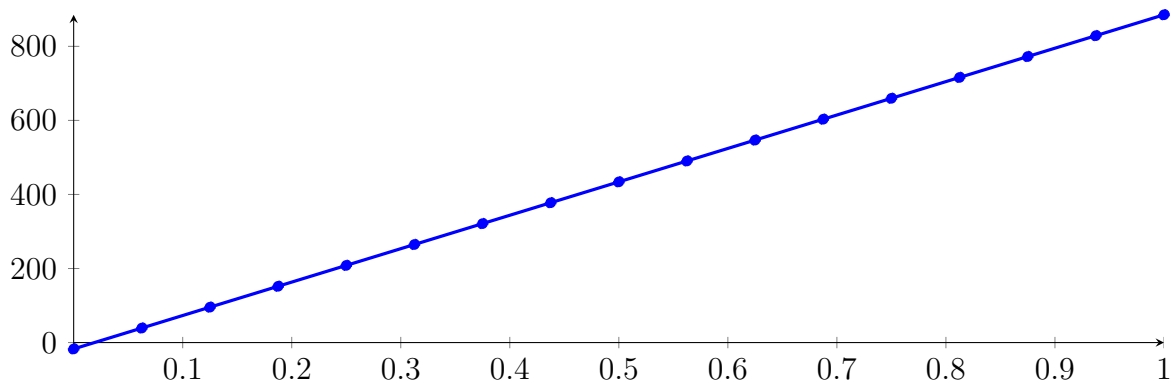
Longest intersection interval: 0.000289554

⇒ Selective recursion: interval 1: [0.333333, 0.333337],

73.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333337]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -5.9692 \cdot 10^{-08} X^{16} + 2.16103 \cdot 10^{-07} X^{15} - 2.28456 \cdot 10^{-07} X^{14} - 1.17238 \cdot 10^{-07} X^{13} \\
 &\quad - 2.29525 \cdot 10^{-06} X^{12} - 8.31778 \cdot 10^{-08} X^{11} - 1.74251 \cdot 10^{-06} X^{10} - 9.42919 \cdot 10^{-08} X^9 \\
 &\quad - 7.38891 \cdot 10^{-08} X^8 + 3.25144 \cdot 10^{-09} X^7 - 2.61741 \cdot 10^{-08} X^6 + 7.44876 \cdot 10^{-10} X^5 \\
 &\quad - 2.58638 \cdot 10^{-10} X^4 - 5.65024 \cdot 10^{-09} X^3 - 0.00382618 X^2 + 902.448 X - 17.178 \\
 &= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
 &\quad + 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
 &\quad + 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
 &\quad + 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

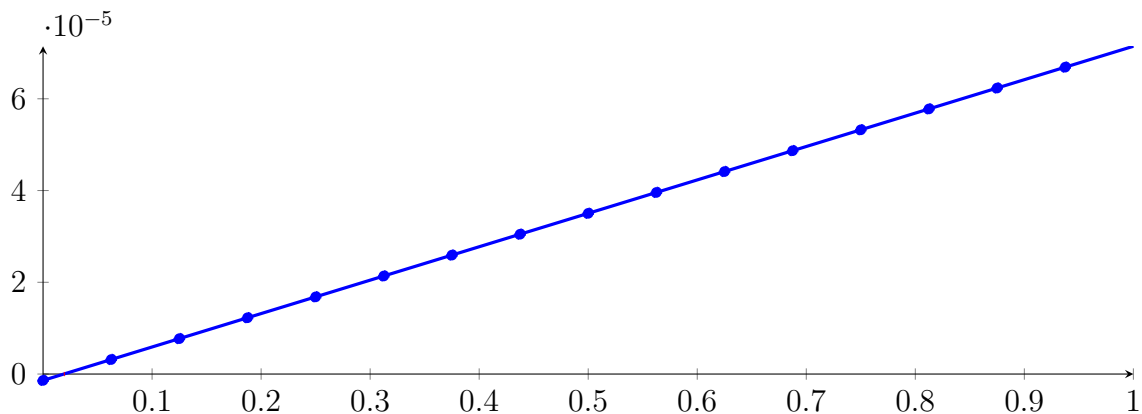
Longest intersection interval: $8.07045 \cdot 10^{-08}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

73.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -4.80379 \cdot 10^{-15} X^{16} + 1.74137 \cdot 10^{-14} X^{15} - 1.72656 \cdot 10^{-14} X^{14} - 6.84186 \cdot 10^{-15} X^{13} \\
 &\quad - 1.91627 \cdot 10^{-13} X^{12} - 4.7358 \cdot 10^{-15} X^{11} - 1.33436 \cdot 10^{-13} X^{10} - 1.97677 \cdot 10^{-15} X^9 \\
 &\quad - 7.4565 \cdot 10^{-15} X^8 - 1.16281 \cdot 10^{-16} X^7 - 1.98065 \cdot 10^{-15} X^6 + 1.47994 \cdot 10^{-17} X^5 \\
 &\quad - 1.84992 \cdot 10^{-17} X^4 - 2.48011 \cdot 10^{-17} X^2 + 7.28316 \cdot 10^{-05} X - 1.38634 \cdot 10^{-06} \\
 &= -1.38634 \cdot 10^{-06} B_{0,16}(X) + 3.16564 \cdot 10^{-06} B_{1,16}(X) + 7.71761 \cdot 10^{-06} B_{2,16}(X) + 1.22696 \\
 &\quad \cdot 10^{-05} B_{3,16}(X) + 1.68216 \cdot 10^{-05} B_{4,16}(X) + 2.13735 \cdot 10^{-05} B_{5,16}(X) + 2.59255 \cdot 10^{-05} B_{6,16}(X) \\
 &\quad + 3.04775 \cdot 10^{-05} B_{7,16}(X) + 3.50295 \cdot 10^{-05} B_{8,16}(X) + 3.95814 \cdot 10^{-05} B_{9,16}(X) + 4.41334 \\
 &\quad \cdot 10^{-05} B_{10,16}(X) + 4.86854 \cdot 10^{-05} B_{11,16}(X) + 5.32374 \cdot 10^{-05} B_{12,16}(X) + 5.77893 \cdot 10^{-05} B_{13,16}(X) \\
 &\quad + 6.23413 \cdot 10^{-05} B_{14,16}(X) + 6.68933 \cdot 10^{-05} B_{15,16}(X) + 7.14453 \cdot 10^{-05} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.50521 \cdot 10^{-15}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

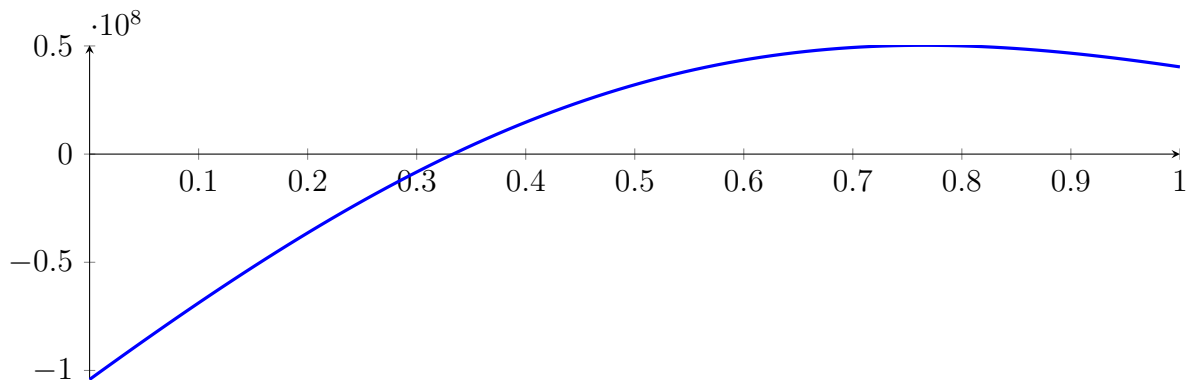
73.6 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

73.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

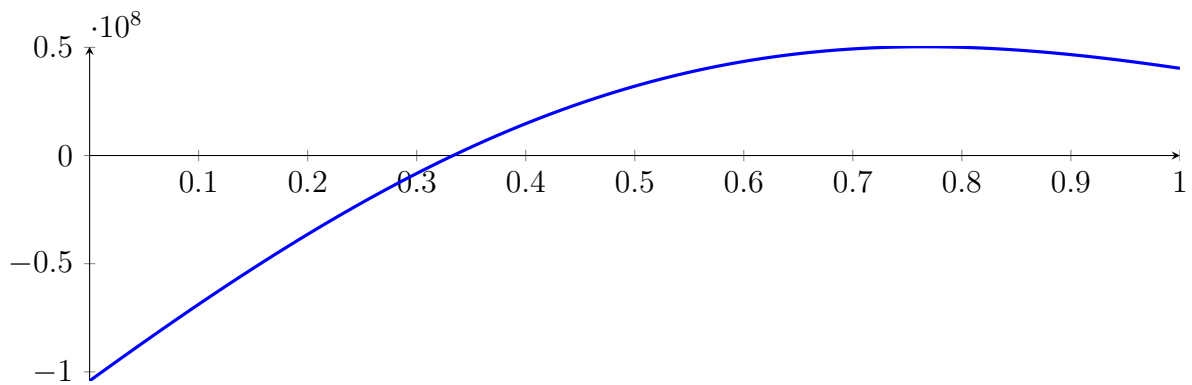
with precision $\varepsilon = 1 \cdot 10^{-16}$.

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$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

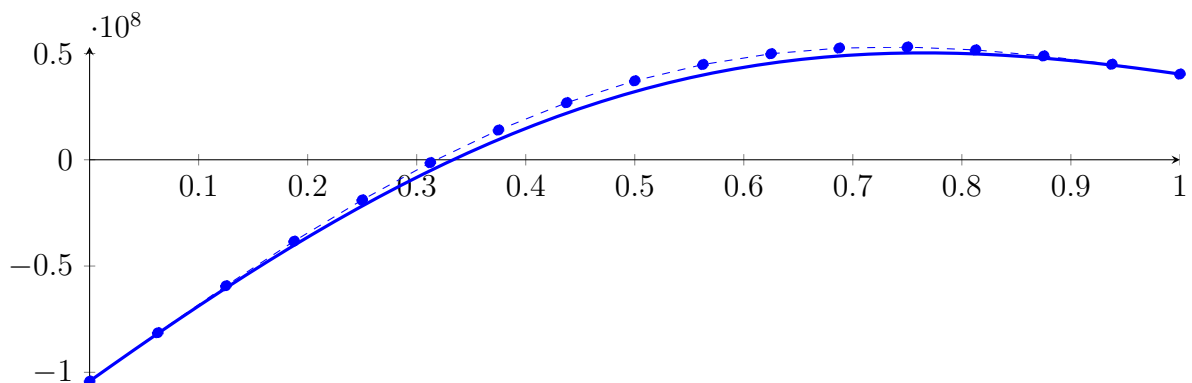
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



74.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = -1.41787 \cdot 10^6 X^{16} + 1.11761 \cdot 10^7 X^{15} - 3.98898 \cdot 10^7 X^{14} + 8.52437 \cdot 10^7 X^{13} - 1.21528 \cdot 10^8 X^{12}$$

$$+ 1.21946 \cdot 10^8 X^{11} - 8.86062 \cdot 10^7 X^{10} + 4.72904 \cdot 10^7 X^9 - 1.86355 \cdot 10^7 X^8 + 5.41059 \cdot 10^6 X^7 - 1.14487$$

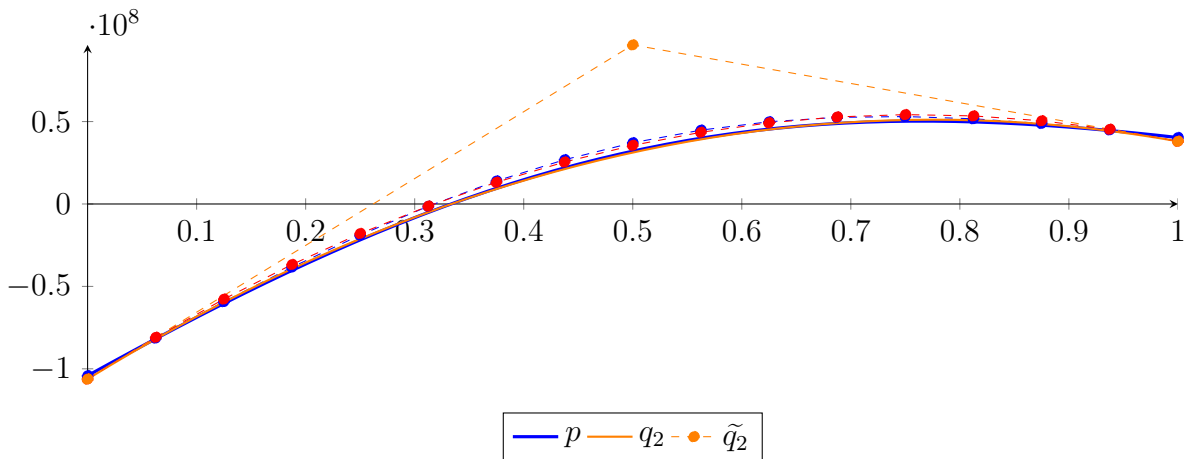
$$\cdot 10^6 X^6 + 172240 X^5 - 17636.6 X^4 + 1159.74 X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26923 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55691 \cdot 10^7 B_{8,16}$$

$$+ 4.34957 \cdot 10^7 B_{9,16} + 4.92455 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

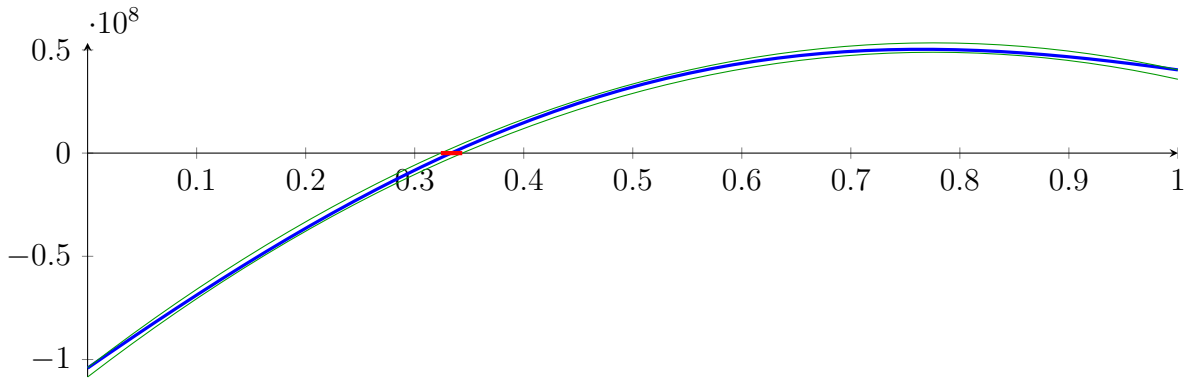
$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\}$$

$$N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

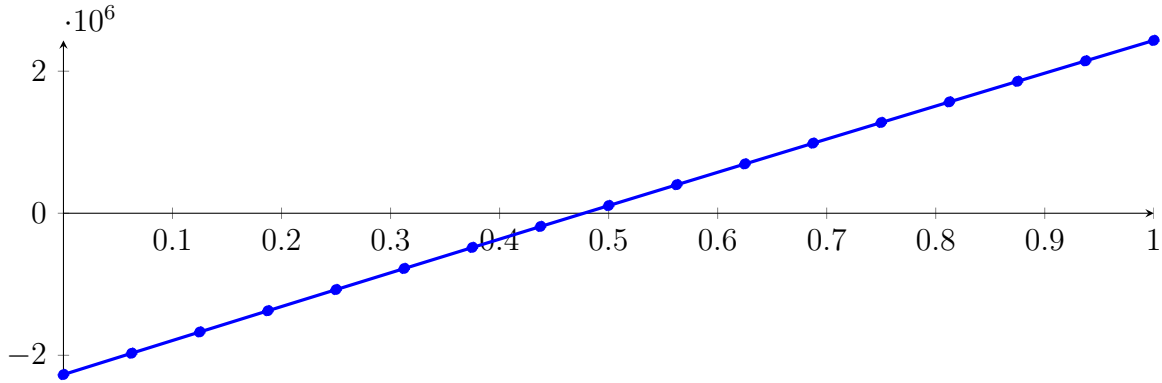
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

74.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

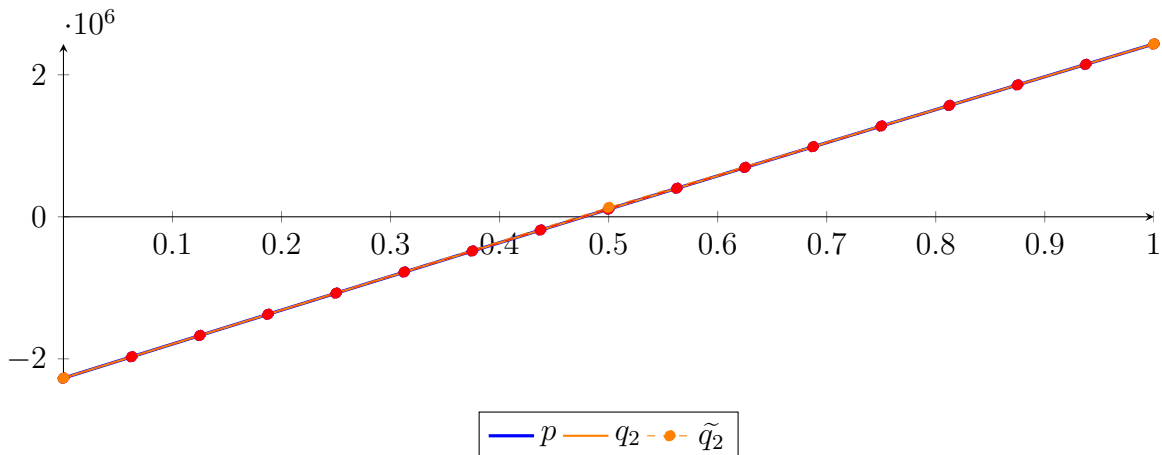
$$\begin{aligned}
 p &= -2.82438 \cdot 10^{-05} X^{16} + 2.90051 \cdot 10^{-05} X^{15} + 0.000231285 X^{14} + 0.000648014 X^{13} + 0.00318916 X^{12} \\
 &\quad + 0.00323204 X^{11} + 0.00460533 X^{10} + 0.00220012 X^9 - 0.000101882 X^8 + 5.32717 \cdot 10^{-06} X^7 \\
 &\quad - 0.00186824 X^6 + 0.132741 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 43556.7 X^{16} - 347904 X^{15} + 1.2616 \cdot 10^6 X^{14} - 2.74835 \cdot 10^6 X^{13} + 4.01042 \cdot 10^6 X^{12} - 4.13709 \\
 &\quad \cdot 10^6 X^{11} + 3.10251 \cdot 10^6 X^{10} - 1.71209 \cdot 10^6 X^9 + 695531 X^8 - 205965 X^7 + 43591.2 X^6 \\
 &\quad - 6402.94 X^5 + 625.678 X^4 - 37.485 X^3 - 104264 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481117 B_{6,16} - 185755 B_{7,16} + 108741 B_{8,16} \\
 &\quad + 402365 B_{9,16} + 695123 B_{10,16} + 987010 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

$$M = -104265 X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

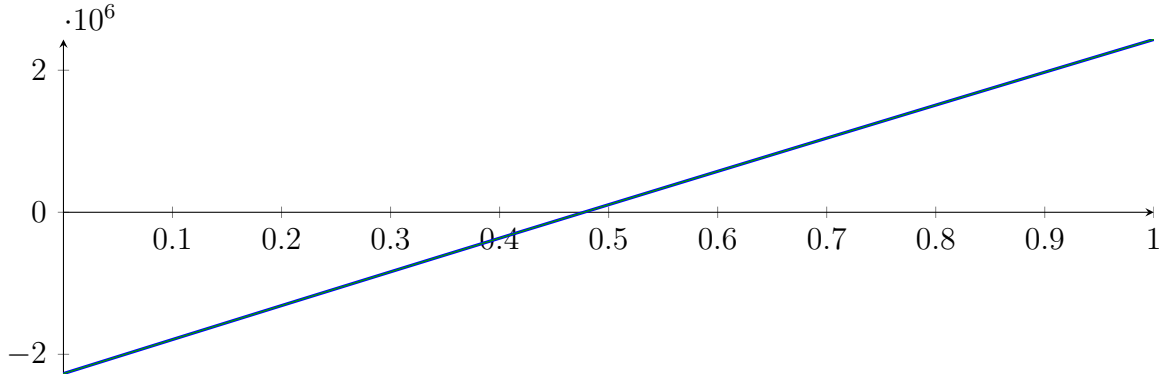
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\}$$

$$N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

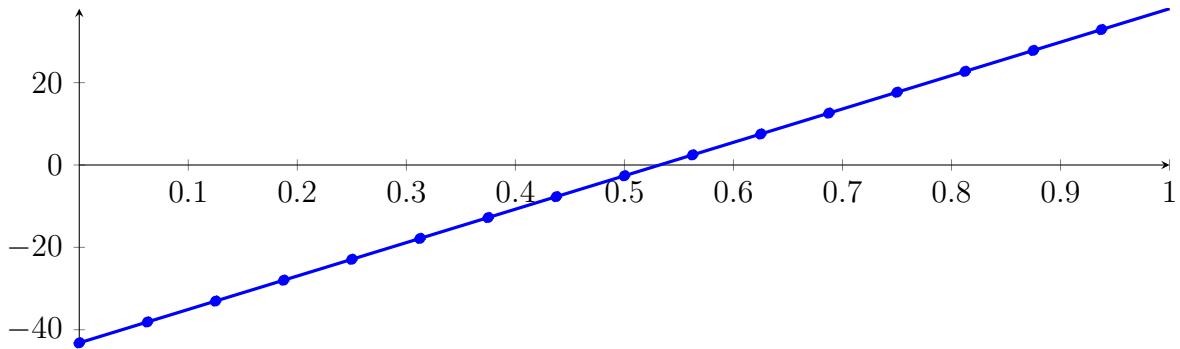
Longest intersection interval: $1.72301 \cdot 10^{-05}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

74.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

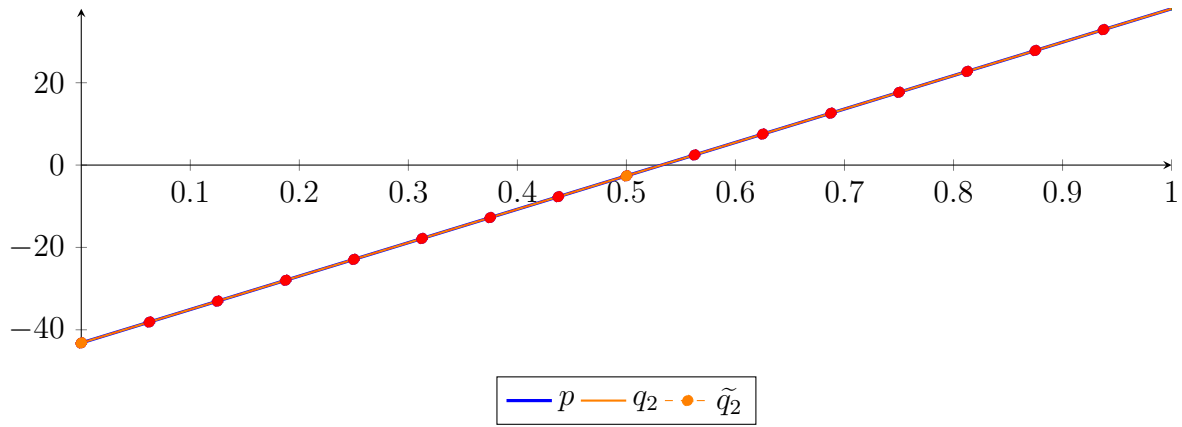
$$\begin{aligned} p &= 8.74252 \cdot 10^{-11} X^{16} - 1.56979 \cdot 10^{-09} X^{15} + 6.68479 \cdot 10^{-09} X^{14} + 1.20008 \cdot 10^{-08} X^{13} + 9.07301 \cdot 10^{-08} X^{12} \\ &+ 5.58657 \cdot 10^{-08} X^{11} + 1.13801 \cdot 10^{-07} X^{10} + 3.70665 \cdot 10^{-08} X^9 + 7.31575 \cdot 10^{-10} X^8 + 1.30058 \cdot 10^{-09} X^7 \\ &+ 5.00722 \cdot 10^{-09} X^6 + 1.24146 \cdot 10^{-10} X^5 + 1.03455 \cdot 10^{-10} X^4 - 3.09388 \cdot 10^{-05} X^2 + 81.1505X - 43.1911 \\ &= -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\ &- 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68777B_{7,16}(X) - 2.61587B_{8,16}(X) \\ &+ 2.45604B_{9,16}(X) + 7.52794B_{10,16}(X) + 12.5998B_{11,16}(X) + 17.6718B_{12,16}(X) \\ &+ 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09388 \cdot 10^{-05} X^2 + 81.1505X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 0.721495X^{16} - 5.74915X^{15} + 20.7933X^{14} - 45.1627X^{13} + 65.6806X^{12} - 67.5044X^{11} \\ &+ 50.4286X^{10} - 27.728X^9 + 11.2318X^8 - 3.32011X^7 + 0.702408X^6 - 0.103415X^5 \\ &+ 0.0102099X^4 - 0.000624725X^3 - 1.10834 \cdot 10^{-05} X^2 + 81.1505X - 43.1911 \\ &= -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} \\ &- 12.7597B_{6,16} - 7.68779B_{7,16} - 2.61585B_{8,16} + 2.45602B_{9,16} + 7.52795B_{10,16} + 12.5998B_{11,16} \\ &+ 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.57956 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -3.09388 \cdot 10^{-05} X^2 + 81.1505 X - 43.1911$$

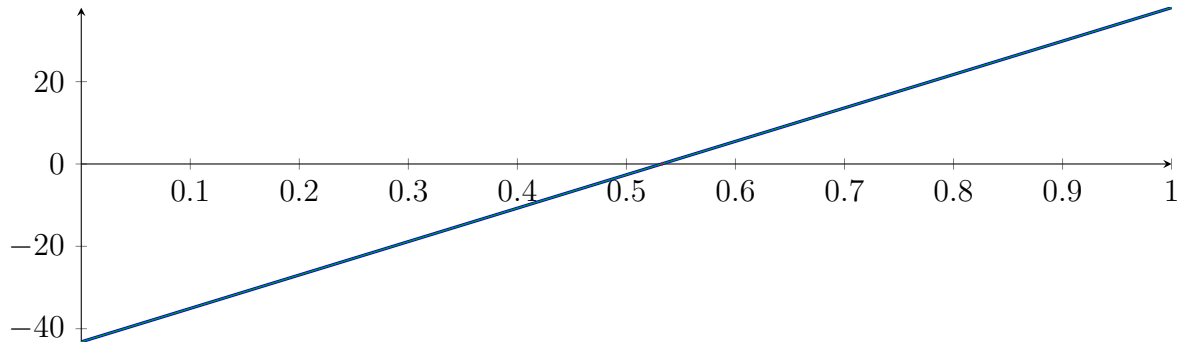
$$m = -3.09388 \cdot 10^{-05} X^2 + 81.1505 X - 43.1911$$

Root of M and m :

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\}$$

$$N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

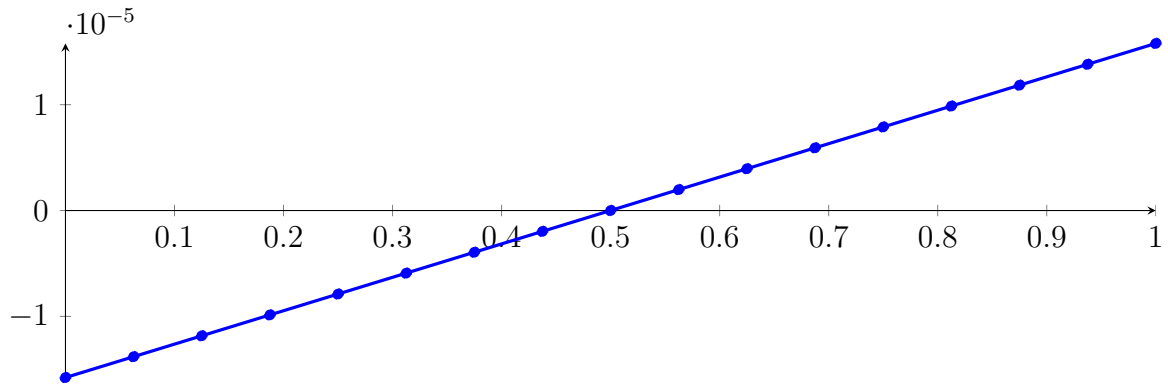
Longest intersection interval: $3.8903 \cdot 10^{-07}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

74.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
p &= -1.04409 \cdot 10^{-16} X^{16} - 1.53089 \cdot 10^{-16} X^{15} + 1.74665 \cdot 10^{-15} X^{14} + 5.46438 \cdot 10^{-15} X^{13} \\
&\quad + 2.56522 \cdot 10^{-14} X^{12} + 2.15479 \cdot 10^{-14} X^{11} + 3.51633 \cdot 10^{-14} X^{10} + 1.67444 \\
&\quad \cdot 10^{-14} X^9 - 8.72105 \cdot 10^{-16} X^8 + 1.41087 \cdot 10^{-15} X^6 + 5.91974 \cdot 10^{-17} X^5 + 4.93312 \\
&\quad \cdot 10^{-17} X^4 + 3.79471 \cdot 10^{-18} X^3 - 4.87891 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05} \\
&= -1.57804 \cdot 10^{-05} B_{0,16}(X) - 1.38073 \cdot 10^{-05} B_{1,16}(X) - 1.18341 \cdot 10^{-05} B_{2,16}(X) - 9.86101 \\
&\quad \cdot 10^{-06} B_{3,16}(X) - 7.88788 \cdot 10^{-06} B_{4,16}(X) - 5.91476 \cdot 10^{-06} B_{5,16}(X) - 3.94163 \cdot 10^{-06} B_{6,16}(X) \\
&\quad - 1.96851 \cdot 10^{-06} B_{7,16}(X) + 4.61501 \cdot 10^{-09} B_{8,16}(X) + 1.97774 \cdot 10^{-06} B_{9,16}(X) + 3.95086 \\
&\quad \cdot 10^{-06} B_{10,16}(X) + 5.92399 \cdot 10^{-06} B_{11,16}(X) + 7.89711 \cdot 10^{-06} B_{12,16}(X) + 9.87024 \cdot 10^{-06} B_{13,16}(X) \\
&\quad + 1.18434 \cdot 10^{-05} B_{14,16}(X) + 1.38165 \cdot 10^{-05} B_{15,16}(X) + 1.57896 \cdot 10^{-05} B_{16,16}(X)
\end{aligned}$$



Degree reduction and raising:

$$q_2 = -4.65868 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

$$= -1.57804 \cdot 10^{-05} B_{0,2} + 4.61501 \cdot 10^{-09} B_{1,2} + 1.57896 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = 2.92413 \cdot 10^{-07} X^{16} - 2.33332 \cdot 10^{-06} X^{15} + 8.45203 \cdot 10^{-06} X^{14} - 1.83895 \cdot 10^{-05} X^{13}$$

$$+ 2.67963 \cdot 10^{-05} X^{12} - 2.75995 \cdot 10^{-05} X^{11} + 2.06638 \cdot 10^{-05} X^{10} - 1.13854 \cdot 10^{-05} X^9$$

$$+ 4.61944 \cdot 10^{-06} X^8 - 1.36687 \cdot 10^{-06} X^7 + 2.89249 \cdot 10^{-07} X^6 - 4.25295 \cdot 10^{-08} X^5 + 4.17283$$

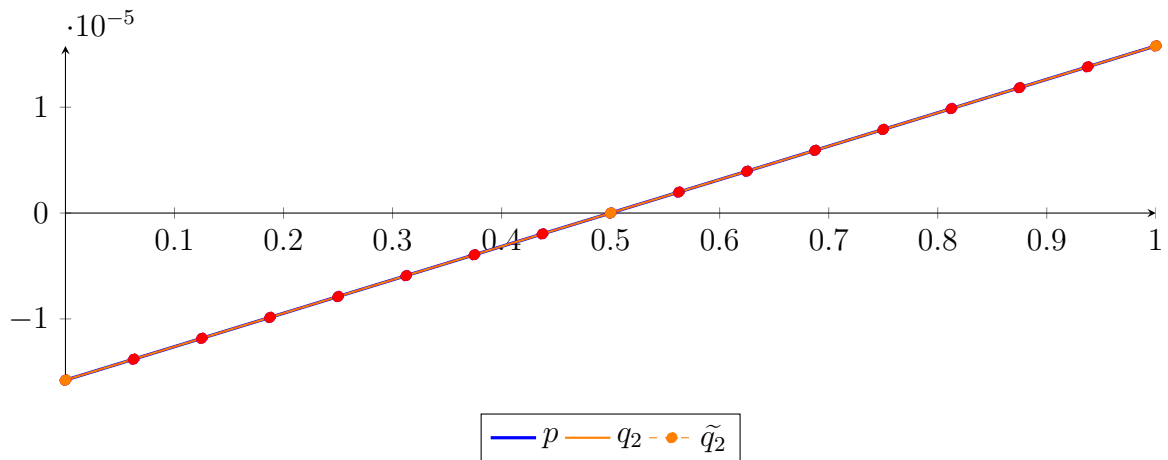
$$\cdot 10^{-09} X^4 - 2.52119 \cdot 10^{-10} X^3 + 7.82992 \cdot 10^{-12} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

$$= -1.57804 \cdot 10^{-05} B_{0,16} - 1.38073 \cdot 10^{-05} B_{1,16} - 1.18341 \cdot 10^{-05} B_{2,16} - 9.86101 \cdot 10^{-06} B_{3,16} - 7.88788$$

$$\cdot 10^{-06} B_{4,16} - 5.91476 \cdot 10^{-06} B_{5,16} - 3.94163 \cdot 10^{-06} B_{6,16} - 1.96851 \cdot 10^{-06} B_{7,16} + 4.62125 \cdot 10^{-09} B_{8,16}$$

$$+ 1.97773 \cdot 10^{-06} B_{9,16} + 3.95087 \cdot 10^{-06} B_{10,16} + 5.92399 \cdot 10^{-06} B_{11,16} + 7.89711 \cdot 10^{-06} B_{12,16}$$

$$+ 9.87024 \cdot 10^{-06} B_{13,16} + 1.18434 \cdot 10^{-05} B_{14,16} + 1.38165 \cdot 10^{-05} B_{15,16} + 1.57896 \cdot 10^{-05} B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 6.24192 \cdot 10^{-12}$.

Bounding polynomials M and m :

$$M = -4.65868 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

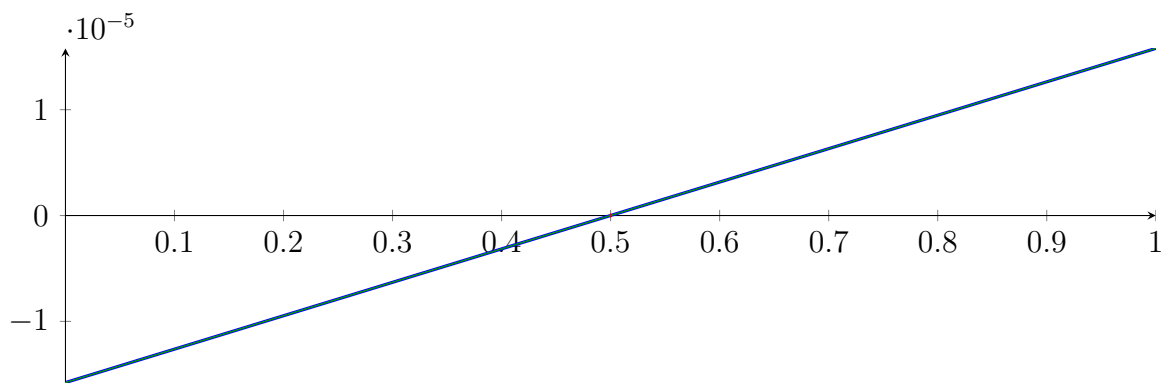
$$m = -4.65868 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.499636, 6.77659 \cdot 10^{12}\}$$

$$N(m) = \{0.500364, 6.77659 \cdot 10^{12}\}$$

Intersection intervals:



[0.499636, 0.500364]

Longest intersection interval: 0.000727273

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

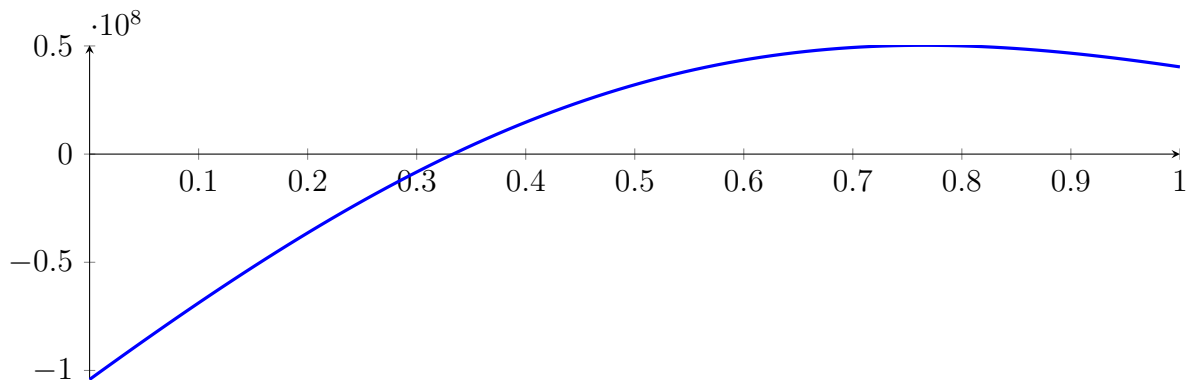
74.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

74.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

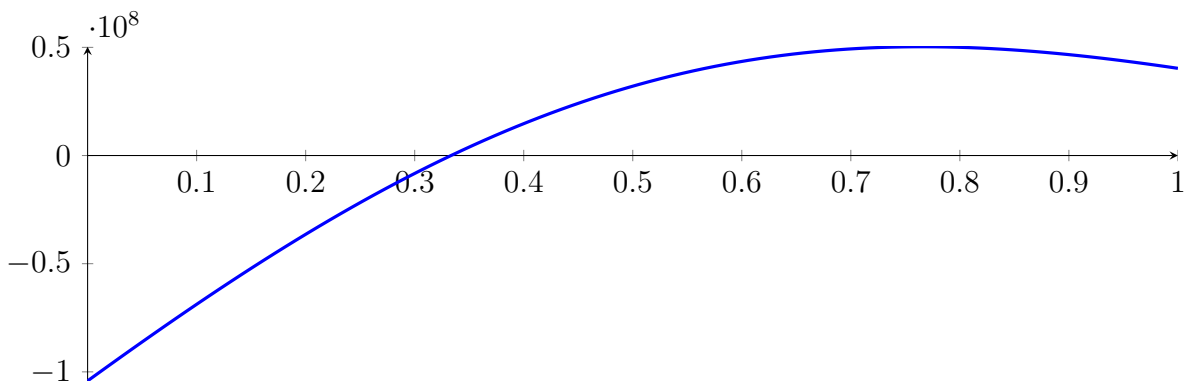
with precision $\varepsilon = 1 \cdot 10^{-16}$.

75 Running CubeClip on f_{16} with epsilon 16

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

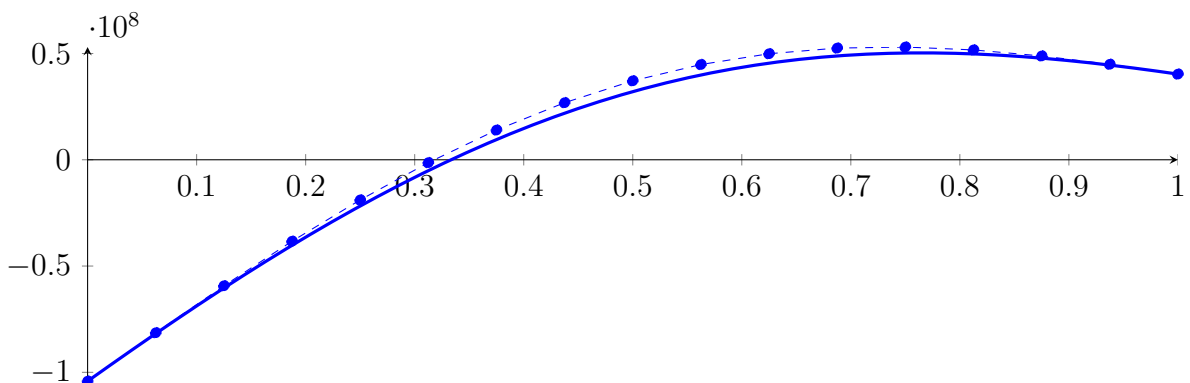
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



75.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 2.11088 \cdot 10^6 X^{16} - 1.70537 \cdot 10^7 X^{15} + 6.25913 \cdot 10^7 X^{14} - 1.38111 \cdot 10^8 X^{13} + 2.043 \cdot 10^8 X^{12} - 2.13799$$

$$\cdot 10^8 X^{11} + 1.62714 \cdot 10^8 X^{10} - 9.11092 \cdot 10^7 X^9 + 3.75254 \cdot 10^7 X^8 - 1.12529 \cdot 10^7 X^7 + 2.40899 \cdot 10^6 X^6$$

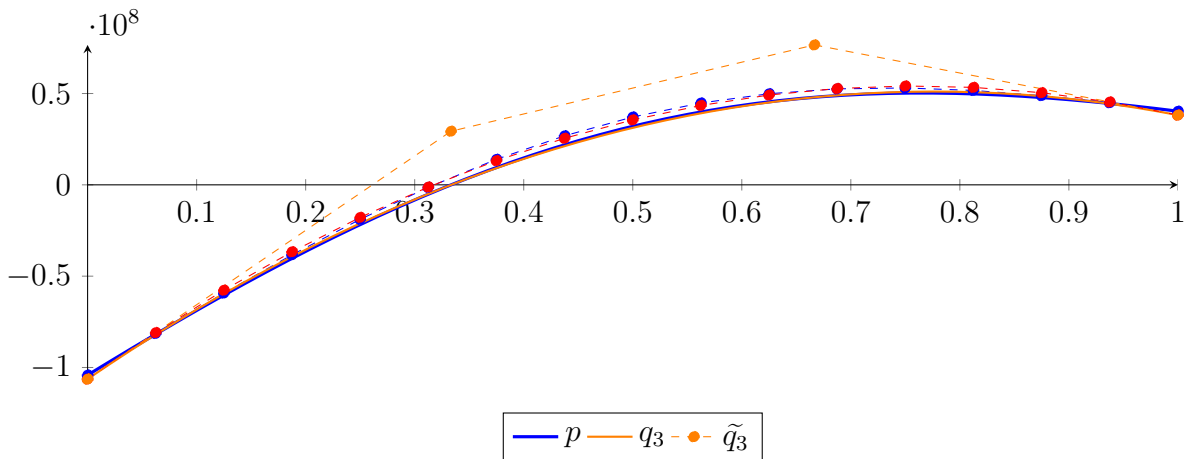
$$- 357156 X^5 + 34986.1 X^4 + 2.75602 \cdot 10^6 X^3 - 2.65318 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18553 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91816 \cdot 10^7 B_{10,16} + 5.27352 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

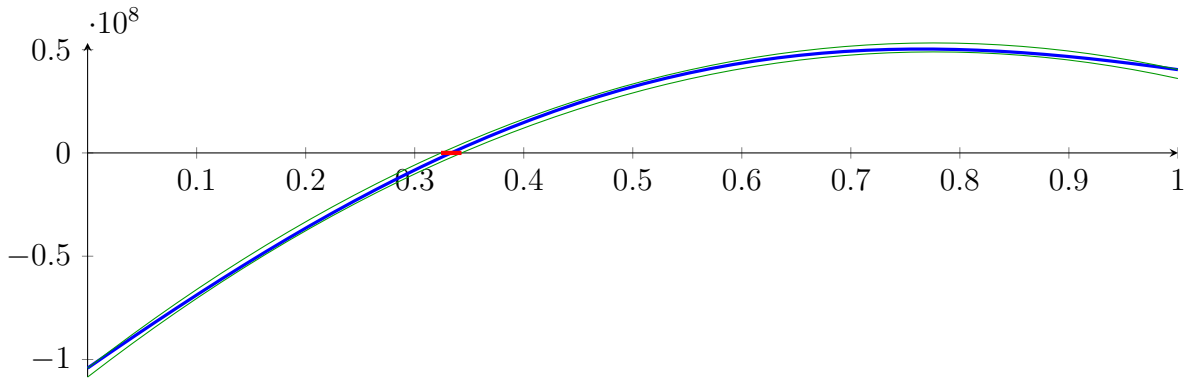
$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\} \quad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

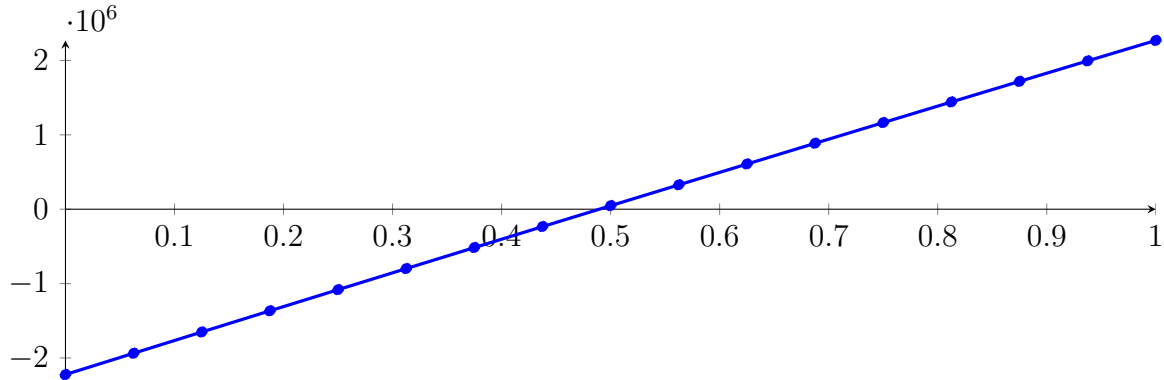
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

75.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

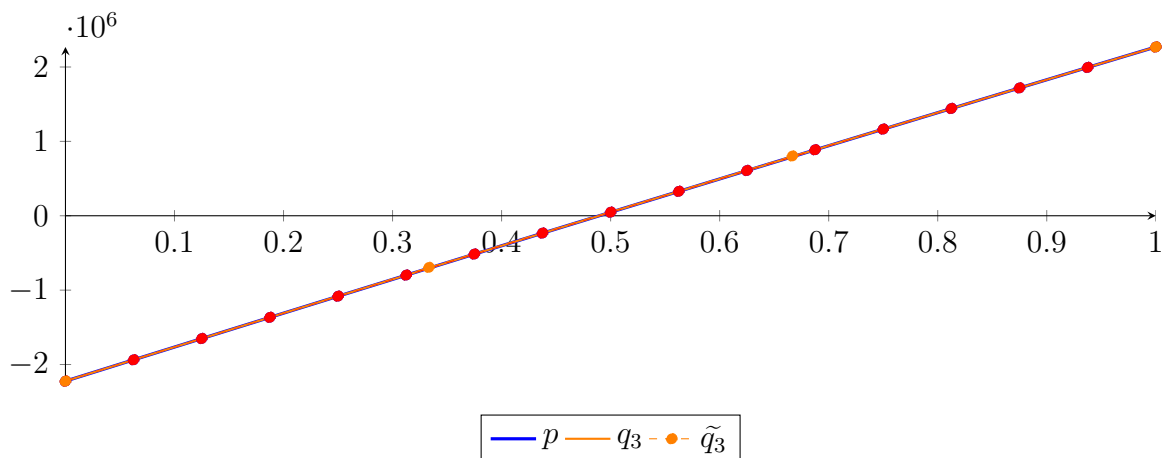
$$\begin{aligned}
 p &= -1.93035 \cdot 10^{-05} X^{16} + 1.08927 \cdot 10^{-05} X^{15} + 0.000255816 X^{14} + 0.000616983 X^{13} + 0.00371715 X^{12} \\
 &+ 0.00325035 X^{11} + 0.00510875 X^{10} + 0.00190713 X^9 + 0.000179792 X^8 + 1.06543 \cdot 10^{-05} X^7 \\
 &- 0.00136482 X^6 + 0.104959 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &- 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &+ 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 15290.6 X^{16} - 125412 X^{15} + 468045 X^{14} - 1.05198 \cdot 10^6 X^{13} + 1.588 \cdot 10^6 X^{12} - 1.69893 \\
 &\cdot 10^6 X^{11} + 1.32382 \cdot 10^6 X^{10} - 759402 X^9 + 320067 X^8 - 97826.5 X^7 + 21197.8 X^6 \\
 &- 3162.8 X^5 + 313.931 X^4 - 720.165 X^3 - 93879.2 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &- 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.7 B_{8,16} \\
 &+ 328649 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.457751$.

Bounding polynomials M and m :

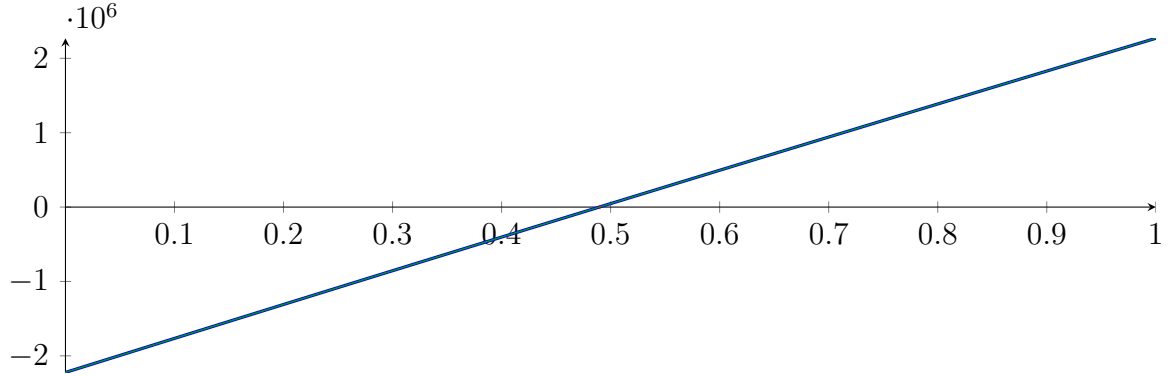
$$M = -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\} \quad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

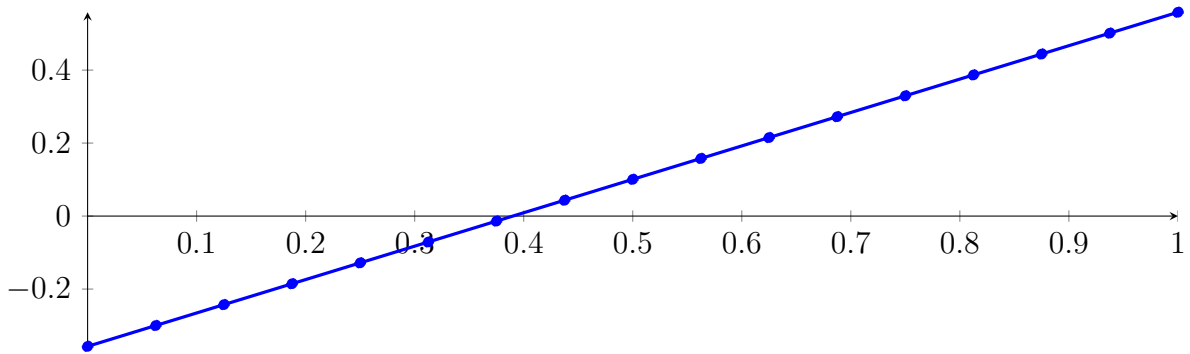
Longest intersection interval: $2.03684 \cdot 10^{-07}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

75.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

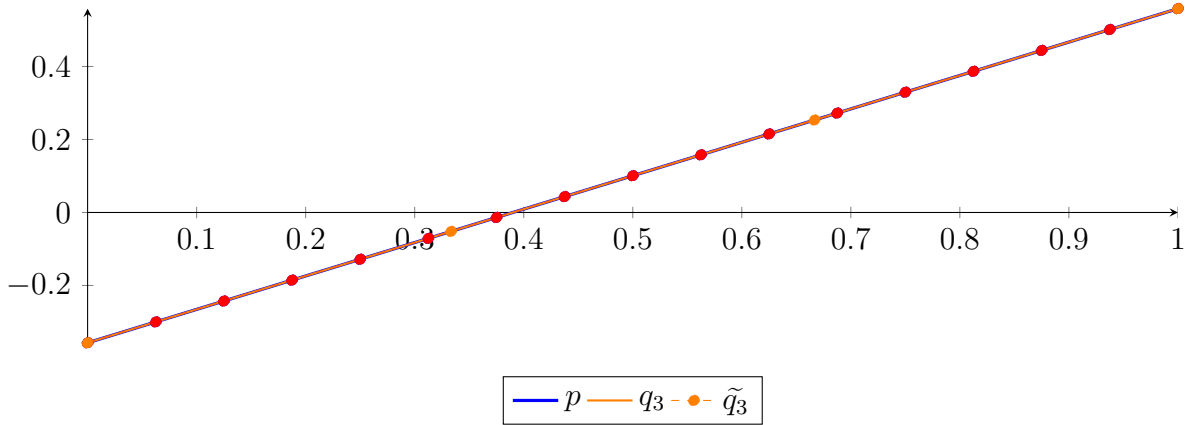
$$\begin{aligned} p &= -1.56399 \cdot 10^{-11} X^{16} + 4.58016 \cdot 10^{-11} X^{15} - 3.19744 \cdot 10^{-12} X^{14} + 6.54055 \cdot 10^{-11} X^{13} \\ &+ 1.05072 \cdot 10^{-10} X^{12} + 4.59728 \cdot 10^{-10} X^{11} + 5.01434 \cdot 10^{-10} X^{10} + 2.99742 \cdot 10^{-10} X^9 \\ &+ 1.14309 \cdot 10^{-11} X^8 - 5.08038 \cdot 10^{-12} X^7 + 3.37845 \cdot 10^{-11} X^6 - 9.69891 \cdot 10^{-13} X^5 \\ &+ 4.04121 \cdot 10^{-13} X^4 + 6.21725 \cdot 10^{-14} X^3 - 3.93767 \cdot 10^{-09} X^2 + 0.915501 X - 0.357072 \\ &= -0.357072 B_{0,16}(X) - 0.299853 B_{1,16}(X) - 0.242635 B_{2,16}(X) - 0.185416 B_{3,16}(X) \\ &- 0.128197 B_{4,16}(X) - 0.0709781 B_{5,16}(X) - 0.0137592 B_{6,16}(X) \\ &+ 0.0434596 B_{7,16}(X) + 0.100678 B_{8,16}(X) + 0.157897 B_{9,16}(X) + 0.215116 B_{10,16}(X) \\ &+ 0.272335 B_{11,16}(X) + 0.329554 B_{12,16}(X) + 0.386773 B_{13,16}(X) \\ &+ 0.443991 B_{14,16}(X) + 0.50121 B_{15,16}(X) + 0.558429 B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 1.05471 \cdot 10^{-15} X^3 - 3.93767 \cdot 10^{-09} X^2 + 0.915501 X - 0.357072 \\ &= -0.357072 B_{0,3} - 0.0519051 B_{1,3} + 0.253262 B_{2,3} + 0.558429 B_{3,3} \end{aligned}$$

$$\begin{aligned}
\tilde{q}_3 &= 0.00291222X^{16} - 0.0241801X^{15} + 0.0914452X^{14} - 0.208537X^{13} + 0.319778X^{12} - 0.347745X^{11} \\
&\quad + 0.275244X^{10} - 0.159971X^9 + 0.0679818X^8 - 0.0208072X^7 + 0.00447629X^6 - 0.000654783X^5 \\
&\quad + 6.22034 \cdot 10^{-05}X^4 - 3.60145 \cdot 10^{-06}X^3 + 9.78811 \cdot 10^{-08}X^2 + 0.915501X - 0.357072 \\
&= -0.357072B_{0,16} - 0.299853B_{1,16} - 0.242635B_{2,16} - 0.185416B_{3,16} - 0.128197B_{4,16} \\
&\quad - 0.0709781B_{5,16} - 0.0137592B_{6,16} + 0.0434595B_{7,16} + 0.100678B_{8,16} \\
&\quad + 0.157897B_{9,16} + 0.215116B_{10,16} + 0.272335B_{11,16} + 0.329554B_{12,16} \\
&\quad + 0.386773B_{13,16} + 0.443991B_{14,16} + 0.50121B_{15,16} + 0.558429B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.5212 \cdot 10^{-08}$.

Bounding polynomials M and m :

$$M = 9.99201 \cdot 10^{-16}X^3 - 3.93767 \cdot 10^{-09}X^2 + 0.915501X - 0.357072$$

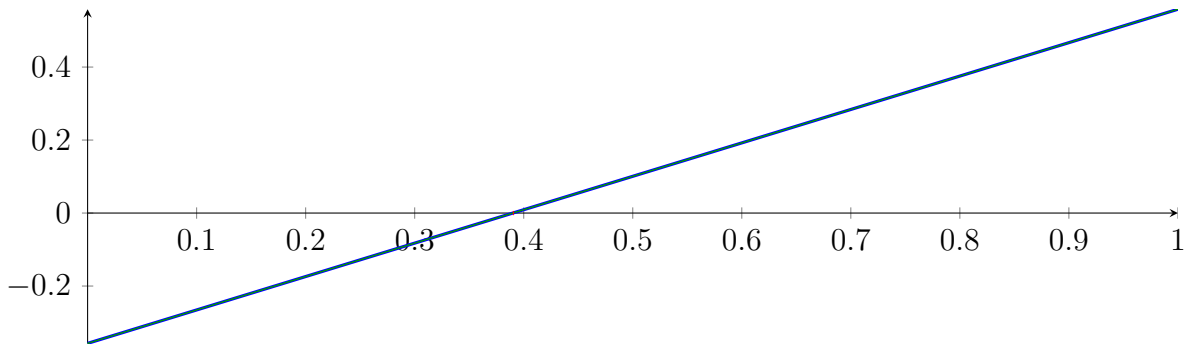
$$m = 1.22125 \cdot 10^{-15}X^3 - 3.93767 \cdot 10^{-09}X^2 + 0.915501X - 0.357072$$

Root of M and m :

$$N(M) = \{0.390029\}$$

$$N(m) = \{0.390029\}$$

Intersection intervals:



$$[0.390029, 0.390029]$$

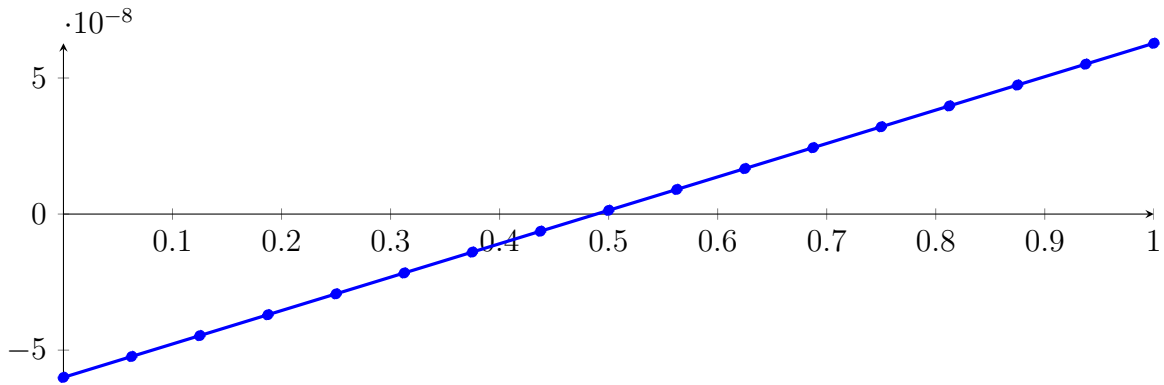
Longest intersection interval: $1.3411 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

75.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

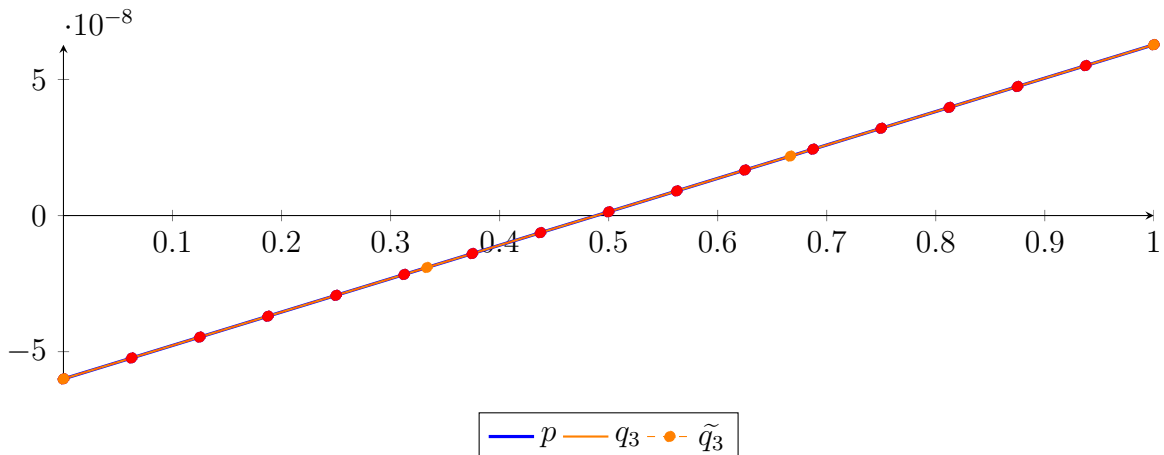
$$\begin{aligned}
 p &= -5.65183 \cdot 10^{-19} X^{16} + 5.13302 \cdot 10^{-19} X^{15} + 6.37816 \cdot 10^{-18} X^{14} + 1.69576 \cdot 10^{-17} X^{13} \\
 &\quad + 9.4423 \cdot 10^{-17} X^{12} + 8.0934 \cdot 10^{-17} X^{11} + 1.37357 \cdot 10^{-16} X^{10} + 6.23797 \cdot 10^{-17} X^9 \\
 &\quad + 1.36266 \cdot 10^{-18} X^8 + 6.05629 \cdot 10^{-19} X^7 + 5.08728 \cdot 10^{-18} X^6 + 2.3124 \cdot 10^{-19} X^5 \\
 &\quad + 1.44525 \cdot 10^{-19} X^4 - 7.41154 \cdot 10^{-21} X^3 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08} \\
 &= -6.00146 \cdot 10^{-08} B_{0,16}(X) - 5.2341 \cdot 10^{-08} B_{1,16}(X) - 4.46674 \cdot 10^{-08} B_{2,16}(X) - 3.69937 \\
 &\quad \cdot 10^{-08} B_{3,16}(X) - 2.93201 \cdot 10^{-08} B_{4,16}(X) - 2.16464 \cdot 10^{-08} B_{5,16}(X) - 1.39728 \cdot 10^{-08} B_{6,16}(X) \\
 &\quad - 6.29913 \cdot 10^{-09} B_{7,16}(X) + 1.37451 \cdot 10^{-09} B_{8,16}(X) + 9.04815 \cdot 10^{-09} B_{9,16}(X) + 1.67218 \\
 &\quad \cdot 10^{-08} B_{10,16}(X) + 2.43954 \cdot 10^{-08} B_{11,16}(X) + 3.20691 \cdot 10^{-08} B_{12,16}(X) + 3.97427 \\
 &\quad \cdot 10^{-08} B_{13,16}(X) + 4.74164 \cdot 10^{-08} B_{14,16}(X) + 5.509 \cdot 10^{-08} B_{15,16}(X) + 6.27637 \cdot 10^{-08} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= 2.51463 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08} \\
 &= -6.00146 \cdot 10^{-08} B_{0,3} - 1.90885 \cdot 10^{-08} B_{1,3} + 2.18376 \cdot 10^{-08} B_{2,3} + 6.27637 \cdot 10^{-08} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 4.01426 \cdot 10^{-10} X^{16} - 3.29623 \cdot 10^{-09} X^{15} + 1.2317 \cdot 10^{-08} X^{14} - 2.77213 \cdot 10^{-08} X^{13} + 4.19074 \\
 &\quad \cdot 10^{-08} X^{12} - 4.49047 \cdot 10^{-08} X^{11} + 3.50457 \cdot 10^{-08} X^{10} - 2.01341 \cdot 10^{-08} X^9 + 8.49676 \\
 &\quad \cdot 10^{-09} X^8 - 2.5992 \cdot 10^{-09} X^7 + 5.63364 \cdot 10^{-10} X^6 - 8.40231 \cdot 10^{-11} X^5 + 8.33259 \\
 &\quad \cdot 10^{-12} X^4 - 5.16965 \cdot 10^{-13} X^3 + 1.7395 \cdot 10^{-14} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08} \\
 &= -6.00146 \cdot 10^{-08} B_{0,16} - 5.2341 \cdot 10^{-08} B_{1,16} - 4.46674 \cdot 10^{-08} B_{2,16} - 3.69937 \cdot 10^{-08} B_{3,16} - 2.93201 \\
 &\quad \cdot 10^{-08} B_{4,16} - 2.16464 \cdot 10^{-08} B_{5,16} - 1.39728 \cdot 10^{-08} B_{6,16} - 6.29914 \cdot 10^{-09} B_{7,16} + 1.37452 \cdot 10^{-09} B_{8,16} \\
 &\quad + 9.04815 \cdot 10^{-09} B_{9,16} + 1.67218 \cdot 10^{-08} B_{10,16} + 2.43954 \cdot 10^{-08} B_{11,16} + 3.20691 \cdot 10^{-08} B_{12,16} \\
 &\quad + 3.97427 \cdot 10^{-08} B_{13,16} + 4.74164 \cdot 10^{-08} B_{14,16} + 5.509 \cdot 10^{-08} B_{15,16} + 6.27637 \cdot 10^{-08} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.90061 \cdot 10^{-15}$.

Bounding polynomials M and m :

$$M = 2.38228 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08}$$

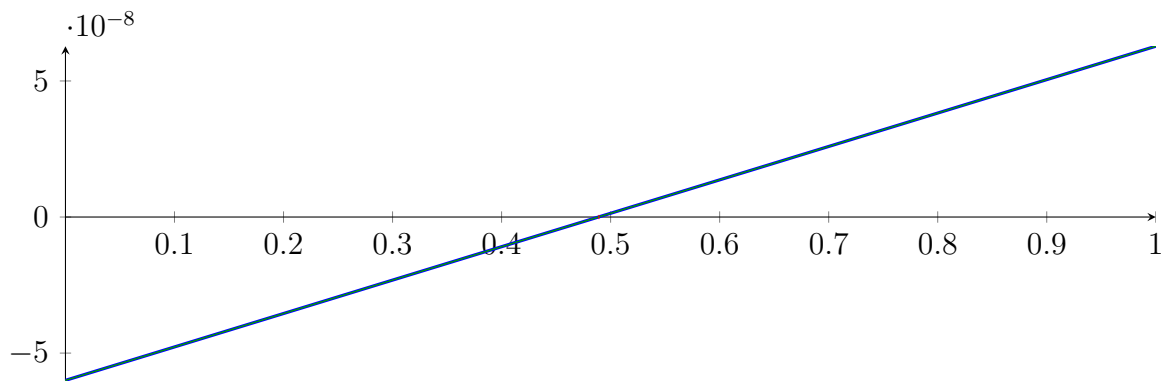
$$m = 2.51463 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08}$$

Root of M and m :

$$N(M) = \{0.488805\}$$

$$N(m) = \{0.488805\}$$

Intersection intervals:



$$[0.488805, 0.488805]$$

Longest intersection interval: $1.3086 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

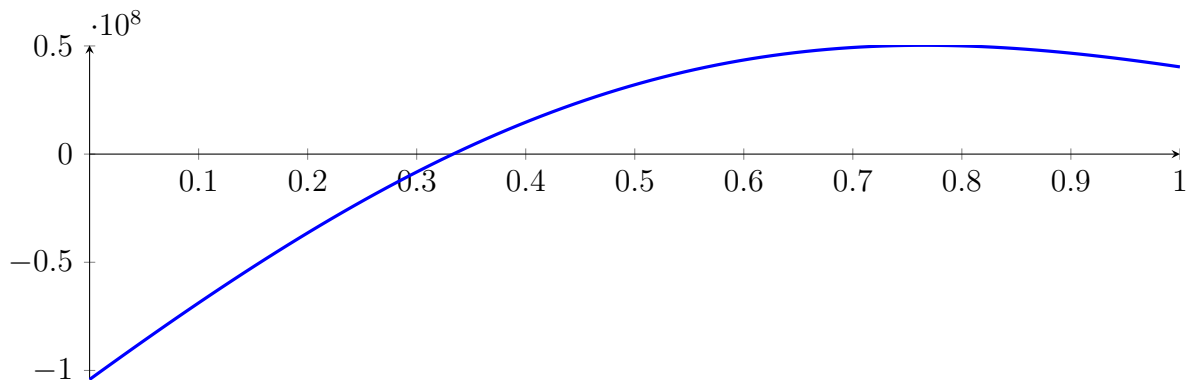
75.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

75.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

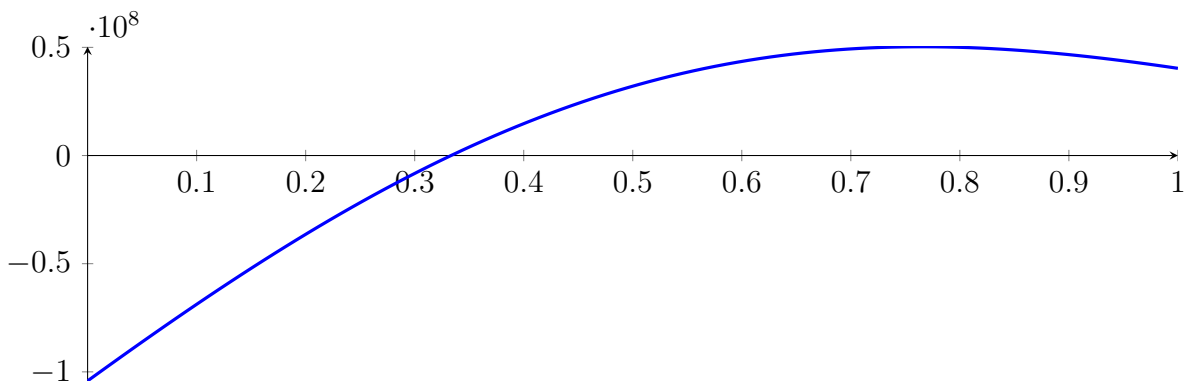
with precision $\varepsilon = 1 \cdot 10^{-16}$.

76 Running BezClip on f_{16} with epsilon 32

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

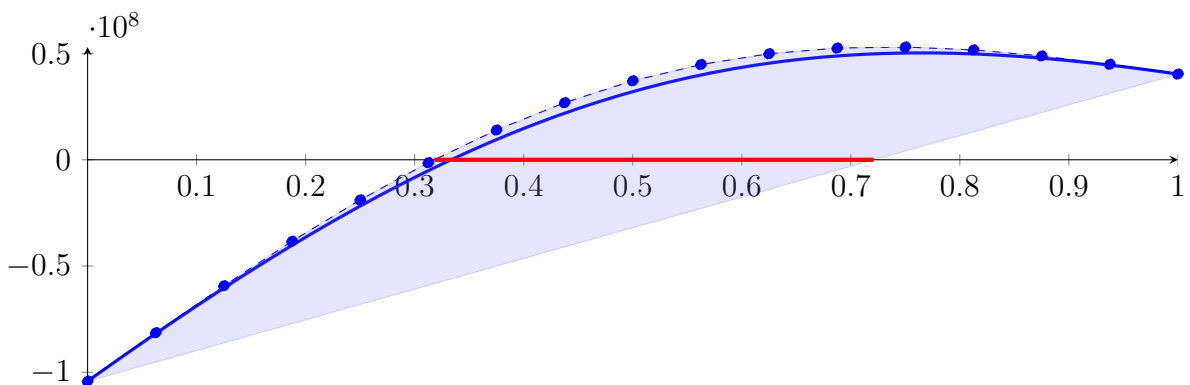
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



76.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

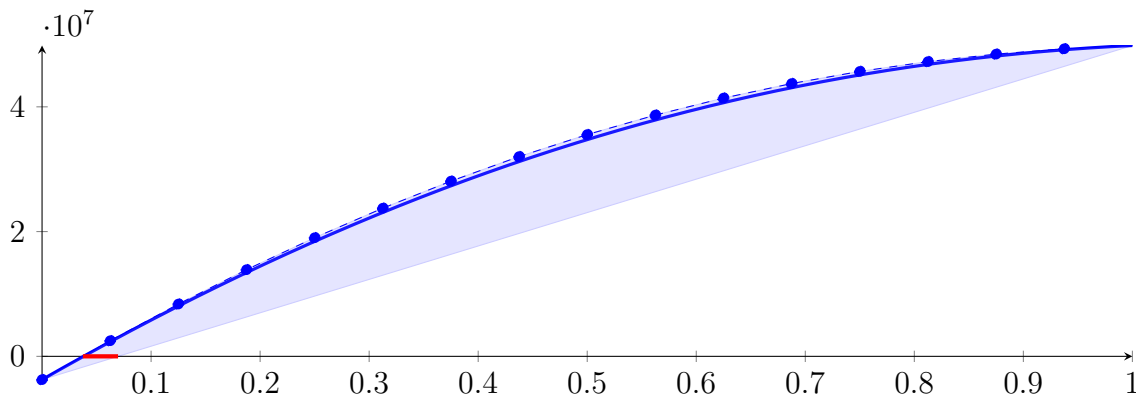
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

76.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.00483322X^{16} + 0.0186809X^{15} - 0.0194312X^{14} - 0.0738695X^{13} - 1.11673X^{12} \\ &\quad - 5.0471X^{11} + 36.3082X^{10} + 692.914X^9 + 1886.96X^8 - 25792X^7 - 149671X^6 + 492605X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

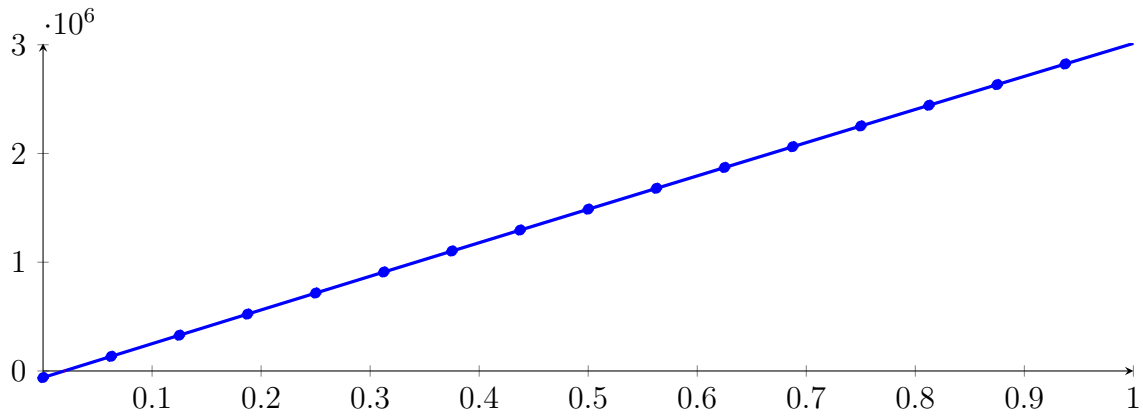
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

76.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.000205056X^{16} + 0.000776167X^{15} - 0.000863333X^{14} - 0.000217499X^{13} - 0.00809276X^{12} \\ &\quad - 5.09773 \cdot 10^{-05} X^{11} - 0.00564923X^{10} - 0.000162811X^9 - 0.000215376X^8 - 3.32948 \cdot 10^{-07} X^7 \\ &\quad - 0.000259866X^6 + 0.0161077X^5 + 4.36155X^4 - 234.216X^3 - 45622.2X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

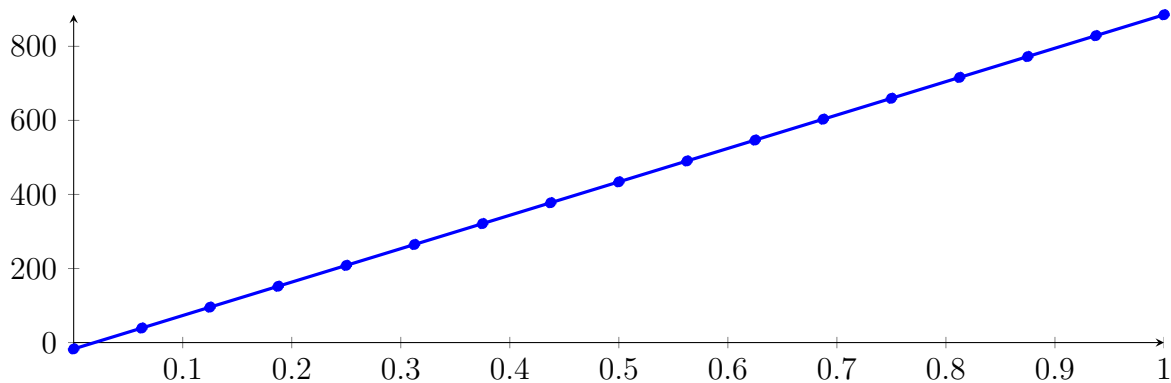
Longest intersection interval: 0.000289554

\Rightarrow Selective recursion: interval 1: [\[0.333333, 0.333337\]](#),

76.4 Recursion Branch 1 1 1 1 in Interval 1: [\[0.333333, 0.333337\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -5.9692 \cdot 10^{-08} X^{16} + 2.16103 \cdot 10^{-07} X^{15} - 2.28456 \cdot 10^{-07} X^{14} - 1.17238 \cdot 10^{-07} X^{13} \\
 &\quad - 2.29525 \cdot 10^{-06} X^{12} - 8.31778 \cdot 10^{-08} X^{11} - 1.74251 \cdot 10^{-06} X^{10} - 9.42919 \cdot 10^{-08} X^9 \\
 &\quad - 7.38891 \cdot 10^{-08} X^8 + 3.25144 \cdot 10^{-09} X^7 - 2.61741 \cdot 10^{-08} X^6 + 7.44876 \cdot 10^{-10} X^5 \\
 &\quad - 2.58638 \cdot 10^{-10} X^4 - 5.65024 \cdot 10^{-09} X^3 - 0.00382618 X^2 + 902.448 X - 17.178 \\
 &= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
 &\quad + 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
 &\quad + 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
 &\quad + 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

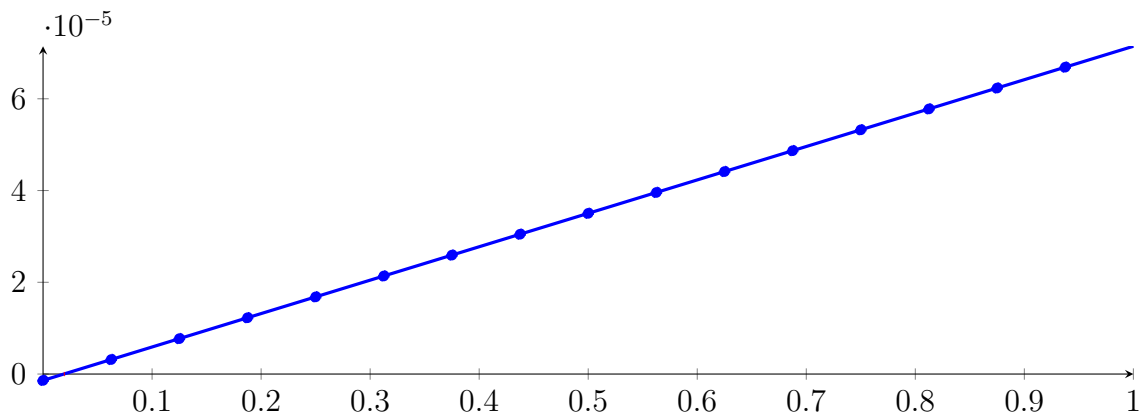
Longest intersection interval: $8.07045 \cdot 10^{-08}$

\Rightarrow Selective recursion: interval 1: [\[0.333333, 0.333333\]](#),

76.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -4.80379 \cdot 10^{-15} X^{16} + 1.74137 \cdot 10^{-14} X^{15} - 1.72656 \cdot 10^{-14} X^{14} - 6.84186 \cdot 10^{-15} X^{13} \\
 &\quad - 1.91627 \cdot 10^{-13} X^{12} - 4.7358 \cdot 10^{-15} X^{11} - 1.33436 \cdot 10^{-13} X^{10} - 1.97677 \cdot 10^{-15} X^9 \\
 &\quad - 7.4565 \cdot 10^{-15} X^8 - 1.16281 \cdot 10^{-16} X^7 - 1.98065 \cdot 10^{-15} X^6 + 1.47994 \cdot 10^{-17} X^5 \\
 &\quad - 1.84992 \cdot 10^{-17} X^4 - 2.48011 \cdot 10^{-17} X^2 + 7.28316 \cdot 10^{-05} X - 1.38634 \cdot 10^{-06} \\
 &= -1.38634 \cdot 10^{-06} B_{0,16}(X) + 3.16564 \cdot 10^{-06} B_{1,16}(X) + 7.71761 \cdot 10^{-06} B_{2,16}(X) + 1.22696 \\
 &\quad \cdot 10^{-05} B_{3,16}(X) + 1.68216 \cdot 10^{-05} B_{4,16}(X) + 2.13735 \cdot 10^{-05} B_{5,16}(X) + 2.59255 \cdot 10^{-05} B_{6,16}(X) \\
 &\quad + 3.04775 \cdot 10^{-05} B_{7,16}(X) + 3.50295 \cdot 10^{-05} B_{8,16}(X) + 3.95814 \cdot 10^{-05} B_{9,16}(X) + 4.41334 \\
 &\quad \cdot 10^{-05} B_{10,16}(X) + 4.86854 \cdot 10^{-05} B_{11,16}(X) + 5.32374 \cdot 10^{-05} B_{12,16}(X) + 5.77893 \cdot 10^{-05} B_{13,16}(X) \\
 &\quad + 6.23413 \cdot 10^{-05} B_{14,16}(X) + 6.68933 \cdot 10^{-05} B_{15,16}(X) + 7.14453 \cdot 10^{-05} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.50521 \cdot 10^{-15}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

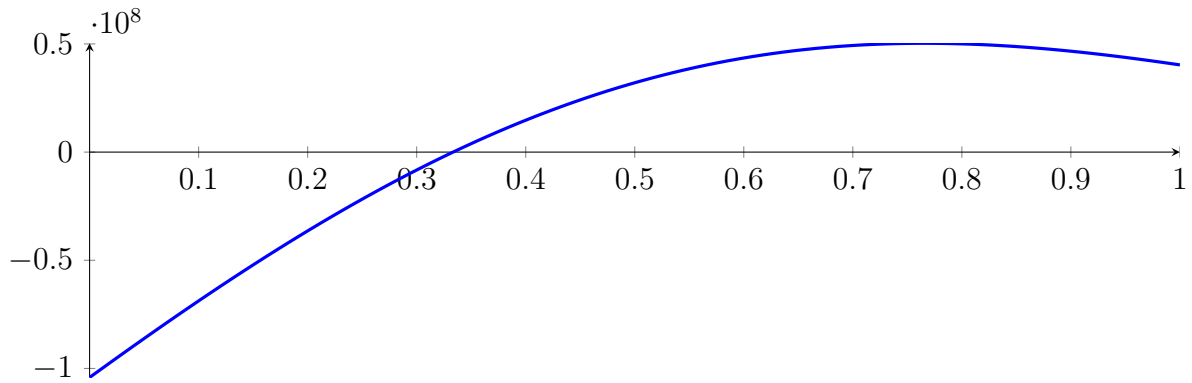
76.6 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

76.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

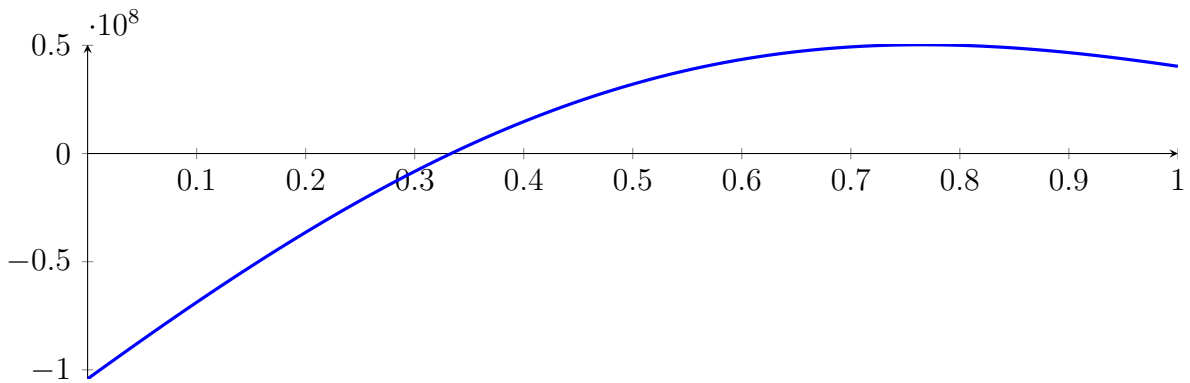
with precision $\varepsilon = 1 \cdot 10^{-32}$.

77 Running QuadClip on f_{16} with epsilon 32

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

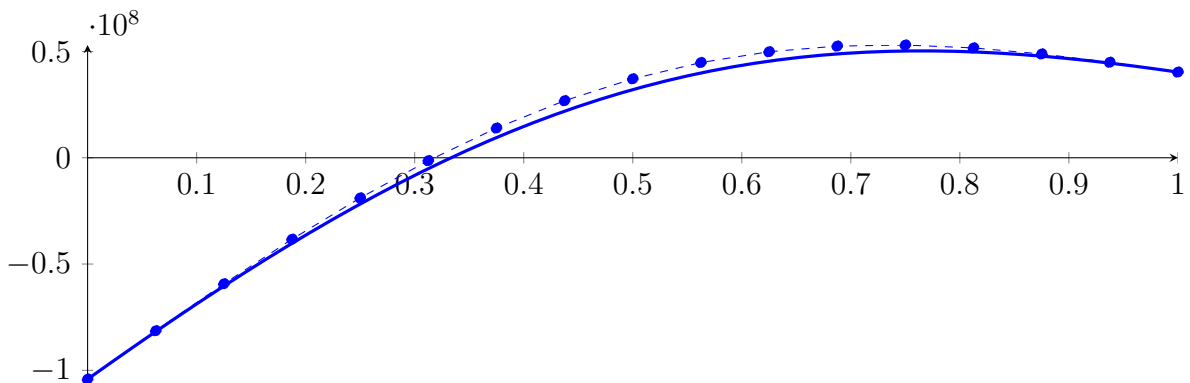
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



77.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = -1.41787 \cdot 10^6 X^{16} + 1.11761 \cdot 10^7 X^{15} - 3.98898 \cdot 10^7 X^{14} + 8.52437 \cdot 10^7 X^{13} - 1.21528 \cdot 10^8 X^{12}$$

$$+ 1.21946 \cdot 10^8 X^{11} - 8.86062 \cdot 10^7 X^{10} + 4.72904 \cdot 10^7 X^9 - 1.86355 \cdot 10^7 X^8 + 5.41059 \cdot 10^6 X^7 - 1.14487$$

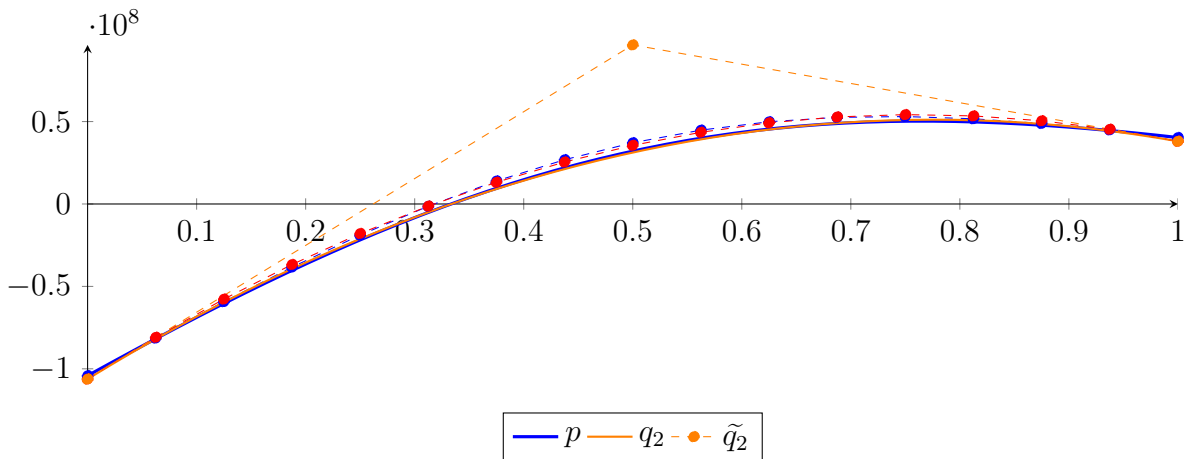
$$\cdot 10^6 X^6 + 172240 X^5 - 17636.6 X^4 + 1159.74 X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26923 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55691 \cdot 10^7 B_{8,16}$$

$$+ 4.34957 \cdot 10^7 B_{9,16} + 4.92455 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

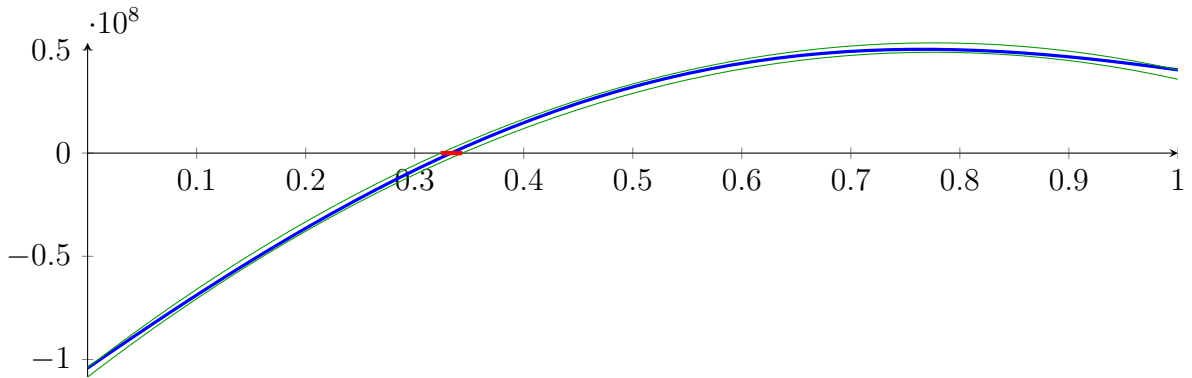
$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\}$$

$$N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

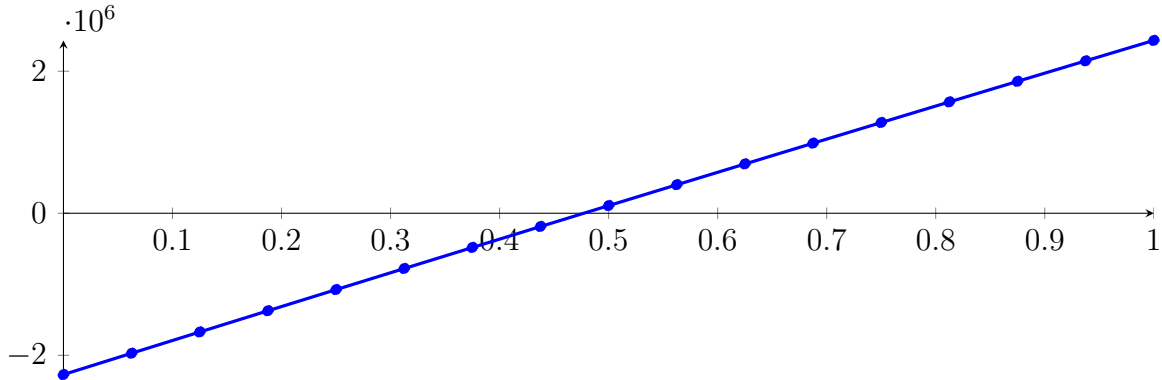
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

77.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

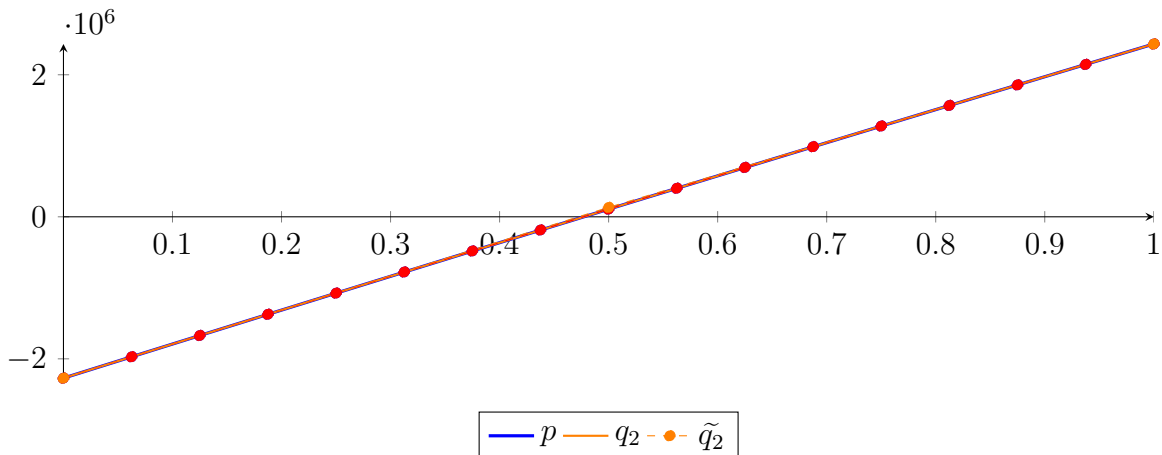
$$\begin{aligned}
 p &= -2.82438 \cdot 10^{-05} X^{16} + 2.90051 \cdot 10^{-05} X^{15} + 0.000231285 X^{14} + 0.000648014 X^{13} + 0.00318916 X^{12} \\
 &\quad + 0.00323204 X^{11} + 0.00460533 X^{10} + 0.00220012 X^9 - 0.000101882 X^8 + 5.32717 \cdot 10^{-06} X^7 \\
 &\quad - 0.00186824 X^6 + 0.132741 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 43556.7 X^{16} - 347904 X^{15} + 1.2616 \cdot 10^6 X^{14} - 2.74835 \cdot 10^6 X^{13} + 4.01042 \cdot 10^6 X^{12} - 4.13709 \\
 &\quad \cdot 10^6 X^{11} + 3.10251 \cdot 10^6 X^{10} - 1.71209 \cdot 10^6 X^9 + 695531 X^8 - 205965 X^7 + 43591.2 X^6 \\
 &\quad - 6402.94 X^5 + 625.678 X^4 - 37.485 X^3 - 104264 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481117 B_{6,16} - 185755 B_{7,16} + 108741 B_{8,16} \\
 &\quad + 402365 B_{9,16} + 695123 B_{10,16} + 987010 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

$$M = -104265 X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

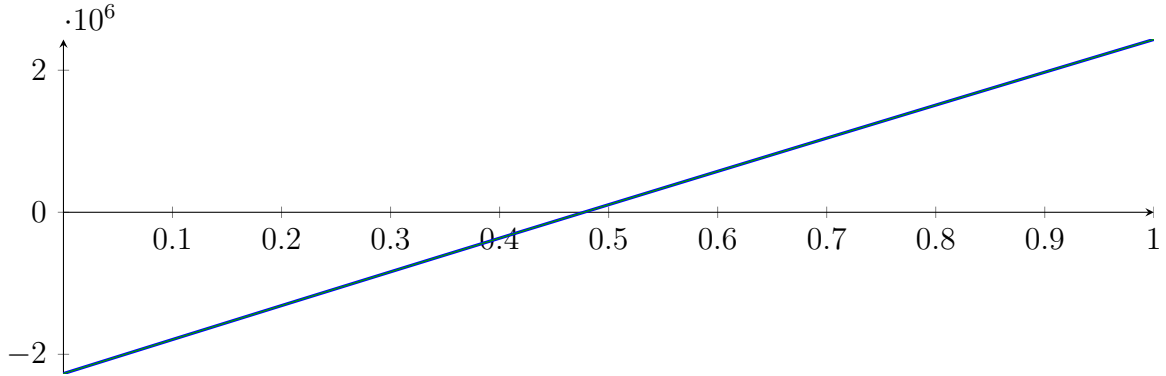
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\}$$

$$N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

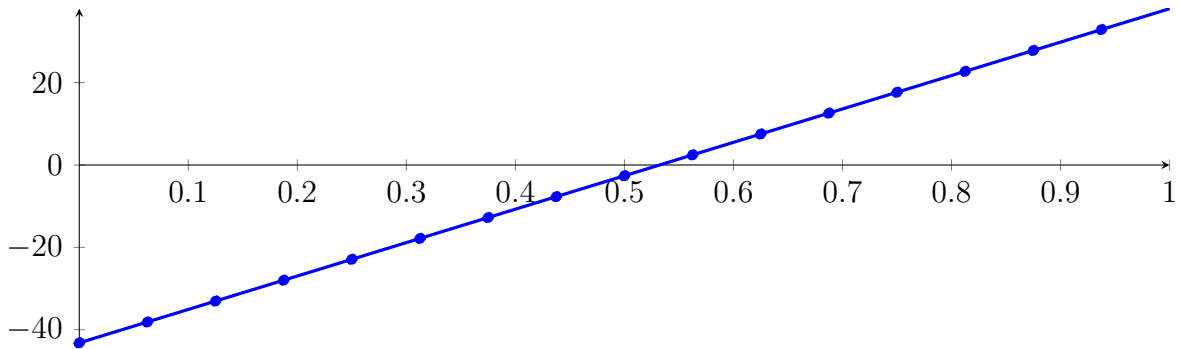
Longest intersection interval: $1.72301 \cdot 10^{-05}$

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333333]$,

77.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

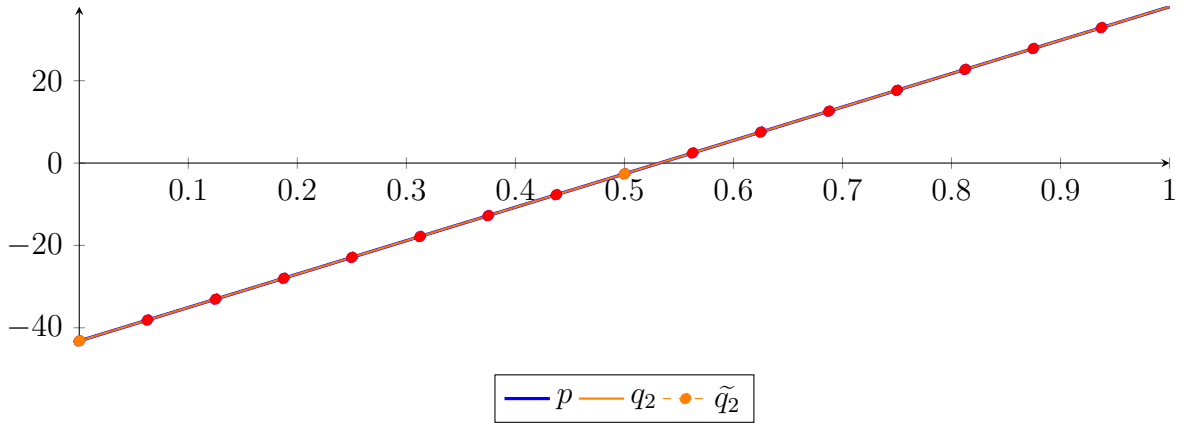
$$\begin{aligned} p &= 8.74252 \cdot 10^{-11} X^{16} - 1.56979 \cdot 10^{-09} X^{15} + 6.68479 \cdot 10^{-09} X^{14} + 1.20008 \cdot 10^{-08} X^{13} + 9.07301 \cdot 10^{-08} X^{12} \\ &+ 5.58657 \cdot 10^{-08} X^{11} + 1.13801 \cdot 10^{-07} X^{10} + 3.70665 \cdot 10^{-08} X^9 + 7.31575 \cdot 10^{-10} X^8 + 1.30058 \cdot 10^{-09} X^7 \\ &+ 5.00722 \cdot 10^{-09} X^6 + 1.24146 \cdot 10^{-10} X^5 + 1.03455 \cdot 10^{-10} X^4 - 3.09388 \cdot 10^{-05} X^2 + 81.1505 X - 43.1911 \\ &= -43.1911 B_{0,16}(X) - 38.1192 B_{1,16}(X) - 33.0473 B_{2,16}(X) - 27.9754 B_{3,16}(X) - 22.9035 B_{4,16}(X) \\ &- 17.8316 B_{5,16}(X) - 12.7597 B_{6,16}(X) - 7.68777 B_{7,16}(X) - 2.61587 B_{8,16}(X) \\ &+ 2.45604 B_{9,16}(X) + 7.52794 B_{10,16}(X) + 12.5998 B_{11,16}(X) + 17.6718 B_{12,16}(X) \\ &+ 22.7437 B_{13,16}(X) + 27.8156 B_{14,16}(X) + 32.8875 B_{15,16}(X) + 37.9594 B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09388 \cdot 10^{-05} X^2 + 81.1505 X - 43.1911 \\ &= -43.1911 B_{0,2} - 2.61586 B_{1,2} + 37.9594 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 0.721495 X^{16} - 5.74915 X^{15} + 20.7933 X^{14} - 45.1627 X^{13} + 65.6806 X^{12} - 67.5044 X^{11} \\ &+ 50.4286 X^{10} - 27.728 X^9 + 11.2318 X^8 - 3.32011 X^7 + 0.702408 X^6 - 0.103415 X^5 \\ &+ 0.0102099 X^4 - 0.000624725 X^3 - 1.10834 \cdot 10^{-05} X^2 + 81.1505 X - 43.1911 \\ &= -43.1911 B_{0,16} - 38.1192 B_{1,16} - 33.0473 B_{2,16} - 27.9754 B_{3,16} - 22.9035 B_{4,16} - 17.8316 B_{5,16} \\ &- 12.7597 B_{6,16} - 7.68779 B_{7,16} - 2.61585 B_{8,16} + 2.45602 B_{9,16} + 7.52795 B_{10,16} + 12.5998 B_{11,16} \\ &+ 17.6718 B_{12,16} + 22.7437 B_{13,16} + 27.8156 B_{14,16} + 32.8875 B_{15,16} + 37.9594 B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.57956 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -3.09388 \cdot 10^{-05} X^2 + 81.1505 X - 43.1911$$

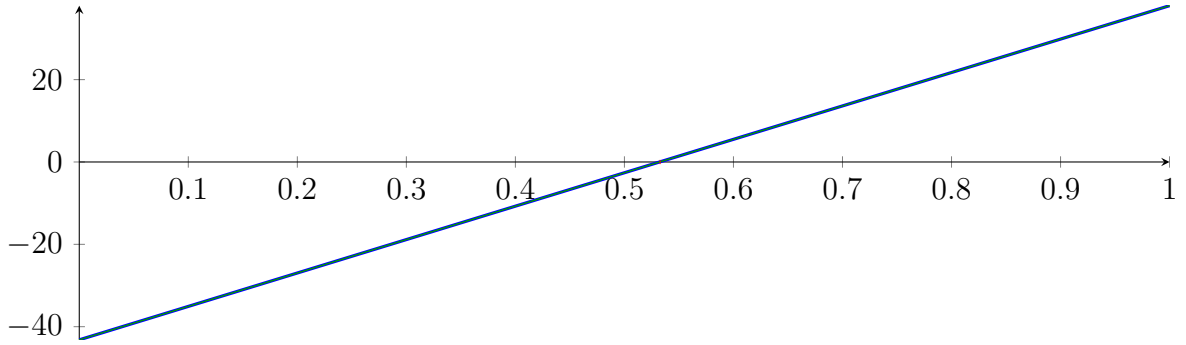
$$m = -3.09388 \cdot 10^{-05} X^2 + 81.1505 X - 43.1911$$

Root of M and m :

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\}$$

$$N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

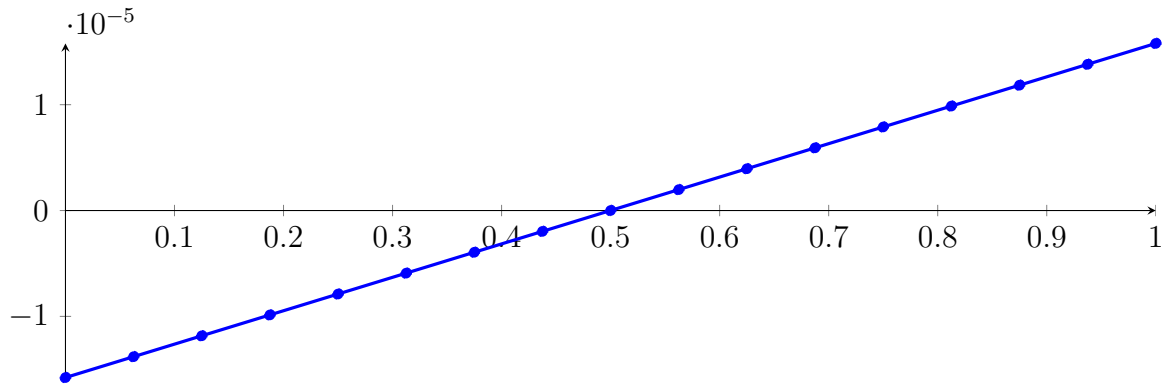
Longest intersection interval: $3.8903 \cdot 10^{-07}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

77.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
p &= -1.04409 \cdot 10^{-16} X^{16} - 1.53089 \cdot 10^{-16} X^{15} + 1.74665 \cdot 10^{-15} X^{14} + 5.46438 \cdot 10^{-15} X^{13} \\
&\quad + 2.56522 \cdot 10^{-14} X^{12} + 2.15479 \cdot 10^{-14} X^{11} + 3.51633 \cdot 10^{-14} X^{10} + 1.67444 \\
&\quad \cdot 10^{-14} X^9 - 8.72105 \cdot 10^{-16} X^8 + 1.41087 \cdot 10^{-15} X^6 + 5.91974 \cdot 10^{-17} X^5 + 4.93312 \\
&\quad \cdot 10^{-17} X^4 + 3.79471 \cdot 10^{-18} X^3 - 4.87891 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05} \\
&= -1.57804 \cdot 10^{-05} B_{0,16}(X) - 1.38073 \cdot 10^{-05} B_{1,16}(X) - 1.18341 \cdot 10^{-05} B_{2,16}(X) - 9.86101 \\
&\quad \cdot 10^{-06} B_{3,16}(X) - 7.88788 \cdot 10^{-06} B_{4,16}(X) - 5.91476 \cdot 10^{-06} B_{5,16}(X) - 3.94163 \cdot 10^{-06} B_{6,16}(X) \\
&\quad - 1.96851 \cdot 10^{-06} B_{7,16}(X) + 4.61501 \cdot 10^{-09} B_{8,16}(X) + 1.97774 \cdot 10^{-06} B_{9,16}(X) + 3.95086 \\
&\quad \cdot 10^{-06} B_{10,16}(X) + 5.92399 \cdot 10^{-06} B_{11,16}(X) + 7.89711 \cdot 10^{-06} B_{12,16}(X) + 9.87024 \cdot 10^{-06} B_{13,16}(X) \\
&\quad + 1.18434 \cdot 10^{-05} B_{14,16}(X) + 1.38165 \cdot 10^{-05} B_{15,16}(X) + 1.57896 \cdot 10^{-05} B_{16,16}(X)
\end{aligned}$$



Degree reduction and raising:

$$q_2 = -4.65868 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

$$= -1.57804 \cdot 10^{-05} B_{0,2} + 4.61501 \cdot 10^{-09} B_{1,2} + 1.57896 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = 2.92413 \cdot 10^{-07} X^{16} - 2.33332 \cdot 10^{-06} X^{15} + 8.45203 \cdot 10^{-06} X^{14} - 1.83895 \cdot 10^{-05} X^{13}$$

$$+ 2.67963 \cdot 10^{-05} X^{12} - 2.75995 \cdot 10^{-05} X^{11} + 2.06638 \cdot 10^{-05} X^{10} - 1.13854 \cdot 10^{-05} X^9$$

$$+ 4.61944 \cdot 10^{-06} X^8 - 1.36687 \cdot 10^{-06} X^7 + 2.89249 \cdot 10^{-07} X^6 - 4.25295 \cdot 10^{-08} X^5 + 4.17283$$

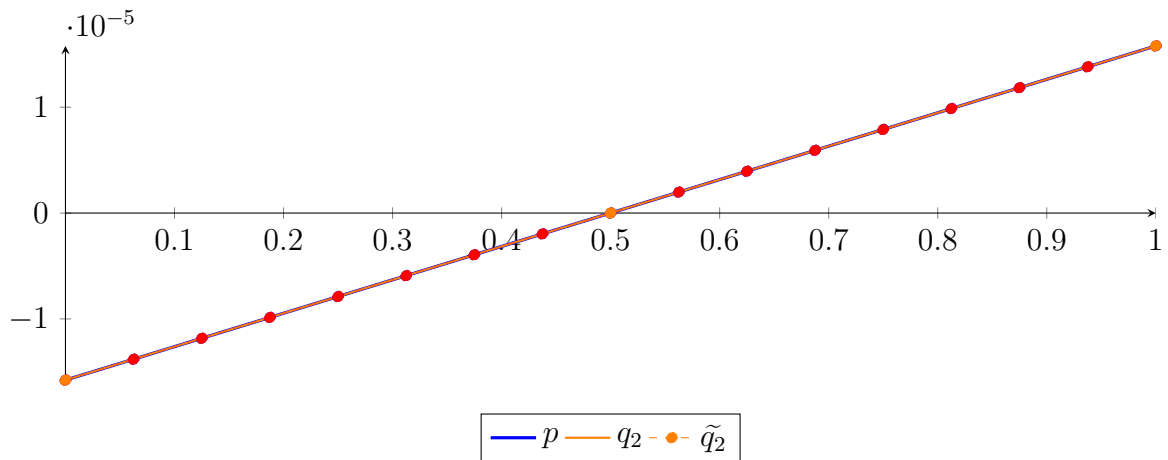
$$\cdot 10^{-09} X^4 - 2.52119 \cdot 10^{-10} X^3 + 7.82992 \cdot 10^{-12} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

$$= -1.57804 \cdot 10^{-05} B_{0,16} - 1.38073 \cdot 10^{-05} B_{1,16} - 1.18341 \cdot 10^{-05} B_{2,16} - 9.86101 \cdot 10^{-06} B_{3,16} - 7.88788$$

$$\cdot 10^{-06} B_{4,16} - 5.91476 \cdot 10^{-06} B_{5,16} - 3.94163 \cdot 10^{-06} B_{6,16} - 1.96851 \cdot 10^{-06} B_{7,16} + 4.62125 \cdot 10^{-09} B_{8,16}$$

$$+ 1.97773 \cdot 10^{-06} B_{9,16} + 3.95087 \cdot 10^{-06} B_{10,16} + 5.92399 \cdot 10^{-06} B_{11,16} + 7.89711 \cdot 10^{-06} B_{12,16}$$

$$+ 9.87024 \cdot 10^{-06} B_{13,16} + 1.18434 \cdot 10^{-05} B_{14,16} + 1.38165 \cdot 10^{-05} B_{15,16} + 1.57896 \cdot 10^{-05} B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 6.24192 \cdot 10^{-12}$.

Bounding polynomials M and m :

$$M = -4.65868 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

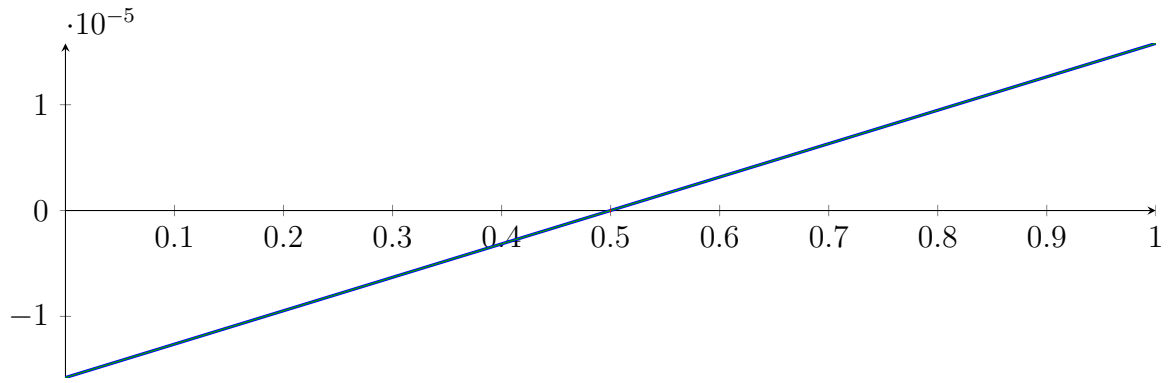
$$m = -4.65868 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.499636, 6.77659 \cdot 10^{12}\}$$

$$N(m) = \{0.500364, 6.77659 \cdot 10^{12}\}$$

Intersection intervals:



[0.499636, 0.500364]

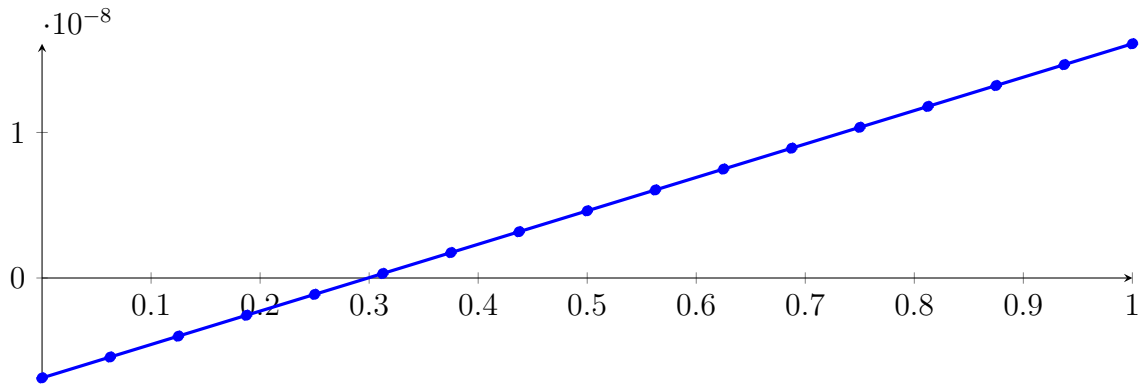
Longest intersection interval: 0.000727273

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

77.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

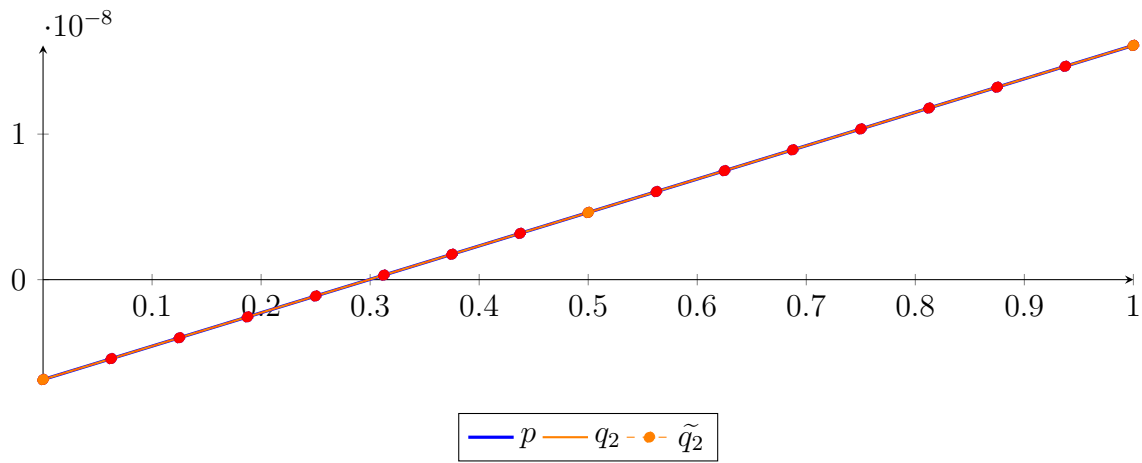
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -6.76104 \cdot 10^{-19} X^{16} + 2.21478 \cdot 10^{-18} X^{15} - 1.59295 \cdot 10^{-18} X^{14} + 1.03762 \cdot 10^{-18} X^{13} \\
 &\quad - 1.34649 \cdot 10^{-17} X^{12} + 9.65427 \cdot 10^{-18} X^{11} - 2.75561 \cdot 10^{-18} X^{10} + 5.90488 \cdot 10^{-18} X^9 \\
 &\quad - 8.51665 \cdot 10^{-19} X^8 + 3.02814 \cdot 10^{-19} X^7 + 2.64962 \cdot 10^{-19} X^6 + 2.8905 \cdot 10^{-20} X^5 \\
 &\quad + 6.02187 \cdot 10^{-21} X^4 + 9.26442 \cdot 10^{-22} X^3 + 2.296 \cdot 10^{-08} X - 6.86499 \cdot 10^{-09} \\
 &= -6.86499 \cdot 10^{-09} B_{0,16}(X) - 5.42999 \cdot 10^{-09} B_{1,16}(X) - 3.99499 \cdot 10^{-09} B_{2,16}(X) \\
 &\quad - 2.55999 \cdot 10^{-09} B_{3,16}(X) - 1.12499 \cdot 10^{-09} B_{4,16}(X) + 3.10008 \cdot 10^{-10} B_{5,16}(X) + 1.74501 \\
 &\quad \cdot 10^{-09} B_{6,16}(X) + 3.18001 \cdot 10^{-09} B_{7,16}(X) + 4.61501 \cdot 10^{-09} B_{8,16}(X) + 6.05001 \cdot 10^{-09} B_{9,16}(X) \\
 &\quad + 7.48501 \cdot 10^{-09} B_{10,16}(X) + 8.92001 \cdot 10^{-09} B_{11,16}(X) + 1.0355 \cdot 10^{-08} B_{12,16}(X) + 1.179 \\
 &\quad \cdot 10^{-08} B_{13,16}(X) + 1.3225 \cdot 10^{-08} B_{14,16}(X) + 1.466 \cdot 10^{-08} B_{15,16}(X) + 1.6095 \cdot 10^{-08} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= 1.40621 \cdot 10^{-23} X^2 + 2.296 \cdot 10^{-08} X - 6.86499 \cdot 10^{-09} \\
 &= -6.86499 \cdot 10^{-09} B_{0,2} + 4.61501 \cdot 10^{-09} B_{1,2} + 1.6095 \cdot 10^{-08} B_{2,2} \\
 \tilde{q}_2 &= 2.65578 \cdot 10^{-10} X^{16} - 2.13329 \cdot 10^{-09} X^{15} + 7.78369 \cdot 10^{-09} X^{14} - 1.70728 \cdot 10^{-08} X^{13} \\
 &\quad + 2.51029 \cdot 10^{-08} X^{12} - 2.61095 \cdot 10^{-08} X^{11} + 1.97446 \cdot 10^{-08} X^{10} - 1.09796 \cdot 10^{-08} X^9 \\
 &\quad + 4.48709 \cdot 10^{-09} X^8 - 1.33355 \cdot 10^{-09} X^7 + 2.82501 \cdot 10^{-10} X^6 - 4.12963 \cdot 10^{-11} X^5 + 3.94088 \\
 &\quad \cdot 10^{-12} X^4 - 2.24328 \cdot 10^{-13} X^3 + 6.17064 \cdot 10^{-15} X^2 + 2.296 \cdot 10^{-08} X - 6.86499 \cdot 10^{-09} \\
 &= -6.86499 \cdot 10^{-09} B_{0,16} - 5.42999 \cdot 10^{-09} B_{1,16} - 3.99499 \cdot 10^{-09} B_{2,16} - 2.55999 \cdot 10^{-09} B_{3,16} \\
 &\quad - 1.12499 \cdot 10^{-09} B_{4,16} + 3.10006 \cdot 10^{-10} B_{5,16} + 1.74501 \cdot 10^{-09} B_{6,16} + 3.18 \cdot 10^{-09} B_{7,16} + 4.61501 \\
 &\quad \cdot 10^{-09} B_{8,16} + 6.05 \cdot 10^{-09} B_{9,16} + 7.48501 \cdot 10^{-09} B_{10,16} + 8.92 \cdot 10^{-09} B_{11,16} + 1.0355 \cdot 10^{-08} B_{12,16} \\
 &\quad + 1.179 \cdot 10^{-08} B_{13,16} + 1.3225 \cdot 10^{-08} B_{14,16} + 1.466 \cdot 10^{-08} B_{15,16} + 1.6095 \cdot 10^{-08} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.35405 \cdot 10^{-15}$.

Bounding polynomials M and m :

$$M = 1.32349 \cdot 10^{-23} X^2 + 2.296 \cdot 10^{-08} X - 6.86498 \cdot 10^{-09}$$

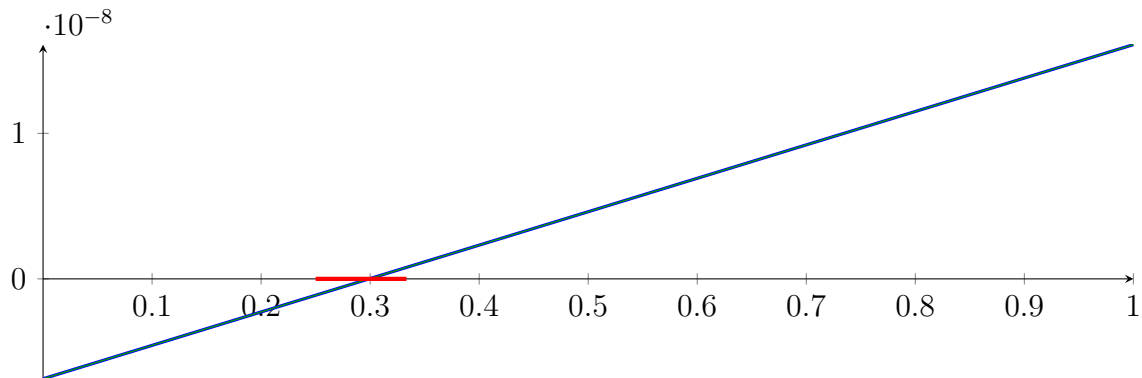
$$m = 1.48893 \cdot 10^{-23} X^2 + 2.296 \cdot 10^{-08} X - 6.865 \cdot 10^{-09}$$

Root of M and m :

$$N(M) = \{-1.73481 \cdot 10^{15}, 0.25\}$$

$$N(m) = \{-1.54205 \cdot 10^{15}, 0.333333\}$$

Intersection intervals:



$$[0.333333, 0.25]$$

Longest intersection interval: -0.0833333

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

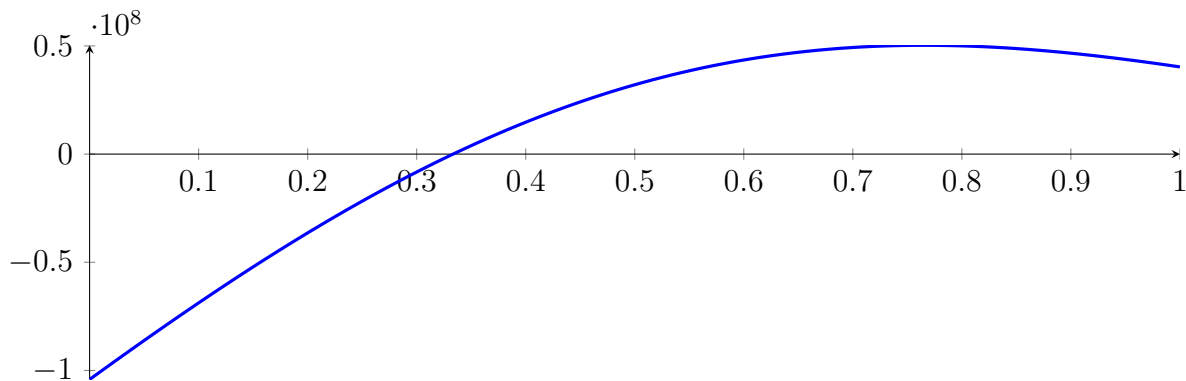
77.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

77.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

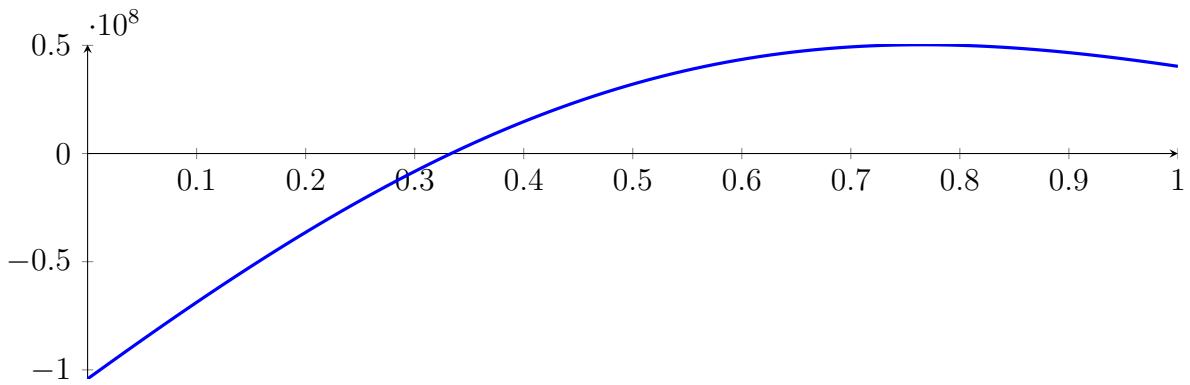
with precision $\varepsilon = 1 \cdot 10^{-32}$.

78 Running CubeClip on f_{16} with epsilon 32

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

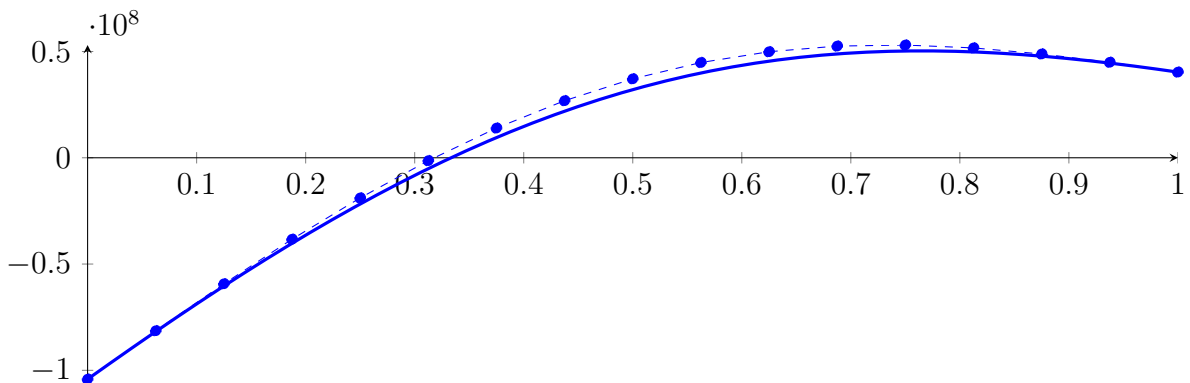
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



78.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 2.11088 \cdot 10^6 X^{16} - 1.70537 \cdot 10^7 X^{15} + 6.25913 \cdot 10^7 X^{14} - 1.38111 \cdot 10^8 X^{13} + 2.043 \cdot 10^8 X^{12} - 2.13799$$

$$\cdot 10^8 X^{11} + 1.62714 \cdot 10^8 X^{10} - 9.11092 \cdot 10^7 X^9 + 3.75254 \cdot 10^7 X^8 - 1.12529 \cdot 10^7 X^7 + 2.40899 \cdot 10^6 X^6$$

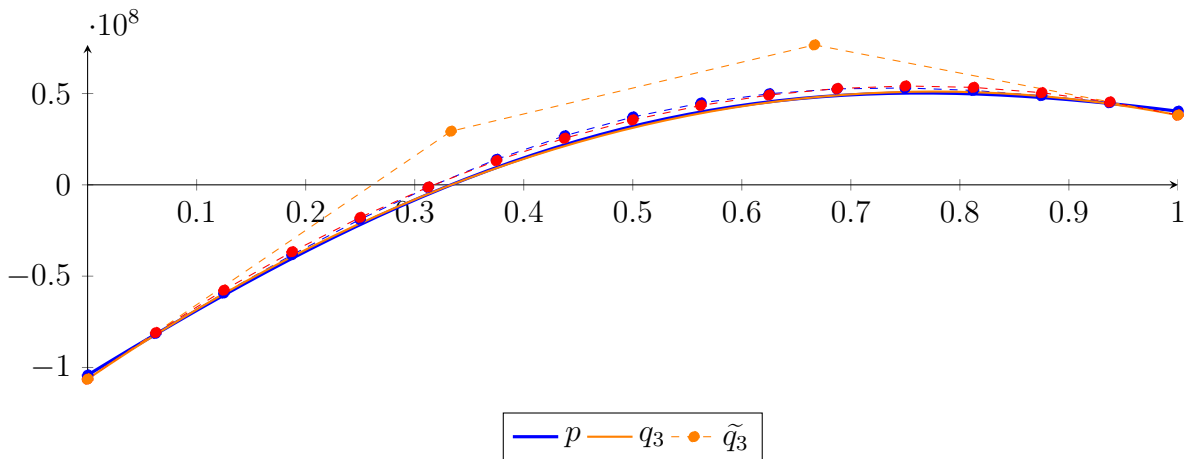
$$- 357156 X^5 + 34986.1 X^4 + 2.75602 \cdot 10^6 X^3 - 2.65318 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18553 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91816 \cdot 10^7 B_{10,16} + 5.27352 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

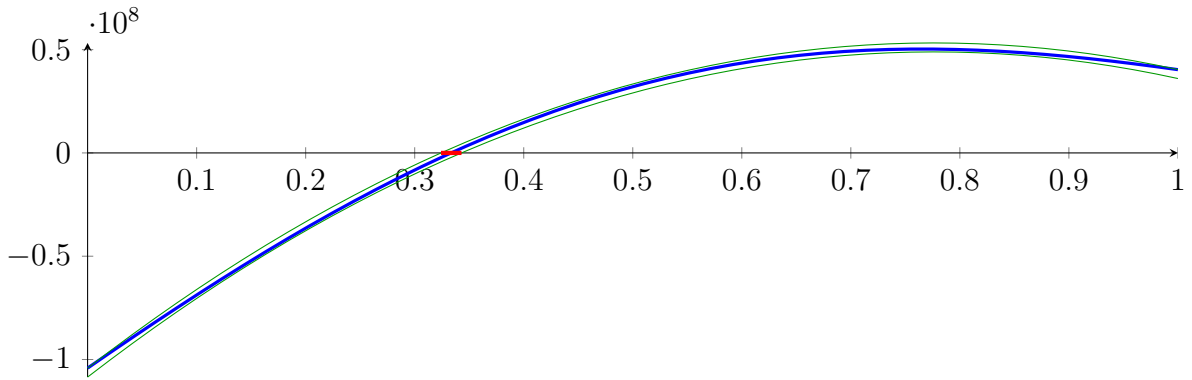
$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\}$$

$$N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

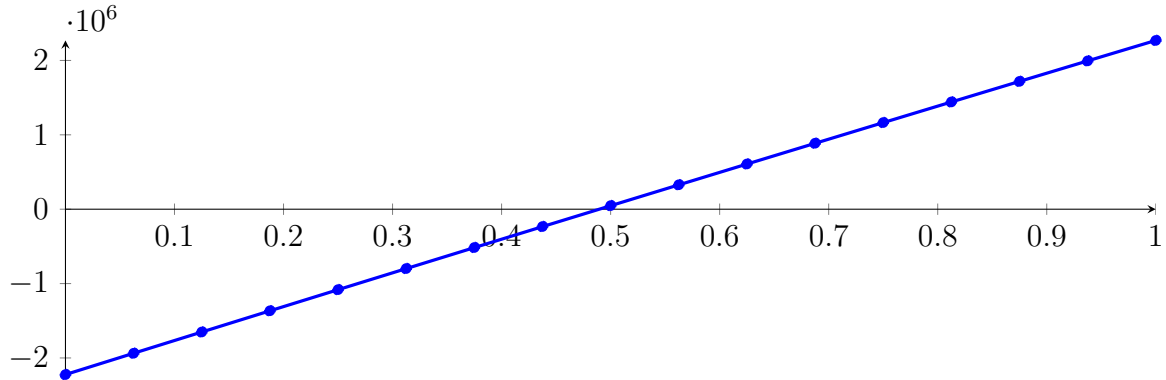
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

78.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

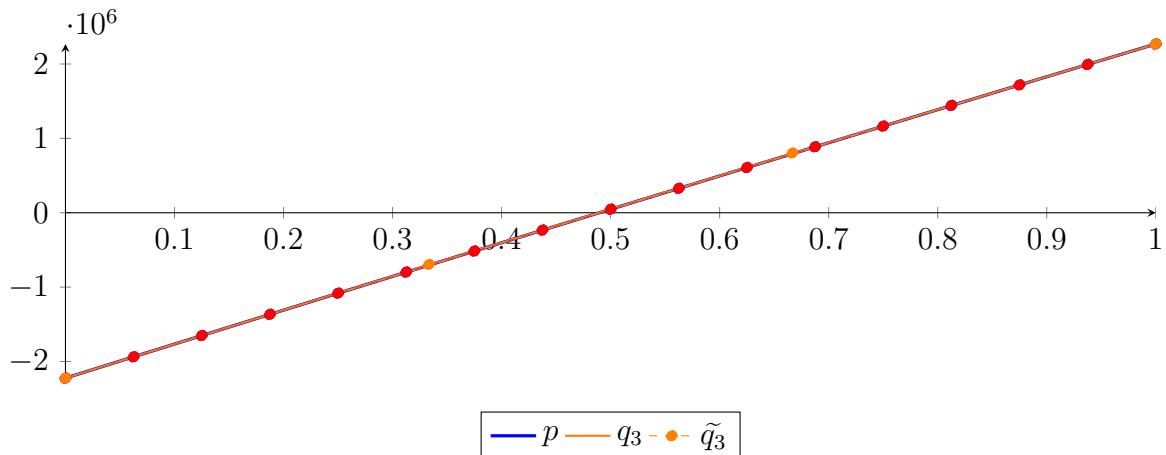
$$\begin{aligned}
 p &= -1.93035 \cdot 10^{-05} X^{16} + 1.08927 \cdot 10^{-05} X^{15} + 0.000255816 X^{14} + 0.000616983 X^{13} + 0.00371715 X^{12} \\
 &\quad + 0.00325035 X^{11} + 0.00510875 X^{10} + 0.00190713 X^9 + 0.000179792 X^8 + 1.06543 \cdot 10^{-05} X^7 \\
 &\quad - 0.00136482 X^6 + 0.104959 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 15290.6 X^{16} - 125412 X^{15} + 468045 X^{14} - 1.05198 \cdot 10^6 X^{13} + 1.588 \cdot 10^6 X^{12} - 1.69893 \\
 &\quad \cdot 10^6 X^{11} + 1.32382 \cdot 10^6 X^{10} - 759402 X^9 + 320067 X^8 - 97826.5 X^7 + 21197.8 X^6 \\
 &\quad - 3162.8 X^5 + 313.931 X^4 - 720.165 X^3 - 93879.2 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.7 B_{8,16} \\
 &\quad + 328649 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.457751$.

Bounding polynomials M and m :

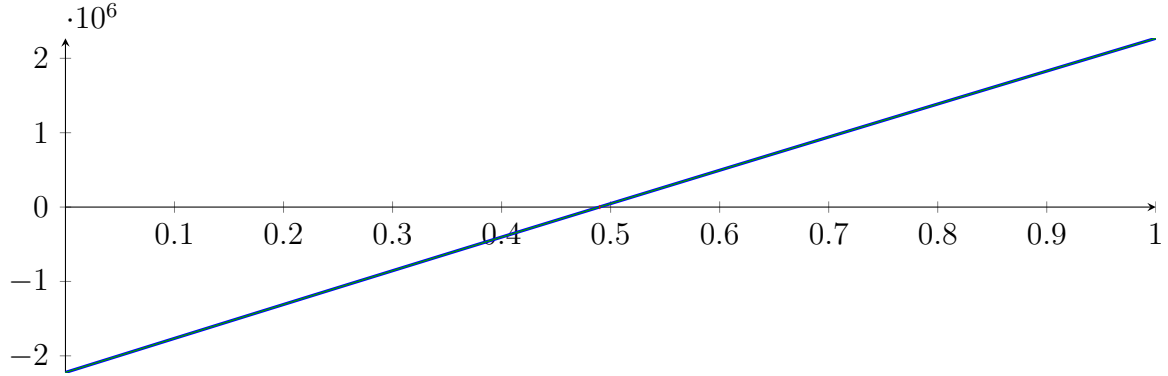
$$M = -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\} \quad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

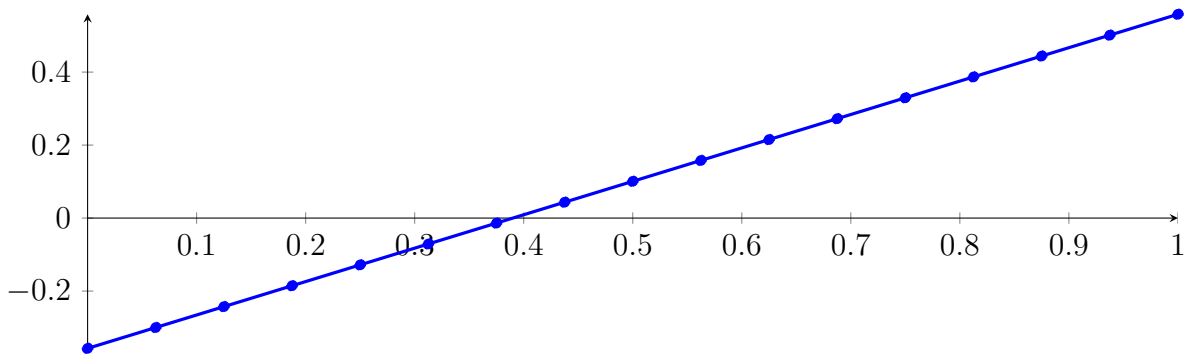
Longest intersection interval: $2.03684 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

78.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

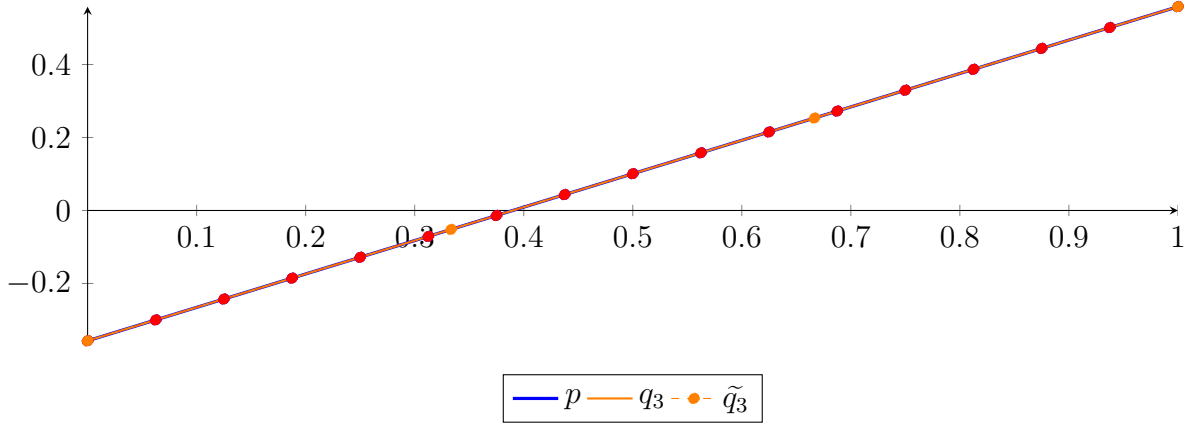
$$\begin{aligned} p &= -1.56399 \cdot 10^{-11} X^{16} + 4.58016 \cdot 10^{-11} X^{15} - 3.19744 \cdot 10^{-12} X^{14} + 6.54055 \cdot 10^{-11} X^{13} \\ &\quad + 1.05072 \cdot 10^{-10} X^{12} + 4.59728 \cdot 10^{-10} X^{11} + 5.01434 \cdot 10^{-10} X^{10} + 2.99742 \cdot 10^{-10} X^9 \\ &\quad + 1.14309 \cdot 10^{-11} X^8 - 5.08038 \cdot 10^{-12} X^7 + 3.37845 \cdot 10^{-11} X^6 - 9.69891 \cdot 10^{-13} X^5 \\ &\quad + 4.04121 \cdot 10^{-13} X^4 + 6.21725 \cdot 10^{-14} X^3 - 3.93767 \cdot 10^{-09} X^2 + 0.915501 X - 0.357072 \\ &= -0.357072 B_{0,16}(X) - 0.299853 B_{1,16}(X) - 0.242635 B_{2,16}(X) - 0.185416 B_{3,16}(X) \\ &\quad - 0.128197 B_{4,16}(X) - 0.0709781 B_{5,16}(X) - 0.0137592 B_{6,16}(X) \\ &\quad + 0.0434596 B_{7,16}(X) + 0.100678 B_{8,16}(X) + 0.157897 B_{9,16}(X) + 0.215116 B_{10,16}(X) \\ &\quad + 0.272335 B_{11,16}(X) + 0.329554 B_{12,16}(X) + 0.386773 B_{13,16}(X) \\ &\quad + 0.443991 B_{14,16}(X) + 0.50121 B_{15,16}(X) + 0.558429 B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 1.05471 \cdot 10^{-15} X^3 - 3.93767 \cdot 10^{-09} X^2 + 0.915501 X - 0.357072 \\ &= -0.357072 B_{0,3} - 0.0519051 B_{1,3} + 0.253262 B_{2,3} + 0.558429 B_{3,3} \end{aligned}$$

$$\begin{aligned}
\tilde{q}_3 &= 0.00291222X^{16} - 0.0241801X^{15} + 0.0914452X^{14} - 0.208537X^{13} + 0.319778X^{12} - 0.347745X^{11} \\
&\quad + 0.275244X^{10} - 0.159971X^9 + 0.0679818X^8 - 0.0208072X^7 + 0.00447629X^6 - 0.000654783X^5 \\
&\quad + 6.22034 \cdot 10^{-05}X^4 - 3.60145 \cdot 10^{-06}X^3 + 9.78811 \cdot 10^{-08}X^2 + 0.915501X - 0.357072 \\
&= -0.357072B_{0,16} - 0.299853B_{1,16} - 0.242635B_{2,16} - 0.185416B_{3,16} - 0.128197B_{4,16} \\
&\quad - 0.0709781B_{5,16} - 0.0137592B_{6,16} + 0.0434595B_{7,16} + 0.100678B_{8,16} \\
&\quad + 0.157897B_{9,16} + 0.215116B_{10,16} + 0.272335B_{11,16} + 0.329554B_{12,16} \\
&\quad + 0.386773B_{13,16} + 0.443991B_{14,16} + 0.50121B_{15,16} + 0.558429B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.5212 \cdot 10^{-08}$.

Bounding polynomials M and m :

$$M = 9.99201 \cdot 10^{-16}X^3 - 3.93767 \cdot 10^{-09}X^2 + 0.915501X - 0.357072$$

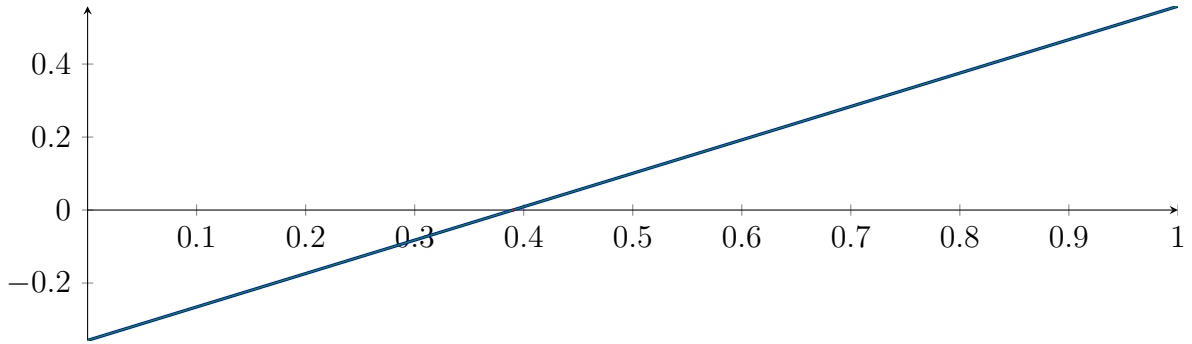
$$m = 1.22125 \cdot 10^{-15}X^3 - 3.93767 \cdot 10^{-09}X^2 + 0.915501X - 0.357072$$

Root of M and m :

$$N(M) = \{0.390029\}$$

$$N(m) = \{0.390029\}$$

Intersection intervals:



$$[0.390029, 0.390029]$$

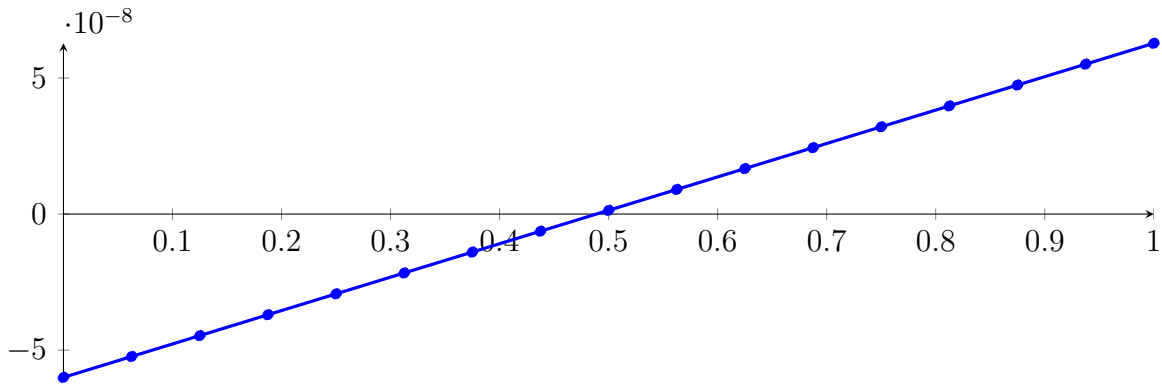
Longest intersection interval: $1.3411 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

78.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

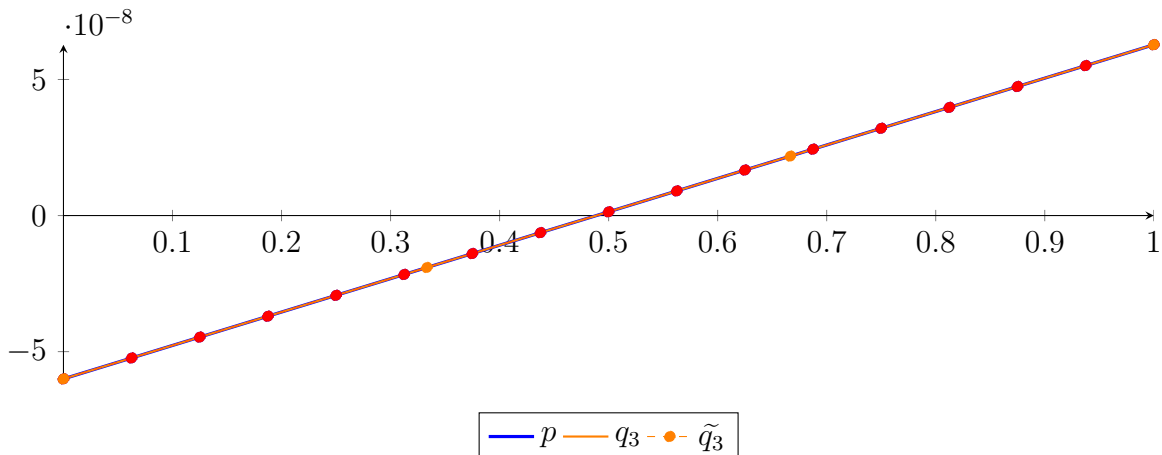
$$\begin{aligned}
 p &= -5.65183 \cdot 10^{-19} X^{16} + 5.13302 \cdot 10^{-19} X^{15} + 6.37816 \cdot 10^{-18} X^{14} + 1.69576 \cdot 10^{-17} X^{13} \\
 &\quad + 9.4423 \cdot 10^{-17} X^{12} + 8.0934 \cdot 10^{-17} X^{11} + 1.37357 \cdot 10^{-16} X^{10} + 6.23797 \cdot 10^{-17} X^9 \\
 &\quad + 1.36266 \cdot 10^{-18} X^8 + 6.05629 \cdot 10^{-19} X^7 + 5.08728 \cdot 10^{-18} X^6 + 2.3124 \cdot 10^{-19} X^5 \\
 &\quad + 1.44525 \cdot 10^{-19} X^4 - 7.41154 \cdot 10^{-21} X^3 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08} \\
 &= -6.00146 \cdot 10^{-08} B_{0,16}(X) - 5.2341 \cdot 10^{-08} B_{1,16}(X) - 4.46674 \cdot 10^{-08} B_{2,16}(X) - 3.69937 \\
 &\quad \cdot 10^{-08} B_{3,16}(X) - 2.93201 \cdot 10^{-08} B_{4,16}(X) - 2.16464 \cdot 10^{-08} B_{5,16}(X) - 1.39728 \cdot 10^{-08} B_{6,16}(X) \\
 &\quad - 6.29913 \cdot 10^{-09} B_{7,16}(X) + 1.37451 \cdot 10^{-09} B_{8,16}(X) + 9.04815 \cdot 10^{-09} B_{9,16}(X) + 1.67218 \\
 &\quad \cdot 10^{-08} B_{10,16}(X) + 2.43954 \cdot 10^{-08} B_{11,16}(X) + 3.20691 \cdot 10^{-08} B_{12,16}(X) + 3.97427 \\
 &\quad \cdot 10^{-08} B_{13,16}(X) + 4.74164 \cdot 10^{-08} B_{14,16}(X) + 5.509 \cdot 10^{-08} B_{15,16}(X) + 6.27637 \cdot 10^{-08} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= 2.51463 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08} \\
 &= -6.00146 \cdot 10^{-08} B_{0,3} - 1.90885 \cdot 10^{-08} B_{1,3} + 2.18376 \cdot 10^{-08} B_{2,3} + 6.27637 \cdot 10^{-08} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 4.01426 \cdot 10^{-10} X^{16} - 3.29623 \cdot 10^{-09} X^{15} + 1.2317 \cdot 10^{-08} X^{14} - 2.77213 \cdot 10^{-08} X^{13} + 4.19074 \\
 &\quad \cdot 10^{-08} X^{12} - 4.49047 \cdot 10^{-08} X^{11} + 3.50457 \cdot 10^{-08} X^{10} - 2.01341 \cdot 10^{-08} X^9 + 8.49676 \\
 &\quad \cdot 10^{-09} X^8 - 2.5992 \cdot 10^{-09} X^7 + 5.63364 \cdot 10^{-10} X^6 - 8.40231 \cdot 10^{-11} X^5 + 8.33259 \\
 &\quad \cdot 10^{-12} X^4 - 5.16965 \cdot 10^{-13} X^3 + 1.7395 \cdot 10^{-14} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08} \\
 &= -6.00146 \cdot 10^{-08} B_{0,16} - 5.2341 \cdot 10^{-08} B_{1,16} - 4.46674 \cdot 10^{-08} B_{2,16} - 3.69937 \cdot 10^{-08} B_{3,16} - 2.93201 \\
 &\quad \cdot 10^{-08} B_{4,16} - 2.16464 \cdot 10^{-08} B_{5,16} - 1.39728 \cdot 10^{-08} B_{6,16} - 6.29914 \cdot 10^{-09} B_{7,16} + 1.37452 \cdot 10^{-09} B_{8,16} \\
 &\quad + 9.04815 \cdot 10^{-09} B_{9,16} + 1.67218 \cdot 10^{-08} B_{10,16} + 2.43954 \cdot 10^{-08} B_{11,16} + 3.20691 \cdot 10^{-08} B_{12,16} \\
 &\quad + 3.97427 \cdot 10^{-08} B_{13,16} + 4.74164 \cdot 10^{-08} B_{14,16} + 5.509 \cdot 10^{-08} B_{15,16} + 6.27637 \cdot 10^{-08} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.90061 \cdot 10^{-15}$.

Bounding polynomials M and m :

$$M = 2.38228 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08}$$

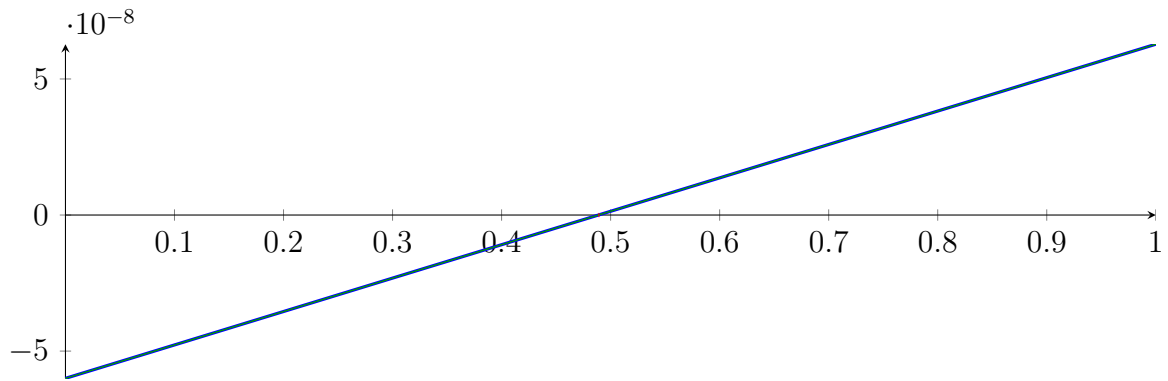
$$m = 2.51463 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08}$$

Root of M and m :

$$N(M) = \{0.488805\}$$

$$N(m) = \{0.488805\}$$

Intersection intervals:



$$[0.488805, 0.488805]$$

Longest intersection interval: $1.3086 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

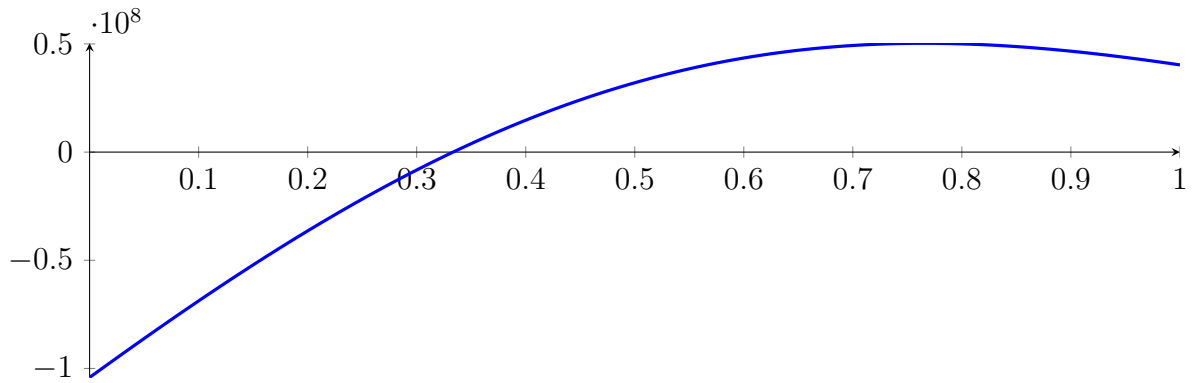
78.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

78.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

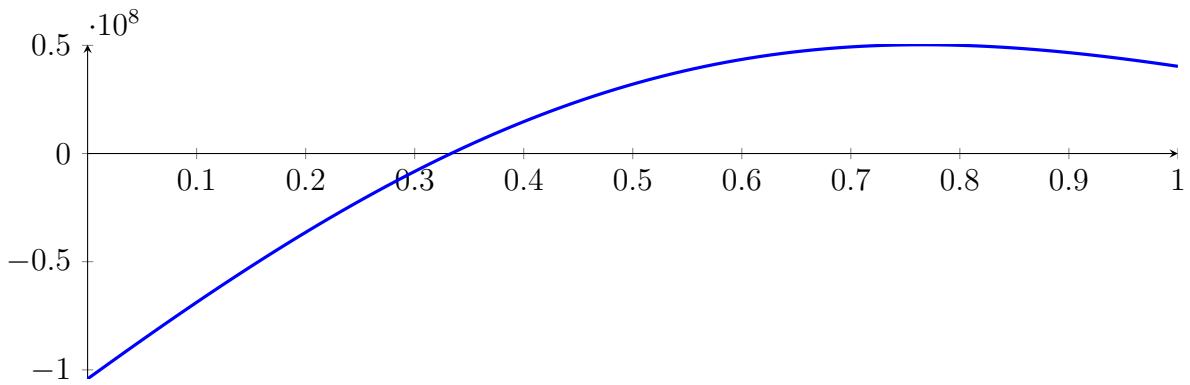
with precision $\varepsilon = 1 \cdot 10^{-32}$.

79 Running BezClip on f_{16} with epsilon 64

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

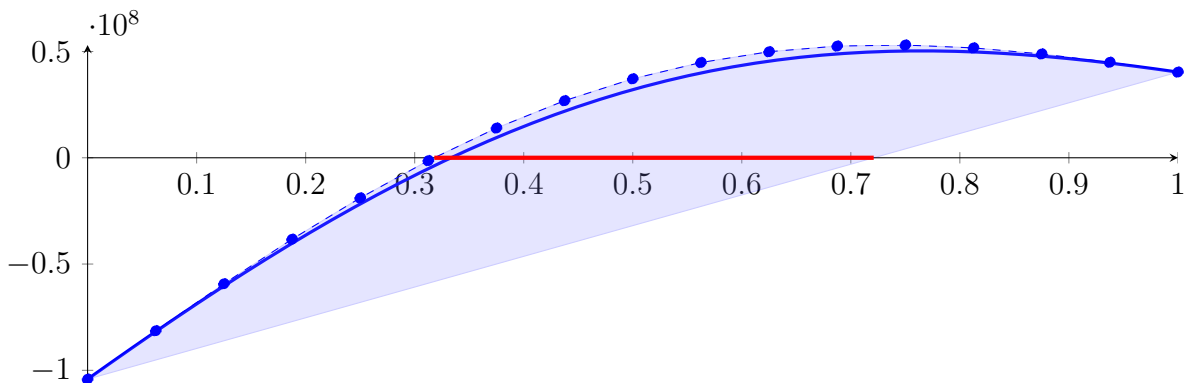
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



79.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

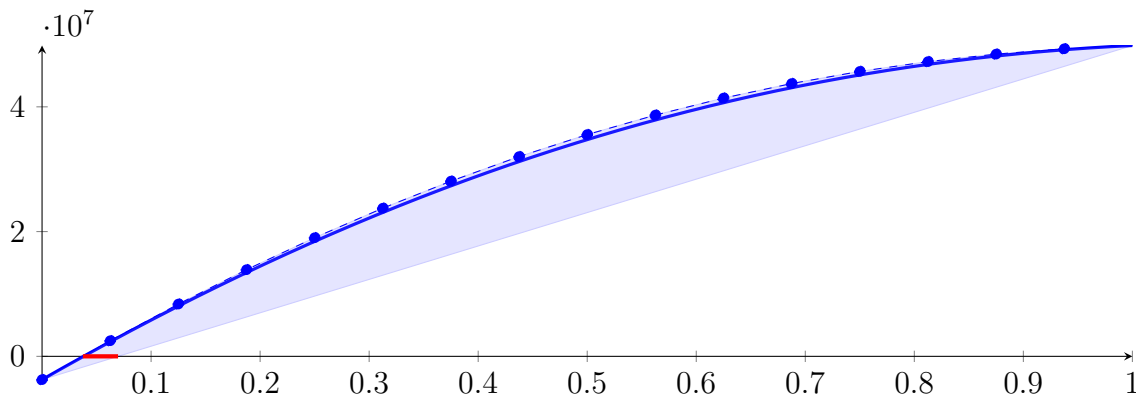
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

79.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.00483322X^{16} + 0.0186809X^{15} - 0.0194312X^{14} - 0.0738695X^{13} - 1.11673X^{12} \\ &\quad - 5.0471X^{11} + 36.3082X^{10} + 692.914X^9 + 1886.96X^8 - 25792X^7 - 149671X^6 + 492605X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

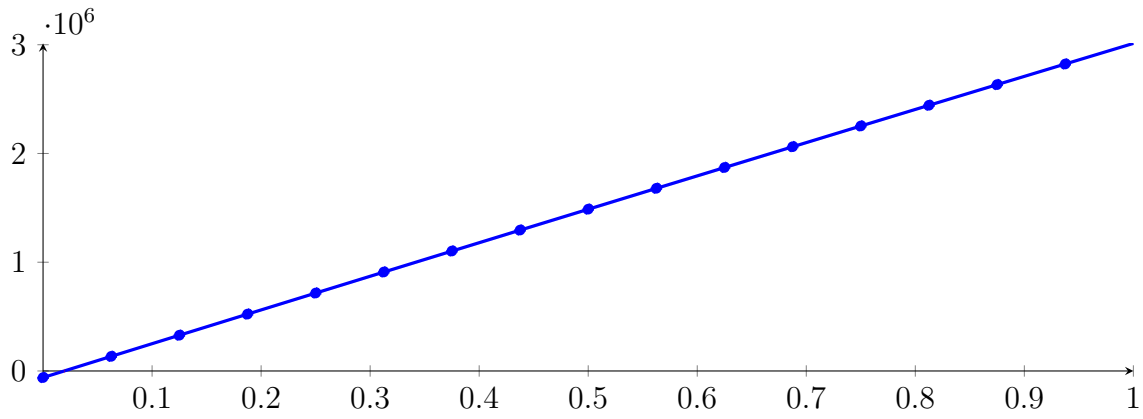
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

79.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.000205056X^{16} + 0.000776167X^{15} - 0.000863333X^{14} - 0.000217499X^{13} - 0.00809276X^{12} \\ &\quad - 5.09773 \cdot 10^{-05} X^{11} - 0.00564923X^{10} - 0.000162811X^9 - 0.000215376X^8 - 3.32948 \cdot 10^{-07} X^7 \\ &\quad - 0.000259866X^6 + 0.0161077X^5 + 4.36155X^4 - 234.216X^3 - 45622.2X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

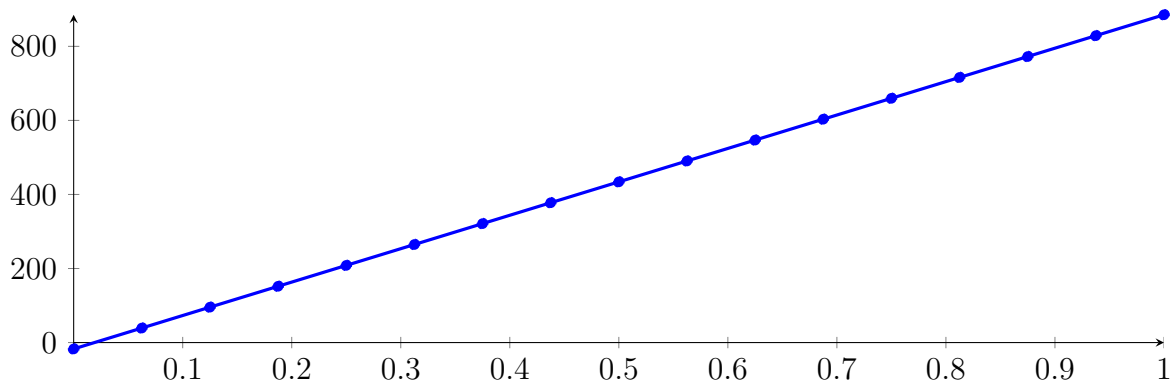
Longest intersection interval: 0.000289554

⇒ Selective recursion: interval 1: [0.333333, 0.333337],

79.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333337]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -5.9692 \cdot 10^{-08} X^{16} + 2.16103 \cdot 10^{-07} X^{15} - 2.28456 \cdot 10^{-07} X^{14} - 1.17238 \cdot 10^{-07} X^{13} \\
 &\quad - 2.29525 \cdot 10^{-06} X^{12} - 8.31778 \cdot 10^{-08} X^{11} - 1.74251 \cdot 10^{-06} X^{10} - 9.42919 \cdot 10^{-08} X^9 \\
 &\quad - 7.38891 \cdot 10^{-08} X^8 + 3.25144 \cdot 10^{-09} X^7 - 2.61741 \cdot 10^{-08} X^6 + 7.44876 \cdot 10^{-10} X^5 \\
 &\quad - 2.58638 \cdot 10^{-10} X^4 - 5.65024 \cdot 10^{-09} X^3 - 0.00382618 X^2 + 902.448 X - 17.178 \\
 &= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
 &\quad + 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
 &\quad + 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
 &\quad + 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

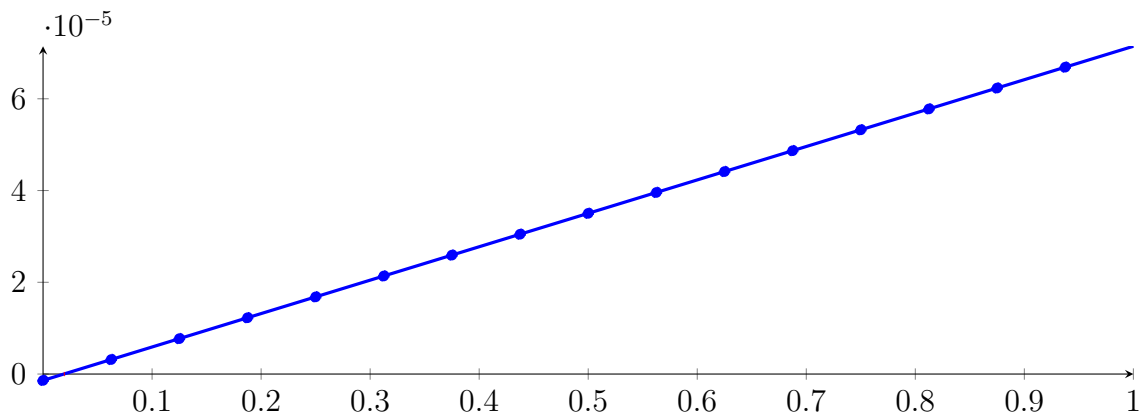
Longest intersection interval: $8.07045 \cdot 10^{-08}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

79.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -4.80379 \cdot 10^{-15} X^{16} + 1.74137 \cdot 10^{-14} X^{15} - 1.72656 \cdot 10^{-14} X^{14} - 6.84186 \cdot 10^{-15} X^{13} \\
 &\quad - 1.91627 \cdot 10^{-13} X^{12} - 4.7358 \cdot 10^{-15} X^{11} - 1.33436 \cdot 10^{-13} X^{10} - 1.97677 \cdot 10^{-15} X^9 \\
 &\quad - 7.4565 \cdot 10^{-15} X^8 - 1.16281 \cdot 10^{-16} X^7 - 1.98065 \cdot 10^{-15} X^6 + 1.47994 \cdot 10^{-17} X^5 \\
 &\quad - 1.84992 \cdot 10^{-17} X^4 - 2.48011 \cdot 10^{-17} X^2 + 7.28316 \cdot 10^{-05} X - 1.38634 \cdot 10^{-06} \\
 &= -1.38634 \cdot 10^{-06} B_{0,16}(X) + 3.16564 \cdot 10^{-06} B_{1,16}(X) + 7.71761 \cdot 10^{-06} B_{2,16}(X) + 1.22696 \\
 &\quad \cdot 10^{-05} B_{3,16}(X) + 1.68216 \cdot 10^{-05} B_{4,16}(X) + 2.13735 \cdot 10^{-05} B_{5,16}(X) + 2.59255 \cdot 10^{-05} B_{6,16}(X) \\
 &\quad + 3.04775 \cdot 10^{-05} B_{7,16}(X) + 3.50295 \cdot 10^{-05} B_{8,16}(X) + 3.95814 \cdot 10^{-05} B_{9,16}(X) + 4.41334 \\
 &\quad \cdot 10^{-05} B_{10,16}(X) + 4.86854 \cdot 10^{-05} B_{11,16}(X) + 5.32374 \cdot 10^{-05} B_{12,16}(X) + 5.77893 \cdot 10^{-05} B_{13,16}(X) \\
 &\quad + 6.23413 \cdot 10^{-05} B_{14,16}(X) + 6.68933 \cdot 10^{-05} B_{15,16}(X) + 7.14453 \cdot 10^{-05} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.50521 \cdot 10^{-15}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

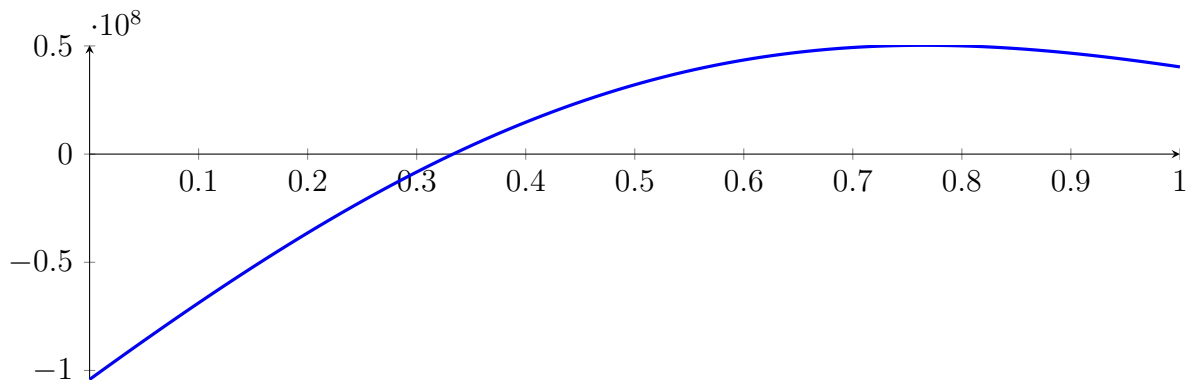
79.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

79.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

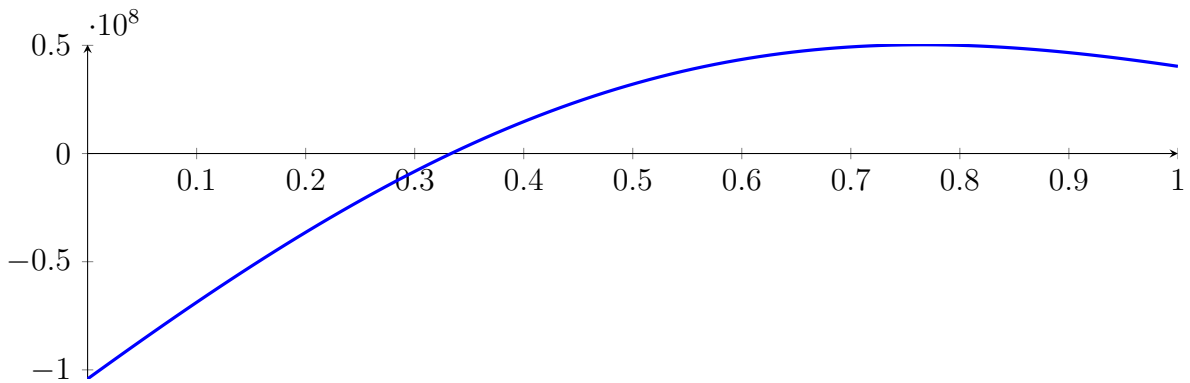
with precision $\varepsilon = 1 \cdot 10^{-64}$.

80 Running QuadClip on f_{16} with epsilon 64

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

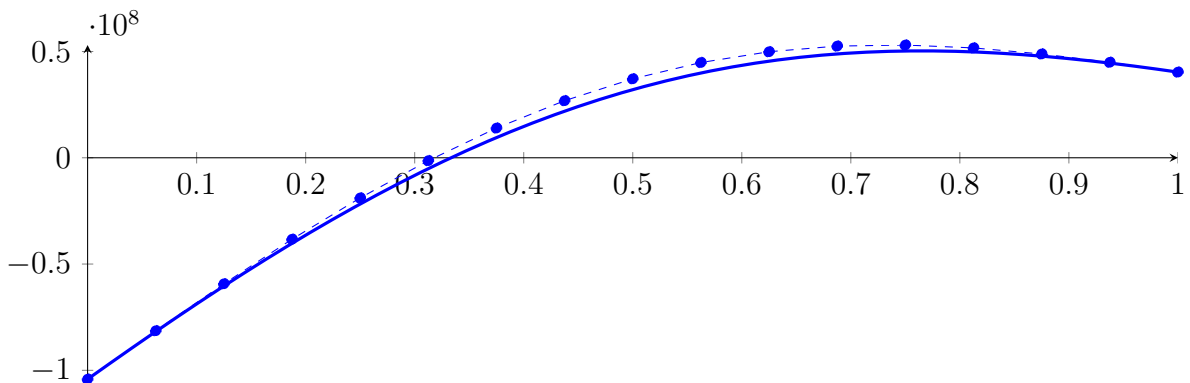
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



80.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = -1.41787 \cdot 10^6 X^{16} + 1.11761 \cdot 10^7 X^{15} - 3.98898 \cdot 10^7 X^{14} + 8.52437 \cdot 10^7 X^{13} - 1.21528 \cdot 10^8 X^{12}$$

$$+ 1.21946 \cdot 10^8 X^{11} - 8.86062 \cdot 10^7 X^{10} + 4.72904 \cdot 10^7 X^9 - 1.86355 \cdot 10^7 X^8 + 5.41059 \cdot 10^6 X^7 - 1.14487$$

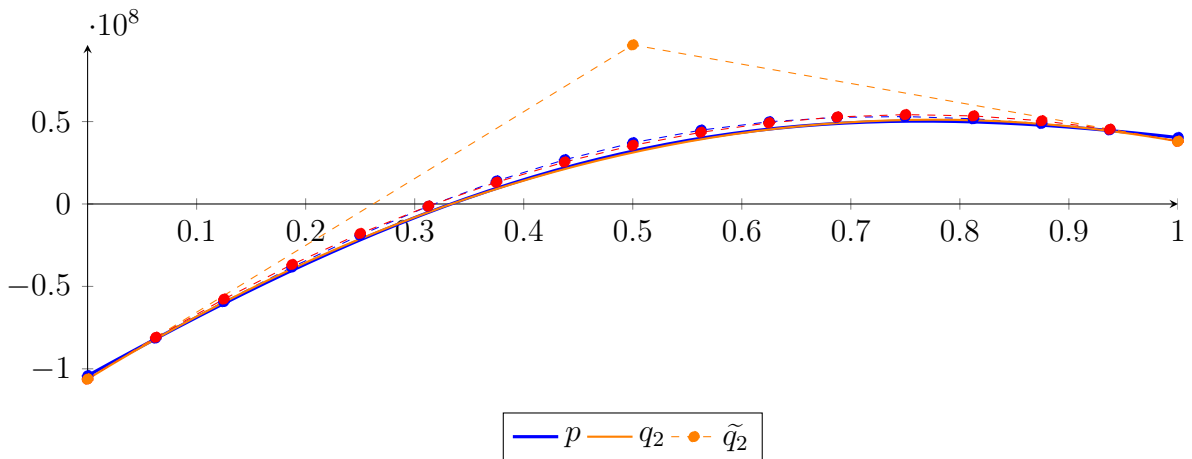
$$\cdot 10^6 X^6 + 172240 X^5 - 17636.6 X^4 + 1159.74 X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26923 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55691 \cdot 10^7 B_{8,16}$$

$$+ 4.34957 \cdot 10^7 B_{9,16} + 4.92455 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

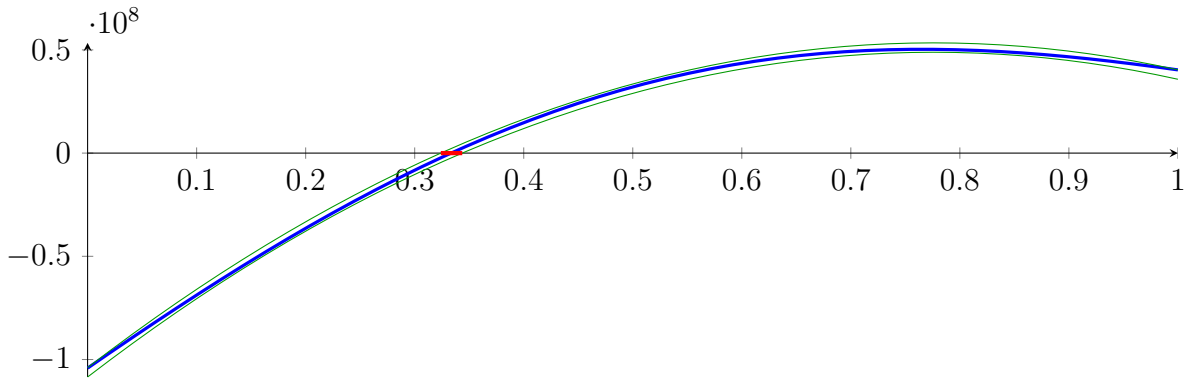
$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\}$$

$$N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

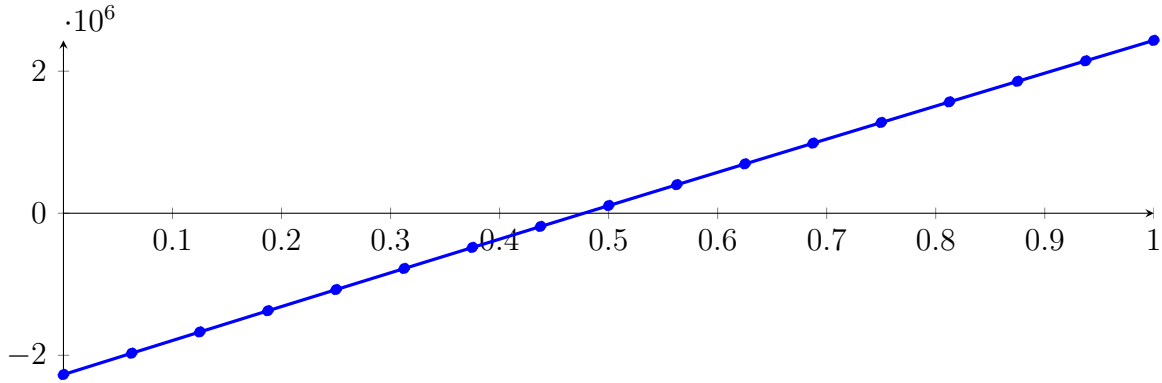
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

80.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

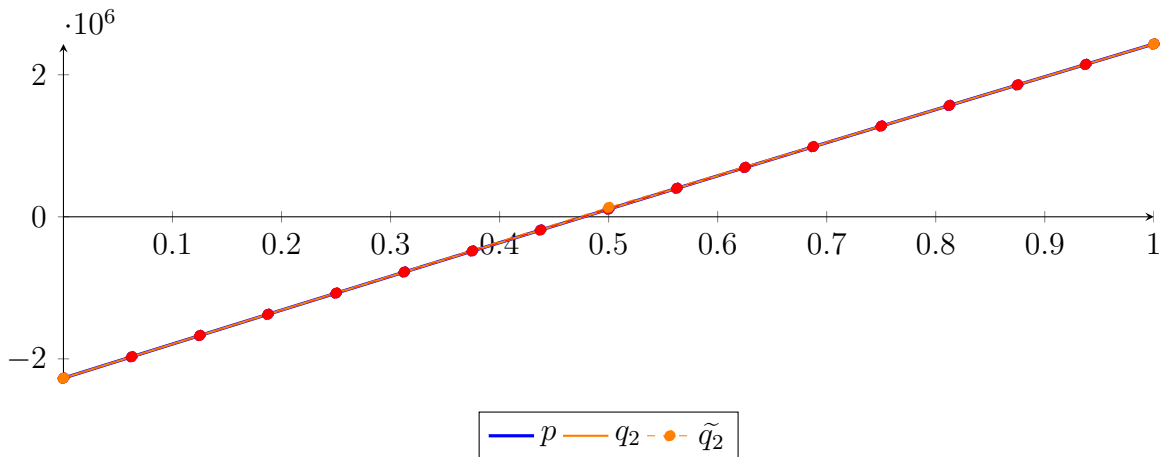
$$\begin{aligned}
 p &= -2.82438 \cdot 10^{-05} X^{16} + 2.90051 \cdot 10^{-05} X^{15} + 0.000231285 X^{14} + 0.000648014 X^{13} + 0.00318916 X^{12} \\
 &\quad + 0.00323204 X^{11} + 0.00460533 X^{10} + 0.00220012 X^9 - 0.000101882 X^8 + 5.32717 \cdot 10^{-06} X^7 \\
 &\quad - 0.00186824 X^6 + 0.132741 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 43556.7 X^{16} - 347904 X^{15} + 1.2616 \cdot 10^6 X^{14} - 2.74835 \cdot 10^6 X^{13} + 4.01042 \cdot 10^6 X^{12} - 4.13709 \\
 &\quad \cdot 10^6 X^{11} + 3.10251 \cdot 10^6 X^{10} - 1.71209 \cdot 10^6 X^9 + 695531 X^8 - 205965 X^7 + 43591.2 X^6 \\
 &\quad - 6402.94 X^5 + 625.678 X^4 - 37.485 X^3 - 104264 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481117 B_{6,16} - 185755 B_{7,16} + 108741 B_{8,16} \\
 &\quad + 402365 B_{9,16} + 695123 B_{10,16} + 987010 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

$$M = -104265 X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

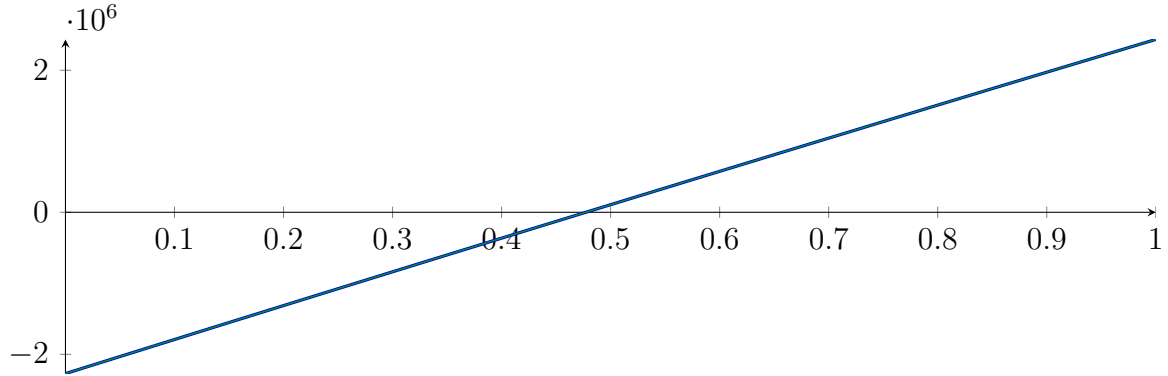
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\}$$

$$N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

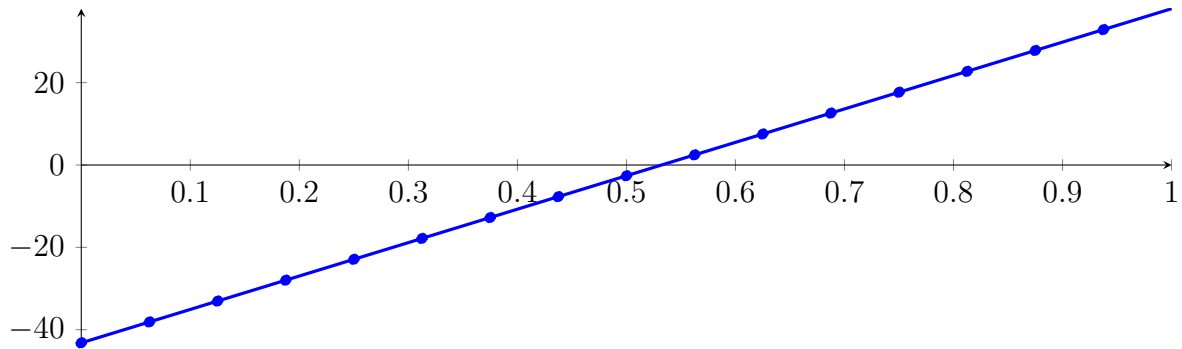
Longest intersection interval: $1.72301 \cdot 10^{-05}$

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333333]$,

80.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

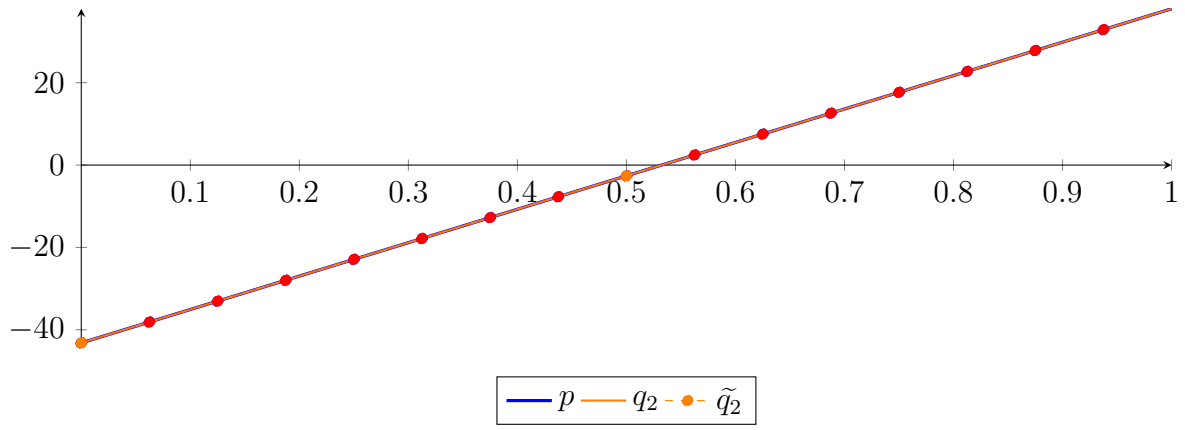
$$\begin{aligned} p &= 8.74252 \cdot 10^{-11} X^{16} - 1.56979 \cdot 10^{-09} X^{15} + 6.68479 \cdot 10^{-09} X^{14} + 1.20008 \cdot 10^{-08} X^{13} + 9.07301 \cdot 10^{-08} X^{12} \\ &+ 5.58657 \cdot 10^{-08} X^{11} + 1.13801 \cdot 10^{-07} X^{10} + 3.70665 \cdot 10^{-08} X^9 + 7.31575 \cdot 10^{-10} X^8 + 1.30058 \cdot 10^{-09} X^7 \\ &+ 5.00722 \cdot 10^{-09} X^6 + 1.24146 \cdot 10^{-10} X^5 + 1.03455 \cdot 10^{-10} X^4 - 3.09388 \cdot 10^{-05} X^2 + 81.1505X - 43.1911 \\ &= -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\ &- 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68777B_{7,16}(X) - 2.61587B_{8,16}(X) \\ &+ 2.45604B_{9,16}(X) + 7.52794B_{10,16}(X) + 12.5998B_{11,16}(X) + 17.6718B_{12,16}(X) \\ &+ 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09388 \cdot 10^{-05} X^2 + 81.1505X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 0.721495X^{16} - 5.74915X^{15} + 20.7933X^{14} - 45.1627X^{13} + 65.6806X^{12} - 67.5044X^{11} \\ &+ 50.4286X^{10} - 27.728X^9 + 11.2318X^8 - 3.32011X^7 + 0.702408X^6 - 0.103415X^5 \\ &+ 0.0102099X^4 - 0.000624725X^3 - 1.10834 \cdot 10^{-05} X^2 + 81.1505X - 43.1911 \\ &= -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} \\ &- 12.7597B_{6,16} - 7.68779B_{7,16} - 2.61585B_{8,16} + 2.45602B_{9,16} + 7.52795B_{10,16} + 12.5998B_{11,16} \\ &+ 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.57956 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -3.09388 \cdot 10^{-05} X^2 + 81.1505 X - 43.1911$$

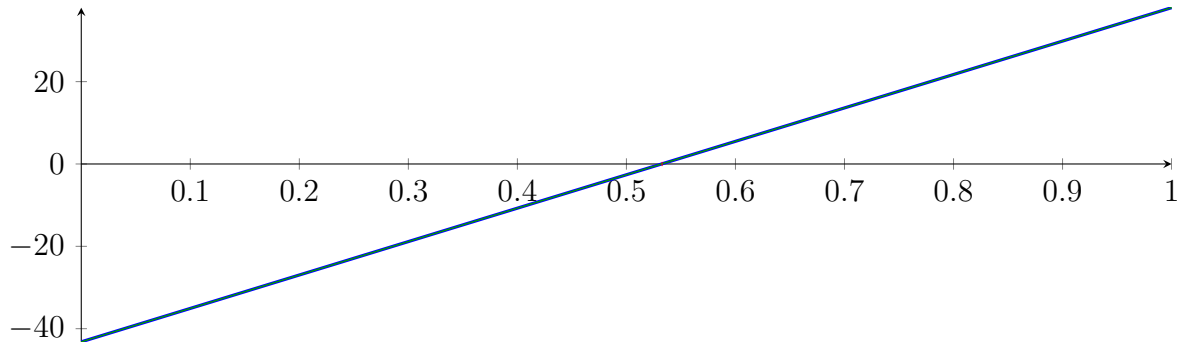
$$m = -3.09388 \cdot 10^{-05} X^2 + 81.1505 X - 43.1911$$

Root of M and m :

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\}$$

$$N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

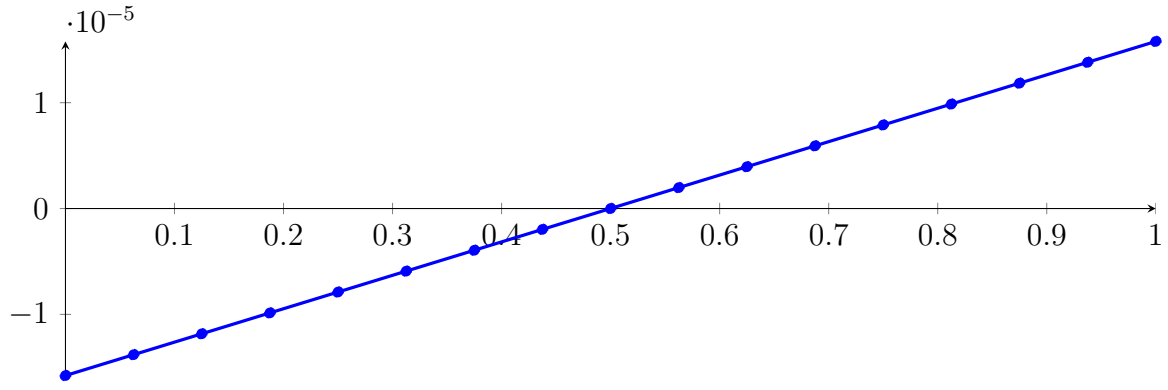
Longest intersection interval: $3.8903 \cdot 10^{-07}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

80.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
p &= -1.04409 \cdot 10^{-16} X^{16} - 1.53089 \cdot 10^{-16} X^{15} + 1.74665 \cdot 10^{-15} X^{14} + 5.46438 \cdot 10^{-15} X^{13} \\
&\quad + 2.56522 \cdot 10^{-14} X^{12} + 2.15479 \cdot 10^{-14} X^{11} + 3.51633 \cdot 10^{-14} X^{10} + 1.67444 \\
&\quad \cdot 10^{-14} X^9 - 8.72105 \cdot 10^{-16} X^8 + 1.41087 \cdot 10^{-15} X^6 + 5.91974 \cdot 10^{-17} X^5 + 4.93312 \\
&\quad \cdot 10^{-17} X^4 + 3.79471 \cdot 10^{-18} X^3 - 4.87891 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05} \\
&= -1.57804 \cdot 10^{-05} B_{0,16}(X) - 1.38073 \cdot 10^{-05} B_{1,16}(X) - 1.18341 \cdot 10^{-05} B_{2,16}(X) - 9.86101 \\
&\quad \cdot 10^{-06} B_{3,16}(X) - 7.88788 \cdot 10^{-06} B_{4,16}(X) - 5.91476 \cdot 10^{-06} B_{5,16}(X) - 3.94163 \cdot 10^{-06} B_{6,16}(X) \\
&\quad - 1.96851 \cdot 10^{-06} B_{7,16}(X) + 4.61501 \cdot 10^{-09} B_{8,16}(X) + 1.97774 \cdot 10^{-06} B_{9,16}(X) + 3.95086 \\
&\quad \cdot 10^{-06} B_{10,16}(X) + 5.92399 \cdot 10^{-06} B_{11,16}(X) + 7.89711 \cdot 10^{-06} B_{12,16}(X) + 9.87024 \cdot 10^{-06} B_{13,16}(X) \\
&\quad + 1.18434 \cdot 10^{-05} B_{14,16}(X) + 1.38165 \cdot 10^{-05} B_{15,16}(X) + 1.57896 \cdot 10^{-05} B_{16,16}(X)
\end{aligned}$$



Degree reduction and raising:

$$q_2 = -4.65868 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

$$= -1.57804 \cdot 10^{-05} B_{0,2} + 4.61501 \cdot 10^{-09} B_{1,2} + 1.57896 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = 2.92413 \cdot 10^{-07} X^{16} - 2.33332 \cdot 10^{-06} X^{15} + 8.45203 \cdot 10^{-06} X^{14} - 1.83895 \cdot 10^{-05} X^{13}$$

$$+ 2.67963 \cdot 10^{-05} X^{12} - 2.75995 \cdot 10^{-05} X^{11} + 2.06638 \cdot 10^{-05} X^{10} - 1.13854 \cdot 10^{-05} X^9$$

$$+ 4.61944 \cdot 10^{-06} X^8 - 1.36687 \cdot 10^{-06} X^7 + 2.89249 \cdot 10^{-07} X^6 - 4.25295 \cdot 10^{-08} X^5 + 4.17283$$

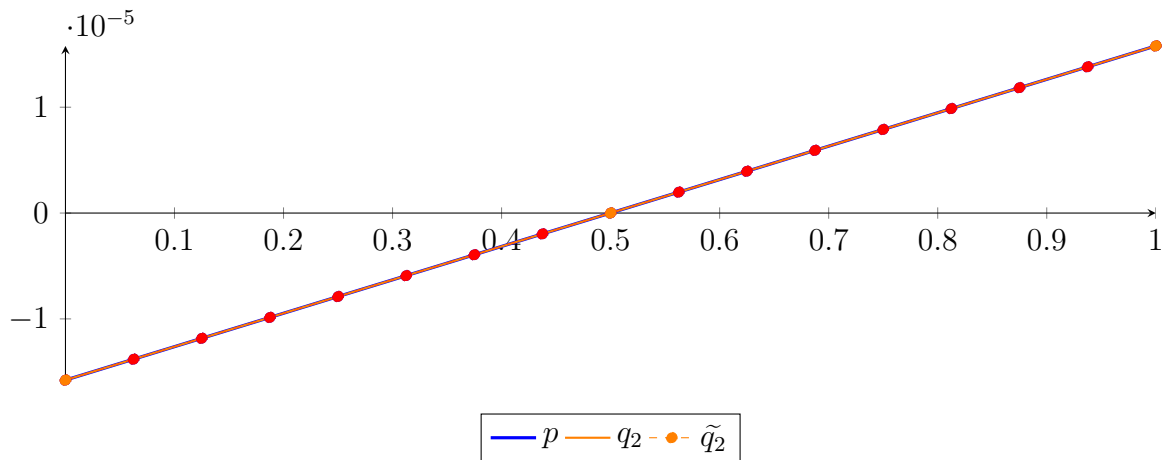
$$\cdot 10^{-09} X^4 - 2.52119 \cdot 10^{-10} X^3 + 7.82992 \cdot 10^{-12} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

$$= -1.57804 \cdot 10^{-05} B_{0,16} - 1.38073 \cdot 10^{-05} B_{1,16} - 1.18341 \cdot 10^{-05} B_{2,16} - 9.86101 \cdot 10^{-06} B_{3,16} - 7.88788$$

$$\cdot 10^{-06} B_{4,16} - 5.91476 \cdot 10^{-06} B_{5,16} - 3.94163 \cdot 10^{-06} B_{6,16} - 1.96851 \cdot 10^{-06} B_{7,16} + 4.62125 \cdot 10^{-09} B_{8,16}$$

$$+ 1.97773 \cdot 10^{-06} B_{9,16} + 3.95087 \cdot 10^{-06} B_{10,16} + 5.92399 \cdot 10^{-06} B_{11,16} + 7.89711 \cdot 10^{-06} B_{12,16}$$

$$+ 9.87024 \cdot 10^{-06} B_{13,16} + 1.18434 \cdot 10^{-05} B_{14,16} + 1.38165 \cdot 10^{-05} B_{15,16} + 1.57896 \cdot 10^{-05} B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 6.24192 \cdot 10^{-12}$.

Bounding polynomials M and m :

$$M = -4.65868 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

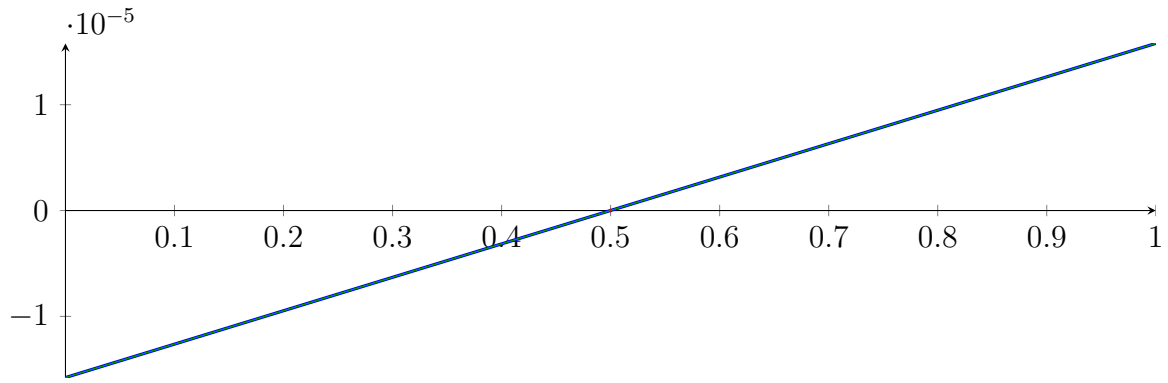
$$m = -4.65868 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.499636, 6.77659 \cdot 10^{12}\}$$

$$N(m) = \{0.500364, 6.77659 \cdot 10^{12}\}$$

Intersection intervals:



$$[0.499636, 0.500364]$$

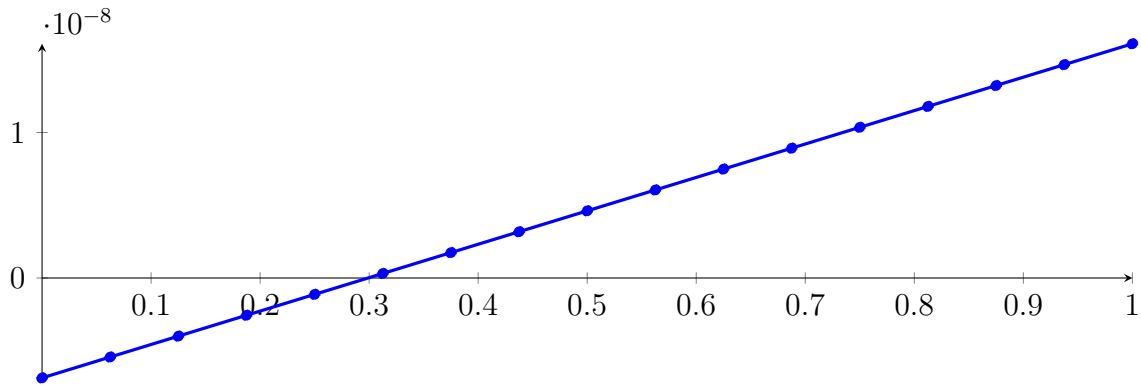
Longest intersection interval: 0.000727273

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

80.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

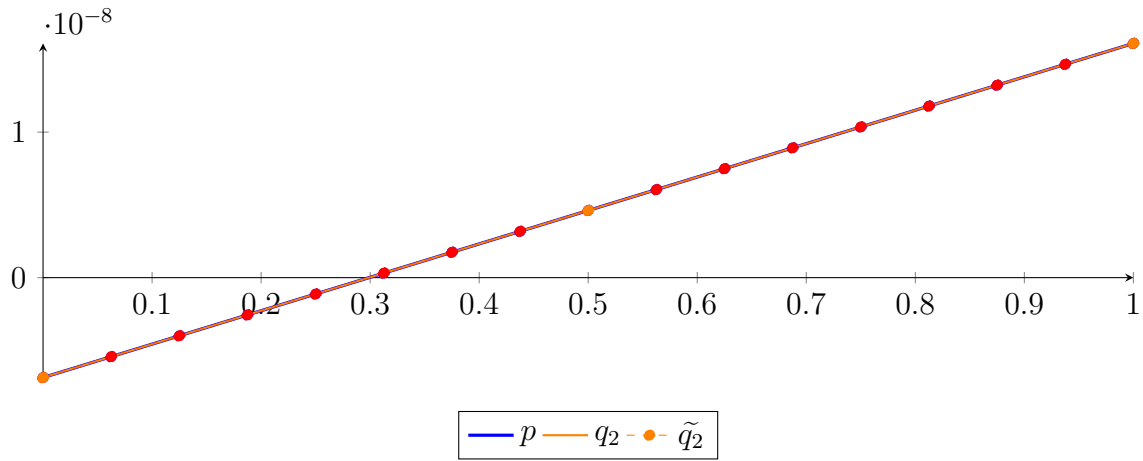
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
p &= -6.76104 \cdot 10^{-19} X^{16} + 2.21478 \cdot 10^{-18} X^{15} - 1.59295 \cdot 10^{-18} X^{14} + 1.03762 \cdot 10^{-18} X^{13} \\
&\quad - 1.34649 \cdot 10^{-17} X^{12} + 9.65427 \cdot 10^{-18} X^{11} - 2.75561 \cdot 10^{-18} X^{10} + 5.90488 \cdot 10^{-18} X^9 \\
&\quad - 8.51665 \cdot 10^{-19} X^8 + 3.02814 \cdot 10^{-19} X^7 + 2.64962 \cdot 10^{-19} X^6 + 2.8905 \cdot 10^{-20} X^5 \\
&\quad + 6.02187 \cdot 10^{-21} X^4 + 9.26442 \cdot 10^{-22} X^3 + 2.296 \cdot 10^{-08} X - 6.86499 \cdot 10^{-09} \\
&= -6.86499 \cdot 10^{-09} B_{0,16}(X) - 5.42999 \cdot 10^{-09} B_{1,16}(X) - 3.99499 \cdot 10^{-09} B_{2,16}(X) \\
&\quad - 2.55999 \cdot 10^{-09} B_{3,16}(X) - 1.12499 \cdot 10^{-09} B_{4,16}(X) + 3.10008 \cdot 10^{-10} B_{5,16}(X) + 1.74501 \\
&\quad \cdot 10^{-09} B_{6,16}(X) + 3.18001 \cdot 10^{-09} B_{7,16}(X) + 4.61501 \cdot 10^{-09} B_{8,16}(X) + 6.05001 \cdot 10^{-09} B_{9,16}(X) \\
&\quad + 7.48501 \cdot 10^{-09} B_{10,16}(X) + 8.92001 \cdot 10^{-09} B_{11,16}(X) + 1.0355 \cdot 10^{-08} B_{12,16}(X) + 1.179 \\
&\quad \cdot 10^{-08} B_{13,16}(X) + 1.3225 \cdot 10^{-08} B_{14,16}(X) + 1.466 \cdot 10^{-08} B_{15,16}(X) + 1.6095 \cdot 10^{-08} B_{16,16}(X)
\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
q_2 &= 1.40621 \cdot 10^{-23} X^2 + 2.296 \cdot 10^{-08} X - 6.86499 \cdot 10^{-09} \\
&= -6.86499 \cdot 10^{-09} B_{0,2} + 4.61501 \cdot 10^{-09} B_{1,2} + 1.6095 \cdot 10^{-08} B_{2,2} \\
\tilde{q}_2 &= 2.65578 \cdot 10^{-10} X^{16} - 2.13329 \cdot 10^{-09} X^{15} + 7.78369 \cdot 10^{-09} X^{14} - 1.70728 \cdot 10^{-08} X^{13} \\
&\quad + 2.51029 \cdot 10^{-08} X^{12} - 2.61095 \cdot 10^{-08} X^{11} + 1.97446 \cdot 10^{-08} X^{10} - 1.09796 \cdot 10^{-08} X^9 \\
&\quad + 4.48709 \cdot 10^{-09} X^8 - 1.33355 \cdot 10^{-09} X^7 + 2.82501 \cdot 10^{-10} X^6 - 4.12963 \cdot 10^{-11} X^5 + 3.94088 \\
&\quad \cdot 10^{-12} X^4 - 2.24328 \cdot 10^{-13} X^3 + 6.17064 \cdot 10^{-15} X^2 + 2.296 \cdot 10^{-08} X - 6.86499 \cdot 10^{-09} \\
&= -6.86499 \cdot 10^{-09} B_{0,16} - 5.42999 \cdot 10^{-09} B_{1,16} - 3.99499 \cdot 10^{-09} B_{2,16} - 2.55999 \cdot 10^{-09} B_{3,16} \\
&\quad - 1.12499 \cdot 10^{-09} B_{4,16} + 3.10006 \cdot 10^{-10} B_{5,16} + 1.74501 \cdot 10^{-09} B_{6,16} + 3.18 \cdot 10^{-09} B_{7,16} + 4.61501 \\
&\quad \cdot 10^{-09} B_{8,16} + 6.05 \cdot 10^{-09} B_{9,16} + 7.48501 \cdot 10^{-09} B_{10,16} + 8.92 \cdot 10^{-09} B_{11,16} + 1.0355 \cdot 10^{-08} B_{12,16} \\
&\quad + 1.179 \cdot 10^{-08} B_{13,16} + 1.3225 \cdot 10^{-08} B_{14,16} + 1.466 \cdot 10^{-08} B_{15,16} + 1.6095 \cdot 10^{-08} B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.35405 \cdot 10^{-15}$.

Bounding polynomials M and m :

$$M = 1.32349 \cdot 10^{-23} X^2 + 2.296 \cdot 10^{-08} X - 6.86498 \cdot 10^{-09}$$

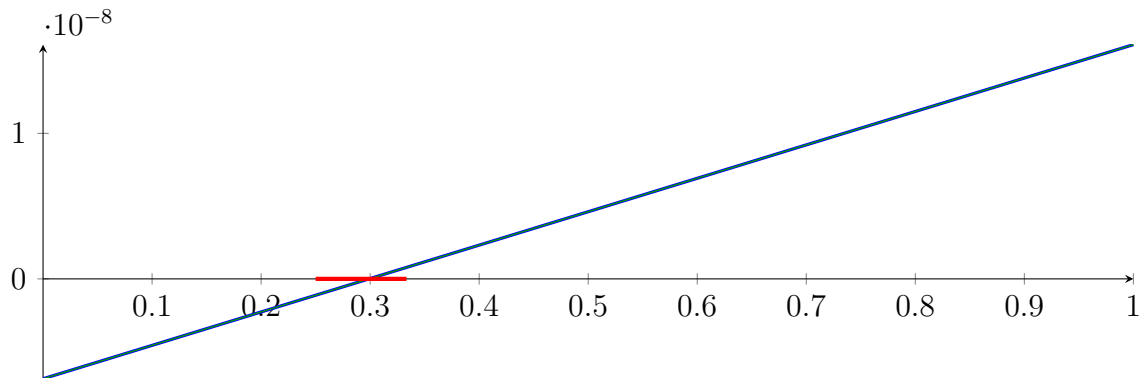
$$m = 1.48893 \cdot 10^{-23} X^2 + 2.296 \cdot 10^{-08} X - 6.865 \cdot 10^{-09}$$

Root of M and m :

$$N(M) = \{-1.73481 \cdot 10^{15}, 0.25\}$$

$$N(m) = \{-1.54205 \cdot 10^{15}, 0.333333\}$$

Intersection intervals:



$$[0.333333, 0.25]$$

Longest intersection interval: -0.0833333

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

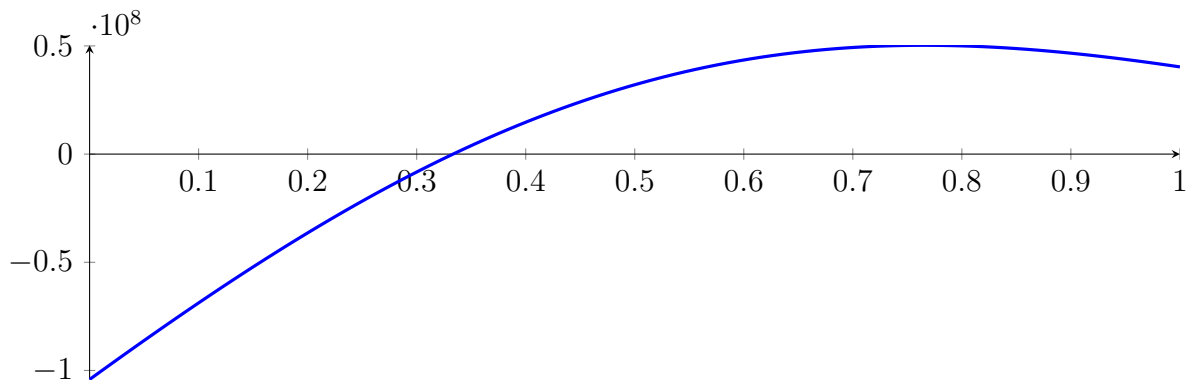
80.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

80.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

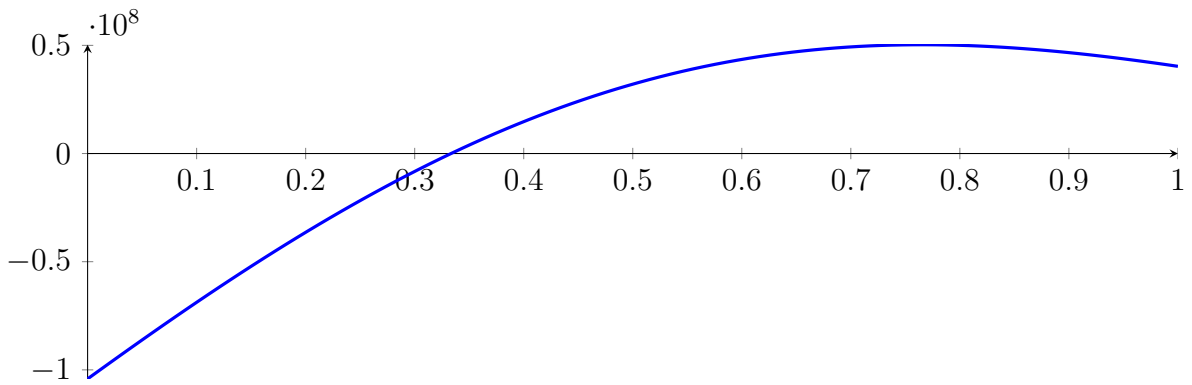
with precision $\varepsilon = 1 \cdot 10^{-64}$.

81 Running CubeClip on f_{16} with epsilon 64

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

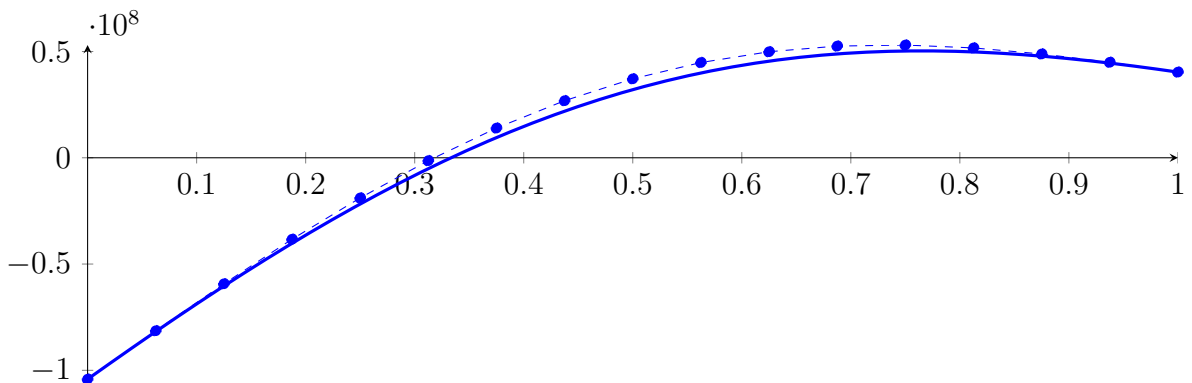
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



81.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 2.11088 \cdot 10^6 X^{16} - 1.70537 \cdot 10^7 X^{15} + 6.25913 \cdot 10^7 X^{14} - 1.38111 \cdot 10^8 X^{13} + 2.043 \cdot 10^8 X^{12} - 2.13799$$

$$\cdot 10^8 X^{11} + 1.62714 \cdot 10^8 X^{10} - 9.11092 \cdot 10^7 X^9 + 3.75254 \cdot 10^7 X^8 - 1.12529 \cdot 10^7 X^7 + 2.40899 \cdot 10^6 X^6$$

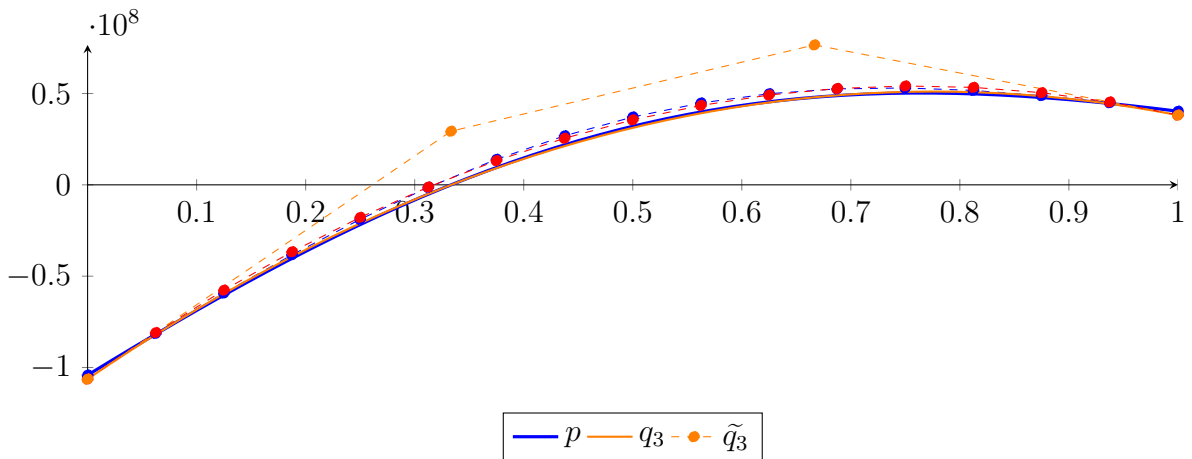
$$- 357156 X^5 + 34986.1 X^4 + 2.75602 \cdot 10^6 X^3 - 2.65318 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18553 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91816 \cdot 10^7 B_{10,16} + 5.27352 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

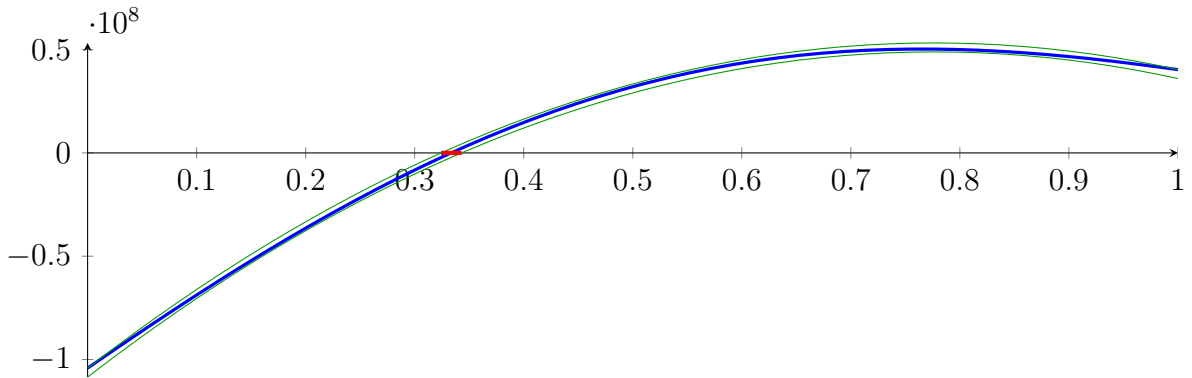
$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\}$$

$$N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

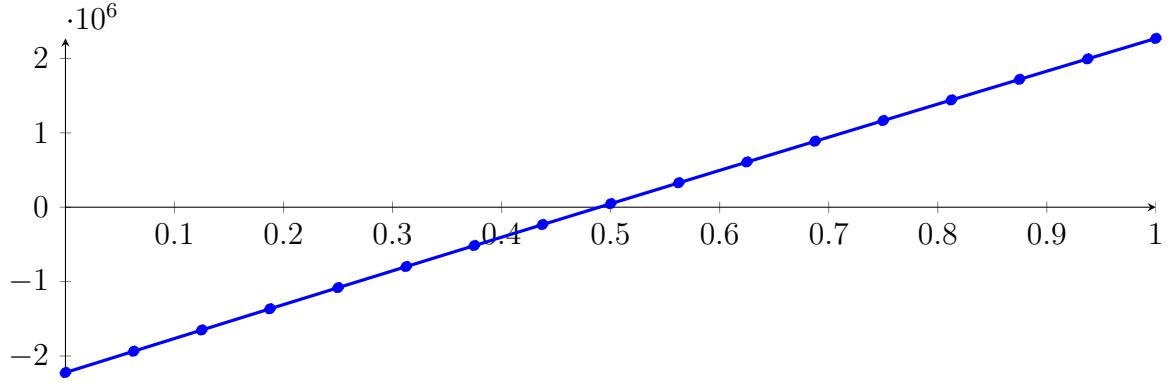
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

81.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

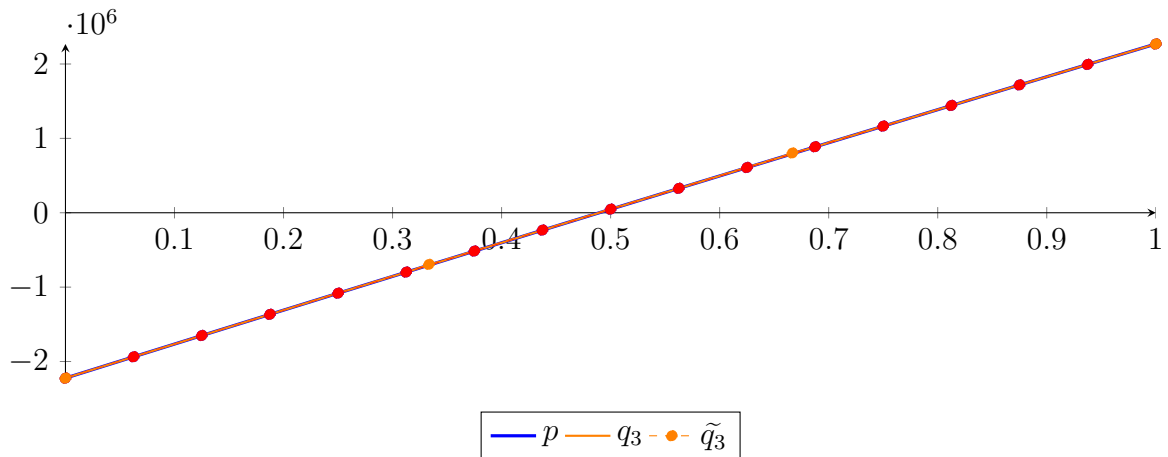
$$\begin{aligned}
 p &= -1.93035 \cdot 10^{-05} X^{16} + 1.08927 \cdot 10^{-05} X^{15} + 0.000255816 X^{14} + 0.000616983 X^{13} + 0.00371715 X^{12} \\
 &+ 0.00325035 X^{11} + 0.00510875 X^{10} + 0.00190713 X^9 + 0.000179792 X^8 + 1.06543 \cdot 10^{-05} X^7 \\
 &- 0.00136482 X^6 + 0.104959 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &- 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &+ 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 15290.6 X^{16} - 125412 X^{15} + 468045 X^{14} - 1.05198 \cdot 10^6 X^{13} + 1.588 \cdot 10^6 X^{12} - 1.69893 \\
 &\cdot 10^6 X^{11} + 1.32382 \cdot 10^6 X^{10} - 759402 X^9 + 320067 X^8 - 97826.5 X^7 + 21197.8 X^6 \\
 &- 3162.8 X^5 + 313.931 X^4 - 720.165 X^3 - 93879.2 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &- 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.7 B_{8,16} \\
 &+ 328649 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.457751$.

Bounding polynomials M and m :

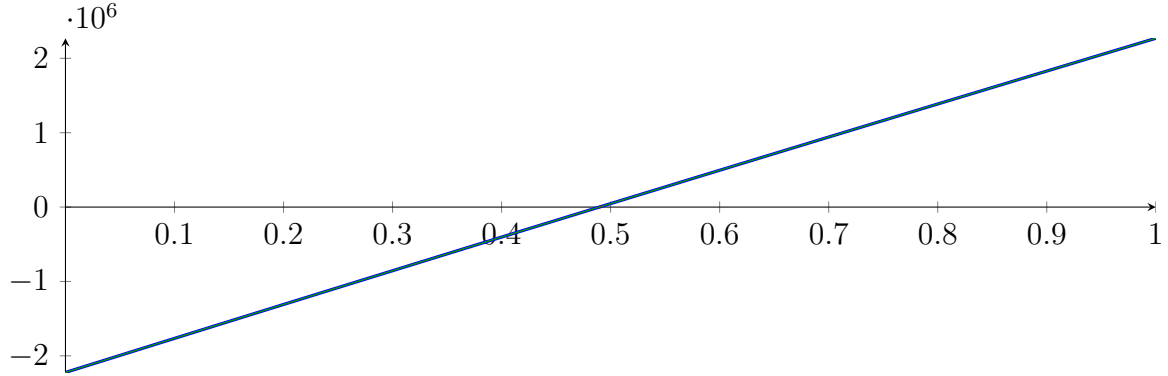
$$M = -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\} \quad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

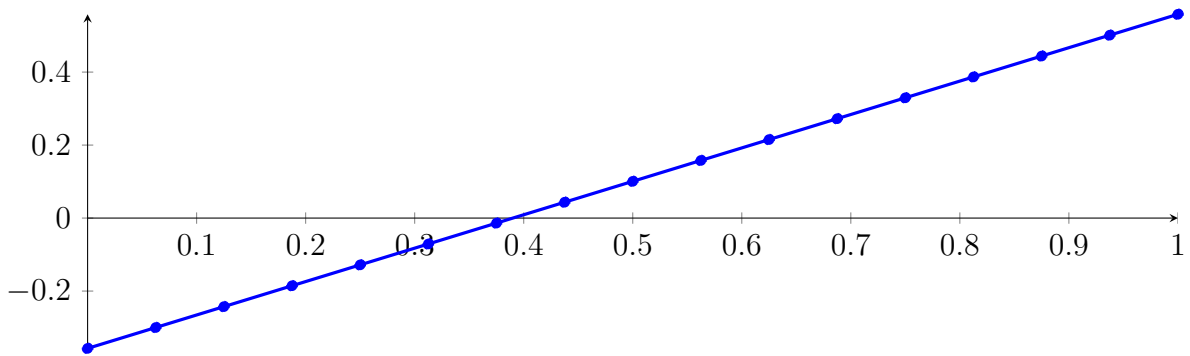
Longest intersection interval: $2.03684 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

81.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

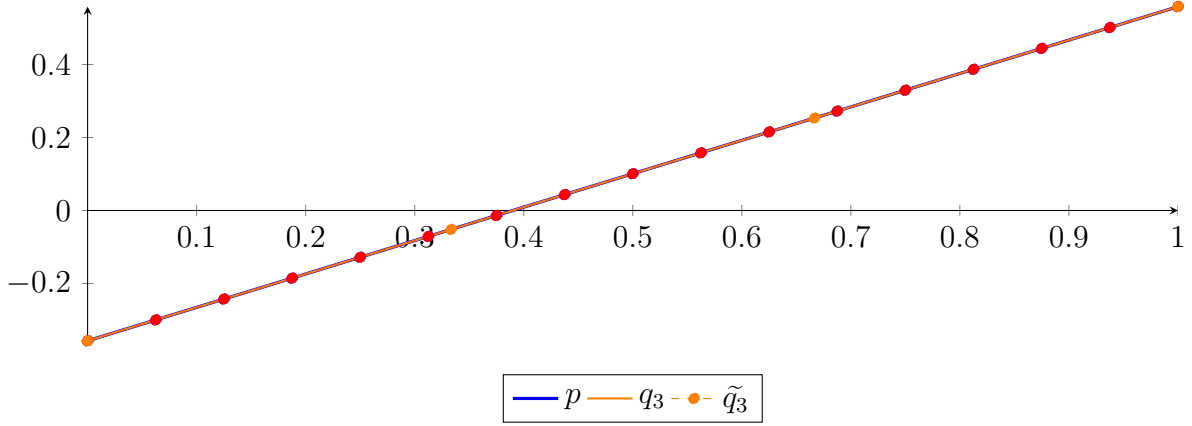
$$\begin{aligned} p &= -1.56399 \cdot 10^{-11} X^{16} + 4.58016 \cdot 10^{-11} X^{15} - 3.19744 \cdot 10^{-12} X^{14} + 6.54055 \cdot 10^{-11} X^{13} \\ &\quad + 1.05072 \cdot 10^{-10} X^{12} + 4.59728 \cdot 10^{-10} X^{11} + 5.01434 \cdot 10^{-10} X^{10} + 2.99742 \cdot 10^{-10} X^9 \\ &\quad + 1.14309 \cdot 10^{-11} X^8 - 5.08038 \cdot 10^{-12} X^7 + 3.37845 \cdot 10^{-11} X^6 - 9.69891 \cdot 10^{-13} X^5 \\ &\quad + 4.04121 \cdot 10^{-13} X^4 + 6.21725 \cdot 10^{-14} X^3 - 3.93767 \cdot 10^{-09} X^2 + 0.915501 X - 0.357072 \\ &= -0.357072 B_{0,16}(X) - 0.299853 B_{1,16}(X) - 0.242635 B_{2,16}(X) - 0.185416 B_{3,16}(X) \\ &\quad - 0.128197 B_{4,16}(X) - 0.0709781 B_{5,16}(X) - 0.0137592 B_{6,16}(X) \\ &\quad + 0.0434596 B_{7,16}(X) + 0.100678 B_{8,16}(X) + 0.157897 B_{9,16}(X) + 0.215116 B_{10,16}(X) \\ &\quad + 0.272335 B_{11,16}(X) + 0.329554 B_{12,16}(X) + 0.386773 B_{13,16}(X) \\ &\quad + 0.443991 B_{14,16}(X) + 0.50121 B_{15,16}(X) + 0.558429 B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 1.05471 \cdot 10^{-15} X^3 - 3.93767 \cdot 10^{-09} X^2 + 0.915501 X - 0.357072 \\ &= -0.357072 B_{0,3} - 0.0519051 B_{1,3} + 0.253262 B_{2,3} + 0.558429 B_{3,3} \end{aligned}$$

$$\begin{aligned}
\tilde{q}_3 &= 0.00291222X^{16} - 0.0241801X^{15} + 0.0914452X^{14} - 0.208537X^{13} + 0.319778X^{12} - 0.347745X^{11} \\
&\quad + 0.275244X^{10} - 0.159971X^9 + 0.0679818X^8 - 0.0208072X^7 + 0.00447629X^6 - 0.000654783X^5 \\
&\quad + 6.22034 \cdot 10^{-05}X^4 - 3.60145 \cdot 10^{-06}X^3 + 9.78811 \cdot 10^{-08}X^2 + 0.915501X - 0.357072 \\
&= -0.357072B_{0,16} - 0.299853B_{1,16} - 0.242635B_{2,16} - 0.185416B_{3,16} - 0.128197B_{4,16} \\
&\quad - 0.0709781B_{5,16} - 0.0137592B_{6,16} + 0.0434595B_{7,16} + 0.100678B_{8,16} \\
&\quad + 0.157897B_{9,16} + 0.215116B_{10,16} + 0.272335B_{11,16} + 0.329554B_{12,16} \\
&\quad + 0.386773B_{13,16} + 0.443991B_{14,16} + 0.50121B_{15,16} + 0.558429B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.5212 \cdot 10^{-08}$.

Bounding polynomials M and m :

$$M = 9.99201 \cdot 10^{-16}X^3 - 3.93767 \cdot 10^{-09}X^2 + 0.915501X - 0.357072$$

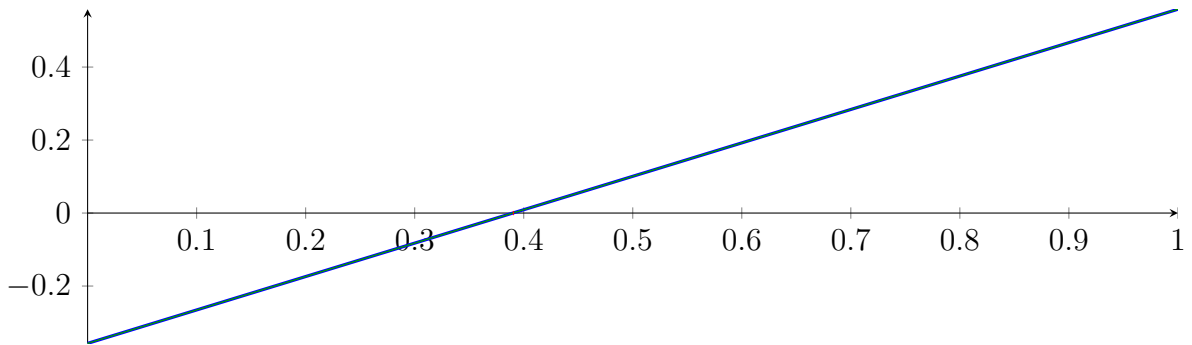
$$m = 1.22125 \cdot 10^{-15}X^3 - 3.93767 \cdot 10^{-09}X^2 + 0.915501X - 0.357072$$

Root of M and m :

$$N(M) = \{0.390029\}$$

$$N(m) = \{0.390029\}$$

Intersection intervals:



$$[0.390029, 0.390029]$$

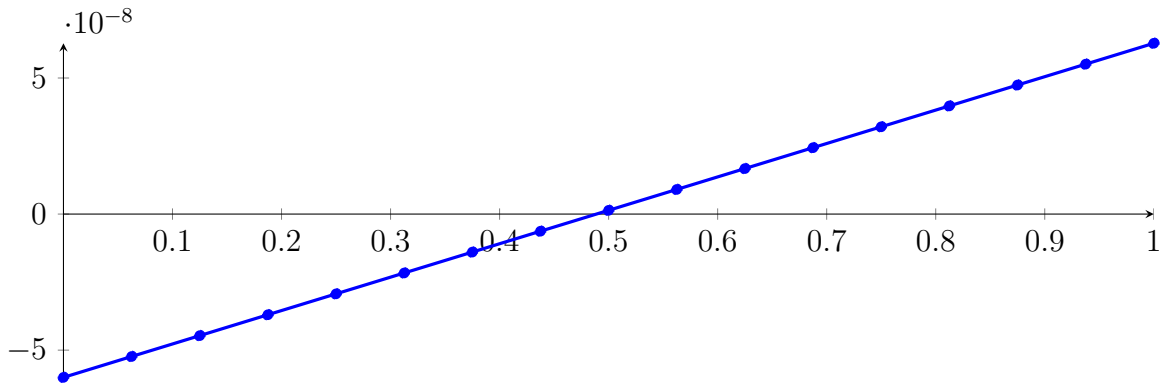
Longest intersection interval: $1.3411 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

81.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

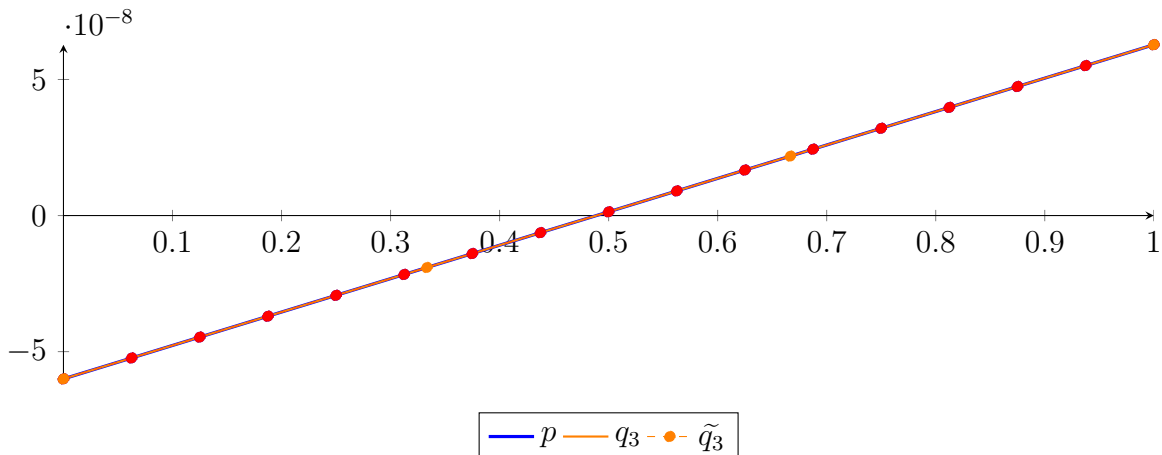
$$\begin{aligned}
 p &= -5.65183 \cdot 10^{-19} X^{16} + 5.13302 \cdot 10^{-19} X^{15} + 6.37816 \cdot 10^{-18} X^{14} + 1.69576 \cdot 10^{-17} X^{13} \\
 &\quad + 9.4423 \cdot 10^{-17} X^{12} + 8.0934 \cdot 10^{-17} X^{11} + 1.37357 \cdot 10^{-16} X^{10} + 6.23797 \cdot 10^{-17} X^9 \\
 &\quad + 1.36266 \cdot 10^{-18} X^8 + 6.05629 \cdot 10^{-19} X^7 + 5.08728 \cdot 10^{-18} X^6 + 2.3124 \cdot 10^{-19} X^5 \\
 &\quad + 1.44525 \cdot 10^{-19} X^4 - 7.41154 \cdot 10^{-21} X^3 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08} \\
 &= -6.00146 \cdot 10^{-08} B_{0,16}(X) - 5.2341 \cdot 10^{-08} B_{1,16}(X) - 4.46674 \cdot 10^{-08} B_{2,16}(X) - 3.69937 \\
 &\quad \cdot 10^{-08} B_{3,16}(X) - 2.93201 \cdot 10^{-08} B_{4,16}(X) - 2.16464 \cdot 10^{-08} B_{5,16}(X) - 1.39728 \cdot 10^{-08} B_{6,16}(X) \\
 &\quad - 6.29913 \cdot 10^{-09} B_{7,16}(X) + 1.37451 \cdot 10^{-09} B_{8,16}(X) + 9.04815 \cdot 10^{-09} B_{9,16}(X) + 1.67218 \\
 &\quad \cdot 10^{-08} B_{10,16}(X) + 2.43954 \cdot 10^{-08} B_{11,16}(X) + 3.20691 \cdot 10^{-08} B_{12,16}(X) + 3.97427 \\
 &\quad \cdot 10^{-08} B_{13,16}(X) + 4.74164 \cdot 10^{-08} B_{14,16}(X) + 5.509 \cdot 10^{-08} B_{15,16}(X) + 6.27637 \cdot 10^{-08} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= 2.51463 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08} \\
 &= -6.00146 \cdot 10^{-08} B_{0,3} - 1.90885 \cdot 10^{-08} B_{1,3} + 2.18376 \cdot 10^{-08} B_{2,3} + 6.27637 \cdot 10^{-08} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 4.01426 \cdot 10^{-10} X^{16} - 3.29623 \cdot 10^{-09} X^{15} + 1.2317 \cdot 10^{-08} X^{14} - 2.77213 \cdot 10^{-08} X^{13} + 4.19074 \\
 &\quad \cdot 10^{-08} X^{12} - 4.49047 \cdot 10^{-08} X^{11} + 3.50457 \cdot 10^{-08} X^{10} - 2.01341 \cdot 10^{-08} X^9 + 8.49676 \\
 &\quad \cdot 10^{-09} X^8 - 2.5992 \cdot 10^{-09} X^7 + 5.63364 \cdot 10^{-10} X^6 - 8.40231 \cdot 10^{-11} X^5 + 8.33259 \\
 &\quad \cdot 10^{-12} X^4 - 5.16965 \cdot 10^{-13} X^3 + 1.7395 \cdot 10^{-14} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08} \\
 &= -6.00146 \cdot 10^{-08} B_{0,16} - 5.2341 \cdot 10^{-08} B_{1,16} - 4.46674 \cdot 10^{-08} B_{2,16} - 3.69937 \cdot 10^{-08} B_{3,16} - 2.93201 \\
 &\quad \cdot 10^{-08} B_{4,16} - 2.16464 \cdot 10^{-08} B_{5,16} - 1.39728 \cdot 10^{-08} B_{6,16} - 6.29914 \cdot 10^{-09} B_{7,16} + 1.37452 \cdot 10^{-09} B_{8,16} \\
 &\quad + 9.04815 \cdot 10^{-09} B_{9,16} + 1.67218 \cdot 10^{-08} B_{10,16} + 2.43954 \cdot 10^{-08} B_{11,16} + 3.20691 \cdot 10^{-08} B_{12,16} \\
 &\quad + 3.97427 \cdot 10^{-08} B_{13,16} + 4.74164 \cdot 10^{-08} B_{14,16} + 5.509 \cdot 10^{-08} B_{15,16} + 6.27637 \cdot 10^{-08} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.90061 \cdot 10^{-15}$.

Bounding polynomials M and m :

$$M = 2.38228 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08}$$

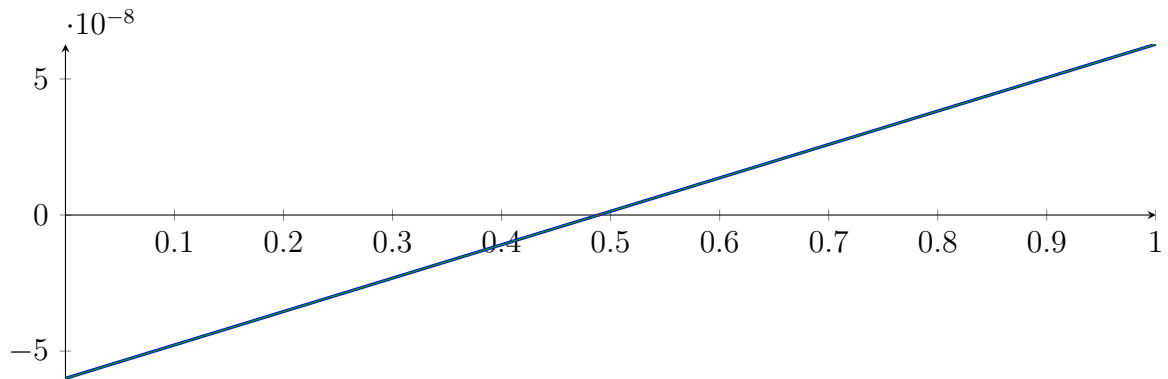
$$m = 2.51463 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08}$$

Root of M and m :

$$N(M) = \{0.488805\}$$

$$N(m) = \{0.488805\}$$

Intersection intervals:



$$[0.488805, 0.488805]$$

Longest intersection interval: $1.3086 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

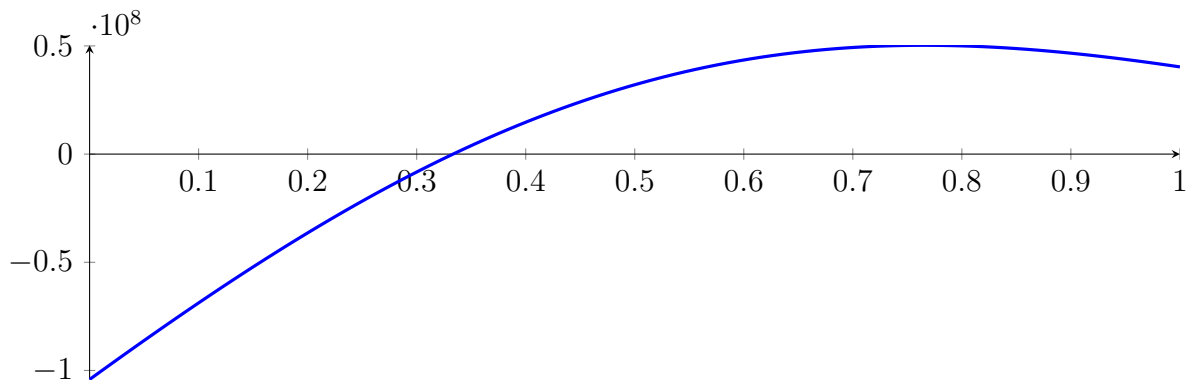
81.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

81.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

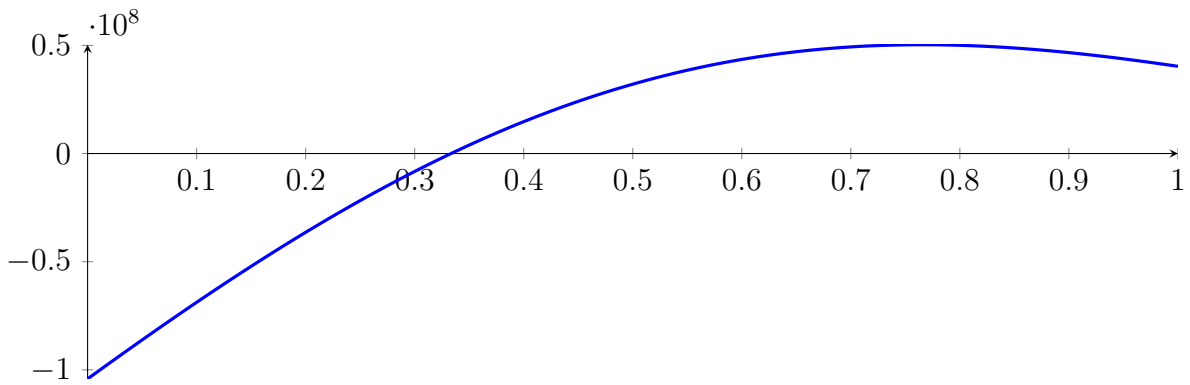
with precision $\varepsilon = 1 \cdot 10^{-64}$.

82 Running BezClip on f_{16} with epsilon 128

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

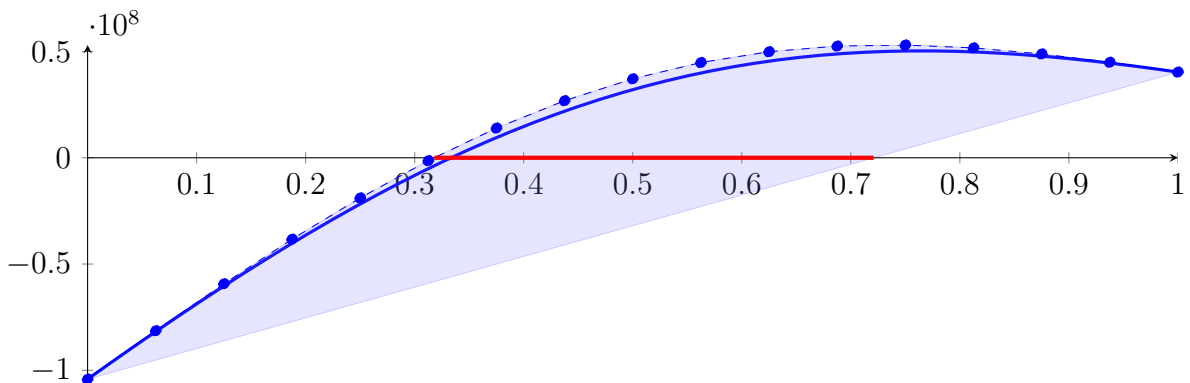
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



82.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

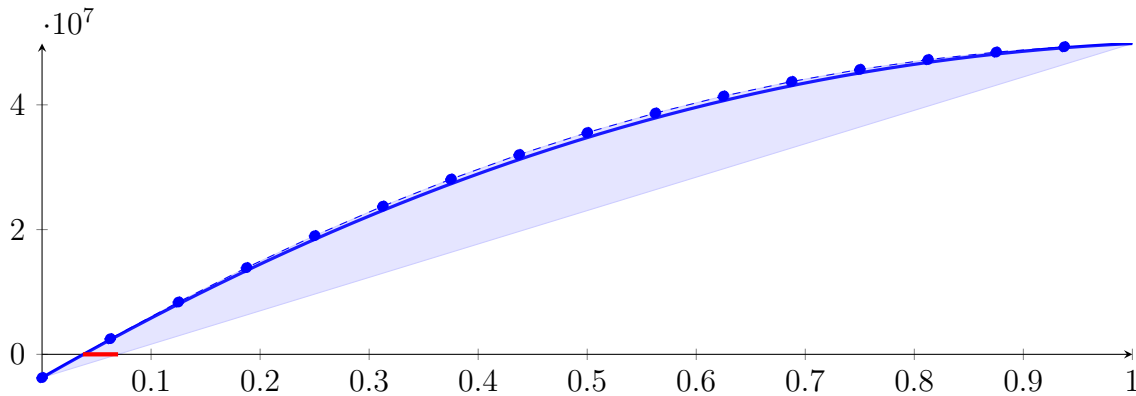
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

82.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.00483322X^{16} + 0.0186809X^{15} - 0.0194312X^{14} - 0.0738695X^{13} - 1.11673X^{12} \\ &\quad - 5.0471X^{11} + 36.3082X^{10} + 692.914X^9 + 1886.96X^8 - 25792X^7 - 149671X^6 + 492605X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

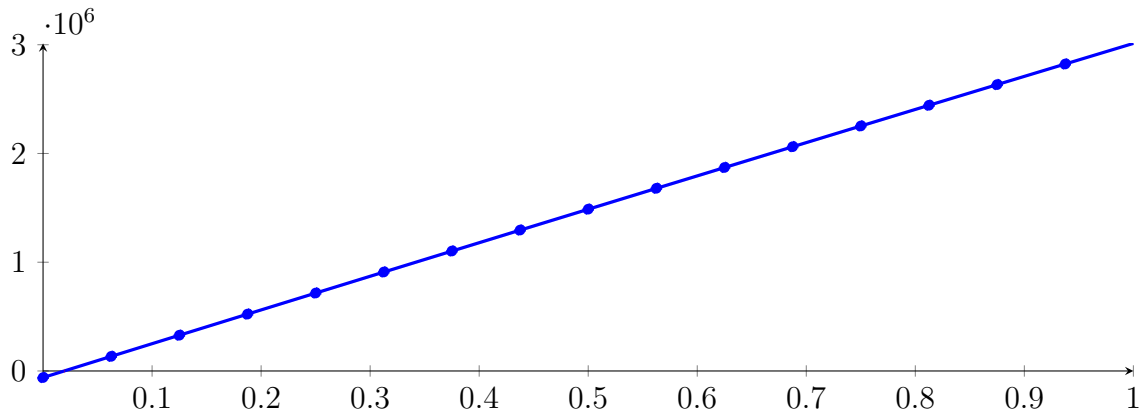
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

82.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.000205056X^{16} + 0.000776167X^{15} - 0.000863333X^{14} - 0.000217499X^{13} - 0.00809276X^{12} \\ &\quad - 5.09773 \cdot 10^{-05} X^{11} - 0.00564923X^{10} - 0.000162811X^9 - 0.000215376X^8 - 3.32948 \cdot 10^{-07} X^7 \\ &\quad - 0.000259866X^6 + 0.0161077X^5 + 4.36155X^4 - 234.216X^3 - 45622.2X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

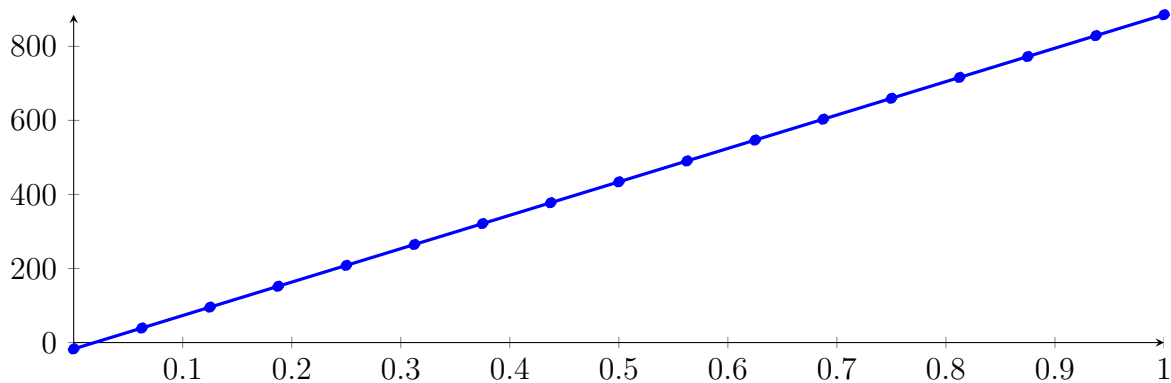
Longest intersection interval: 0.000289554

⇒ Selective recursion: interval 1: [\[0.333333, 0.333337\]](#),

82.4 Recursion Branch 1 1 1 1 in Interval 1: [\[0.333333, 0.333337\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -5.9692 \cdot 10^{-08} X^{16} + 2.16103 \cdot 10^{-07} X^{15} - 2.28456 \cdot 10^{-07} X^{14} - 1.17238 \cdot 10^{-07} X^{13} \\
 &\quad - 2.29525 \cdot 10^{-06} X^{12} - 8.31778 \cdot 10^{-08} X^{11} - 1.74251 \cdot 10^{-06} X^{10} - 9.42919 \cdot 10^{-08} X^9 \\
 &\quad - 7.38891 \cdot 10^{-08} X^8 + 3.25144 \cdot 10^{-09} X^7 - 2.61741 \cdot 10^{-08} X^6 + 7.44876 \cdot 10^{-10} X^5 \\
 &\quad - 2.58638 \cdot 10^{-10} X^4 - 5.65024 \cdot 10^{-09} X^3 - 0.00382618 X^2 + 902.448 X - 17.178 \\
 &= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
 &\quad + 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
 &\quad + 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
 &\quad + 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

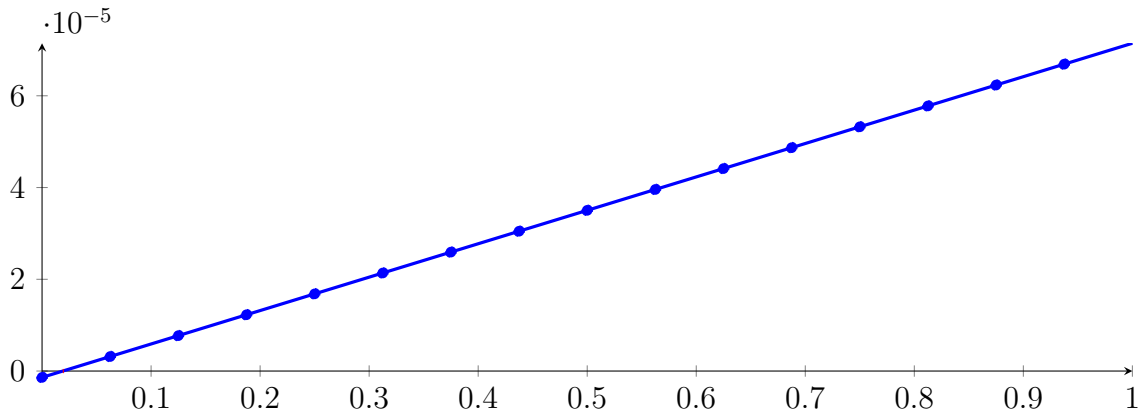
Longest intersection interval: $8.07045 \cdot 10^{-08}$

⇒ Selective recursion: interval 1: [\[0.333333, 0.333333\]](#),

82.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -4.80379 \cdot 10^{-15} X^{16} + 1.74137 \cdot 10^{-14} X^{15} - 1.72656 \cdot 10^{-14} X^{14} - 6.84186 \cdot 10^{-15} X^{13} \\
 &\quad - 1.91627 \cdot 10^{-13} X^{12} - 4.7358 \cdot 10^{-15} X^{11} - 1.33436 \cdot 10^{-13} X^{10} - 1.97677 \cdot 10^{-15} X^9 \\
 &\quad - 7.4565 \cdot 10^{-15} X^8 - 1.16281 \cdot 10^{-16} X^7 - 1.98065 \cdot 10^{-15} X^6 + 1.47994 \cdot 10^{-17} X^5 \\
 &\quad - 1.84992 \cdot 10^{-17} X^4 - 2.48011 \cdot 10^{-17} X^2 + 7.28316 \cdot 10^{-05} X - 1.38634 \cdot 10^{-06} \\
 &= -1.38634 \cdot 10^{-06} B_{0,16}(X) + 3.16564 \cdot 10^{-06} B_{1,16}(X) + 7.71761 \cdot 10^{-06} B_{2,16}(X) + 1.22696 \\
 &\quad \cdot 10^{-05} B_{3,16}(X) + 1.68216 \cdot 10^{-05} B_{4,16}(X) + 2.13735 \cdot 10^{-05} B_{5,16}(X) + 2.59255 \cdot 10^{-05} B_{6,16}(X) \\
 &\quad + 3.04775 \cdot 10^{-05} B_{7,16}(X) + 3.50295 \cdot 10^{-05} B_{8,16}(X) + 3.95814 \cdot 10^{-05} B_{9,16}(X) + 4.41334 \\
 &\quad \cdot 10^{-05} B_{10,16}(X) + 4.86854 \cdot 10^{-05} B_{11,16}(X) + 5.32374 \cdot 10^{-05} B_{12,16}(X) + 5.77893 \cdot 10^{-05} B_{13,16}(X) \\
 &\quad + 6.23413 \cdot 10^{-05} B_{14,16}(X) + 6.68933 \cdot 10^{-05} B_{15,16}(X) + 7.14453 \cdot 10^{-05} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.50521 \cdot 10^{-15}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

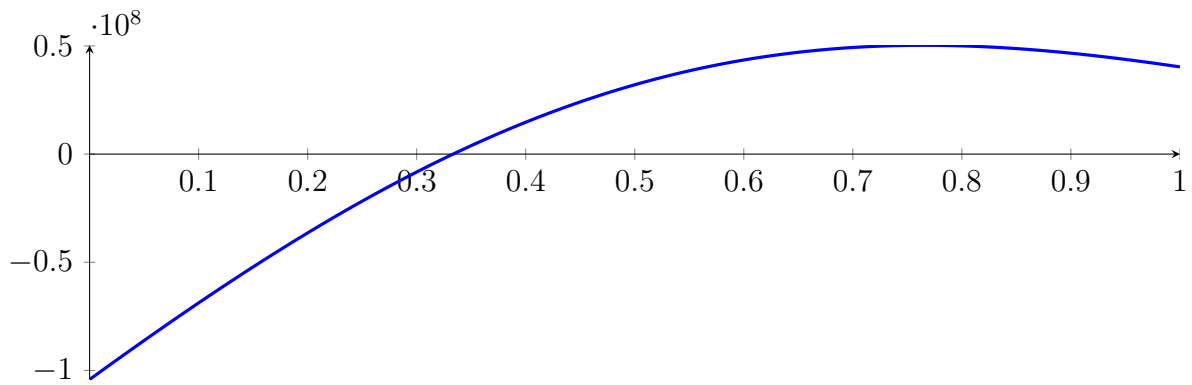
82.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

82.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

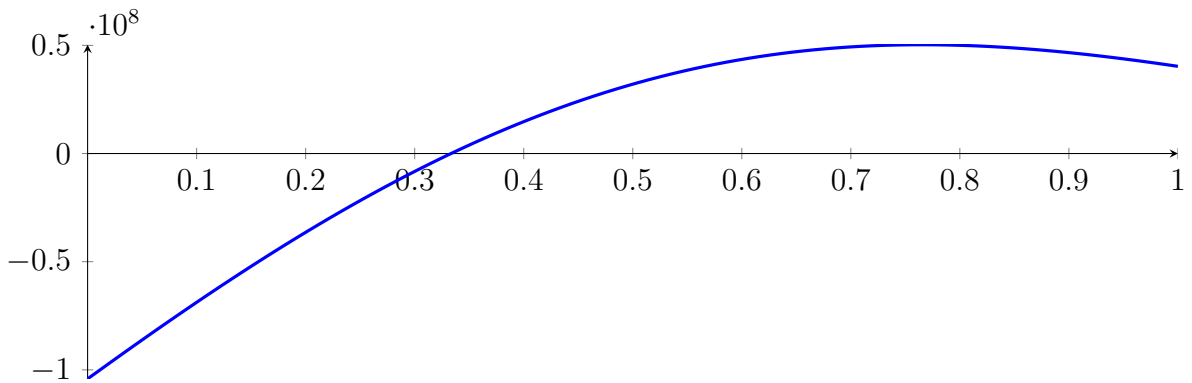
with precision $\varepsilon = 1 \cdot 10^{-128}$.

83 Running QuadClip on f_{16} with epsilon 128

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

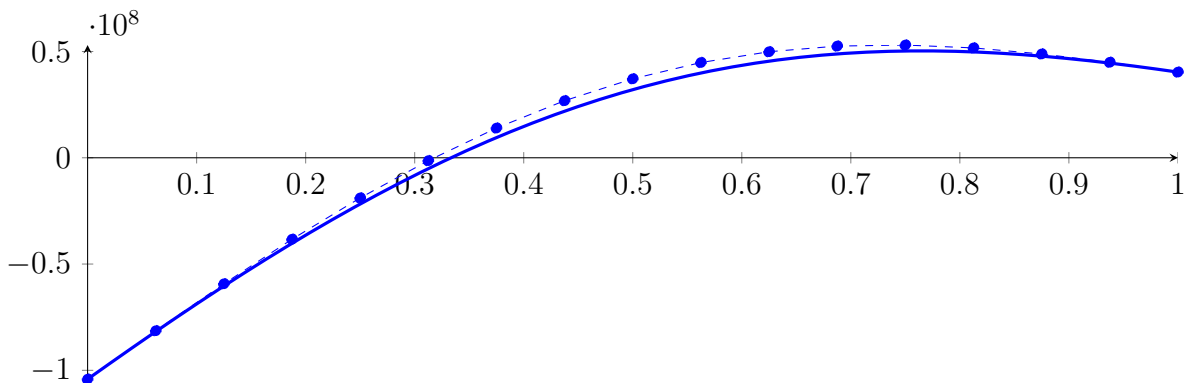
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



83.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = -1.41787 \cdot 10^6 X^{16} + 1.11761 \cdot 10^7 X^{15} - 3.98898 \cdot 10^7 X^{14} + 8.52437 \cdot 10^7 X^{13} - 1.21528 \cdot 10^8 X^{12}$$

$$+ 1.21946 \cdot 10^8 X^{11} - 8.86062 \cdot 10^7 X^{10} + 4.72904 \cdot 10^7 X^9 - 1.86355 \cdot 10^7 X^8 + 5.41059 \cdot 10^6 X^7 - 1.14487$$

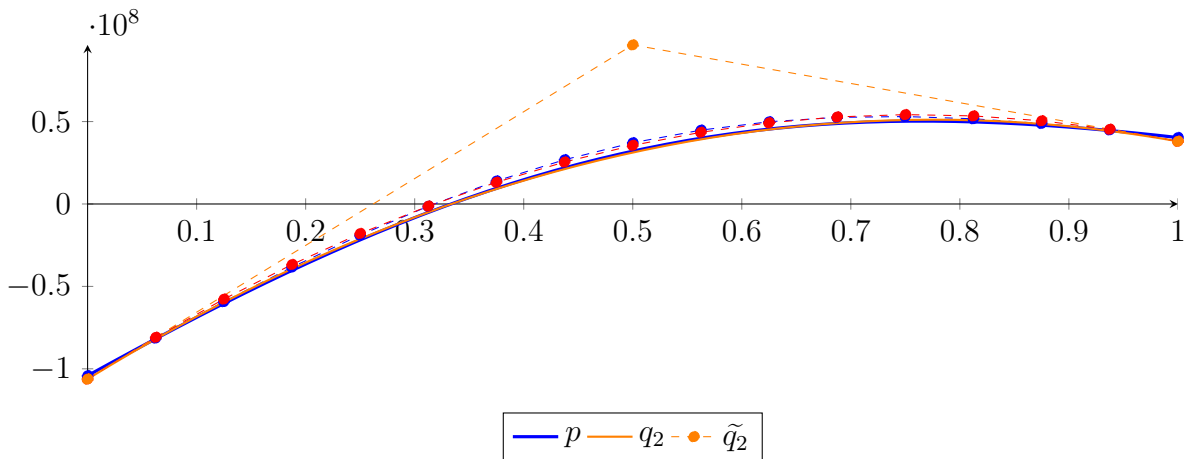
$$\cdot 10^6 X^6 + 172240 X^5 - 17636.6 X^4 + 1159.74 X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26923 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55691 \cdot 10^7 B_{8,16}$$

$$+ 4.34957 \cdot 10^7 B_{9,16} + 4.92455 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

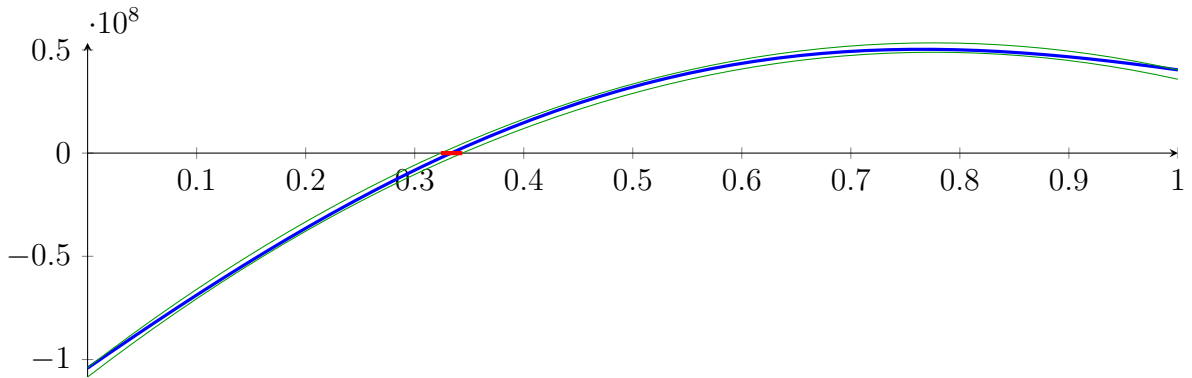
$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\}$$

$$N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

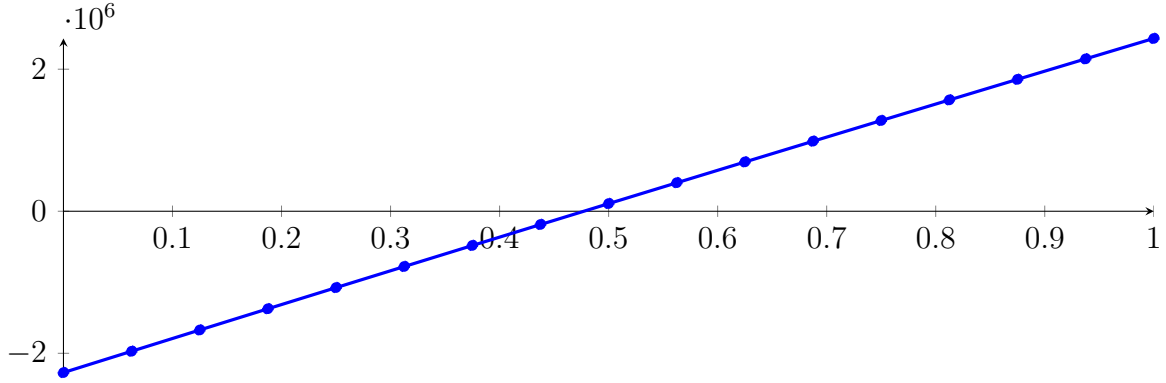
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

83.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

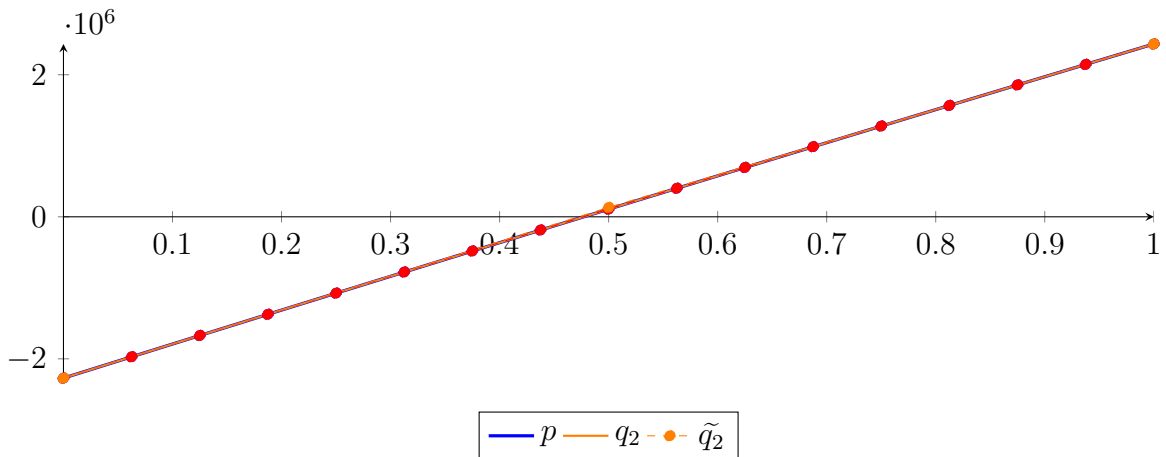
$$\begin{aligned}
 p &= -2.82438 \cdot 10^{-05} X^{16} + 2.90051 \cdot 10^{-05} X^{15} + 0.000231285 X^{14} + 0.000648014 X^{13} + 0.00318916 X^{12} \\
 &\quad + 0.00323204 X^{11} + 0.00460533 X^{10} + 0.00220012 X^9 - 0.000101882 X^8 + 5.32717 \cdot 10^{-06} X^7 \\
 &\quad - 0.00186824 X^6 + 0.132741 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 43556.7 X^{16} - 347904 X^{15} + 1.2616 \cdot 10^6 X^{14} - 2.74835 \cdot 10^6 X^{13} + 4.01042 \cdot 10^6 X^{12} - 4.13709 \\
 &\quad \cdot 10^6 X^{11} + 3.10251 \cdot 10^6 X^{10} - 1.71209 \cdot 10^6 X^9 + 695531 X^8 - 205965 X^7 + 43591.2 X^6 \\
 &\quad - 6402.94 X^5 + 625.678 X^4 - 37.485 X^3 - 104264 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481117 B_{6,16} - 185755 B_{7,16} + 108741 B_{8,16} \\
 &\quad + 402365 B_{9,16} + 695123 B_{10,16} + 987010 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

$$M = -104265 X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

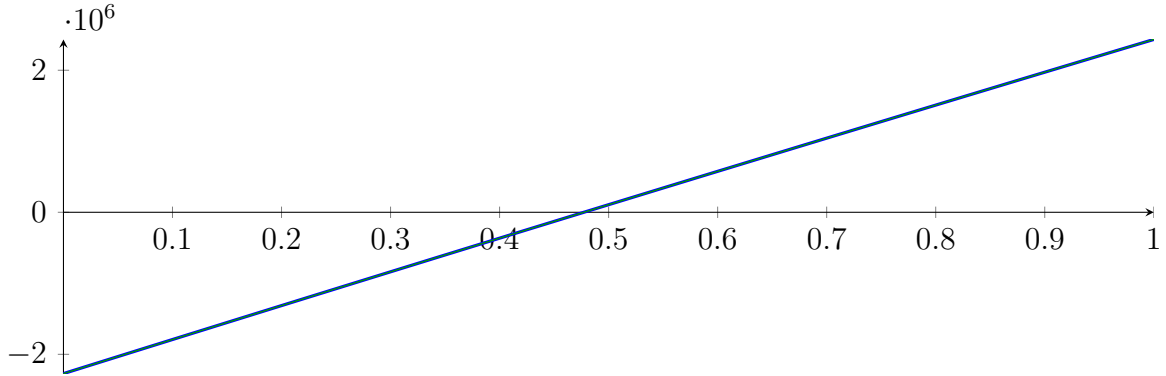
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\}$$

$$N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

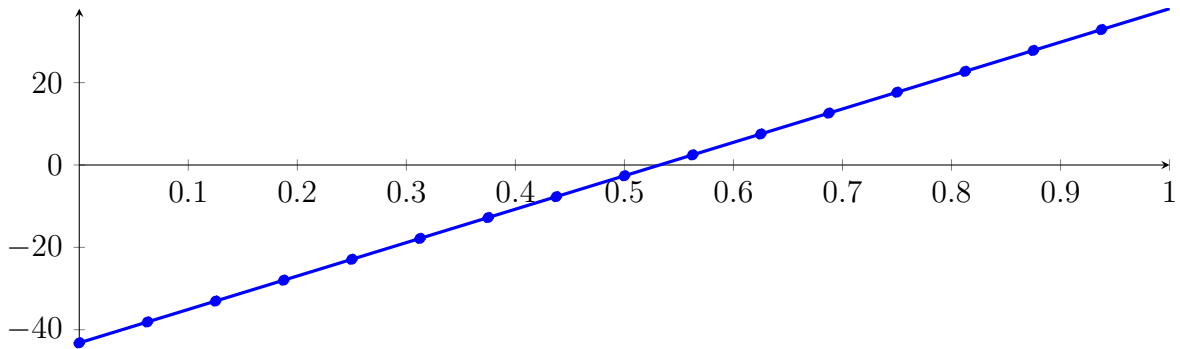
Longest intersection interval: $1.72301 \cdot 10^{-05}$

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333333]$,

83.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

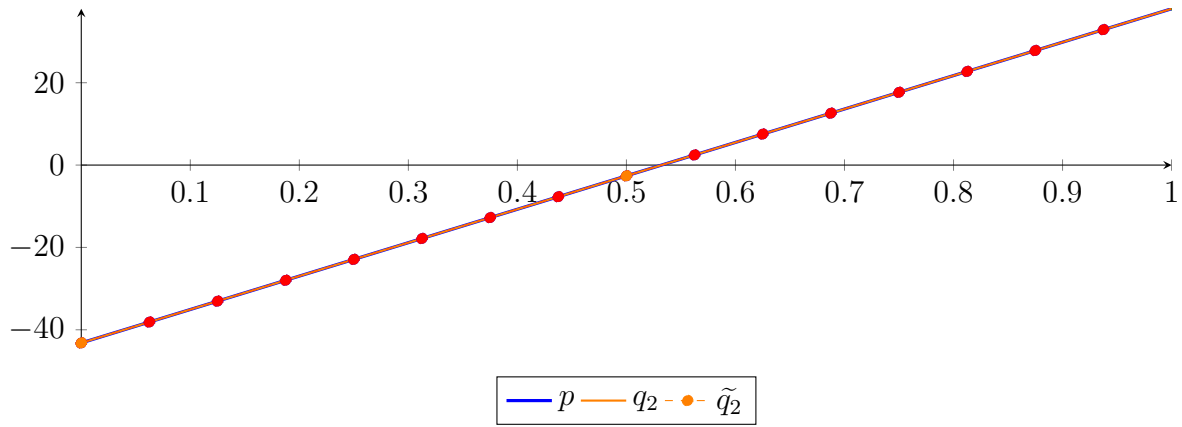
$$\begin{aligned} p &= 8.74252 \cdot 10^{-11} X^{16} - 1.56979 \cdot 10^{-09} X^{15} + 6.68479 \cdot 10^{-09} X^{14} + 1.20008 \cdot 10^{-08} X^{13} + 9.07301 \cdot 10^{-08} X^{12} \\ &+ 5.58657 \cdot 10^{-08} X^{11} + 1.13801 \cdot 10^{-07} X^{10} + 3.70665 \cdot 10^{-08} X^9 + 7.31575 \cdot 10^{-10} X^8 + 1.30058 \cdot 10^{-09} X^7 \\ &+ 5.00722 \cdot 10^{-09} X^6 + 1.24146 \cdot 10^{-10} X^5 + 1.03455 \cdot 10^{-10} X^4 - 3.09388 \cdot 10^{-05} X^2 + 81.1505X - 43.1911 \\ &= -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\ &- 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68777B_{7,16}(X) - 2.61587B_{8,16}(X) \\ &+ 2.45604B_{9,16}(X) + 7.52794B_{10,16}(X) + 12.5998B_{11,16}(X) + 17.6718B_{12,16}(X) \\ &+ 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09388 \cdot 10^{-05} X^2 + 81.1505X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 0.721495X^{16} - 5.74915X^{15} + 20.7933X^{14} - 45.1627X^{13} + 65.6806X^{12} - 67.5044X^{11} \\ &+ 50.4286X^{10} - 27.728X^9 + 11.2318X^8 - 3.32011X^7 + 0.702408X^6 - 0.103415X^5 \\ &+ 0.0102099X^4 - 0.000624725X^3 - 1.10834 \cdot 10^{-05} X^2 + 81.1505X - 43.1911 \\ &= -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} \\ &- 12.7597B_{6,16} - 7.68779B_{7,16} - 2.61585B_{8,16} + 2.45602B_{9,16} + 7.52795B_{10,16} + 12.5998B_{11,16} \\ &+ 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.57956 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -3.09388 \cdot 10^{-05} X^2 + 81.1505 X - 43.1911$$

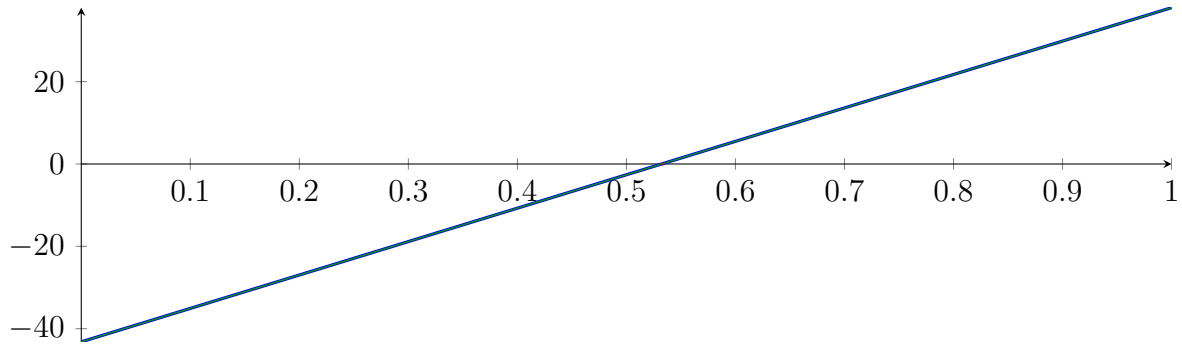
$$m = -3.09388 \cdot 10^{-05} X^2 + 81.1505 X - 43.1911$$

Root of M and m :

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\}$$

$$N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

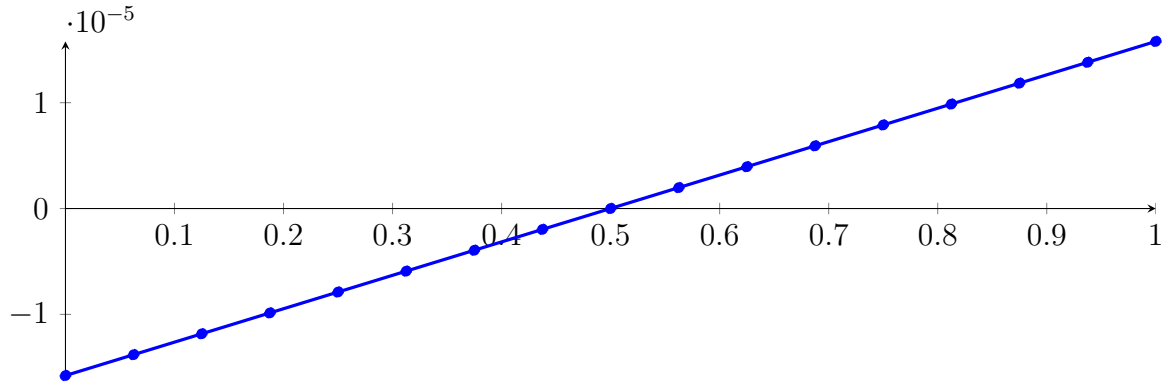
Longest intersection interval: $3.8903 \cdot 10^{-07}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

83.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
p &= -1.04409 \cdot 10^{-16} X^{16} - 1.53089 \cdot 10^{-16} X^{15} + 1.74665 \cdot 10^{-15} X^{14} + 5.46438 \cdot 10^{-15} X^{13} \\
&\quad + 2.56522 \cdot 10^{-14} X^{12} + 2.15479 \cdot 10^{-14} X^{11} + 3.51633 \cdot 10^{-14} X^{10} + 1.67444 \\
&\quad \cdot 10^{-14} X^9 - 8.72105 \cdot 10^{-16} X^8 + 1.41087 \cdot 10^{-15} X^6 + 5.91974 \cdot 10^{-17} X^5 + 4.93312 \\
&\quad \cdot 10^{-17} X^4 + 3.79471 \cdot 10^{-18} X^3 - 4.87891 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05} \\
&= -1.57804 \cdot 10^{-05} B_{0,16}(X) - 1.38073 \cdot 10^{-05} B_{1,16}(X) - 1.18341 \cdot 10^{-05} B_{2,16}(X) - 9.86101 \\
&\quad \cdot 10^{-06} B_{3,16}(X) - 7.88788 \cdot 10^{-06} B_{4,16}(X) - 5.91476 \cdot 10^{-06} B_{5,16}(X) - 3.94163 \cdot 10^{-06} B_{6,16}(X) \\
&\quad - 1.96851 \cdot 10^{-06} B_{7,16}(X) + 4.61501 \cdot 10^{-09} B_{8,16}(X) + 1.97774 \cdot 10^{-06} B_{9,16}(X) + 3.95086 \\
&\quad \cdot 10^{-06} B_{10,16}(X) + 5.92399 \cdot 10^{-06} B_{11,16}(X) + 7.89711 \cdot 10^{-06} B_{12,16}(X) + 9.87024 \cdot 10^{-06} B_{13,16}(X) \\
&\quad + 1.18434 \cdot 10^{-05} B_{14,16}(X) + 1.38165 \cdot 10^{-05} B_{15,16}(X) + 1.57896 \cdot 10^{-05} B_{16,16}(X)
\end{aligned}$$



Degree reduction and raising:

$$q_2 = -4.65868 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

$$= -1.57804 \cdot 10^{-05} B_{0,2} + 4.61501 \cdot 10^{-09} B_{1,2} + 1.57896 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = 2.92413 \cdot 10^{-07} X^{16} - 2.33332 \cdot 10^{-06} X^{15} + 8.45203 \cdot 10^{-06} X^{14} - 1.83895 \cdot 10^{-05} X^{13}$$

$$+ 2.67963 \cdot 10^{-05} X^{12} - 2.75995 \cdot 10^{-05} X^{11} + 2.06638 \cdot 10^{-05} X^{10} - 1.13854 \cdot 10^{-05} X^9$$

$$+ 4.61944 \cdot 10^{-06} X^8 - 1.36687 \cdot 10^{-06} X^7 + 2.89249 \cdot 10^{-07} X^6 - 4.25295 \cdot 10^{-08} X^5 + 4.17283$$

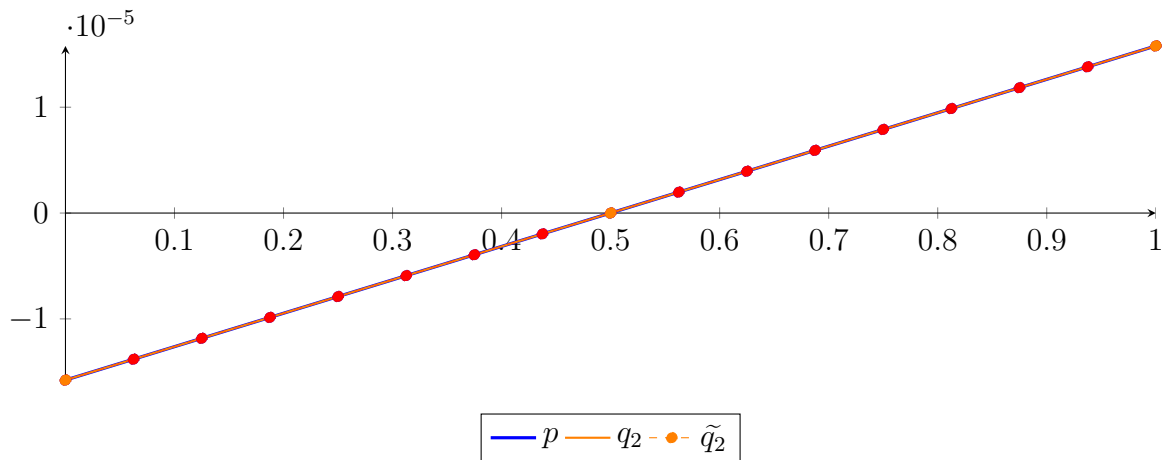
$$\cdot 10^{-09} X^4 - 2.52119 \cdot 10^{-10} X^3 + 7.82992 \cdot 10^{-12} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

$$= -1.57804 \cdot 10^{-05} B_{0,16} - 1.38073 \cdot 10^{-05} B_{1,16} - 1.18341 \cdot 10^{-05} B_{2,16} - 9.86101 \cdot 10^{-06} B_{3,16} - 7.88788$$

$$\cdot 10^{-06} B_{4,16} - 5.91476 \cdot 10^{-06} B_{5,16} - 3.94163 \cdot 10^{-06} B_{6,16} - 1.96851 \cdot 10^{-06} B_{7,16} + 4.62125 \cdot 10^{-09} B_{8,16}$$

$$+ 1.97773 \cdot 10^{-06} B_{9,16} + 3.95087 \cdot 10^{-06} B_{10,16} + 5.92399 \cdot 10^{-06} B_{11,16} + 7.89711 \cdot 10^{-06} B_{12,16}$$

$$+ 9.87024 \cdot 10^{-06} B_{13,16} + 1.18434 \cdot 10^{-05} B_{14,16} + 1.38165 \cdot 10^{-05} B_{15,16} + 1.57896 \cdot 10^{-05} B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 6.24192 \cdot 10^{-12}$.

Bounding polynomials M and m :

$$M = -4.65868 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

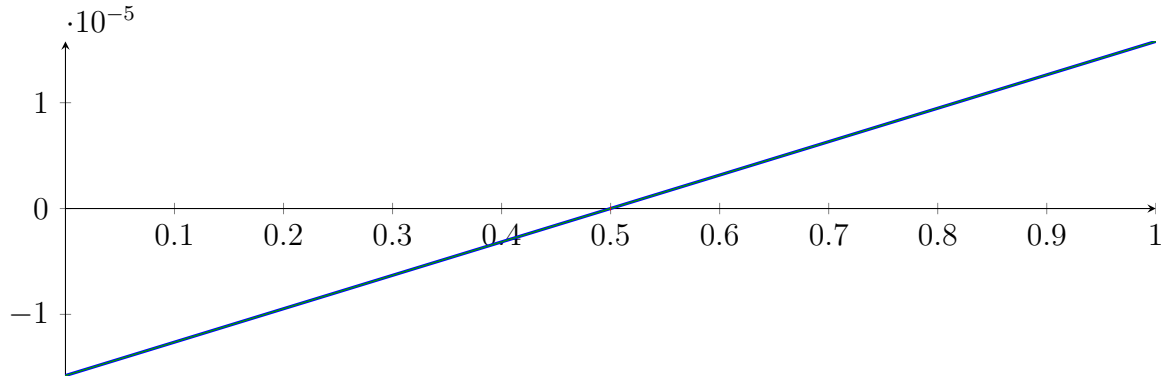
$$m = -4.65868 \cdot 10^{-18} X^2 + 3.157 \cdot 10^{-05} X - 1.57804 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.499636, 6.77659 \cdot 10^{12}\}$$

$$N(m) = \{0.500364, 6.77659 \cdot 10^{12}\}$$

Intersection intervals:



[0.499636, 0.500364]

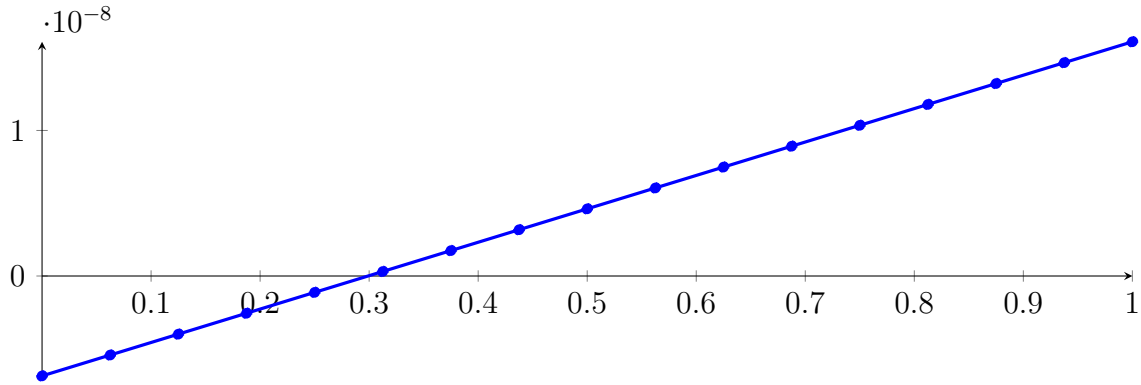
Longest intersection interval: 0.000727273

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

83.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

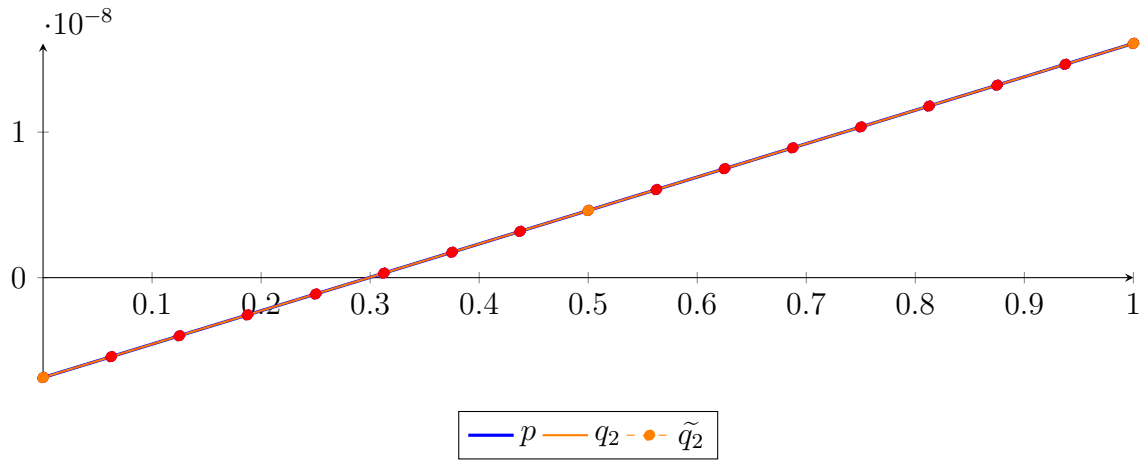
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -6.76104 \cdot 10^{-19} X^{16} + 2.21478 \cdot 10^{-18} X^{15} - 1.59295 \cdot 10^{-18} X^{14} + 1.03762 \cdot 10^{-18} X^{13} \\
 &\quad - 1.34649 \cdot 10^{-17} X^{12} + 9.65427 \cdot 10^{-18} X^{11} - 2.75561 \cdot 10^{-18} X^{10} + 5.90488 \cdot 10^{-18} X^9 \\
 &\quad - 8.51665 \cdot 10^{-19} X^8 + 3.02814 \cdot 10^{-19} X^7 + 2.64962 \cdot 10^{-19} X^6 + 2.8905 \cdot 10^{-20} X^5 \\
 &\quad + 6.02187 \cdot 10^{-21} X^4 + 9.26442 \cdot 10^{-22} X^3 + 2.296 \cdot 10^{-08} X - 6.86499 \cdot 10^{-09} \\
 &= -6.86499 \cdot 10^{-09} B_{0,16}(X) - 5.42999 \cdot 10^{-09} B_{1,16}(X) - 3.99499 \cdot 10^{-09} B_{2,16}(X) \\
 &\quad - 2.55999 \cdot 10^{-09} B_{3,16}(X) - 1.12499 \cdot 10^{-09} B_{4,16}(X) + 3.10008 \cdot 10^{-10} B_{5,16}(X) + 1.74501 \\
 &\quad \cdot 10^{-09} B_{6,16}(X) + 3.18001 \cdot 10^{-09} B_{7,16}(X) + 4.61501 \cdot 10^{-09} B_{8,16}(X) + 6.05001 \cdot 10^{-09} B_{9,16}(X) \\
 &\quad + 7.48501 \cdot 10^{-09} B_{10,16}(X) + 8.92001 \cdot 10^{-09} B_{11,16}(X) + 1.0355 \cdot 10^{-08} B_{12,16}(X) + 1.179 \\
 &\quad \cdot 10^{-08} B_{13,16}(X) + 1.3225 \cdot 10^{-08} B_{14,16}(X) + 1.466 \cdot 10^{-08} B_{15,16}(X) + 1.6095 \cdot 10^{-08} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= 1.40621 \cdot 10^{-23} X^2 + 2.296 \cdot 10^{-08} X - 6.86499 \cdot 10^{-09} \\
 &= -6.86499 \cdot 10^{-09} B_{0,2} + 4.61501 \cdot 10^{-09} B_{1,2} + 1.6095 \cdot 10^{-08} B_{2,2} \\
 \tilde{q}_2 &= 2.65578 \cdot 10^{-10} X^{16} - 2.13329 \cdot 10^{-09} X^{15} + 7.78369 \cdot 10^{-09} X^{14} - 1.70728 \cdot 10^{-08} X^{13} \\
 &\quad + 2.51029 \cdot 10^{-08} X^{12} - 2.61095 \cdot 10^{-08} X^{11} + 1.97446 \cdot 10^{-08} X^{10} - 1.09796 \cdot 10^{-08} X^9 \\
 &\quad + 4.48709 \cdot 10^{-09} X^8 - 1.33355 \cdot 10^{-09} X^7 + 2.82501 \cdot 10^{-10} X^6 - 4.12963 \cdot 10^{-11} X^5 + 3.94088 \\
 &\quad \cdot 10^{-12} X^4 - 2.24328 \cdot 10^{-13} X^3 + 6.17064 \cdot 10^{-15} X^2 + 2.296 \cdot 10^{-08} X - 6.86499 \cdot 10^{-09} \\
 &= -6.86499 \cdot 10^{-09} B_{0,16} - 5.42999 \cdot 10^{-09} B_{1,16} - 3.99499 \cdot 10^{-09} B_{2,16} - 2.55999 \cdot 10^{-09} B_{3,16} \\
 &\quad - 1.12499 \cdot 10^{-09} B_{4,16} + 3.10006 \cdot 10^{-10} B_{5,16} + 1.74501 \cdot 10^{-09} B_{6,16} + 3.18 \cdot 10^{-09} B_{7,16} + 4.61501 \\
 &\quad \cdot 10^{-09} B_{8,16} + 6.05 \cdot 10^{-09} B_{9,16} + 7.48501 \cdot 10^{-09} B_{10,16} + 8.92 \cdot 10^{-09} B_{11,16} + 1.0355 \cdot 10^{-08} B_{12,16} \\
 &\quad + 1.179 \cdot 10^{-08} B_{13,16} + 1.3225 \cdot 10^{-08} B_{14,16} + 1.466 \cdot 10^{-08} B_{15,16} + 1.6095 \cdot 10^{-08} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 5.35405 \cdot 10^{-15}$.

Bounding polynomials M and m :

$$M = 1.32349 \cdot 10^{-23} X^2 + 2.296 \cdot 10^{-08} X - 6.86498 \cdot 10^{-09}$$

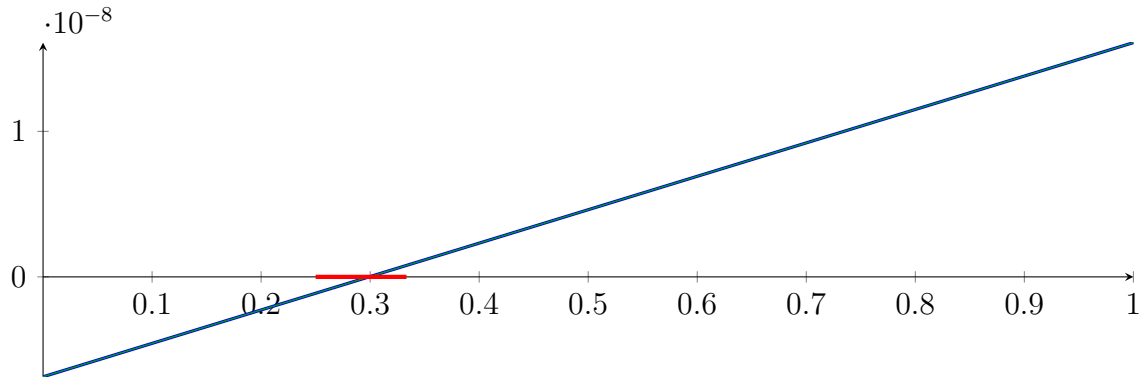
$$m = 1.48893 \cdot 10^{-23} X^2 + 2.296 \cdot 10^{-08} X - 6.865 \cdot 10^{-09}$$

Root of M and m :

$$N(M) = \{-1.73481 \cdot 10^{15}, 0.25\}$$

$$N(m) = \{-1.54205 \cdot 10^{15}, 0.333333\}$$

Intersection intervals:



$$[0.333333, 0.25]$$

Longest intersection interval: -0.0833333

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

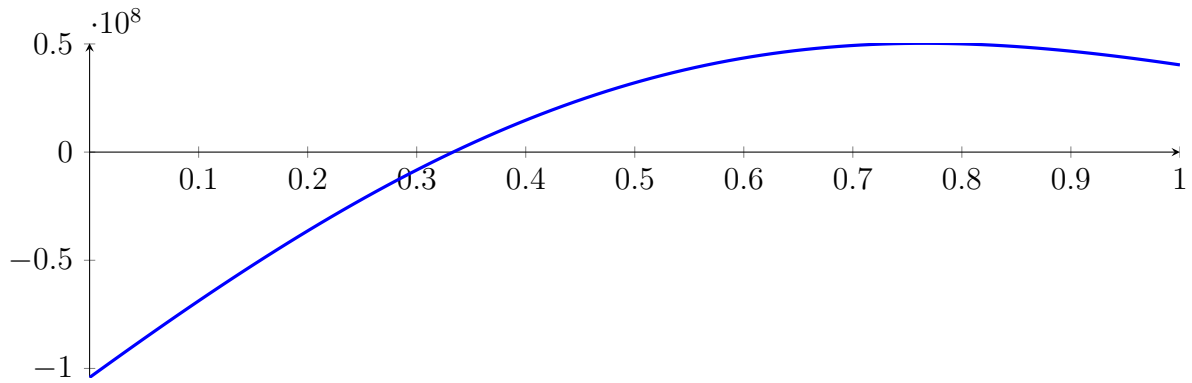
83.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

83.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

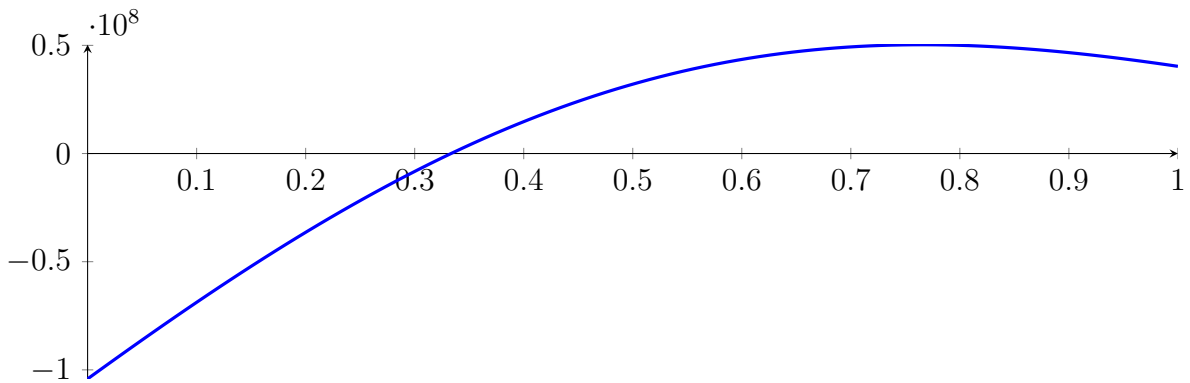
with precision $\varepsilon = 1 \cdot 10^{-128}$.

84 Running CubeClip on f_{16} with epsilon 128

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

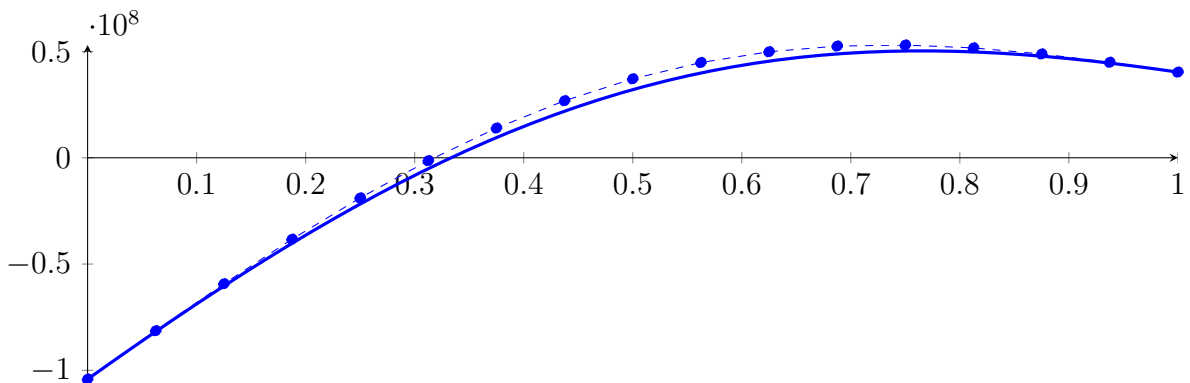
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



84.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -0.994383X^{16} - 39.7274X^{15} - 651.407X^{14} - 5448.89X^{13} - 20439.3X^{12} + 18474.7X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 2.11088 \cdot 10^6 X^{16} - 1.70537 \cdot 10^7 X^{15} + 6.25913 \cdot 10^7 X^{14} - 1.38111 \cdot 10^8 X^{13} + 2.043 \cdot 10^8 X^{12} - 2.13799$$

$$\cdot 10^8 X^{11} + 1.62714 \cdot 10^8 X^{10} - 9.11092 \cdot 10^7 X^9 + 3.75254 \cdot 10^7 X^8 - 1.12529 \cdot 10^7 X^7 + 2.40899 \cdot 10^6 X^6$$

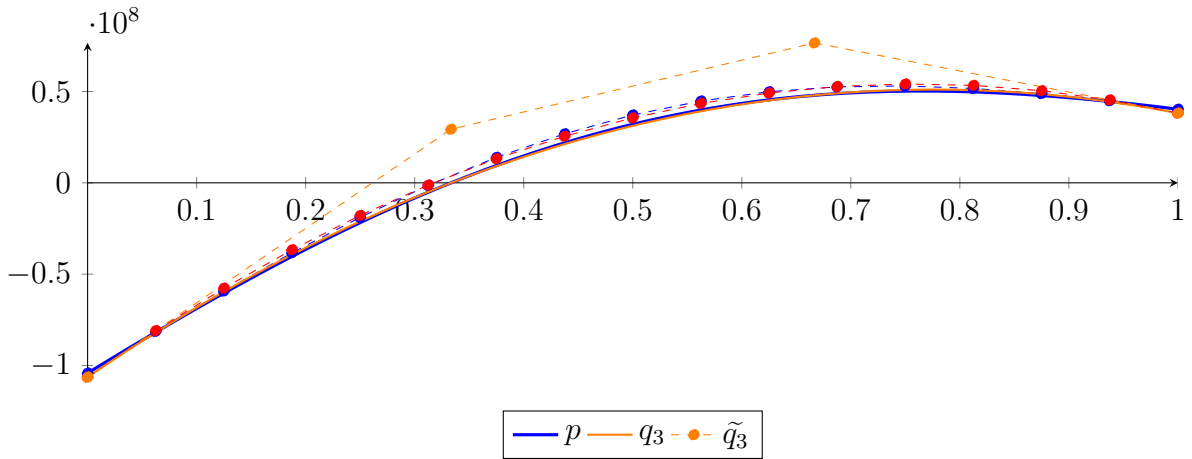
$$- 357156 X^5 + 34986.1 X^4 + 2.75602 \cdot 10^6 X^3 - 2.65318 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18553 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91816 \cdot 10^7 B_{10,16} + 5.27352 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

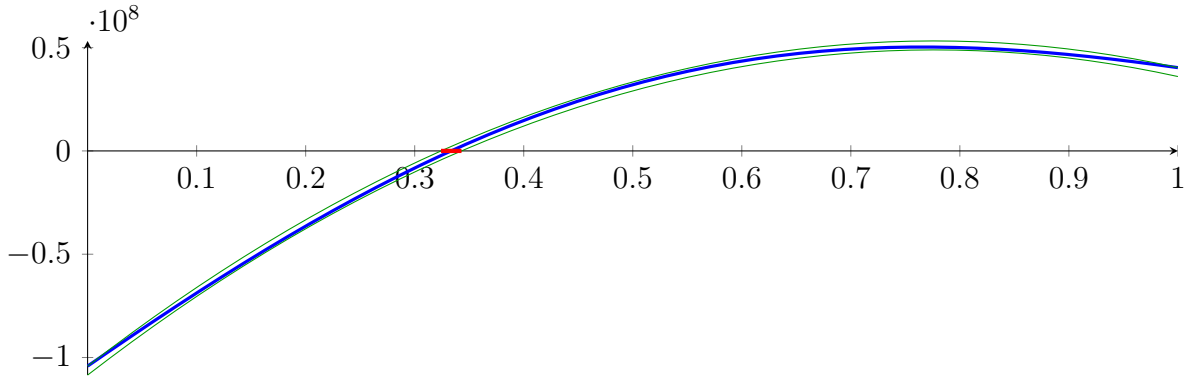
$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\} \quad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

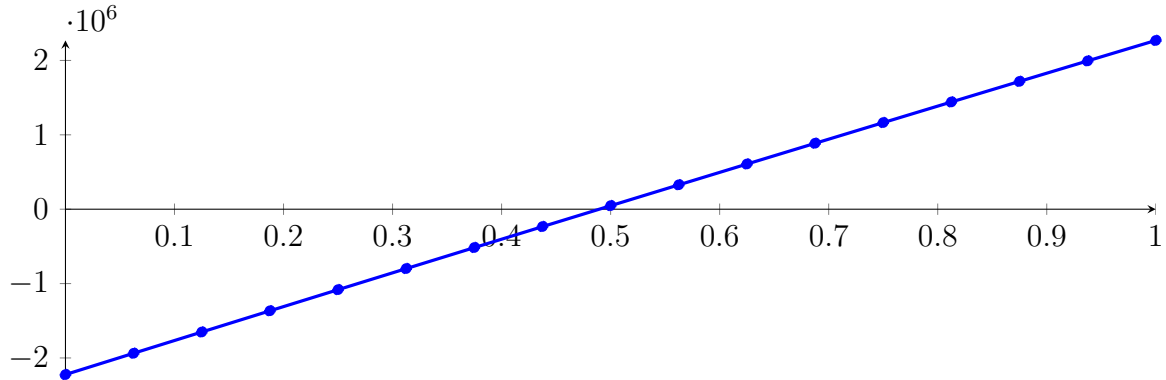
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

84.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

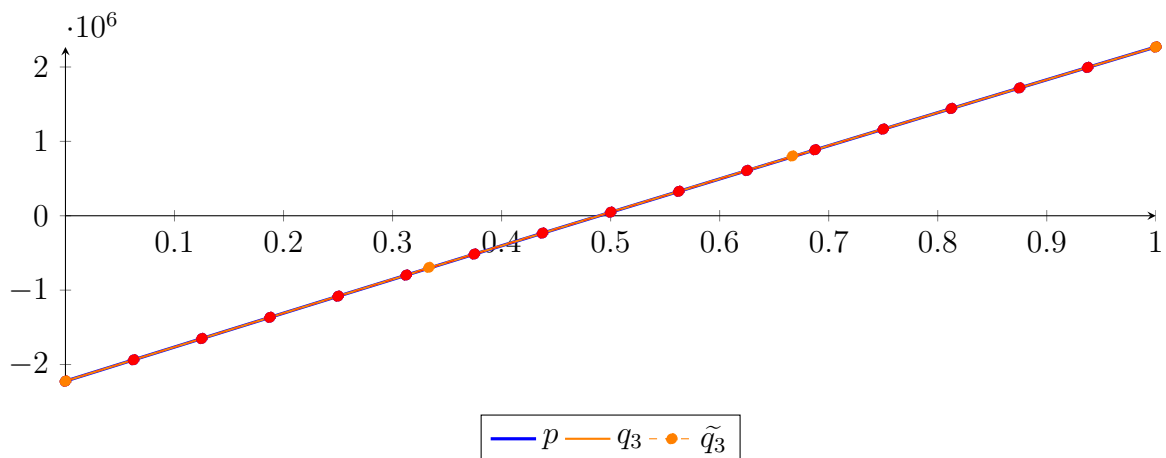
$$\begin{aligned}
 p &= -1.93035 \cdot 10^{-05} X^{16} + 1.08927 \cdot 10^{-05} X^{15} + 0.000255816 X^{14} + 0.000616983 X^{13} + 0.00371715 X^{12} \\
 &+ 0.00325035 X^{11} + 0.00510875 X^{10} + 0.00190713 X^9 + 0.000179792 X^8 + 1.06543 \cdot 10^{-05} X^7 \\
 &- 0.00136482 X^6 + 0.104959 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &- 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &+ 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 15290.6 X^{16} - 125412 X^{15} + 468045 X^{14} - 1.05198 \cdot 10^6 X^{13} + 1.588 \cdot 10^6 X^{12} - 1.69893 \\
 &\cdot 10^6 X^{11} + 1.32382 \cdot 10^6 X^{10} - 759402 X^9 + 320067 X^8 - 97826.5 X^7 + 21197.8 X^6 \\
 &- 3162.8 X^5 + 313.931 X^4 - 720.165 X^3 - 93879.2 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &- 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.7 B_{8,16} \\
 &+ 328649 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.457751$.

Bounding polynomials M and m :

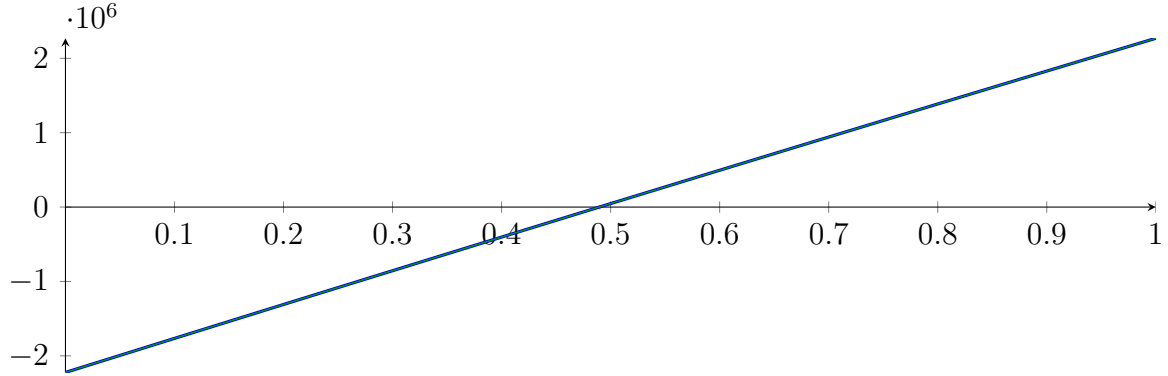
$$M = -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\} \quad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

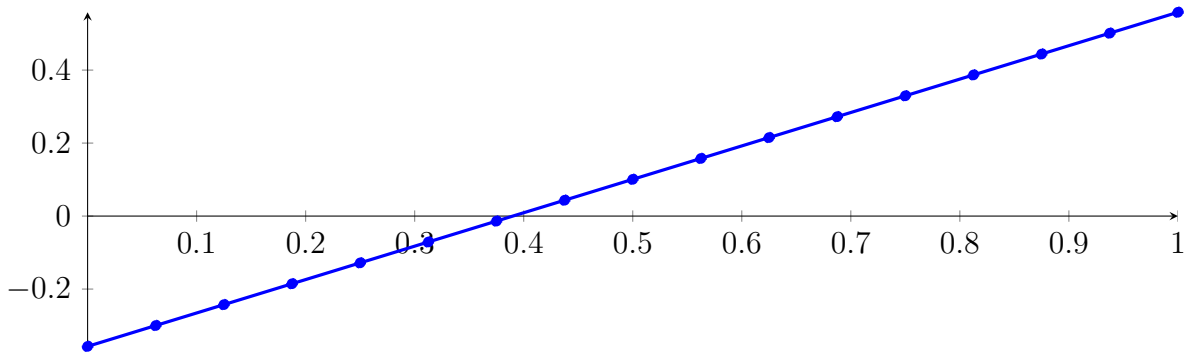
Longest intersection interval: $2.03684 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

84.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

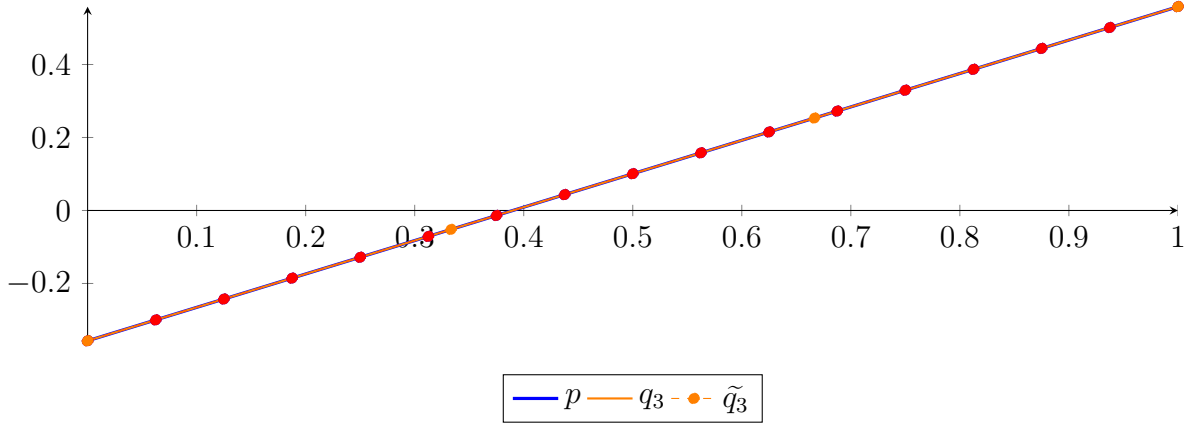
$$\begin{aligned} p &= -1.56399 \cdot 10^{-11} X^{16} + 4.58016 \cdot 10^{-11} X^{15} - 3.19744 \cdot 10^{-12} X^{14} + 6.54055 \cdot 10^{-11} X^{13} \\ &\quad + 1.05072 \cdot 10^{-10} X^{12} + 4.59728 \cdot 10^{-10} X^{11} + 5.01434 \cdot 10^{-10} X^{10} + 2.99742 \cdot 10^{-10} X^9 \\ &\quad + 1.14309 \cdot 10^{-11} X^8 - 5.08038 \cdot 10^{-12} X^7 + 3.37845 \cdot 10^{-11} X^6 - 9.69891 \cdot 10^{-13} X^5 \\ &\quad + 4.04121 \cdot 10^{-13} X^4 + 6.21725 \cdot 10^{-14} X^3 - 3.93767 \cdot 10^{-09} X^2 + 0.915501 X - 0.357072 \\ &= -0.357072 B_{0,16}(X) - 0.299853 B_{1,16}(X) - 0.242635 B_{2,16}(X) - 0.185416 B_{3,16}(X) \\ &\quad - 0.128197 B_{4,16}(X) - 0.0709781 B_{5,16}(X) - 0.0137592 B_{6,16}(X) \\ &\quad + 0.0434596 B_{7,16}(X) + 0.100678 B_{8,16}(X) + 0.157897 B_{9,16}(X) + 0.215116 B_{10,16}(X) \\ &\quad + 0.272335 B_{11,16}(X) + 0.329554 B_{12,16}(X) + 0.386773 B_{13,16}(X) \\ &\quad + 0.443991 B_{14,16}(X) + 0.50121 B_{15,16}(X) + 0.558429 B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 1.05471 \cdot 10^{-15} X^3 - 3.93767 \cdot 10^{-09} X^2 + 0.915501 X - 0.357072 \\ &= -0.357072 B_{0,3} - 0.0519051 B_{1,3} + 0.253262 B_{2,3} + 0.558429 B_{3,3} \end{aligned}$$

$$\begin{aligned}
\tilde{q}_3 &= 0.00291222X^{16} - 0.0241801X^{15} + 0.0914452X^{14} - 0.208537X^{13} + 0.319778X^{12} - 0.347745X^{11} \\
&\quad + 0.275244X^{10} - 0.159971X^9 + 0.0679818X^8 - 0.0208072X^7 + 0.00447629X^6 - 0.000654783X^5 \\
&\quad + 6.22034 \cdot 10^{-05}X^4 - 3.60145 \cdot 10^{-06}X^3 + 9.78811 \cdot 10^{-08}X^2 + 0.915501X - 0.357072 \\
&= -0.357072B_{0,16} - 0.299853B_{1,16} - 0.242635B_{2,16} - 0.185416B_{3,16} - 0.128197B_{4,16} \\
&\quad - 0.0709781B_{5,16} - 0.0137592B_{6,16} + 0.0434595B_{7,16} + 0.100678B_{8,16} \\
&\quad + 0.157897B_{9,16} + 0.215116B_{10,16} + 0.272335B_{11,16} + 0.329554B_{12,16} \\
&\quad + 0.386773B_{13,16} + 0.443991B_{14,16} + 0.50121B_{15,16} + 0.558429B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.5212 \cdot 10^{-08}$.

Bounding polynomials M and m :

$$M = 9.99201 \cdot 10^{-16}X^3 - 3.93767 \cdot 10^{-09}X^2 + 0.915501X - 0.357072$$

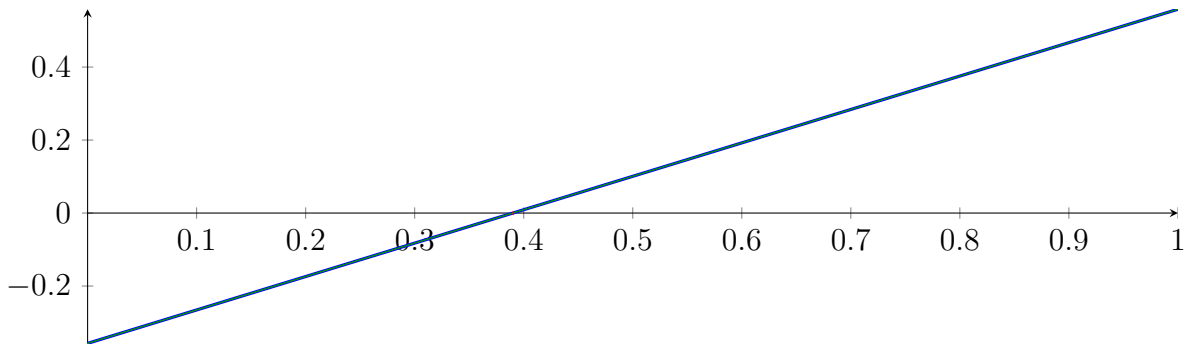
$$m = 1.22125 \cdot 10^{-15}X^3 - 3.93767 \cdot 10^{-09}X^2 + 0.915501X - 0.357072$$

Root of M and m :

$$N(M) = \{0.390029\}$$

$$N(m) = \{0.390029\}$$

Intersection intervals:



$$[0.390029, 0.390029]$$

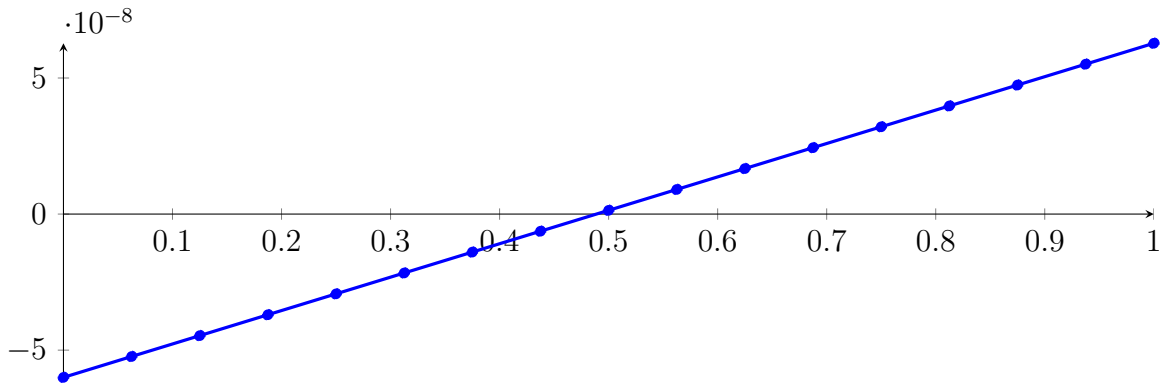
Longest intersection interval: $1.3411 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

84.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

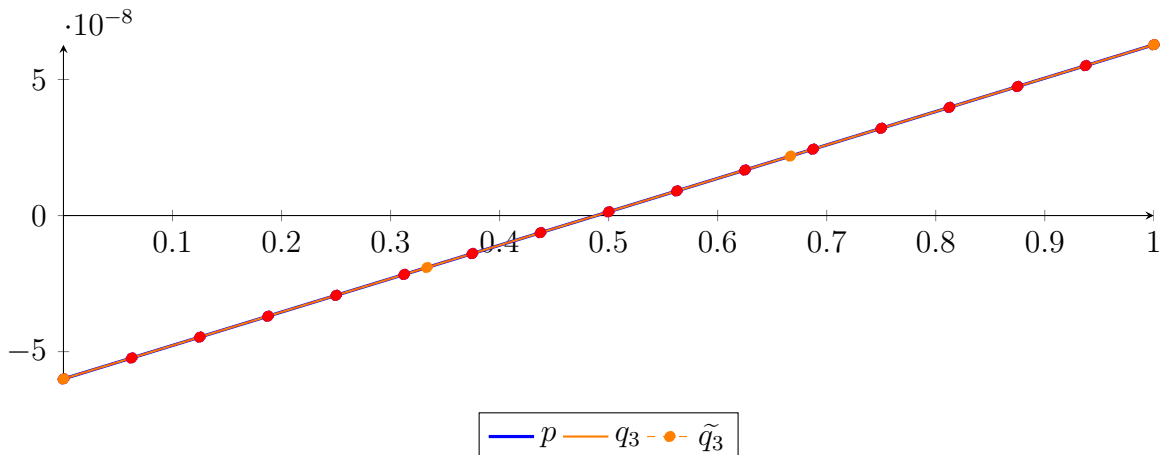
$$\begin{aligned}
 p &= -5.65183 \cdot 10^{-19} X^{16} + 5.13302 \cdot 10^{-19} X^{15} + 6.37816 \cdot 10^{-18} X^{14} + 1.69576 \cdot 10^{-17} X^{13} \\
 &\quad + 9.4423 \cdot 10^{-17} X^{12} + 8.0934 \cdot 10^{-17} X^{11} + 1.37357 \cdot 10^{-16} X^{10} + 6.23797 \cdot 10^{-17} X^9 \\
 &\quad + 1.36266 \cdot 10^{-18} X^8 + 6.05629 \cdot 10^{-19} X^7 + 5.08728 \cdot 10^{-18} X^6 + 2.3124 \cdot 10^{-19} X^5 \\
 &\quad + 1.44525 \cdot 10^{-19} X^4 - 7.41154 \cdot 10^{-21} X^3 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08} \\
 &= -6.00146 \cdot 10^{-08} B_{0,16}(X) - 5.2341 \cdot 10^{-08} B_{1,16}(X) - 4.46674 \cdot 10^{-08} B_{2,16}(X) - 3.69937 \\
 &\quad \cdot 10^{-08} B_{3,16}(X) - 2.93201 \cdot 10^{-08} B_{4,16}(X) - 2.16464 \cdot 10^{-08} B_{5,16}(X) - 1.39728 \cdot 10^{-08} B_{6,16}(X) \\
 &\quad - 6.29913 \cdot 10^{-09} B_{7,16}(X) + 1.37451 \cdot 10^{-09} B_{8,16}(X) + 9.04815 \cdot 10^{-09} B_{9,16}(X) + 1.67218 \\
 &\quad \cdot 10^{-08} B_{10,16}(X) + 2.43954 \cdot 10^{-08} B_{11,16}(X) + 3.20691 \cdot 10^{-08} B_{12,16}(X) + 3.97427 \\
 &\quad \cdot 10^{-08} B_{13,16}(X) + 4.74164 \cdot 10^{-08} B_{14,16}(X) + 5.509 \cdot 10^{-08} B_{15,16}(X) + 6.27637 \cdot 10^{-08} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= 2.51463 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08} \\
 &= -6.00146 \cdot 10^{-08} B_{0,3} - 1.90885 \cdot 10^{-08} B_{1,3} + 2.18376 \cdot 10^{-08} B_{2,3} + 6.27637 \cdot 10^{-08} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 4.01426 \cdot 10^{-10} X^{16} - 3.29623 \cdot 10^{-09} X^{15} + 1.2317 \cdot 10^{-08} X^{14} - 2.77213 \cdot 10^{-08} X^{13} + 4.19074 \\
 &\quad \cdot 10^{-08} X^{12} - 4.49047 \cdot 10^{-08} X^{11} + 3.50457 \cdot 10^{-08} X^{10} - 2.01341 \cdot 10^{-08} X^9 + 8.49676 \\
 &\quad \cdot 10^{-09} X^8 - 2.5992 \cdot 10^{-09} X^7 + 5.63364 \cdot 10^{-10} X^6 - 8.40231 \cdot 10^{-11} X^5 + 8.33259 \\
 &\quad \cdot 10^{-12} X^4 - 5.16965 \cdot 10^{-13} X^3 + 1.7395 \cdot 10^{-14} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08} \\
 &= -6.00146 \cdot 10^{-08} B_{0,16} - 5.2341 \cdot 10^{-08} B_{1,16} - 4.46674 \cdot 10^{-08} B_{2,16} - 3.69937 \cdot 10^{-08} B_{3,16} - 2.93201 \\
 &\quad \cdot 10^{-08} B_{4,16} - 2.16464 \cdot 10^{-08} B_{5,16} - 1.39728 \cdot 10^{-08} B_{6,16} - 6.29914 \cdot 10^{-09} B_{7,16} + 1.37452 \cdot 10^{-09} B_{8,16} \\
 &\quad + 9.04815 \cdot 10^{-09} B_{9,16} + 1.67218 \cdot 10^{-08} B_{10,16} + 2.43954 \cdot 10^{-08} B_{11,16} + 3.20691 \cdot 10^{-08} B_{12,16} \\
 &\quad + 3.97427 \cdot 10^{-08} B_{13,16} + 4.74164 \cdot 10^{-08} B_{14,16} + 5.509 \cdot 10^{-08} B_{15,16} + 6.27637 \cdot 10^{-08} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.90061 \cdot 10^{-15}$.

Bounding polynomials M and m :

$$M = 2.38228 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08}$$

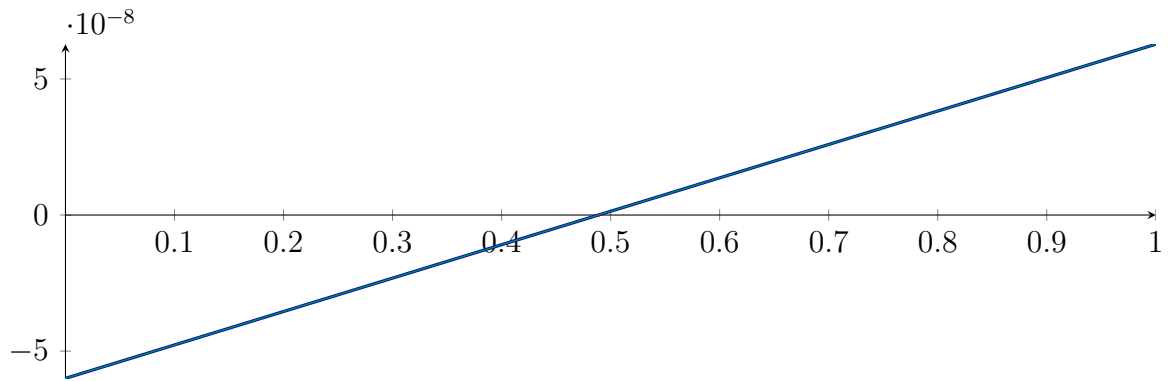
$$m = 2.51463 \cdot 10^{-22} X^3 - 3.17637 \cdot 10^{-22} X^2 + 1.22778 \cdot 10^{-07} X - 6.00146 \cdot 10^{-08}$$

Root of M and m :

$$N(M) = \{0.488805\}$$

$$N(m) = \{0.488805\}$$

Intersection intervals:



$$[0.488805, 0.488805]$$

Longest intersection interval: $1.3086 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

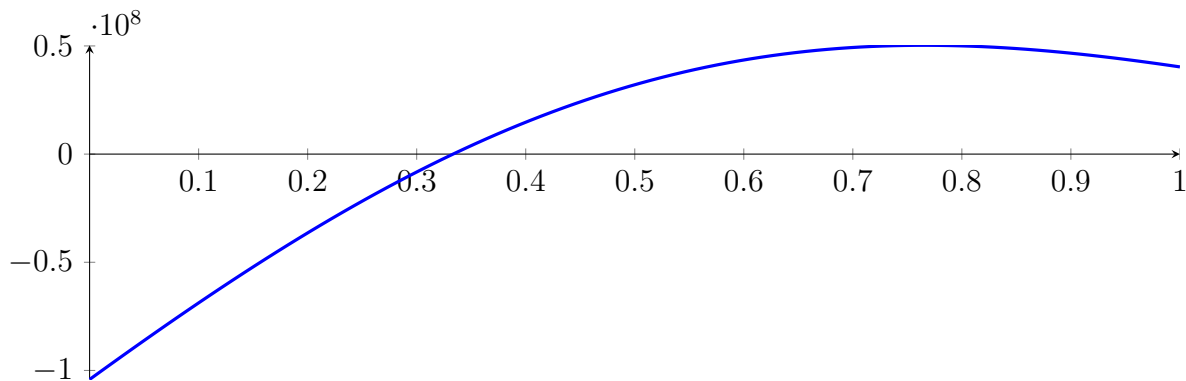
84.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

84.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

with precision $\varepsilon = 1 \cdot 10^{-128}$.

Part II

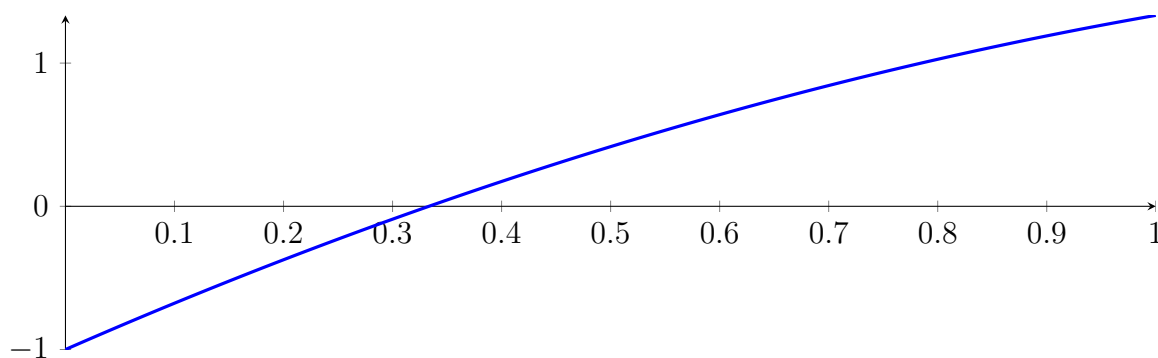
Numeric = long double

85 Running BezClip on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

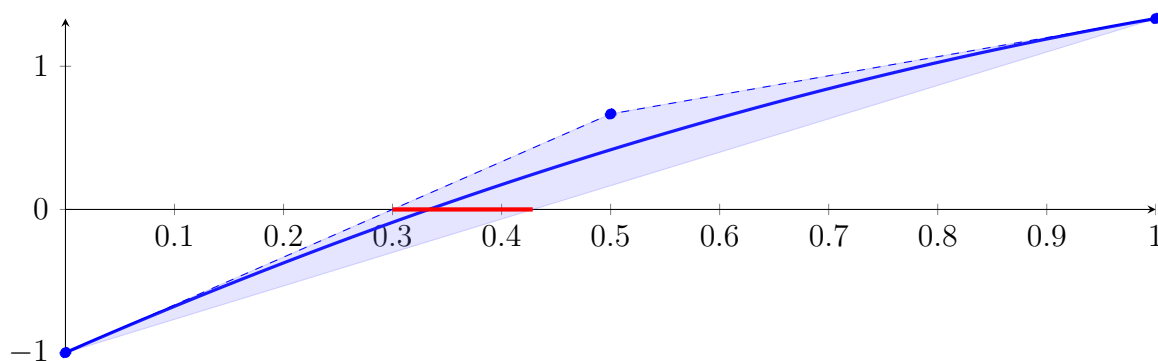
$$p = -1X^2 + 3.33333X - 1$$



85.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

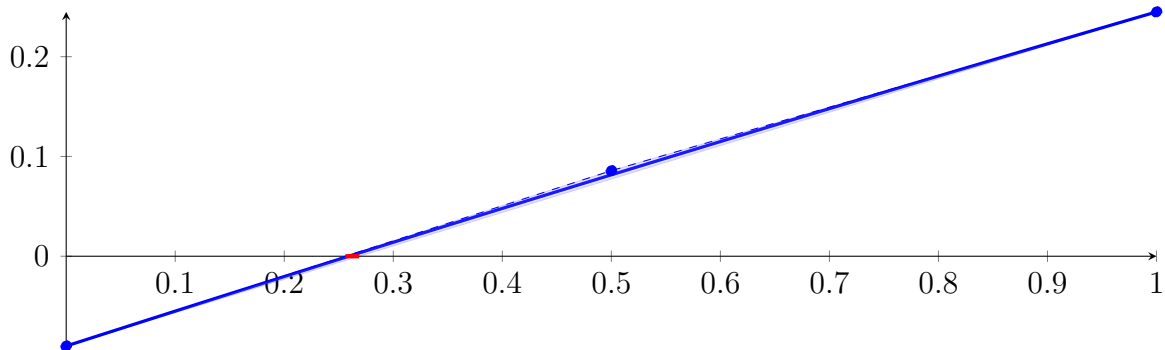
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

85.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641

\implies Selective recursion: interval 1: $[0.332927, 0.334552]$,

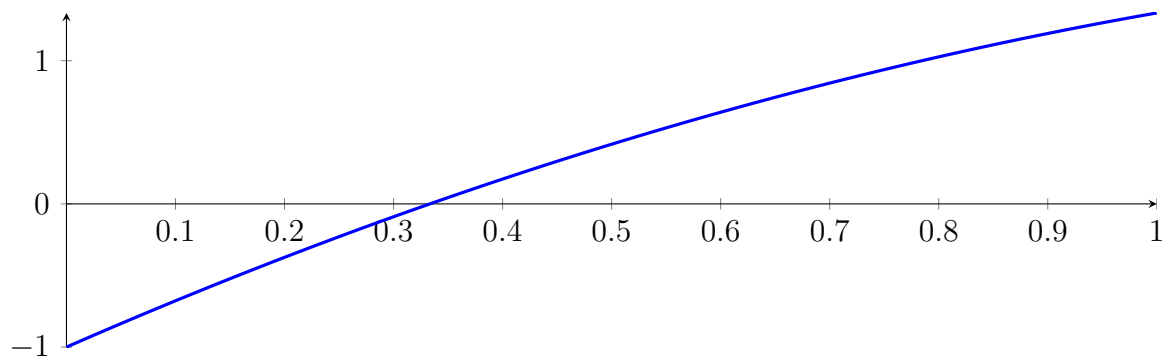
85.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Found root in interval $[0.332927, 0.334552]$ at recursion depth 3!

85.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.332927, 0.334552]$$

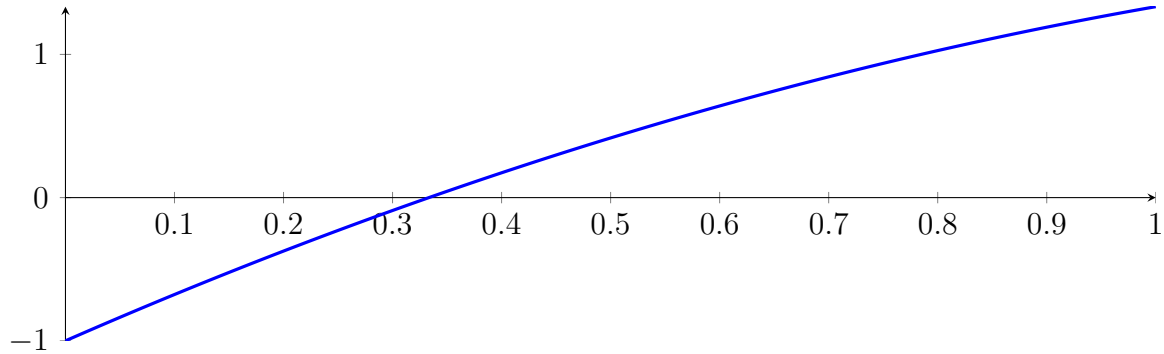
with precision $\varepsilon = 0.01$.

86 Running QuadClip on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

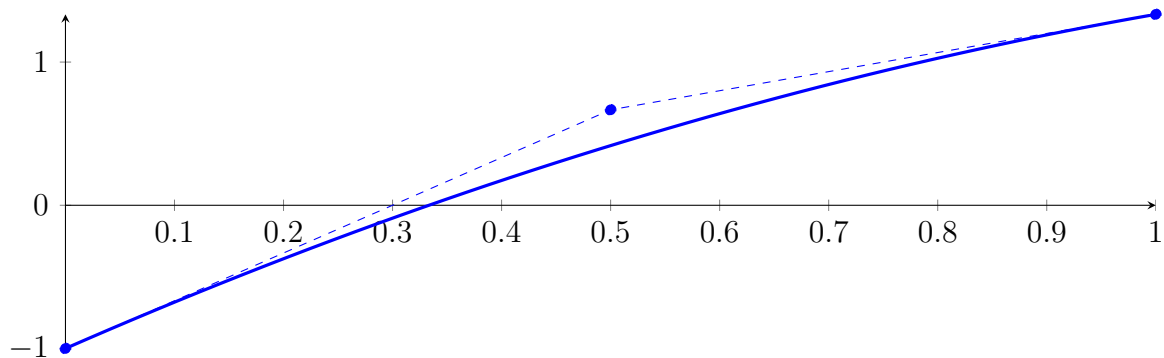
$$p = -1X^2 + 3.33333X - 1$$



86.1 Recursion Branch 1 for Input Interval $[0, 1]$

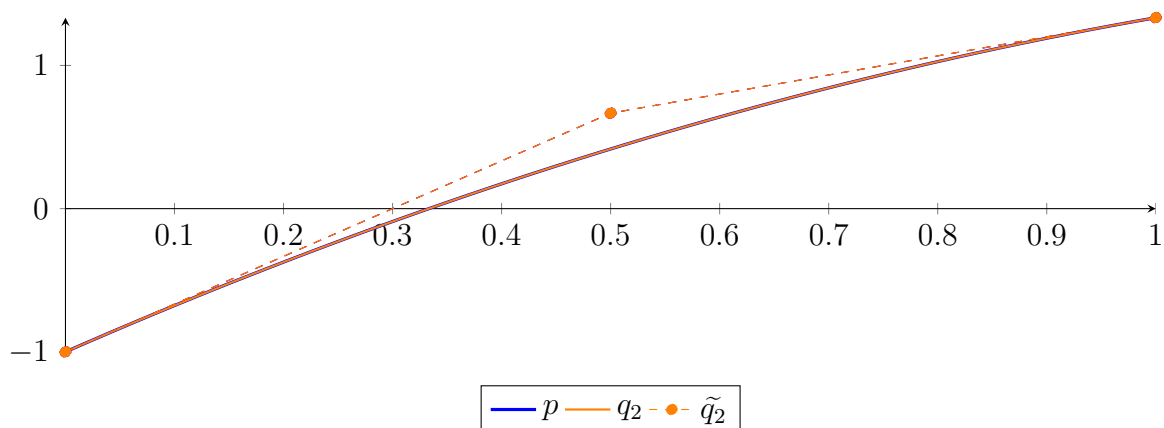
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

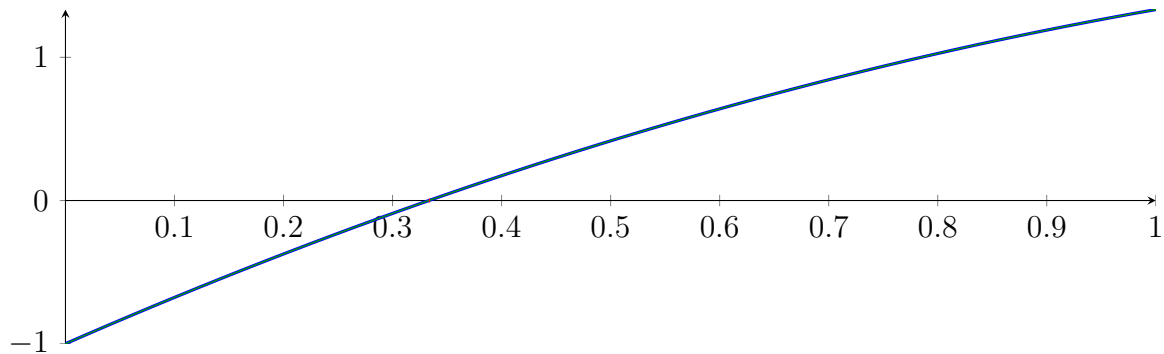
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $3.25261 \cdot 10^{-19}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

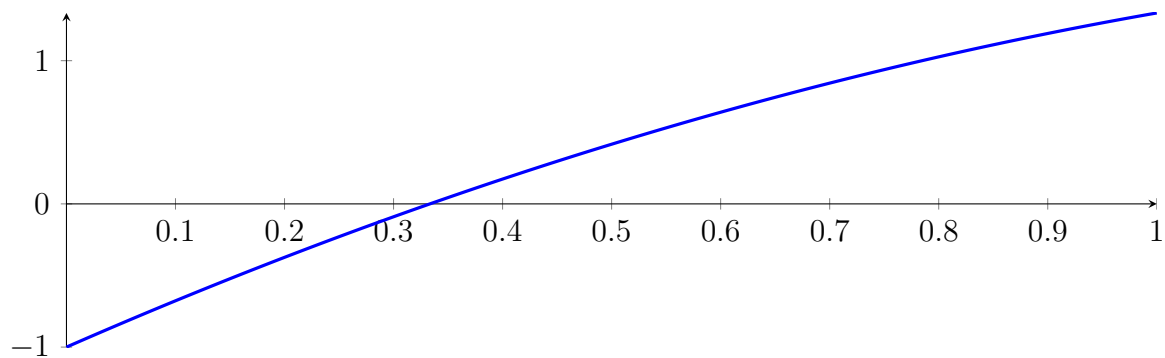
86.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

86.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

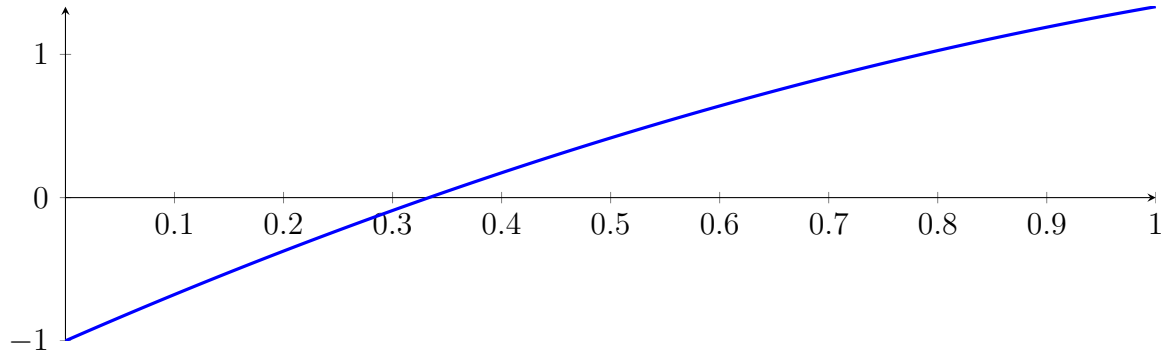
with precision $\varepsilon = 0.01$.

87 Running CubeClip on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

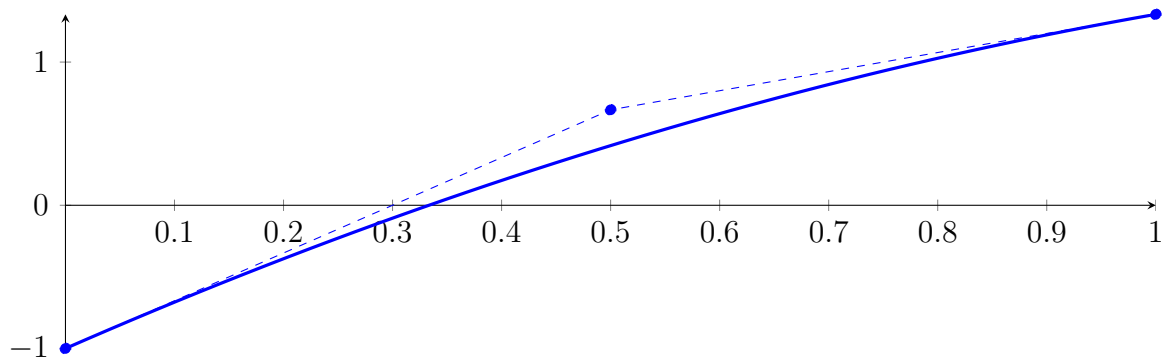
$$p = -1X^2 + 3.33333X - 1$$



87.1 Recursion Branch 1 for Input Interval $[0, 1]$

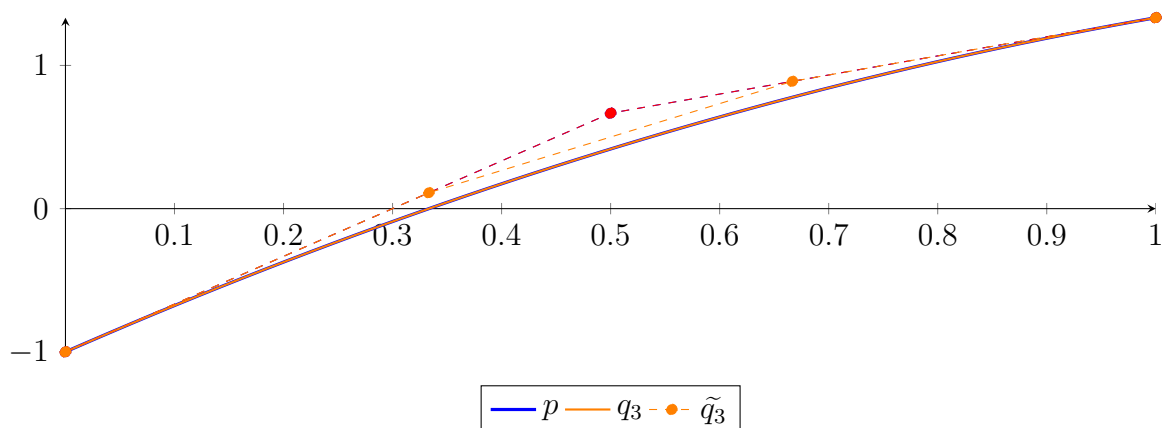
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.63715 \cdot 10^{-17}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

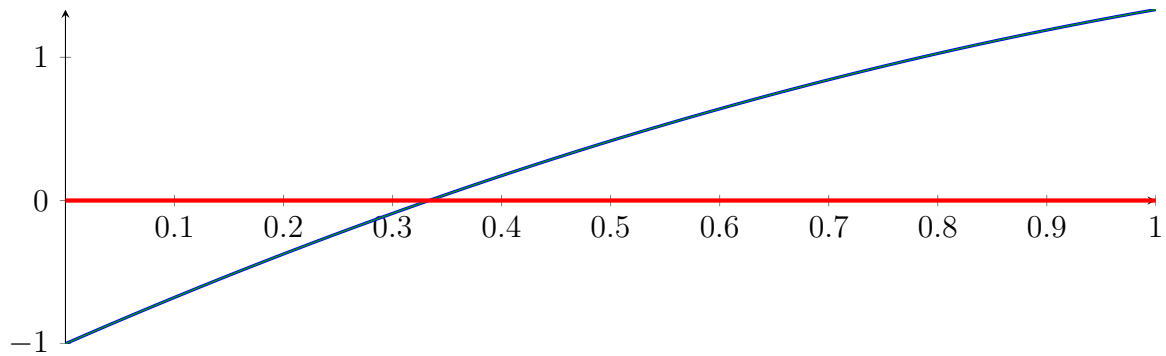
$$m = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

$$N(m) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

Intersection intervals:



$[0, 1]$

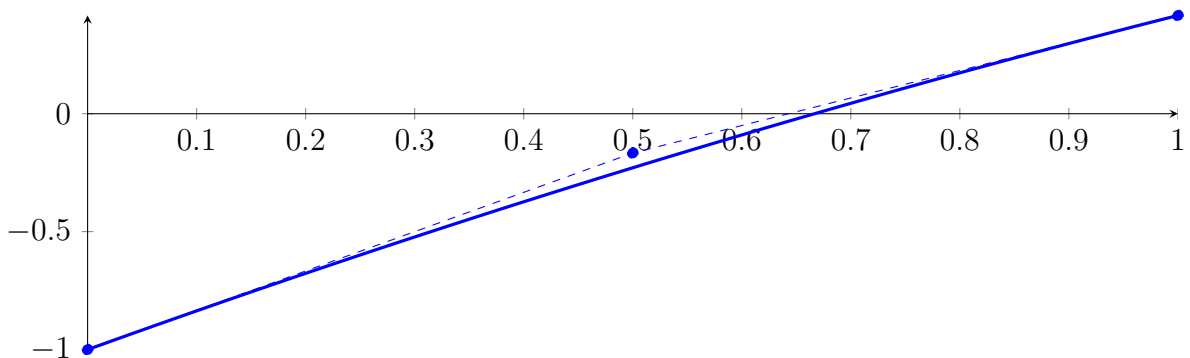
Longest intersection interval: 1

\implies Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

87.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

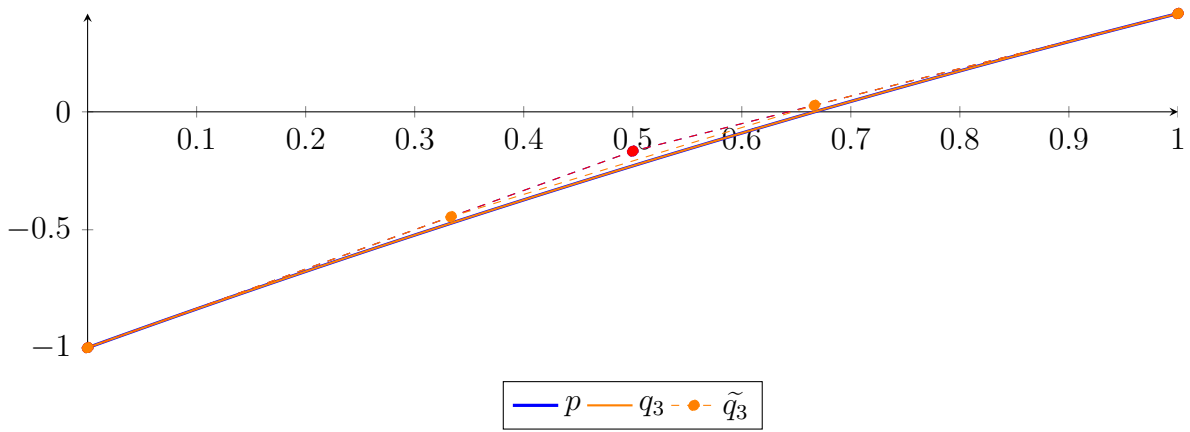
$$\begin{aligned} p &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2}(X) - 0.166667B_{1,2}(X) + 0.416667B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,3} - 0.444444B_{1,3} + 0.0277778B_{2,3} + 0.416667B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2} - 0.166667B_{1,2} + 0.416667B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.58942 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1.07336 \cdot 10^{-17} X^3 - 0.25 X^2 + 1.66667 X - 1$$

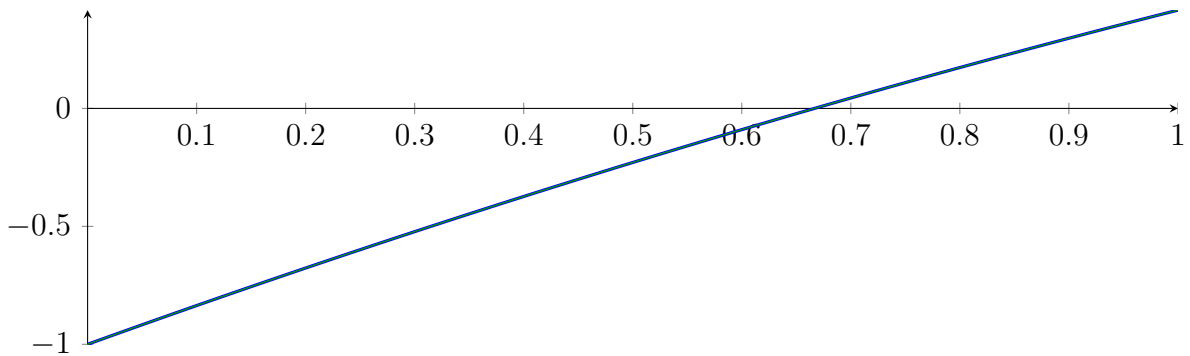
$$m = -1.07336 \cdot 10^{-17} X^3 - 0.25 X^2 + 1.66667 X - 1$$

Root of M and m :

$$N(M) = \{-2.32913 \cdot 10^{16}\}$$

$$N(m) = \{-2.32913 \cdot 10^{16}\}$$

Intersection intervals:

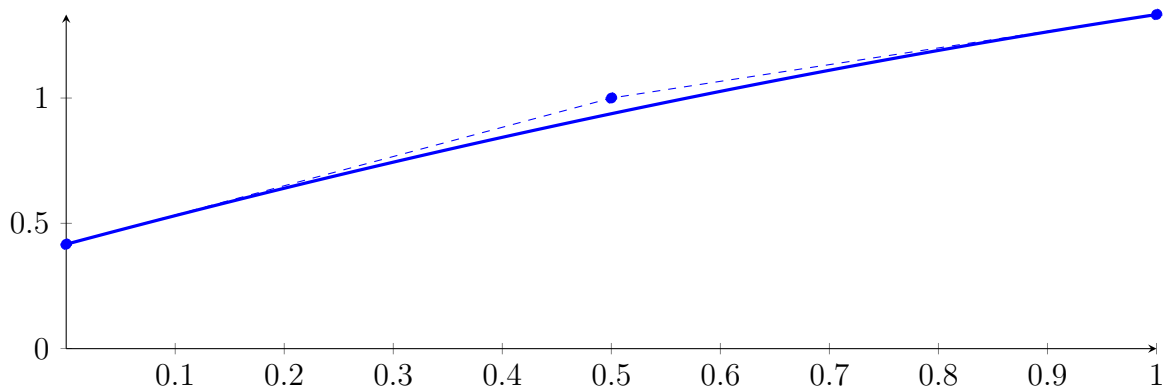


No intersection intervals with the x axis.

87.3 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.25 X^2 + 1.16667 X + 0.416667 \\ &= 0.416667 B_{0,2}(X) + 1 B_{1,2}(X) + 1.333333 B_{2,2}(X) \end{aligned}$$



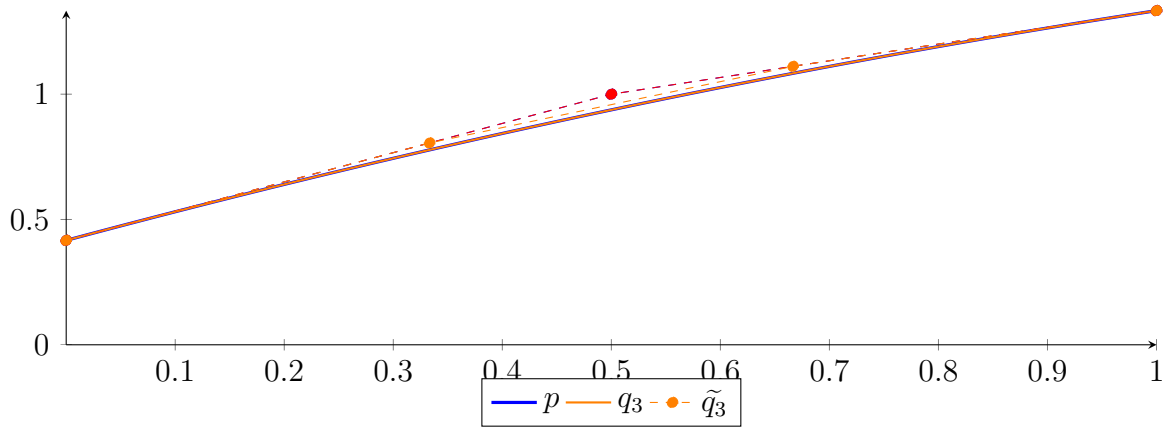
Degree reduction and raising:

$$q_3 = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$= 0.416667B_{0,3} + 0.805556B_{1,3} + 1.111111B_{2,3} + 1.333333B_{3,3}$$

$$\tilde{q}_3 = -0.25X^2 + 1.16667X + 0.416667$$

$$= 0.416667B_{0,2} + 1B_{1,2} + 1.333333B_{2,2}$$



The maximum difference of the Bézier coefficients is $\delta = 1.30104 \cdot 10^{-18}$.

Bounding polynomials M and m :

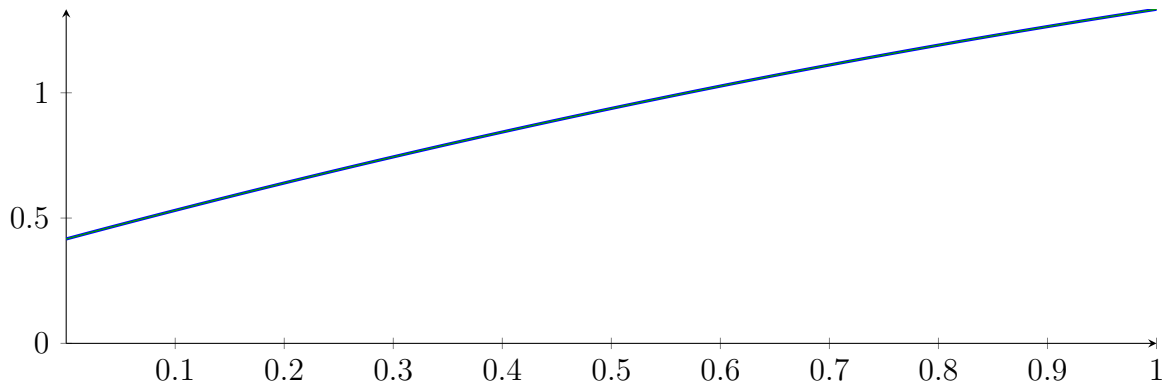
$$M = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$m = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

Root of M and m :

$$N(M) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\} \quad N(m) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\}$$

Intersection intervals:

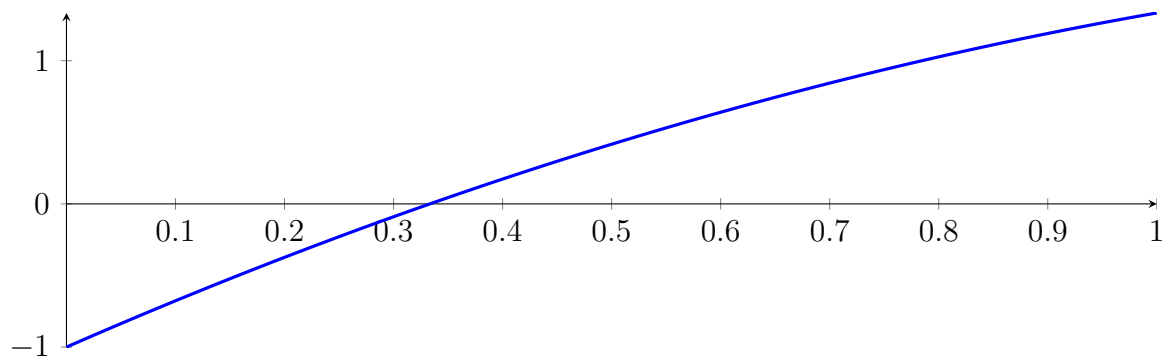


No intersection intervals with the x axis.

87.4 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

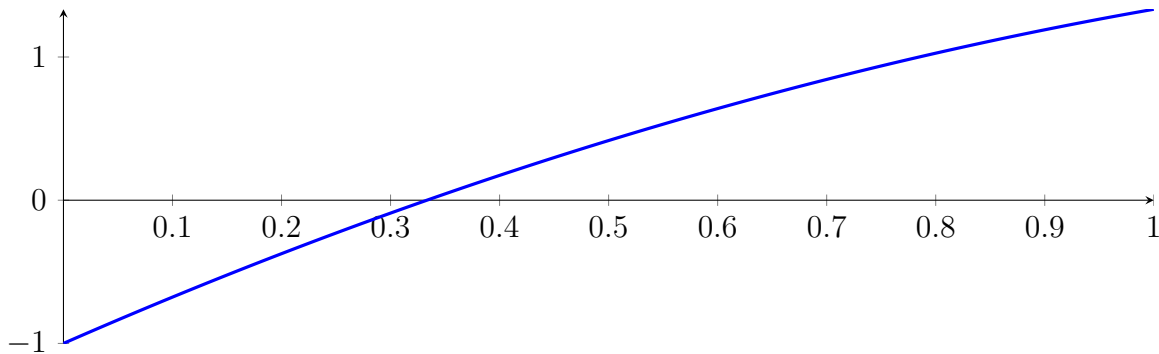
with precision $\varepsilon = 0.01$.

88 Running BezClip on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

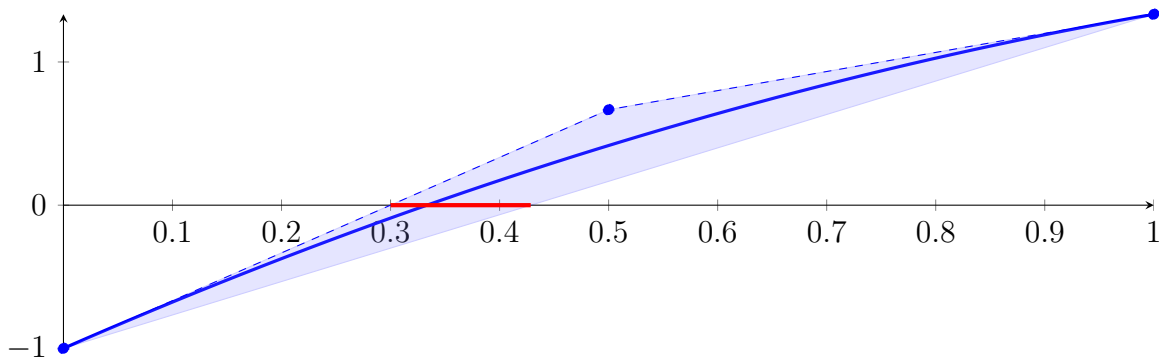
$$p = -1X^2 + 3.33333X - 1$$



88.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

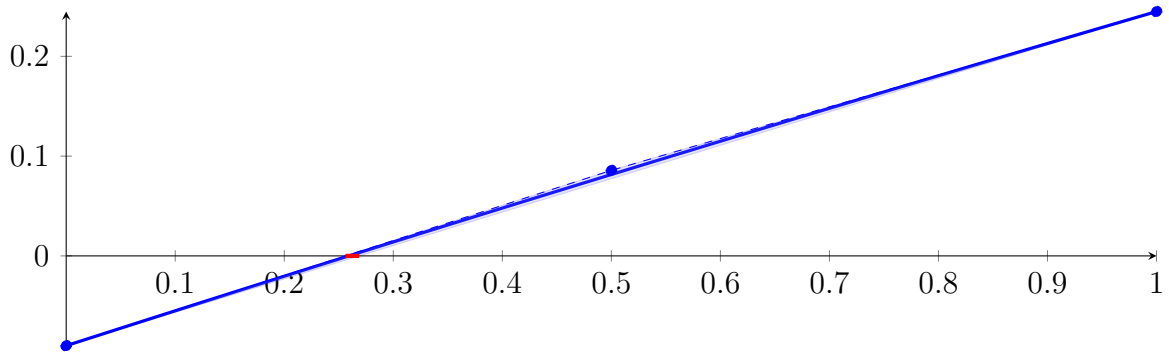
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

88.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

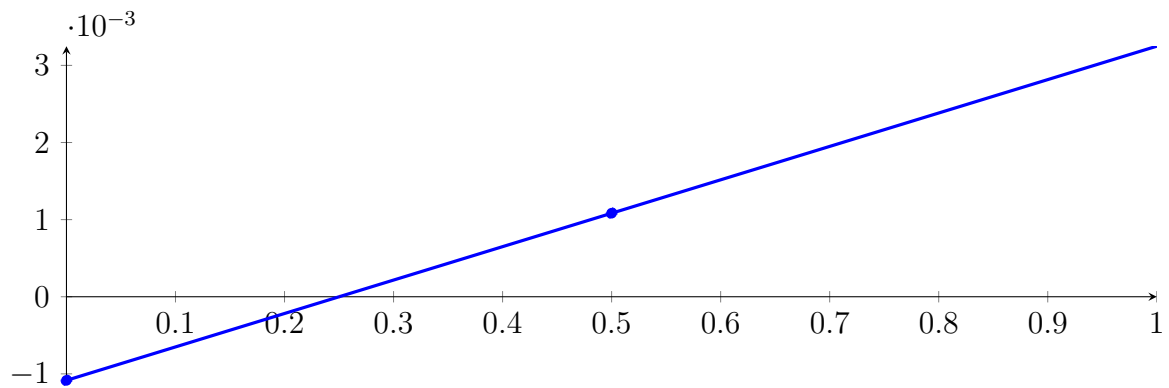
Longest intersection interval: 0.012641

\Rightarrow Selective recursion: interval 1: $[0.332927, 0.334552]$,

88.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333334]$,

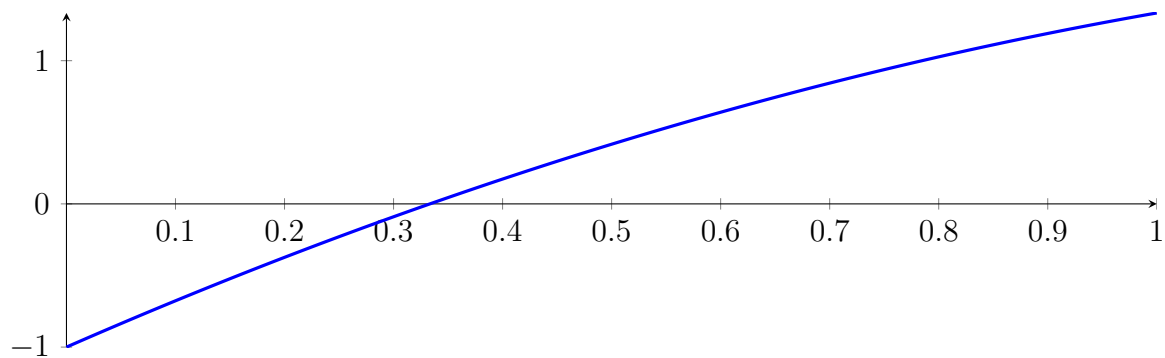
88.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Found root in interval $[0.333333, 0.333334]$ at recursion depth 4!

88.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333334]$$

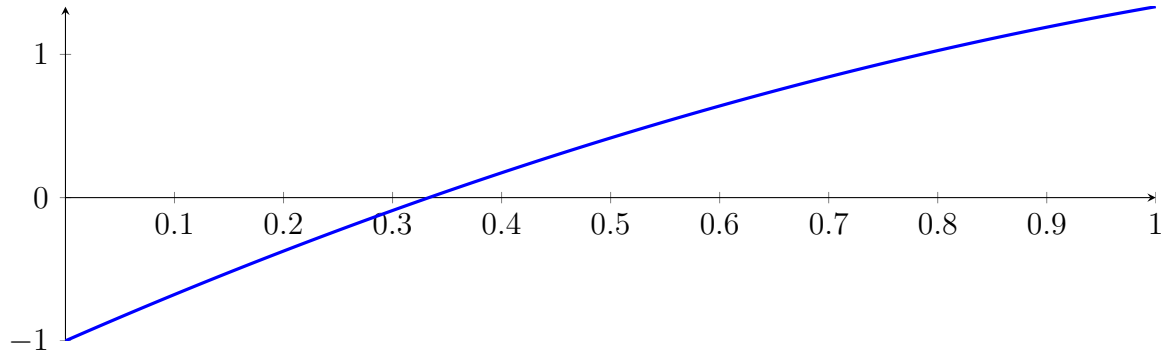
with precision $\varepsilon = 0.0001$.

89 Running QuadClip on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

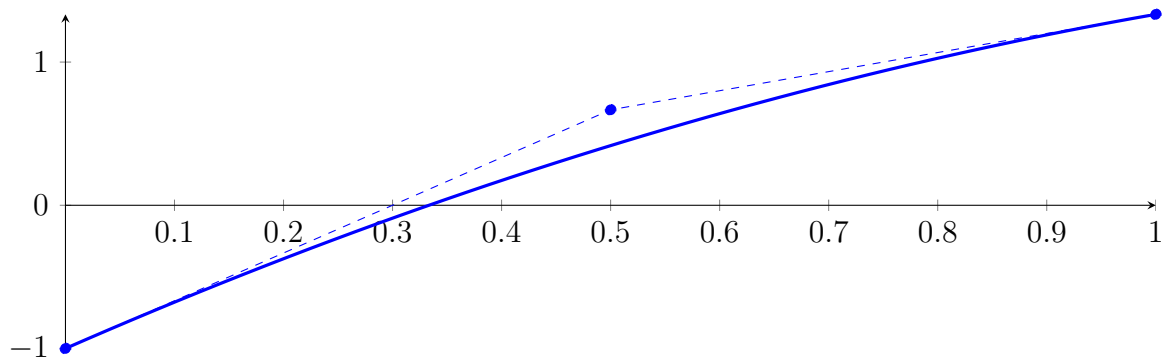
$$p = -1X^2 + 3.33333X - 1$$



89.1 Recursion Branch 1 for Input Interval $[0, 1]$

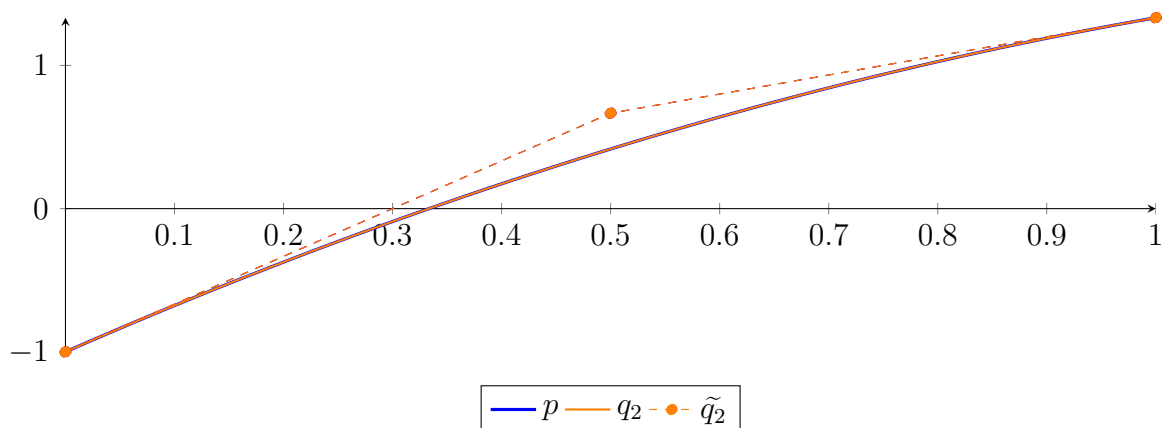
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

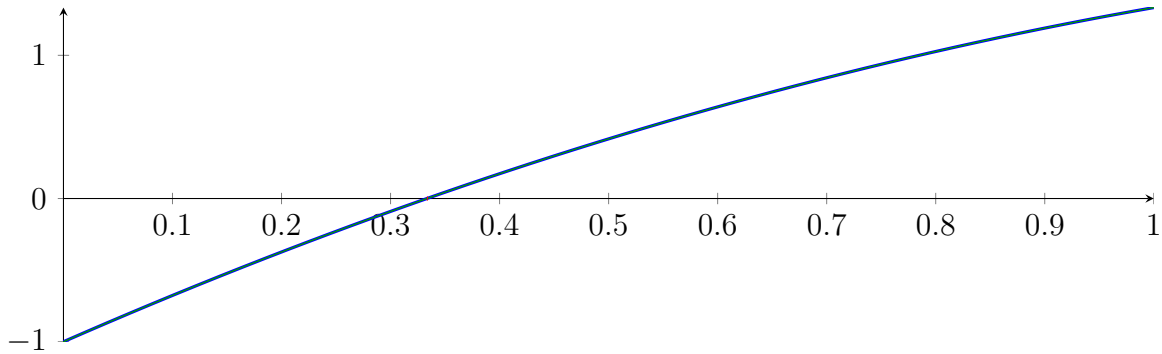
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $3.25261 \cdot 10^{-19}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

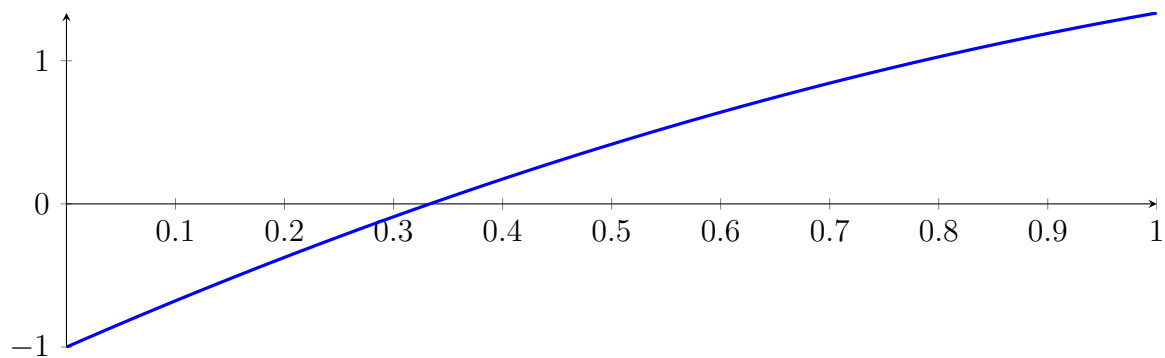
89.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

89.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

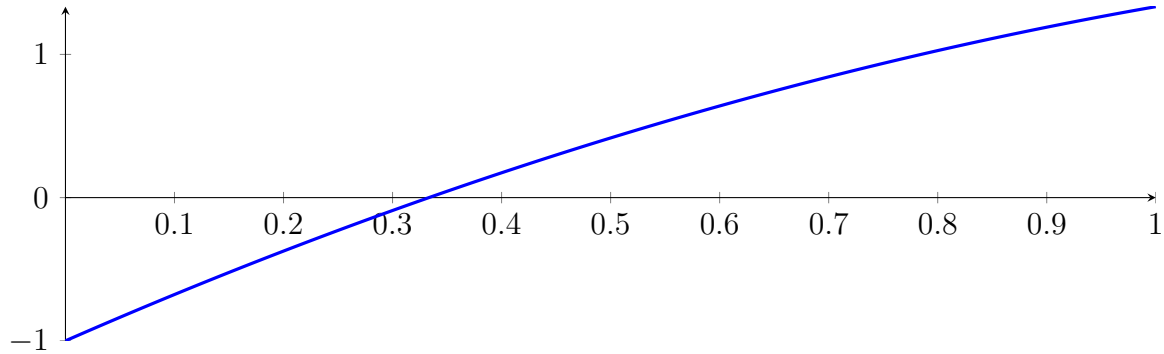
with precision $\varepsilon = 0.0001$.

90 Running CubeClip on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

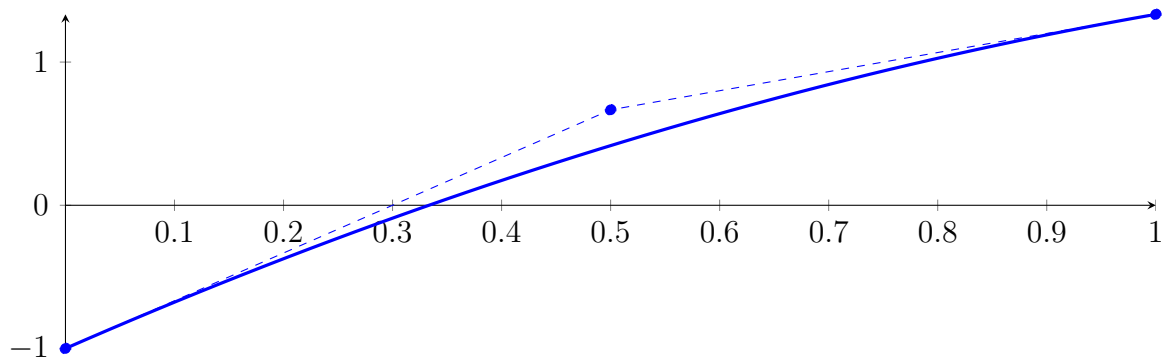
$$p = -1X^2 + 3.33333X - 1$$



90.1 Recursion Branch 1 for Input Interval $[0, 1]$

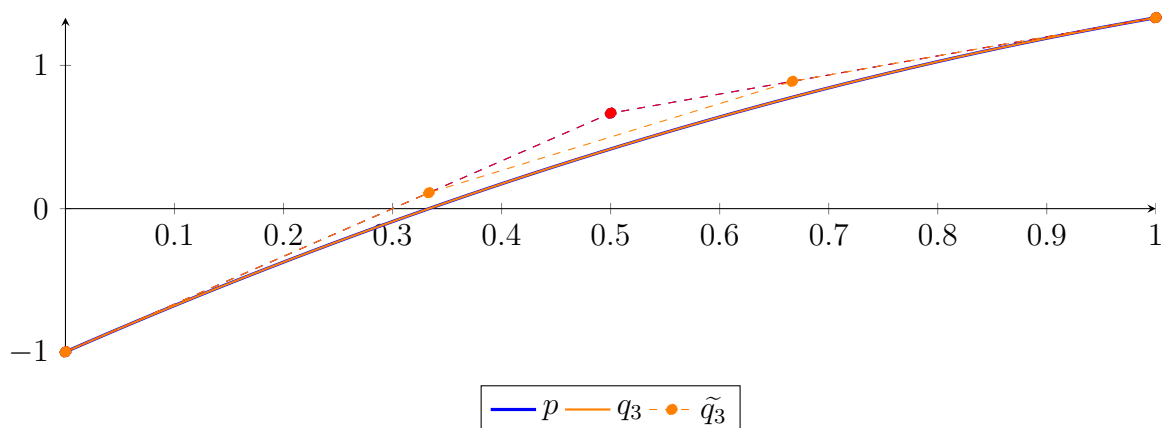
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.63715 \cdot 10^{-17}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

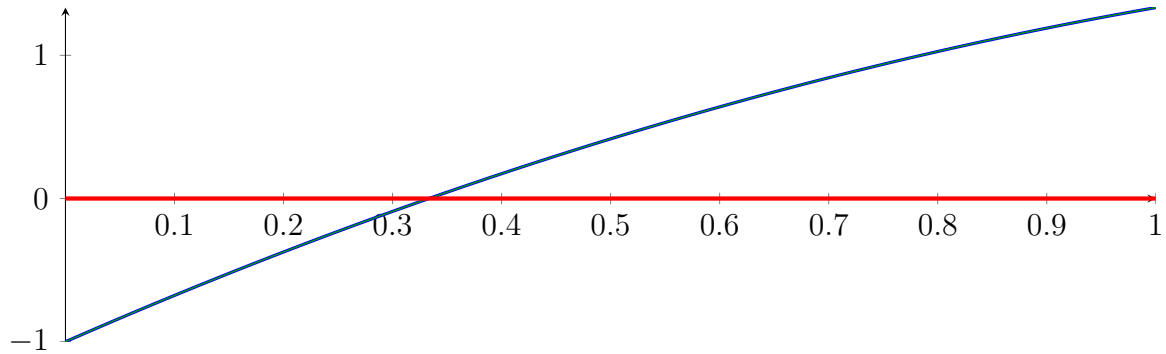
$$m = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

$$N(m) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

Intersection intervals:



$[0, 1]$

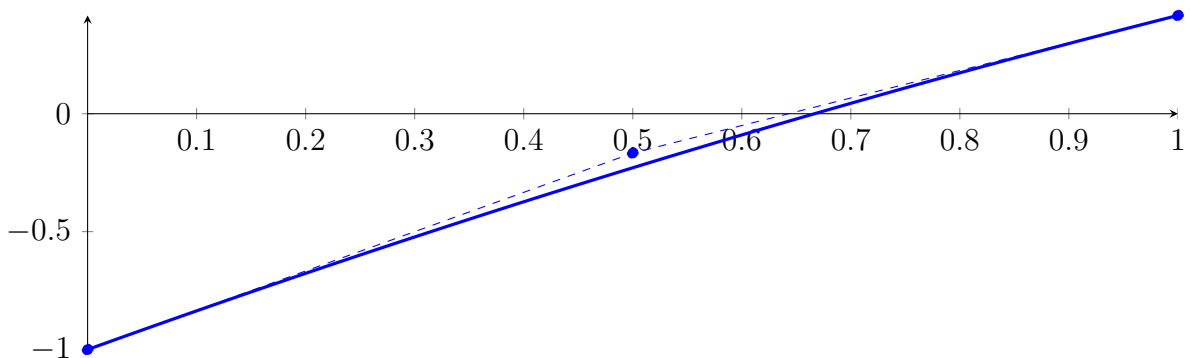
Longest intersection interval: 1

\implies Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

90.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

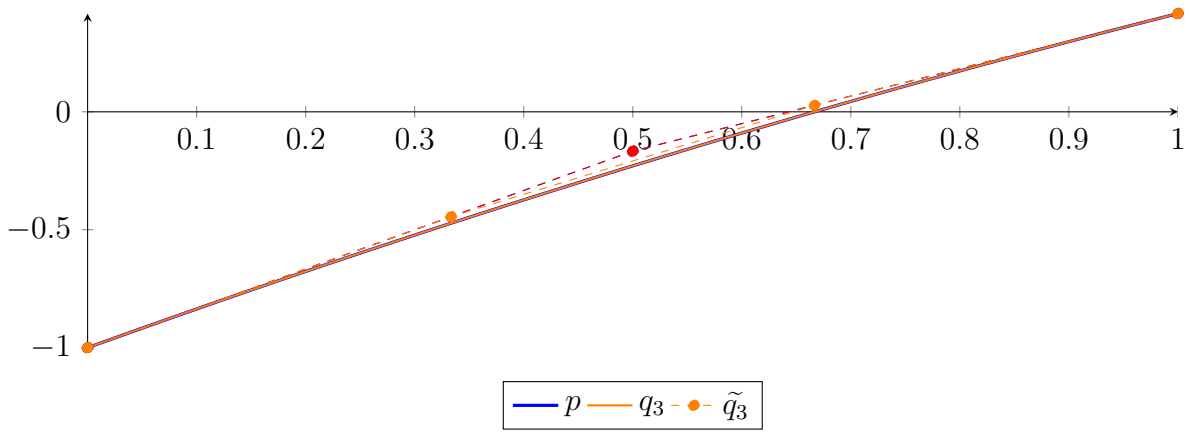
$$\begin{aligned} p &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2}(X) - 0.166667B_{1,2}(X) + 0.416667B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,3} - 0.444444B_{1,3} + 0.0277778B_{2,3} + 0.416667B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2} - 0.166667B_{1,2} + 0.416667B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.58942 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1.07336 \cdot 10^{-17} X^3 - 0.25 X^2 + 1.66667 X - 1$$

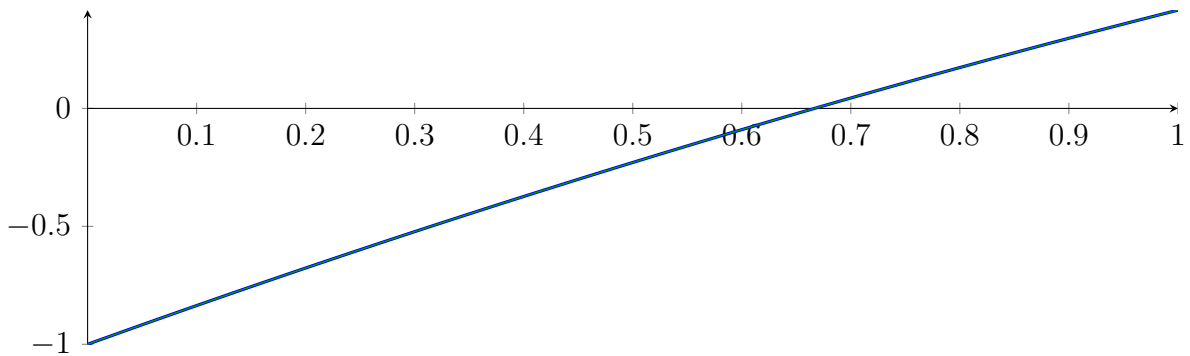
$$m = -1.07336 \cdot 10^{-17} X^3 - 0.25 X^2 + 1.66667 X - 1$$

Root of M and m :

$$N(M) = \{-2.32913 \cdot 10^{16}\}$$

$$N(m) = \{-2.32913 \cdot 10^{16}\}$$

Intersection intervals:

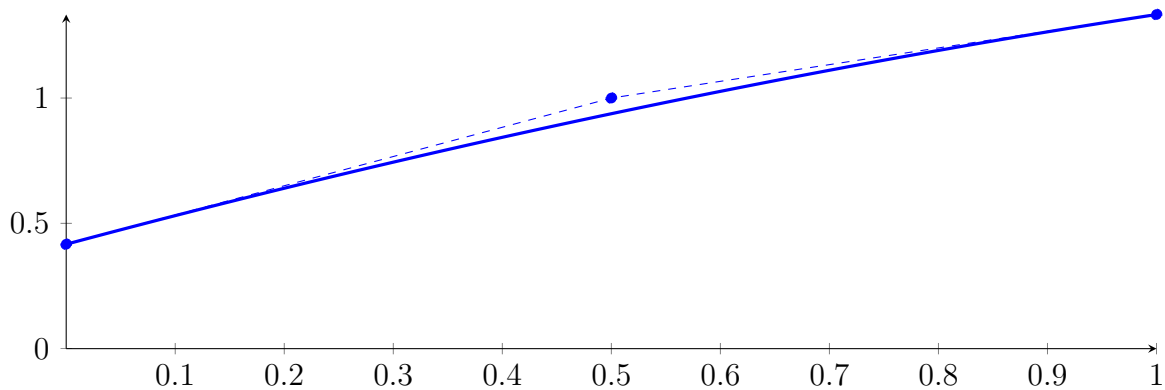


No intersection intervals with the x axis.

90.3 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.25 X^2 + 1.16667 X + 0.416667 \\ &= 0.416667 B_{0,2}(X) + 1 B_{1,2}(X) + 1.333333 B_{2,2}(X) \end{aligned}$$



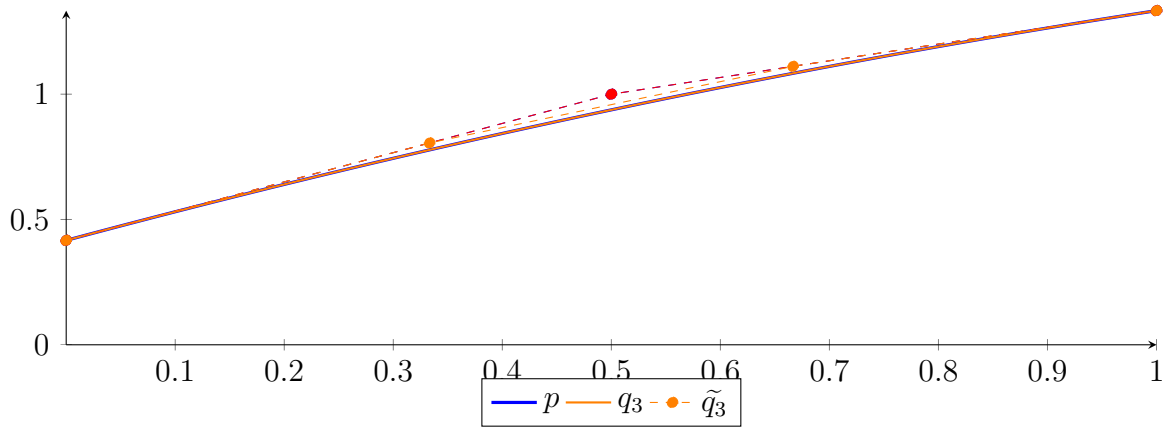
Degree reduction and raising:

$$q_3 = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$= 0.416667B_{0,3} + 0.805556B_{1,3} + 1.111111B_{2,3} + 1.333333B_{3,3}$$

$$\tilde{q}_3 = -0.25X^2 + 1.16667X + 0.416667$$

$$= 0.416667B_{0,2} + 1B_{1,2} + 1.333333B_{2,2}$$



The maximum difference of the Bézier coefficients is $\delta = 1.30104 \cdot 10^{-18}$.

Bounding polynomials M and m :

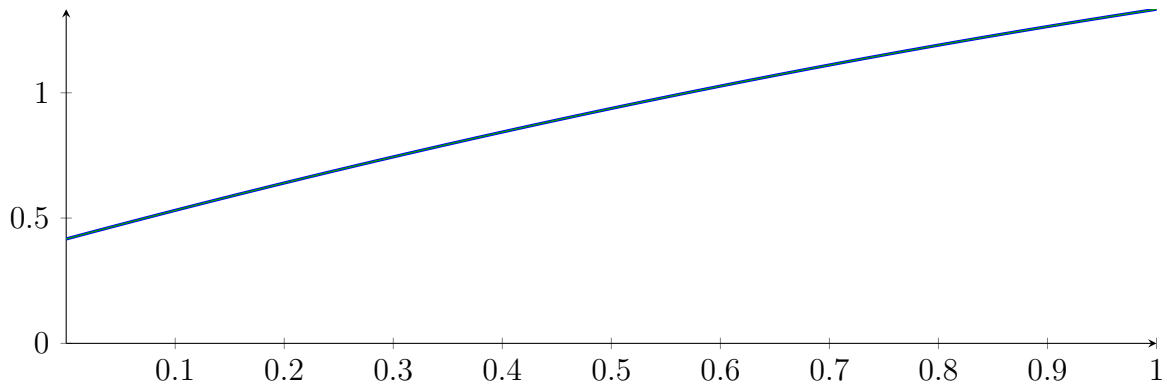
$$M = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$m = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

Root of M and m :

$$N(M) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\} \quad N(m) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\}$$

Intersection intervals:

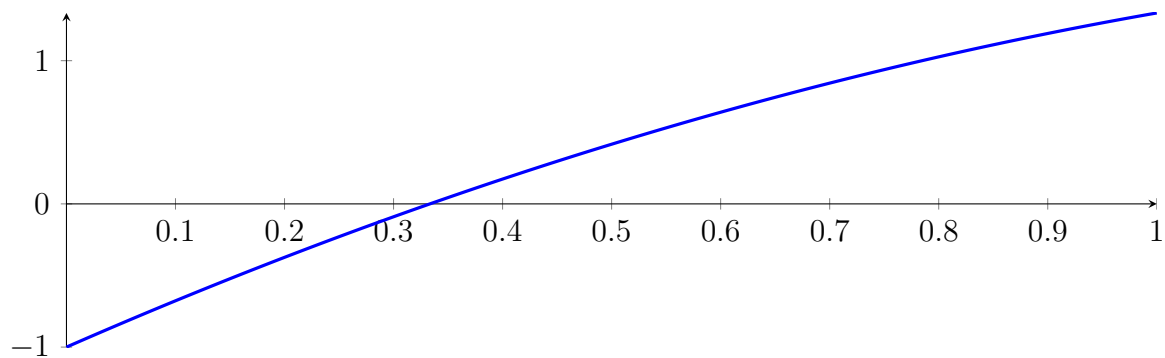


No intersection intervals with the x axis.

90.4 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

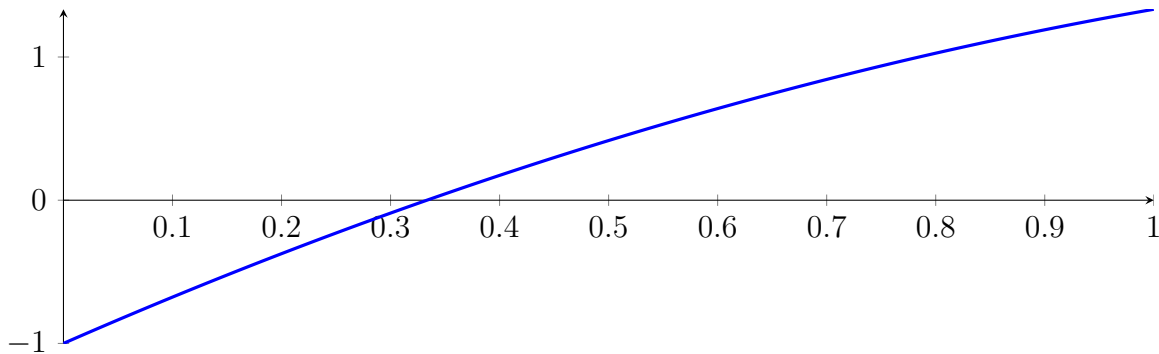
with precision $\varepsilon = 0.0001$.

91 Running BezClip on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

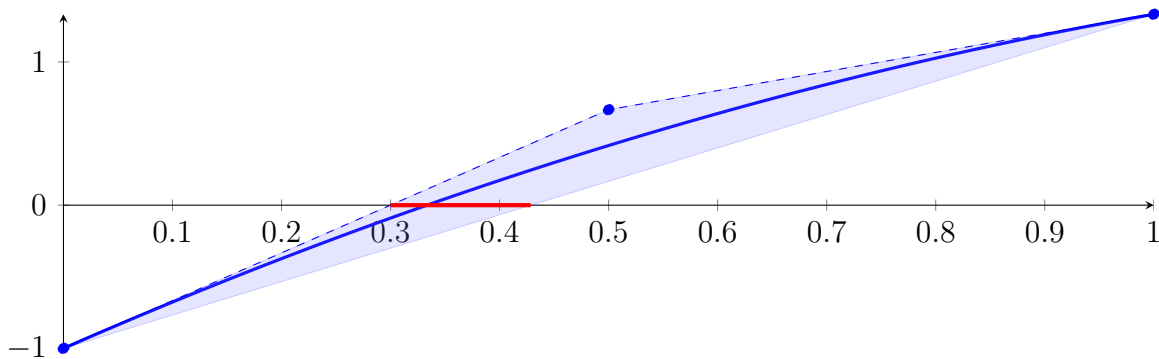
$$p = -1X^2 + 3.33333X - 1$$



91.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

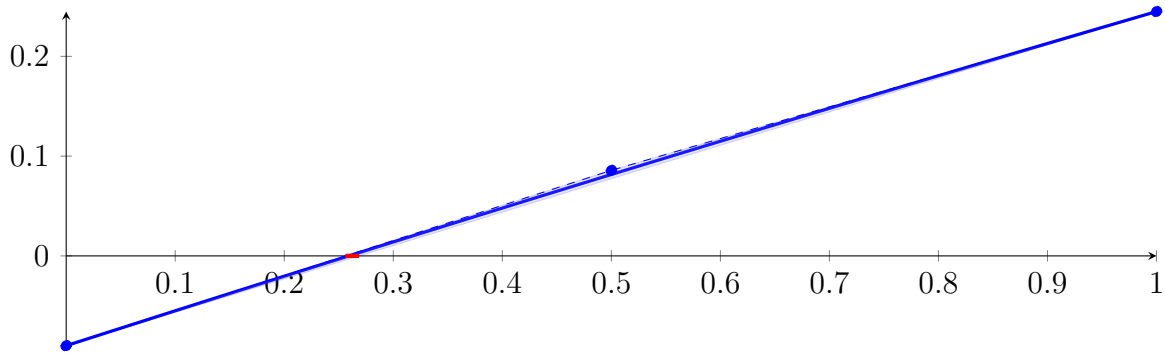
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

91.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

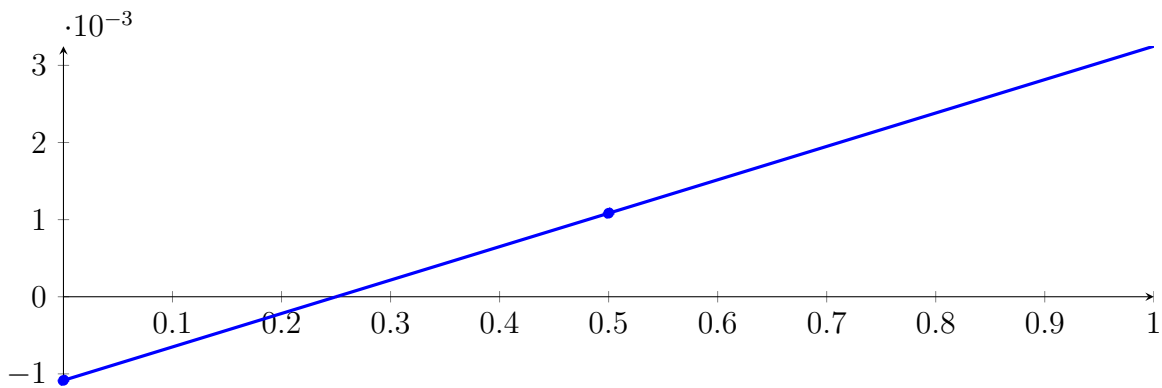
Longest intersection interval: 0.012641

\Rightarrow Selective recursion: interval 1: $[0.332927, 0.334552]$,

91.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

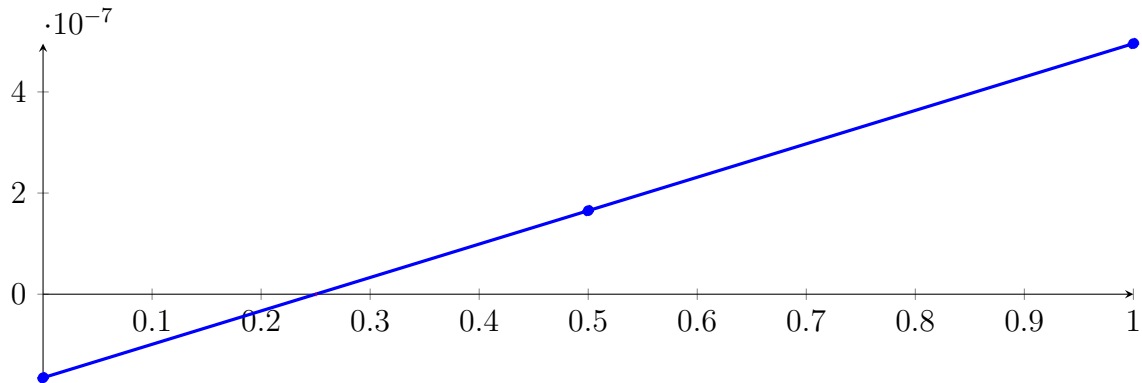
Longest intersection interval: 0.000152462

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333334]$,

91.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07} \\ &= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $2.32306 \cdot 10^{-08}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

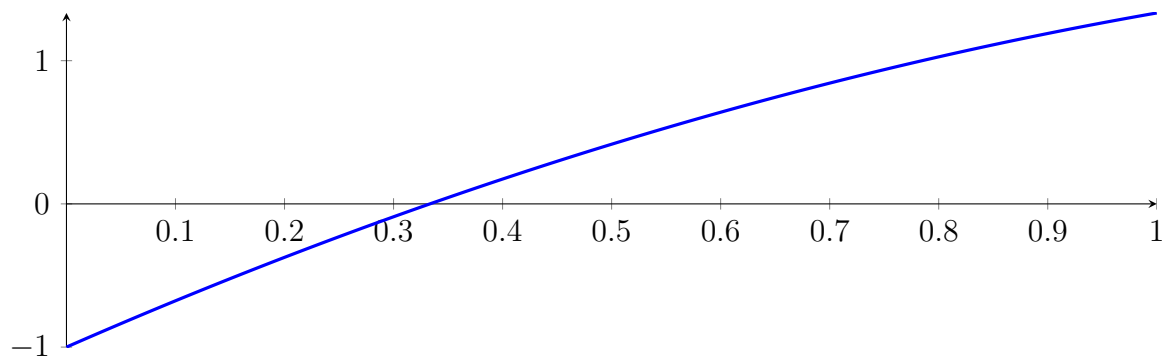
91.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

91.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

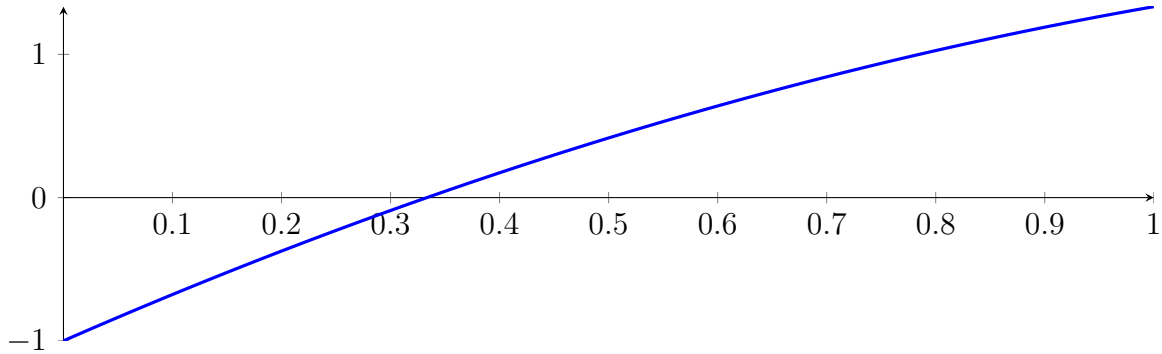
with precision $\varepsilon = 1 \cdot 10^{-08}$.

92 Running QuadClip on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

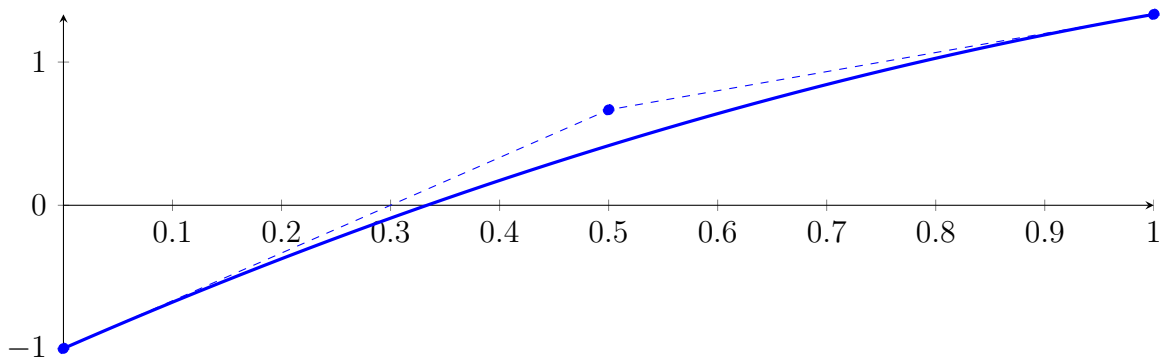
$$p = -1X^2 + 3.33333X - 1$$



92.1 Recursion Branch 1 for Input Interval $[0, 1]$

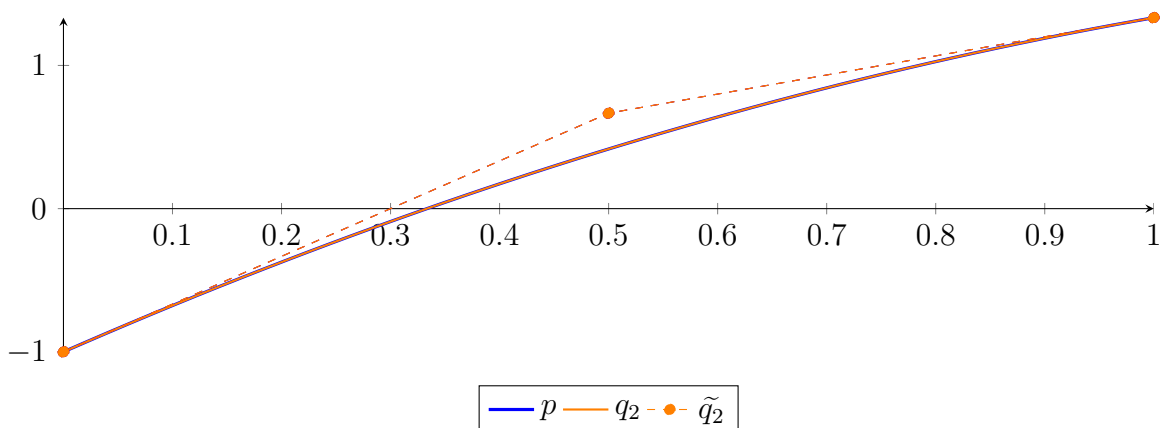
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

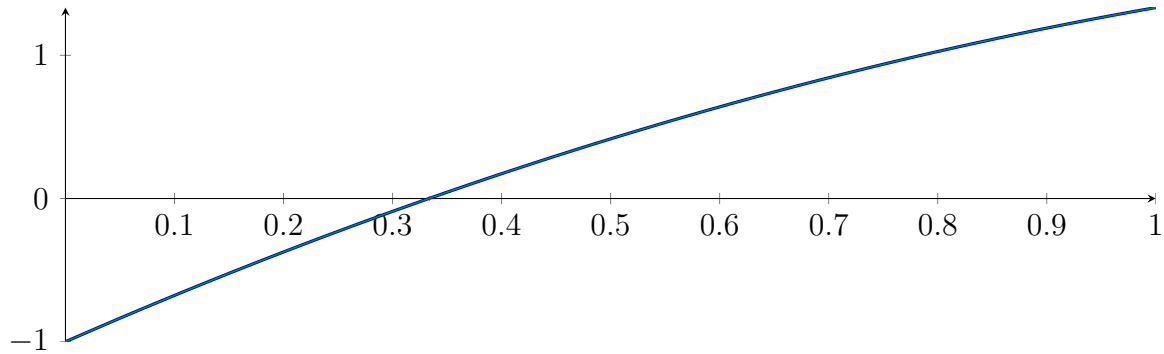
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $3.25261 \cdot 10^{-19}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

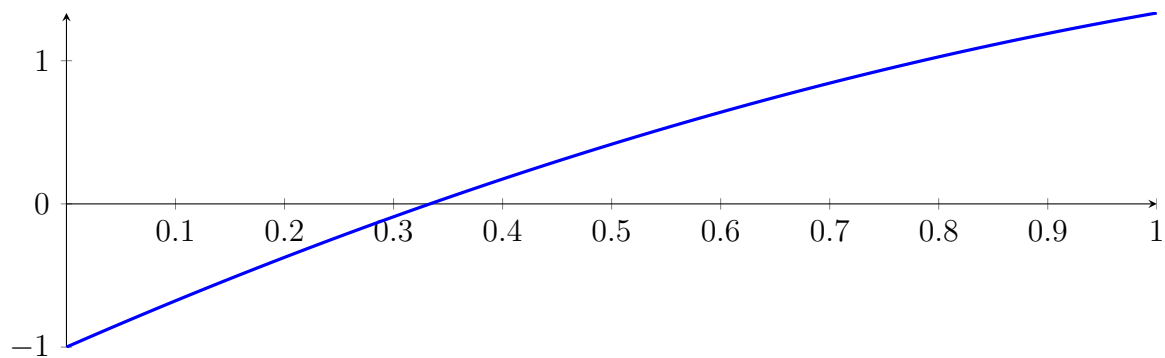
92.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

92.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

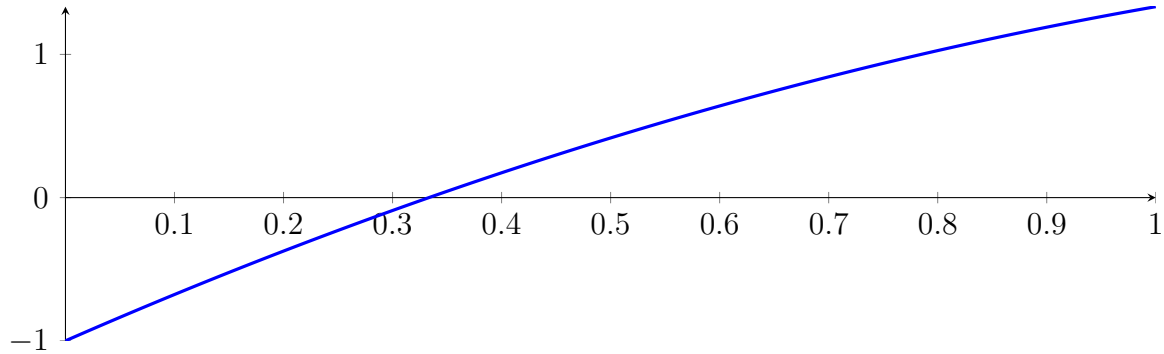
with precision $\varepsilon = 1 \cdot 10^{-08}$.

93 Running CubeClip on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

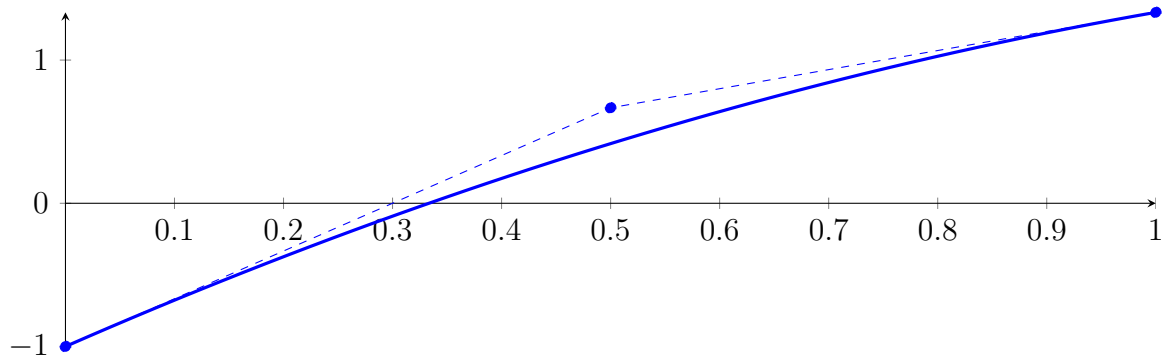
$$p = -1X^2 + 3.33333X - 1$$



93.1 Recursion Branch 1 for Input Interval $[0, 1]$

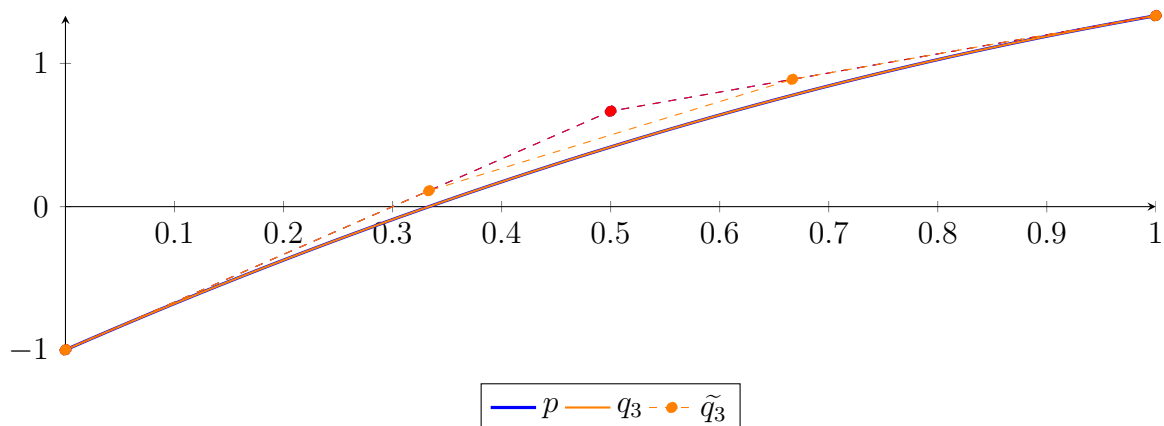
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.63715 \cdot 10^{-17}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

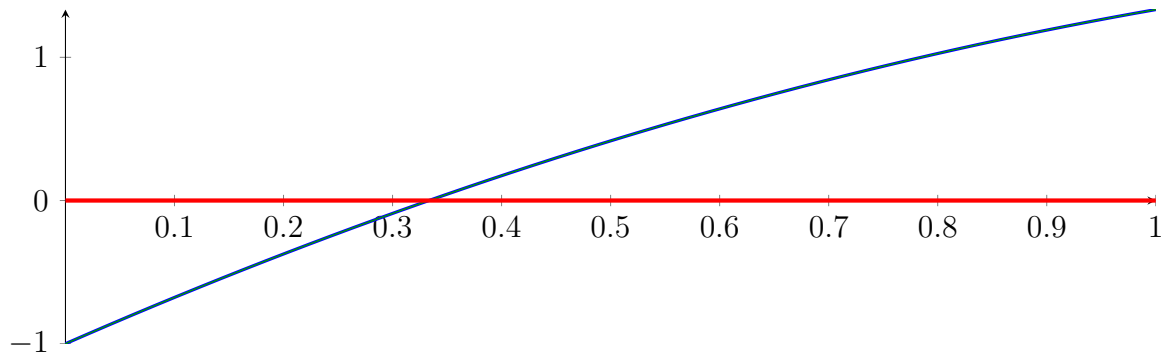
$$m = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

$$N(m) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

Intersection intervals:



$[0, 1]$

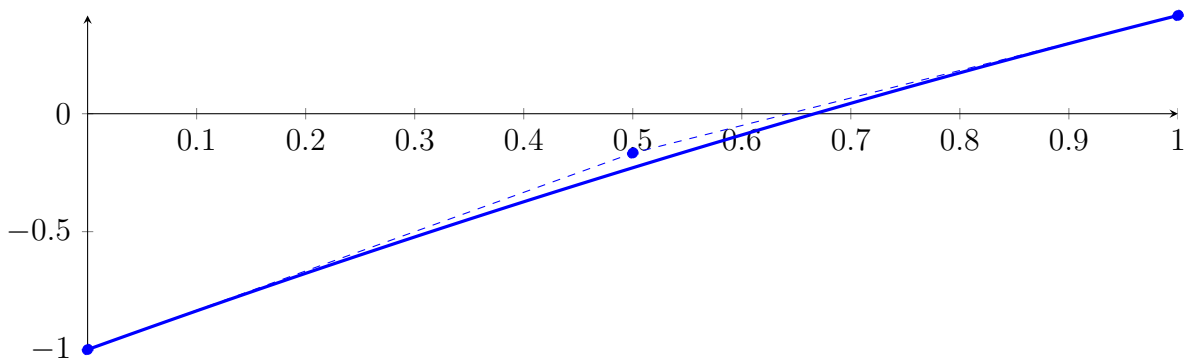
Longest intersection interval: 1

\implies Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

93.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

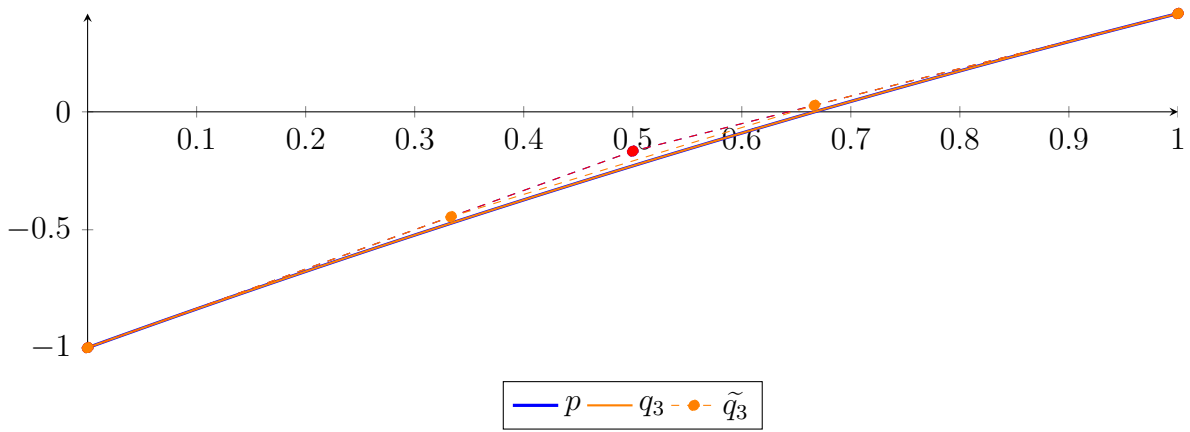
$$\begin{aligned} p &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2}(X) - 0.166667B_{1,2}(X) + 0.416667B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,3} - 0.444444B_{1,3} + 0.0277778B_{2,3} + 0.416667B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2} - 0.166667B_{1,2} + 0.416667B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.58942 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1.07336 \cdot 10^{-17} X^3 - 0.25 X^2 + 1.66667 X - 1$$

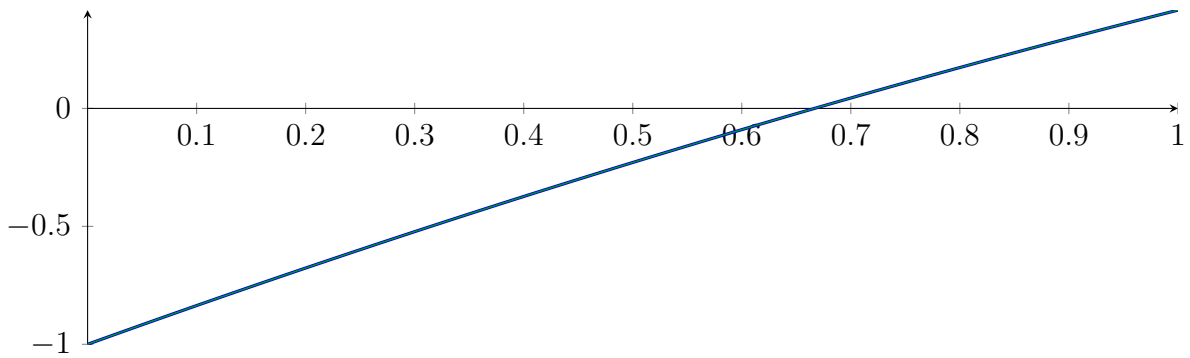
$$m = -1.07336 \cdot 10^{-17} X^3 - 0.25 X^2 + 1.66667 X - 1$$

Root of M and m :

$$N(M) = \{-2.32913 \cdot 10^{16}\}$$

$$N(m) = \{-2.32913 \cdot 10^{16}\}$$

Intersection intervals:

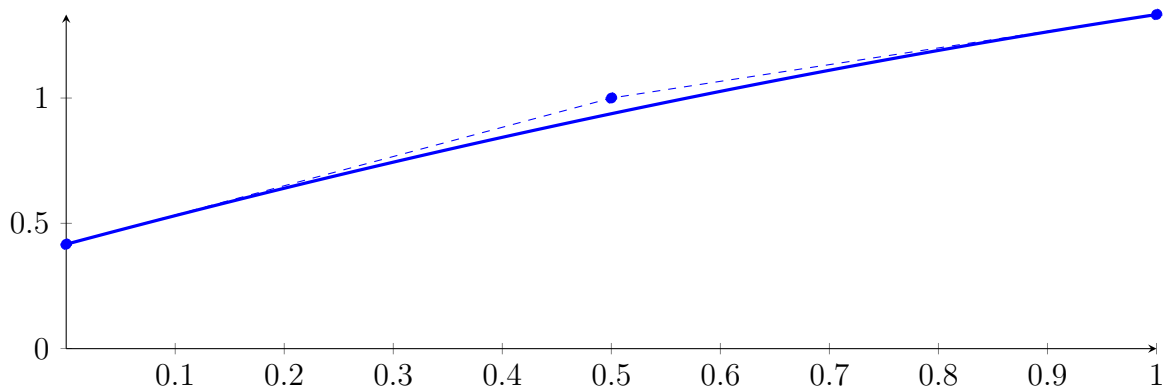


No intersection intervals with the x axis.

93.3 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.25 X^2 + 1.16667 X + 0.416667 \\ &= 0.416667 B_{0,2}(X) + 1 B_{1,2}(X) + 1.333333 B_{2,2}(X) \end{aligned}$$



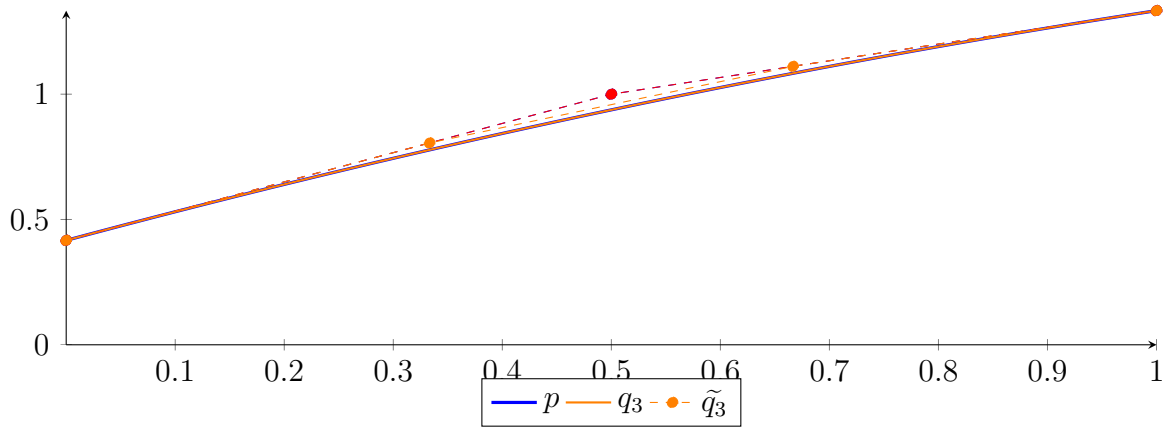
Degree reduction and raising:

$$q_3 = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$= 0.416667B_{0,3} + 0.805556B_{1,3} + 1.111111B_{2,3} + 1.333333B_{3,3}$$

$$\tilde{q}_3 = -0.25X^2 + 1.16667X + 0.416667$$

$$= 0.416667B_{0,2} + 1B_{1,2} + 1.333333B_{2,2}$$



The maximum difference of the Bézier coefficients is $\delta = 1.30104 \cdot 10^{-18}$.

Bounding polynomials M and m :

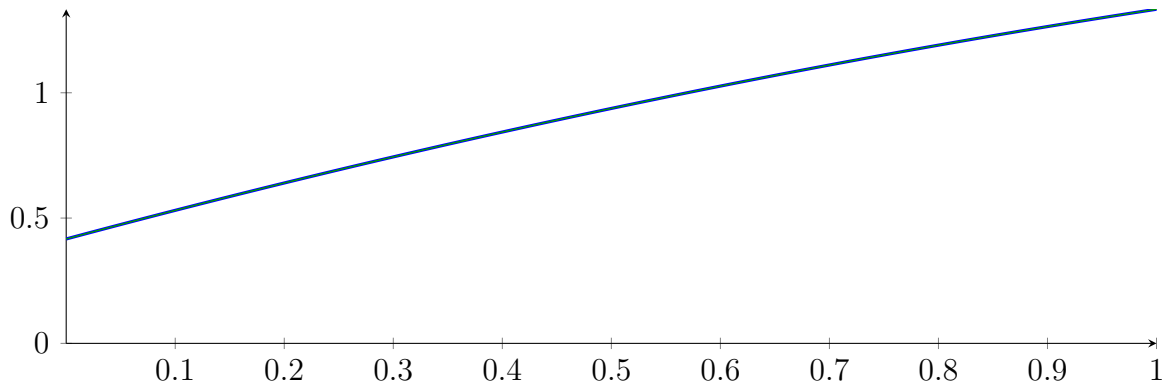
$$M = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$m = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

Root of M and m :

$$N(M) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\} \quad N(m) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\}$$

Intersection intervals:

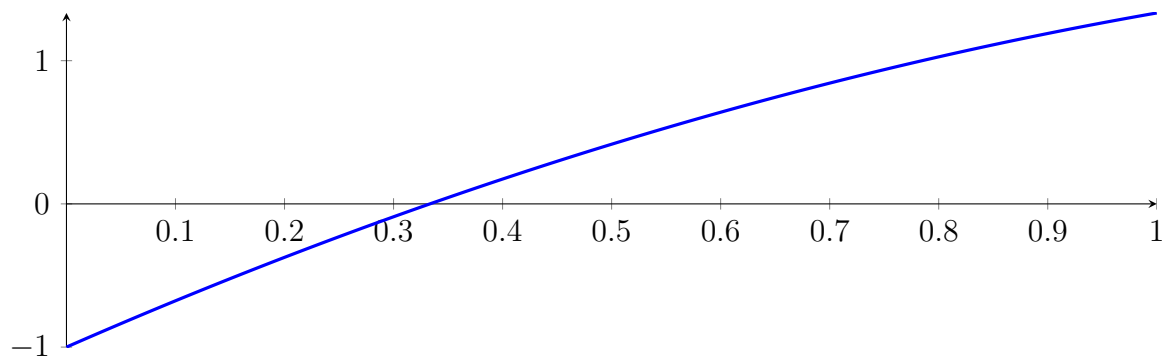


No intersection intervals with the x axis.

93.4 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

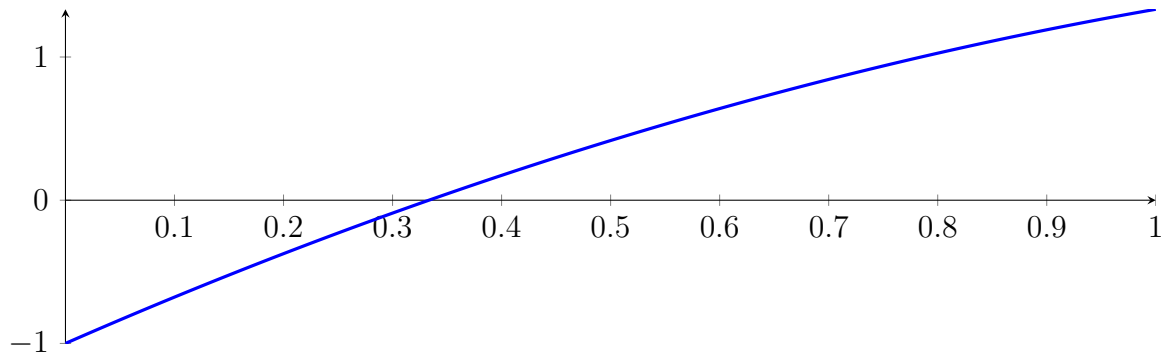
with precision $\varepsilon = 1 \cdot 10^{-08}$.

94 Running BezClip on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

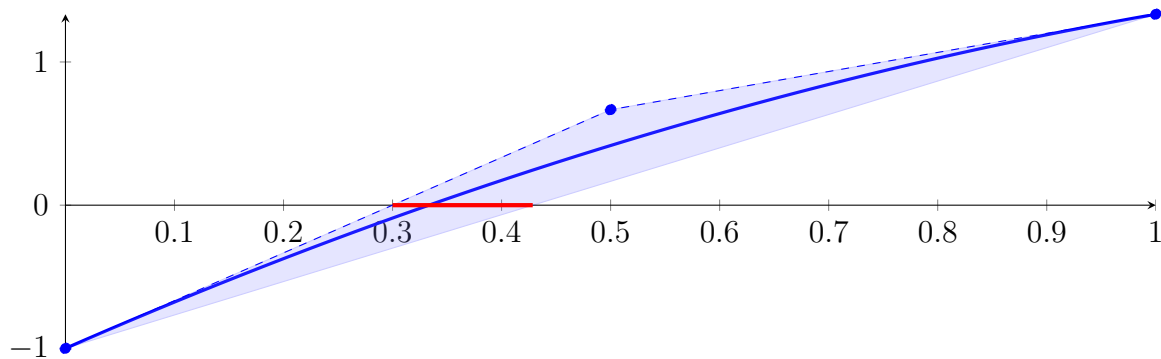
$$p = -1X^2 + 3.33333X - 1$$



94.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

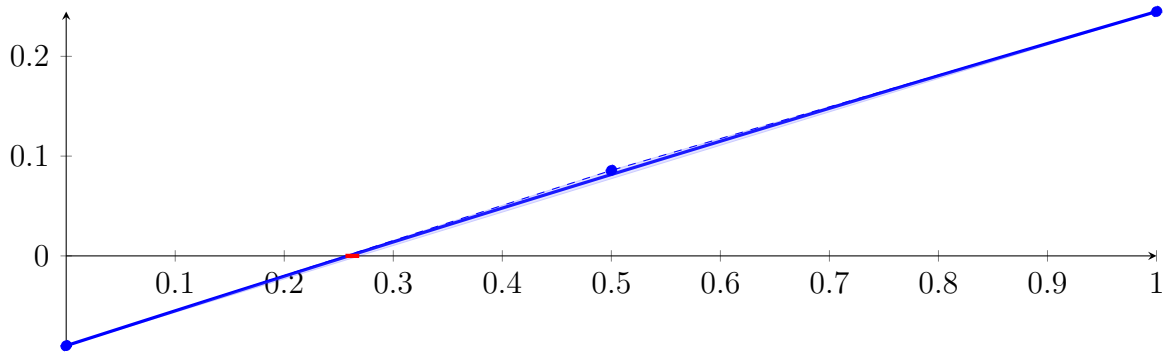
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

94.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

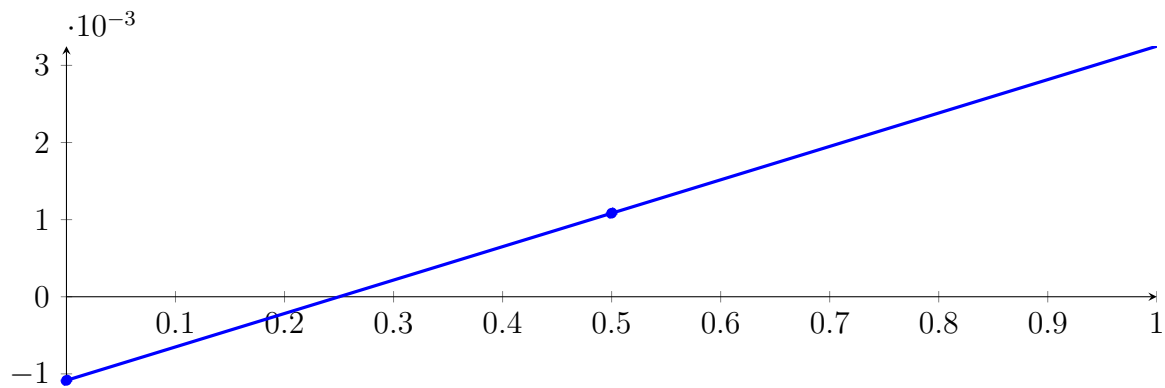
Longest intersection interval: 0.012641

\Rightarrow Selective recursion: interval 1: $[0.332927, 0.334552]$,

94.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

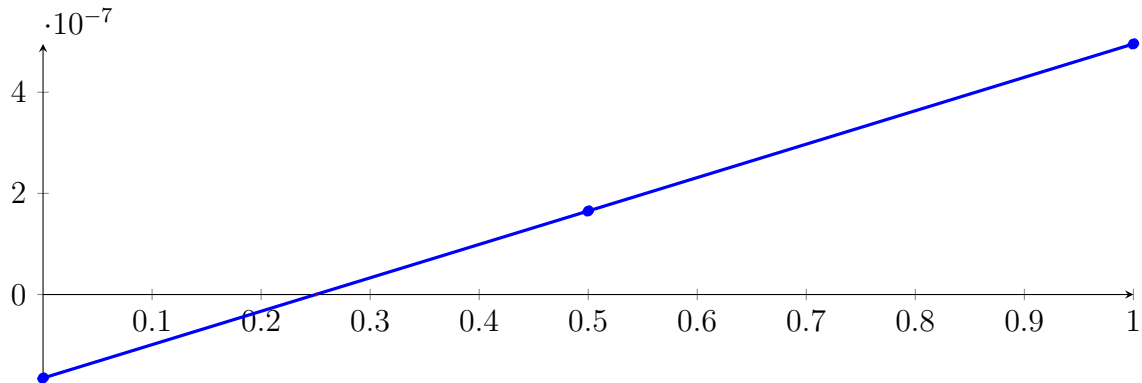
Longest intersection interval: 0.000152462

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333334]$,

94.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07} \\
 &= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

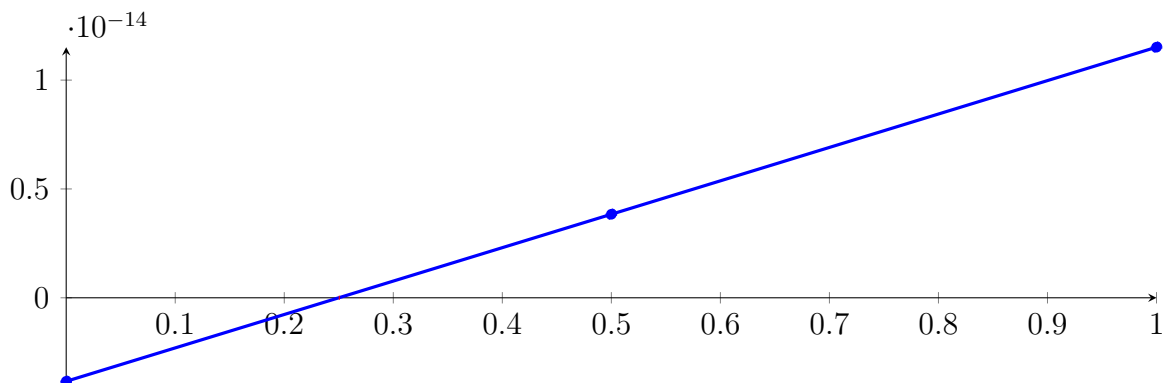
Longest intersection interval: $2.32306 \cdot 10^{-08}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

94.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.31352 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15} \\
 &= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.39635 \cdot 10^{-16}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

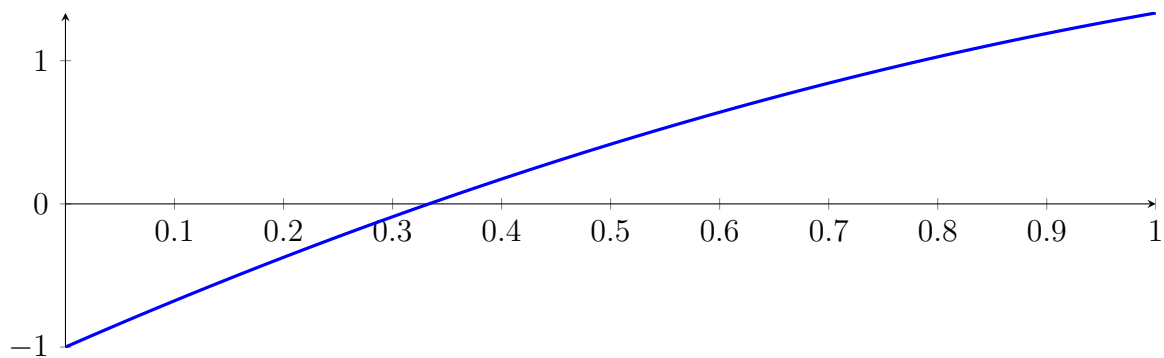
94.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

94.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

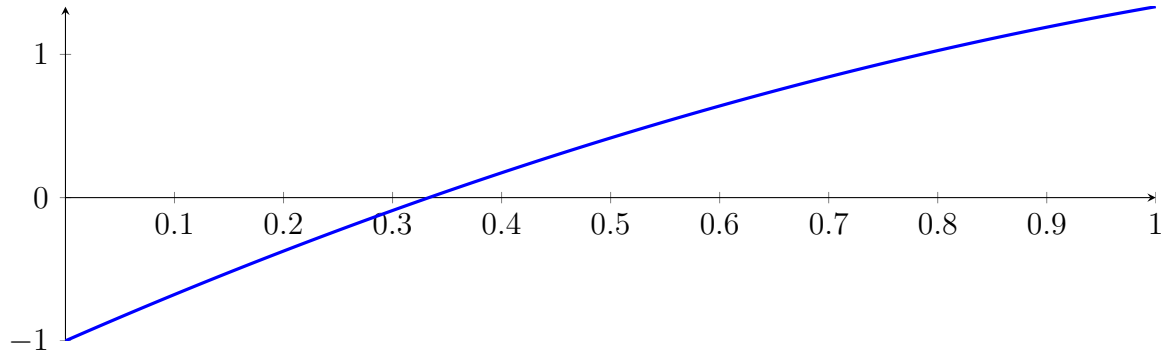
with precision $\varepsilon = 1 \cdot 10^{-16}$.

95 Running QuadClip on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

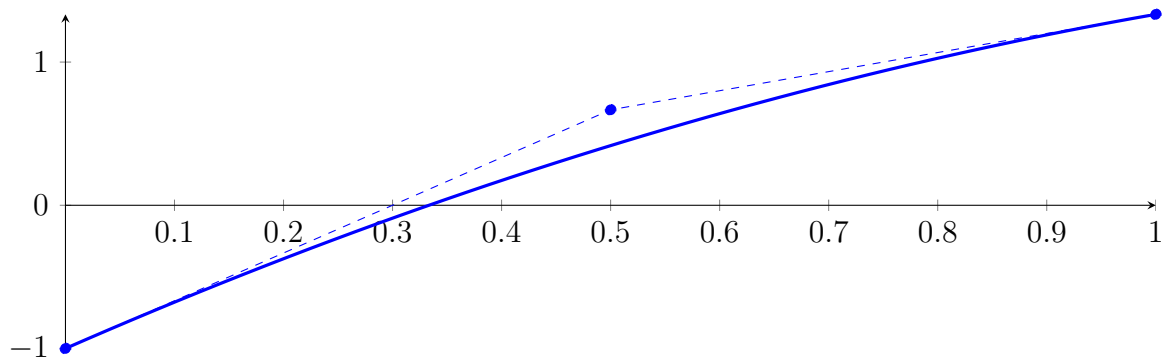
$$p = -1X^2 + 3.33333X - 1$$



95.1 Recursion Branch 1 for Input Interval $[0, 1]$

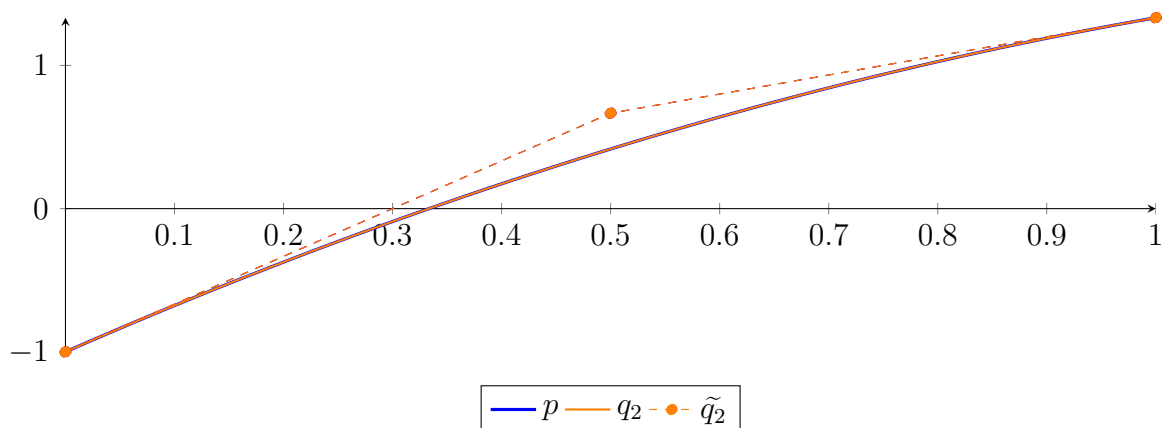
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

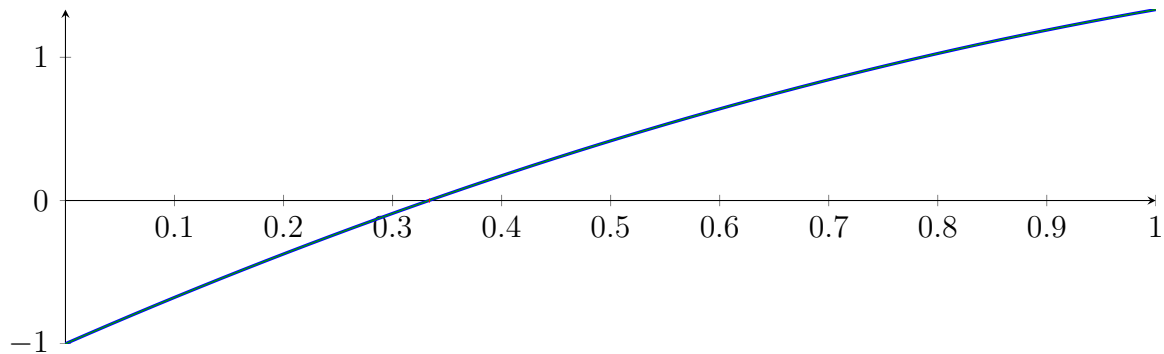
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $3.25261 \cdot 10^{-19}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

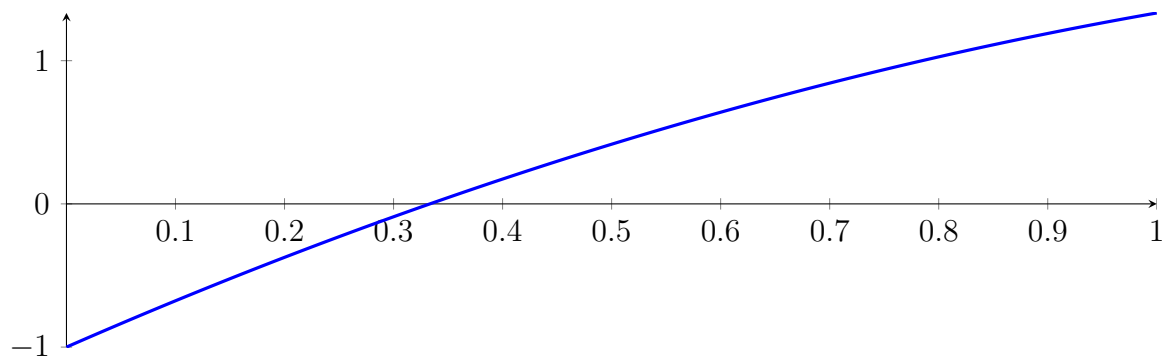
95.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

95.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

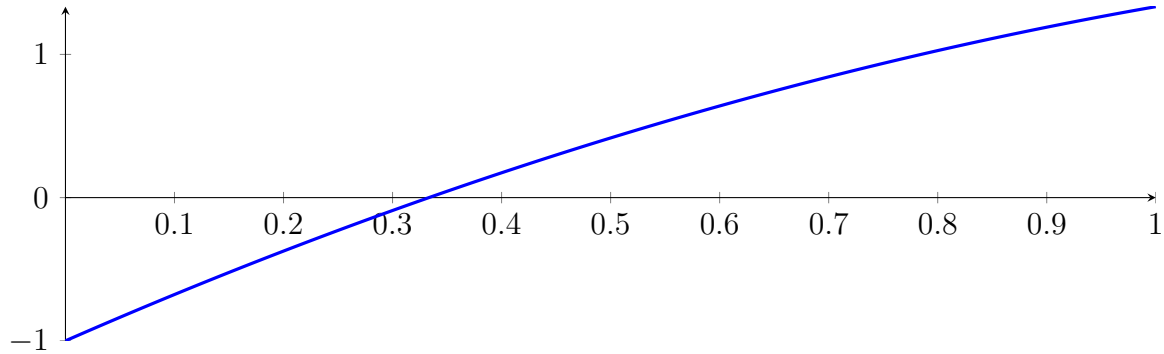
with precision $\varepsilon = 1 \cdot 10^{-16}$.

96 Running CubeClip on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

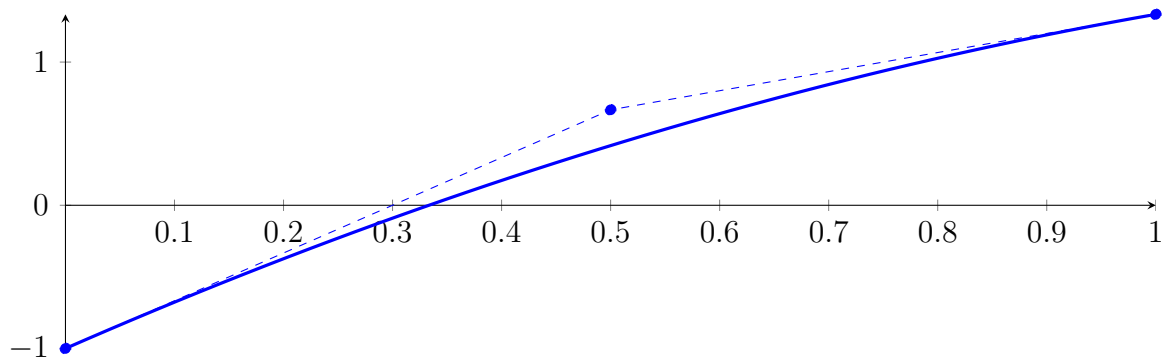
$$p = -1X^2 + 3.33333X - 1$$



96.1 Recursion Branch 1 for Input Interval $[0, 1]$

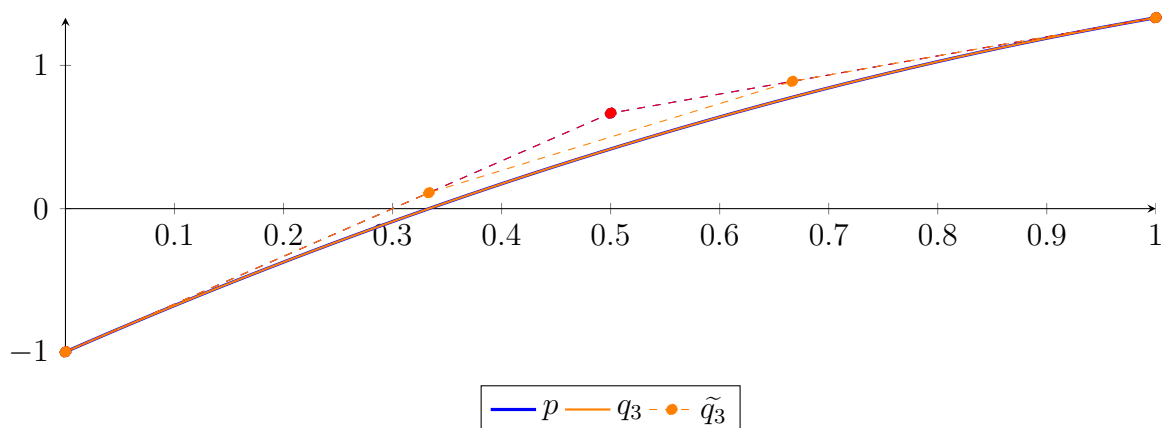
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.63715 \cdot 10^{-17}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

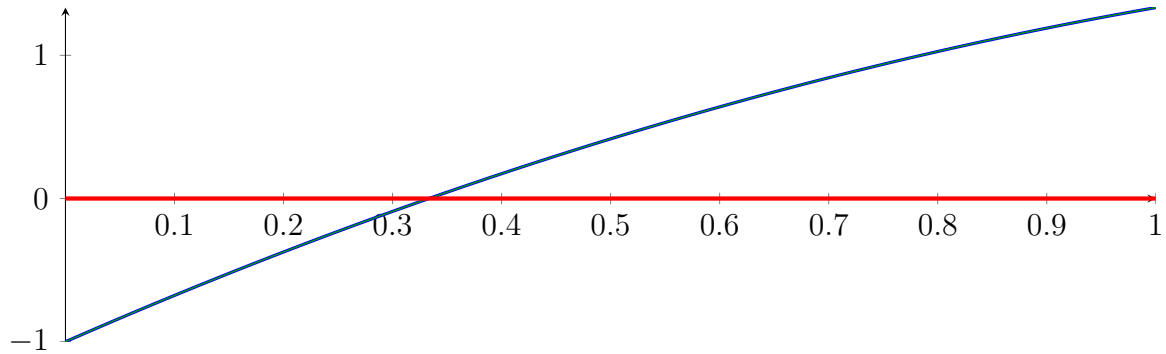
$$m = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

$$N(m) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

Intersection intervals:



$[0, 1]$

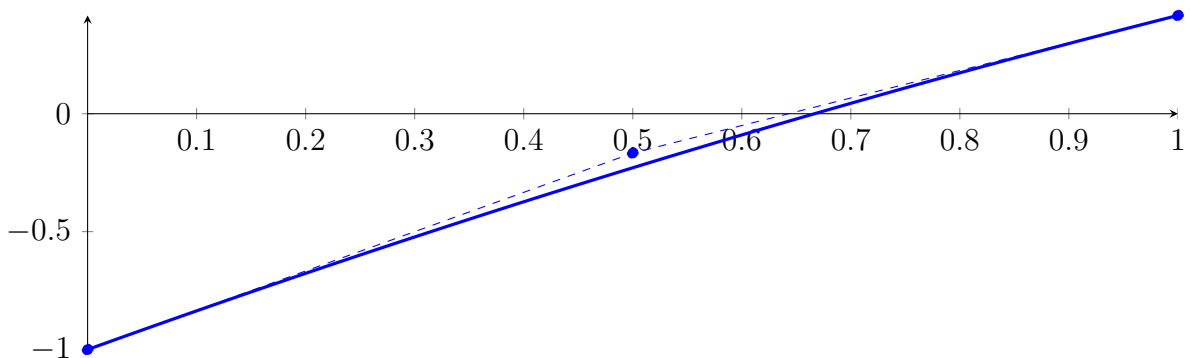
Longest intersection interval: 1

\implies Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

96.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

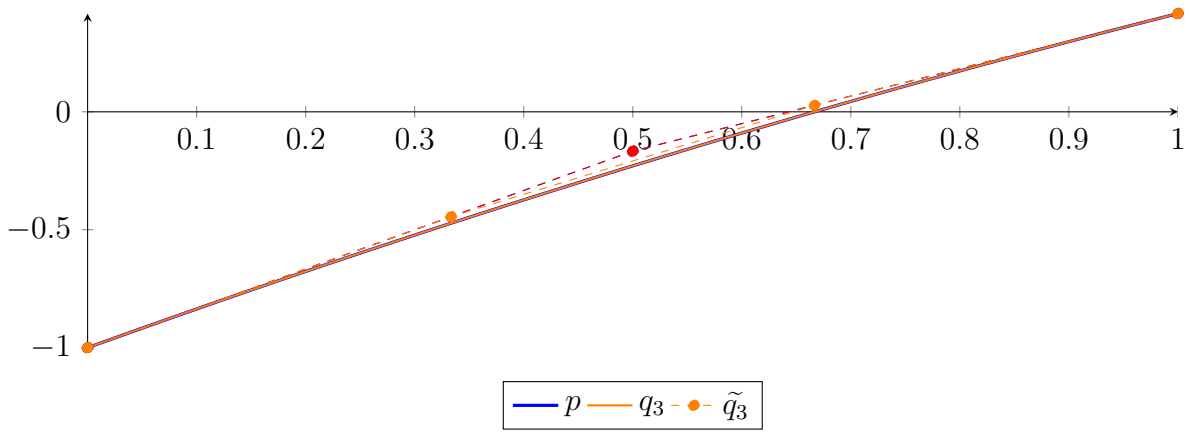
$$\begin{aligned} p &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2}(X) - 0.166667B_{1,2}(X) + 0.416667B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,3} - 0.444444B_{1,3} + 0.0277778B_{2,3} + 0.416667B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2} - 0.166667B_{1,2} + 0.416667B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.58942 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1.07336 \cdot 10^{-17} X^3 - 0.25 X^2 + 1.66667 X - 1$$

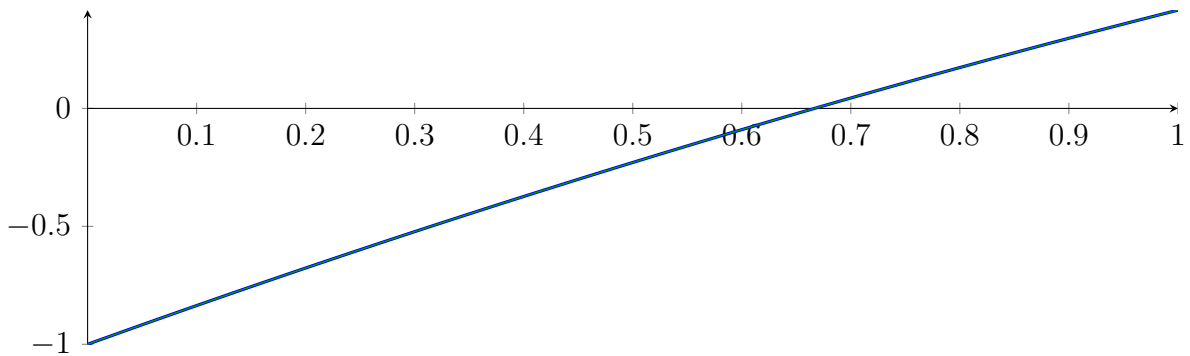
$$m = -1.07336 \cdot 10^{-17} X^3 - 0.25 X^2 + 1.66667 X - 1$$

Root of M and m :

$$N(M) = \{-2.32913 \cdot 10^{16}\}$$

$$N(m) = \{-2.32913 \cdot 10^{16}\}$$

Intersection intervals:

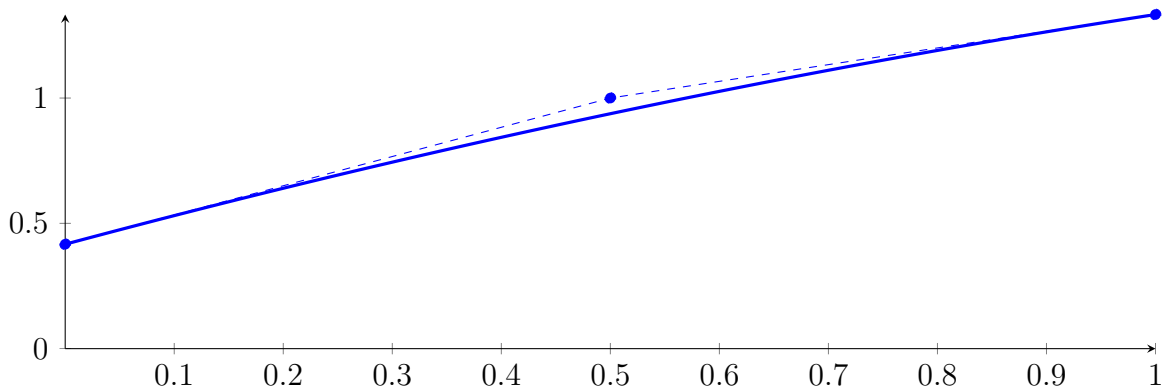


No intersection intervals with the x axis.

96.3 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.25 X^2 + 1.16667 X + 0.416667 \\ &= 0.416667 B_{0,2}(X) + 1 B_{1,2}(X) + 1.333333 B_{2,2}(X) \end{aligned}$$



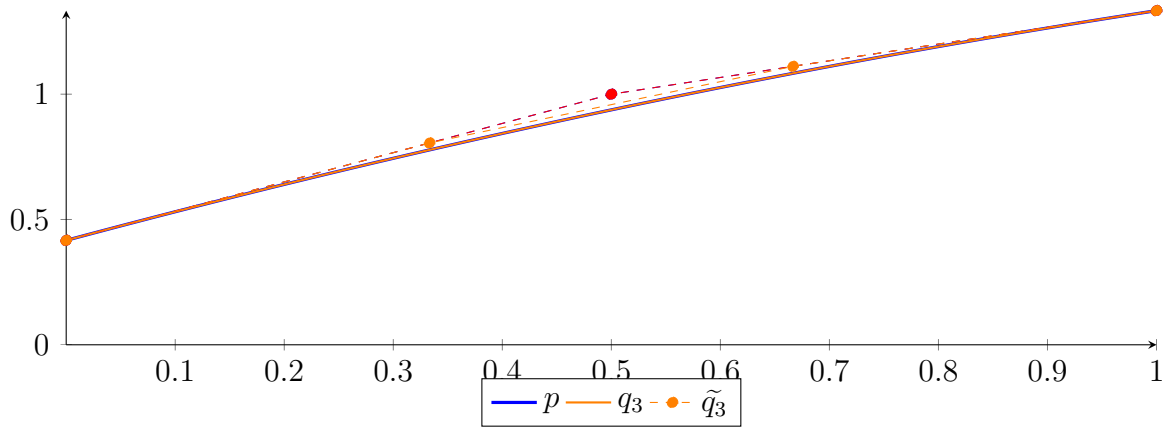
Degree reduction and raising:

$$q_3 = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$= 0.416667B_{0,3} + 0.805556B_{1,3} + 1.111111B_{2,3} + 1.333333B_{3,3}$$

$$\tilde{q}_3 = -0.25X^2 + 1.16667X + 0.416667$$

$$= 0.416667B_{0,2} + 1B_{1,2} + 1.333333B_{2,2}$$



The maximum difference of the Bézier coefficients is $\delta = 1.30104 \cdot 10^{-18}$.

Bounding polynomials M and m :

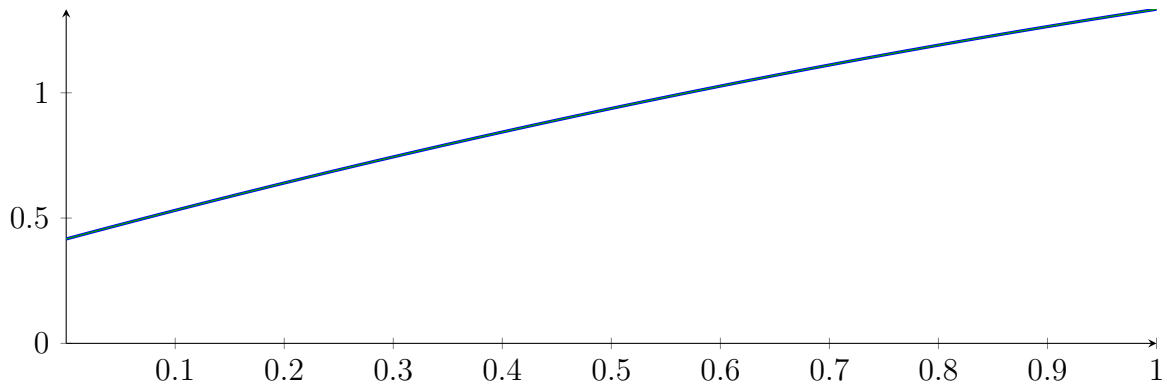
$$M = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$m = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

Root of M and m :

$$N(M) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\} \quad N(m) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\}$$

Intersection intervals:

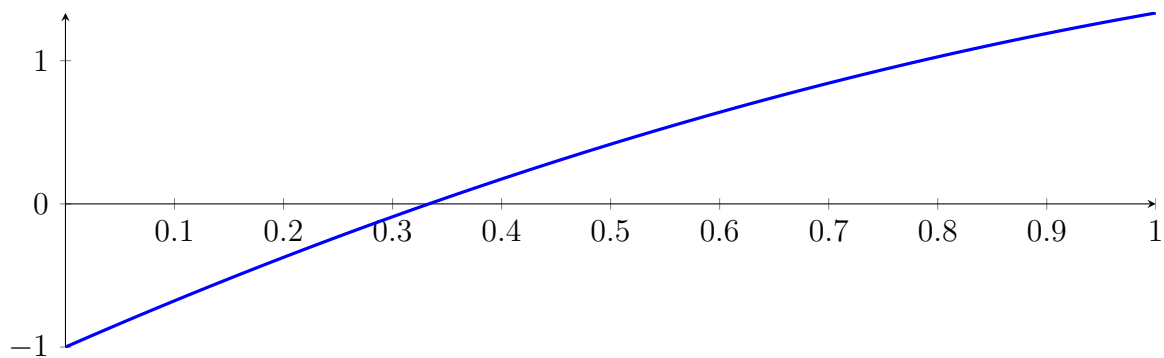


No intersection intervals with the x axis.

96.4 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

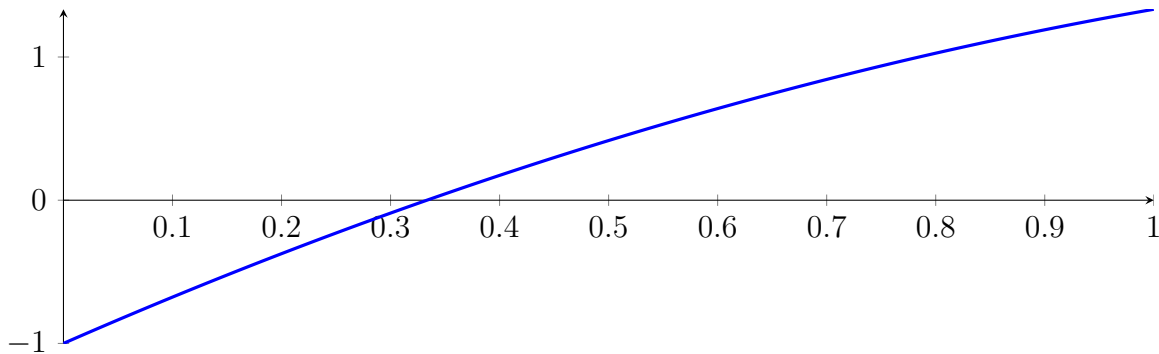
with precision $\varepsilon = 1 \cdot 10^{-16}$.

97 Running BezClip on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

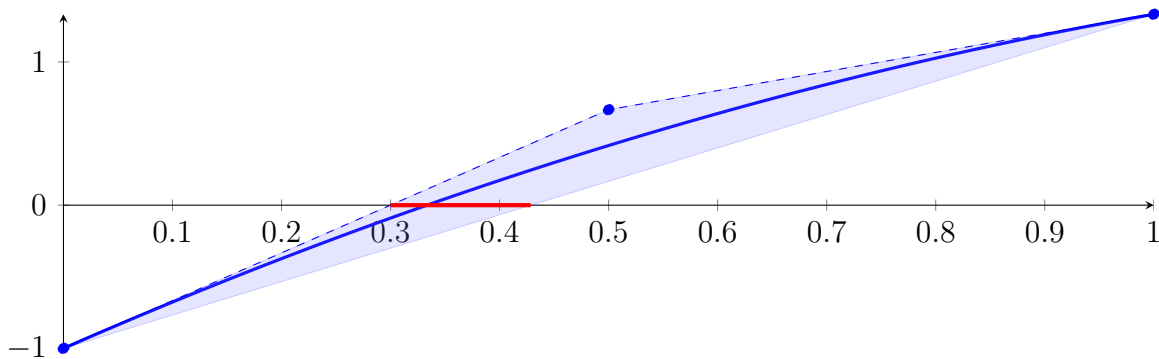
$$p = -1X^2 + 3.33333X - 1$$



97.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

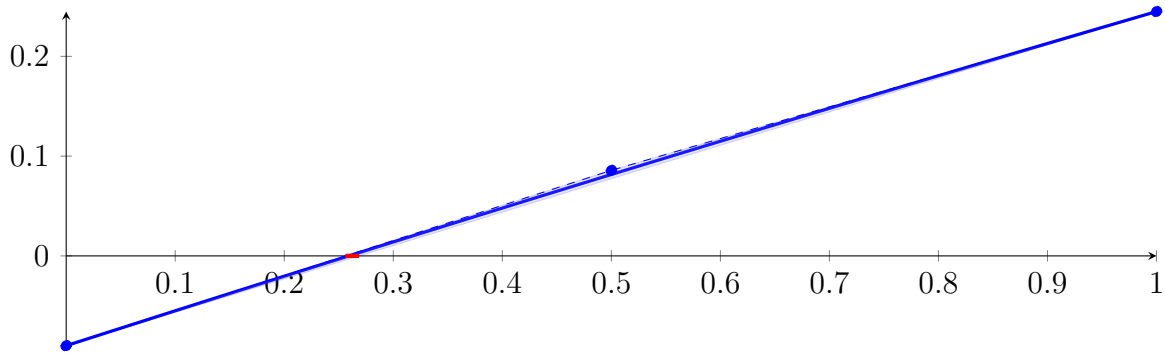
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

97.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

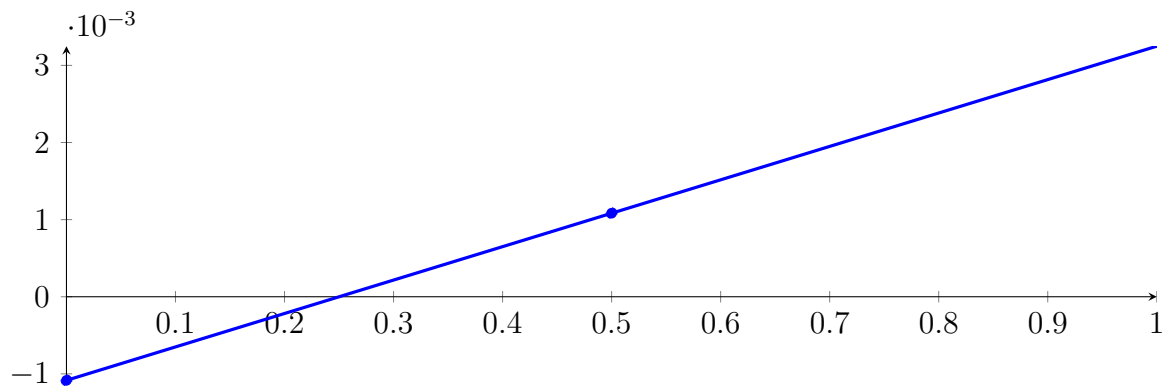
Longest intersection interval: 0.012641

\Rightarrow Selective recursion: interval 1: $[0.332927, 0.334552]$,

97.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

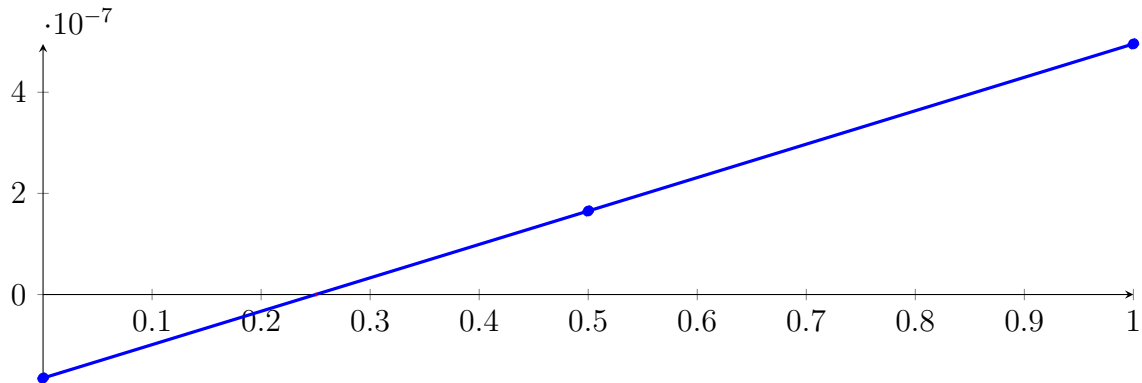
Longest intersection interval: 0.000152462

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333334]$,

97.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07} \\
 &= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

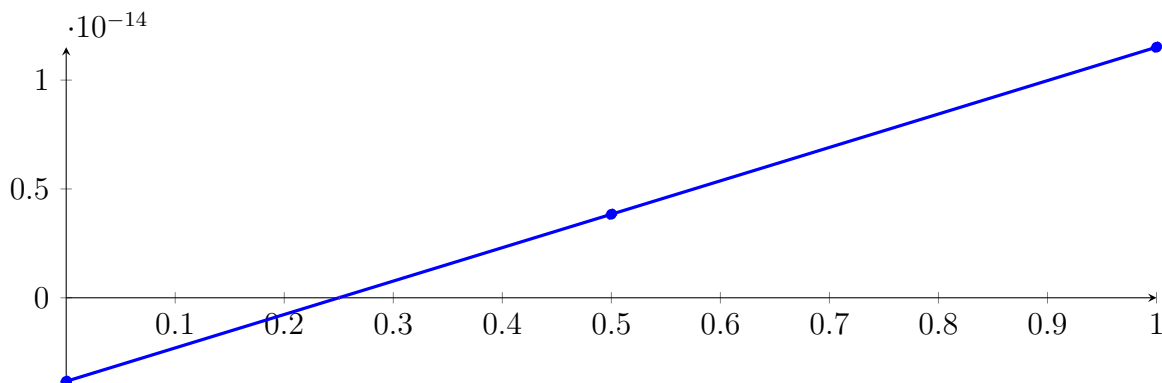
Longest intersection interval: $2.32306 \cdot 10^{-08}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

97.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.31352 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15} \\
 &= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.39635 \cdot 10^{-16}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

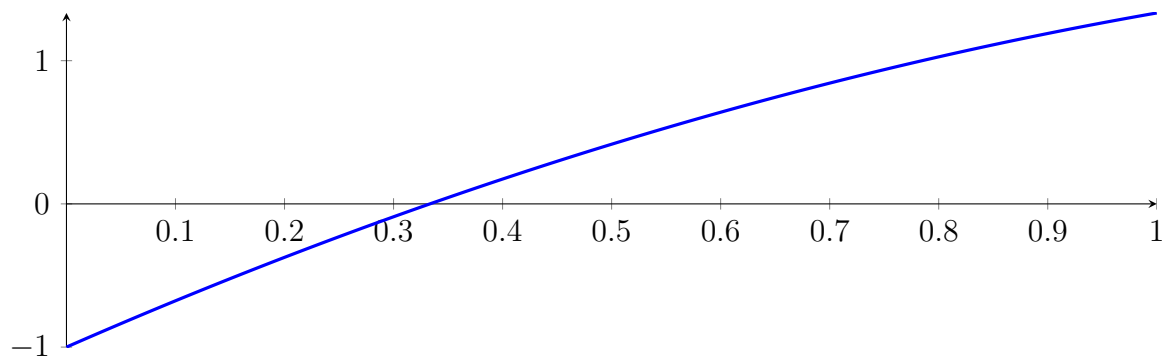
97.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

97.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

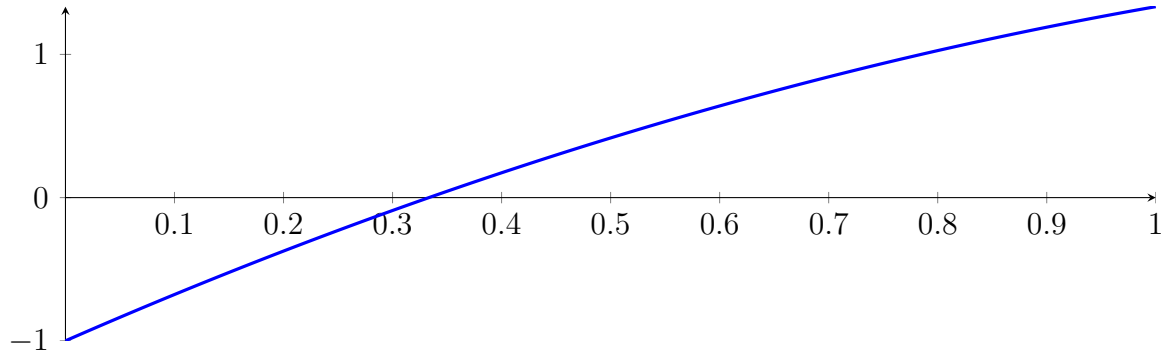
with precision $\varepsilon = 1 \cdot 10^{-32}$.

98 Running QuadClip on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

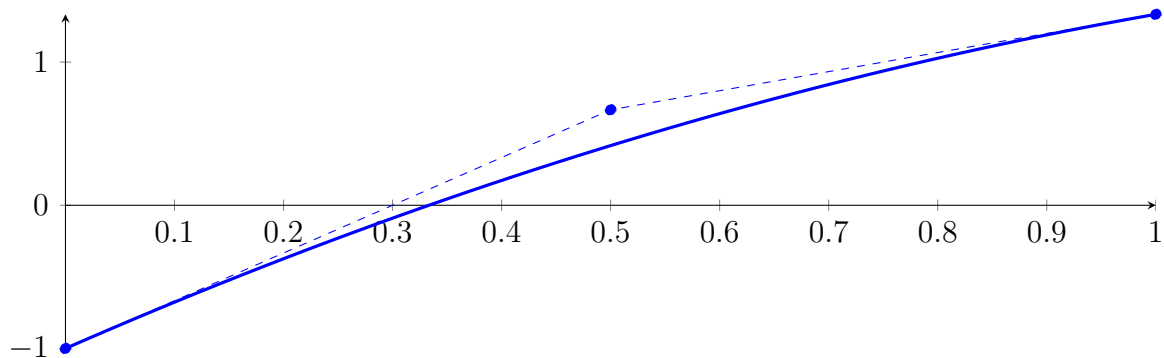
$$p = -1X^2 + 3.33333X - 1$$



98.1 Recursion Branch 1 for Input Interval $[0, 1]$

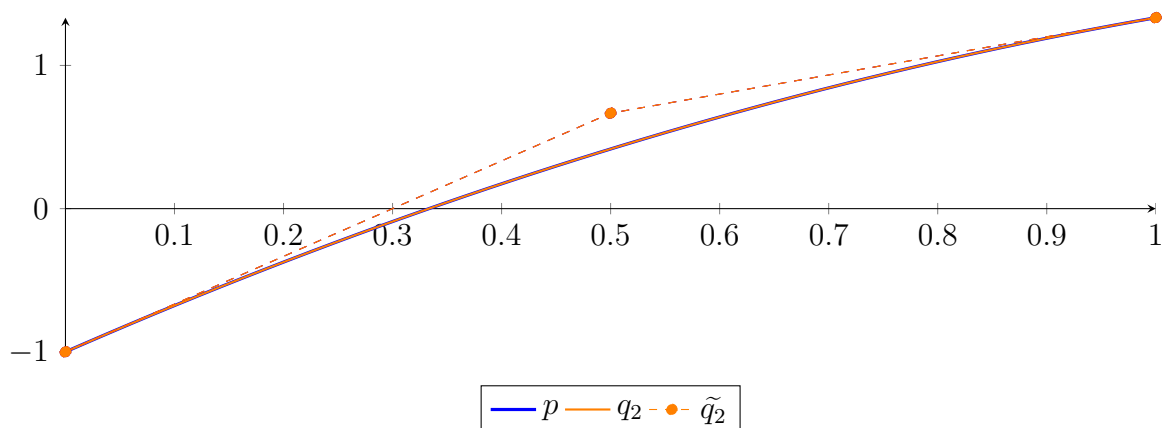
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

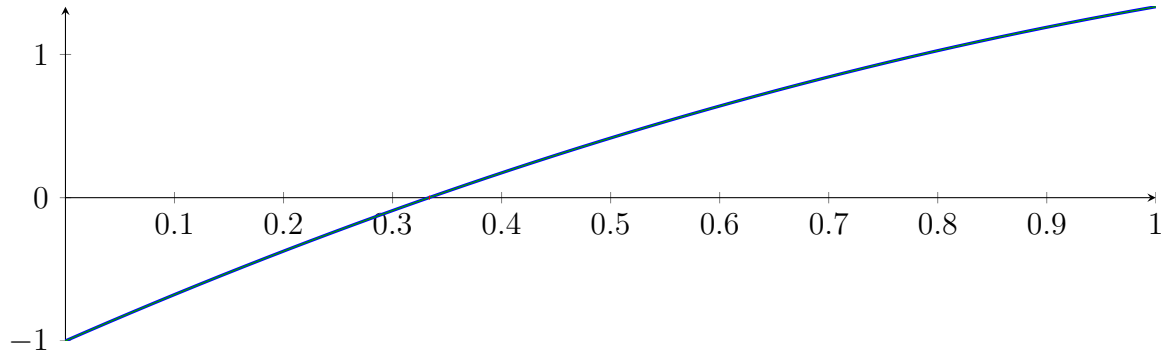
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

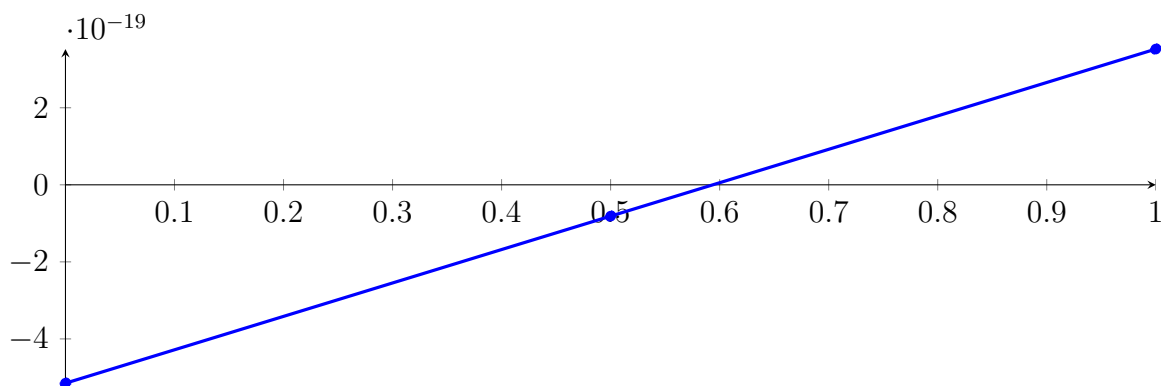
Longest intersection interval: $3.25261 \cdot 10^{-19}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

98.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

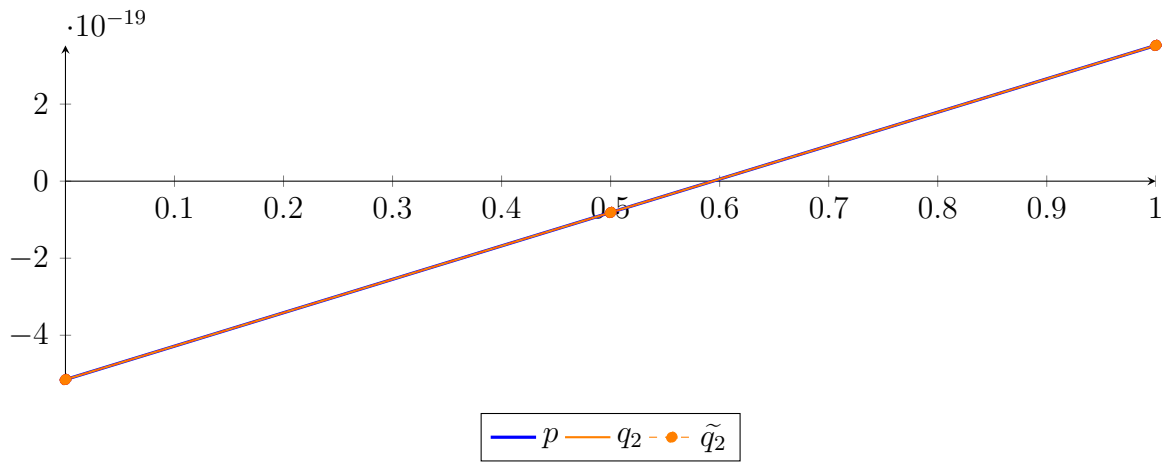
$$\begin{aligned} p &= -9.40395 \cdot 10^{-38} X^2 + 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2}(X) - 8.13152 \cdot 10^{-20} B_{1,2}(X) + 3.52366 \cdot 10^{-19} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2} - 8.13152 \cdot 10^{-20} B_{1,2} + 3.52366 \cdot 10^{-19} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 9.40395 \cdot 10^{-38} X^2 + 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2} - 8.13152 \cdot 10^{-20} B_{1,2} + 3.52366 \cdot 10^{-19} B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.29304 \cdot 10^{-37}$.

Bounding polynomials M and m :

$$M = 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19}$$

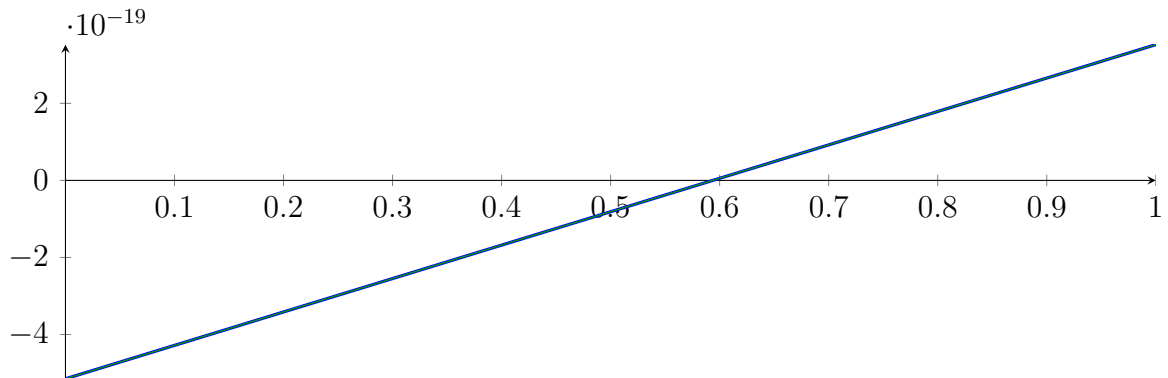
$$m = 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19}$$

Root of M and m :

$$N(M) = \{\}$$

$$N(m) = \{\}$$

Intersection intervals:

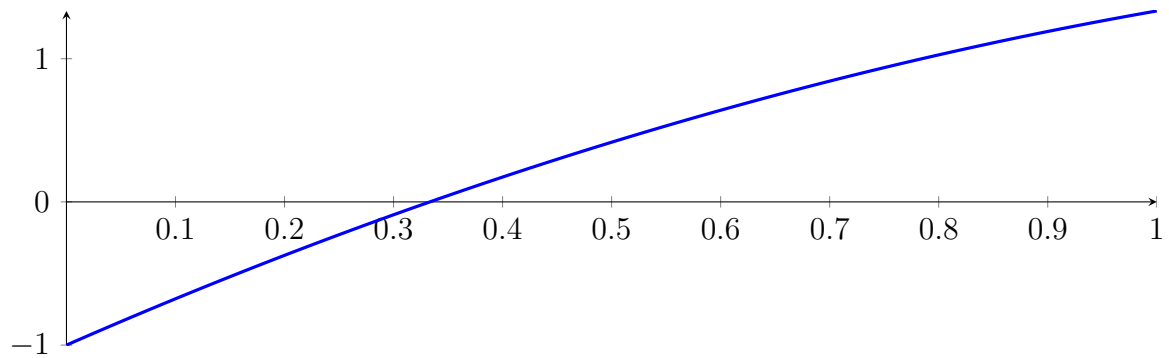


No intersection intervals with the x axis.

98.3 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

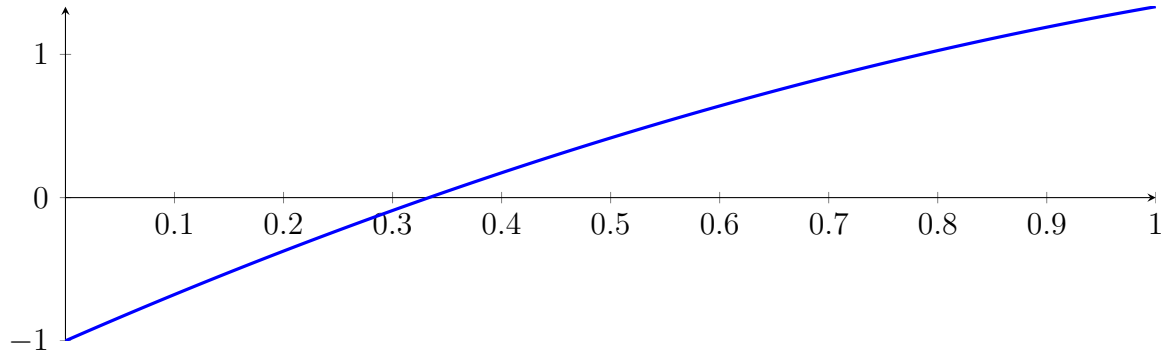
with precision $\varepsilon = 1 \cdot 10^{-32}$.

99 Running CubeClip on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

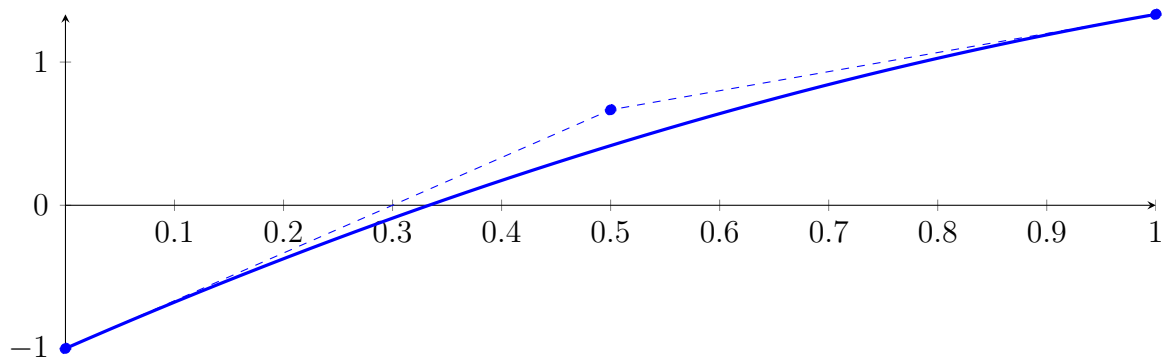
$$p = -1X^2 + 3.33333X - 1$$



99.1 Recursion Branch 1 for Input Interval $[0, 1]$

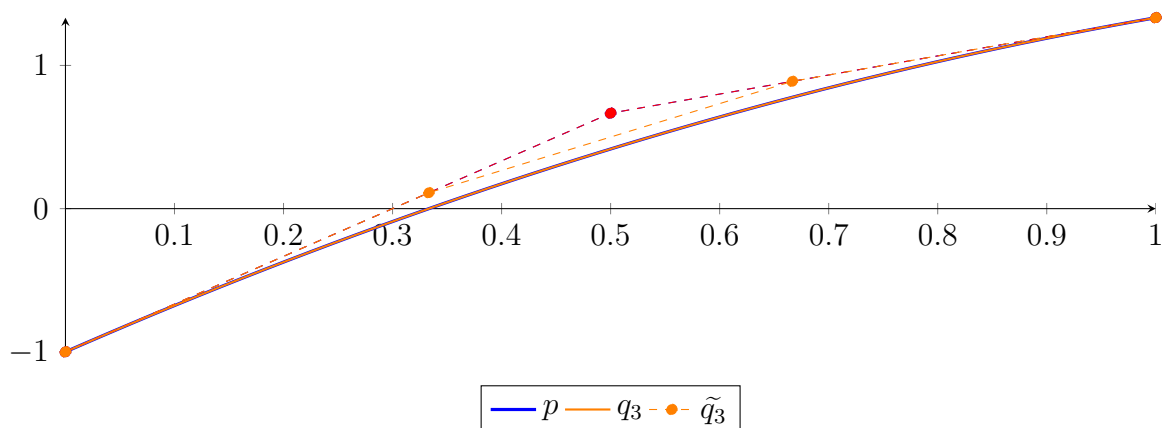
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.63715 \cdot 10^{-17}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

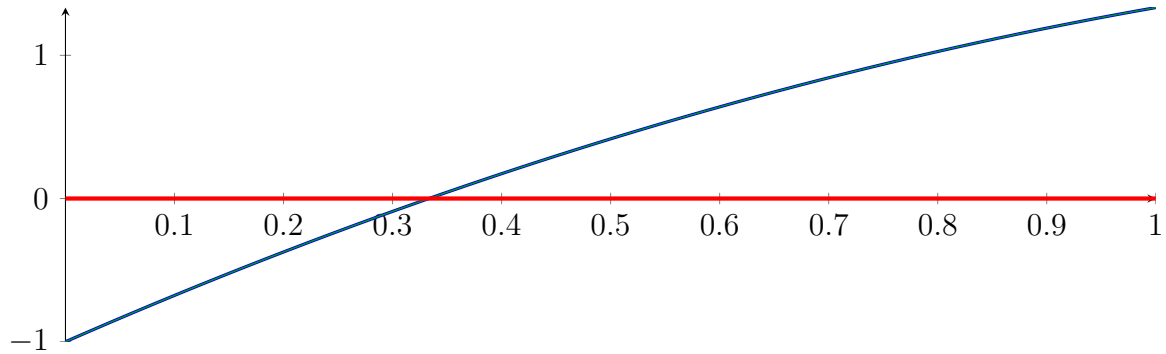
$$m = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

$$N(m) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

Intersection intervals:



$[0, 1]$

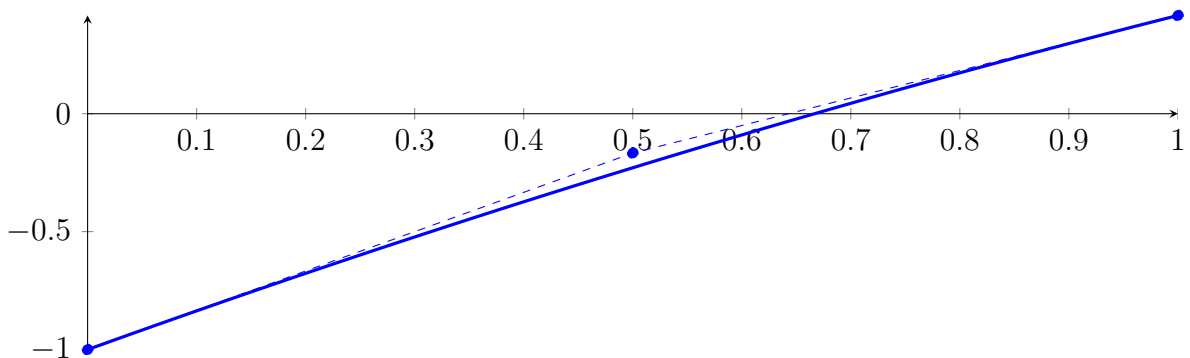
Longest intersection interval: 1

\implies Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

99.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

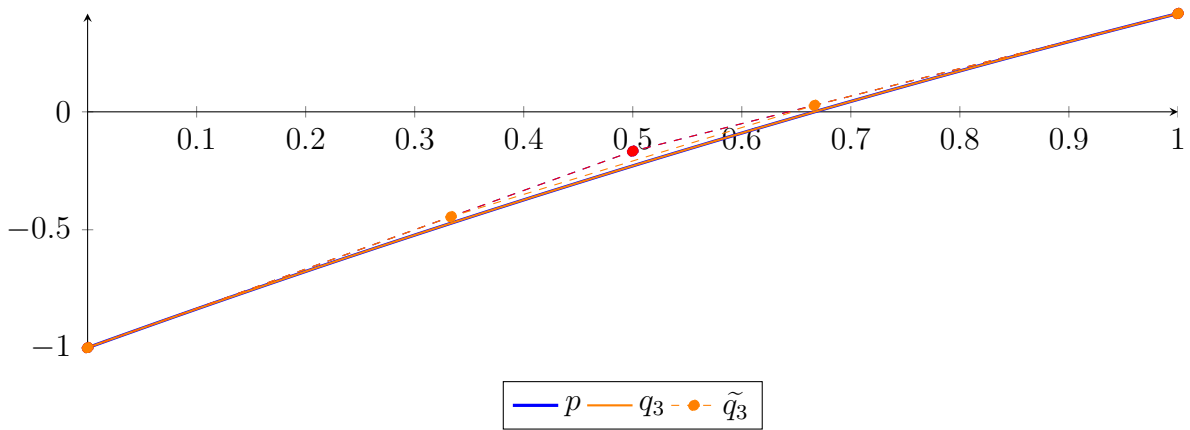
$$\begin{aligned} p &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2}(X) - 0.166667B_{1,2}(X) + 0.416667B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,3} - 0.444444B_{1,3} + 0.0277778B_{2,3} + 0.416667B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2} - 0.166667B_{1,2} + 0.416667B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.58942 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1.07336 \cdot 10^{-17} X^3 - 0.25 X^2 + 1.66667 X - 1$$

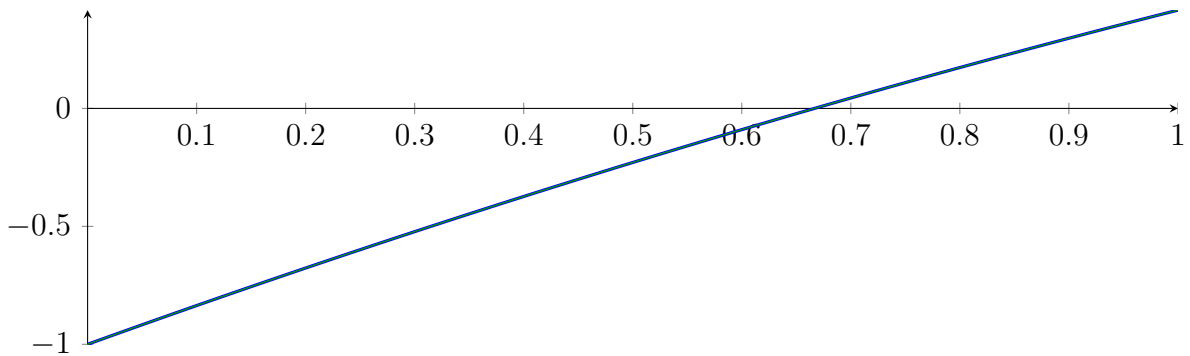
$$m = -1.07336 \cdot 10^{-17} X^3 - 0.25 X^2 + 1.66667 X - 1$$

Root of M and m :

$$N(M) = \{-2.32913 \cdot 10^{16}\}$$

$$N(m) = \{-2.32913 \cdot 10^{16}\}$$

Intersection intervals:

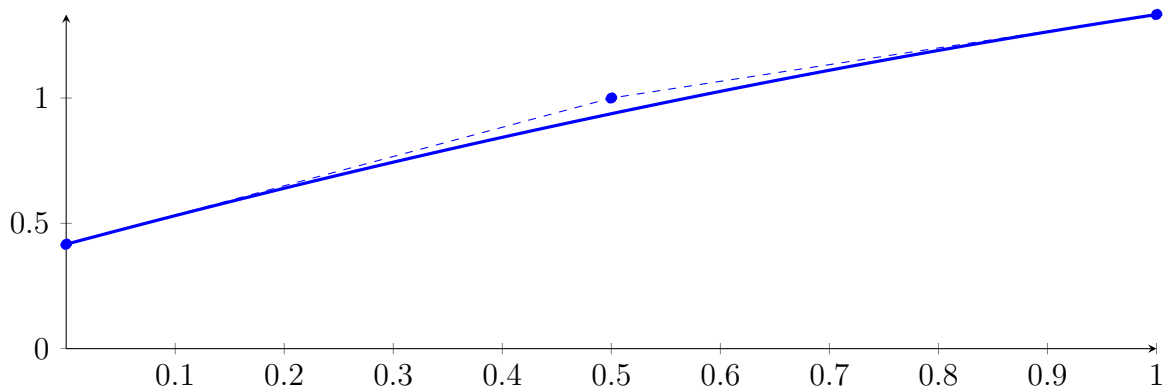


No intersection intervals with the x axis.

99.3 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.25 X^2 + 1.16667 X + 0.416667 \\ &= 0.416667 B_{0,2}(X) + 1 B_{1,2}(X) + 1.333333 B_{2,2}(X) \end{aligned}$$



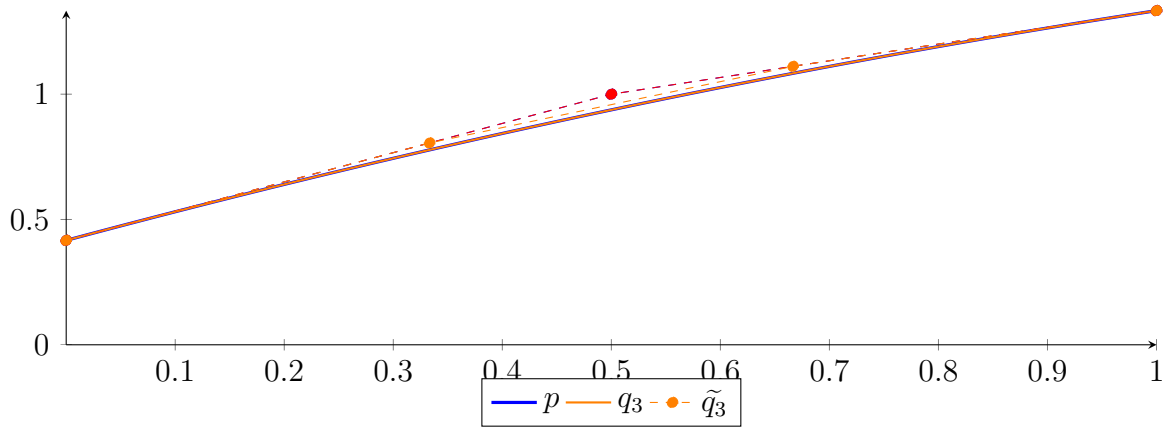
Degree reduction and raising:

$$q_3 = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$= 0.416667B_{0,3} + 0.805556B_{1,3} + 1.111111B_{2,3} + 1.333333B_{3,3}$$

$$\tilde{q}_3 = -0.25X^2 + 1.16667X + 0.416667$$

$$= 0.416667B_{0,2} + 1B_{1,2} + 1.333333B_{2,2}$$



The maximum difference of the Bézier coefficients is $\delta = 1.30104 \cdot 10^{-18}$.

Bounding polynomials M and m :

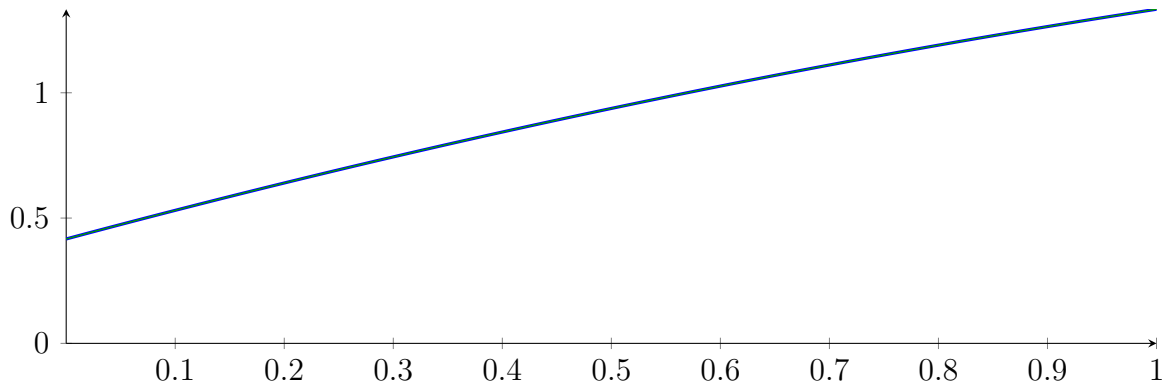
$$M = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$m = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

Root of M and m :

$$N(M) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\} \quad N(m) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\}$$

Intersection intervals:

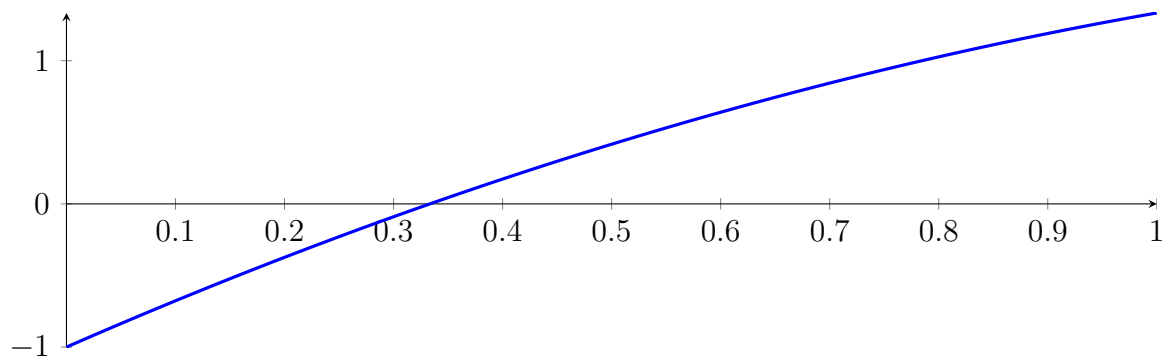


No intersection intervals with the x axis.

99.4 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

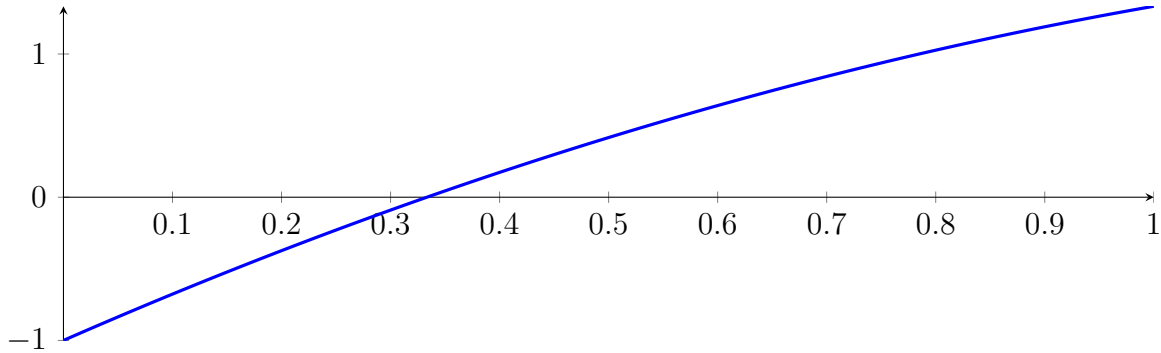
with precision $\varepsilon = 1 \cdot 10^{-32}$.

100 Running BezClip on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

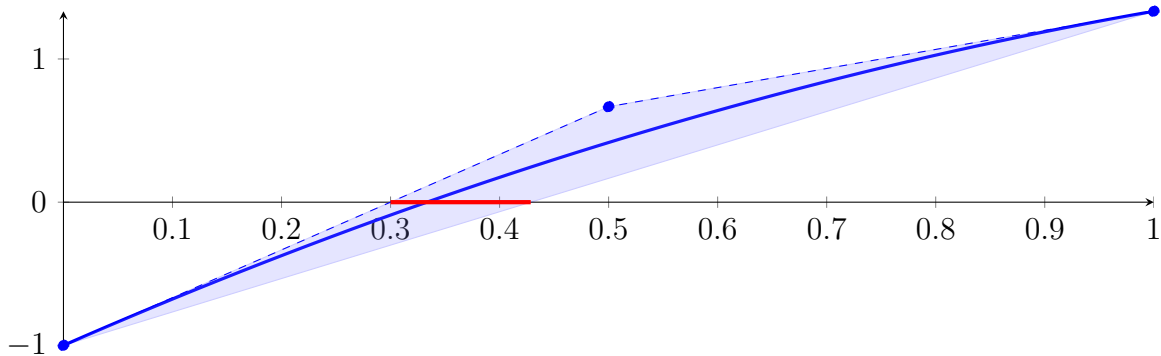
$$p = -1X^2 + 3.33333X - 1$$



100.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

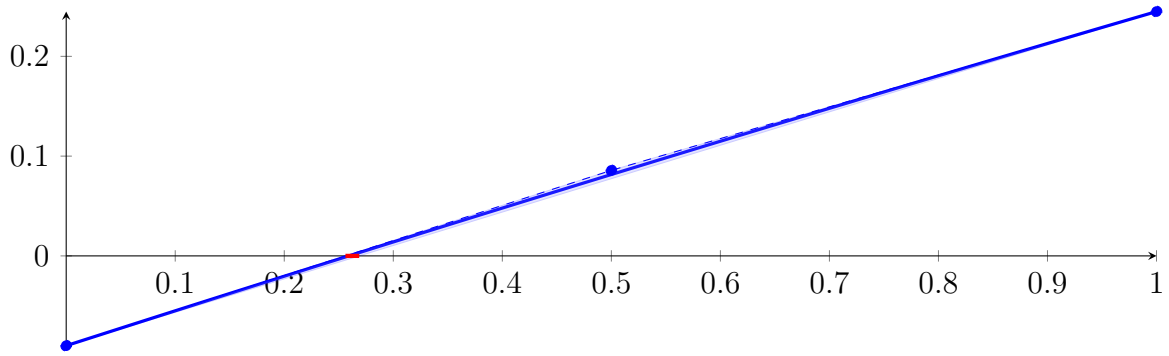
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

100.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

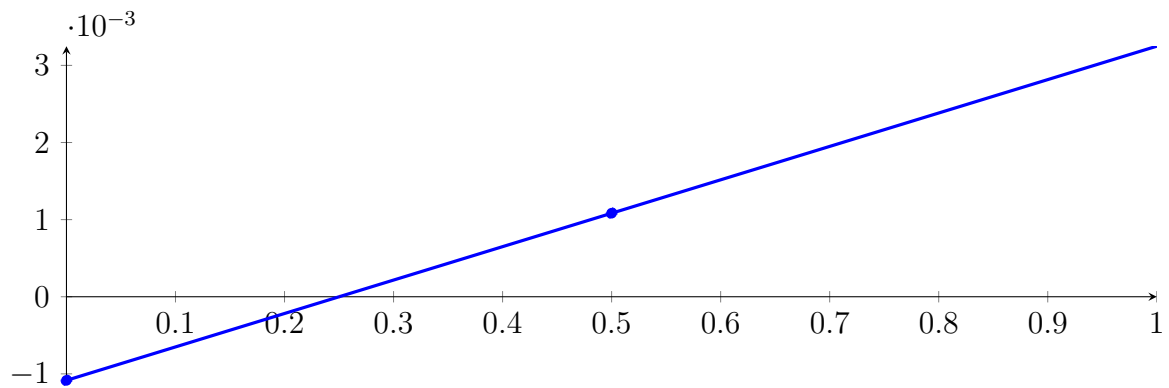
Longest intersection interval: 0.012641

\Rightarrow Selective recursion: interval 1: $[0.332927, 0.334552]$,

100.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

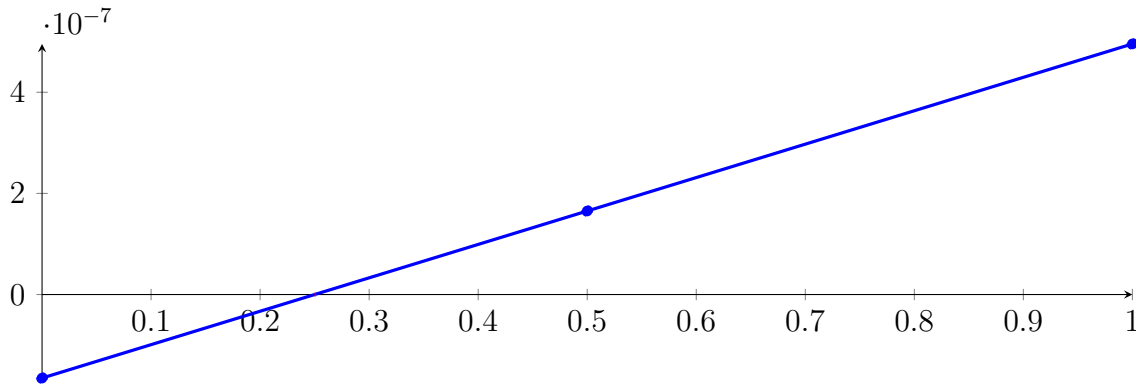
Longest intersection interval: 0.000152462

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333334]$,

100.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07} \\
 &= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

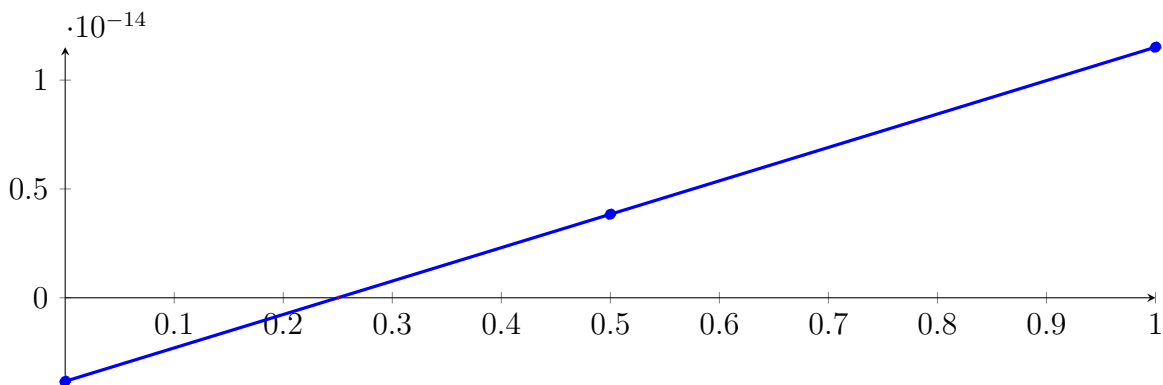
Longest intersection interval: $2.32306 \cdot 10^{-08}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

100.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.31352 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15} \\
 &= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.39635 \cdot 10^{-16}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

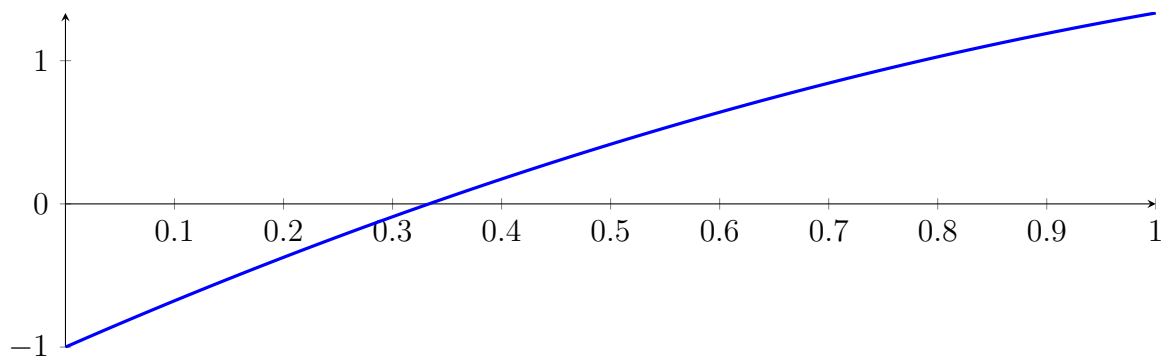
100.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

100.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

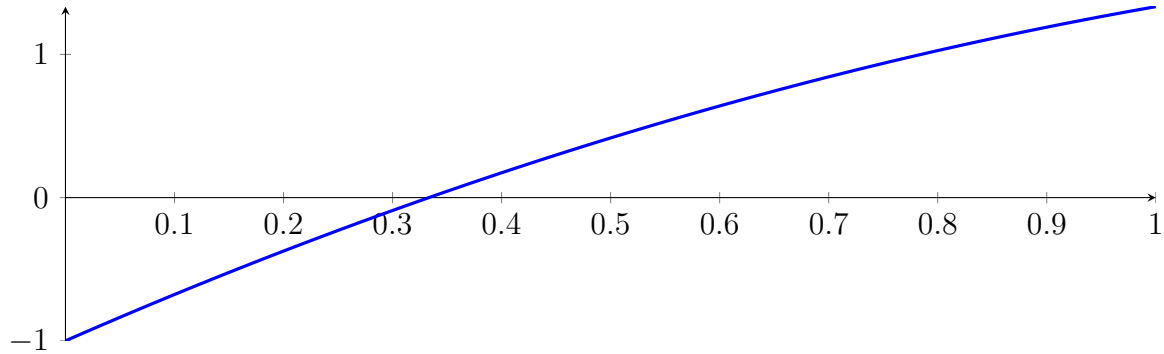
with precision $\varepsilon = 1 \cdot 10^{-64}$.

101 Running QuadClip on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

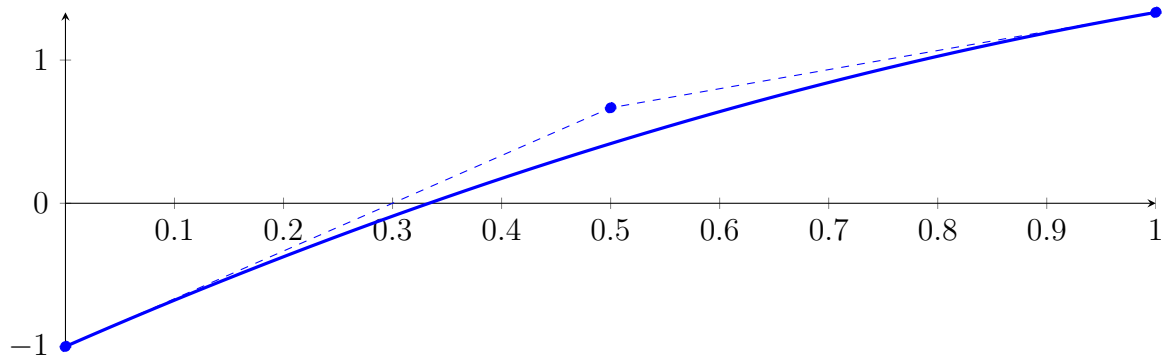
$$p = -1X^2 + 3.33333X - 1$$



101.1 Recursion Branch 1 for Input Interval $[0, 1]$

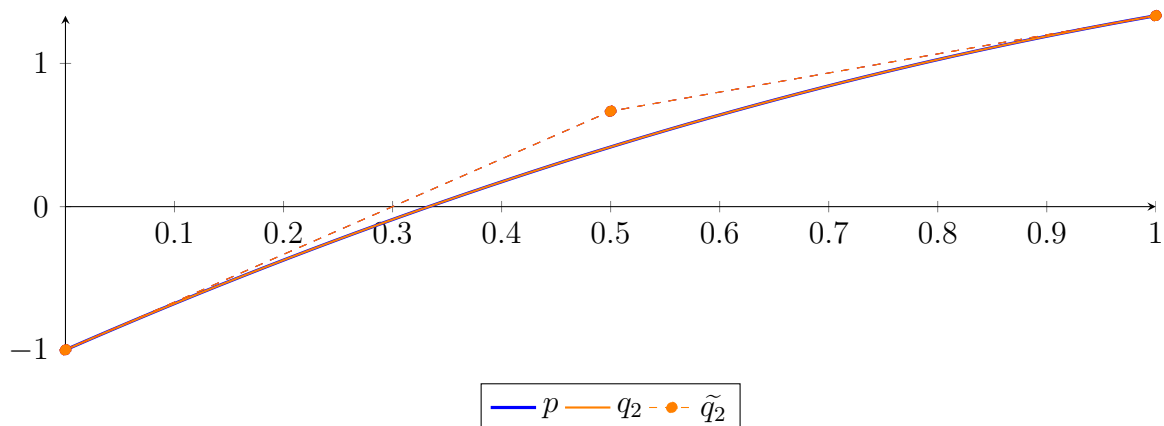
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

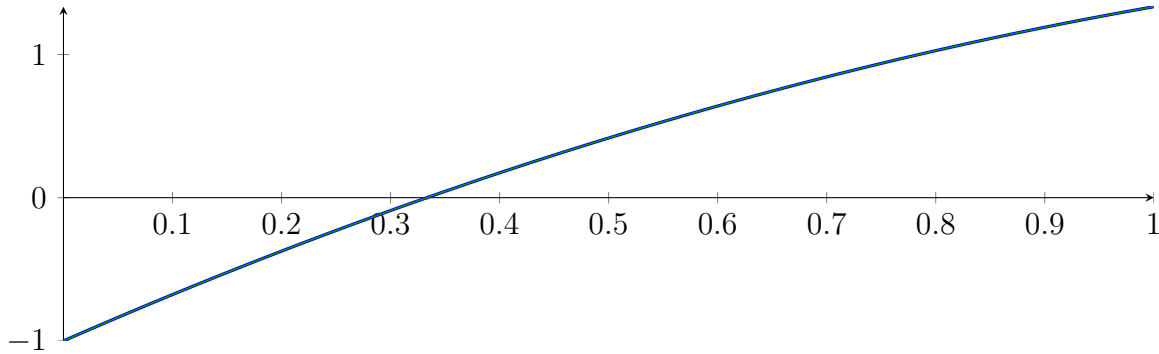
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

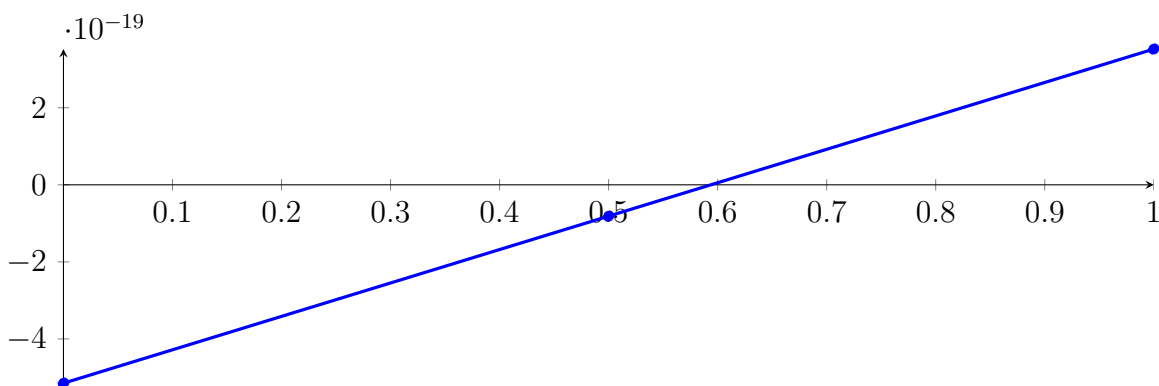
Longest intersection interval: $3.25261 \cdot 10^{-19}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

101.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

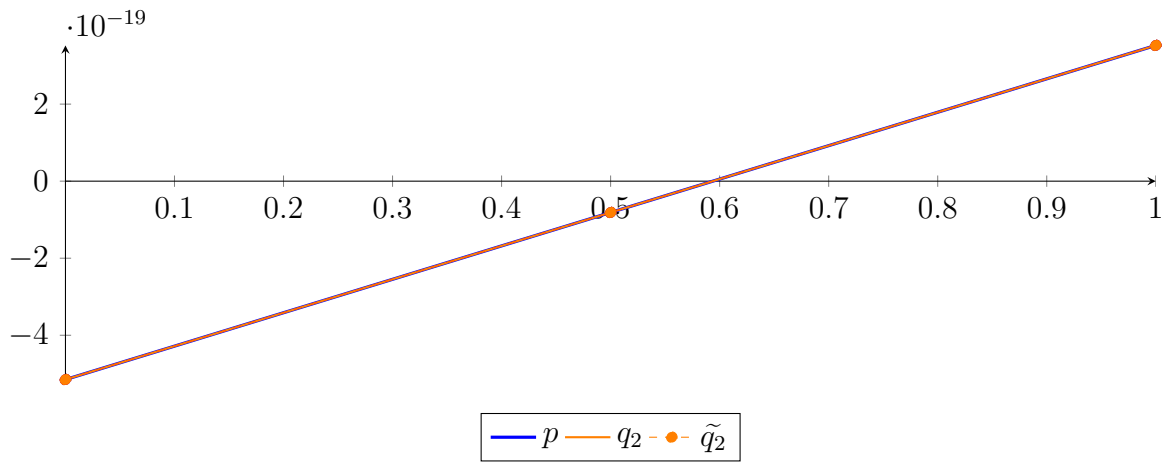
$$\begin{aligned} p &= -9.40395 \cdot 10^{-38} X^2 + 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2}(X) - 8.13152 \cdot 10^{-20} B_{1,2}(X) + 3.52366 \cdot 10^{-19} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2} - 8.13152 \cdot 10^{-20} B_{1,2} + 3.52366 \cdot 10^{-19} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 9.40395 \cdot 10^{-38} X^2 + 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2} - 8.13152 \cdot 10^{-20} B_{1,2} + 3.52366 \cdot 10^{-19} B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.29304 \cdot 10^{-37}$.

Bounding polynomials M and m :

$$M = 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19}$$

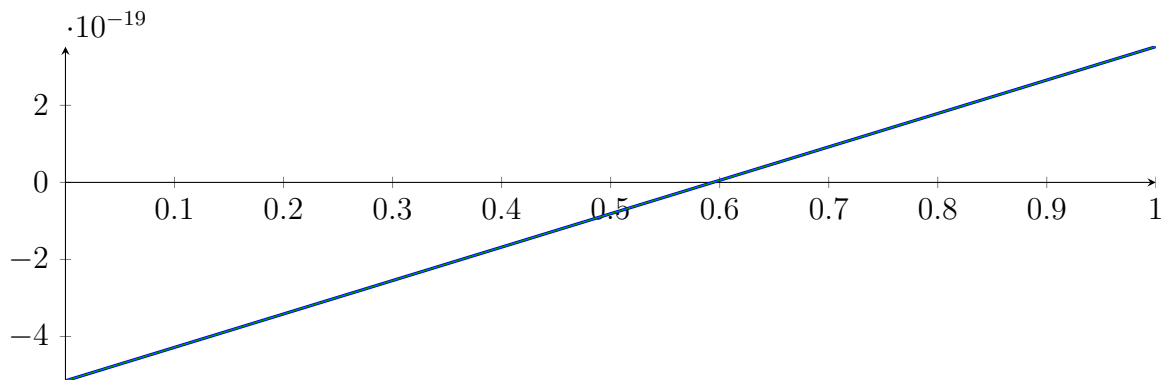
$$m = 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19}$$

Root of M and m :

$$N(M) = \{ \}$$

$$N(m) = \{ \}$$

Intersection intervals:

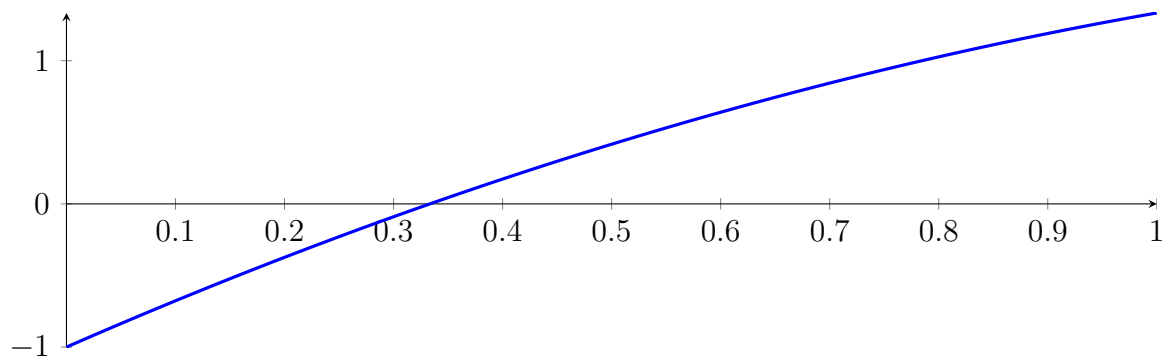


No intersection intervals with the x axis.

101.3 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

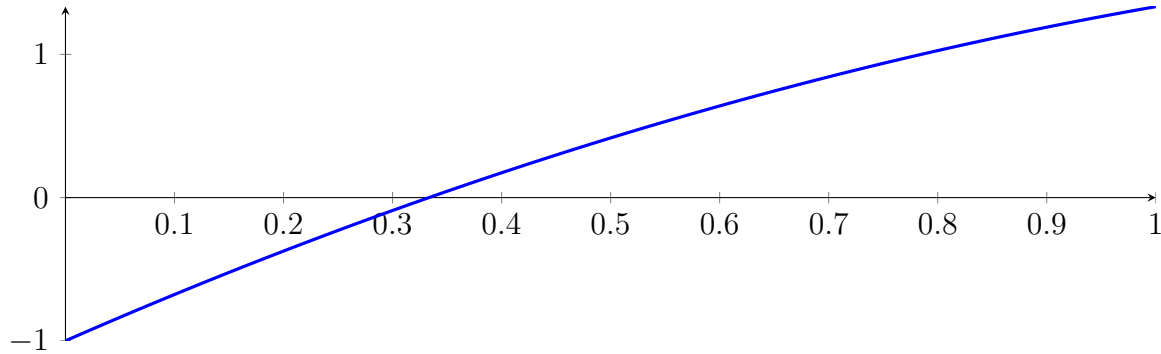
with precision $\varepsilon = 1 \cdot 10^{-64}$.

102 Running CubeClip on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

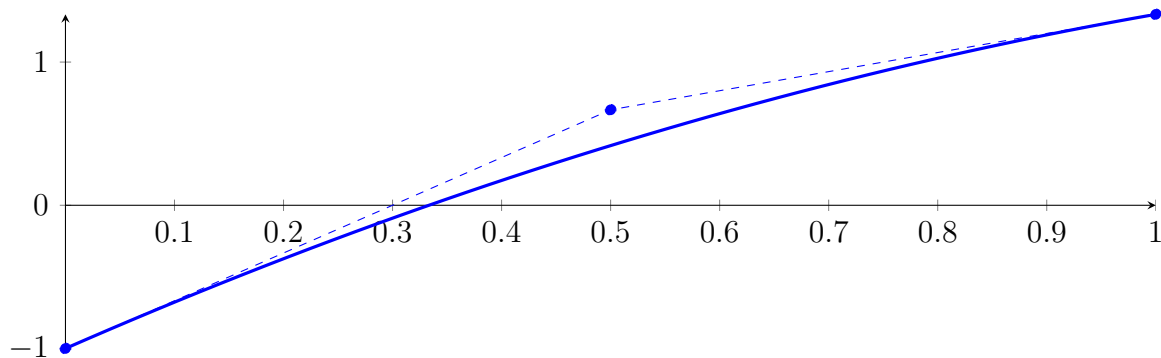
$$p = -1X^2 + 3.33333X - 1$$



102.1 Recursion Branch 1 for Input Interval $[0, 1]$

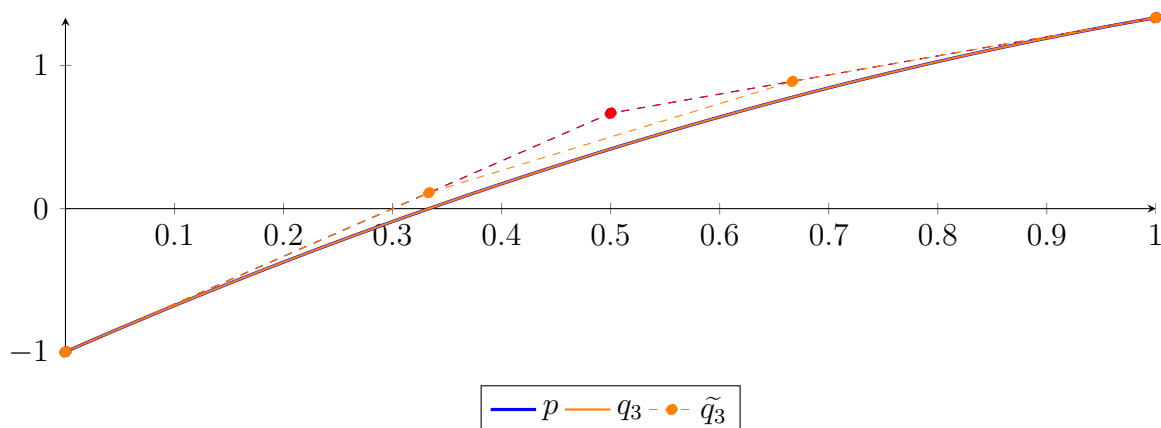
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.63715 \cdot 10^{-17}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

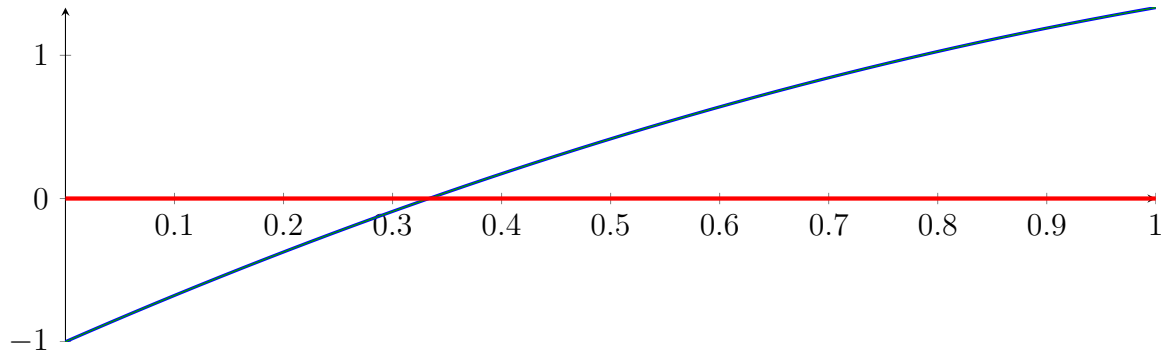
$$m = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

$$N(m) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

Intersection intervals:



[0, 1]

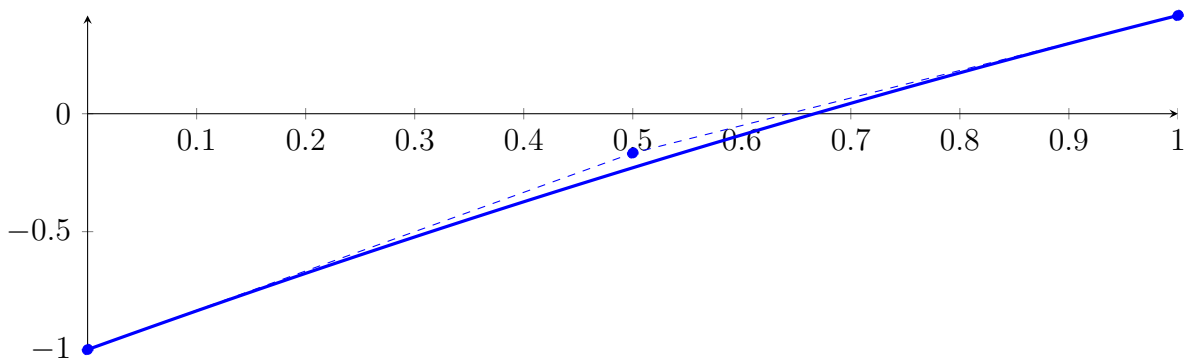
Longest intersection interval: 1

\implies Bisection: first half [0, 0.5] und second half [0.5, 1]

102.2 Recursion Branch 1 1 on the First Half [0, 0.5]

Normalized monomial und Bézier representations and the Bézier polygon:

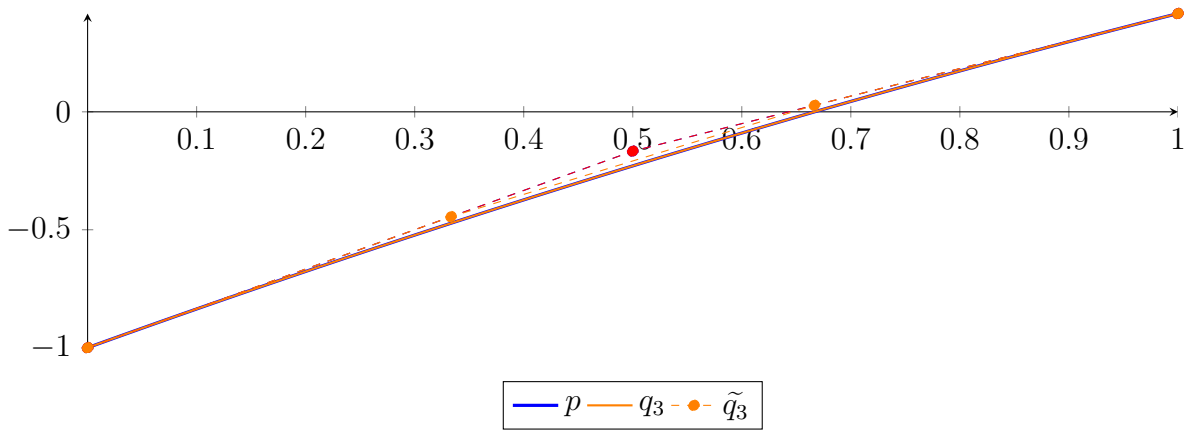
$$\begin{aligned} p &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2}(X) - 0.166667B_{1,2}(X) + 0.416667B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,3} - 0.444444B_{1,3} + 0.0277778B_{2,3} + 0.416667B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2} - 0.166667B_{1,2} + 0.416667B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.58942 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1.07336 \cdot 10^{-17} X^3 - 0.25 X^2 + 1.66667 X - 1$$

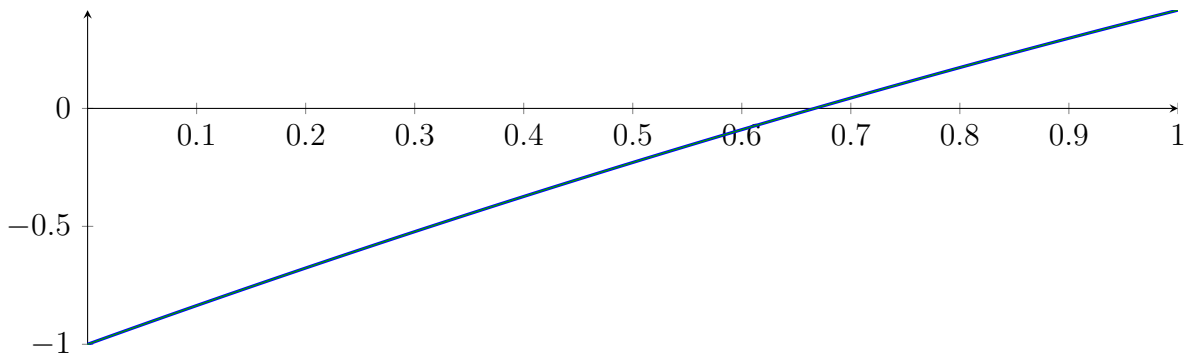
$$m = -1.07336 \cdot 10^{-17} X^3 - 0.25 X^2 + 1.66667 X - 1$$

Root of M and m :

$$N(M) = \{-2.32913 \cdot 10^{16}\}$$

$$N(m) = \{-2.32913 \cdot 10^{16}\}$$

Intersection intervals:

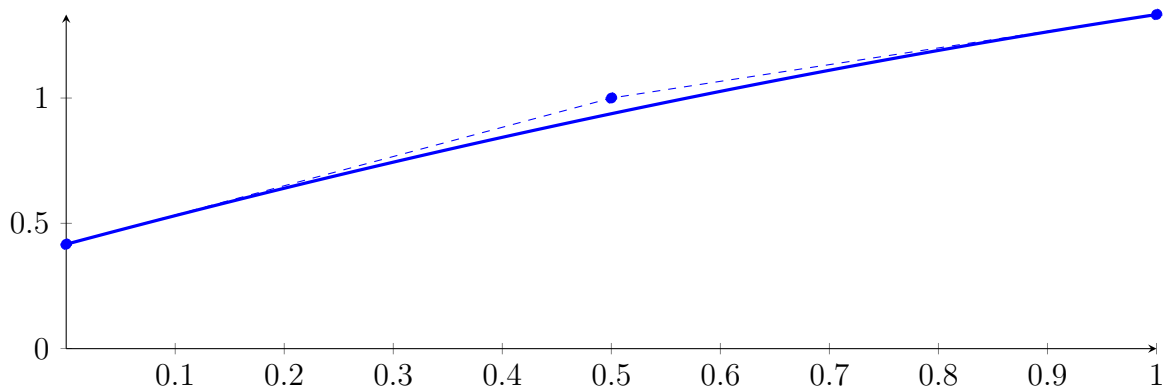


No intersection intervals with the x axis.

102.3 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.25 X^2 + 1.16667 X + 0.416667 \\ &= 0.416667 B_{0,2}(X) + 1 B_{1,2}(X) + 1.333333 B_{2,2}(X) \end{aligned}$$



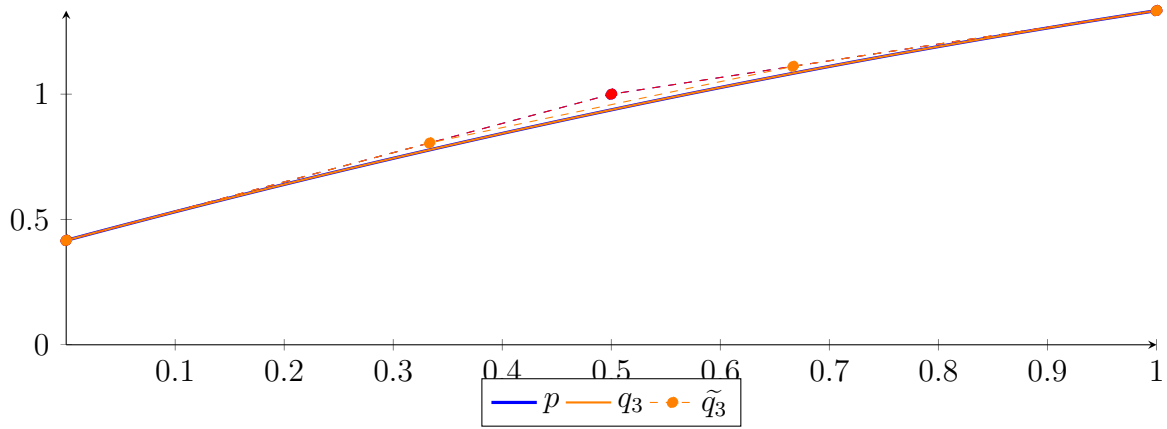
Degree reduction and raising:

$$q_3 = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$= 0.416667B_{0,3} + 0.805556B_{1,3} + 1.111111B_{2,3} + 1.333333B_{3,3}$$

$$\tilde{q}_3 = -0.25X^2 + 1.16667X + 0.416667$$

$$= 0.416667B_{0,2} + 1B_{1,2} + 1.333333B_{2,2}$$



The maximum difference of the Bézier coefficients is $\delta = 1.30104 \cdot 10^{-18}$.

Bounding polynomials M and m :

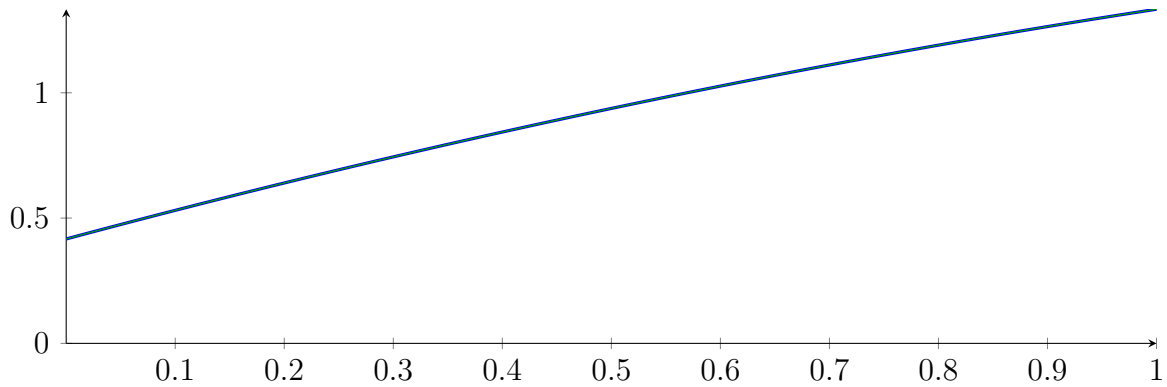
$$M = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$m = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

Root of M and m :

$$N(M) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\} \quad N(m) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\}$$

Intersection intervals:

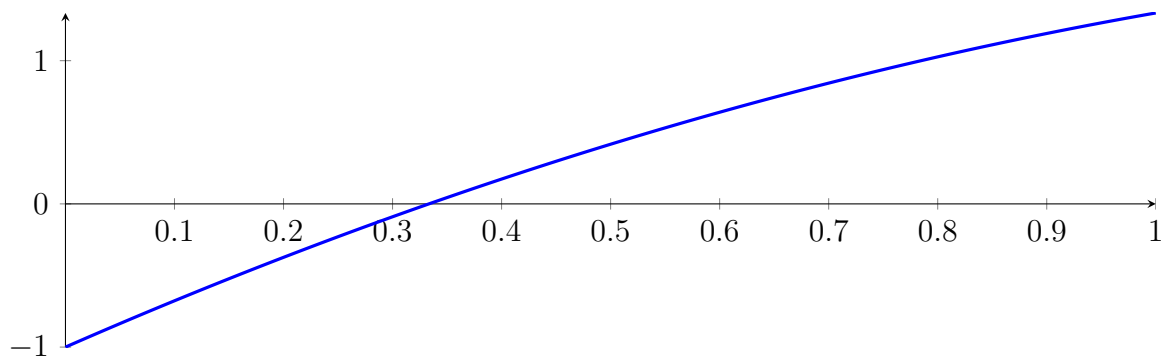


No intersection intervals with the x axis.

102.4 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

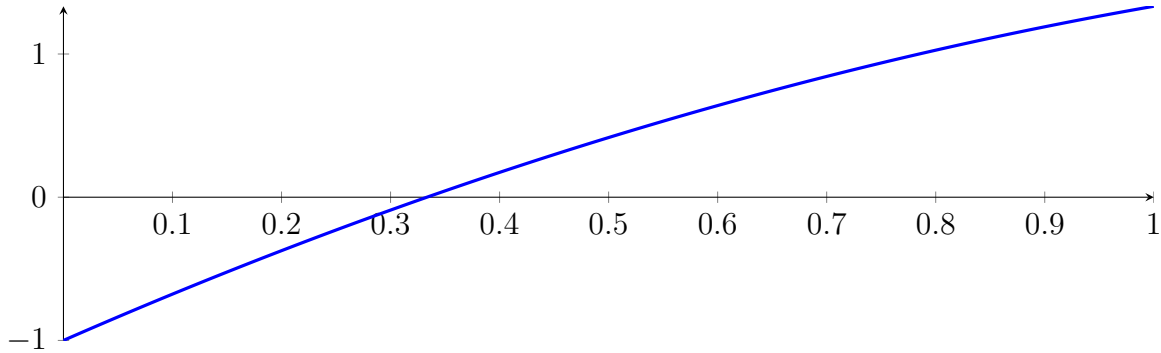
with precision $\varepsilon = 1 \cdot 10^{-64}$.

103 Running BezClip on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

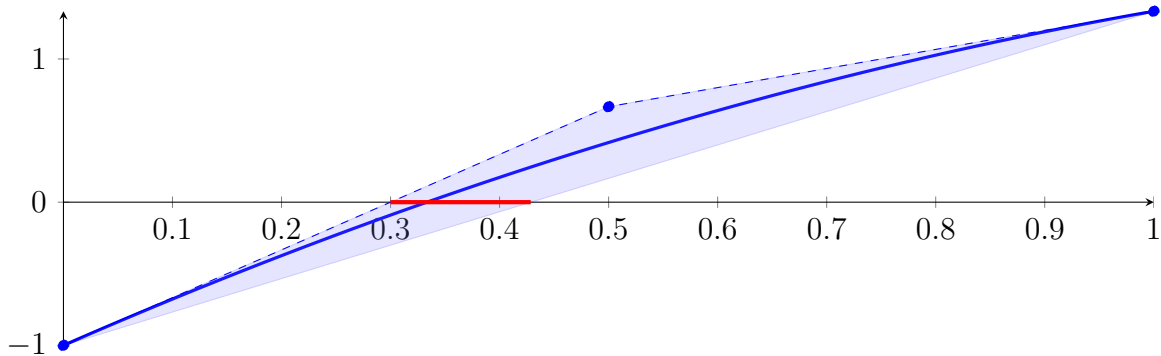
$$p = -1X^2 + 3.33333X - 1$$



103.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

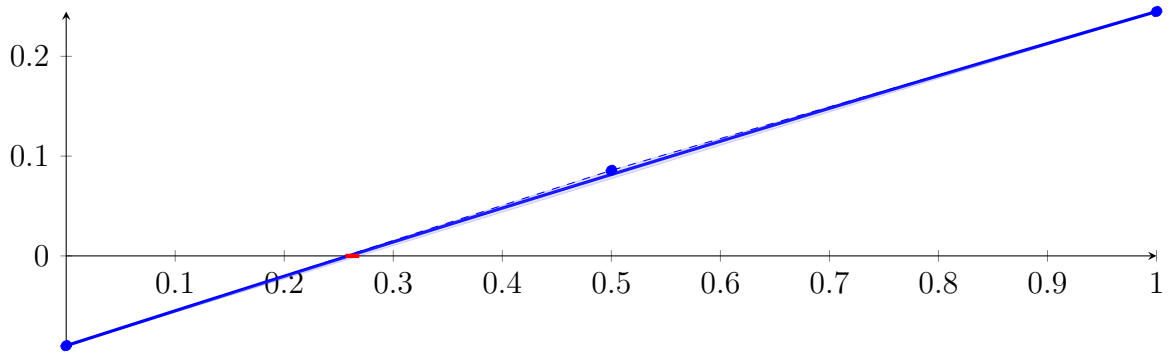
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

103.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

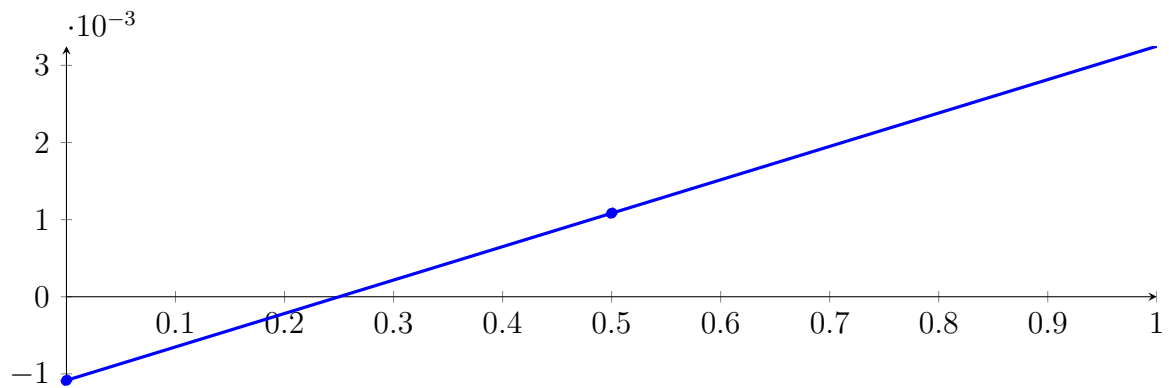
Longest intersection interval: 0.012641

\Rightarrow Selective recursion: interval 1: $[0.332927, 0.334552]$,

103.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

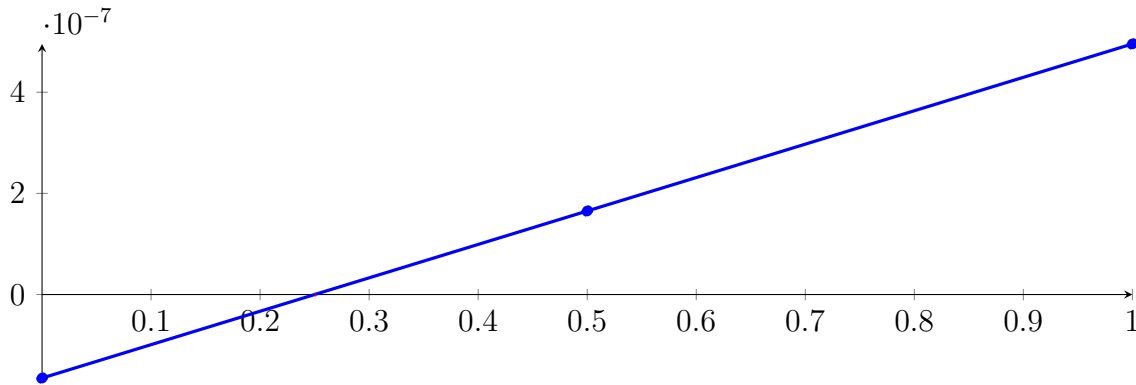
Longest intersection interval: 0.000152462

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333334]$,

103.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07} \\
 &= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

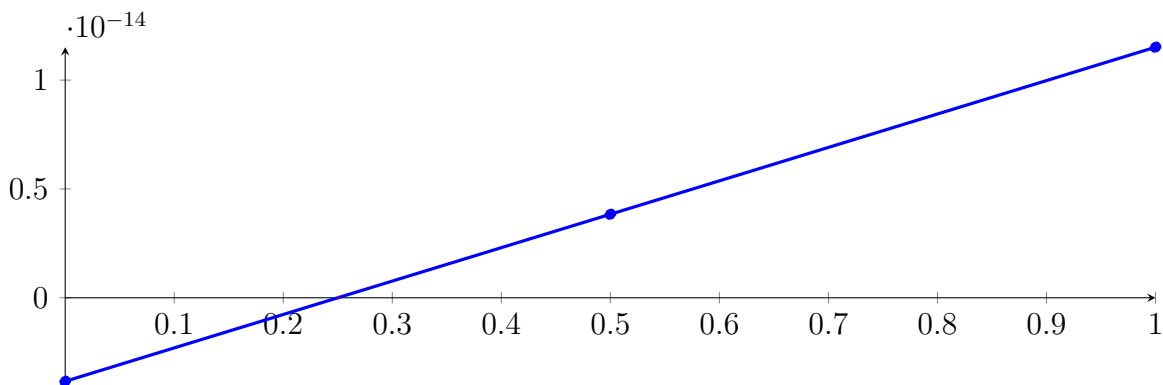
Longest intersection interval: $2.32306 \cdot 10^{-08}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

103.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.31352 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15} \\
 &= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $5.39635 \cdot 10^{-16}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

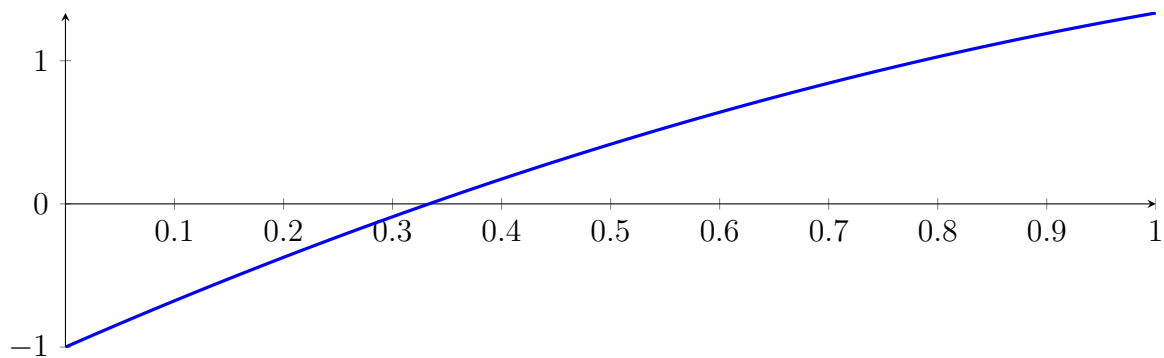
103.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

103.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

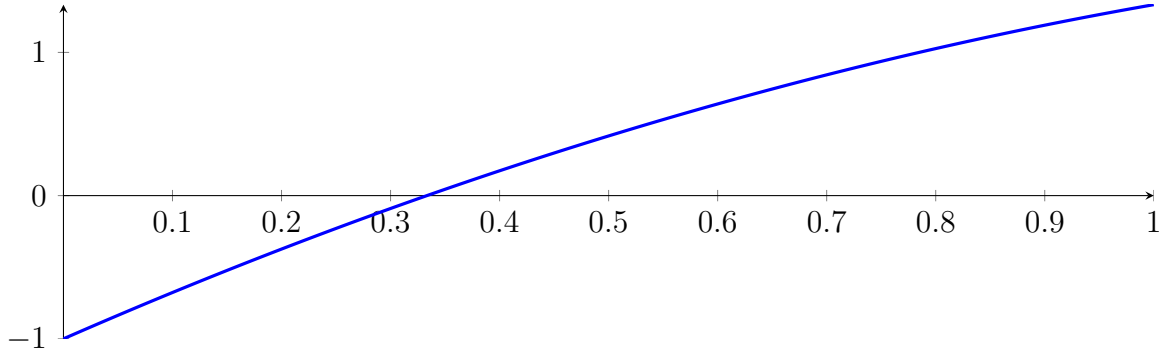
with precision $\varepsilon = 1 \cdot 10^{-128}$.

104 Running QuadClip on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

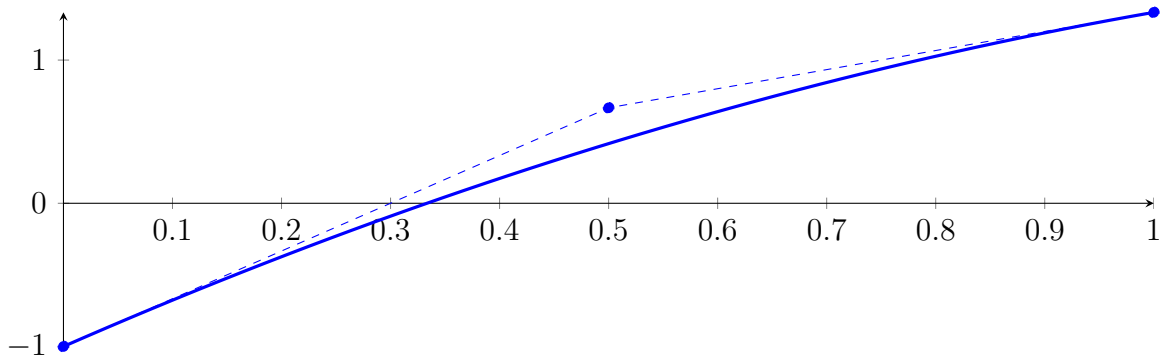
$$p = -1X^2 + 3.33333X - 1$$



104.1 Recursion Branch 1 for Input Interval $[0, 1]$

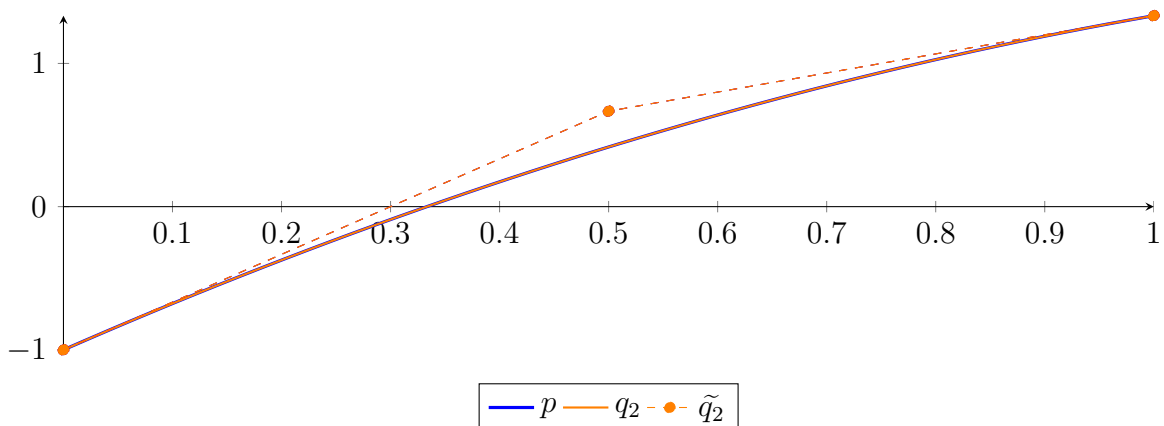
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

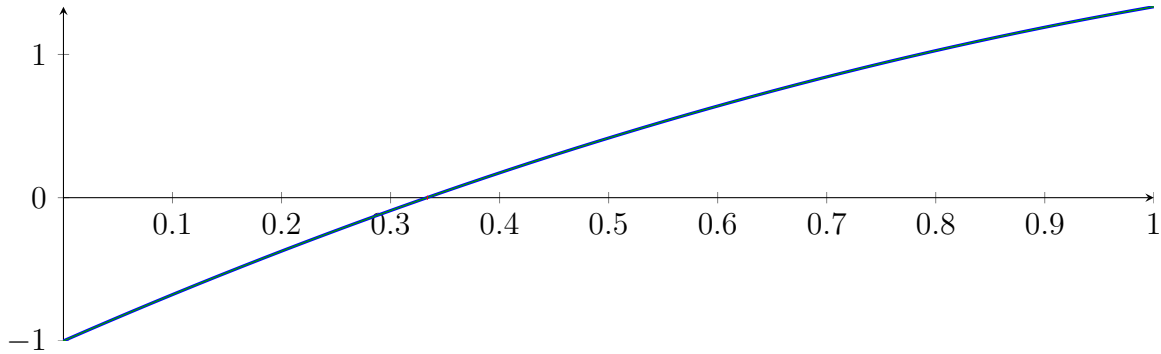
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

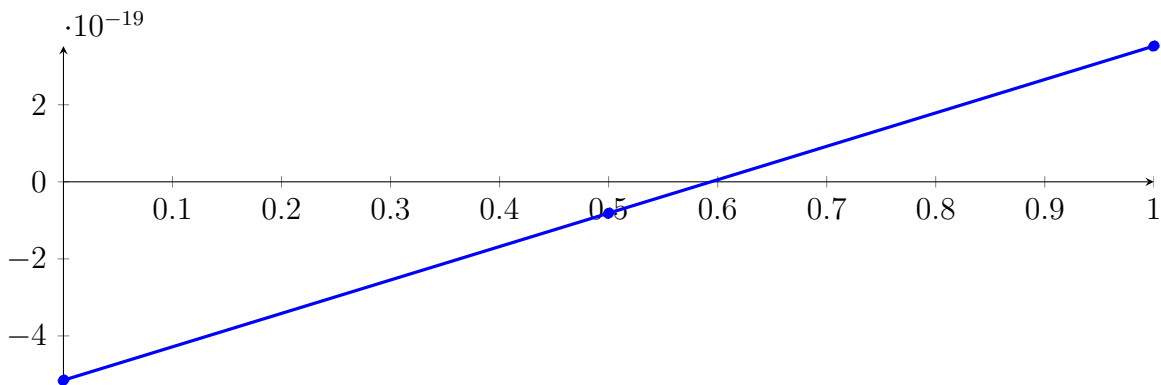
Longest intersection interval: $3.25261 \cdot 10^{-19}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

104.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

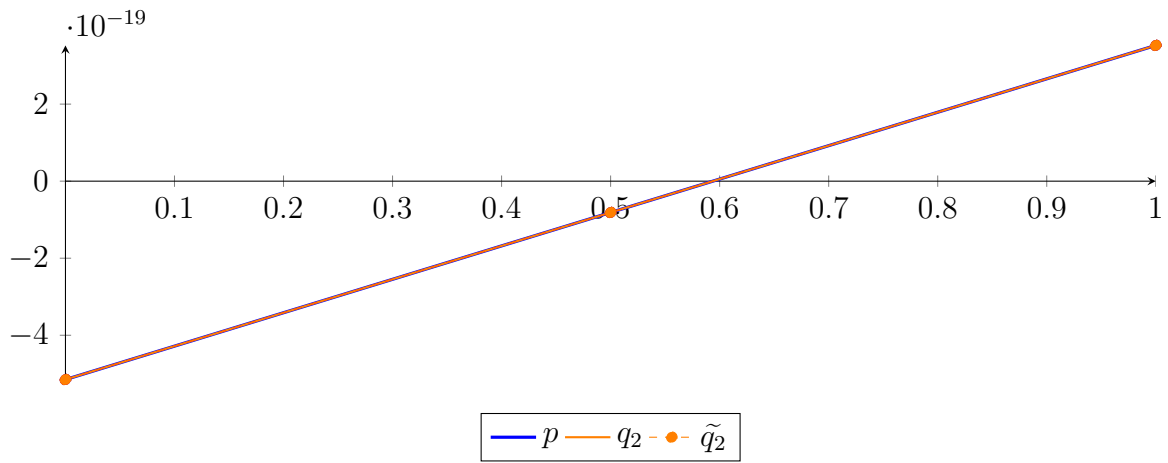
$$\begin{aligned} p &= -9.40395 \cdot 10^{-38} X^2 + 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2}(X) - 8.13152 \cdot 10^{-20} B_{1,2}(X) + 3.52366 \cdot 10^{-19} B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2} - 8.13152 \cdot 10^{-20} B_{1,2} + 3.52366 \cdot 10^{-19} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 9.40395 \cdot 10^{-38} X^2 + 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19} \\ &= -5.14996 \cdot 10^{-19} B_{0,2} - 8.13152 \cdot 10^{-20} B_{1,2} + 3.52366 \cdot 10^{-19} B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.29304 \cdot 10^{-37}$.

Bounding polynomials M and m :

$$M = 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19}$$

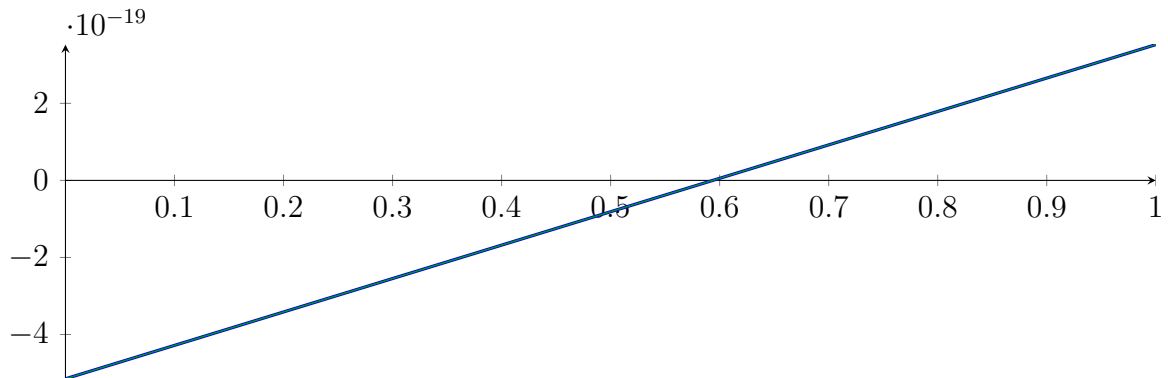
$$m = 8.67362 \cdot 10^{-19} X - 5.14996 \cdot 10^{-19}$$

Root of M and m :

$$N(M) = \{ \}$$

$$N(m) = \{ \}$$

Intersection intervals:

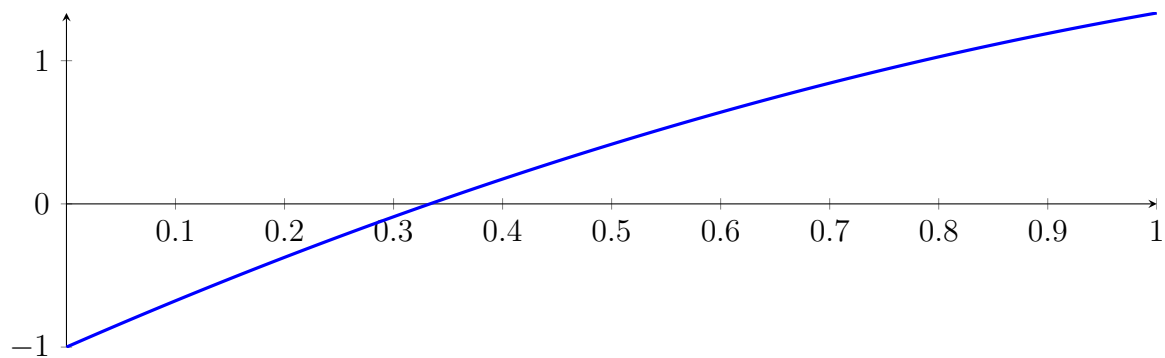


No intersection intervals with the x axis.

104.3 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

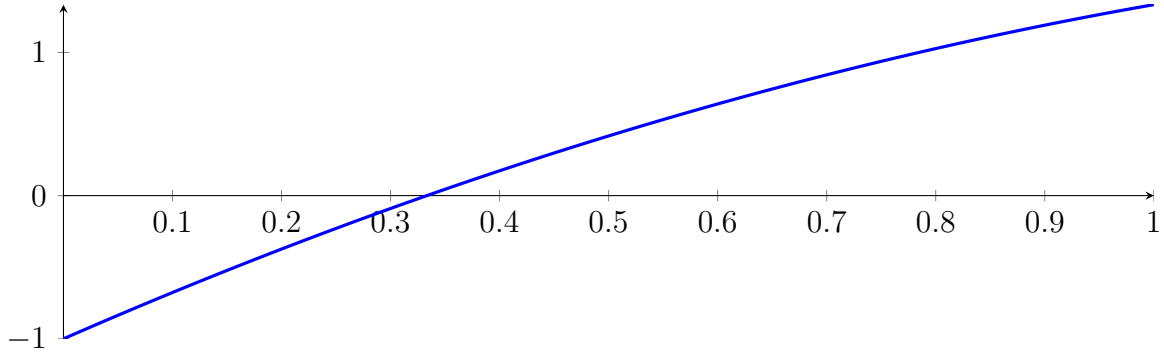
with precision $\varepsilon = 1 \cdot 10^{-128}$.

105 Running CubeClip on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

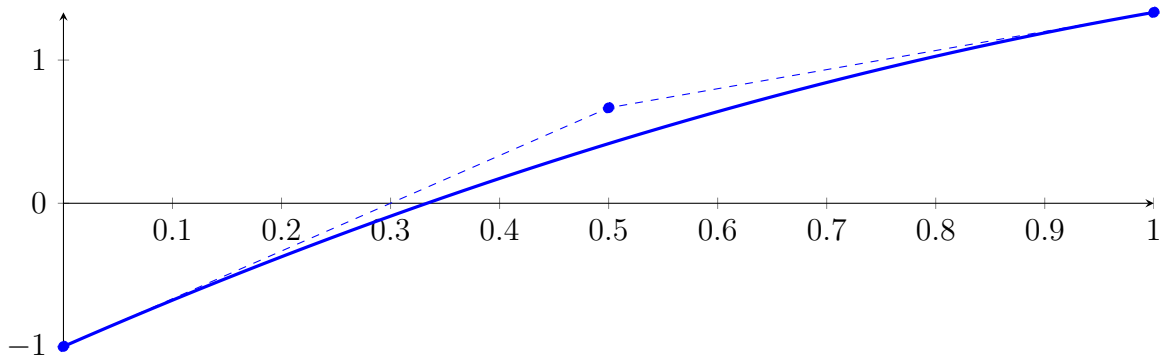
$$p = -1X^2 + 3.33333X - 1$$



105.1 Recursion Branch 1 for Input Interval $[0, 1]$

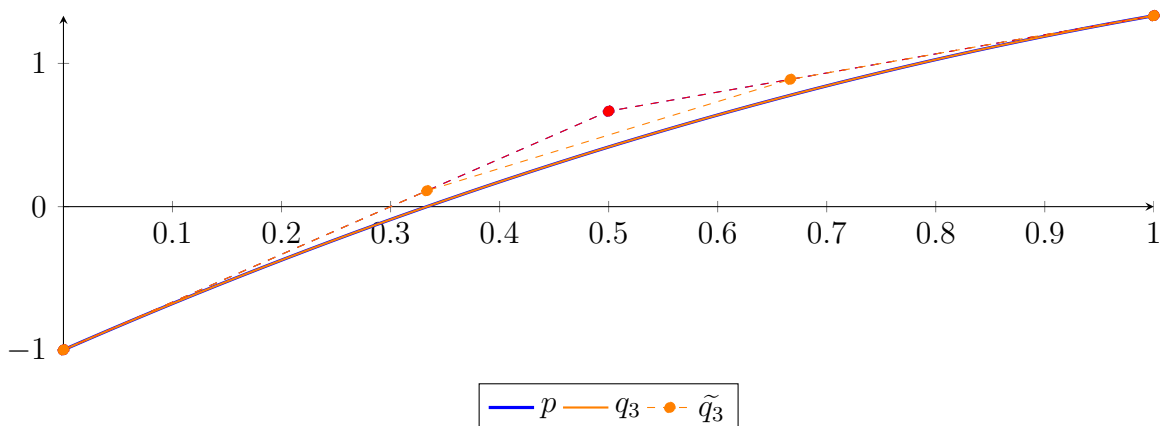
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.63715 \cdot 10^{-17}X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.33681 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

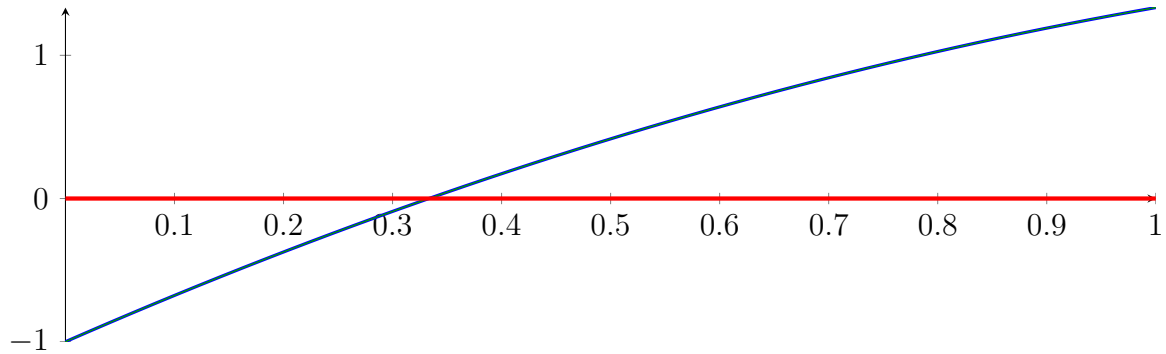
$$m = -1.63715 \cdot 10^{-17} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

$$N(m) = \{-6.10819 \cdot 10^{16}, 1.66602\}$$

Intersection intervals:



$[0, 1]$

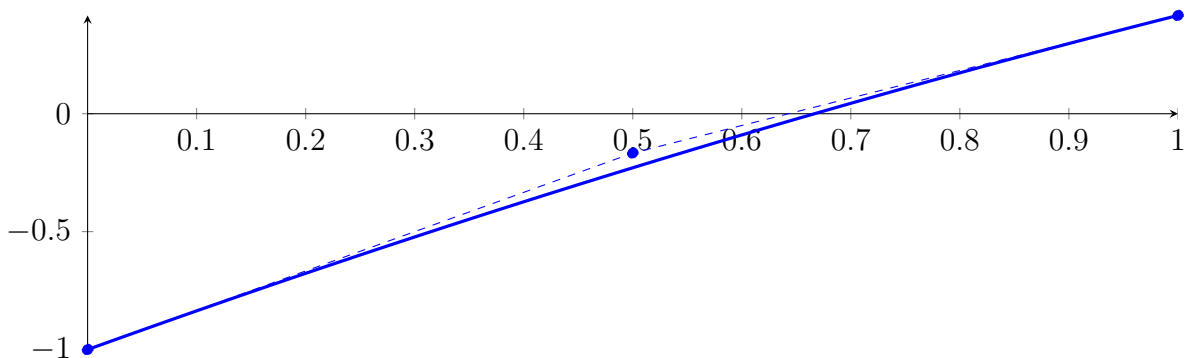
Longest intersection interval: 1

\implies Bisection: first half $[0, 0.5]$ und second half $[0.5, 1]$

105.2 Recursion Branch 1 1 on the First Half $[0, 0.5]$

Normalized monomial und Bézier representations and the Bézier polygon:

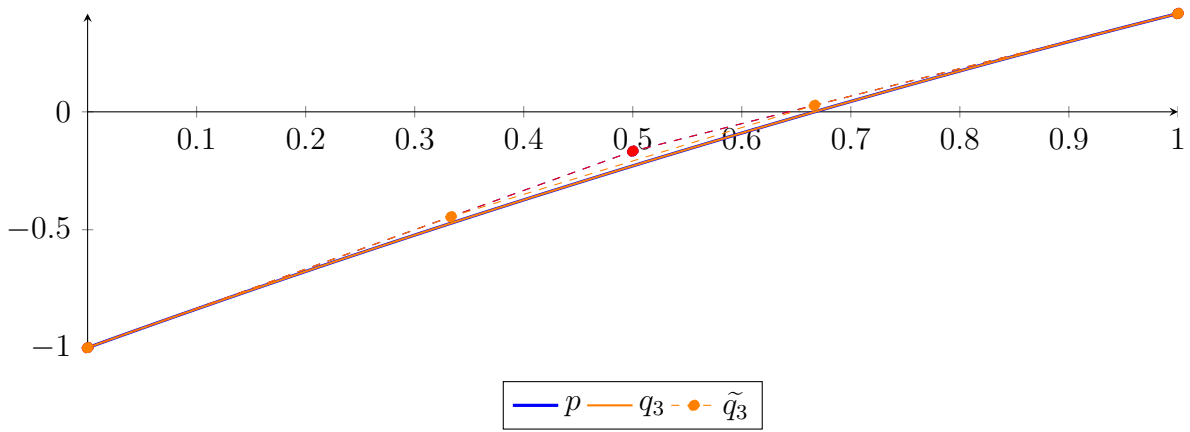
$$\begin{aligned} p &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2}(X) - 0.166667B_{1,2}(X) + 0.416667B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07336 \cdot 10^{-17} X^3 - 0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,3} - 0.444444B_{1,3} + 0.0277778B_{2,3} + 0.416667B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -0.25X^2 + 1.66667X - 1 \\ &= -1B_{0,2} - 0.166667B_{1,2} + 0.416667B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.58942 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -1.07336 \cdot 10^{-17} X^3 - 0.25 X^2 + 1.66667 X - 1$$

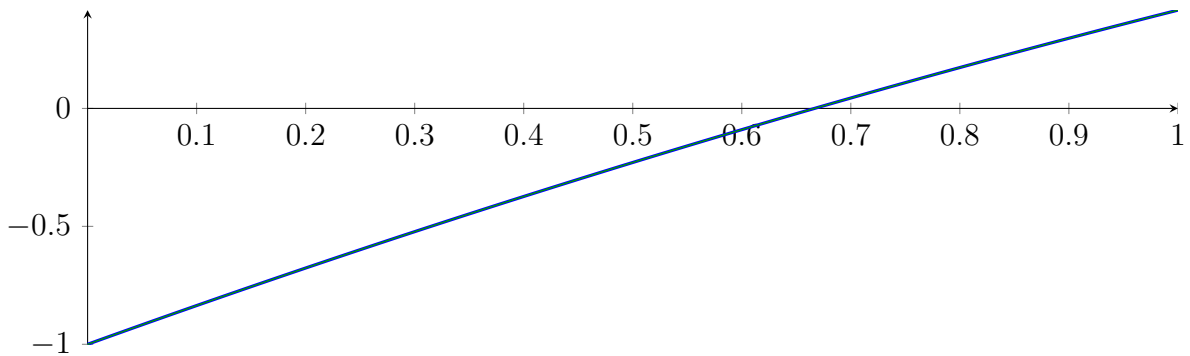
$$m = -1.07336 \cdot 10^{-17} X^3 - 0.25 X^2 + 1.66667 X - 1$$

Root of M and m :

$$N(M) = \{-2.32913 \cdot 10^{16}\}$$

$$N(m) = \{-2.32913 \cdot 10^{16}\}$$

Intersection intervals:

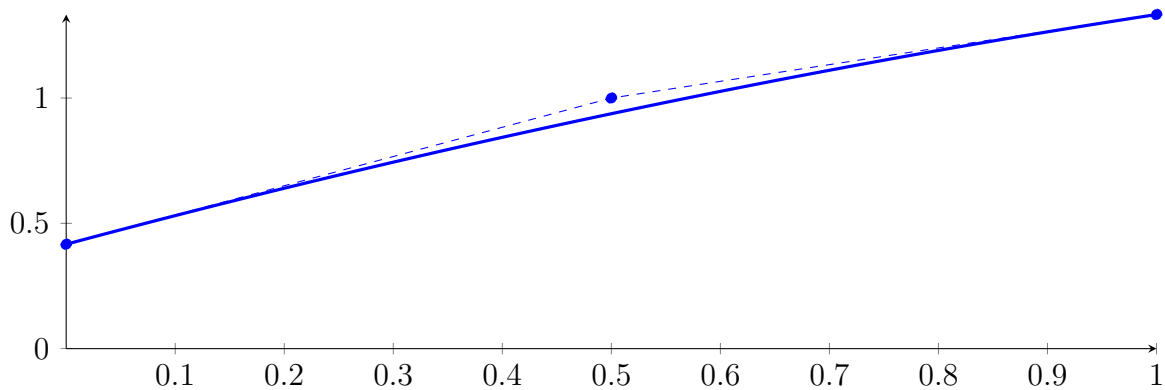


No intersection intervals with the x axis.

105.3 Recursion Branch 1 2 on the Second Half $[0.5, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

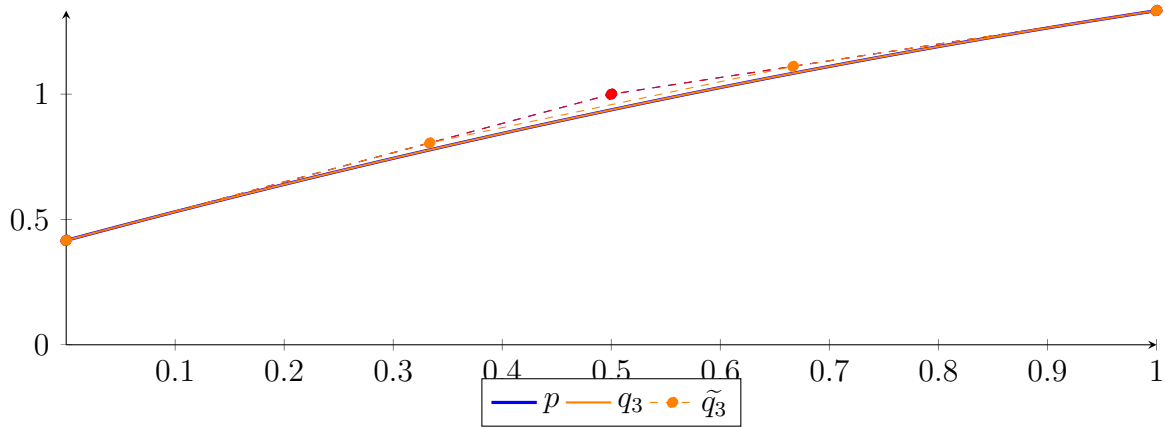
$$\begin{aligned} p &= -0.25 X^2 + 1.16667 X + 0.416667 \\ &= 0.416667 B_{0,2}(X) + 1 B_{1,2}(X) + 1.333333 B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,3} + 0.805556B_{1,3} + 1.111111B_{2,3} + 1.333333B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -0.25X^2 + 1.16667X + 0.416667 \\ &= 0.416667B_{0,2} + 1B_{1,2} + 1.333333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.30104 \cdot 10^{-18}$.

Bounding polynomials M and m :

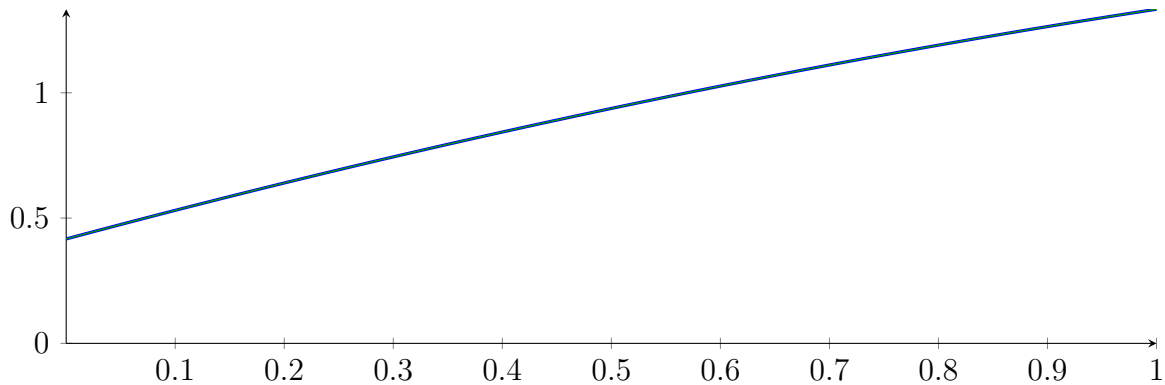
$$M = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

$$m = -2.27682 \cdot 10^{-18} X^3 - 0.25X^2 + 1.16667X + 0.416667$$

Root of M and m :

$$N(M) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\} \quad N(m) = \{-1.09802 \cdot 10^{17}, -8.02734, 2.33594\}$$

Intersection intervals:

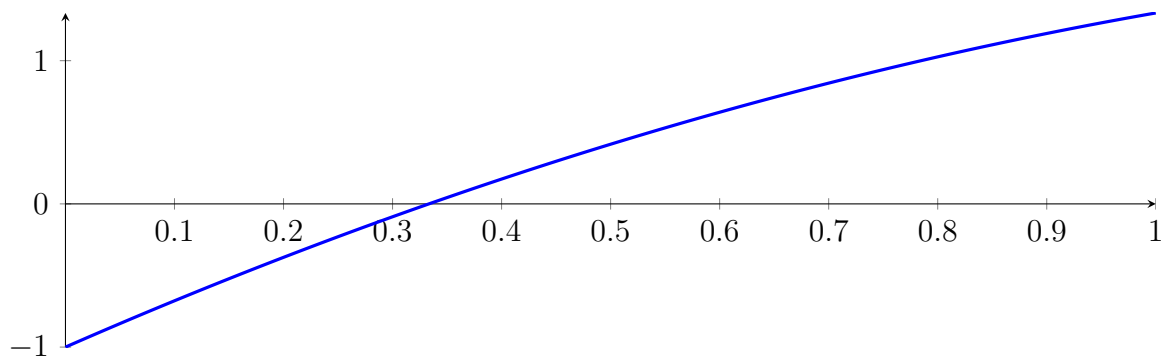


No intersection intervals with the x axis.

105.4 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

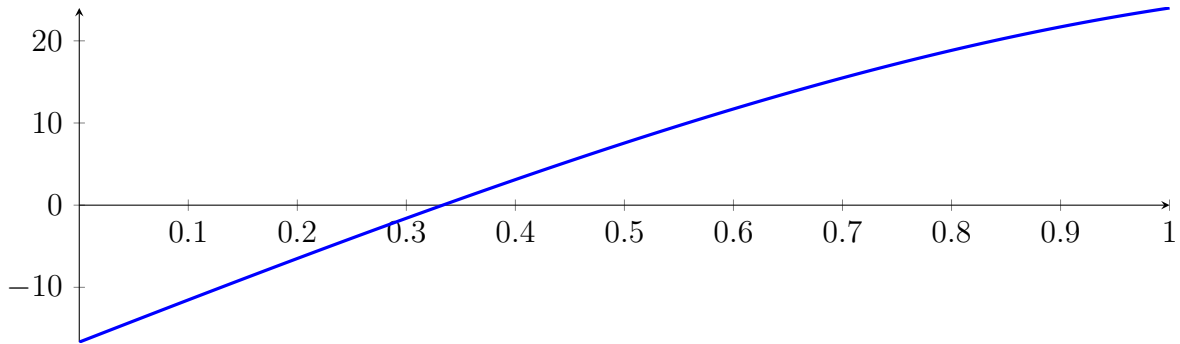
with precision $\varepsilon = 1 \cdot 10^{-128}$.

106 Running BezClip on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

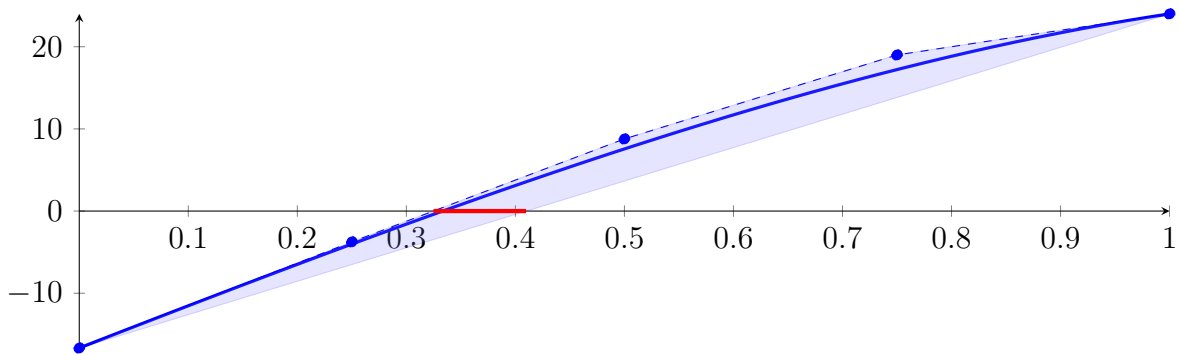
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



106.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

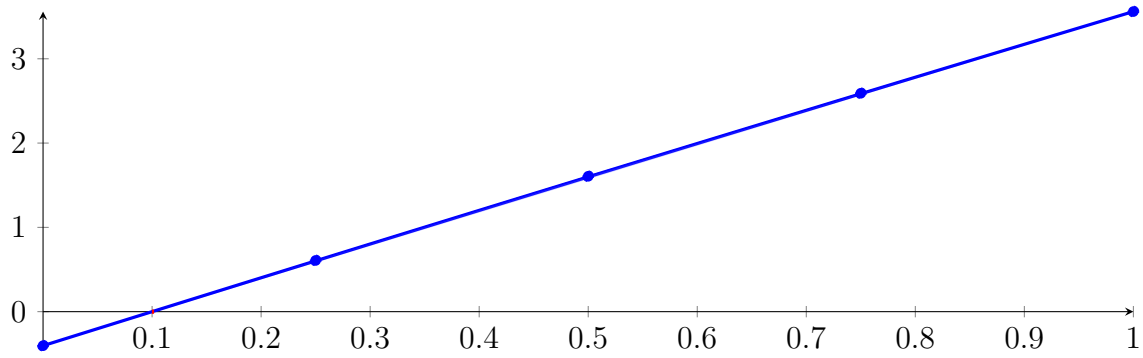
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

106.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877

\implies Selective recursion: interval 1: $[0.333317, 0.333491]$,

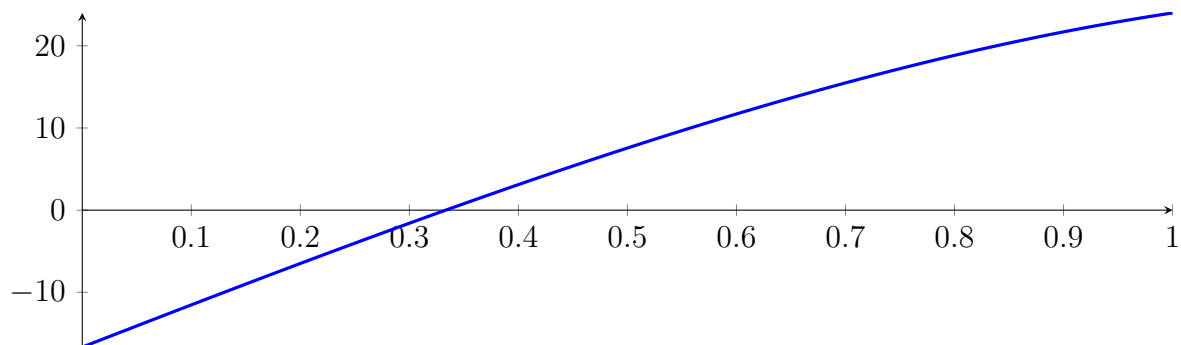
106.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Found root in interval $[0.333317, 0.333491]$ at recursion depth 3!

106.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333317, 0.333491]$$

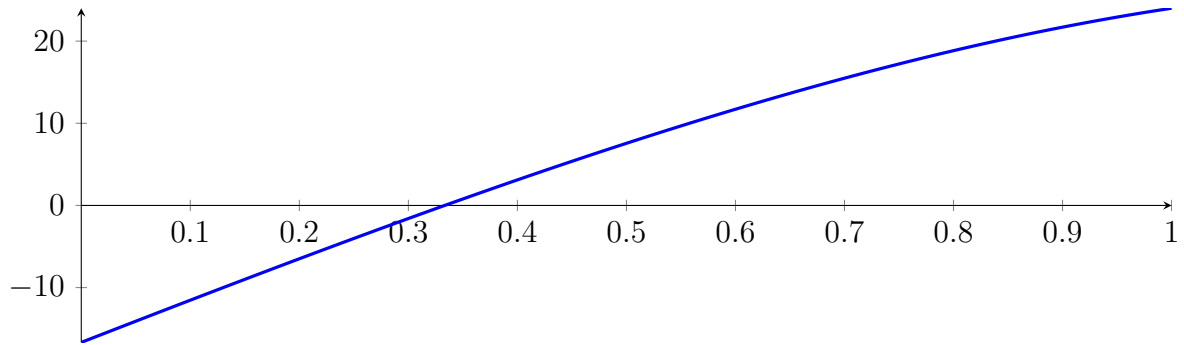
with precision $\varepsilon = 0.01$.

107 Running QuadClip on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

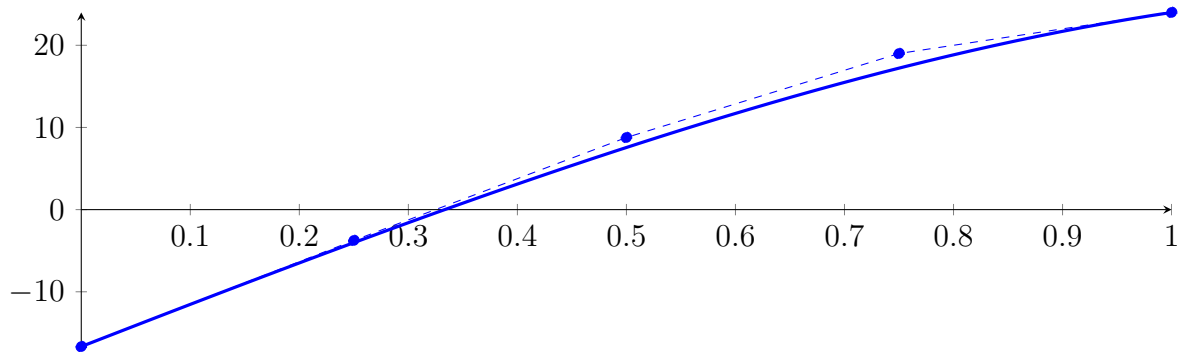
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



107.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

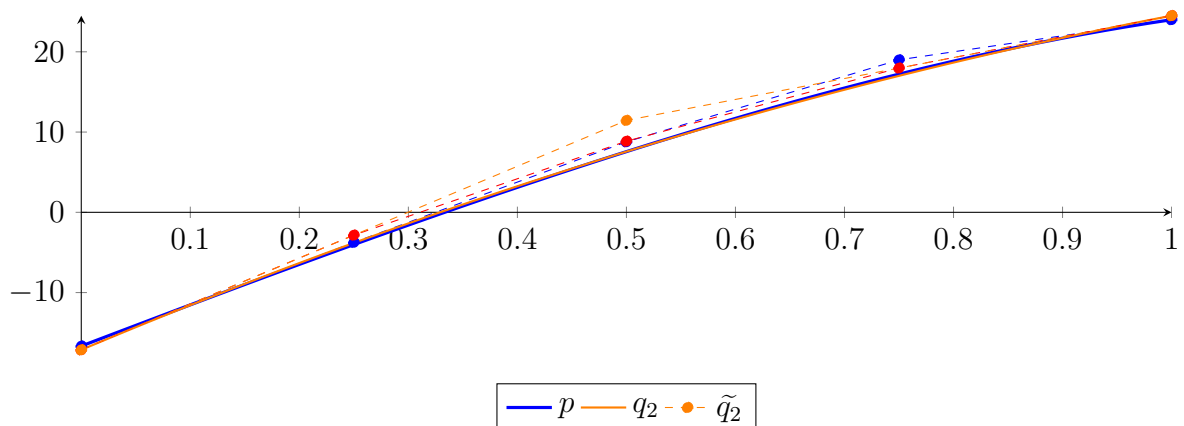
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1.45023 \cdot 10^{-15}X^4 + 4.00374 \cdot 10^{-15}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

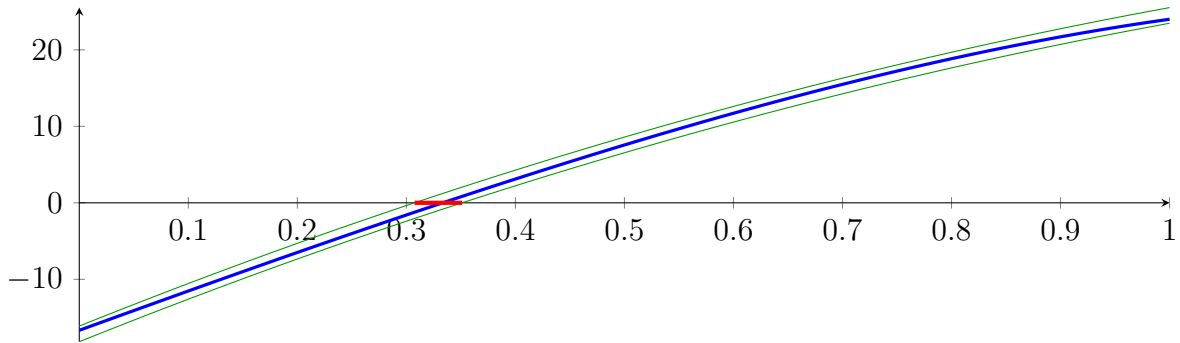
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

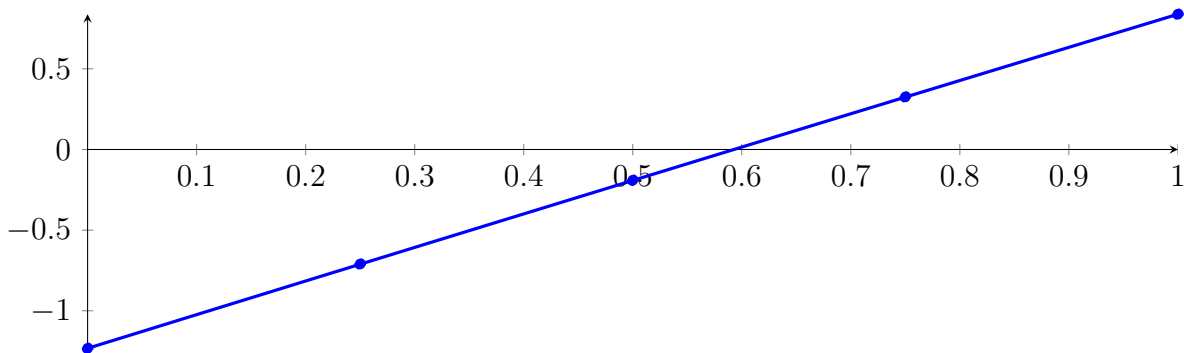
Longest intersection interval: 0.0436205

⇒ Selective recursion: **interval 1:** $[0.307477, 0.351097]$,

107.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

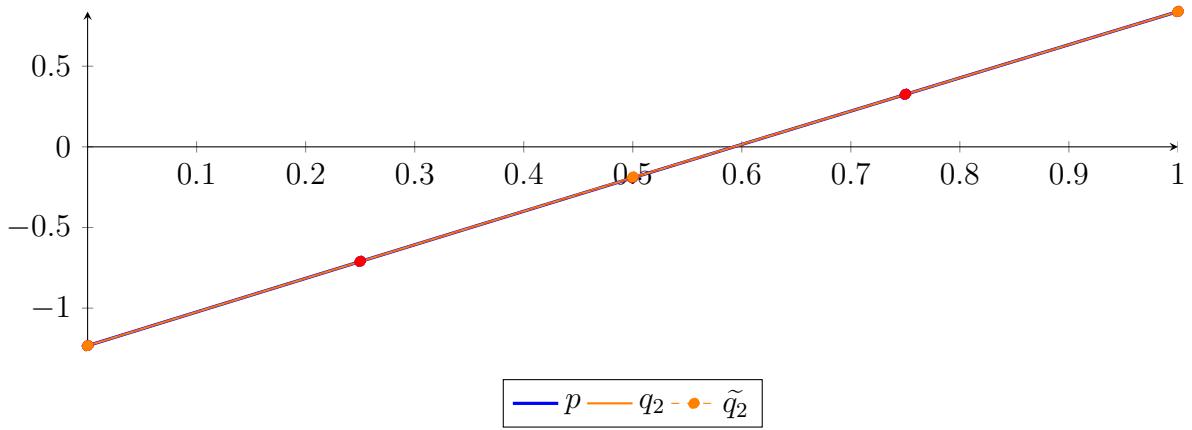
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1.45283 \cdot 10^{-17} X^4 + 3.33934 \cdot 10^{-17} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

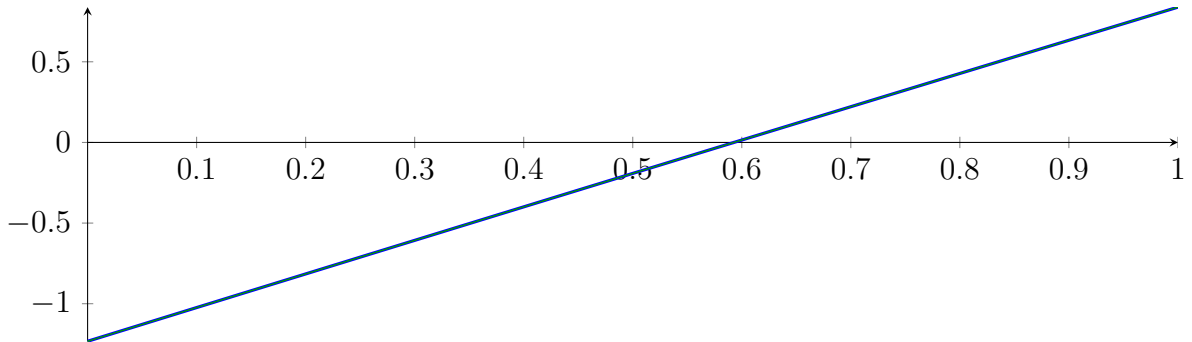
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

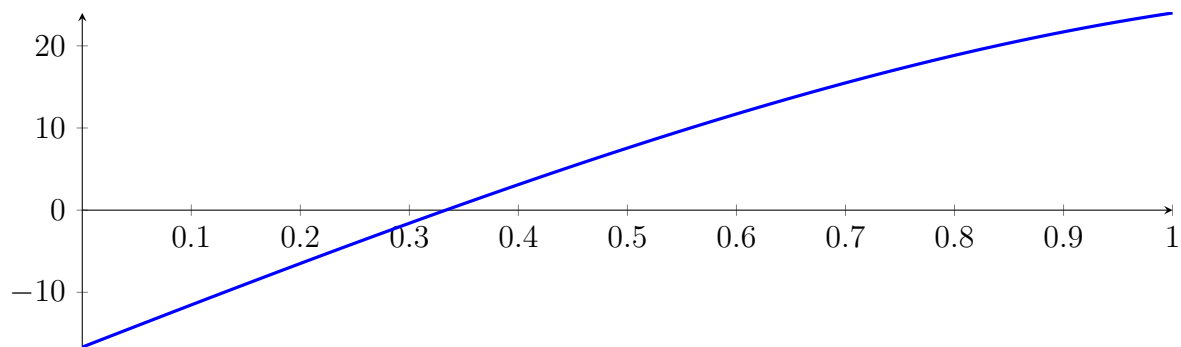
107.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Found root in interval $[0.333332, 0.333335]$ at recursion depth 3!

107.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333332, 0.333335]$$

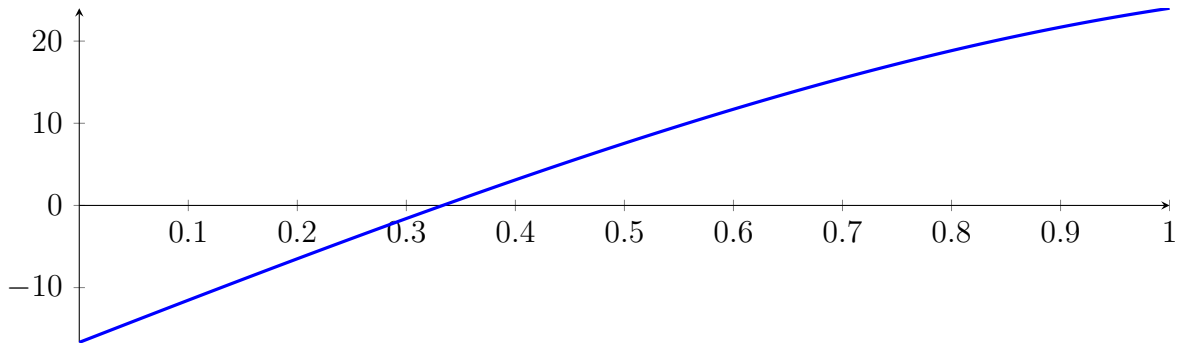
with precision $\varepsilon = 0.01$.

108 Running CubeClip on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

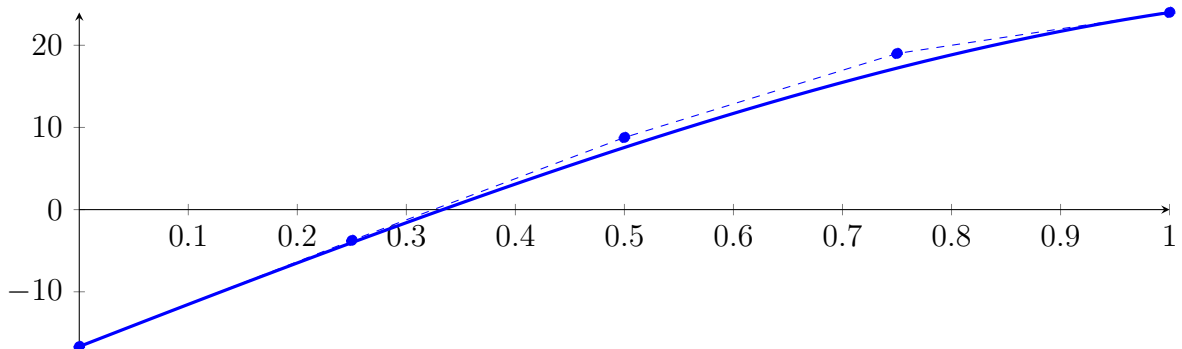
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



108.1 Recursion Branch 1 for Input Interval $[0, 1]$

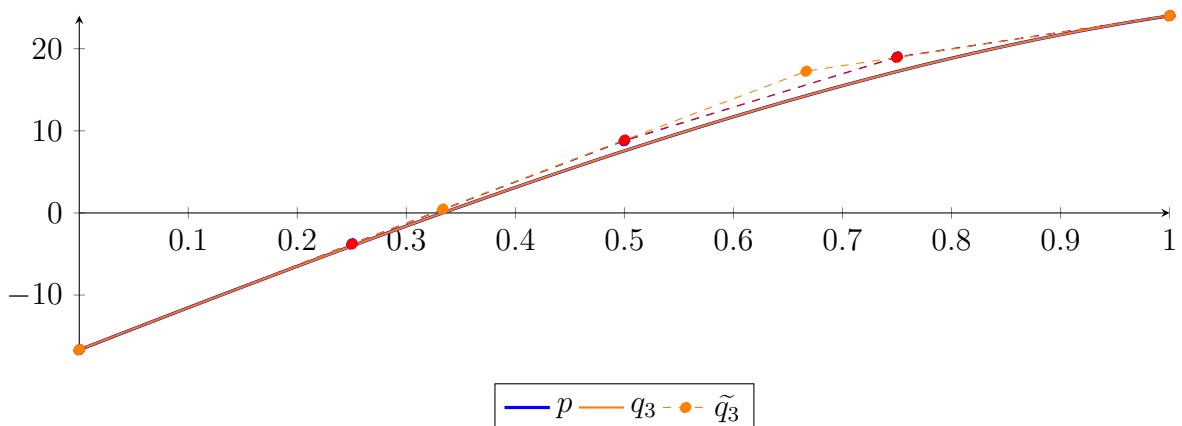
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= 3.34802 \cdot 10^{-16}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

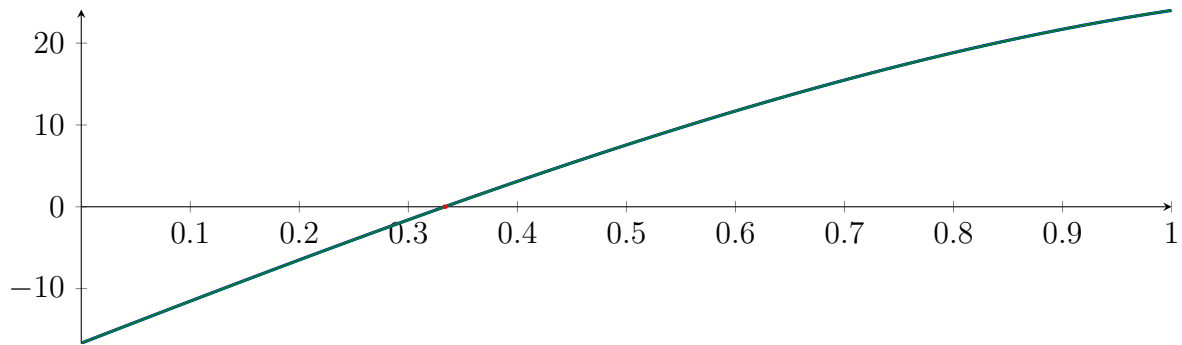
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204

\implies Selective recursion: [interval 1: \[0.331524, 0.335136\]](#),

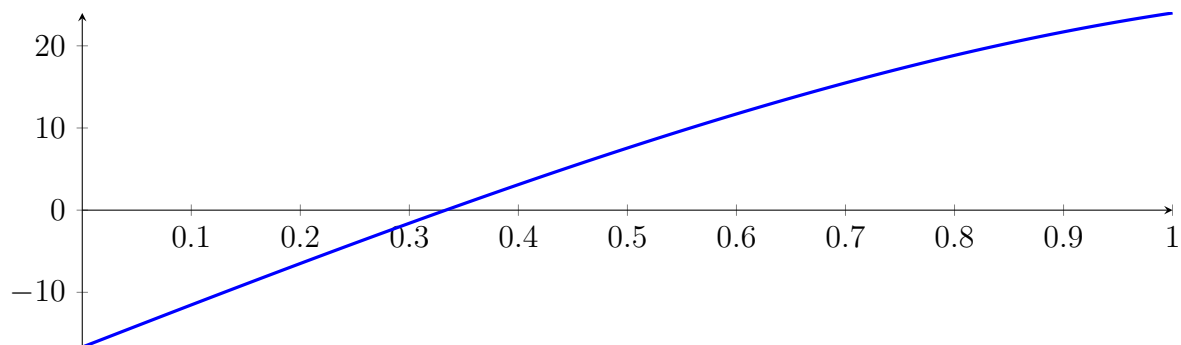
108.2 Recursion Branch 1 1 in Interval 1: [0.331524, 0.335136]

Found root in interval [0.331524, 0.335136] at recursion depth 2!

108.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.331524, 0.335136]$$

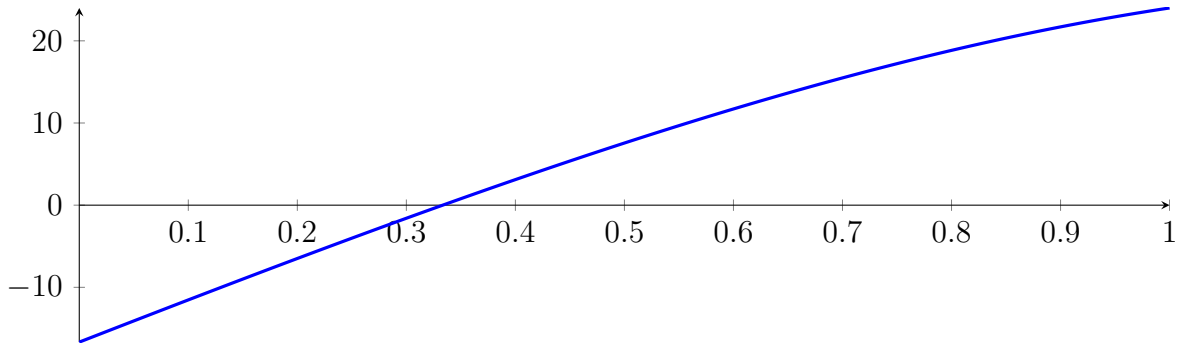
with precision $\varepsilon = 0.01$.

109 Running BezClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

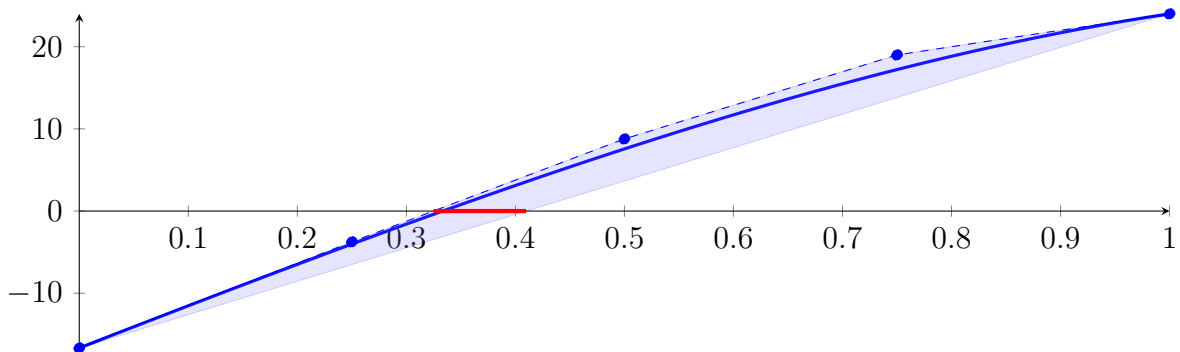
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



109.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

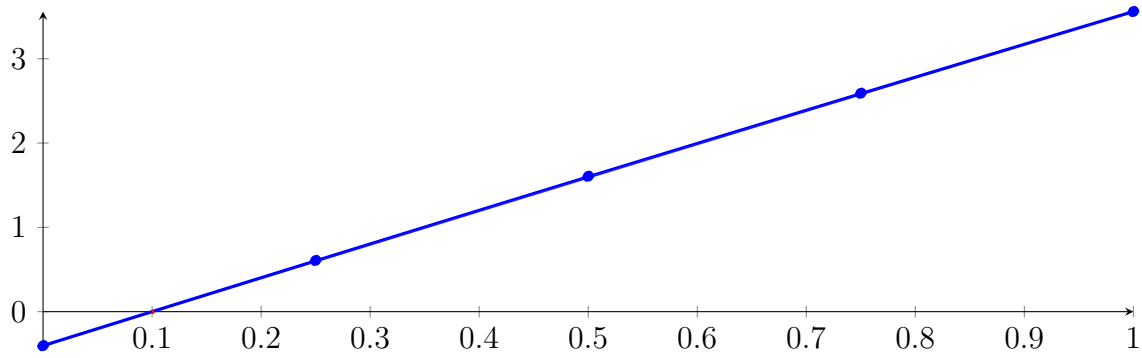
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

109.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

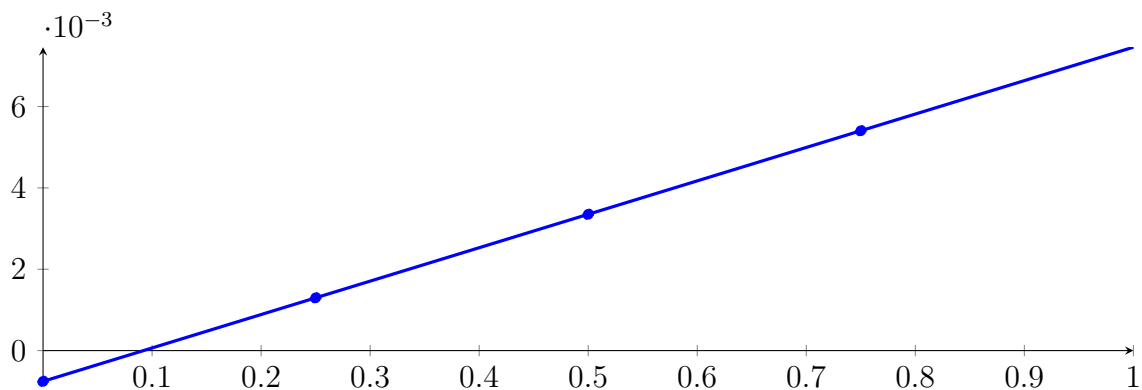
Longest intersection interval: 0.00203877

⇒ Selective recursion: interval 1: $[0.333317, 0.333491]$,

109.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01974 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

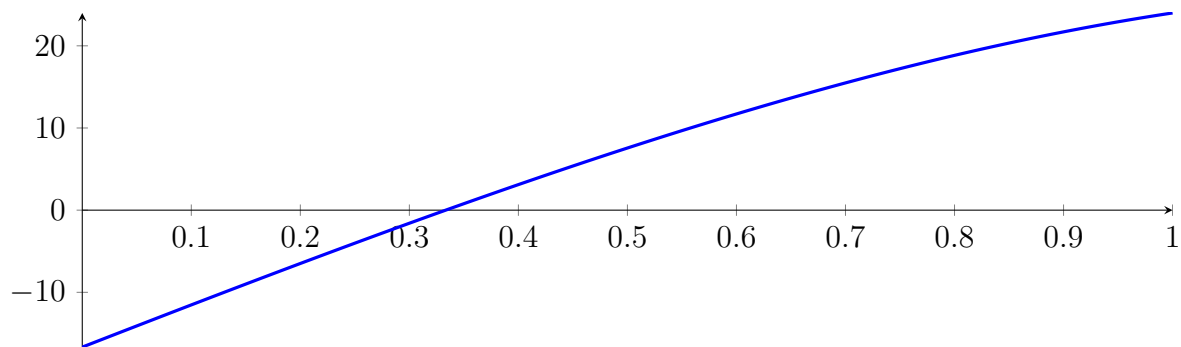
109.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

109.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

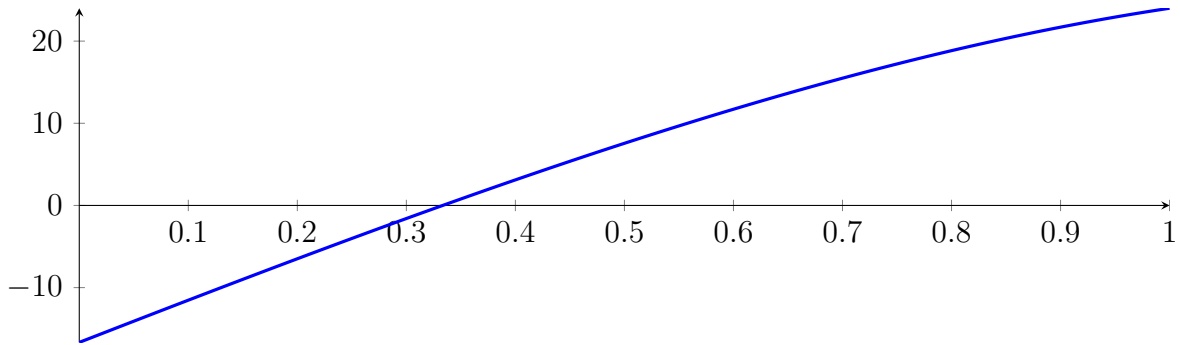
with precision $\varepsilon = 0.0001$.

110 Running QuadClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

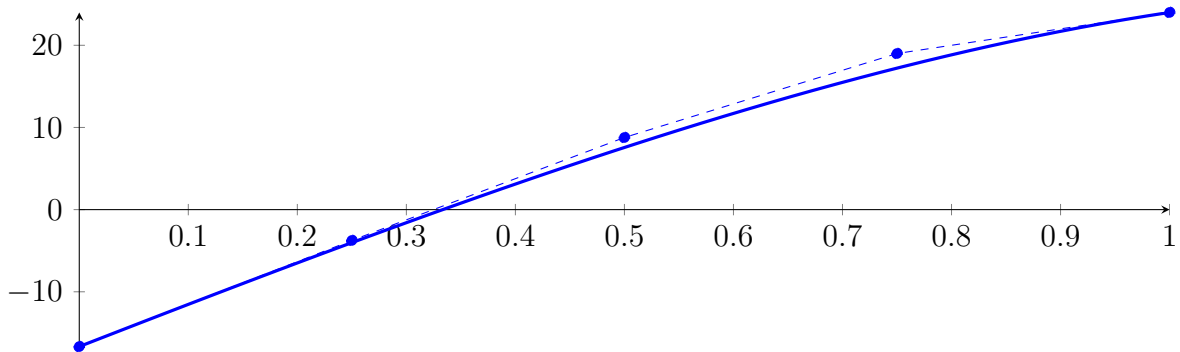
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



110.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

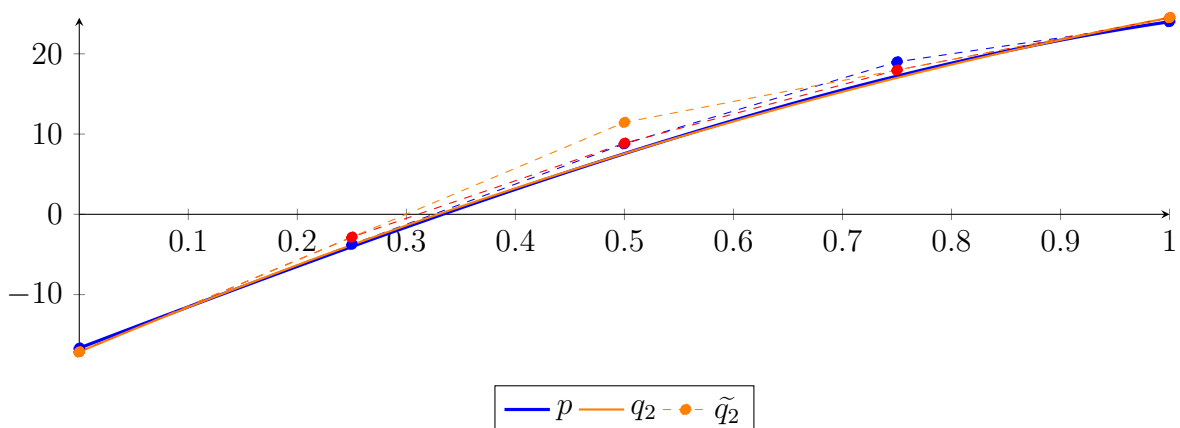
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1.45023 \cdot 10^{-15}X^4 + 4.00374 \cdot 10^{-15}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

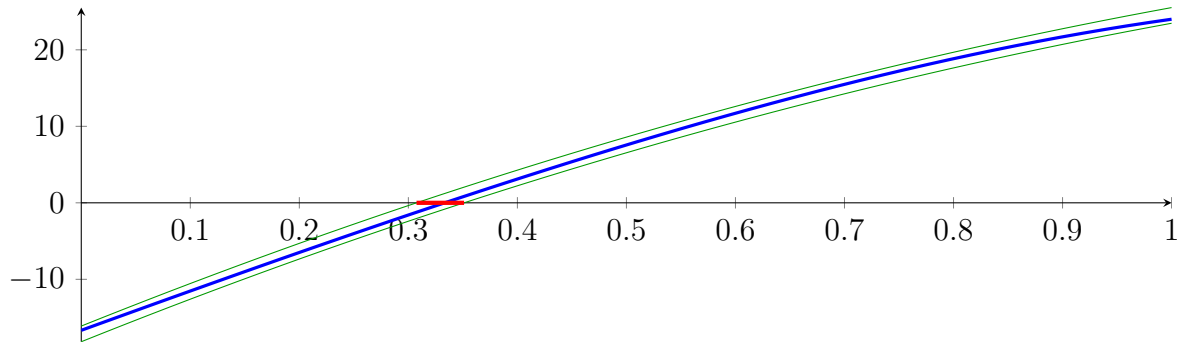
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

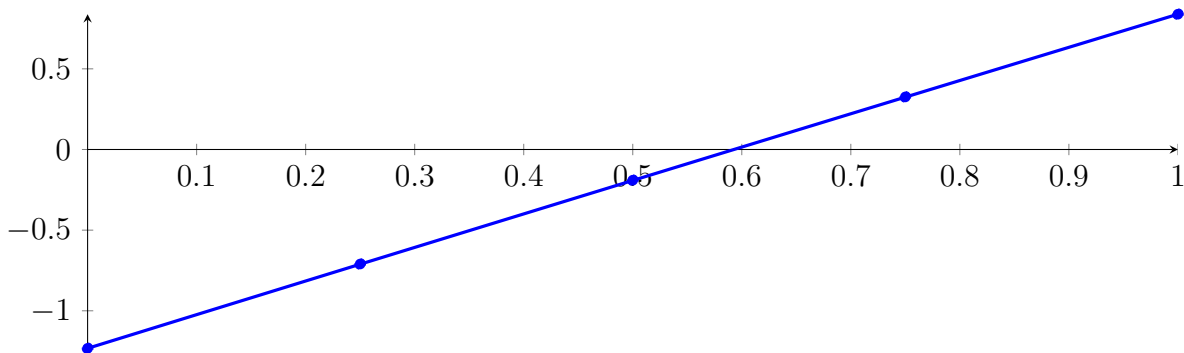
Longest intersection interval: 0.0436205

⇒ Selective recursion: interval 1: $[0.307477, 0.351097]$,

110.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

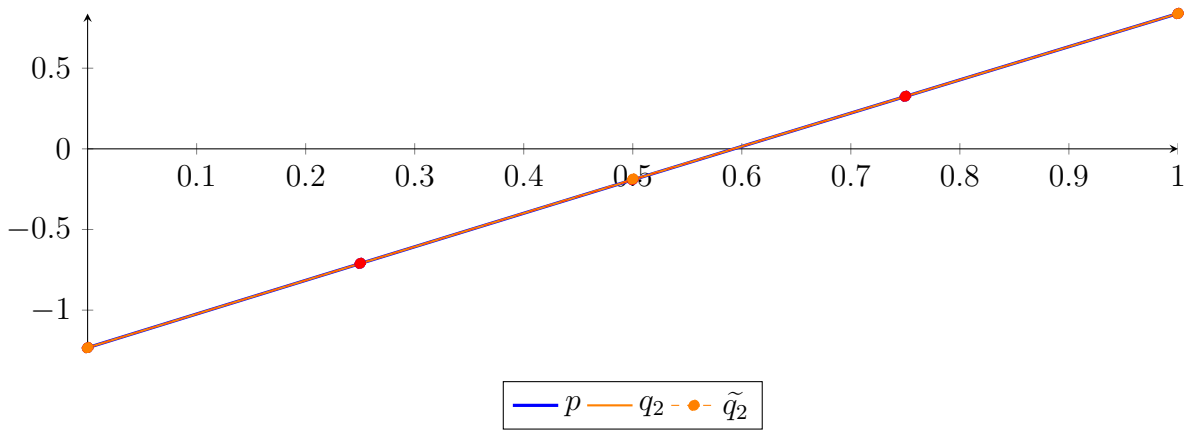
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1.45283 \cdot 10^{-17} X^4 + 3.33934 \cdot 10^{-17} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

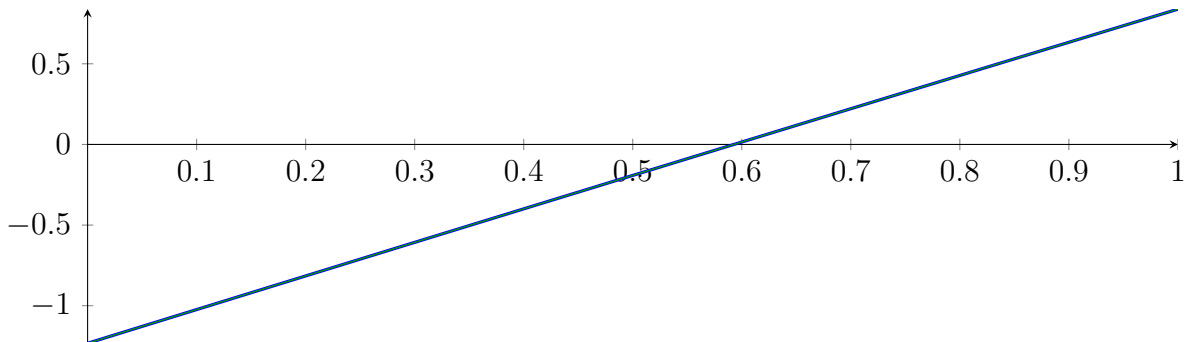
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

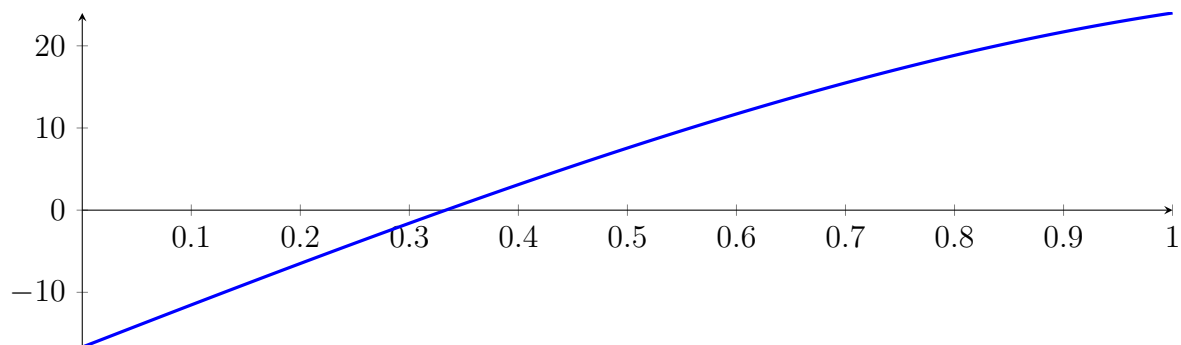
110.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Found root in interval $[0.333332, 0.333335]$ at recursion depth 3!

110.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333332, 0.333335]$$

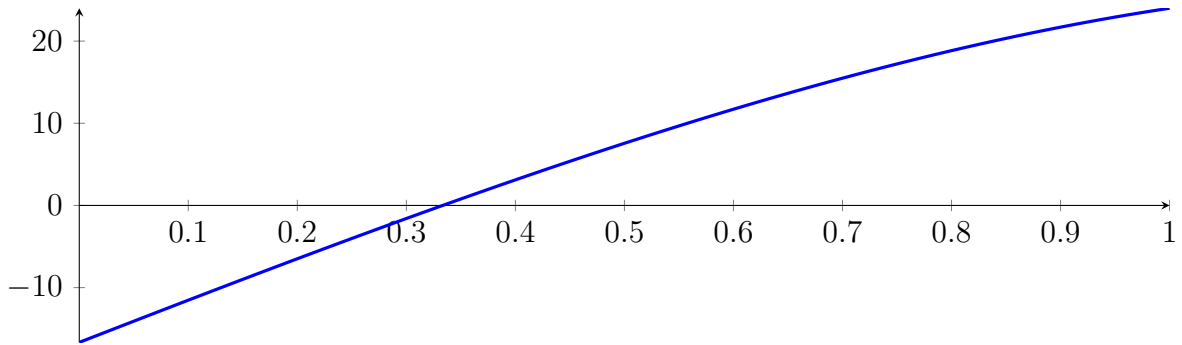
with precision $\varepsilon = 0.0001$.

111 Running CubeClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

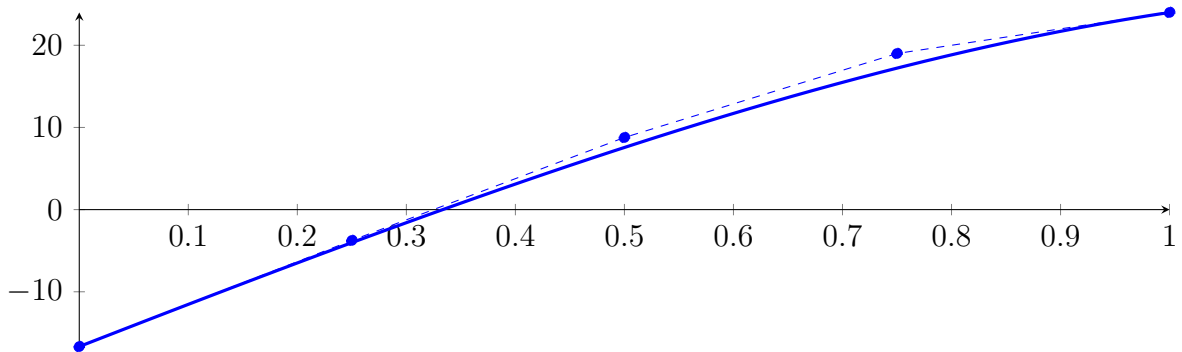
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



111.1 Recursion Branch 1 for Input Interval $[0, 1]$

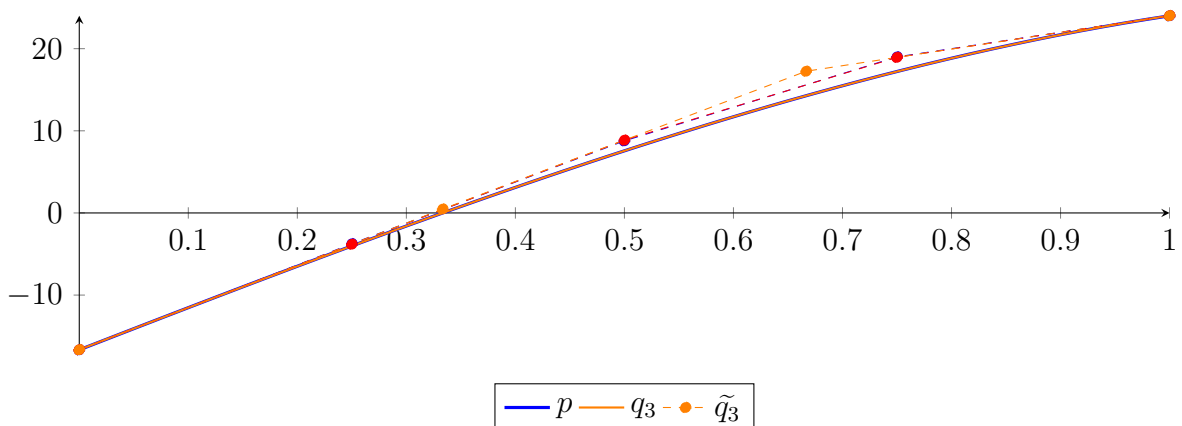
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\
 &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\
 &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\
 \tilde{q}_3 &= 3.34802 \cdot 10^{-16}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\
 &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4}
 \end{aligned}$$



— p — q_3 — \tilde{q}_3

The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

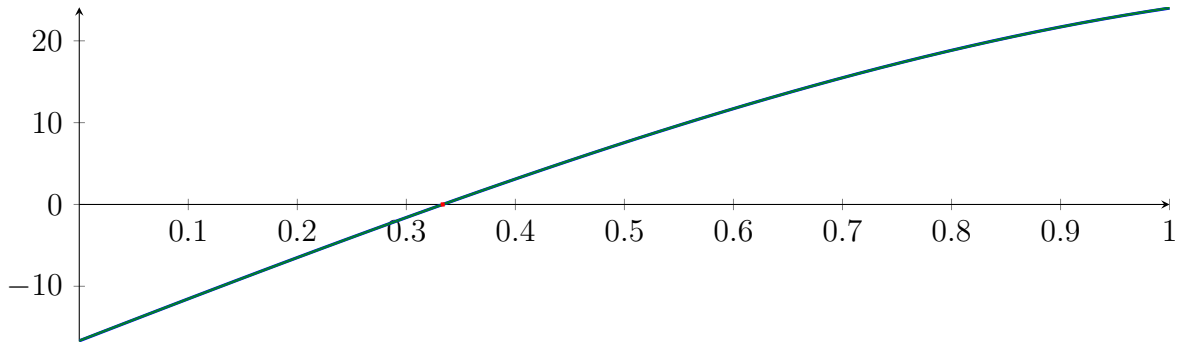
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

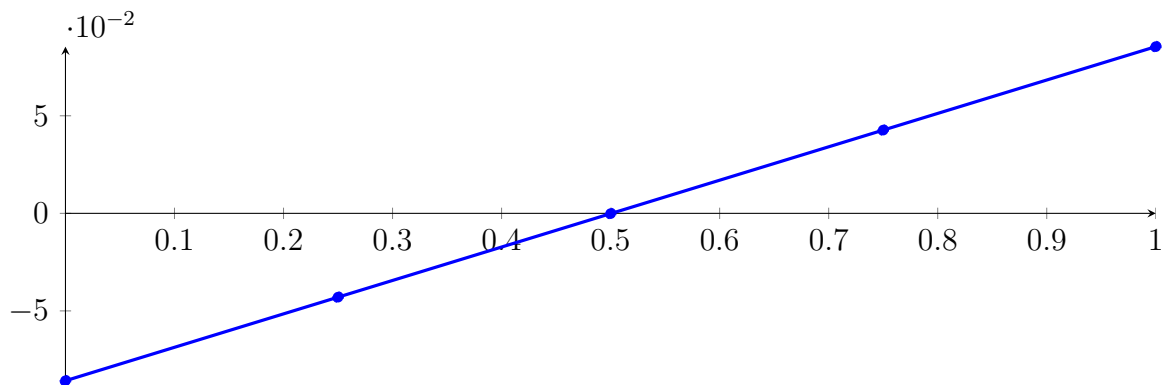
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

111.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

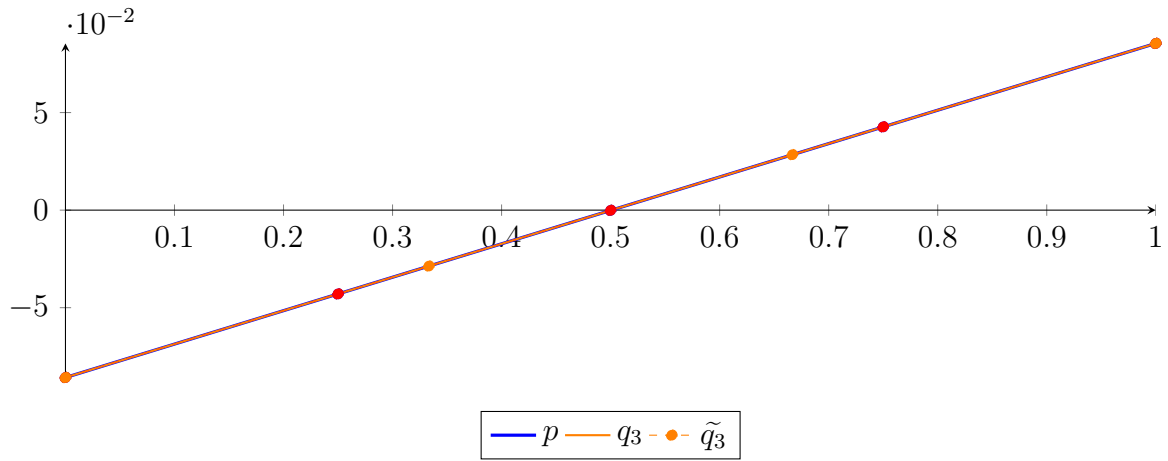
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 1.99222 \cdot 10^{-18} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m :

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

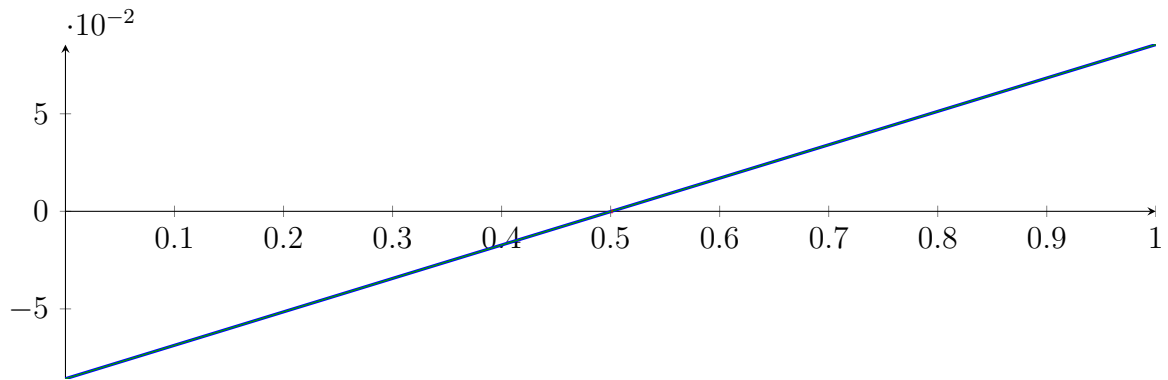
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\}$$

$$N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

Longest intersection interval: $1.7041 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

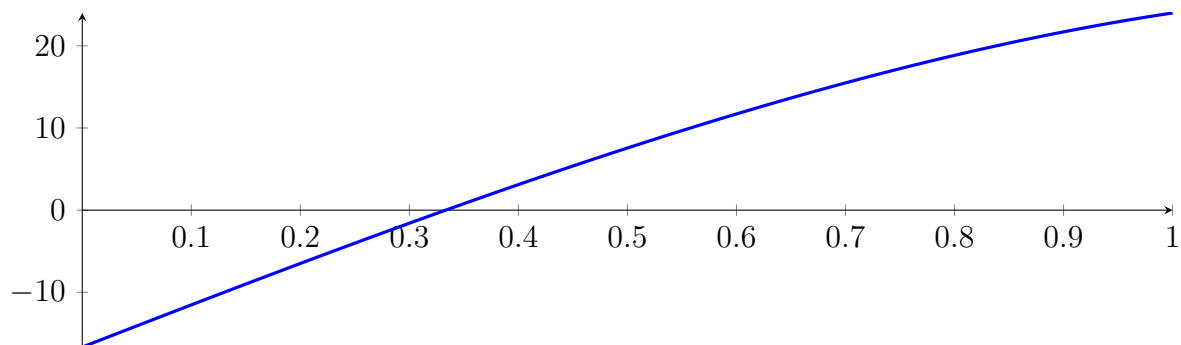
111.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

111.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

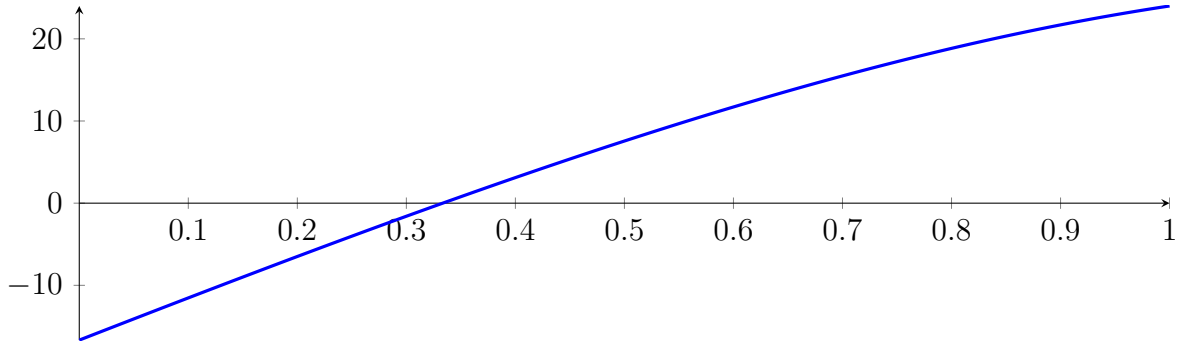
with precision $\varepsilon = 0.0001$.

112 Running BezClip on f_4 with epsilon 8

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

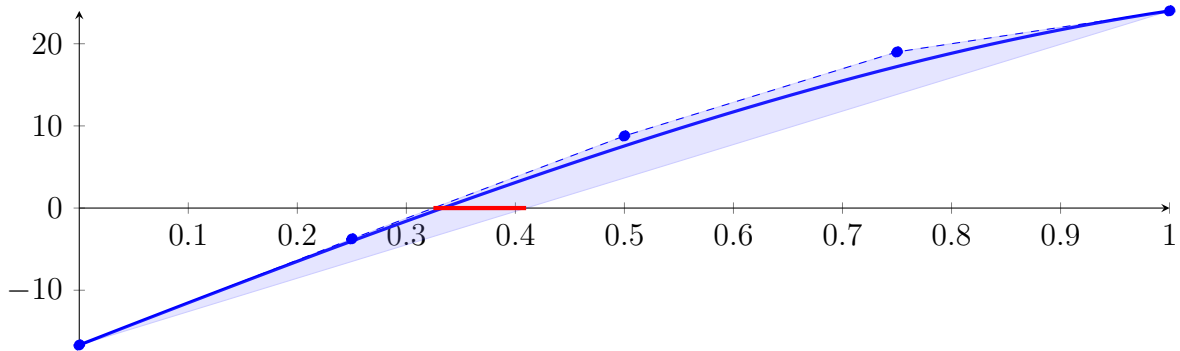
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



112.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

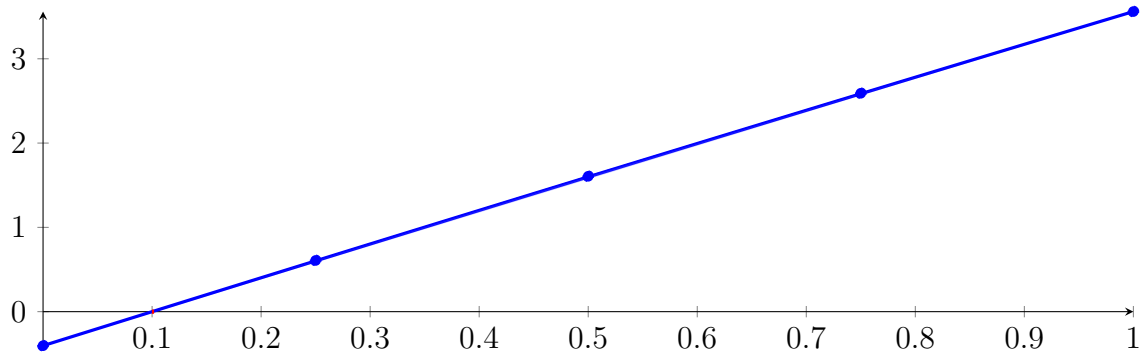
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

112.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

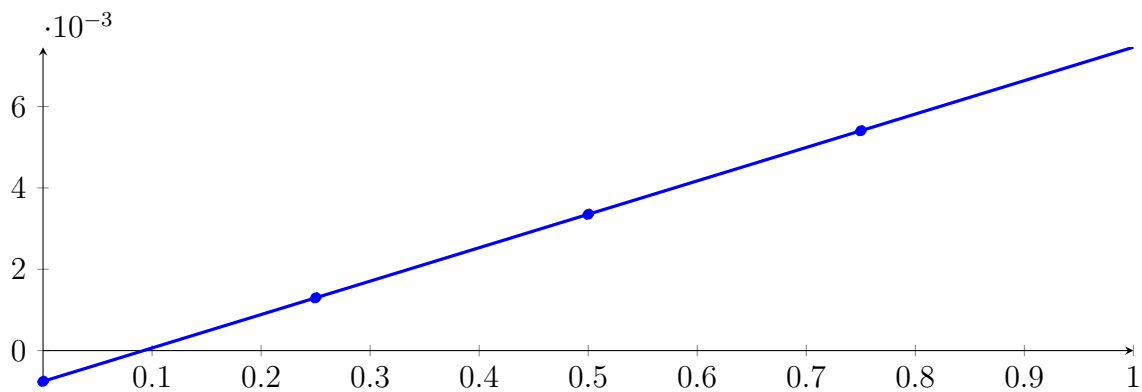
Longest intersection interval: 0.00203877

⇒ Selective recursion: interval 1: $[0.333317, 0.333491]$,

112.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01974 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

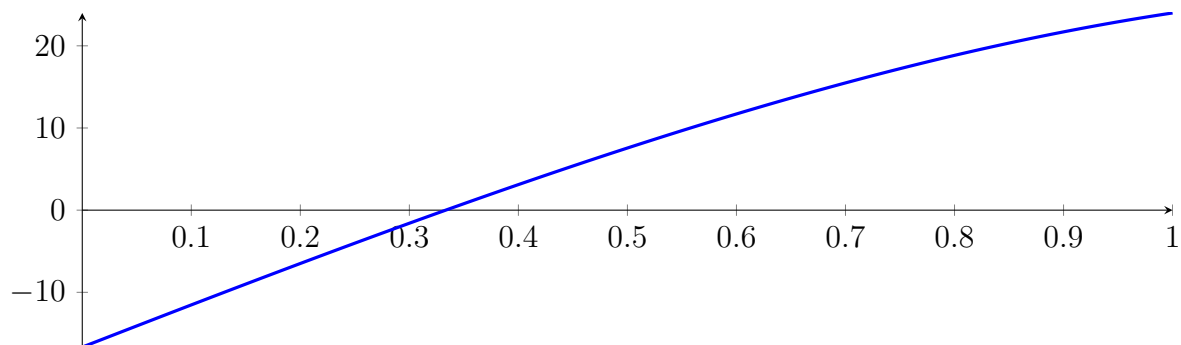
112.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

112.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

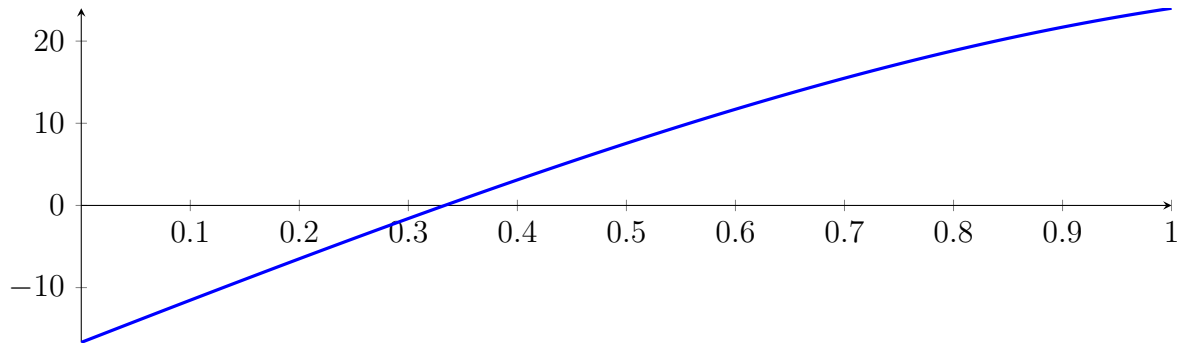
with precision $\varepsilon = 1 \cdot 10^{-08}$.

113 Running QuadClip on f_4 with epsilon 8

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

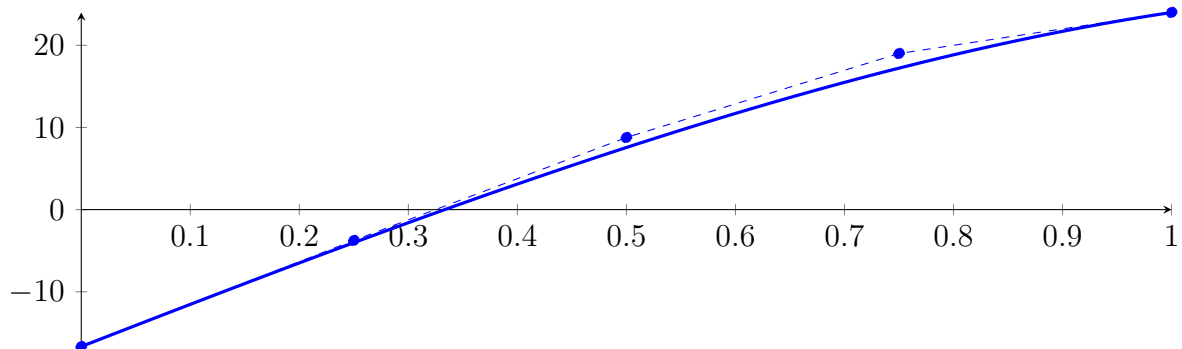
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



113.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

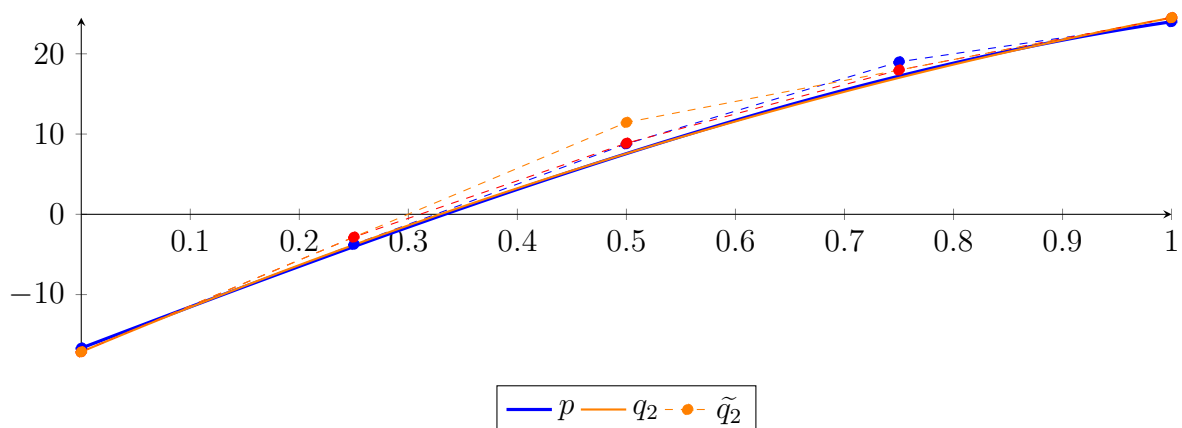
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1.45023 \cdot 10^{-15}X^4 + 4.00374 \cdot 10^{-15}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

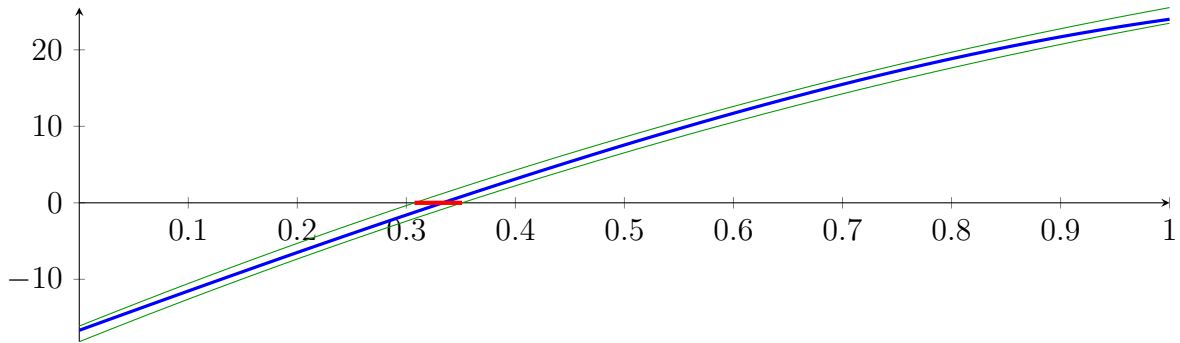
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205

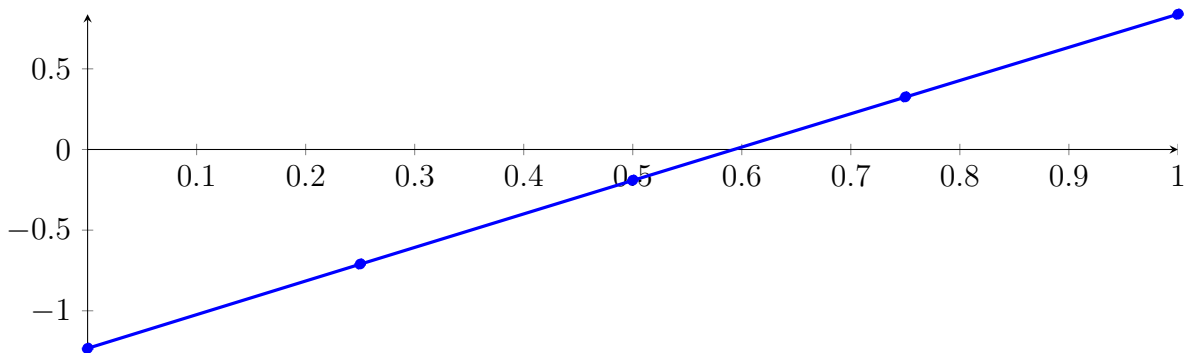
⇒ Selective recursion: interval 1: $[0.307477, 0.351097]$,

113.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278$$

$$= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X)$$



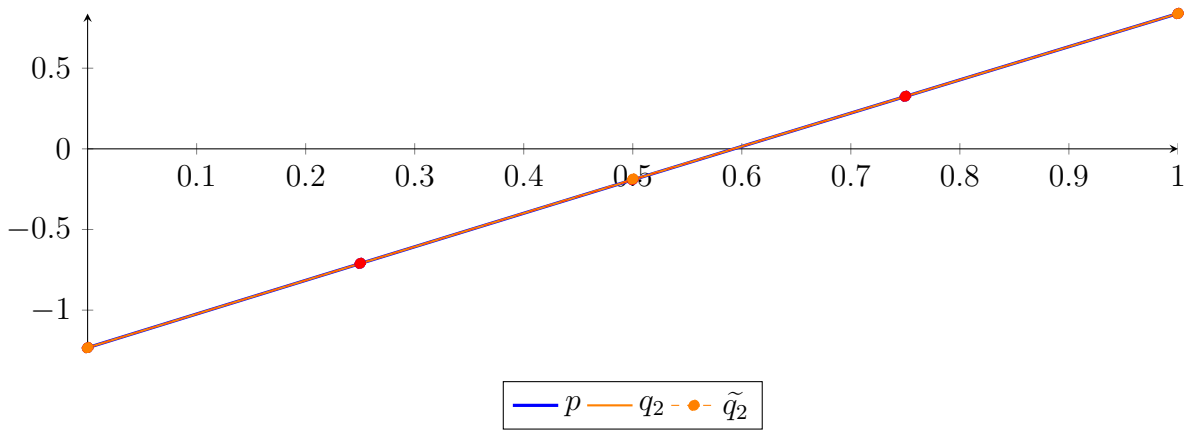
Degree reduction and raising:

$$q_2 = -0.020089X^2 + 2.09166X - 1.23281$$

$$= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2}$$

$$\tilde{q}_2 = -1.45283 \cdot 10^{-17} X^4 + 3.33934 \cdot 10^{-17} X^3 - 0.020089X^2 + 2.09166X - 1.23281$$

$$= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

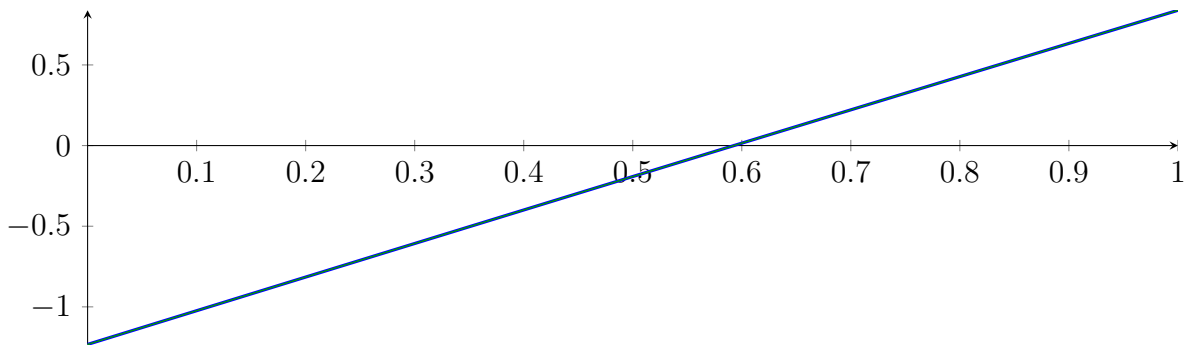
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

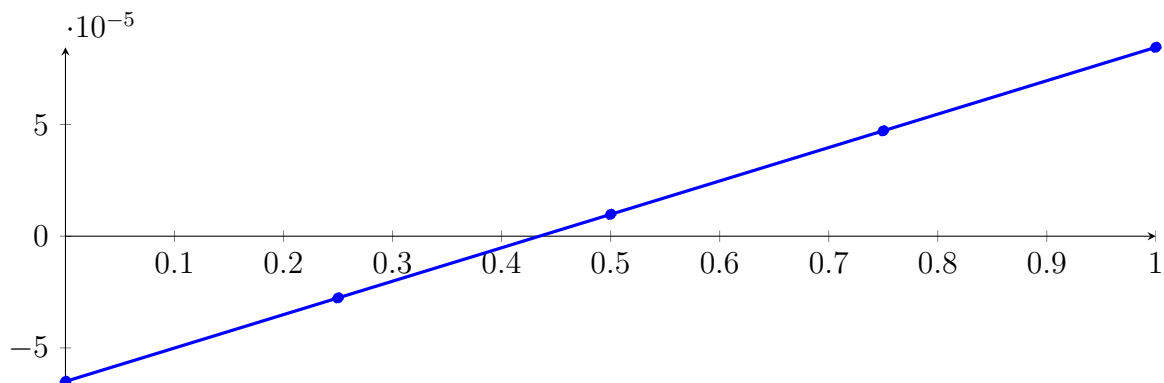
Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

113.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.05879 \cdot 10^{-22} X^4 - 2.82525 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05} \\
 &= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \\
 &\quad \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)
 \end{aligned}$$



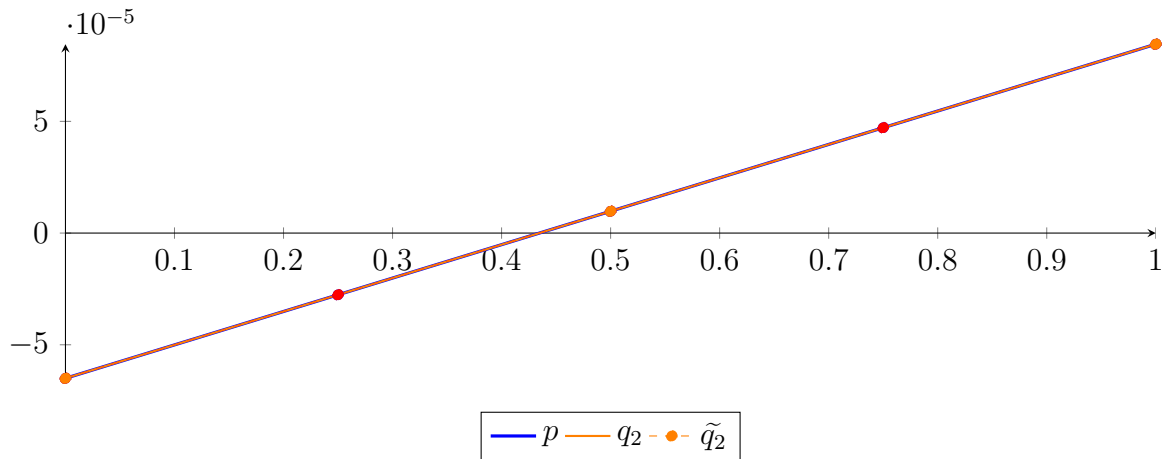
Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = -4.49986 \cdot 10^{-22} X^4 + 3.33519 \cdot 10^{-21} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.82529 \cdot 10^{-17}$.

Bounding polynomials M and m :

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

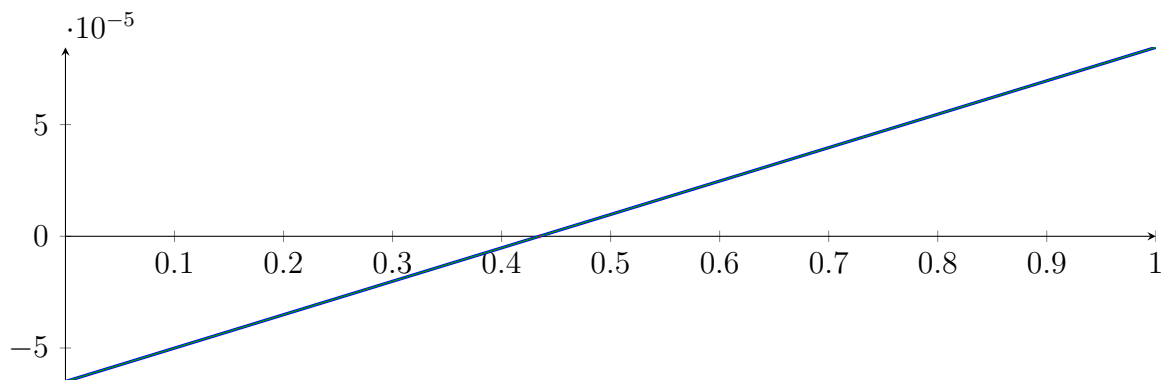
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.434685, 1.4089 \cdot 10^6\}$$

$$N(m) = \{0.434685, 1.4089 \cdot 10^6\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

Longest intersection interval: $3.74055 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

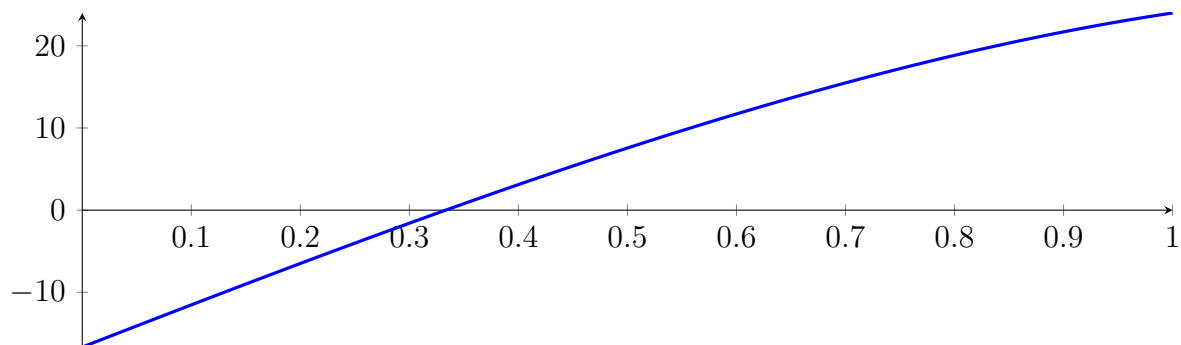
113.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

113.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

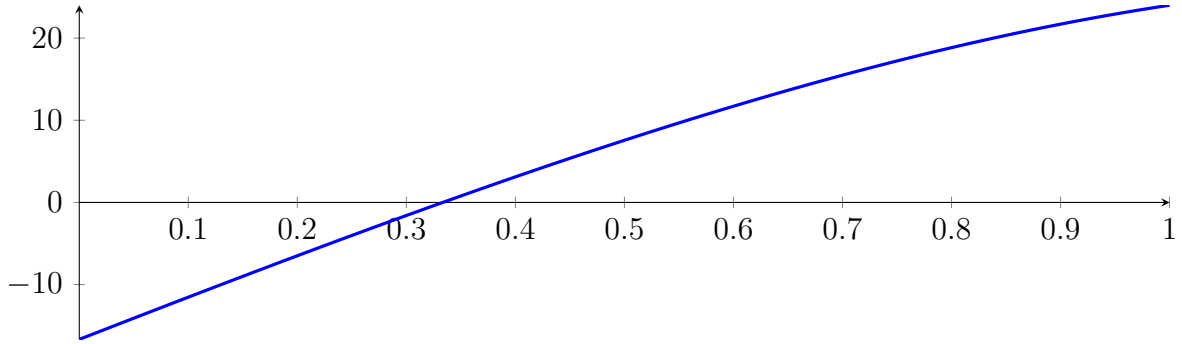
with precision $\varepsilon = 1 \cdot 10^{-08}$.

114 Running CubeClip on f_4 with epsilon 8

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

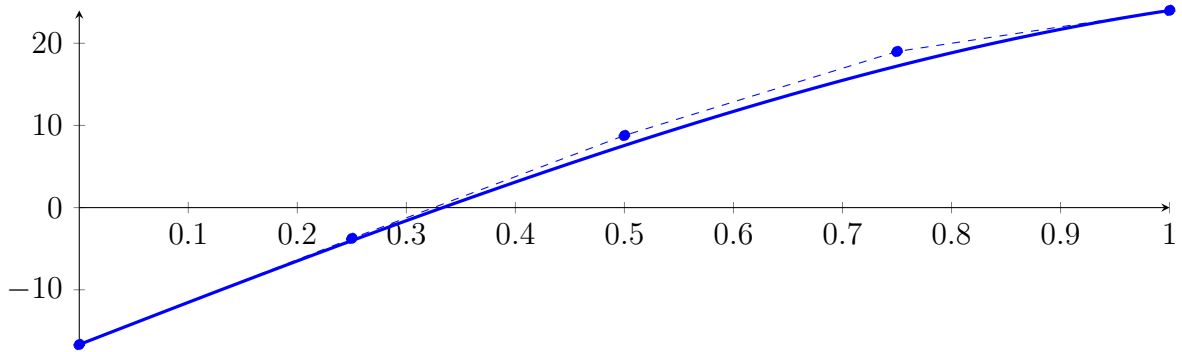


114.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

$$= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X)$$



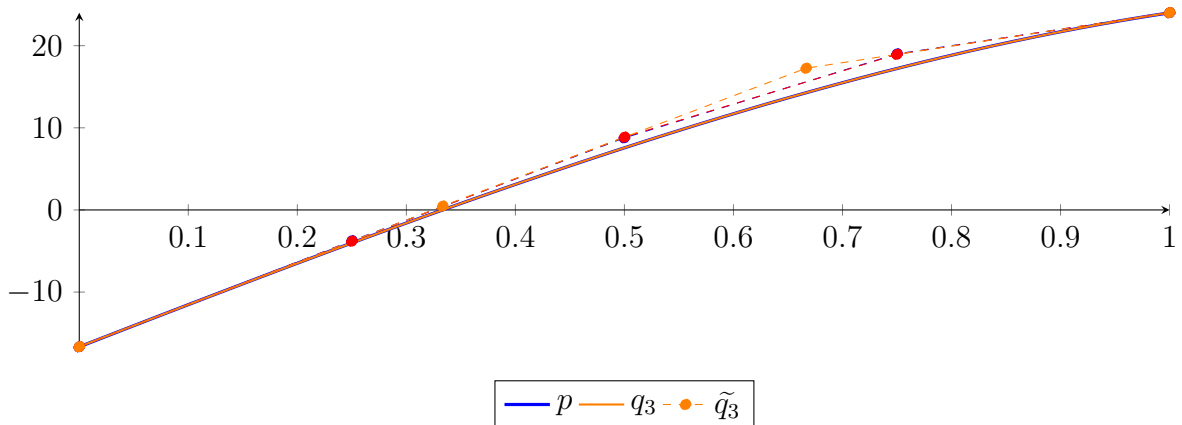
Degree reduction and raising:

$$q_3 = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524$$

$$= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3}$$

$$\tilde{q}_3 = 3.34802 \cdot 10^{-16}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524$$

$$= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4}$$



— p — q_3 - \tilde{q}_3

The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

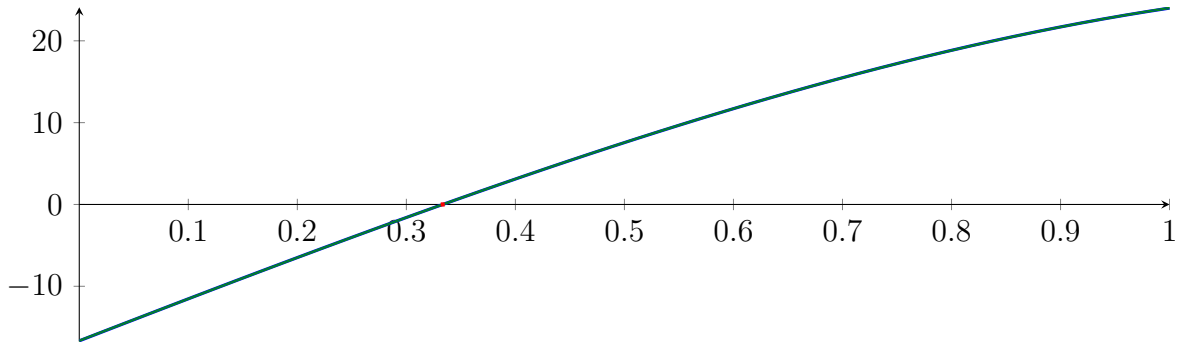
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

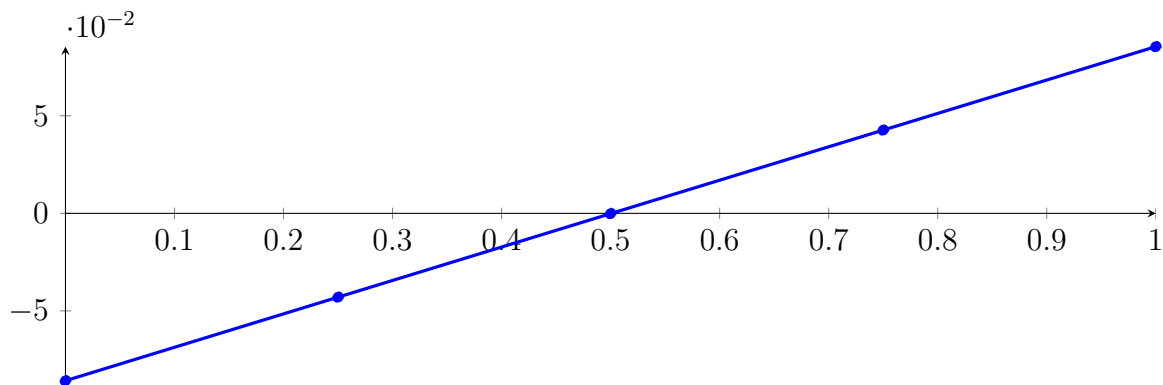
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

114.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

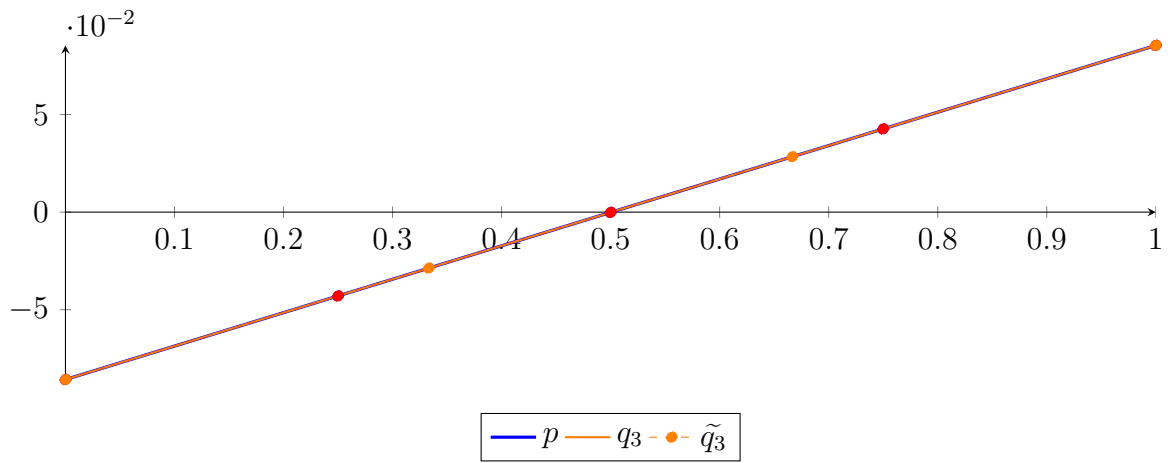
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 1.99222 \cdot 10^{-18} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m :

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

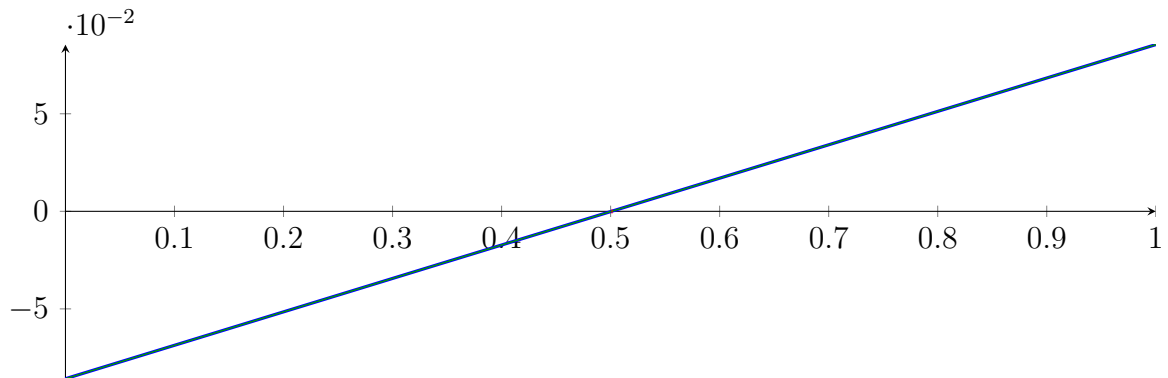
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\}$$

$$N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

Longest intersection interval: $1.7041 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

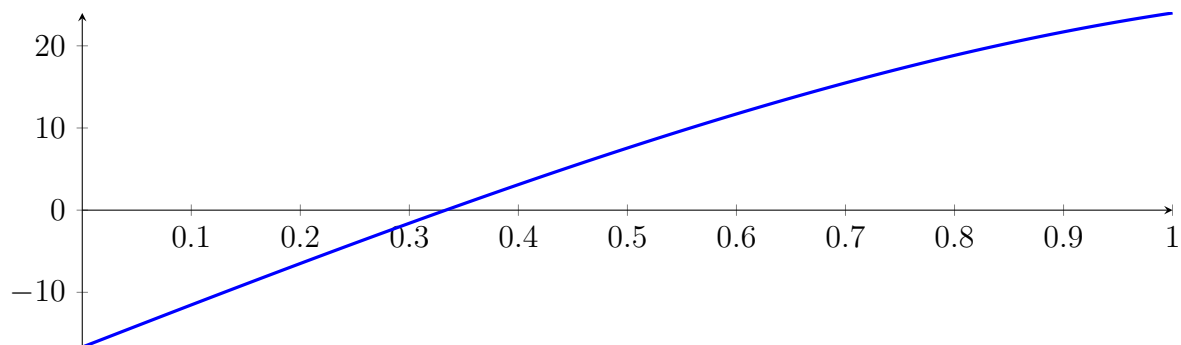
114.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

114.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

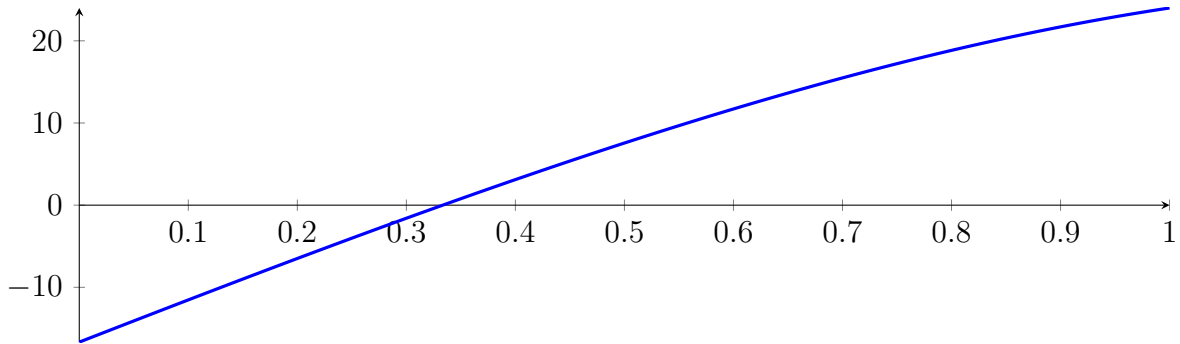
with precision $\varepsilon = 1 \cdot 10^{-08}$.

115 Running BezClip on f_4 with epsilon 16

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

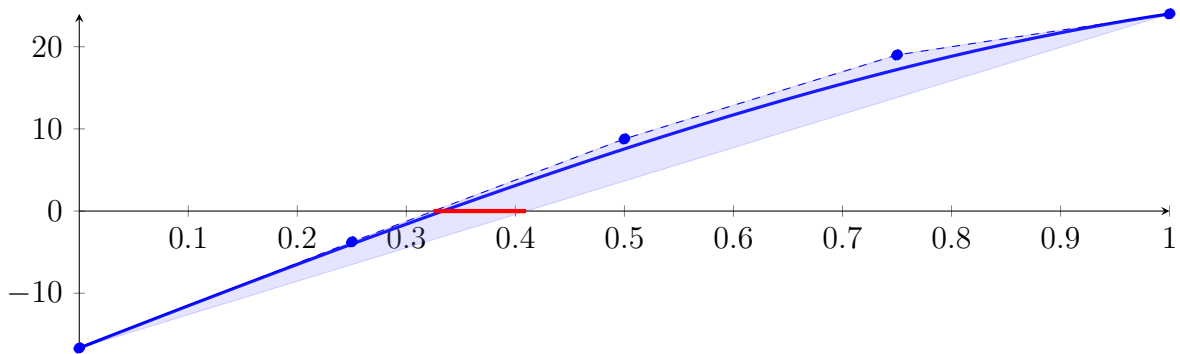
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



115.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

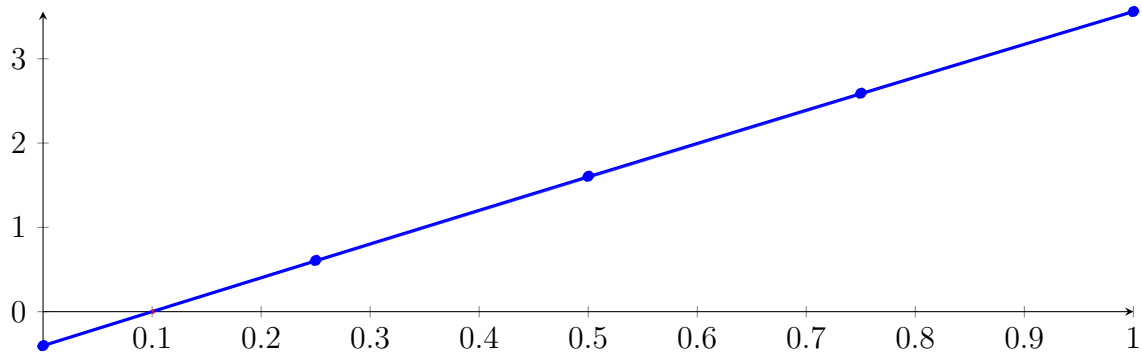
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

115.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

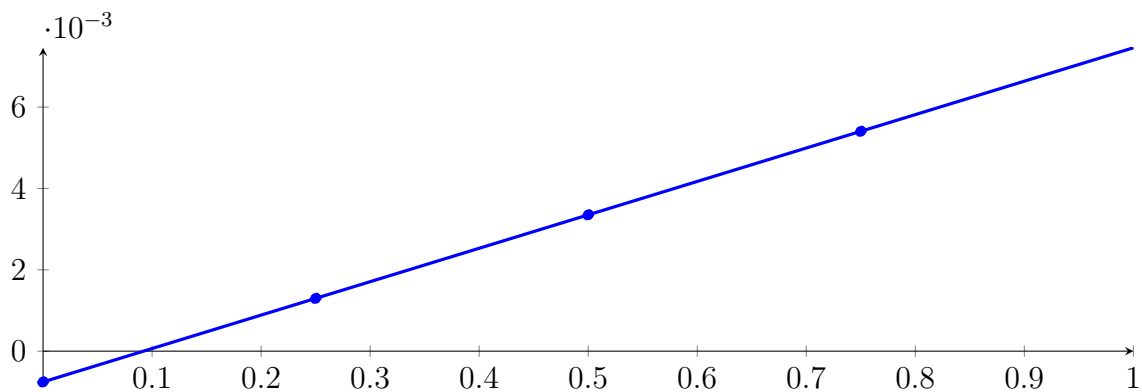
Longest intersection interval: 0.00203877

\implies Selective recursion: interval 1: $[0.333317, 0.333491]$,

115.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01974 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

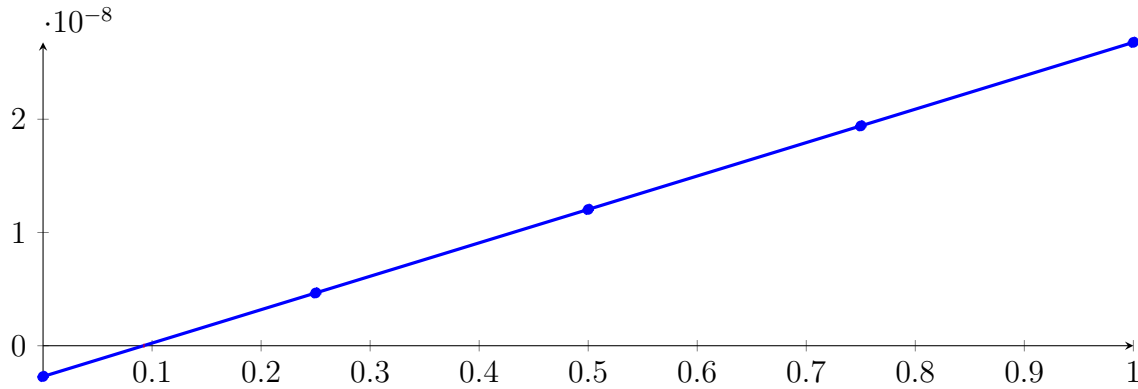
Longest intersection interval: $3.59185 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

115.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.61559 \cdot 10^{-27} X^4 - 3.23117 \cdot 10^{-27} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09} \\ &= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \\ &\quad \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28975 \cdot 10^{-11}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

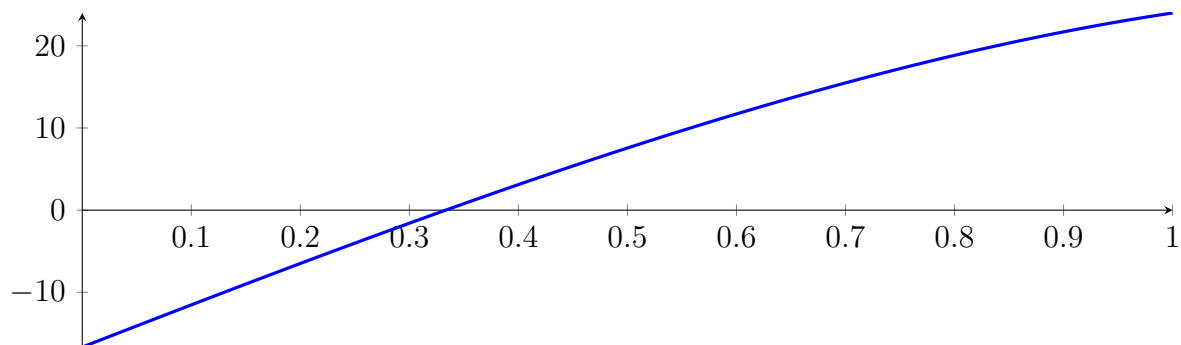
115.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

115.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

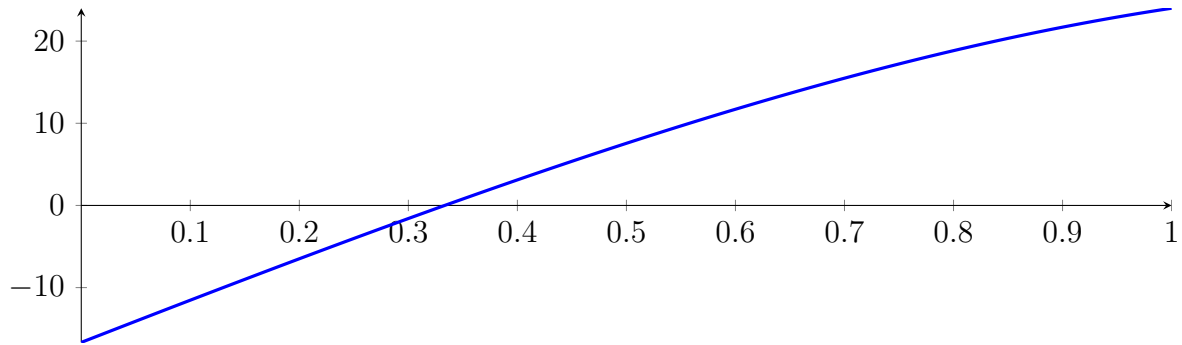
with precision $\varepsilon = 1 \cdot 10^{-16}$.

116 Running QuadClip on f_4 with epsilon 16

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

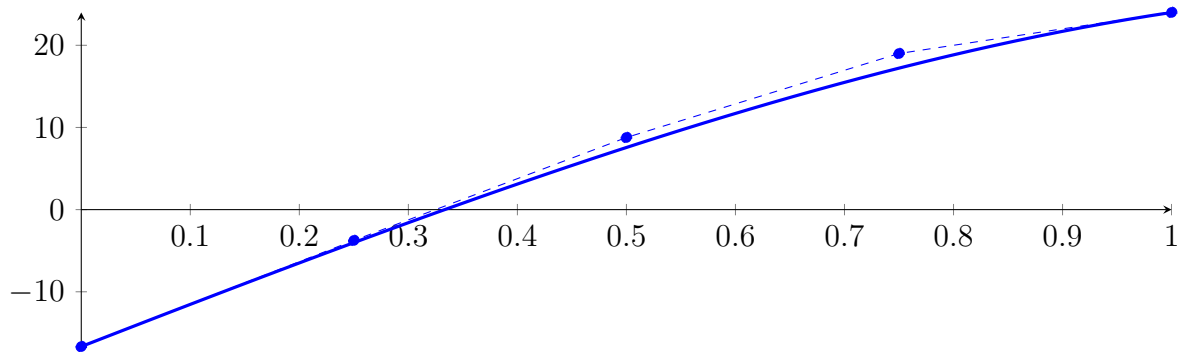
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



116.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

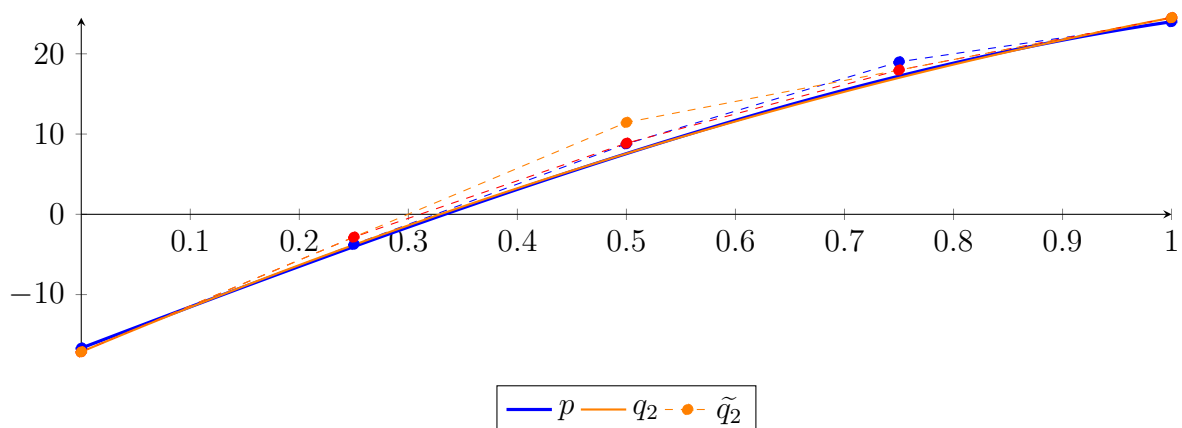
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1.45023 \cdot 10^{-15}X^4 + 4.00374 \cdot 10^{-15}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

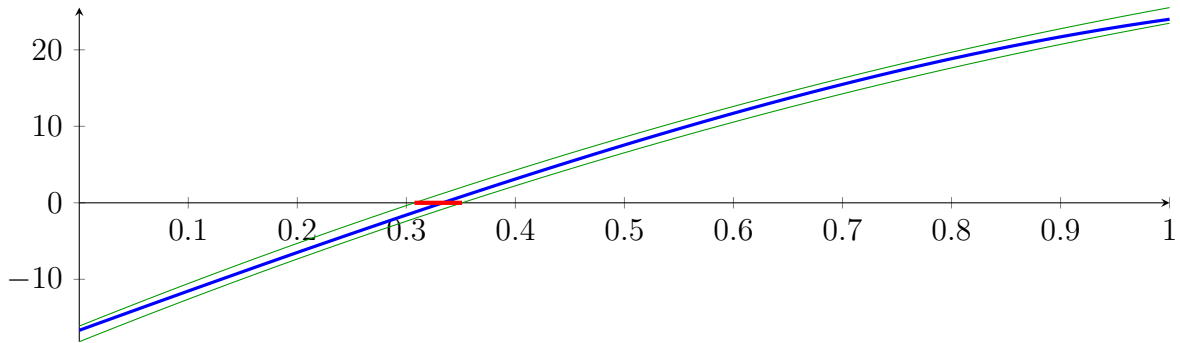
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

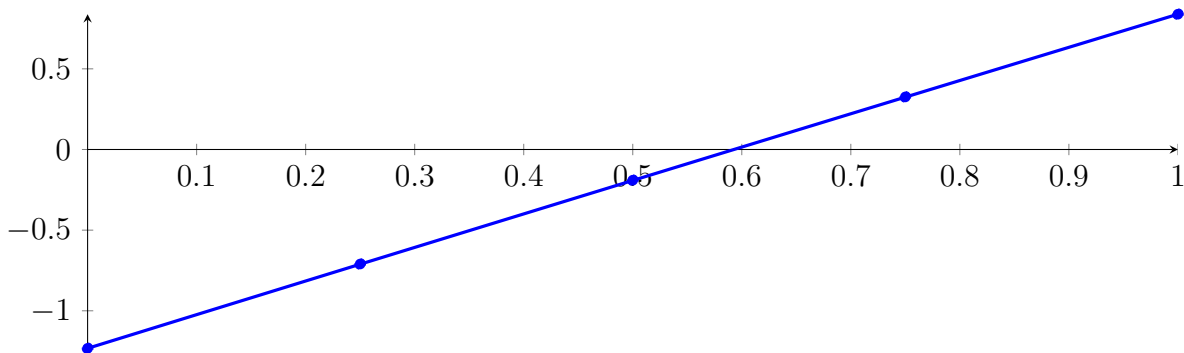
Longest intersection interval: 0.0436205

⇒ Selective recursion: interval 1: $[0.307477, 0.351097]$,

116.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

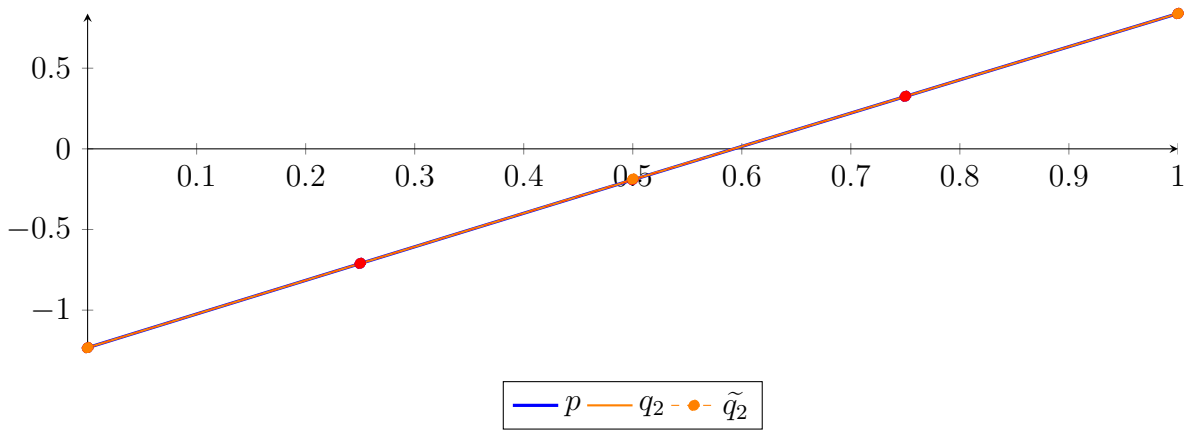
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1.45283 \cdot 10^{-17} X^4 + 3.33934 \cdot 10^{-17} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

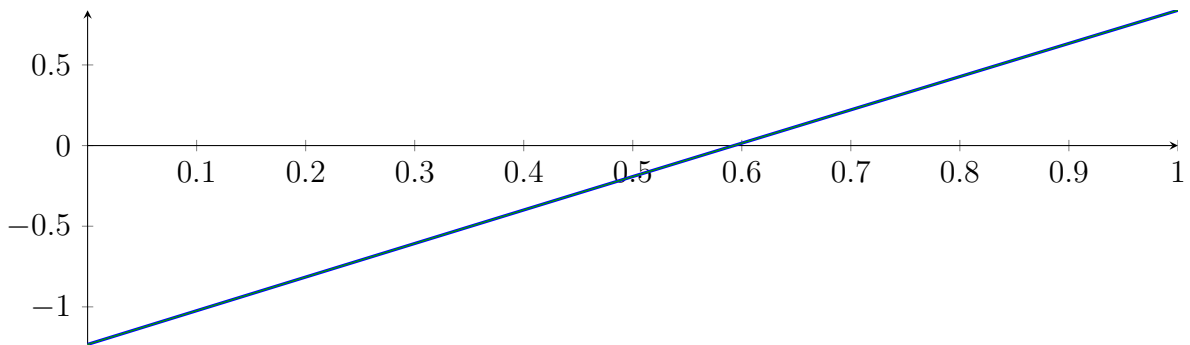
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

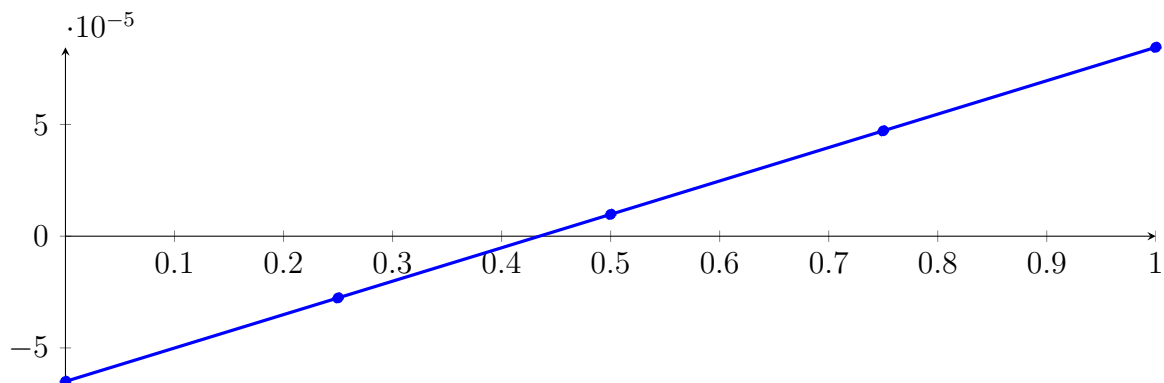
Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: **interval 1:** $[0.333332, 0.333335]$,

116.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.05879 \cdot 10^{-22} X^4 - 2.82525 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05} \\
 &= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \\
 &\quad \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)
 \end{aligned}$$



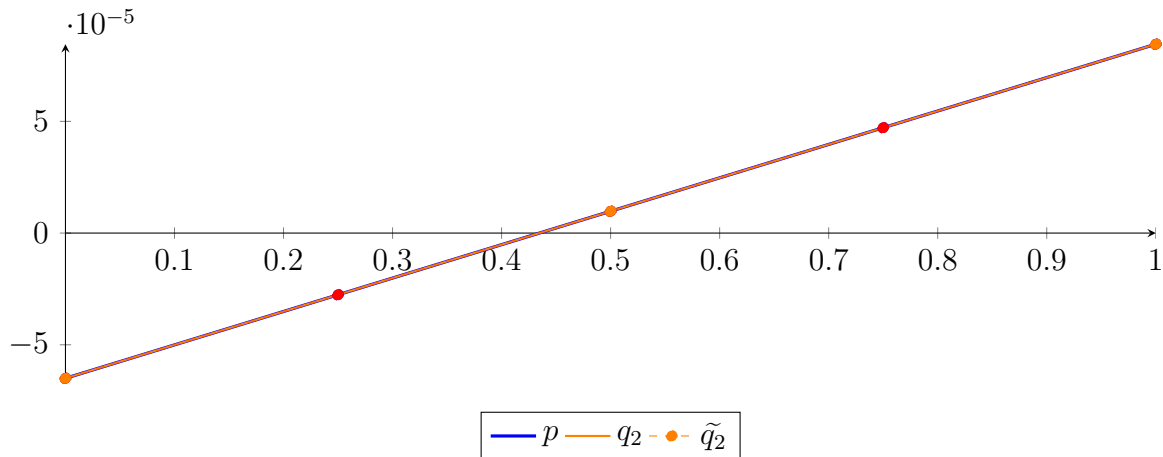
Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = -4.49986 \cdot 10^{-22} X^4 + 3.33519 \cdot 10^{-21} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.82529 \cdot 10^{-17}$.

Bounding polynomials M and m :

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

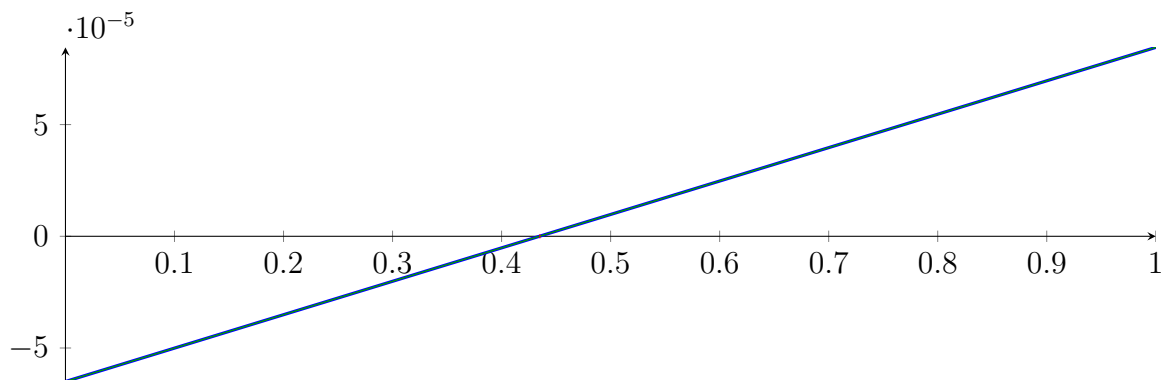
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.434685, 1.4089 \cdot 10^6\}$$

$$N(m) = \{0.434685, 1.4089 \cdot 10^6\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

Longest intersection interval: $3.74055 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

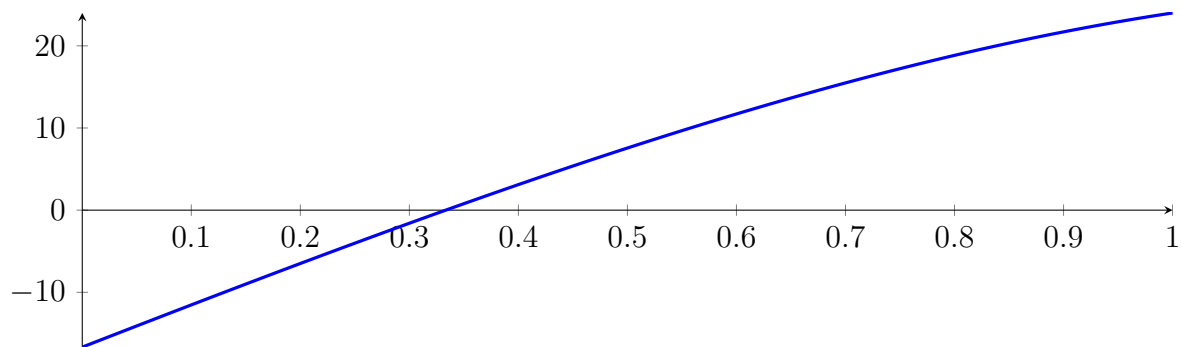
116.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

116.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

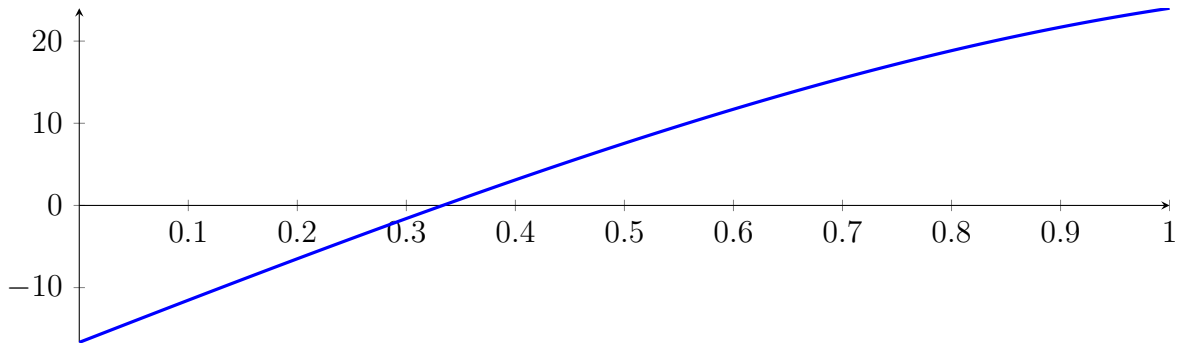
with precision $\varepsilon = 1 \cdot 10^{-16}$.

117 Running CubeClip on f_4 with epsilon 16

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

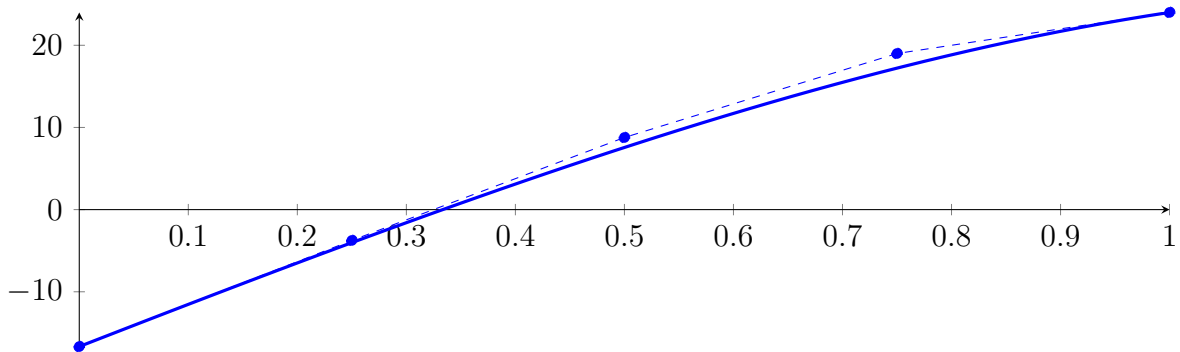
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



117.1 Recursion Branch 1 for Input Interval $[0, 1]$

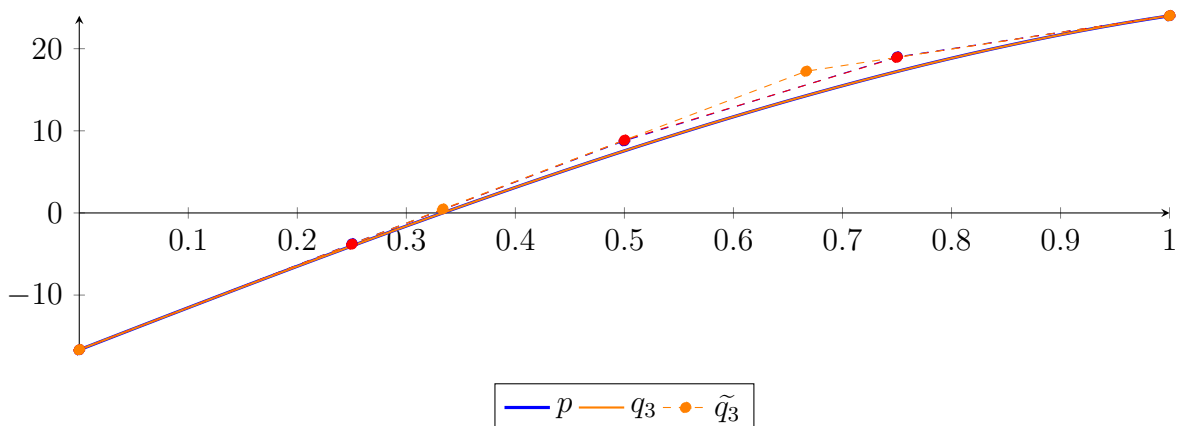
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= 3.34802 \cdot 10^{-16}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

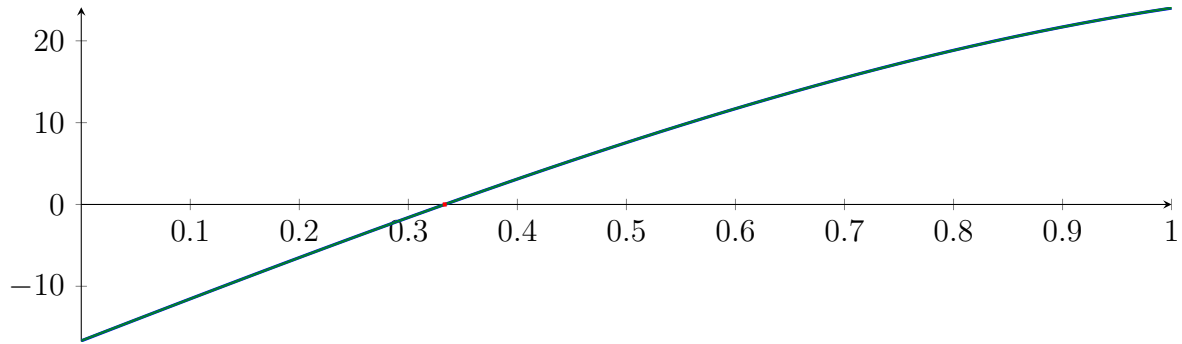
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

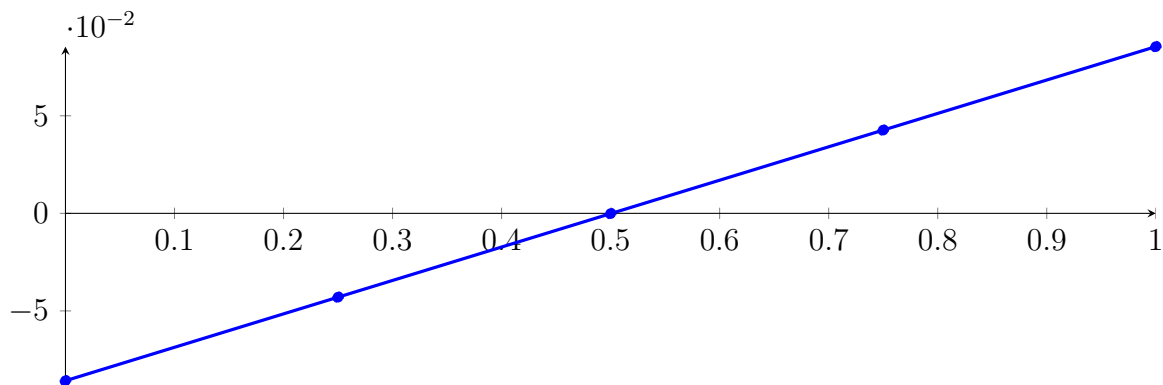
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

117.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

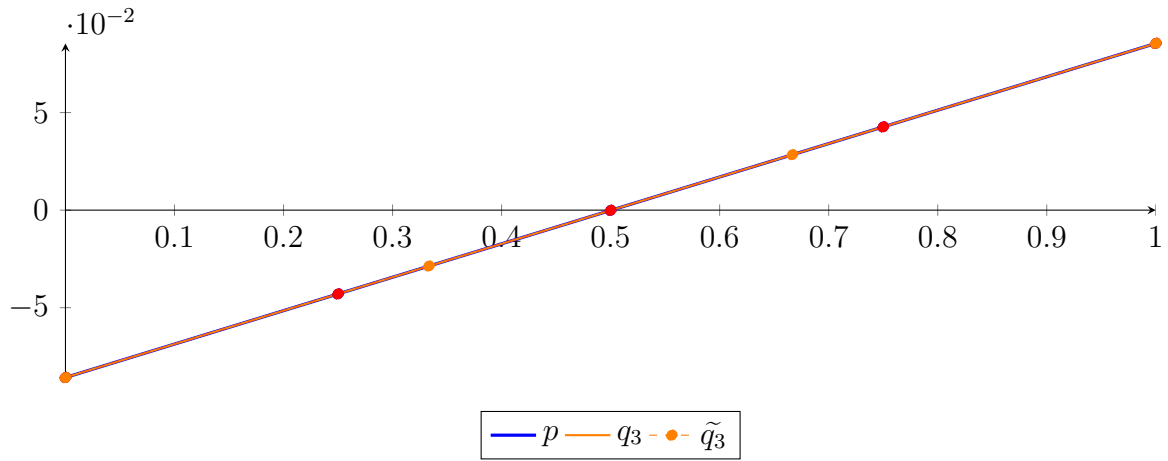
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 1.99222 \cdot 10^{-18} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m :

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

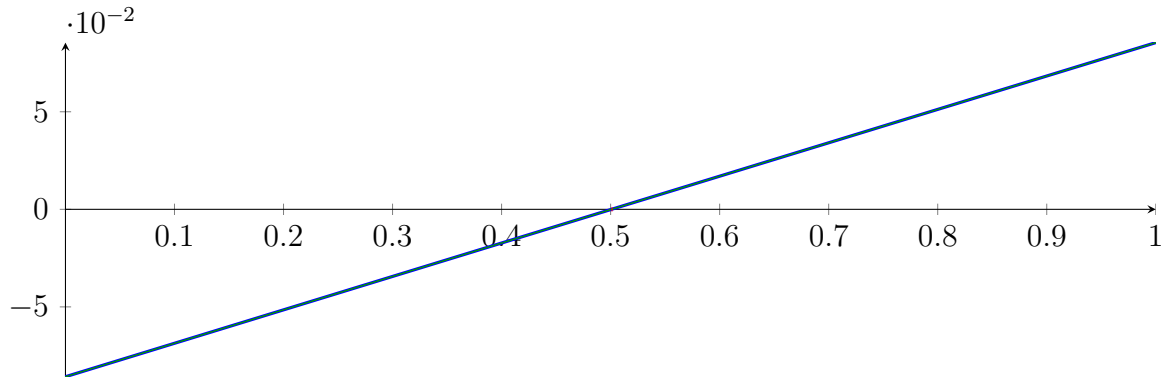
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\}$$

$$N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

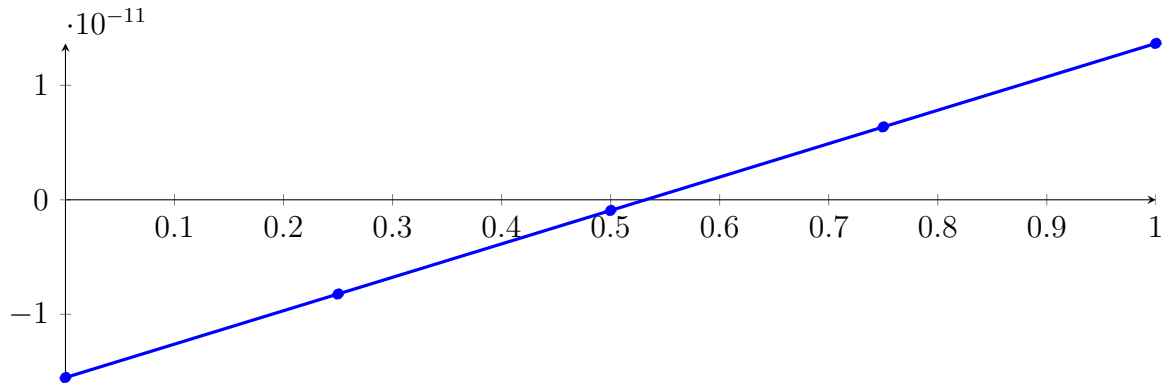
Longest intersection interval: $1.7041 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

117.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

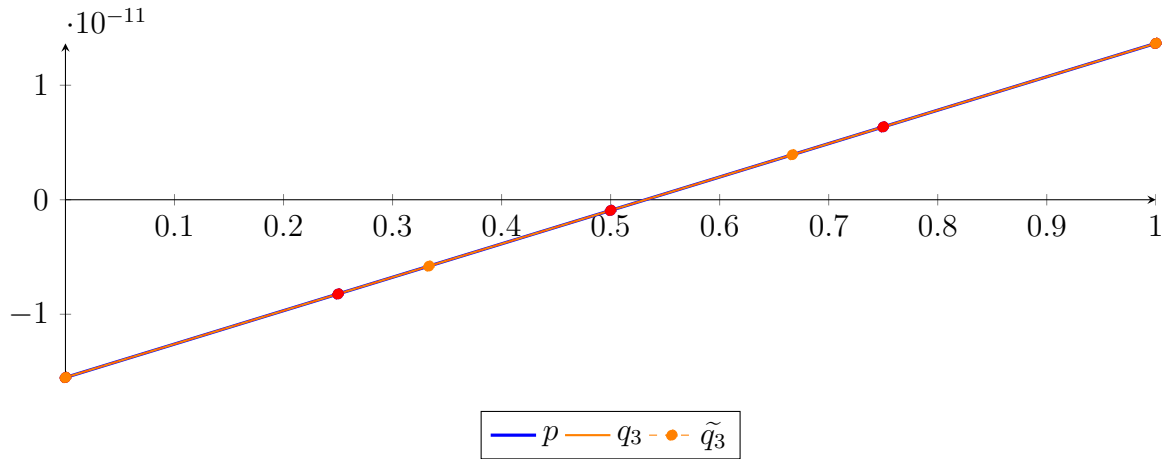
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\ &= -1.55233 \cdot 10^{-11} B_{0,4}(X) - 8.22817 \cdot 10^{-12} B_{1,4}(X) - 9.33054 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36206 \cdot 10^{-12} B_{3,4}(X) + 1.36572 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 &= -1.55233 \cdot 10^{-11} B_{0,3} - 5.79647 \cdot 10^{-12} B_{1,3} + 3.93036 \cdot 10^{-12} B_{2,3} + 1.36572 \cdot 10^{-11} B_{3,3} \\
 \tilde{q}_3 &= 3.84964 \cdot 10^{-28} X^4 - 2.1457 \cdot 10^{-28} X^3 - 4.04147 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 &= -1.55233 \cdot 10^{-11} B_{0,4} - 8.22817 \cdot 10^{-12} B_{1,4} - 9.33054 \cdot 10^{-13} B_{2,4} + 6.36206 \cdot 10^{-12} B_{3,4} + 1.36572 \cdot 10^{-11} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.13596 \cdot 10^{-28}$.

Bounding polynomials M and m :

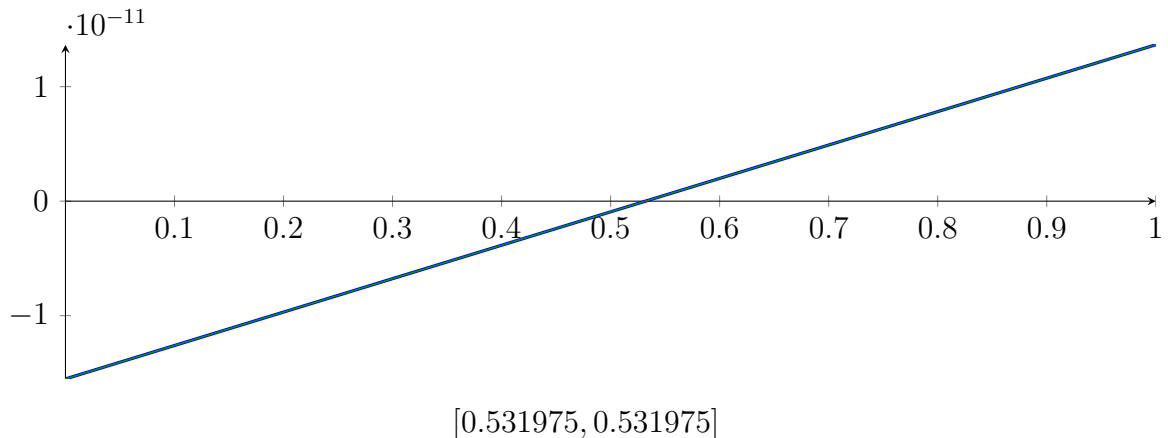
$$\begin{aligned}
 M &= 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 m &= 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.531975\}$$

$$N(m) = \{0.531975\}$$

Intersection intervals:



Longest intersection interval: 0

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333333]$,

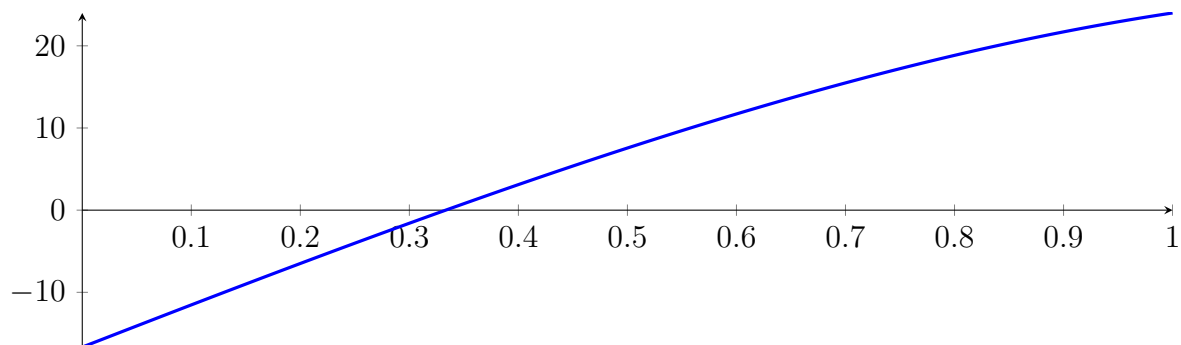
117.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

117.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

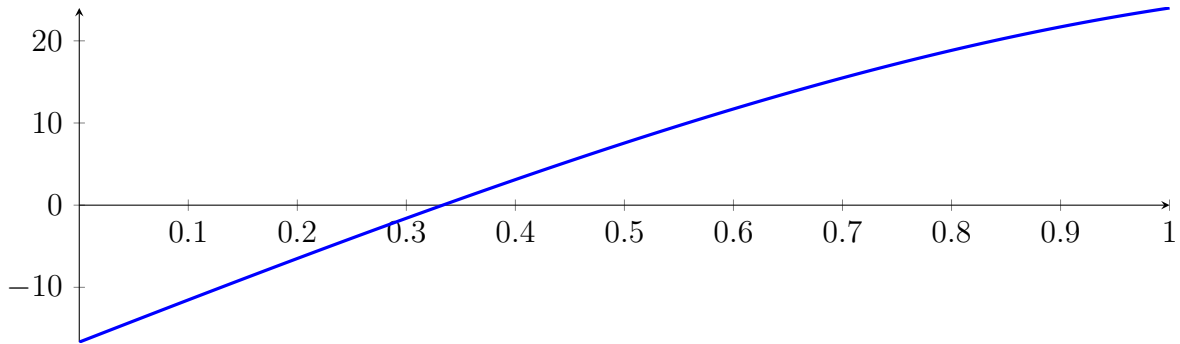
with precision $\varepsilon = 1 \cdot 10^{-16}$.

118 Running BezClip on f_4 with epsilon 32

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

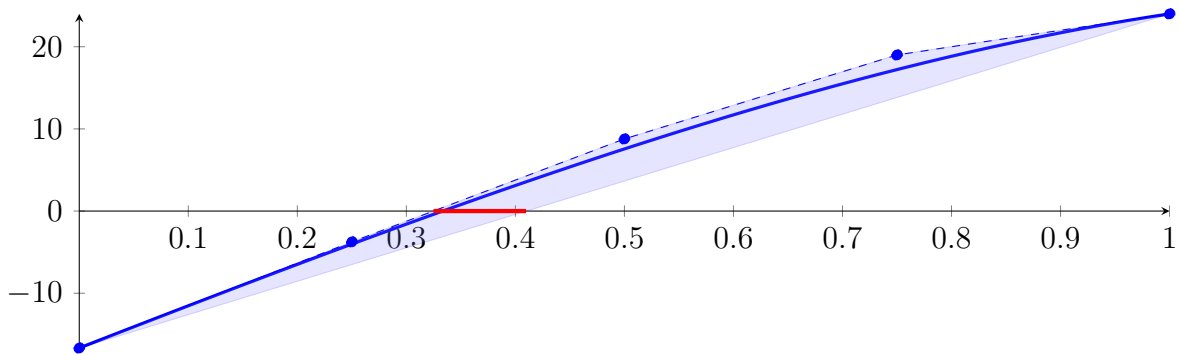
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



118.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

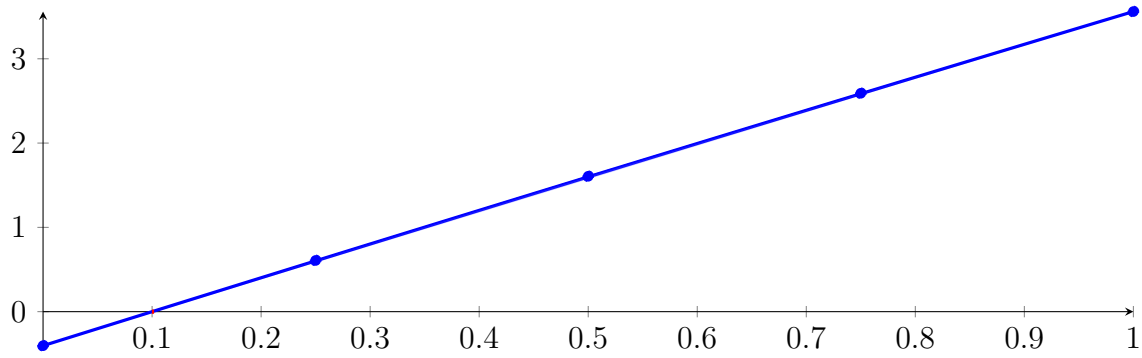
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

118.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

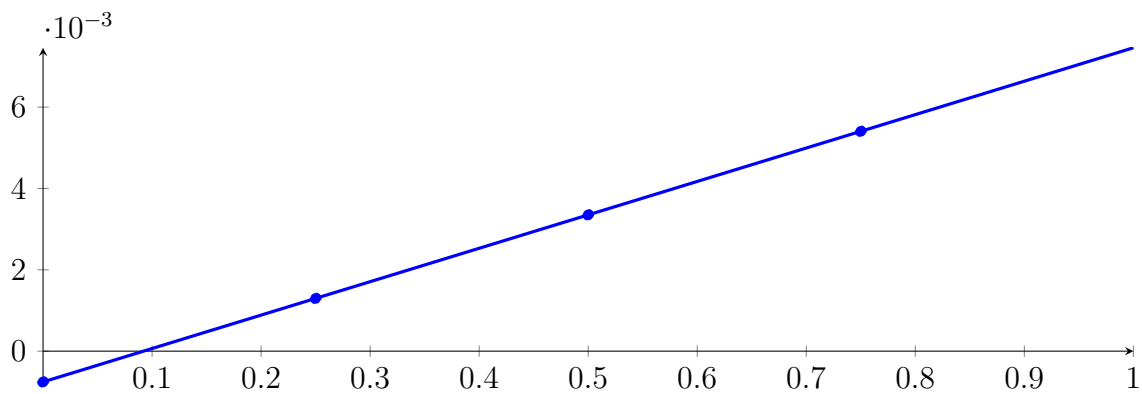
Longest intersection interval: 0.00203877

\implies Selective recursion: interval 1: $[0.333317, 0.333491]$,

118.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01974 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

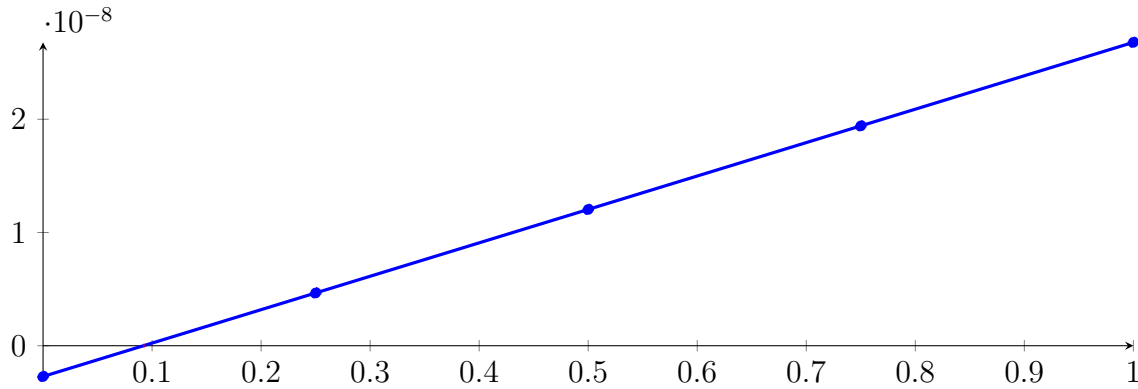
Longest intersection interval: $3.59185 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

118.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.61559 \cdot 10^{-27} X^4 - 3.23117 \cdot 10^{-27} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09} \\ &= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \\ &\quad \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28975 \cdot 10^{-11}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

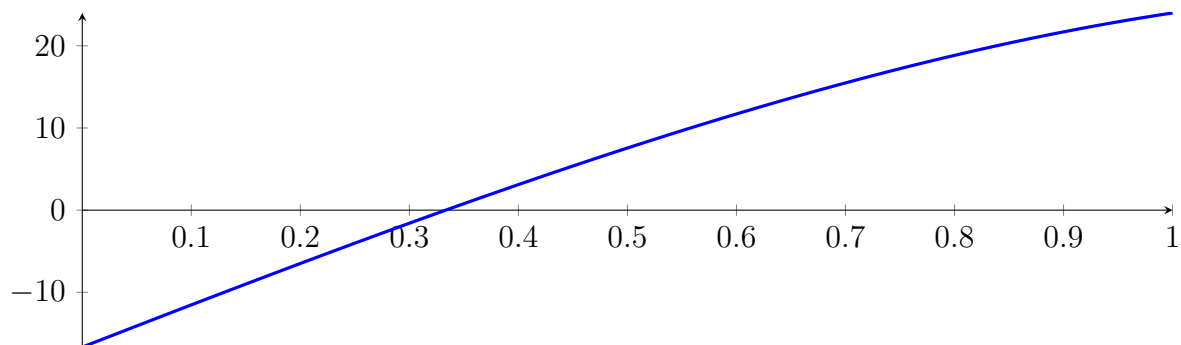
118.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

118.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

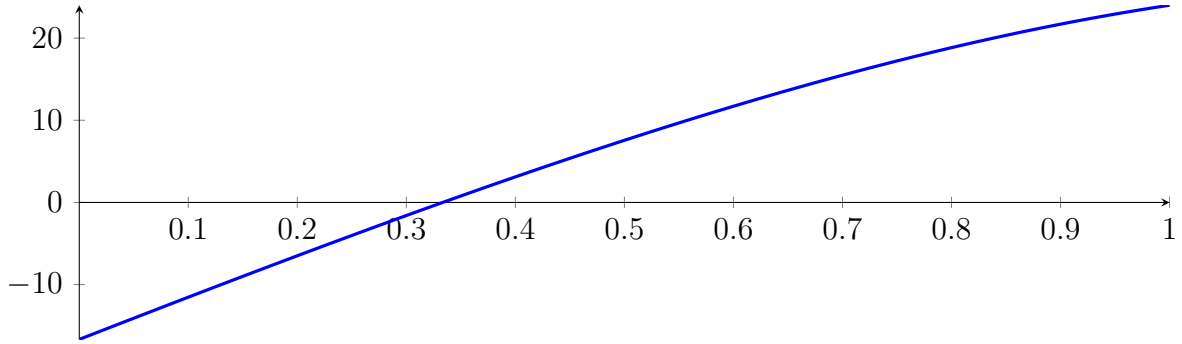
with precision $\varepsilon = 1 \cdot 10^{-32}$.

119 Running QuadClip on f_4 with epsilon 32

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

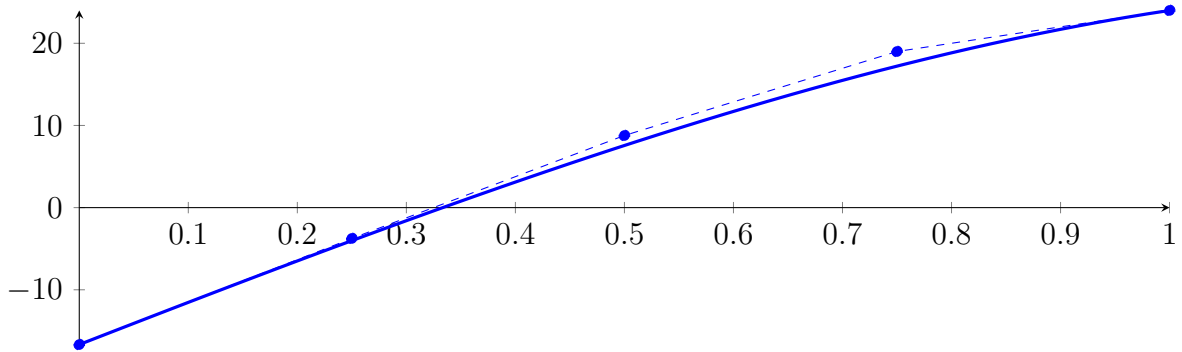


119.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

$$= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X)$$



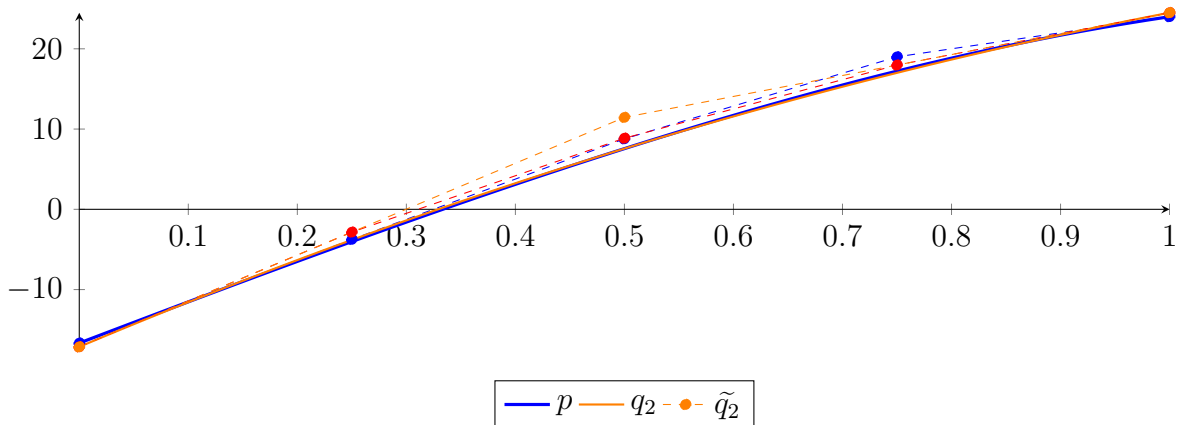
Degree reduction and raising:

$$q_2 = -15.5476X^2 + 57.181X - 17.1357$$

$$= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2}$$

$$\tilde{q}_2 = -1.45023 \cdot 10^{-15}X^4 + 4.00374 \cdot 10^{-15}X^3 - 15.5476X^2 + 57.181X - 17.1357$$

$$= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

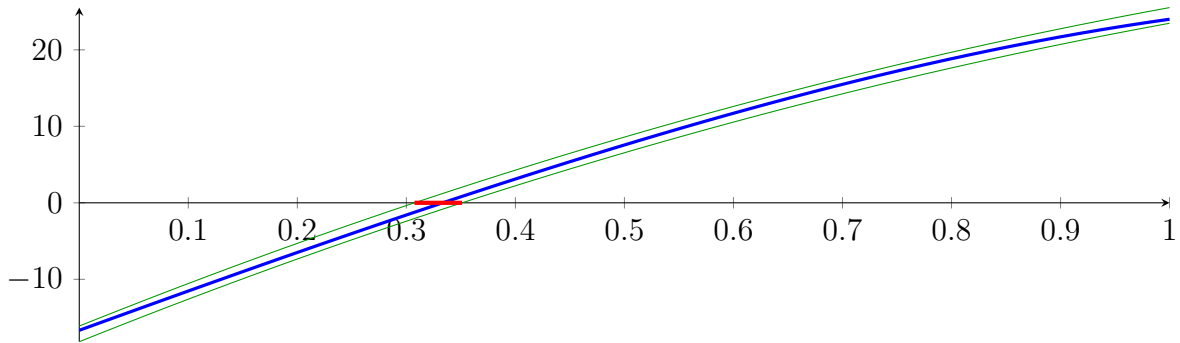
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

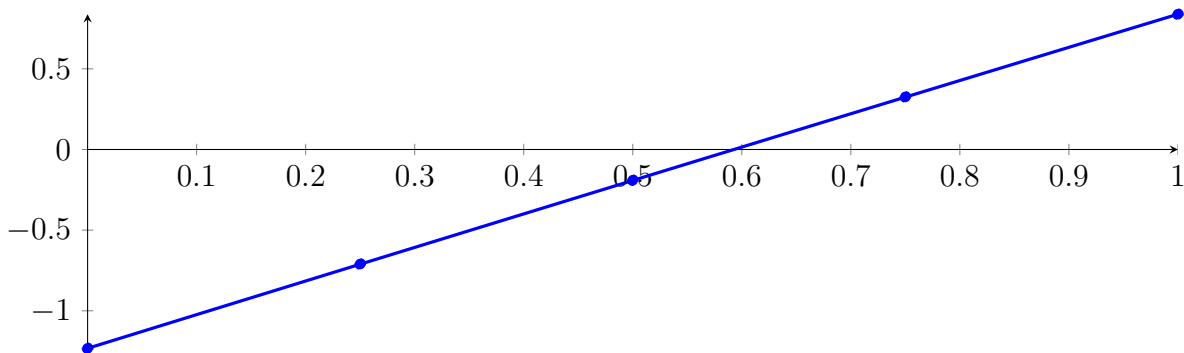
Longest intersection interval: 0.0436205

⇒ Selective recursion: interval 1: $[0.307477, 0.351097]$,

119.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

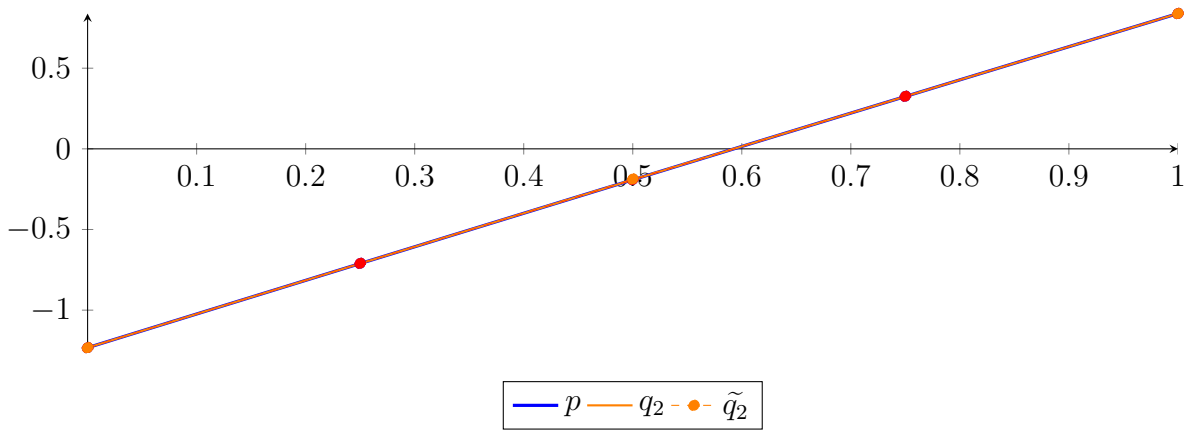
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1.45283 \cdot 10^{-17} X^4 + 3.33934 \cdot 10^{-17} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

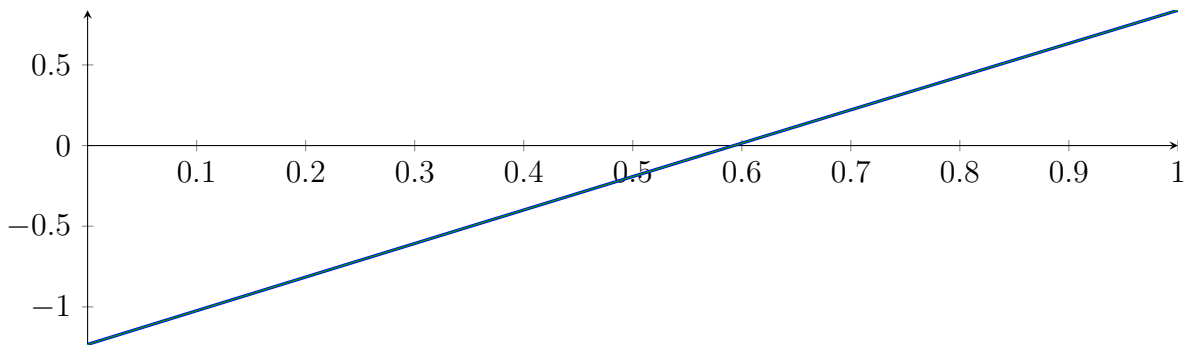
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

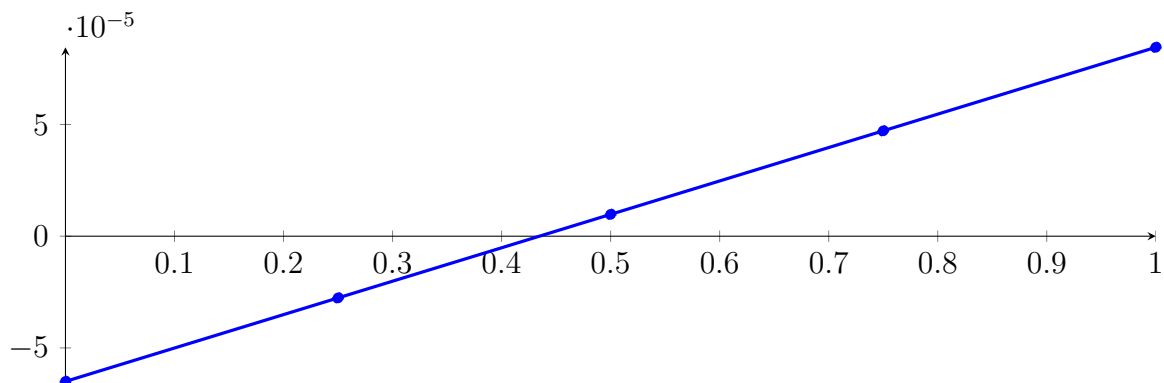
Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

119.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.05879 \cdot 10^{-22} X^4 - 2.82525 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05} \\
 &= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \\
 &\quad \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)
 \end{aligned}$$



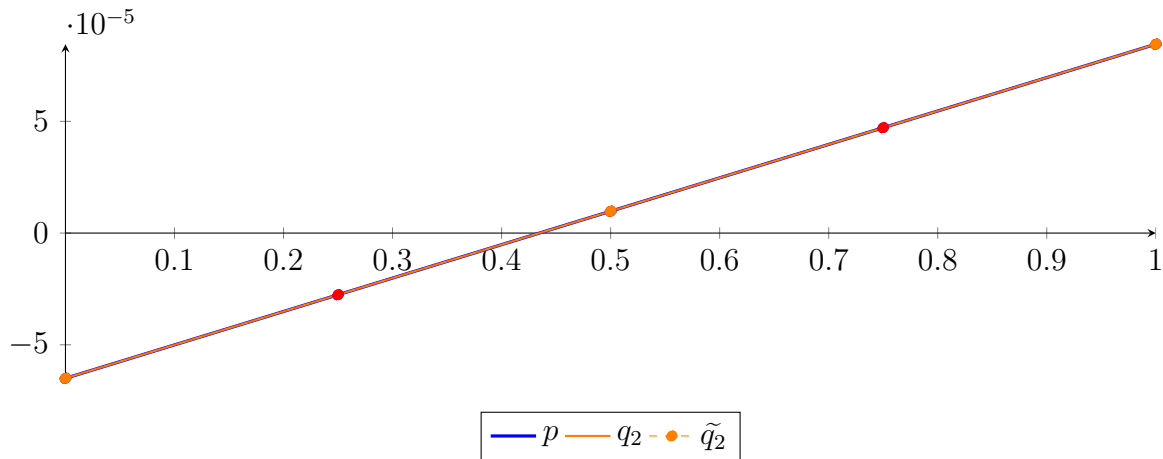
Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = -4.49986 \cdot 10^{-22} X^4 + 3.33519 \cdot 10^{-21} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.82529 \cdot 10^{-17}$.

Bounding polynomials M and m :

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

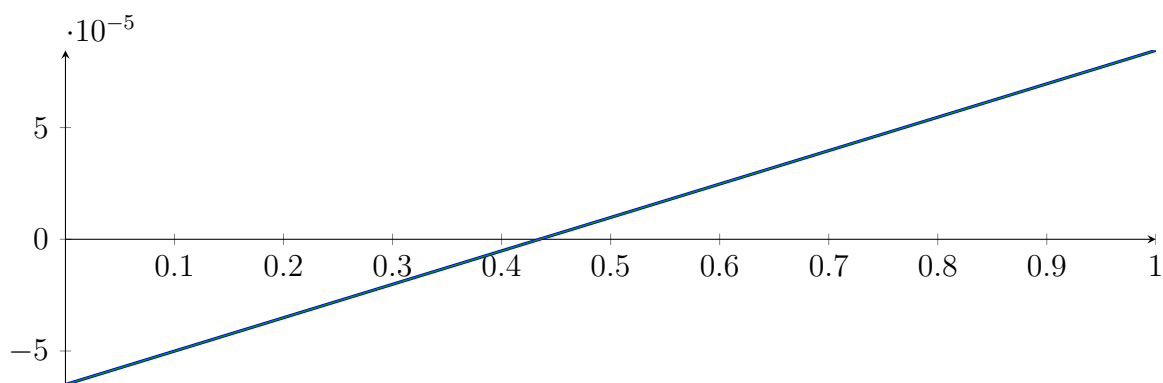
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.434685, 1.4089 \cdot 10^6\}$$

$$N(m) = \{0.434685, 1.4089 \cdot 10^6\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

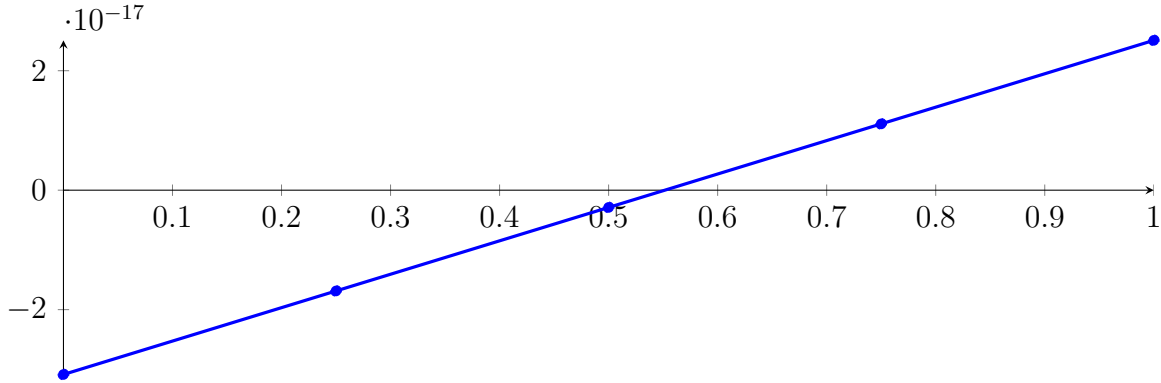
Longest intersection interval: $3.74055 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

119.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

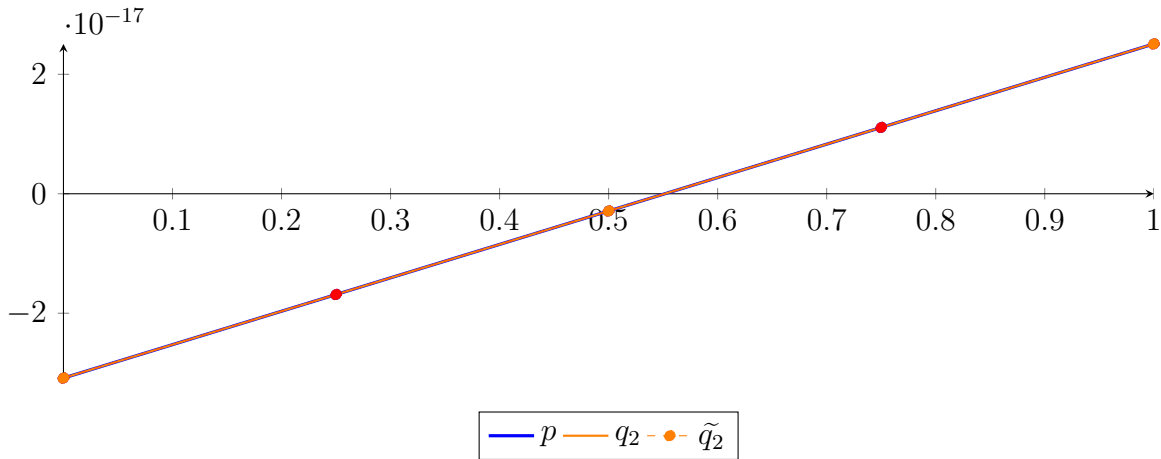
$$\begin{aligned}
 p &= -1.20371 \cdot 10^{-35} X^3 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17} \\
 &= -3.08561 \cdot 10^{-17} B_{0,4}(X) - 1.68712 \cdot 10^{-17} B_{1,4}(X) - 2.88624 \\
 &\quad \cdot 10^{-18} B_{2,4}(X) + 1.10987 \cdot 10^{-17} B_{3,4}(X) + 2.50836 \cdot 10^{-17} B_{4,4}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= 1.50463 \cdot 10^{-35} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17} \\
 &= -3.08561 \cdot 10^{-17} B_{0,2} - 2.88624 \cdot 10^{-18} B_{1,2} + 2.50836 \cdot 10^{-17} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -3.03936 \cdot 10^{-34} X^4 + 9.14817 \cdot 10^{-34} X^3 - 6.31946 \cdot 10^{-34} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17} \\
 &= -3.08561 \cdot 10^{-17} B_{0,4} - 1.68712 \cdot 10^{-17} B_{1,4} - 2.88624 \cdot 10^{-18} B_{2,4} + 1.10987 \cdot 10^{-17} B_{3,4} + 2.50836 \cdot 10^{-17} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.14701 \cdot 10^{-35}$.

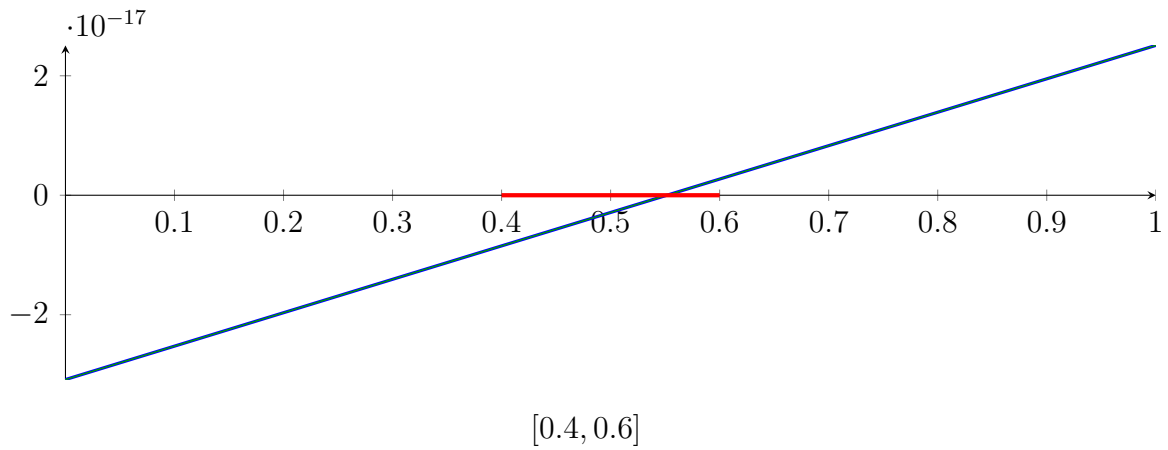
Bounding polynomials M and m :

$$\begin{aligned}
 M &= 1.50463 \cdot 10^{-35} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17} \\
 m &= 1.50463 \cdot 10^{-35} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-3.71783 \cdot 10^{18}, 0.6\} \qquad N(m) = \{-3.71783 \cdot 10^{18}, 0.4\}$$

Intersection intervals:

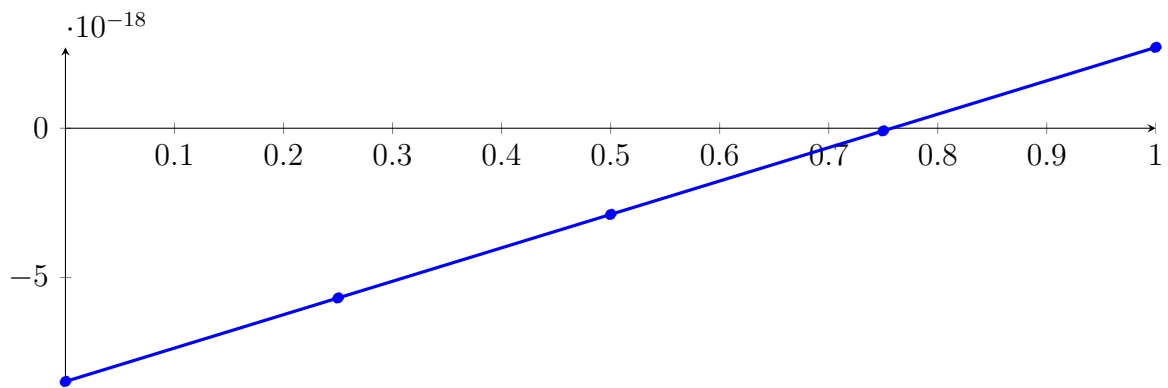


Longest intersection interval: 0.2
 \implies Selective recursion: interval 1: [0.333333, 0.333333],

119.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

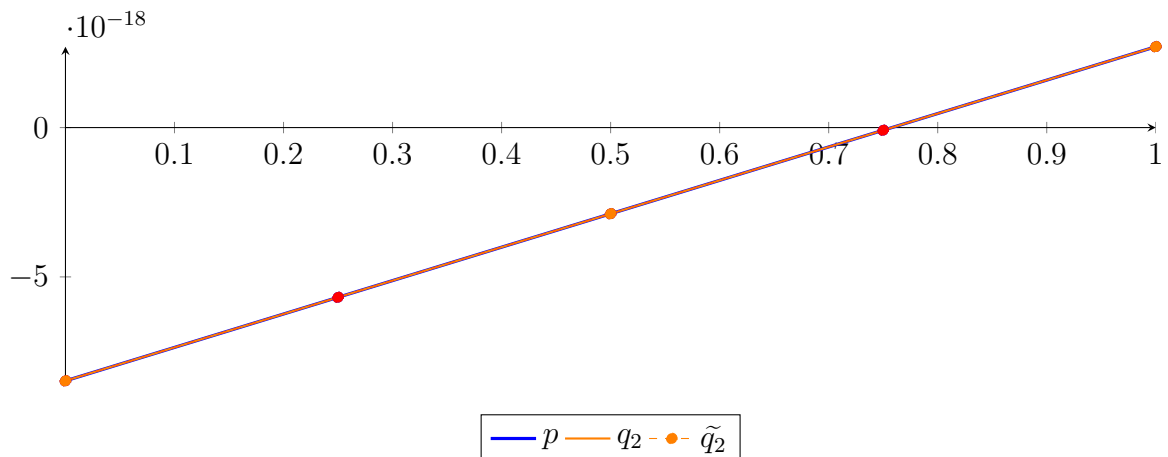
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 1.50463 \cdot 10^{-36} X^4 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18} \\
 &= -8.48022 \cdot 10^{-18} B_{0,4}(X) - 5.68323 \cdot 10^{-18} B_{1,4}(X) - 2.88624 \\
 &\quad \cdot 10^{-18} B_{2,4}(X) - 8.9255 \cdot 10^{-20} B_{3,4}(X) + 2.70773 \cdot 10^{-18} B_{4,4}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= 6.01853 \cdot 10^{-36} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18} \\
 &= -8.48022 \cdot 10^{-18} B_{0,2} - 2.88624 \cdot 10^{-18} B_{1,2} + 2.70773 \cdot 10^{-18} B_{2,2} \\
 \tilde{q}_2 &= -1.06829 \cdot 10^{-34} X^4 + 6.62038 \cdot 10^{-35} X^3 + 7.67363 \cdot 10^{-35} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18} \\
 &= -8.48022 \cdot 10^{-18} B_{0,4} - 5.68323 \cdot 10^{-18} B_{1,4} - 2.88624 \cdot 10^{-18} B_{2,4} - 8.9255 \cdot 10^{-20} B_{3,4} + 2.70773 \cdot 10^{-18} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.36039 \cdot 10^{-35}$.

Bounding polynomials M and m :

$$M = 6.01853 \cdot 10^{-36} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$

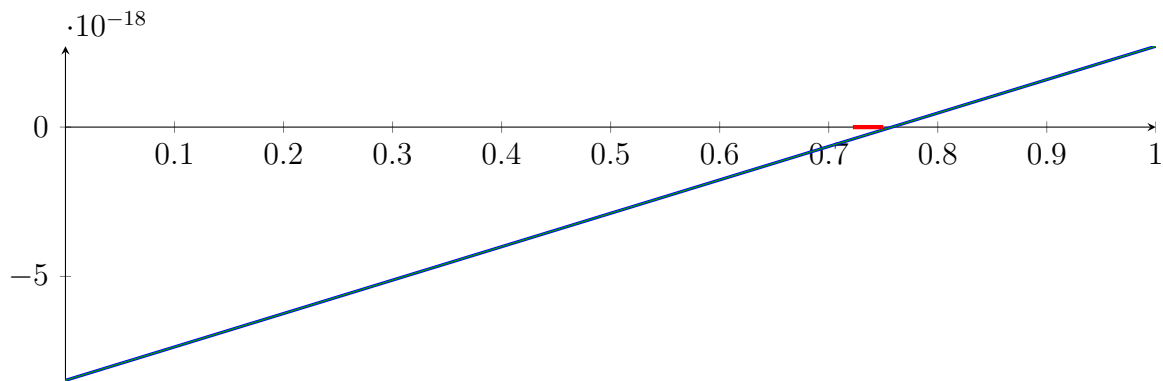
$$m = 6.77085 \cdot 10^{-36} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$

Root of M and m :

$$N(M) = \{-1.85892 \cdot 10^{18}, 0.75\}$$

$$N(m) = \{-1.65237 \cdot 10^{18}, 0.722222\}$$

Intersection intervals:



$$[0.722222, 0.75]$$

Longest intersection interval: 0.0277778

\implies Selective recursion: interval 1: [\[0.333333, 0.333333\]](#),

119.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [\[0.333333, 0.333333\]](#)

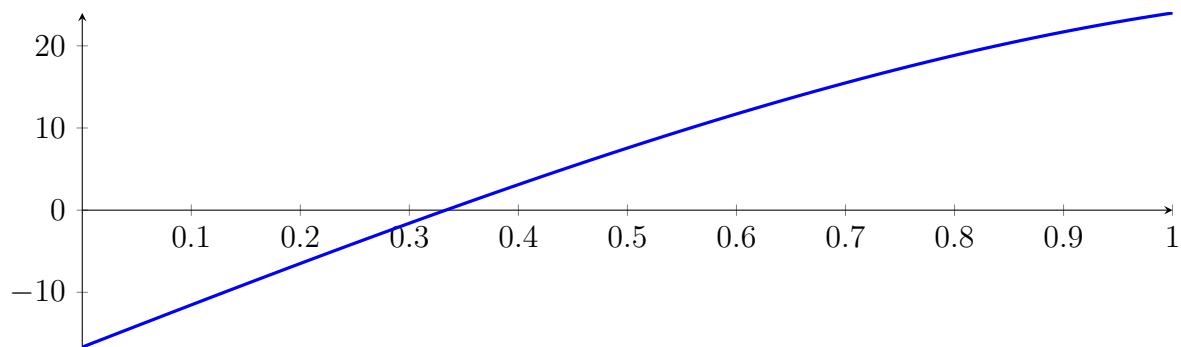
Reached interval [\[0.333333, 0.333333\]](#) **without sign change** at depth 6!

$$p(0) = -4.00031e-19 - p(1) - 8.9255e-20$$

119.7 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

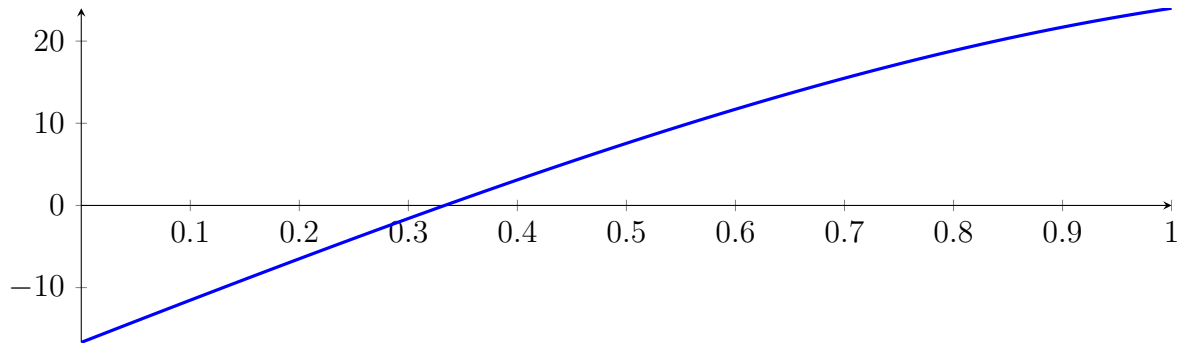
with precision $\varepsilon = 1 \cdot 10^{-32}$.

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$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

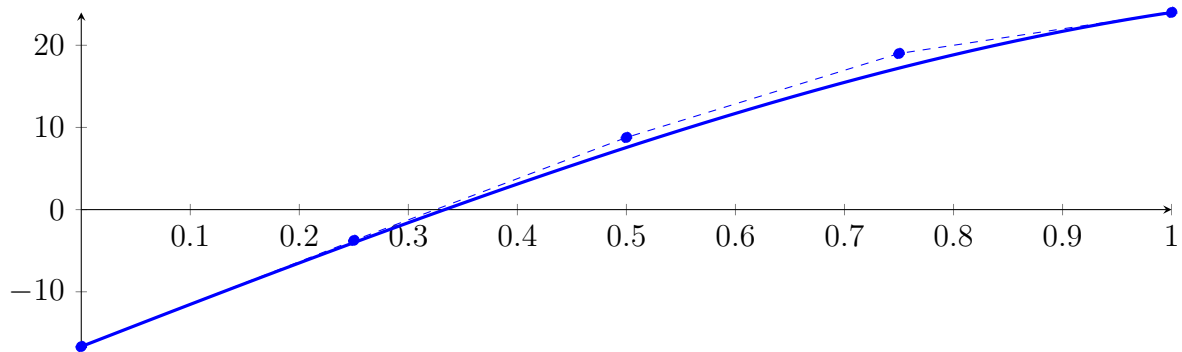
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



120.1 Recursion Branch 1 for Input Interval $[0, 1]$

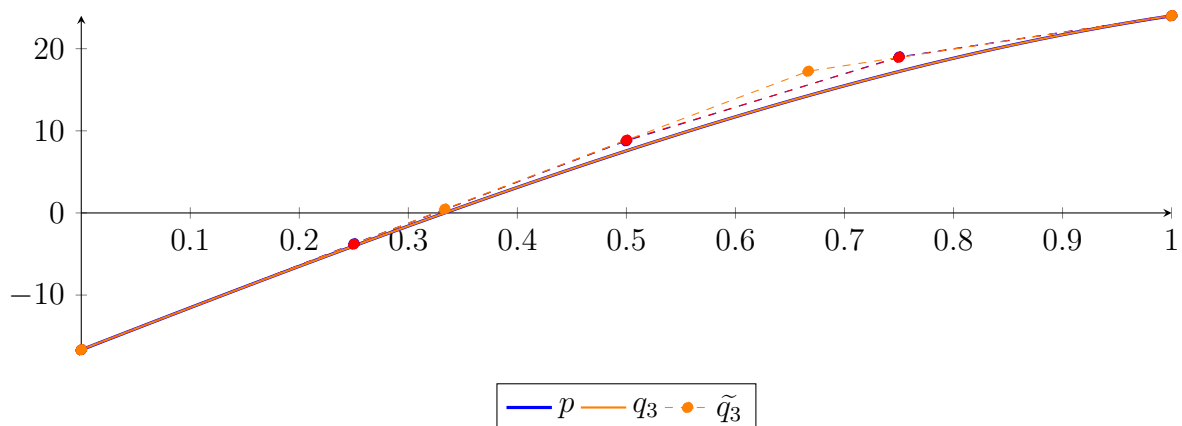
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= 3.34802 \cdot 10^{-16}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

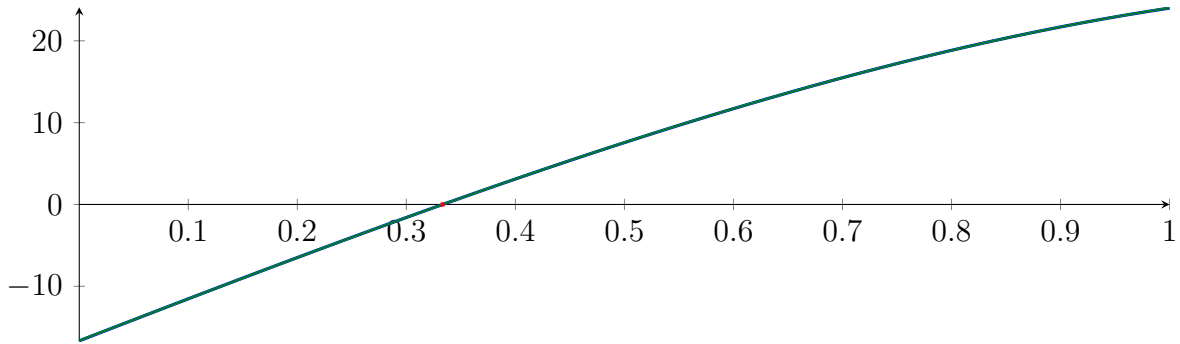
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

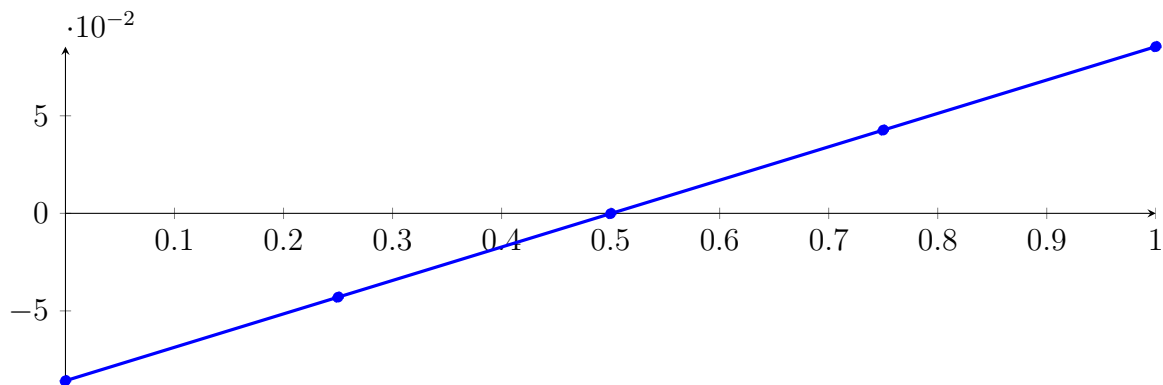
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

120.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

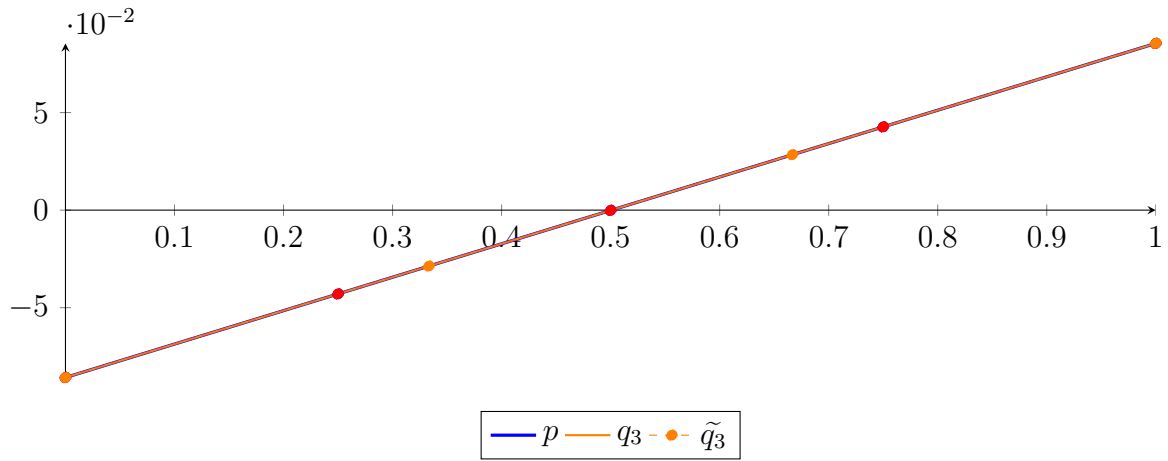
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 1.99222 \cdot 10^{-18} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m :

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

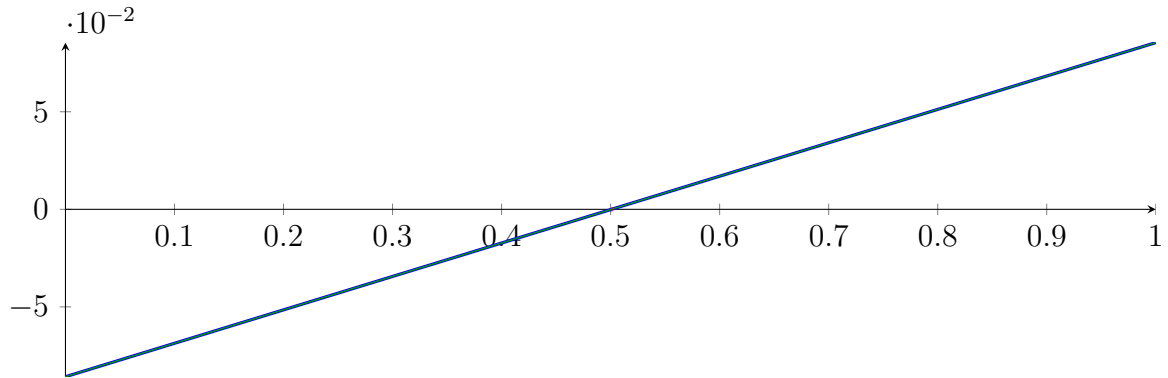
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\}$$

$$N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

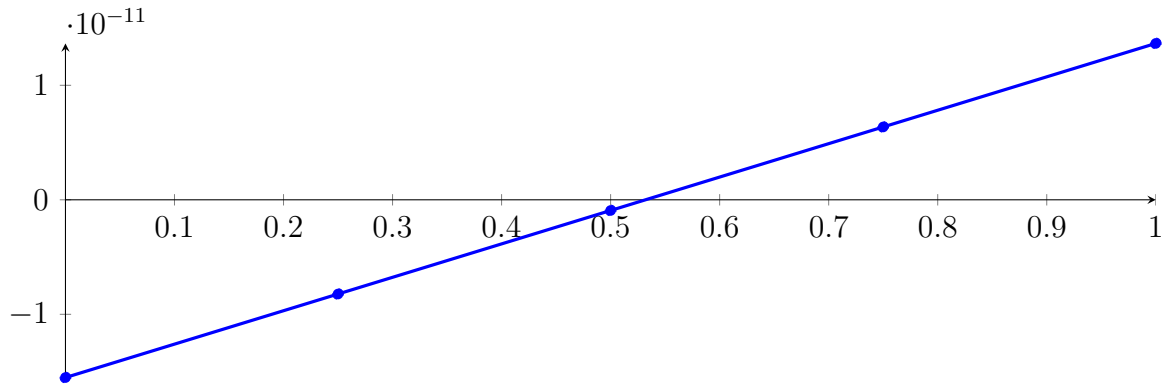
Longest intersection interval: $1.7041 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

120.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

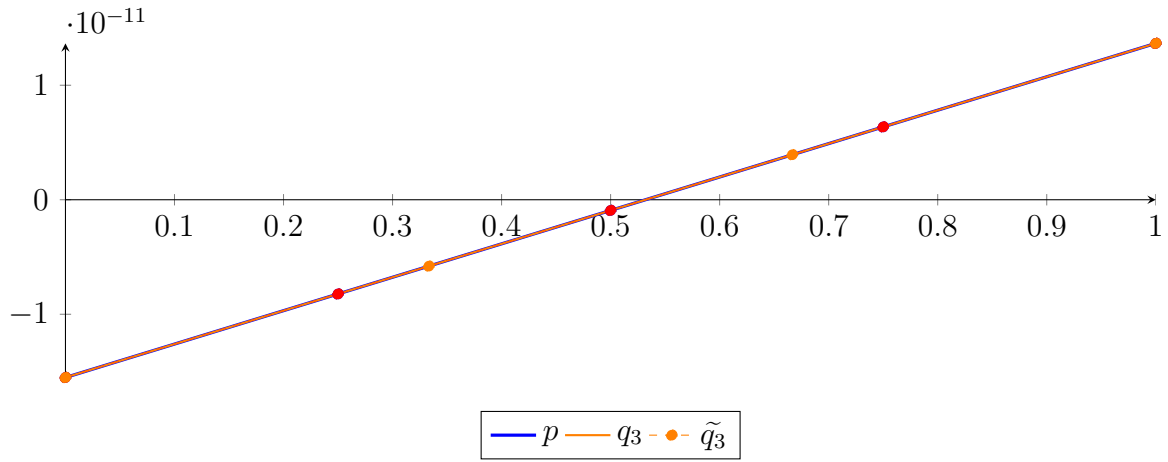
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\ &= -1.55233 \cdot 10^{-11} B_{0,4}(X) - 8.22817 \cdot 10^{-12} B_{1,4}(X) - 9.33054 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36206 \cdot 10^{-12} B_{3,4}(X) + 1.36572 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 &= -1.55233 \cdot 10^{-11} B_{0,3} - 5.79647 \cdot 10^{-12} B_{1,3} + 3.93036 \cdot 10^{-12} B_{2,3} + 1.36572 \cdot 10^{-11} B_{3,3} \\
 \tilde{q}_3 &= 3.84964 \cdot 10^{-28} X^4 - 2.1457 \cdot 10^{-28} X^3 - 4.04147 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 &= -1.55233 \cdot 10^{-11} B_{0,4} - 8.22817 \cdot 10^{-12} B_{1,4} - 9.33054 \cdot 10^{-13} B_{2,4} + 6.36206 \cdot 10^{-12} B_{3,4} + 1.36572 \cdot 10^{-11} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.13596 \cdot 10^{-28}$.

Bounding polynomials M and m :

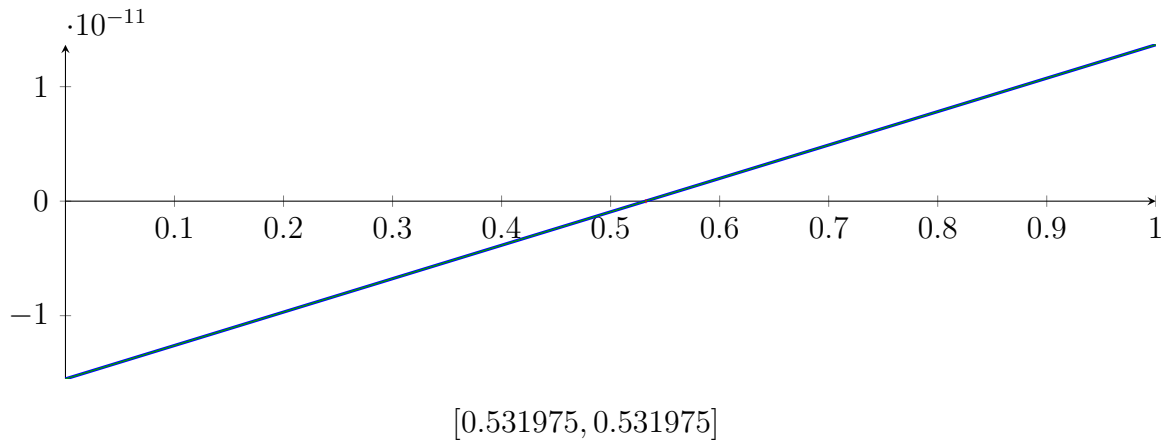
$$\begin{aligned}
 M &= 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 m &= 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.531975\}$$

$$N(m) = \{0.531975\}$$

Intersection intervals:



Longest intersection interval: 0

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333333]$,

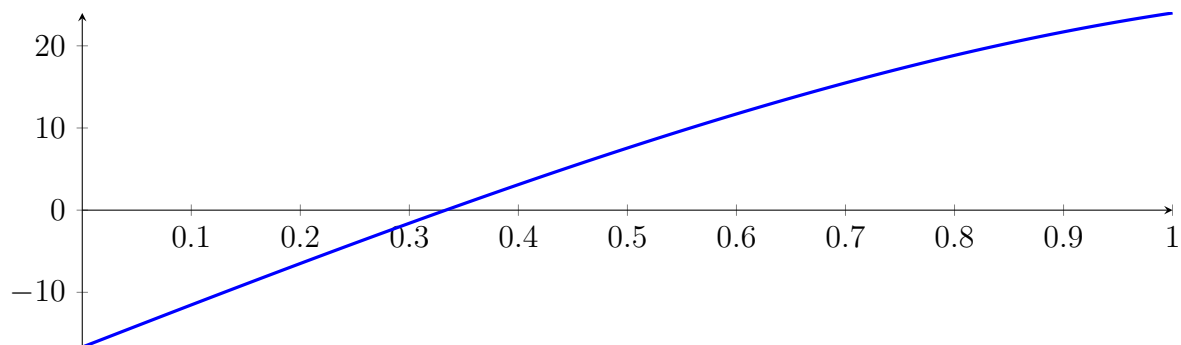
120.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

120.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

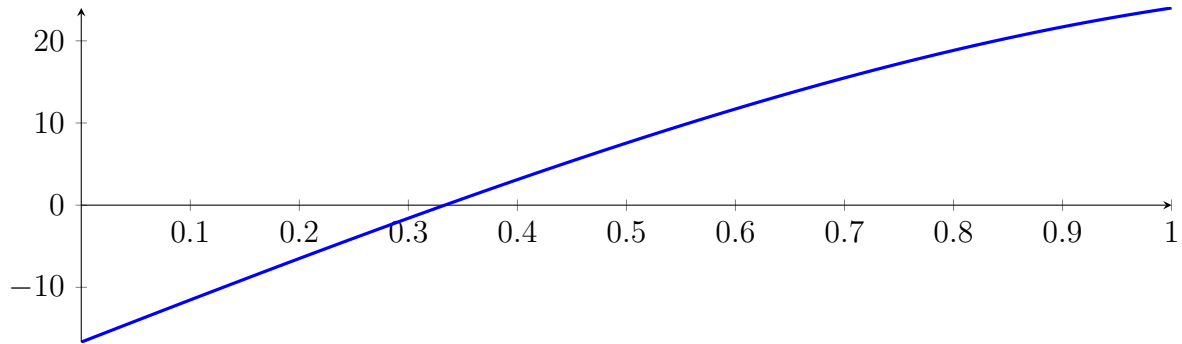
with precision $\varepsilon = 1 \cdot 10^{-32}$.

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$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

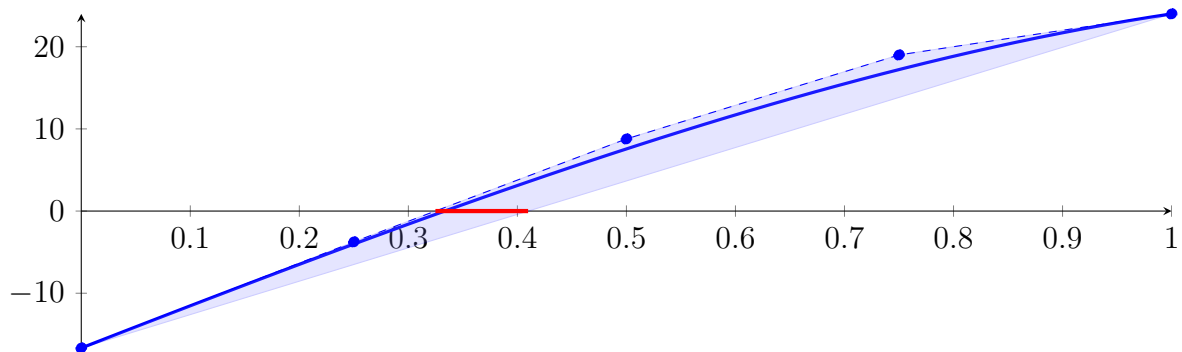
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



121.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

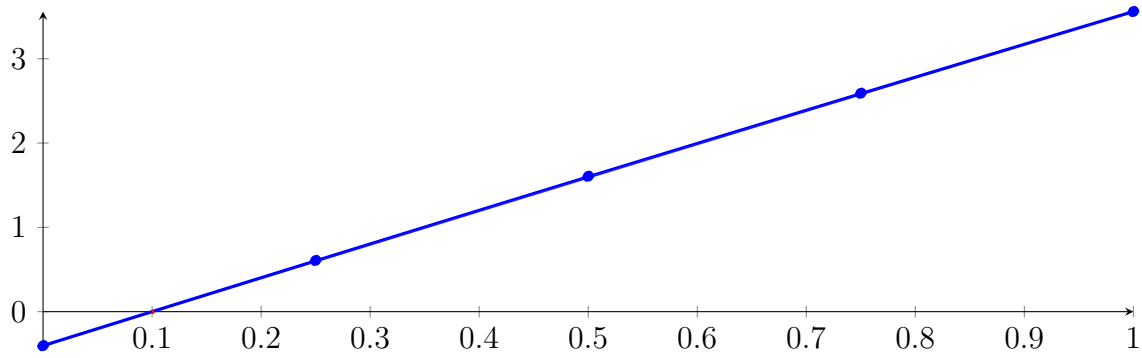
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

121.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711 B_{0,4}(X) + 0.607537 B_{1,4}(X) + 1.60621 B_{2,4}(X) + 2.59095 B_{3,4}(X) + 3.5603 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

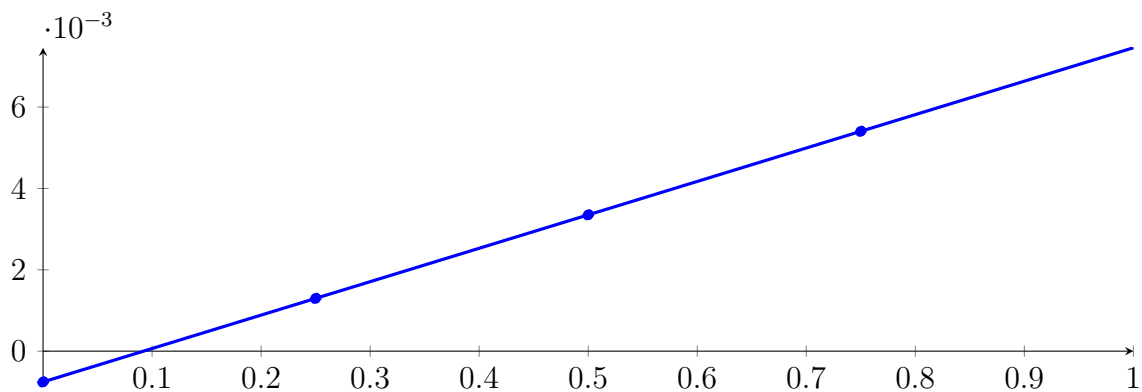
Longest intersection interval: 0.00203877

\implies Selective recursion: interval 1: $[0.333317, 0.333491]$,

121.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01974 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

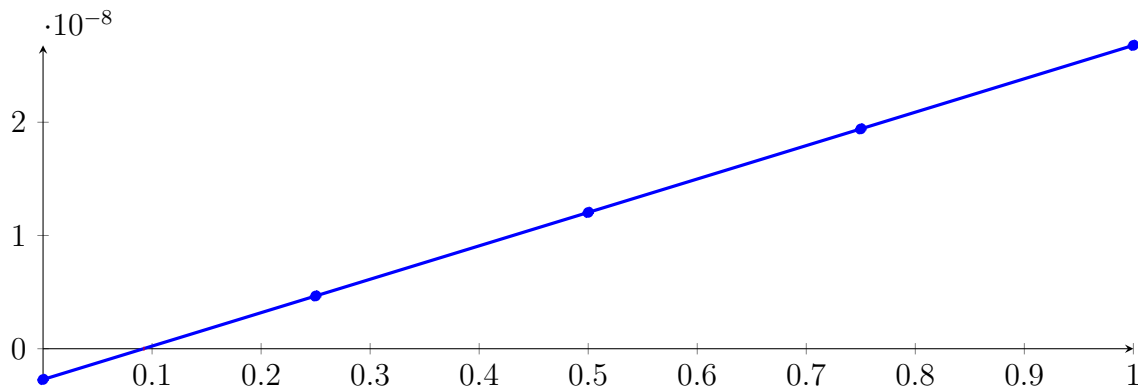
Longest intersection interval: $3.59185 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

121.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.61559 \cdot 10^{-27} X^4 - 3.23117 \cdot 10^{-27} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09} \\ &= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \\ &\quad \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28975 \cdot 10^{-11}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

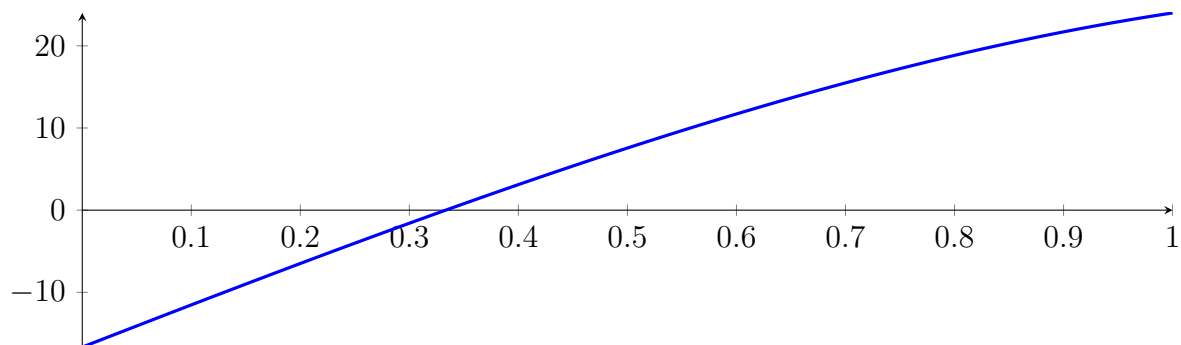
121.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

121.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

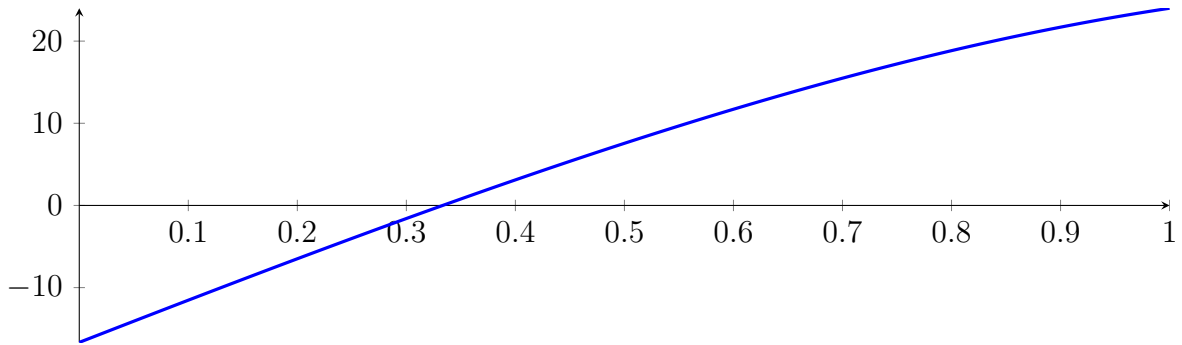
with precision $\varepsilon = 1 \cdot 10^{-64}$.

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$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

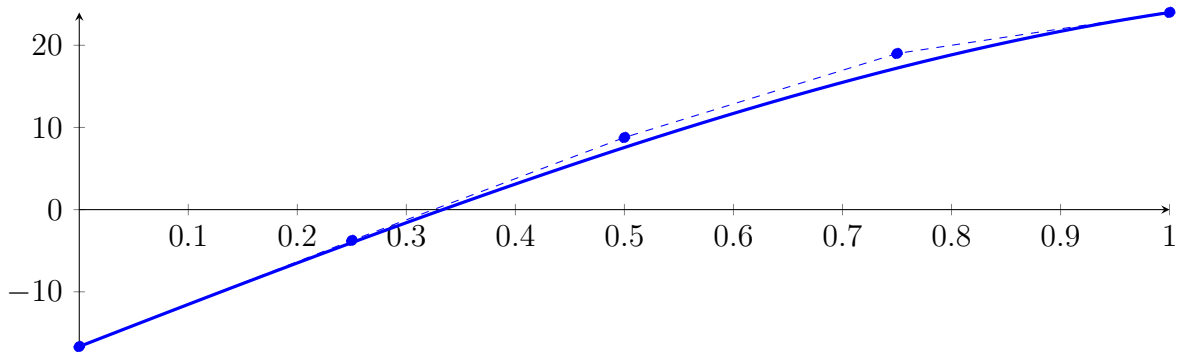
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



122.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

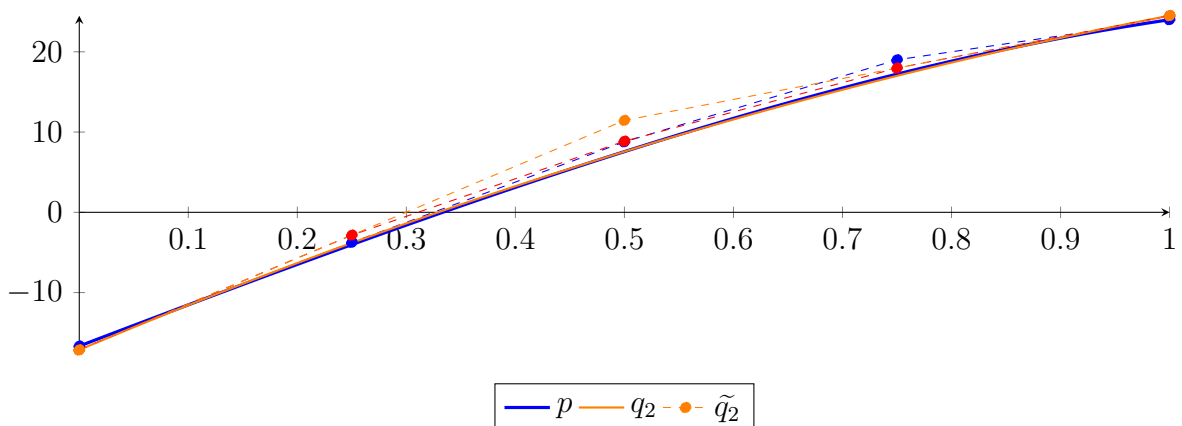
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1.45023 \cdot 10^{-15}X^4 + 4.00374 \cdot 10^{-15}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

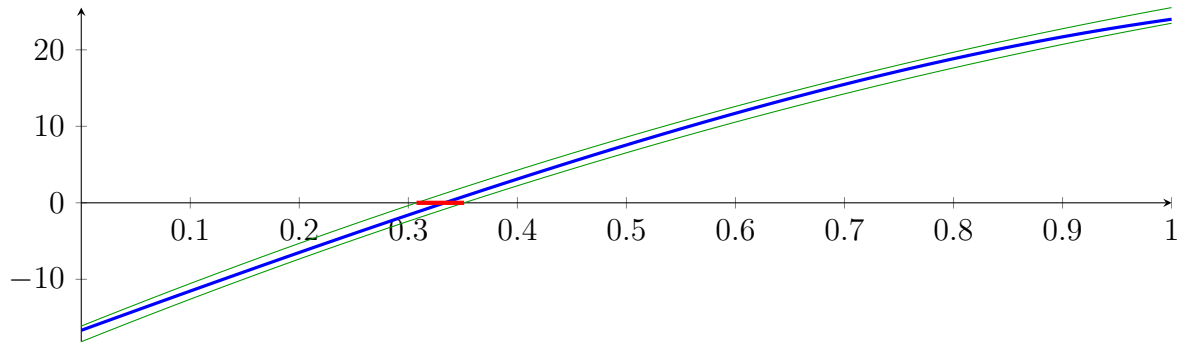
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

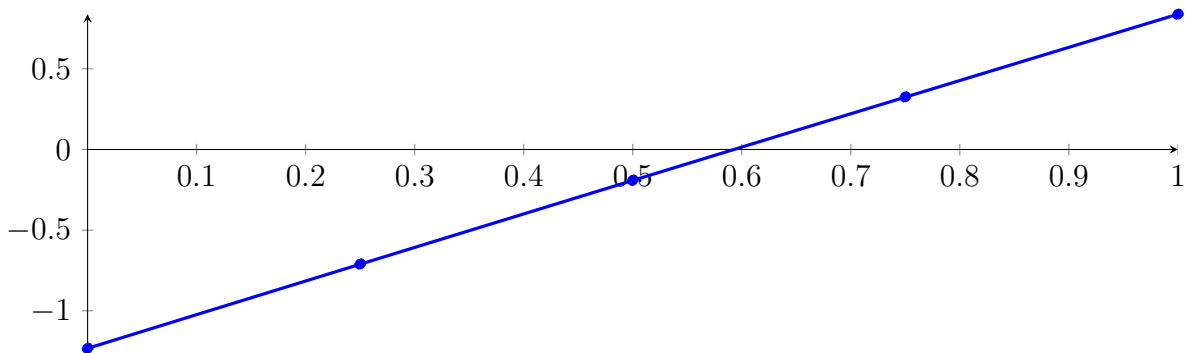
Longest intersection interval: 0.0436205

⇒ Selective recursion: interval 1: $[0.307477, 0.351097]$,

122.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

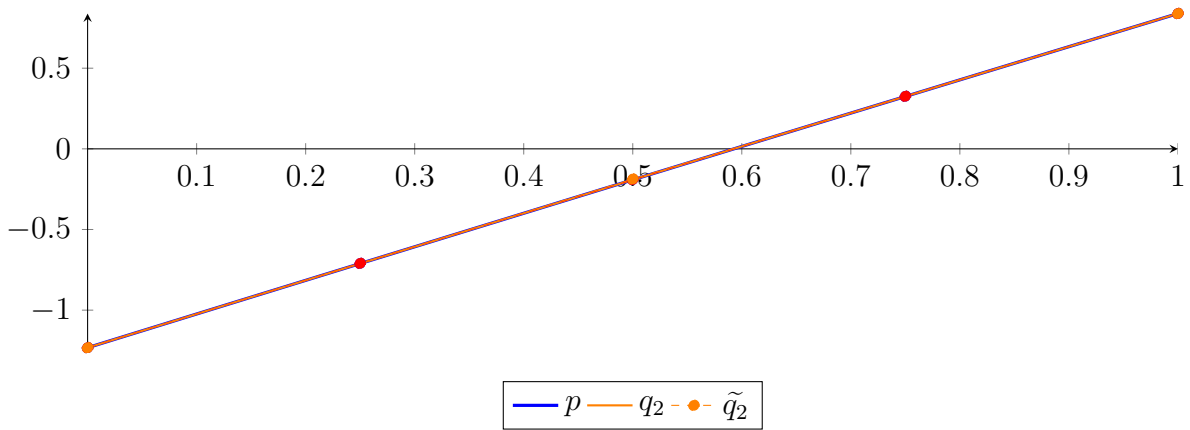
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1.45283 \cdot 10^{-17} X^4 + 3.33934 \cdot 10^{-17} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

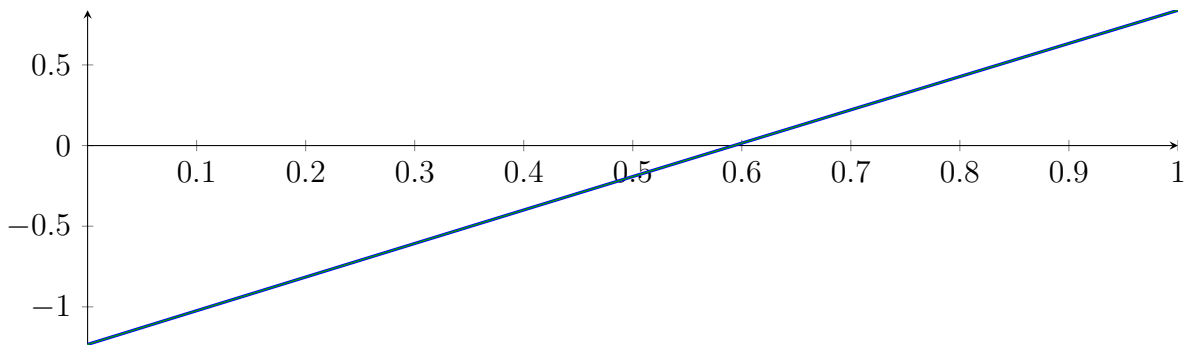
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

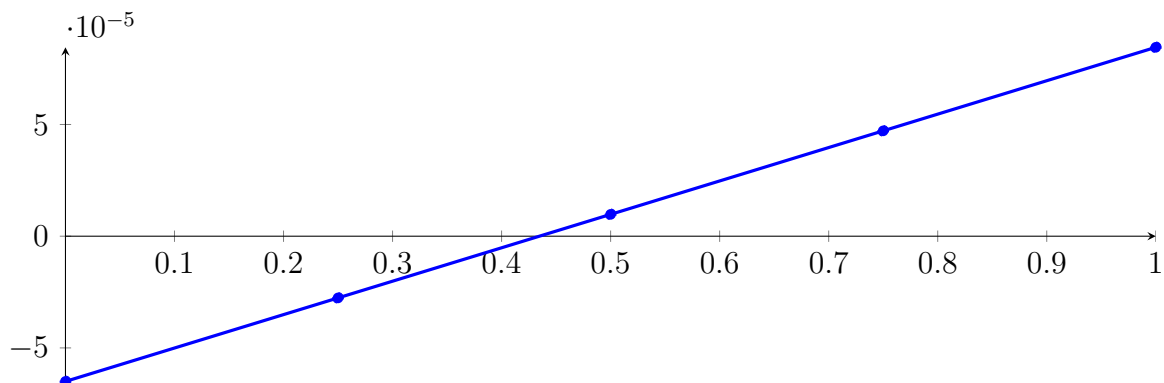
Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

122.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.05879 \cdot 10^{-22} X^4 - 2.82525 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05} \\
 &= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \\
 &\quad \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)
 \end{aligned}$$



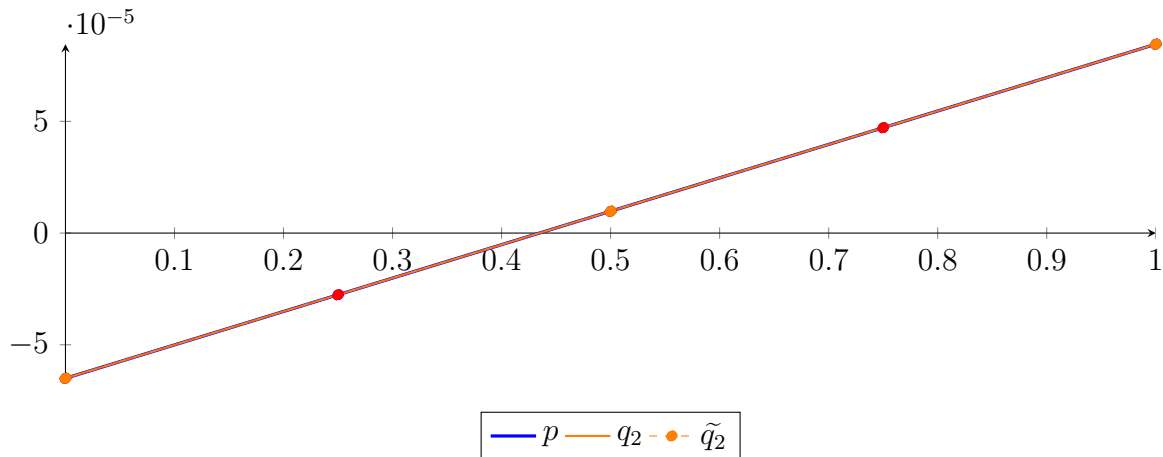
Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = -4.49986 \cdot 10^{-22} X^4 + 3.33519 \cdot 10^{-21} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.82529 \cdot 10^{-17}$.

Bounding polynomials M and m :

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

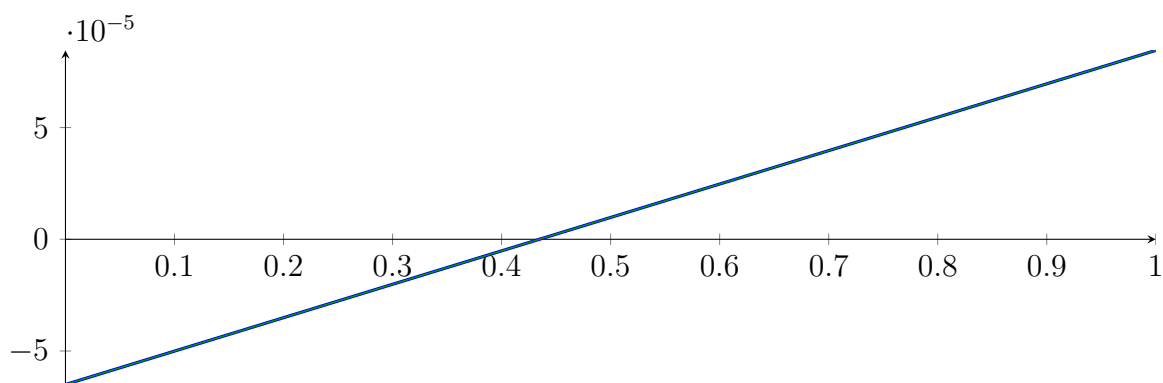
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.434685, 1.4089 \cdot 10^6\}$$

$$N(m) = \{0.434685, 1.4089 \cdot 10^6\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

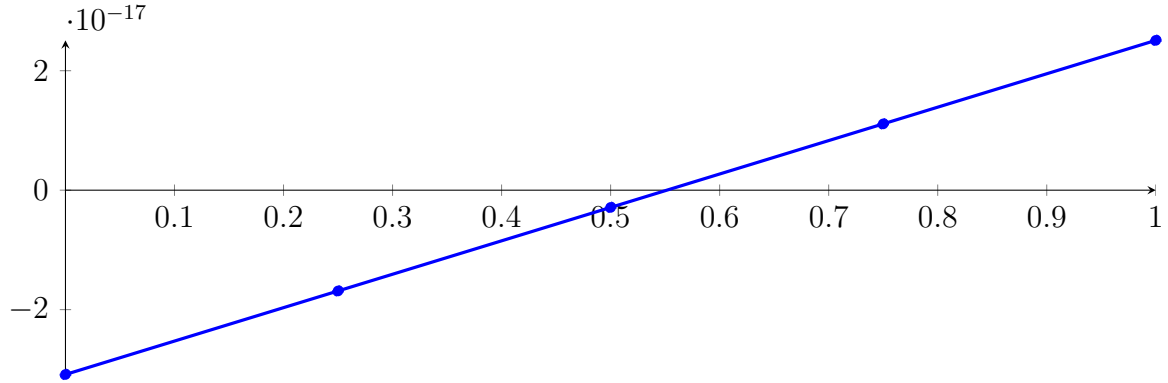
Longest intersection interval: $3.74055 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

122.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

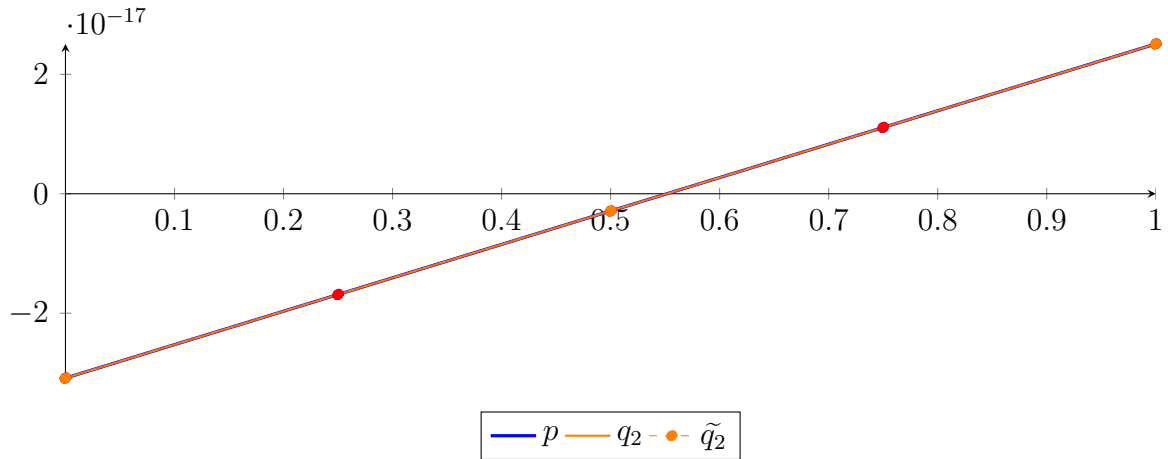
$$\begin{aligned}
 p &= -1.20371 \cdot 10^{-35} X^3 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17} \\
 &= -3.08561 \cdot 10^{-17} B_{0,4}(X) - 1.68712 \cdot 10^{-17} B_{1,4}(X) - 2.88624 \\
 &\quad \cdot 10^{-18} B_{2,4}(X) + 1.10987 \cdot 10^{-17} B_{3,4}(X) + 2.50836 \cdot 10^{-17} B_{4,4}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= 1.50463 \cdot 10^{-35} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17} \\
 &= -3.08561 \cdot 10^{-17} B_{0,2} - 2.88624 \cdot 10^{-18} B_{1,2} + 2.50836 \cdot 10^{-17} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -3.03936 \cdot 10^{-34} X^4 + 9.14817 \cdot 10^{-34} X^3 - 6.31946 \cdot 10^{-34} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17} \\
 &= -3.08561 \cdot 10^{-17} B_{0,4} - 1.68712 \cdot 10^{-17} B_{1,4} - 2.88624 \cdot 10^{-18} B_{2,4} + 1.10987 \cdot 10^{-17} B_{3,4} + 2.50836 \cdot 10^{-17} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.14701 \cdot 10^{-35}$.

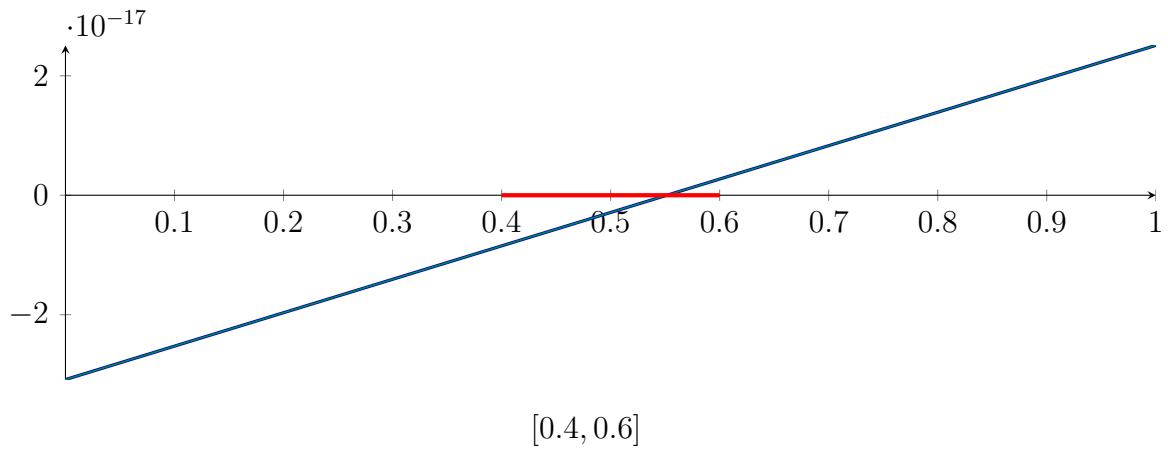
Bounding polynomials M and m :

$$\begin{aligned}
 M &= 1.50463 \cdot 10^{-35} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17} \\
 m &= 1.50463 \cdot 10^{-35} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-3.71783 \cdot 10^{18}, 0.6\} \qquad N(m) = \{-3.71783 \cdot 10^{18}, 0.4\}$$

Intersection intervals:



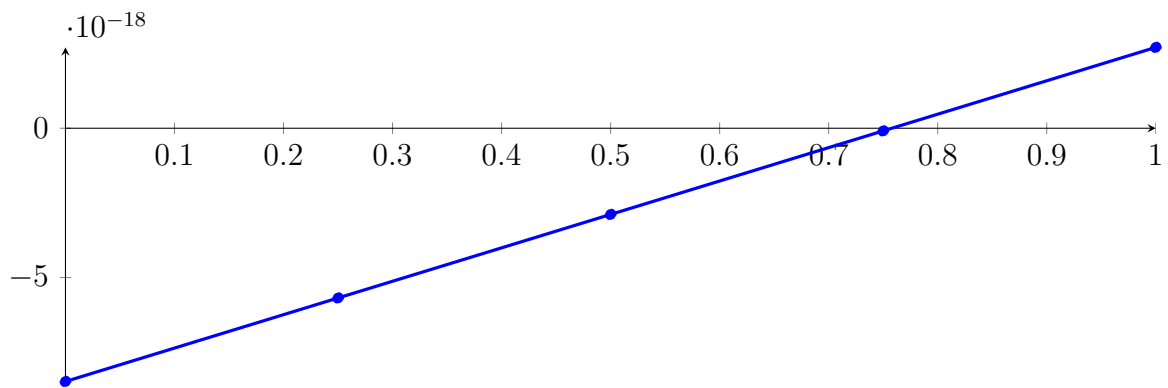
Longest intersection interval: 0.2

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

122.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

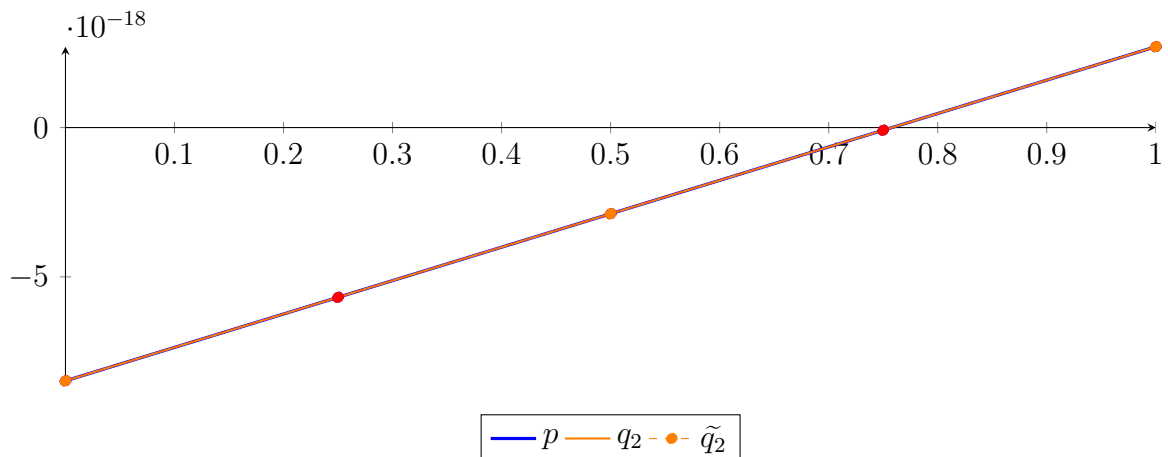
$$\begin{aligned}
 p &= 1.50463 \cdot 10^{-36} X^4 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18} \\
 &= -8.48022 \cdot 10^{-18} B_{0,4}(X) - 5.68323 \cdot 10^{-18} B_{1,4}(X) - 2.88624 \\
 &\quad \cdot 10^{-18} B_{2,4}(X) - 8.9255 \cdot 10^{-20} B_{3,4}(X) + 2.70773 \cdot 10^{-18} B_{4,4}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= 6.01853 \cdot 10^{-36} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18} \\
 &= -8.48022 \cdot 10^{-18} B_{0,2} - 2.88624 \cdot 10^{-18} B_{1,2} + 2.70773 \cdot 10^{-18} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -1.06829 \cdot 10^{-34} X^4 + 6.62038 \cdot 10^{-35} X^3 + 7.67363 \cdot 10^{-35} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18} \\
 &= -8.48022 \cdot 10^{-18} B_{0,4} - 5.68323 \cdot 10^{-18} B_{1,4} - 2.88624 \cdot 10^{-18} B_{2,4} - 8.9255 \cdot 10^{-20} B_{3,4} + 2.70773 \cdot 10^{-18} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.36039 \cdot 10^{-35}$.

Bounding polynomials M and m :

$$M = 6.01853 \cdot 10^{-36} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$

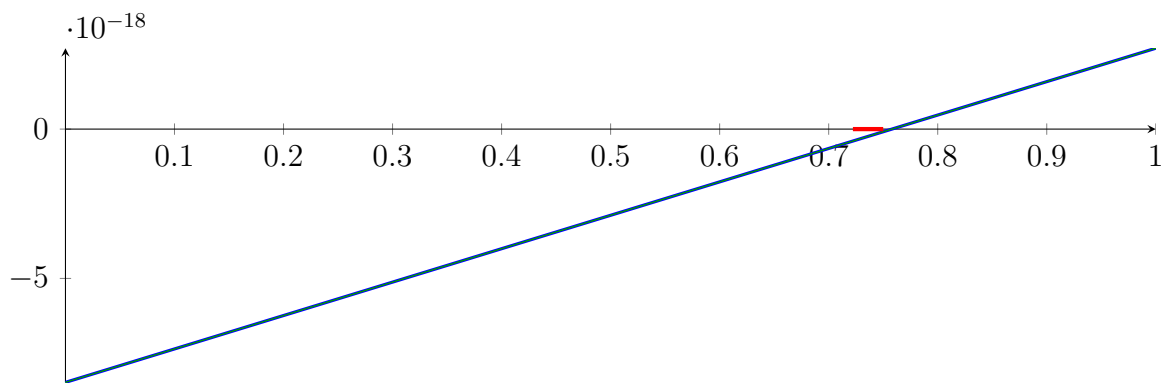
$$m = 6.77085 \cdot 10^{-36} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$

Root of M and m :

$$N(M) = \{-1.85892 \cdot 10^{18}, 0.75\}$$

$$N(m) = \{-1.65237 \cdot 10^{18}, 0.722222\}$$

Intersection intervals:



$$[0.722222, 0.75]$$

Longest intersection interval: 0.0277778

⇒ Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

122.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

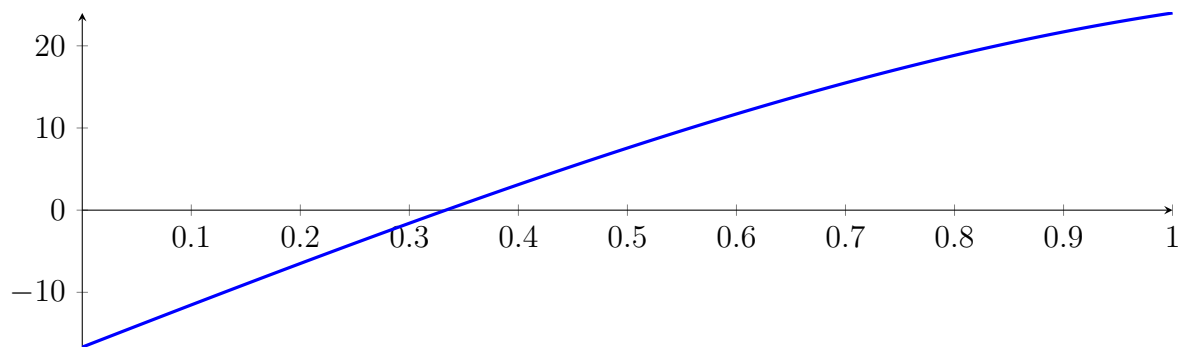
Reached interval [0.333333, 0.333333] **without sign change** at depth 6!

$$p(0) = -4.00031e-19 - p(1) - 8.9255e-20$$

122.7 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

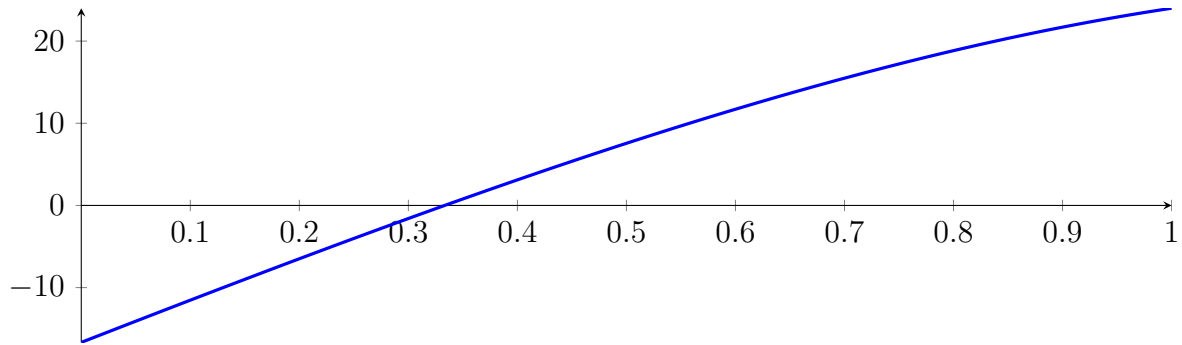
with precision $\varepsilon = 1 \cdot 10^{-64}$.

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$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

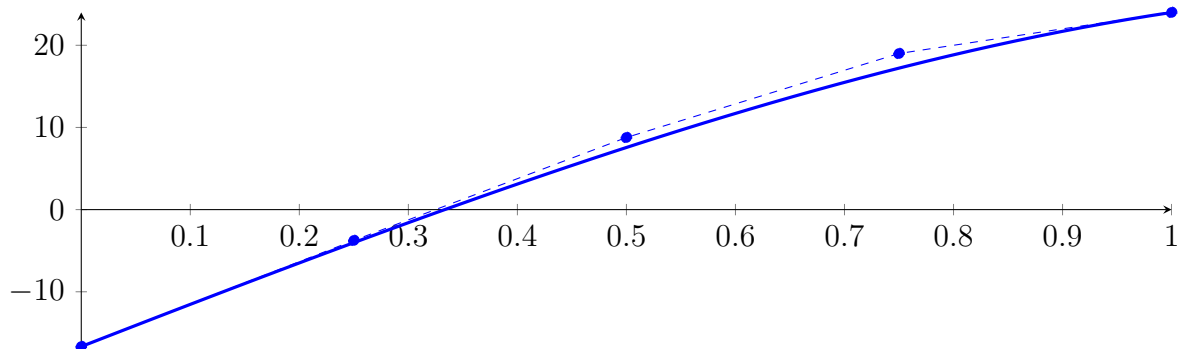
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



123.1 Recursion Branch 1 for Input Interval $[0, 1]$

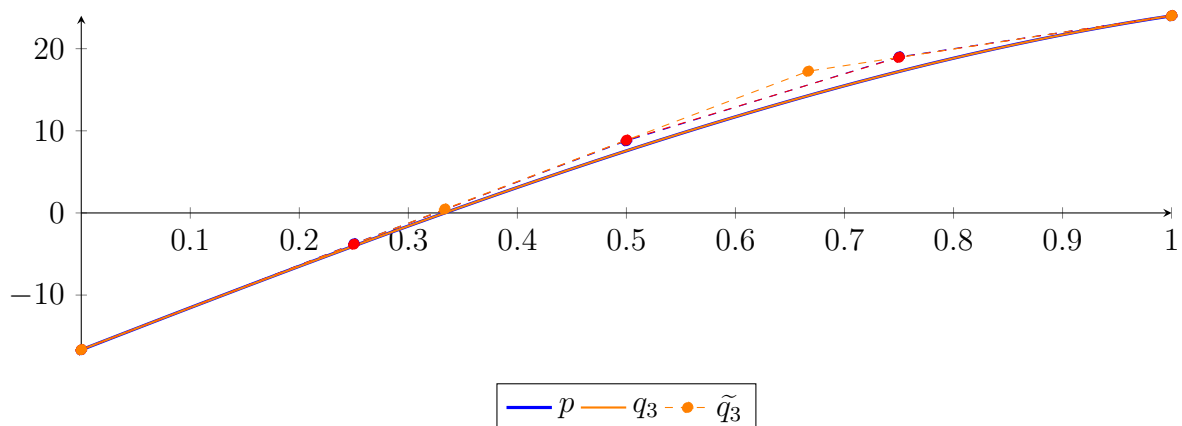
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= 3.34802 \cdot 10^{-16}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



— p — q_3 — \tilde{q}_3

The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

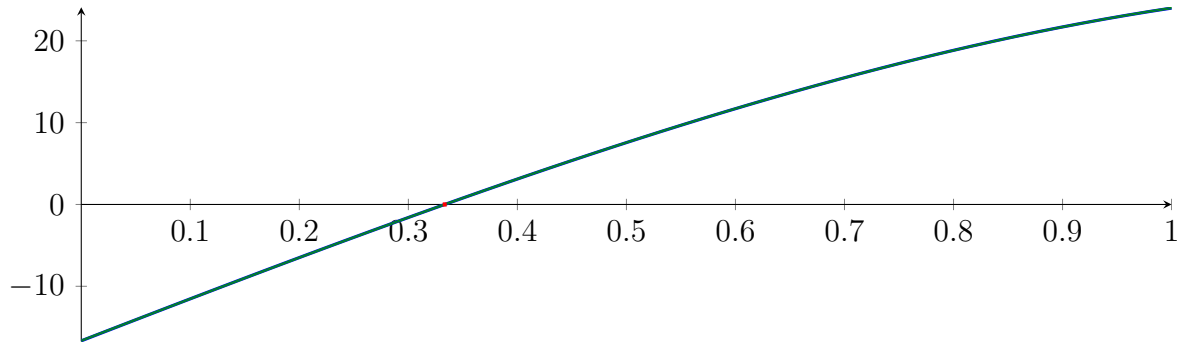
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

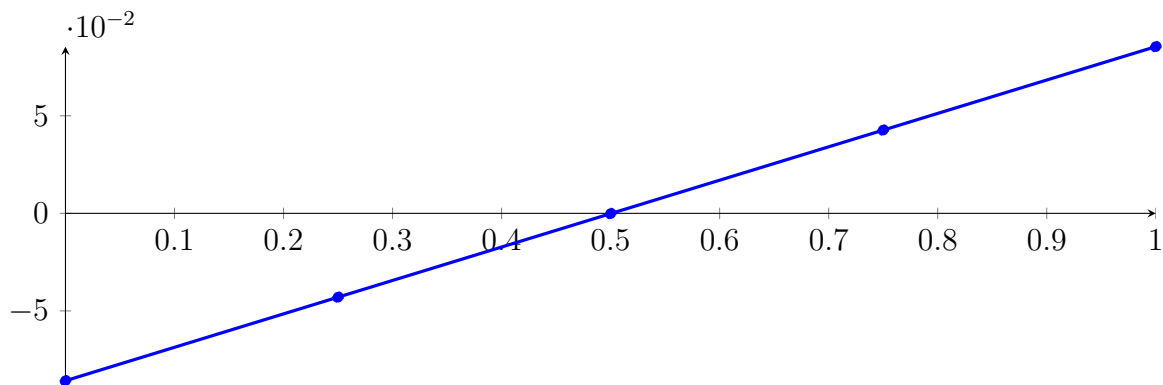
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

123.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

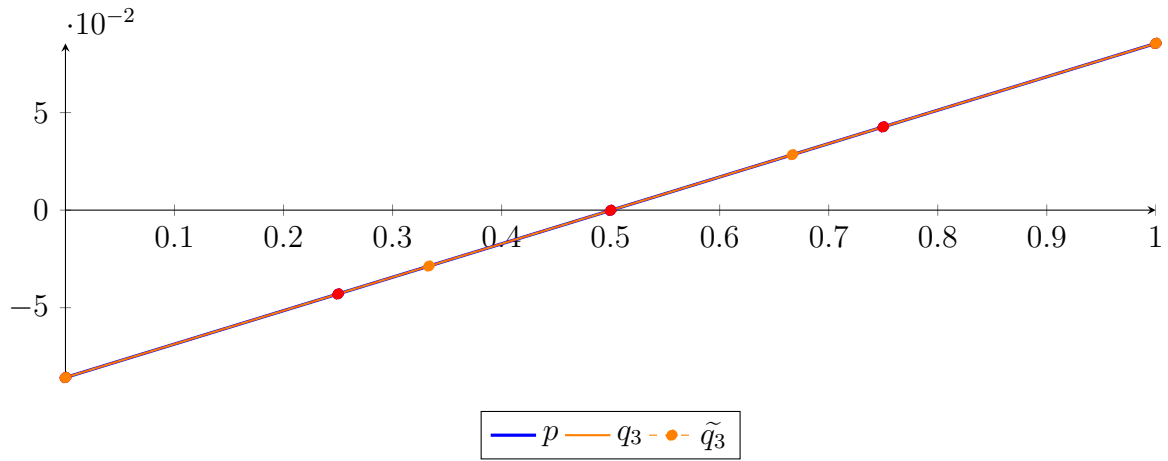
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 1.99222 \cdot 10^{-18} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m :

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

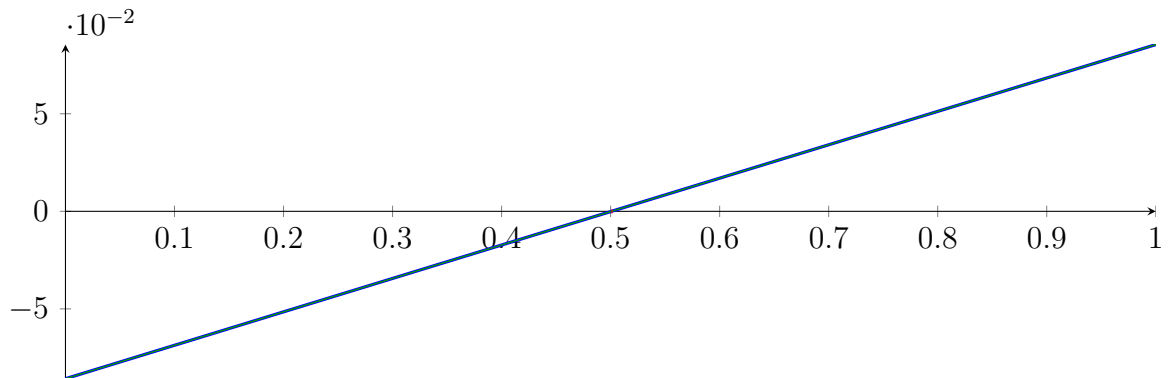
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\}$$

$$N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

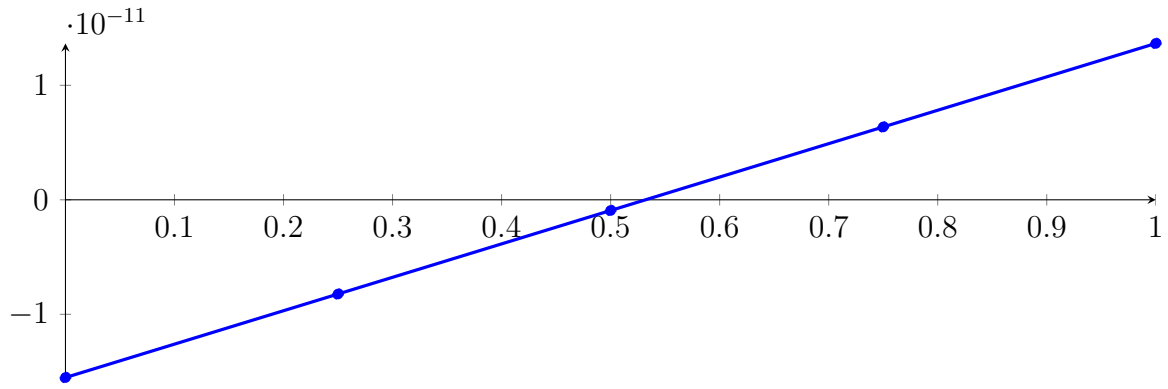
Longest intersection interval: $1.7041 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

123.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

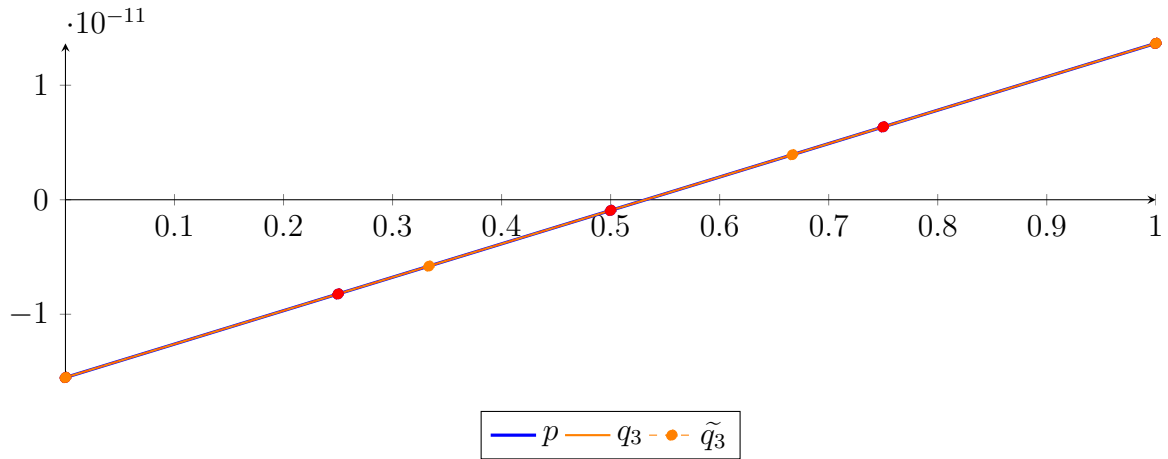
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\ &= -1.55233 \cdot 10^{-11} B_{0,4}(X) - 8.22817 \cdot 10^{-12} B_{1,4}(X) - 9.33054 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36206 \cdot 10^{-12} B_{3,4}(X) + 1.36572 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 &= -1.55233 \cdot 10^{-11} B_{0,3} - 5.79647 \cdot 10^{-12} B_{1,3} + 3.93036 \cdot 10^{-12} B_{2,3} + 1.36572 \cdot 10^{-11} B_{3,3} \\
 \tilde{q}_3 &= 3.84964 \cdot 10^{-28} X^4 - 2.1457 \cdot 10^{-28} X^3 - 4.04147 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 &= -1.55233 \cdot 10^{-11} B_{0,4} - 8.22817 \cdot 10^{-12} B_{1,4} - 9.33054 \cdot 10^{-13} B_{2,4} + 6.36206 \cdot 10^{-12} B_{3,4} + 1.36572 \cdot 10^{-11} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.13596 \cdot 10^{-28}$.

Bounding polynomials M and m :

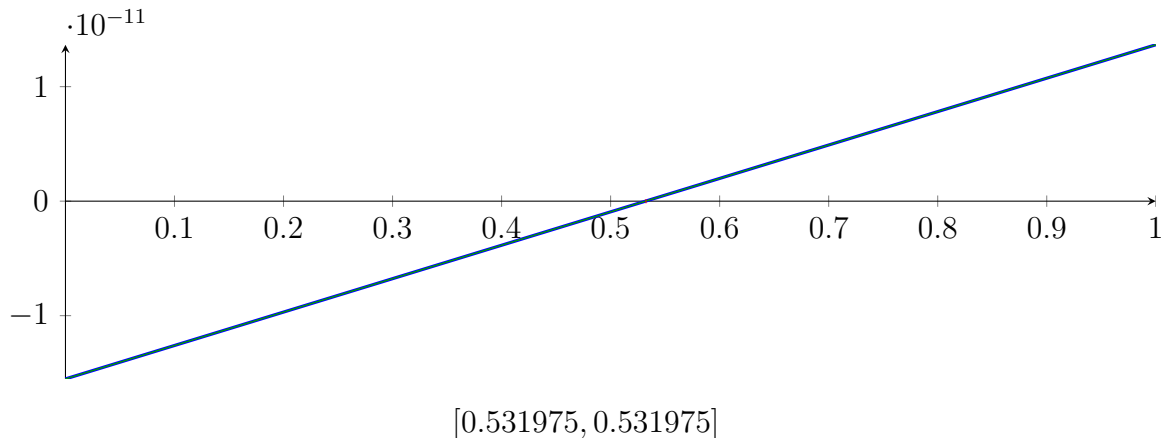
$$\begin{aligned}
 M &= 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 m &= 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.531975\}$$

$$N(m) = \{0.531975\}$$

Intersection intervals:



Longest intersection interval: 0

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

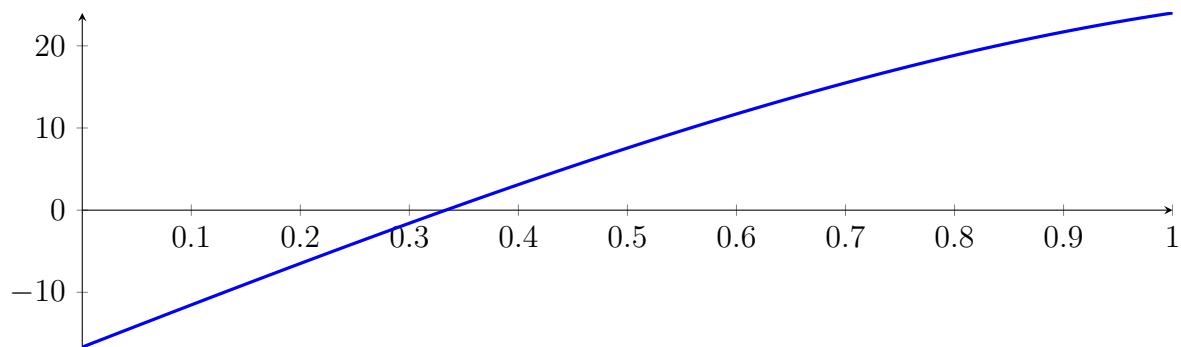
123.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

123.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

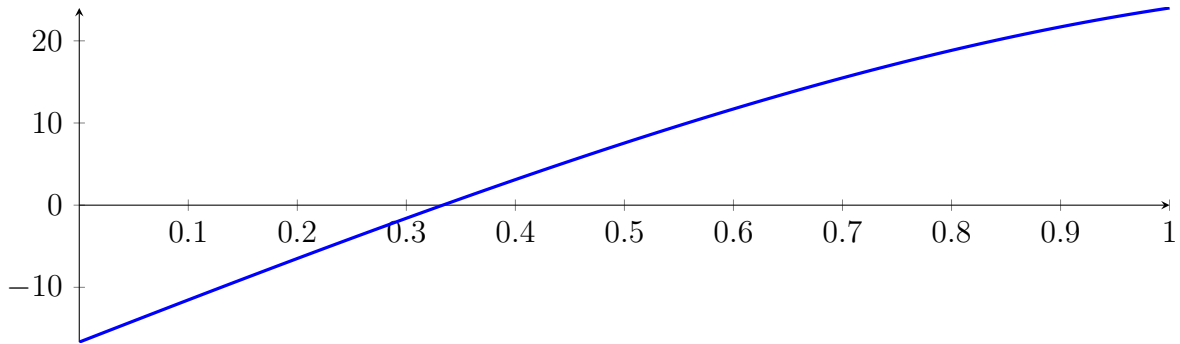
with precision $\varepsilon = 1 \cdot 10^{-64}$.

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$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

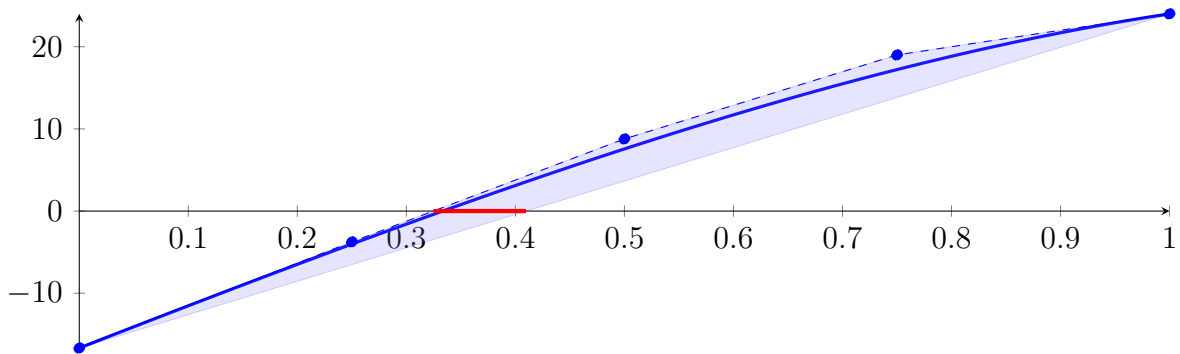
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



124.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

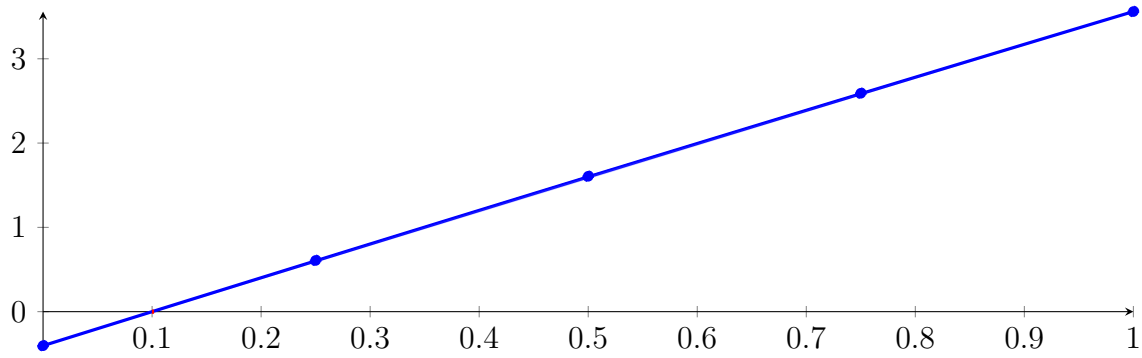
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

124.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

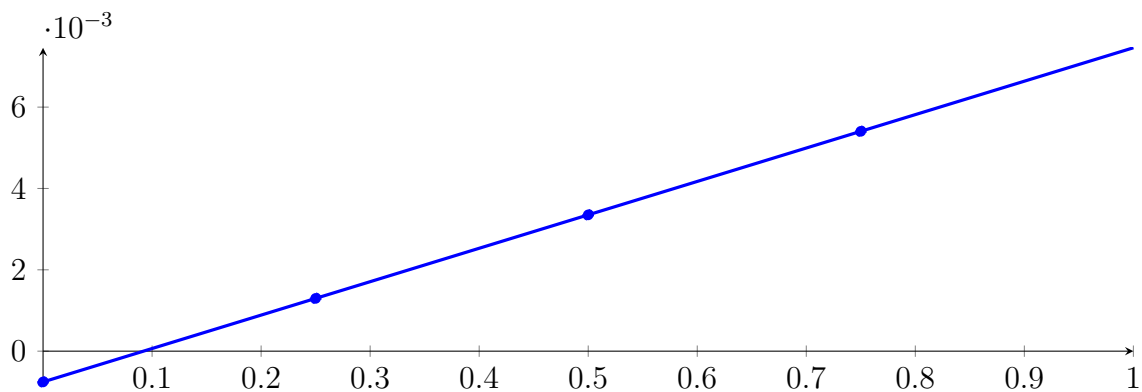
Longest intersection interval: 0.00203877

\implies Selective recursion: interval 1: $[0.333317, 0.333491]$,

124.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01974 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

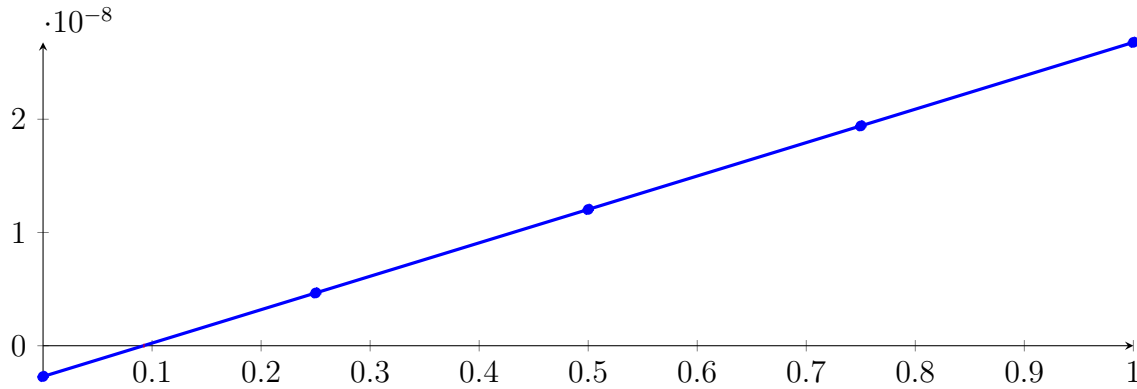
Longest intersection interval: $3.59185 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

124.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.61559 \cdot 10^{-27} X^4 - 3.23117 \cdot 10^{-27} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09} \\ &= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \\ &\quad \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28975 \cdot 10^{-11}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

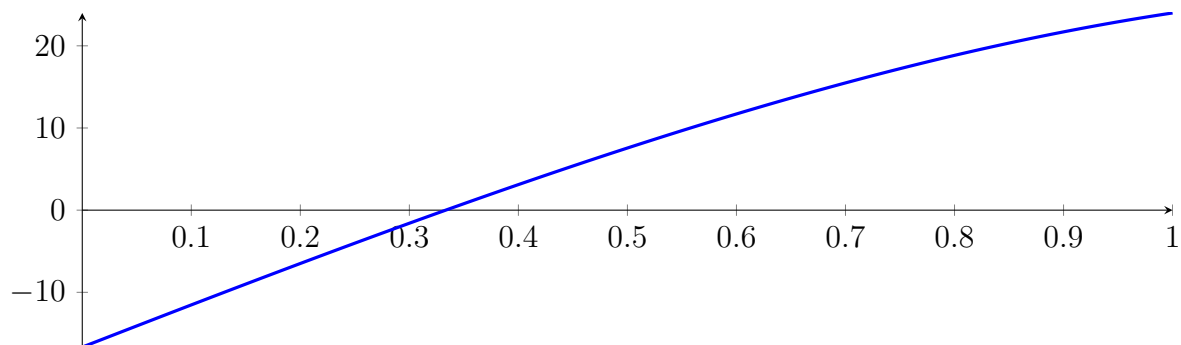
124.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

124.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

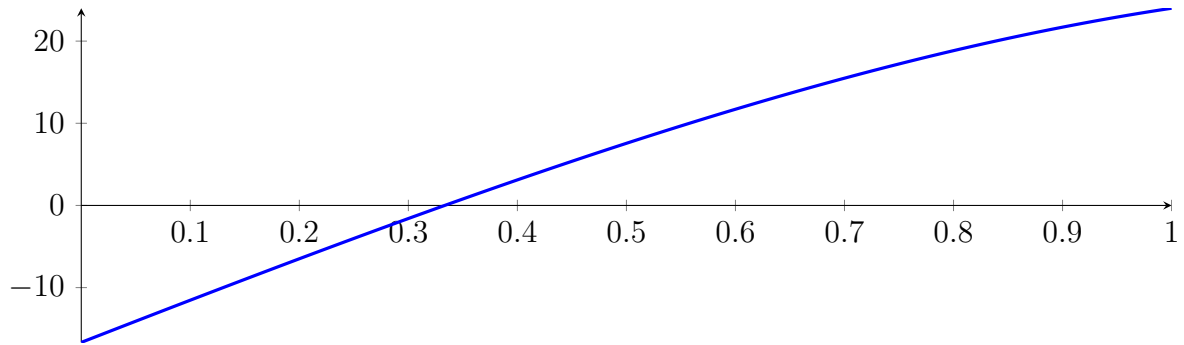
with precision $\varepsilon = 1 \cdot 10^{-128}$.

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$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

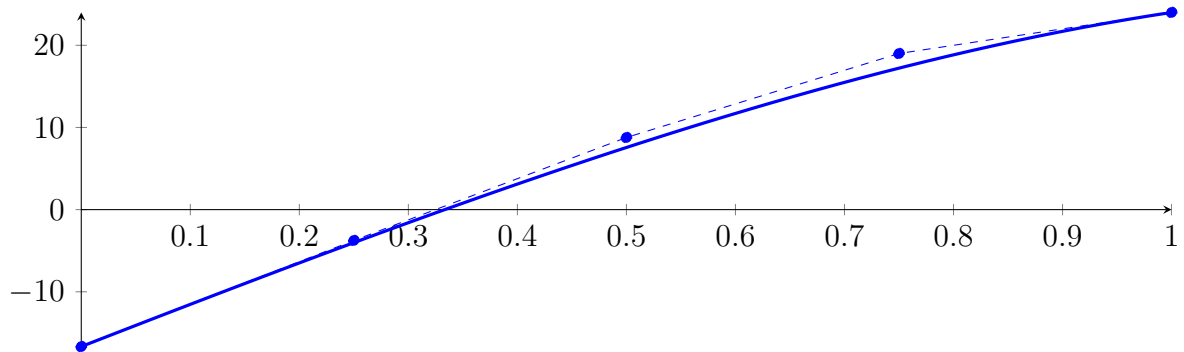
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



125.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

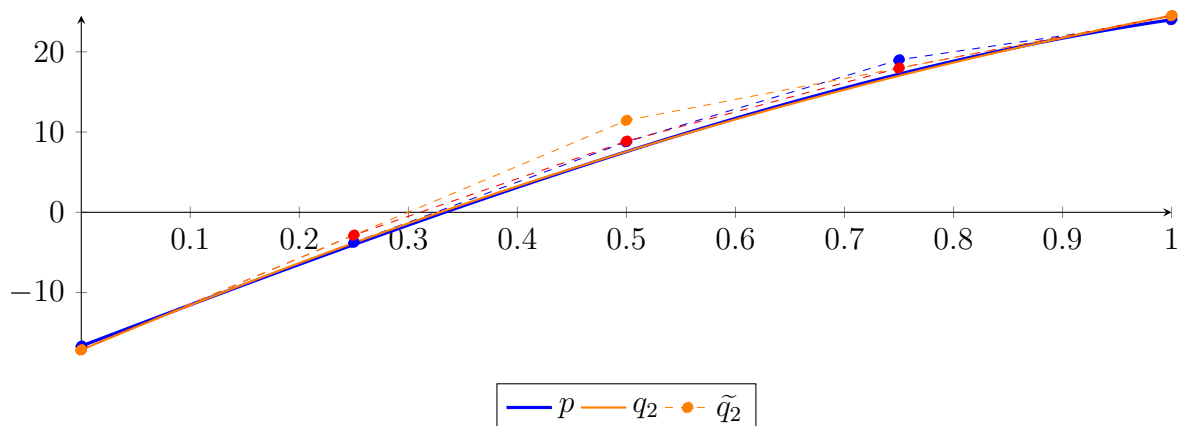
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1.45023 \cdot 10^{-15}X^4 + 4.00374 \cdot 10^{-15}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

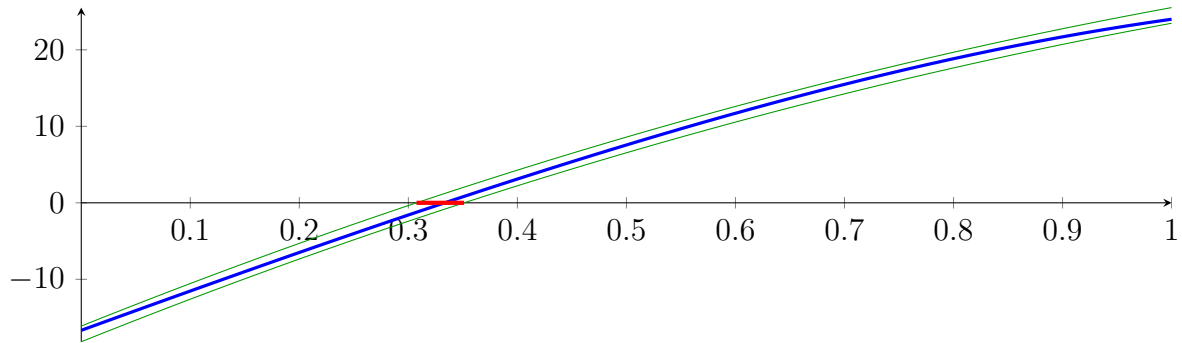
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

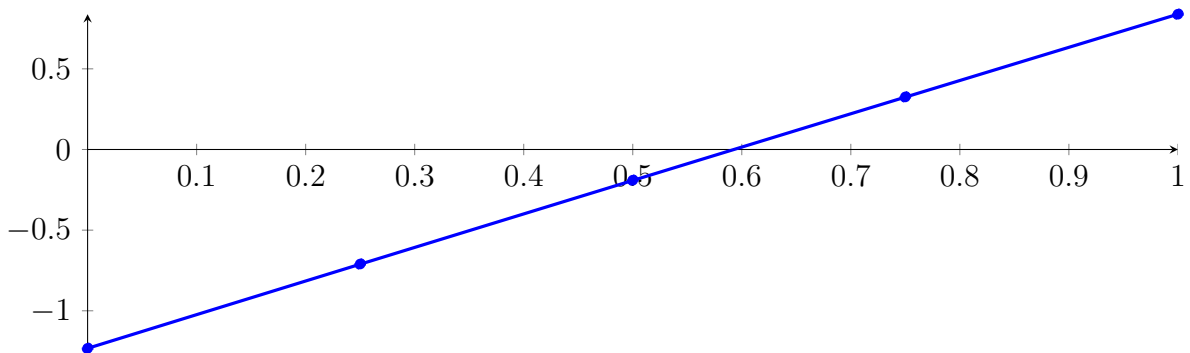
Longest intersection interval: 0.0436205

⇒ Selective recursion: **interval 1:** $[0.307477, 0.351097]$,

125.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

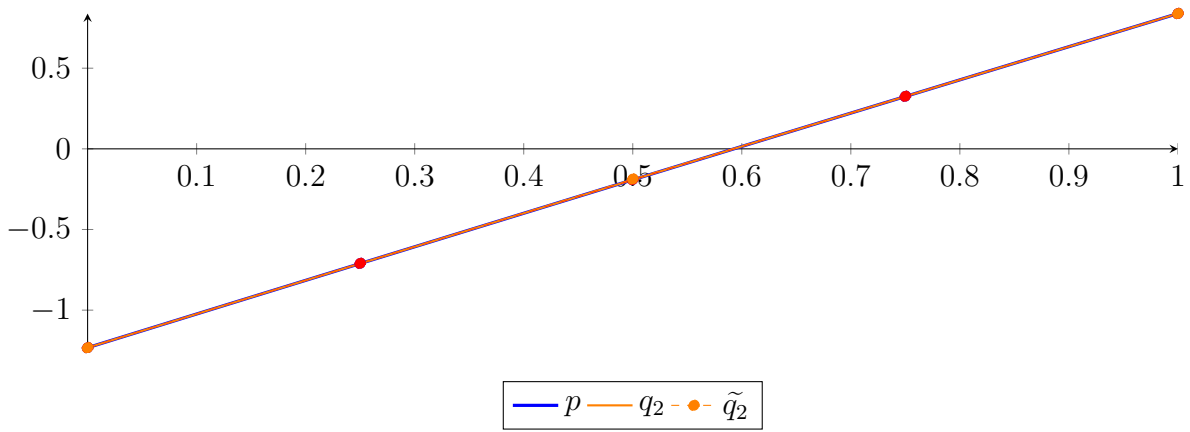
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -1.45283 \cdot 10^{-17} X^4 + 3.33934 \cdot 10^{-17} X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

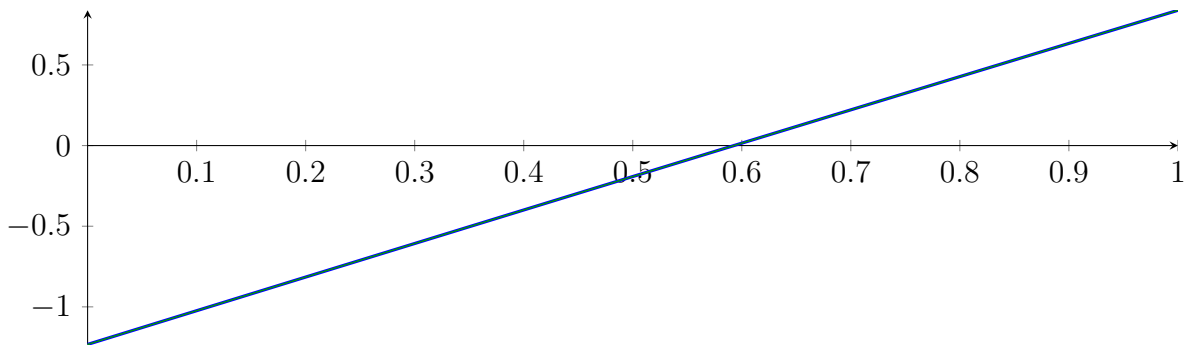
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

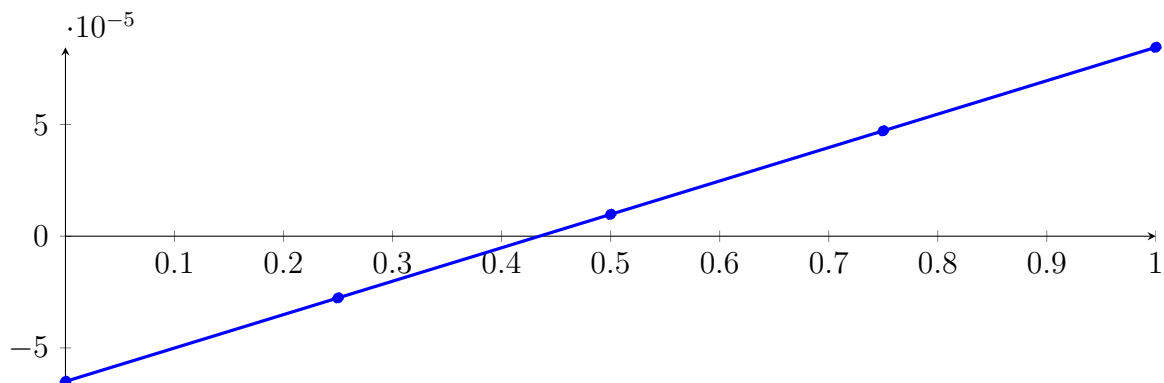
Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

125.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.05879 \cdot 10^{-22} X^4 - 2.82525 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05} \\
 &= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \\
 &\quad \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X)
 \end{aligned}$$



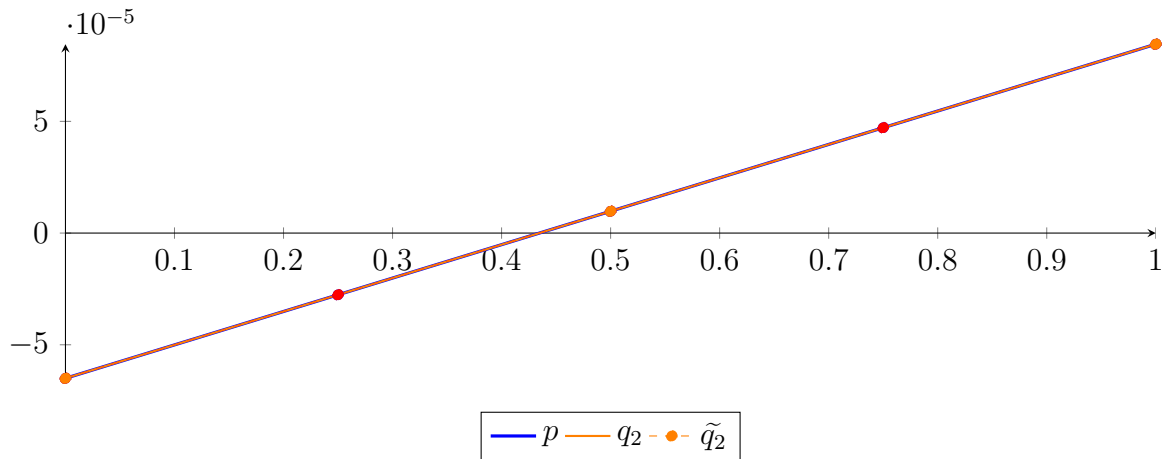
Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = -4.49986 \cdot 10^{-22} X^4 + 3.33519 \cdot 10^{-21} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.82529 \cdot 10^{-17}$.

Bounding polynomials M and m :

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

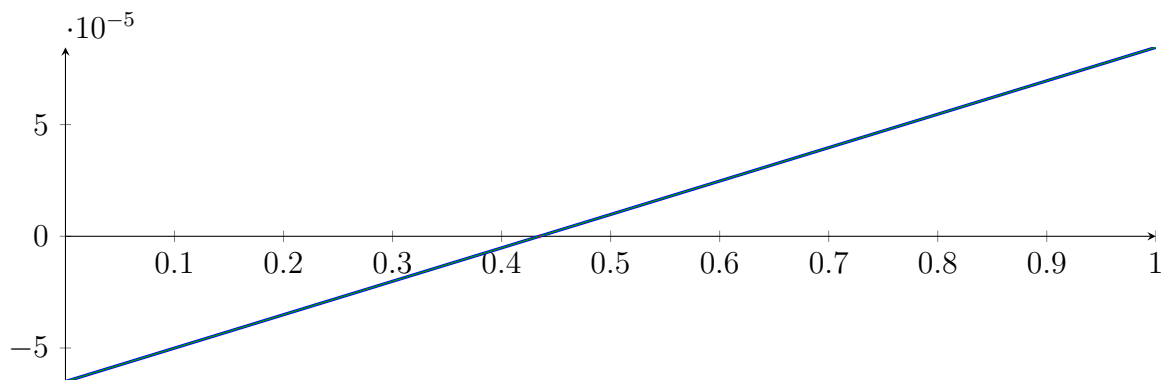
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.434685, 1.4089 \cdot 10^6\}$$

$$N(m) = \{0.434685, 1.4089 \cdot 10^6\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

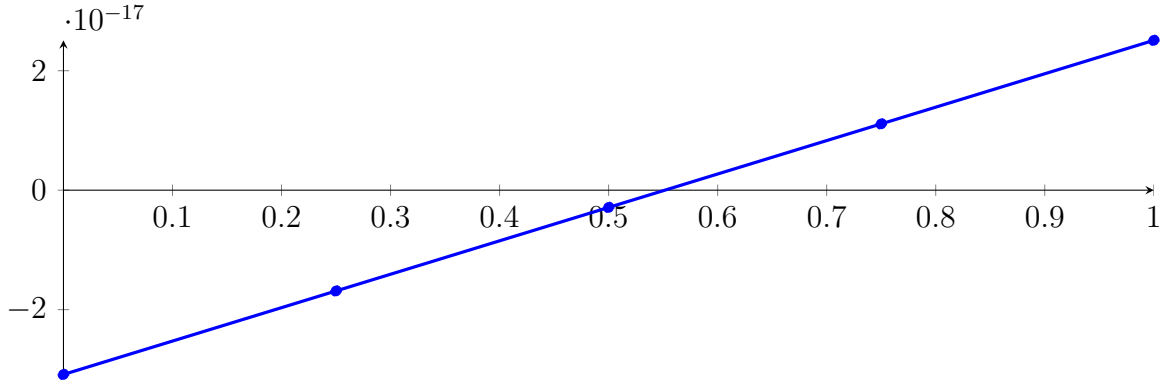
Longest intersection interval: $3.74055 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

125.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

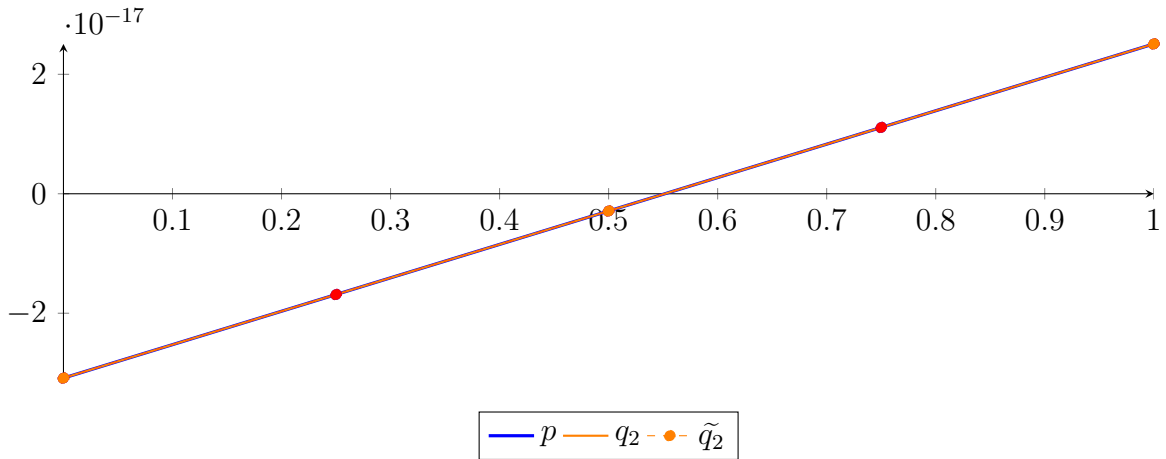
$$\begin{aligned}
 p &= -1.20371 \cdot 10^{-35} X^3 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17} \\
 &= -3.08561 \cdot 10^{-17} B_{0,4}(X) - 1.68712 \cdot 10^{-17} B_{1,4}(X) - 2.88624 \\
 &\quad \cdot 10^{-18} B_{2,4}(X) + 1.10987 \cdot 10^{-17} B_{3,4}(X) + 2.50836 \cdot 10^{-17} B_{4,4}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= 1.50463 \cdot 10^{-35} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17} \\
 &= -3.08561 \cdot 10^{-17} B_{0,2} - 2.88624 \cdot 10^{-18} B_{1,2} + 2.50836 \cdot 10^{-17} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -3.03936 \cdot 10^{-34} X^4 + 9.14817 \cdot 10^{-34} X^3 - 6.31946 \cdot 10^{-34} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17} \\
 &= -3.08561 \cdot 10^{-17} B_{0,4} - 1.68712 \cdot 10^{-17} B_{1,4} - 2.88624 \cdot 10^{-18} B_{2,4} + 1.10987 \cdot 10^{-17} B_{3,4} + 2.50836 \cdot 10^{-17} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.14701 \cdot 10^{-35}$.

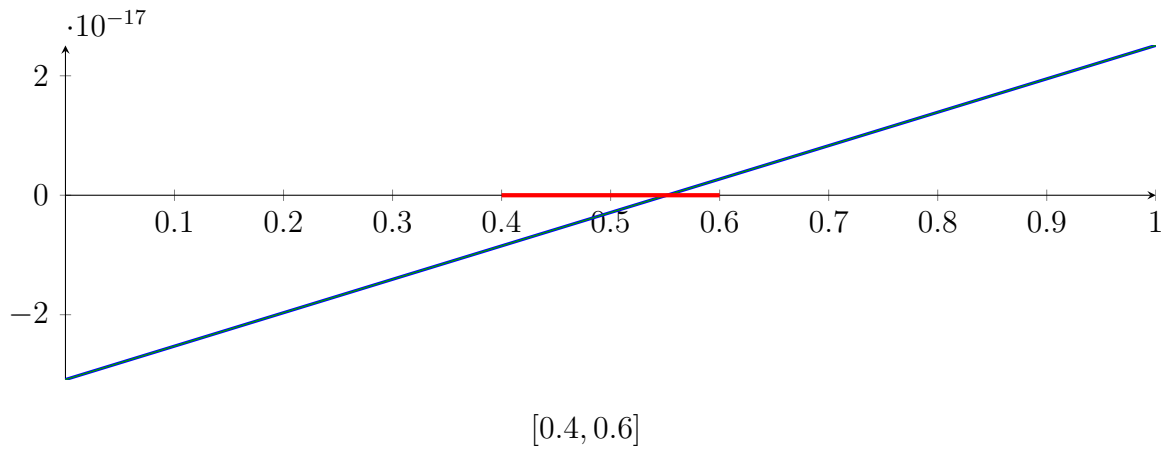
Bounding polynomials M and m :

$$\begin{aligned}
 M &= 1.50463 \cdot 10^{-35} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17} \\
 m &= 1.50463 \cdot 10^{-35} X^2 + 5.59397 \cdot 10^{-17} X - 3.08561 \cdot 10^{-17}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-3.71783 \cdot 10^{18}, 0.6\} \qquad N(m) = \{-3.71783 \cdot 10^{18}, 0.4\}$$

Intersection intervals:

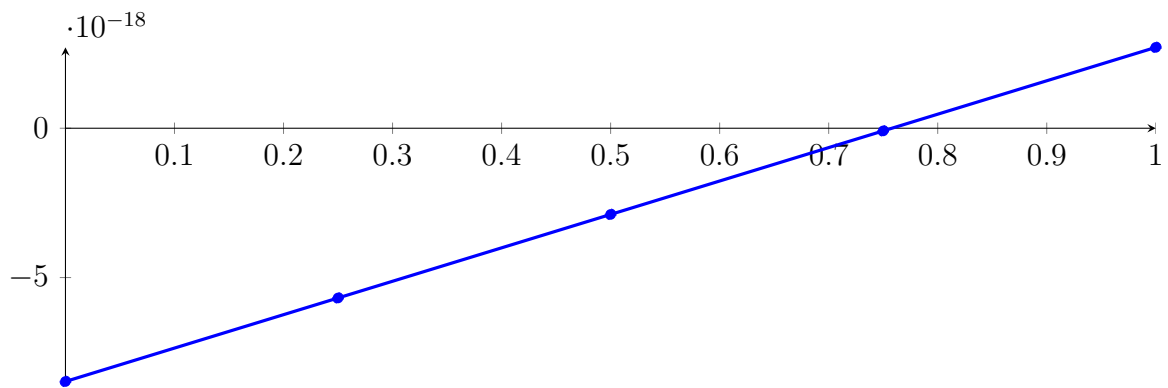


Longest intersection interval: 0.2
 \implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

125.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

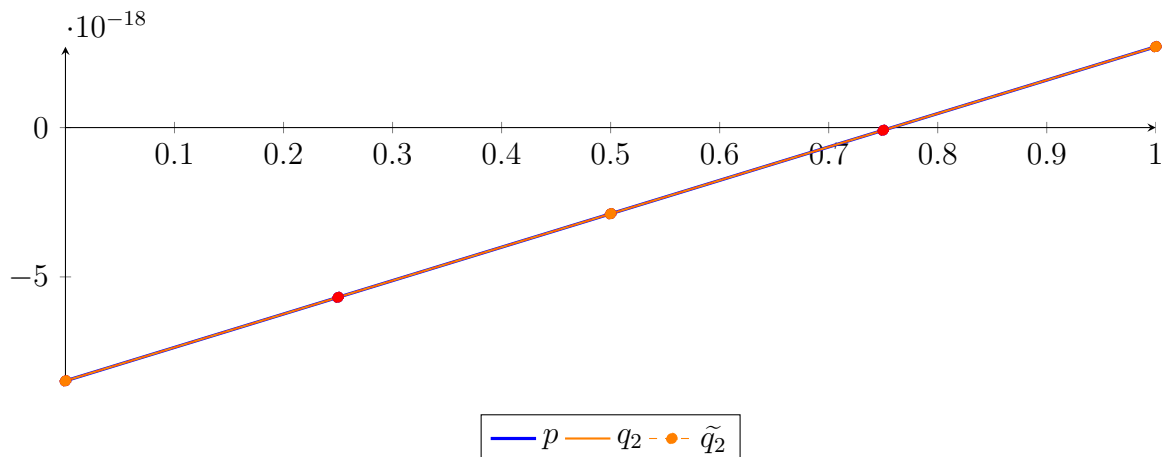
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 1.50463 \cdot 10^{-36} X^4 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18} \\
 &= -8.48022 \cdot 10^{-18} B_{0,4}(X) - 5.68323 \cdot 10^{-18} B_{1,4}(X) - 2.88624 \\
 &\quad \cdot 10^{-18} B_{2,4}(X) - 8.9255 \cdot 10^{-20} B_{3,4}(X) + 2.70773 \cdot 10^{-18} B_{4,4}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= 6.01853 \cdot 10^{-36} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18} \\
 &= -8.48022 \cdot 10^{-18} B_{0,2} - 2.88624 \cdot 10^{-18} B_{1,2} + 2.70773 \cdot 10^{-18} B_{2,2} \\
 \tilde{q}_2 &= -1.06829 \cdot 10^{-34} X^4 + 6.62038 \cdot 10^{-35} X^3 + 7.67363 \cdot 10^{-35} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18} \\
 &= -8.48022 \cdot 10^{-18} B_{0,4} - 5.68323 \cdot 10^{-18} B_{1,4} - 2.88624 \cdot 10^{-18} B_{2,4} - 8.9255 \cdot 10^{-20} B_{3,4} + 2.70773 \cdot 10^{-18} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.36039 \cdot 10^{-35}$.

Bounding polynomials M and m :

$$M = 6.01853 \cdot 10^{-36} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$

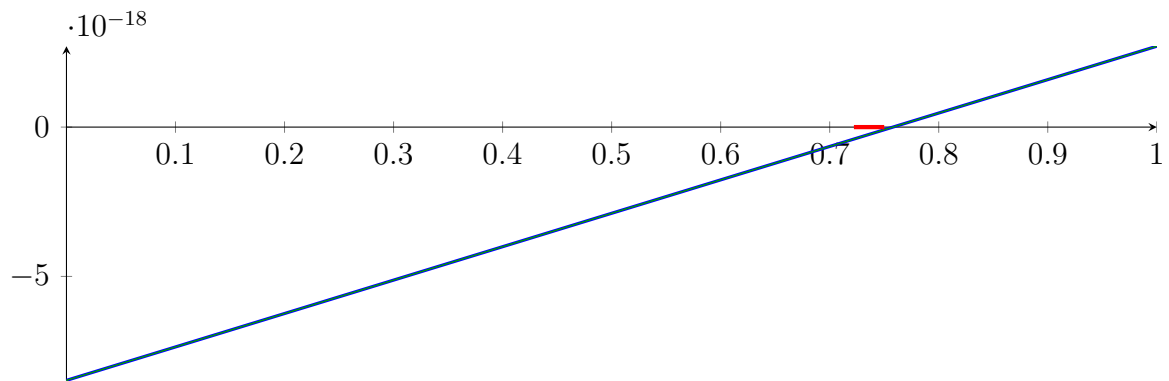
$$m = 6.77085 \cdot 10^{-36} X^2 + 1.11879 \cdot 10^{-17} X - 8.48022 \cdot 10^{-18}$$

Root of M and m :

$$N(M) = \{-1.85892 \cdot 10^{18}, 0.75\}$$

$$N(m) = \{-1.65237 \cdot 10^{18}, 0.722222\}$$

Intersection intervals:



$$[0.722222, 0.75]$$

Longest intersection interval: 0.0277778

⇒ Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

125.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

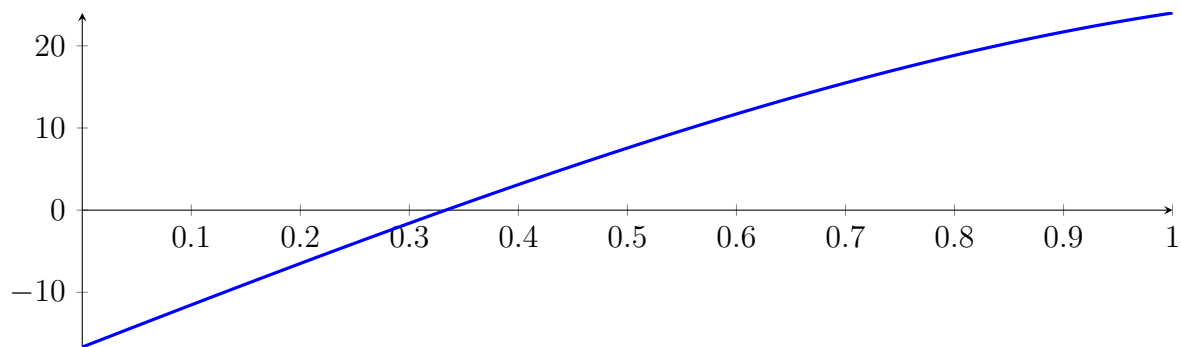
Reached interval [0.333333, 0.333333] **without sign change** at depth 6!

$$p(0) = -4.00031e-19 - p(1) - 8.9255e-20$$

125.7 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

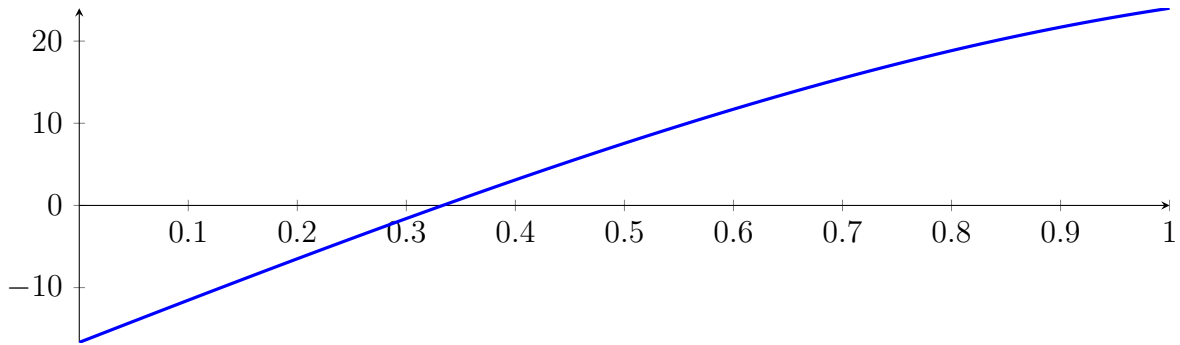
with precision $\varepsilon = 1 \cdot 10^{-128}$.

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$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

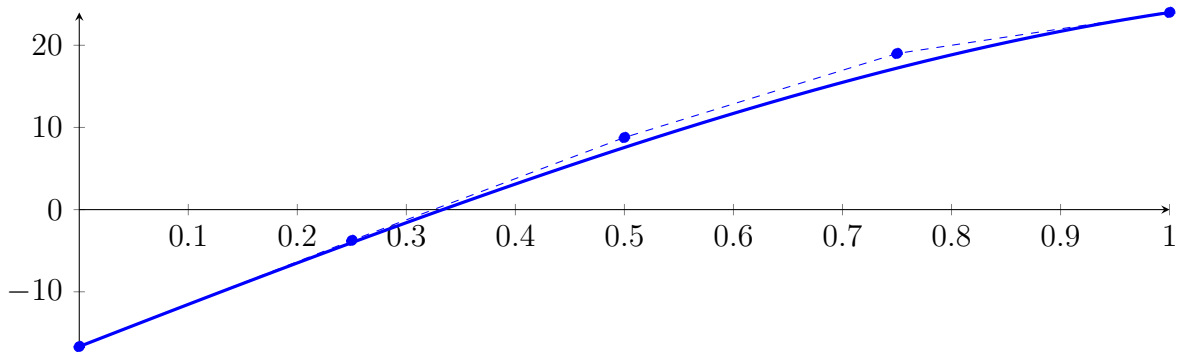
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



126.1 Recursion Branch 1 for Input Interval $[0, 1]$

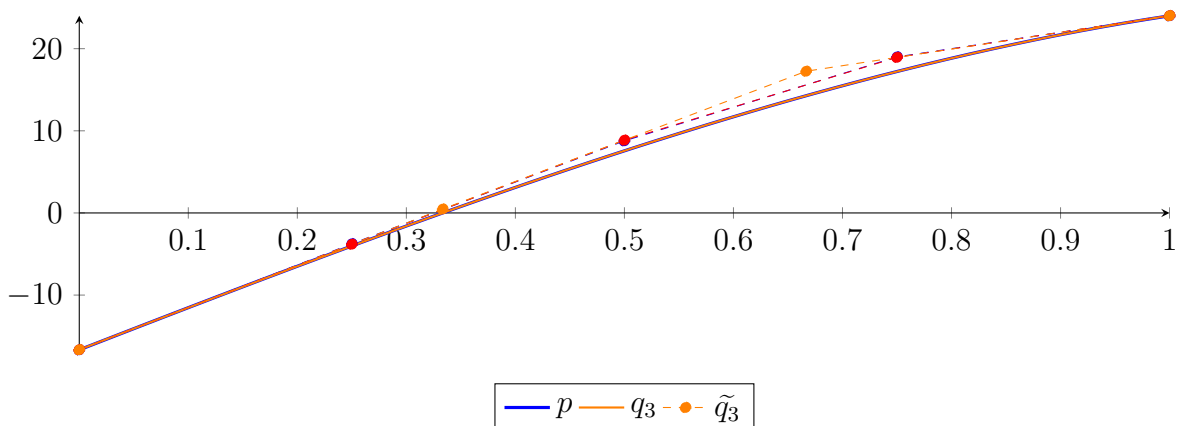
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= 3.34802 \cdot 10^{-16}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

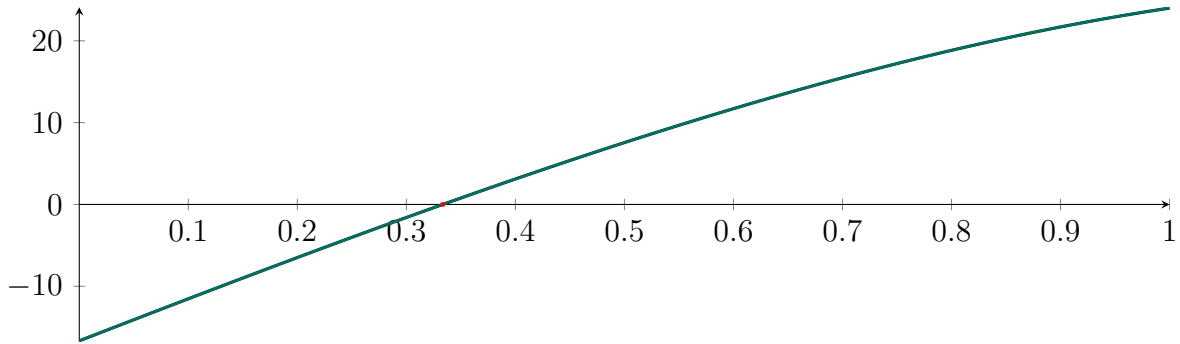
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

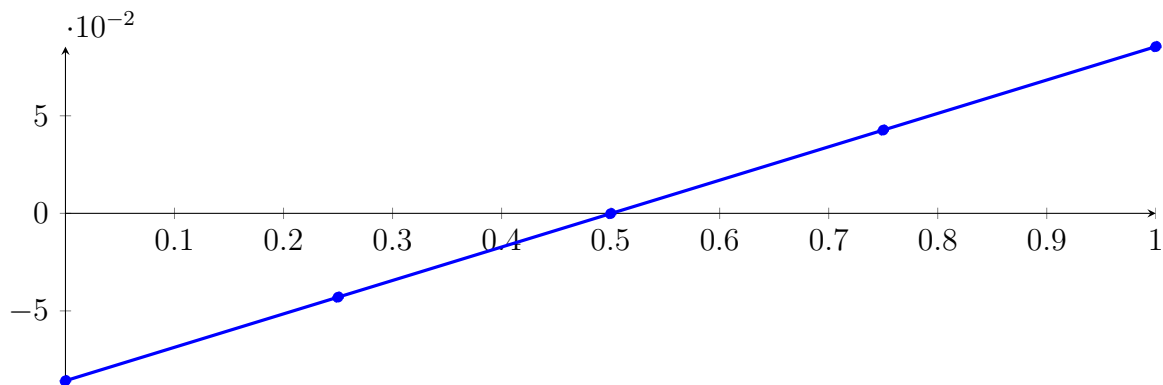
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

126.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

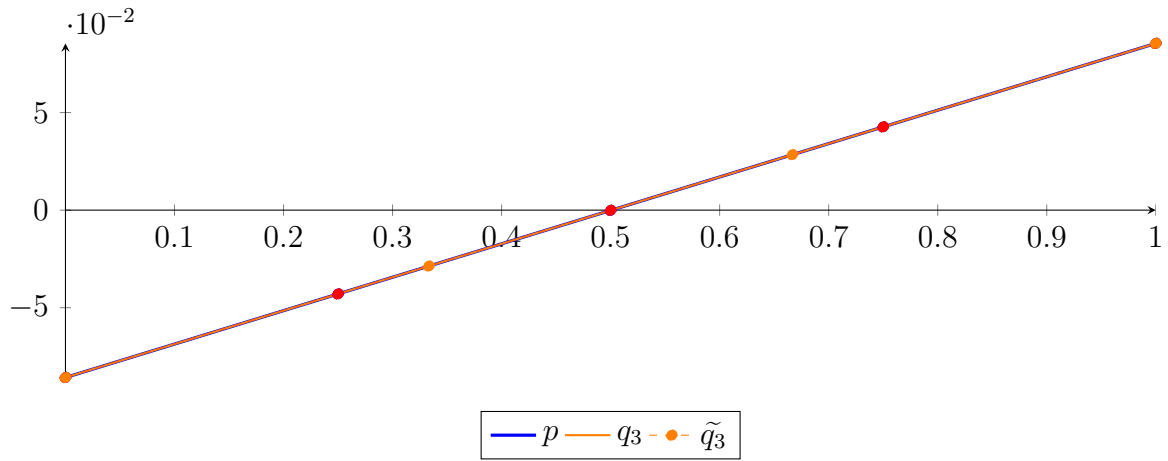
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 1.99222 \cdot 10^{-18} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m :

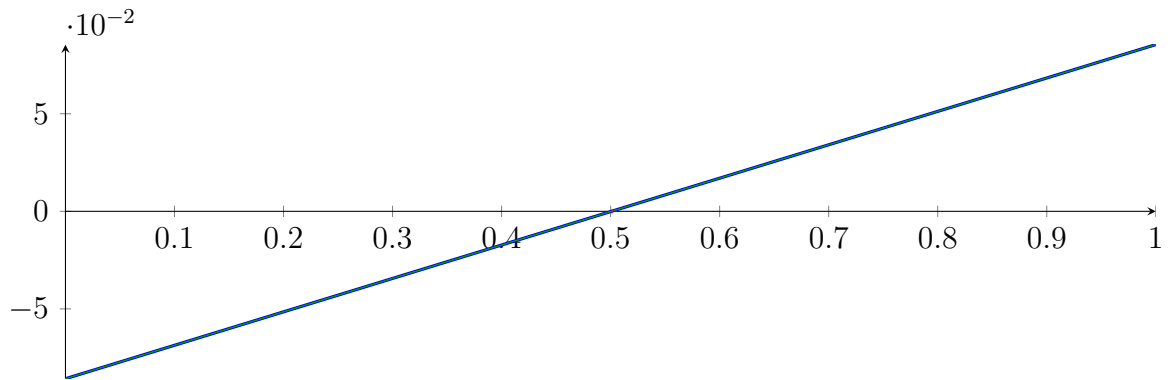
$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\} \quad N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

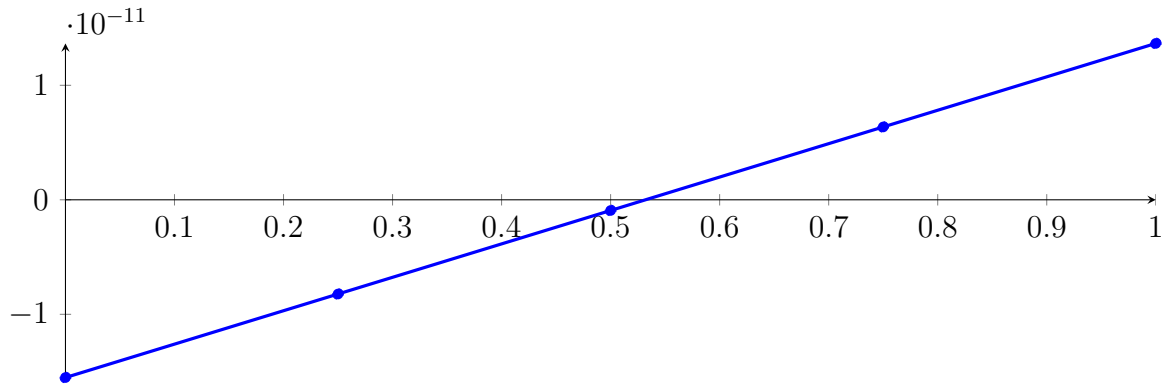
Longest intersection interval: $1.7041 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

126.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

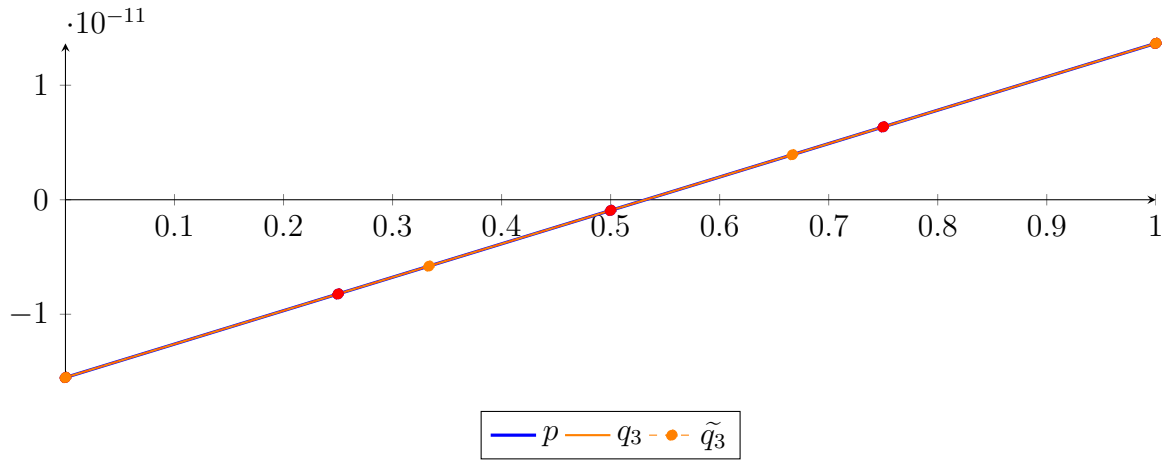
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\ &= -1.55233 \cdot 10^{-11} B_{0,4}(X) - 8.22817 \cdot 10^{-12} B_{1,4}(X) - 9.33054 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36206 \cdot 10^{-12} B_{3,4}(X) + 1.36572 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 &= -1.55233 \cdot 10^{-11} B_{0,3} - 5.79647 \cdot 10^{-12} B_{1,3} + 3.93036 \cdot 10^{-12} B_{2,3} + 1.36572 \cdot 10^{-11} B_{3,3} \\
 \tilde{q}_3 &= 3.84964 \cdot 10^{-28} X^4 - 2.1457 \cdot 10^{-28} X^3 - 4.04147 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 &= -1.55233 \cdot 10^{-11} B_{0,4} - 8.22817 \cdot 10^{-12} B_{1,4} - 9.33054 \cdot 10^{-13} B_{2,4} + 6.36206 \cdot 10^{-12} B_{3,4} + 1.36572 \cdot 10^{-11} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.13596 \cdot 10^{-28}$.

Bounding polynomials M and m :

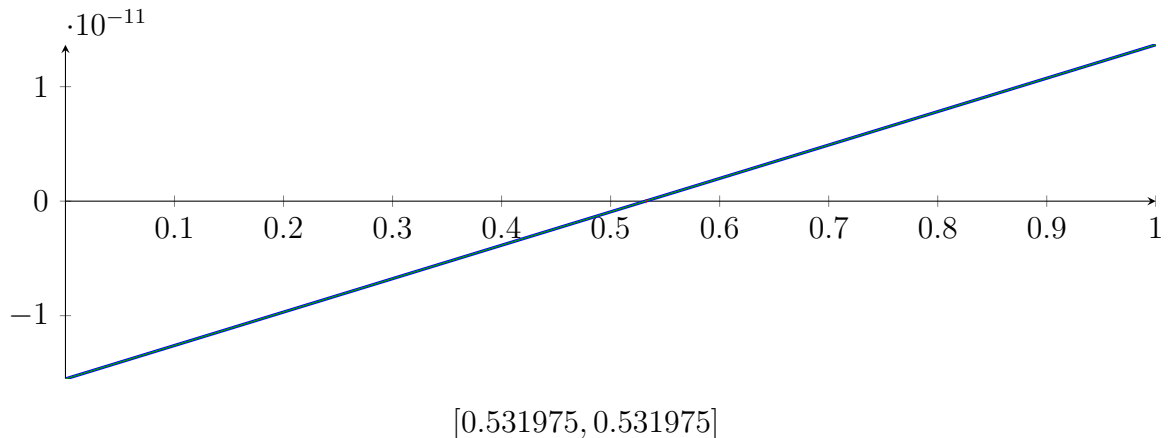
$$\begin{aligned}
 M &= 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 m &= 2.87145 \cdot 10^{-28} X^3 - 4.04172 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.531975\}$$

$$N(m) = \{0.531975\}$$

Intersection intervals:



Longest intersection interval: 0

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

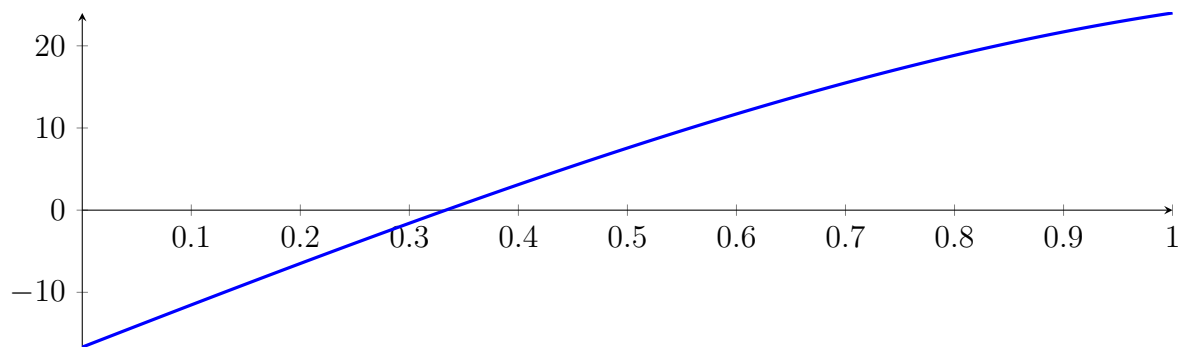
126.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

126.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

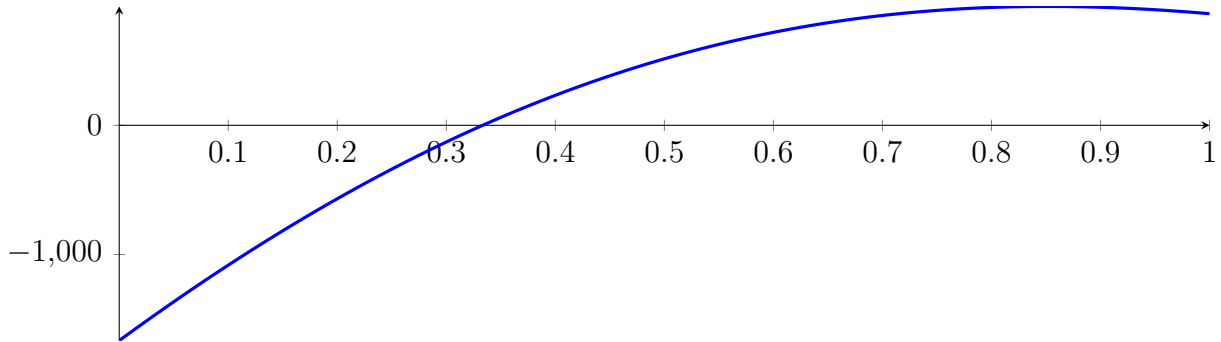
with precision $\varepsilon = 1 \cdot 10^{-128}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

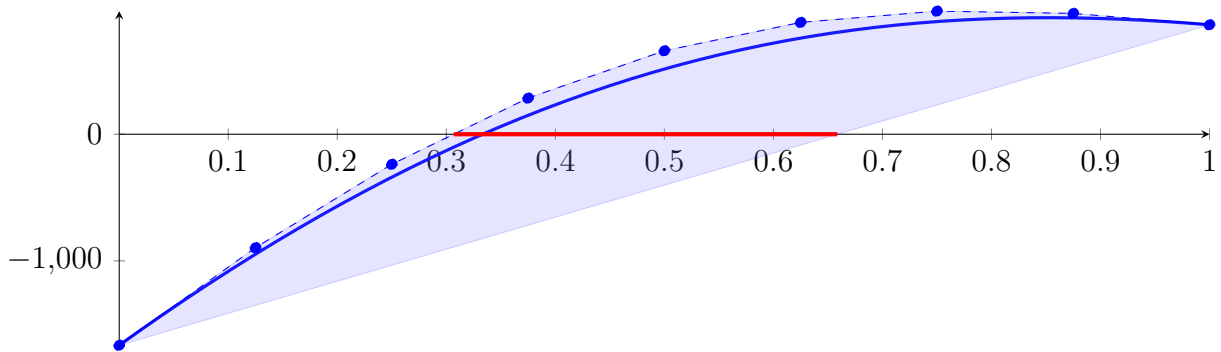
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



127.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

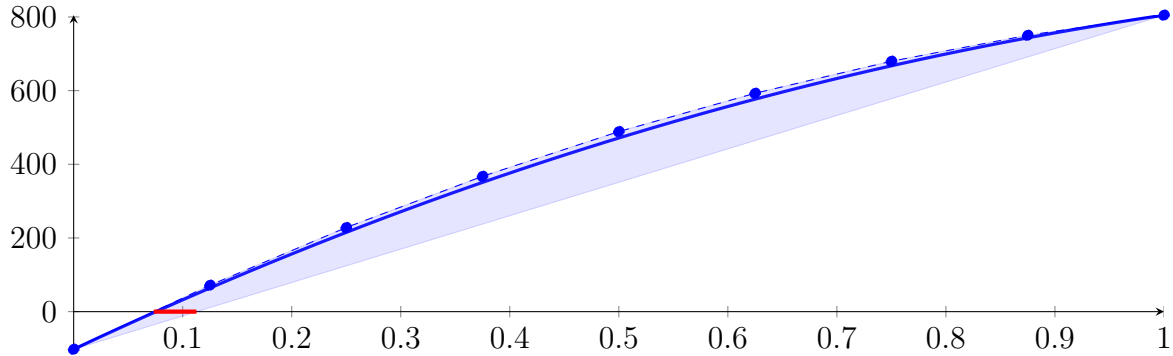
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

127.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

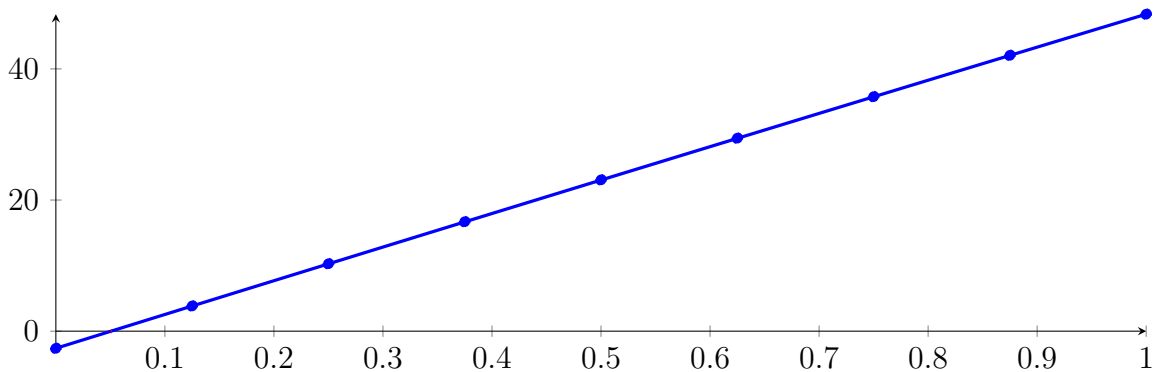
Longest intersection interval: 0.0391855

\implies Selective recursion: interval 1: [0.332635, 0.34642],

127.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.03577 \cdot 10^{-15} X^8 - 1.54633 \cdot 10^{-12} X^7 - 4.95836 \cdot 10^{-10} X^6 + 3.66751 \cdot 10^{-08} X^5 \\
 &\quad + 3.25466 \cdot 10^{-05} X^4 - 0.000586142 X^3 - 0.747315 X^2 + 51.7118 X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589

\implies Selective recursion: interval 1: [0.333333, 0.333343],

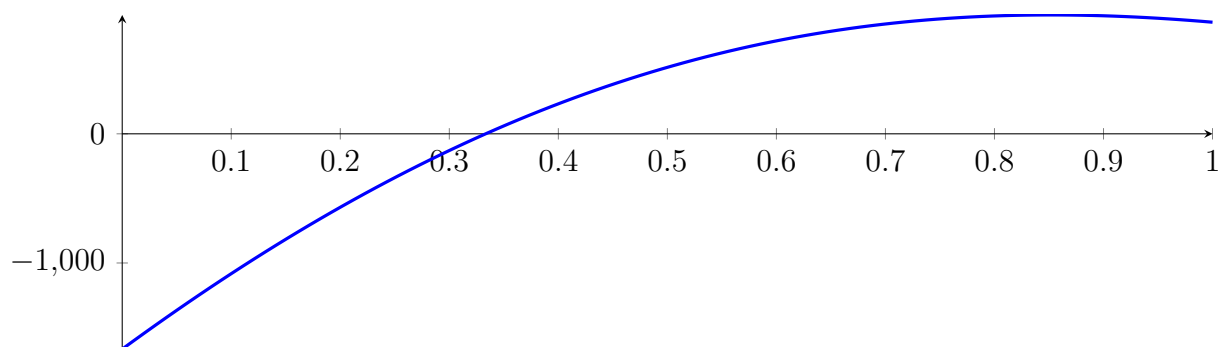
127.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Found root in interval [0.333333, 0.333343] at recursion depth 4!

127.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333343]$$

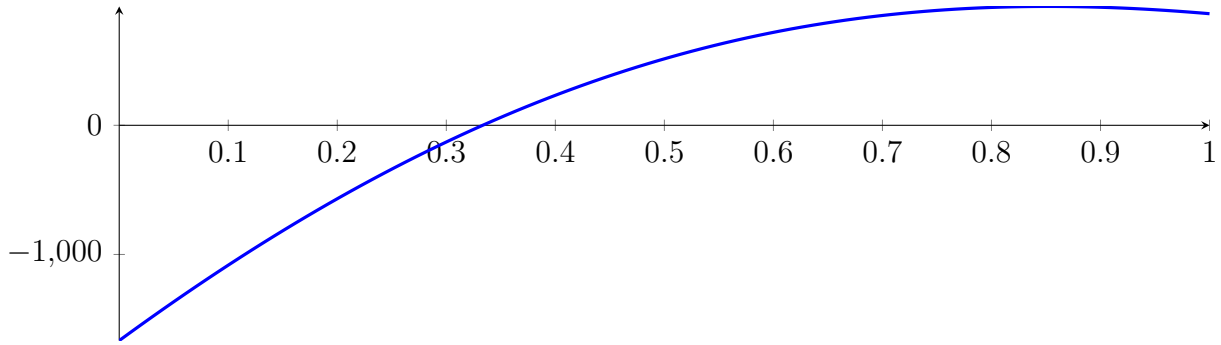
with precision $\varepsilon = 0.01$.

128 Running QuadClip on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

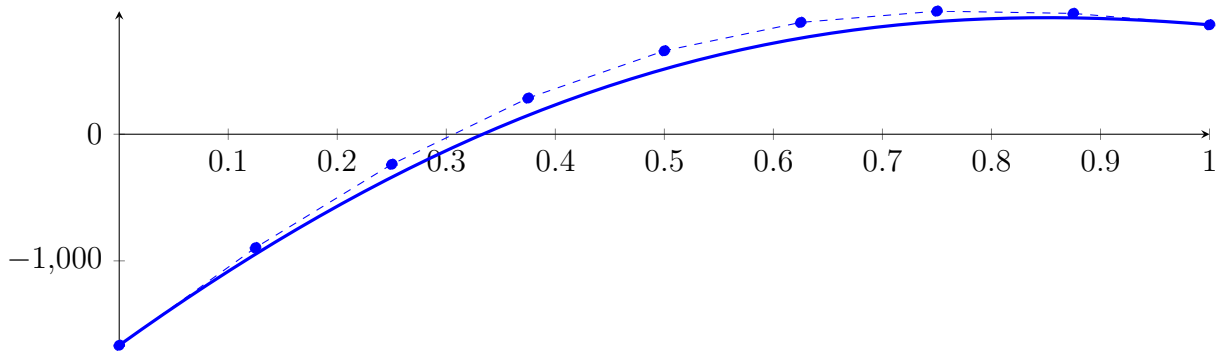
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



128.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

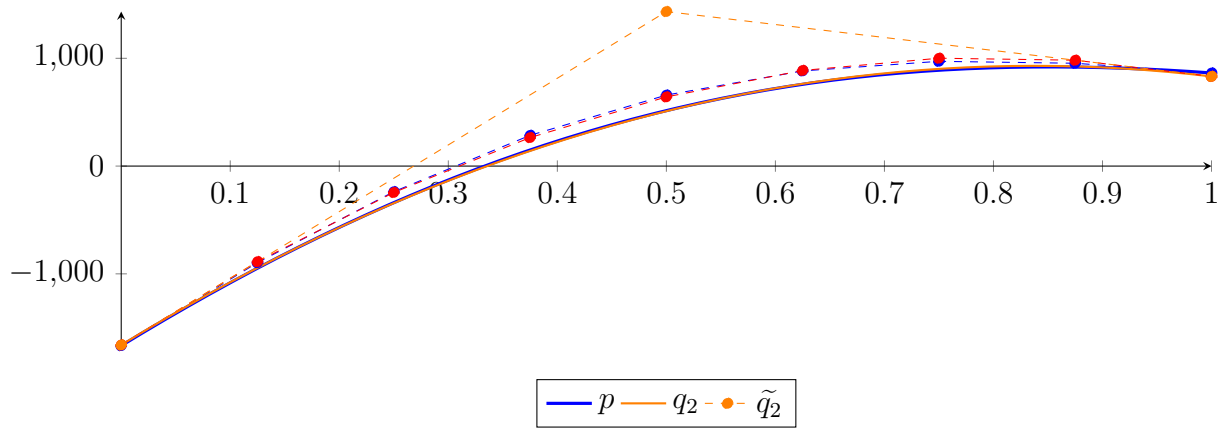
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -5.06304 \cdot 10^{-10}X^8 + 1.93305 \cdot 10^{-09}X^7 - 2.97598 \cdot 10^{-09}X^6 + 2.35116 \cdot 10^{-09}X^5 \\ &\quad - 9.91232 \cdot 10^{-10}X^4 + 2.00705 \cdot 10^{-10}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

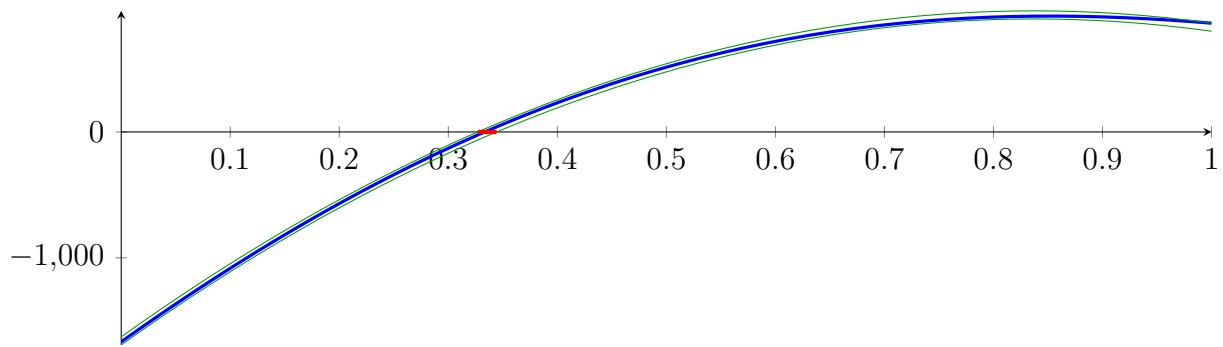
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

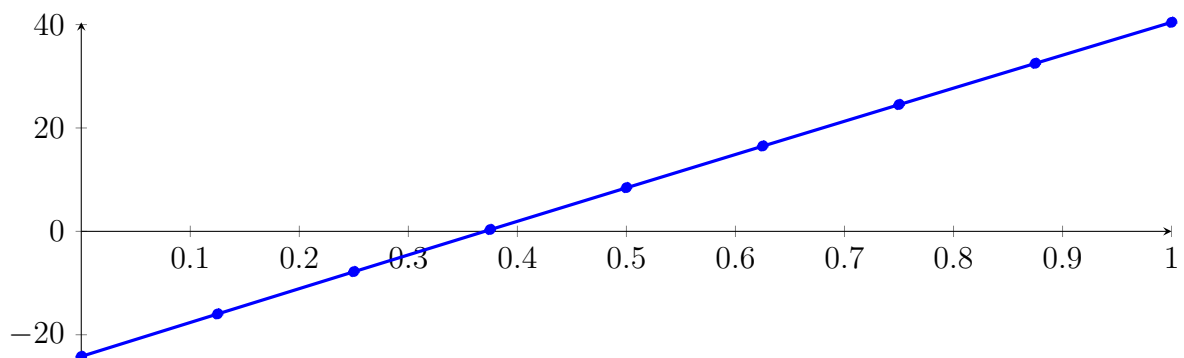
Longest intersection interval: 0.0173372

\implies Selective recursion: interval 1: $[0.326917, 0.344255]$,

128.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -8.80546 \cdot 10^{-15} X^8 - 7.66587 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

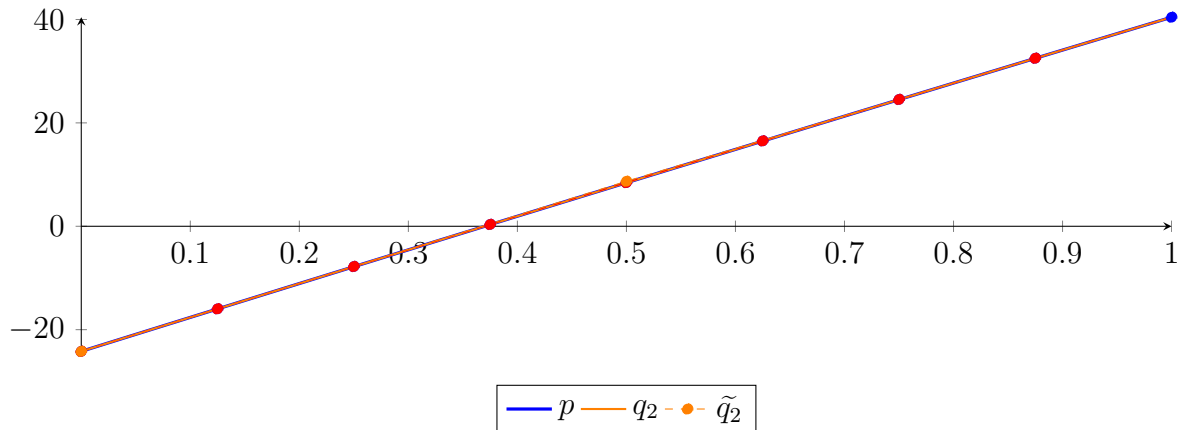
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = -2.59056 \cdot 10^{-11} X^8 + 1.00262 \cdot 10^{-10} X^7 - 1.57692 \cdot 10^{-10} X^6 + 1.29283 \cdot 10^{-10} X^5$$

$$- 5.86775 \cdot 10^{-11} X^4 + 1.42642 \cdot 10^{-11} X^3 - 1.18261 X^2 + 65.8162 X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

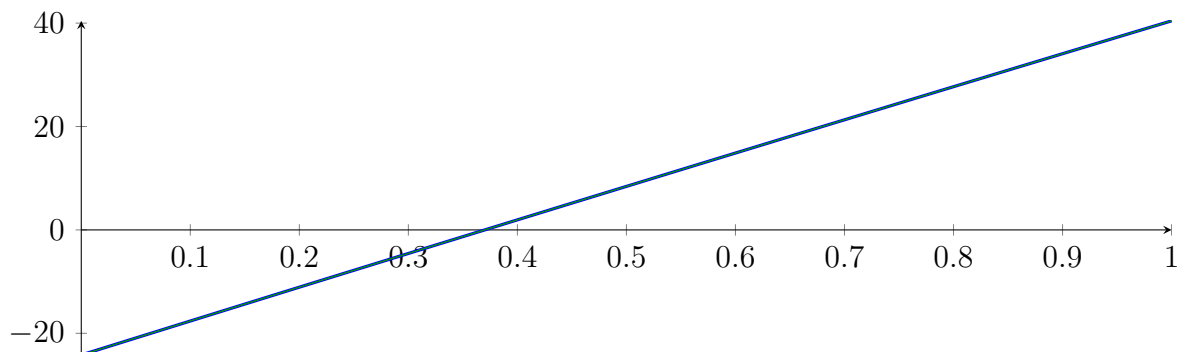
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

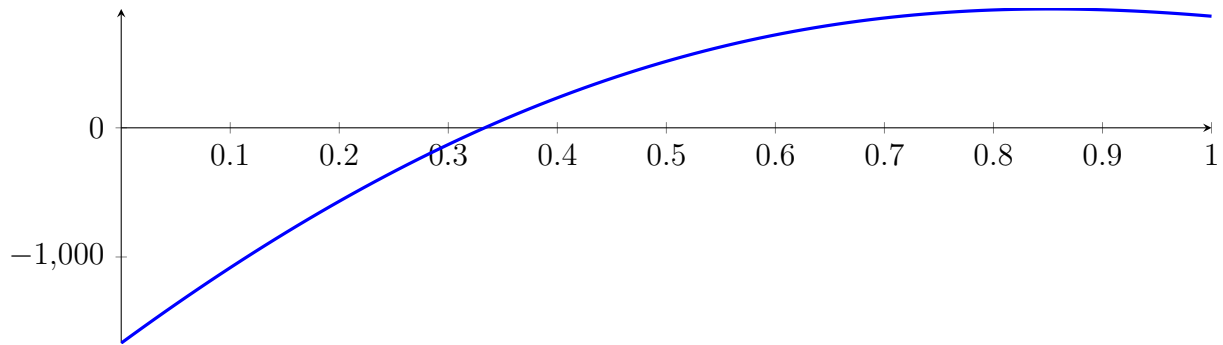
128.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

128.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

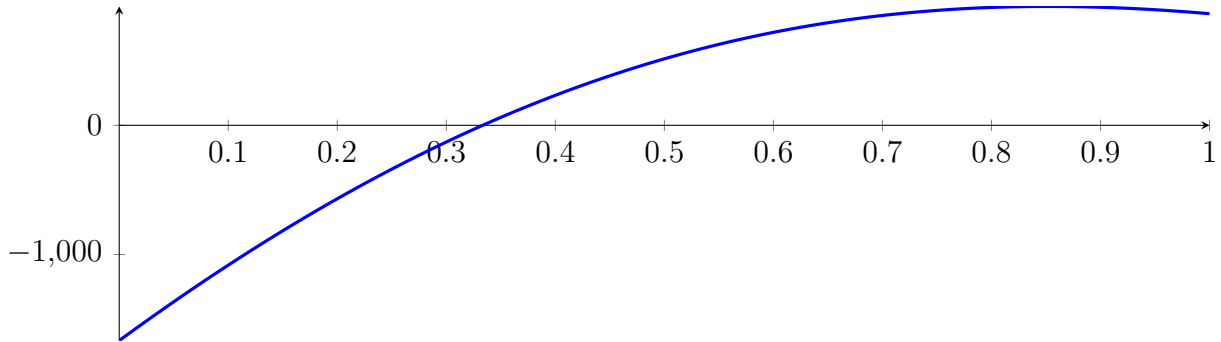
with precision $\varepsilon = 0.01$.

129 Running CubeClip on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

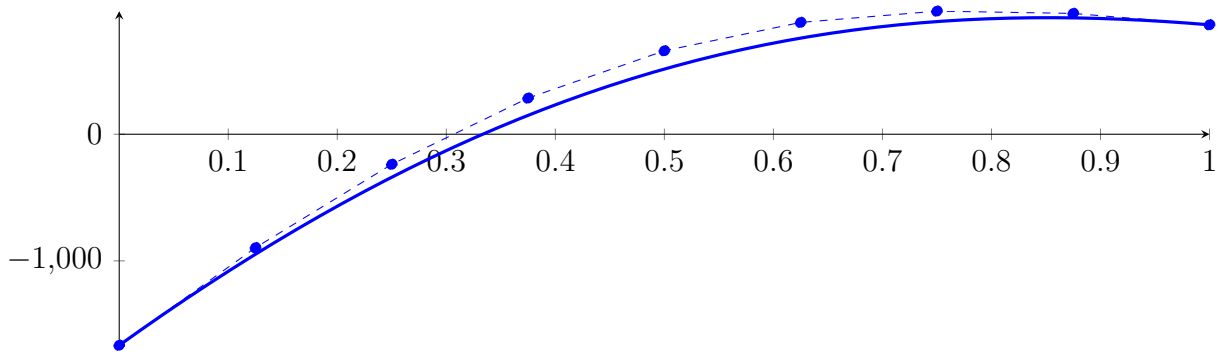
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



129.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

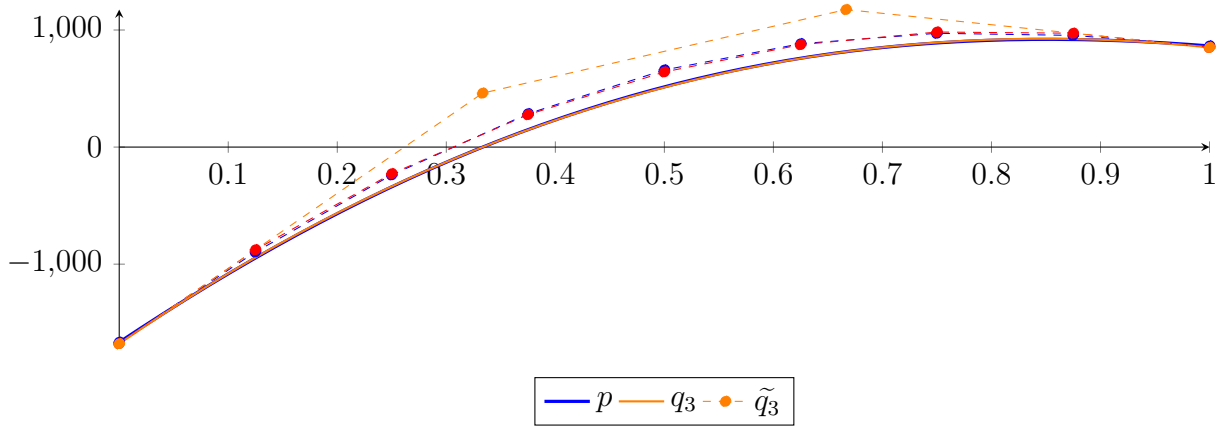
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.39389 \cdot 10^{-09}X^8 + 9.60737 \cdot 10^{-09}X^7 - 1.57477 \cdot 10^{-08}X^6 + 1.35479 \cdot 10^{-08}X^5 \\ &\quad - 6.53397 \cdot 10^{-09}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

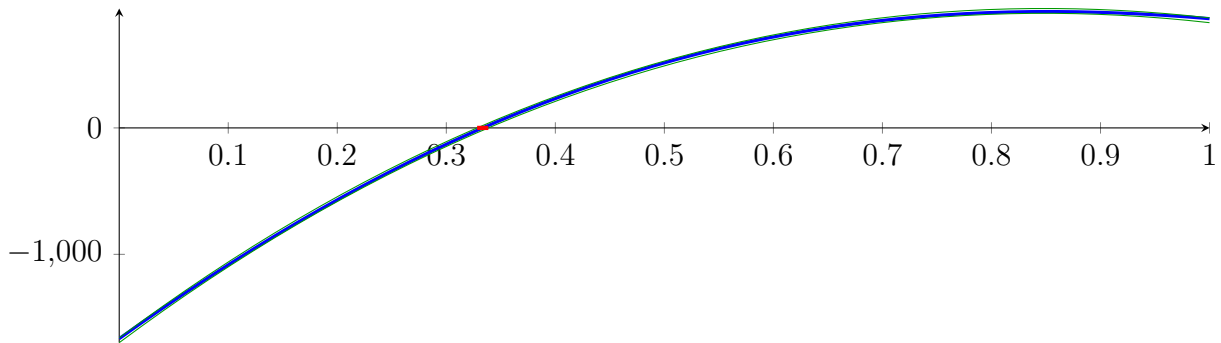
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

129.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

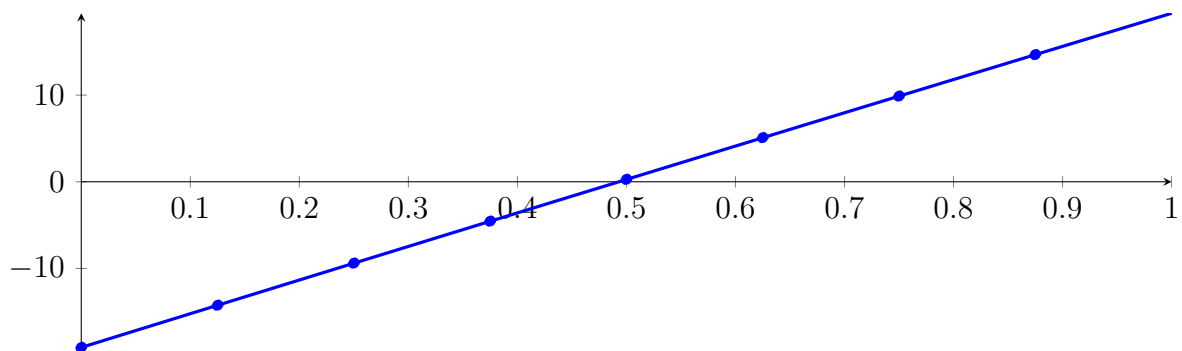
Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.66533 \cdot 10^{-16} X^8 - 1.99007 \cdot 10^{-13} X^7 - 8.53059 \cdot 10^{-11} X^6 + 8.7284 \cdot 10^{-09} X^5$$

$$+ 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124$$

$$= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X)$$

$$+ 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

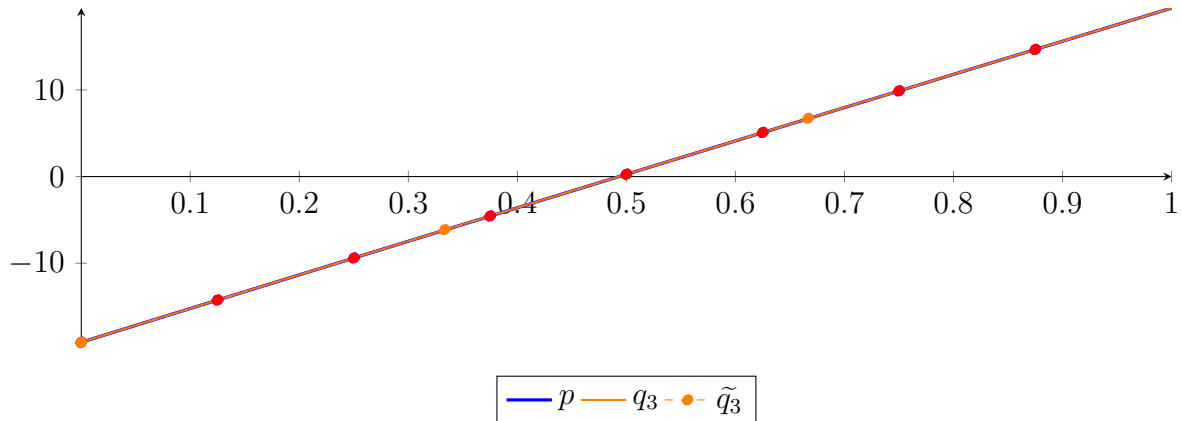
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = 2.82588 \cdot 10^{-12}X^8 - 1.08529 \cdot 10^{-11}X^7 + 1.68738 \cdot 10^{-11}X^6 - 1.34636 \cdot 10^{-11}X^5$$

$$+ 5.72618 \cdot 10^{-12}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m :

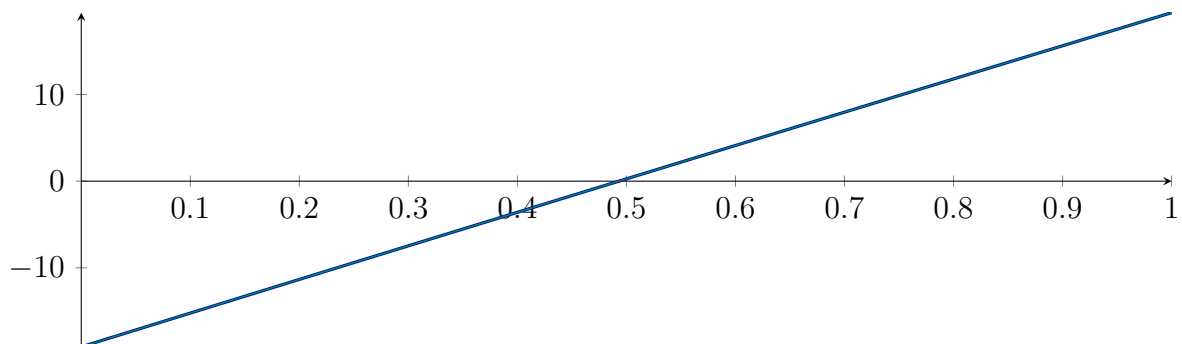
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \quad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.1252 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

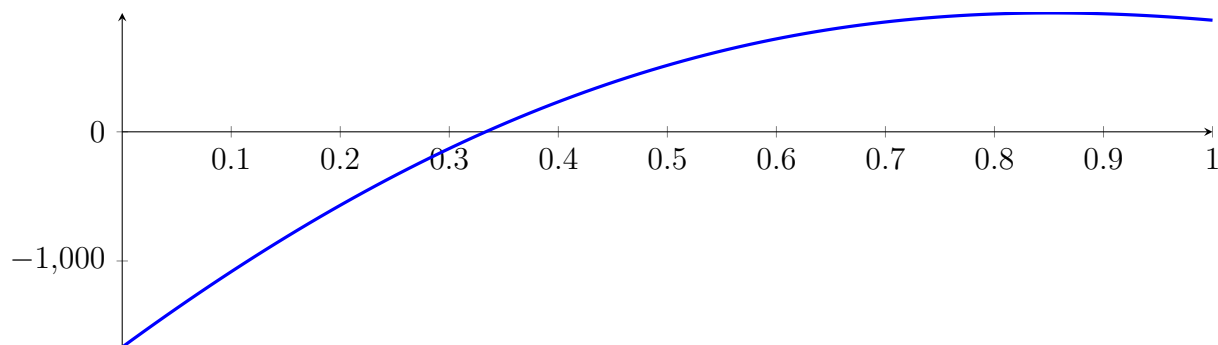
129.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

129.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

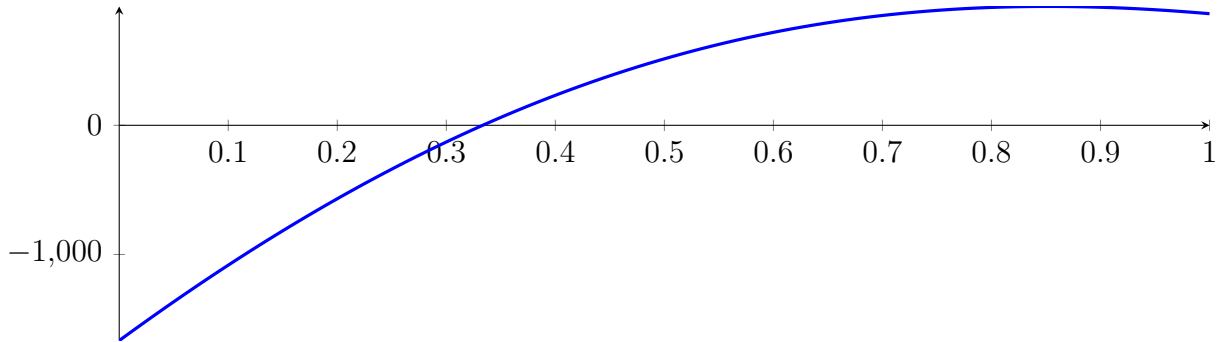
with precision $\varepsilon = 0.01$.

130 Running BezClip on f_8 with epsilon 4

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

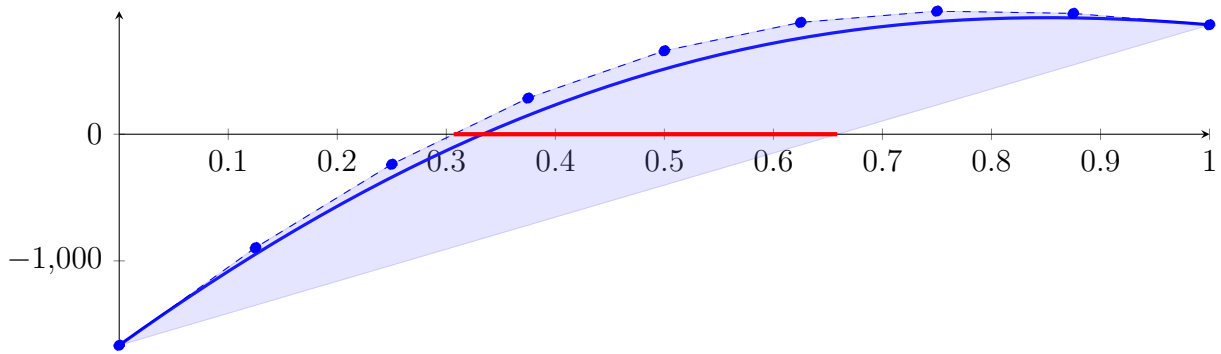
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



130.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

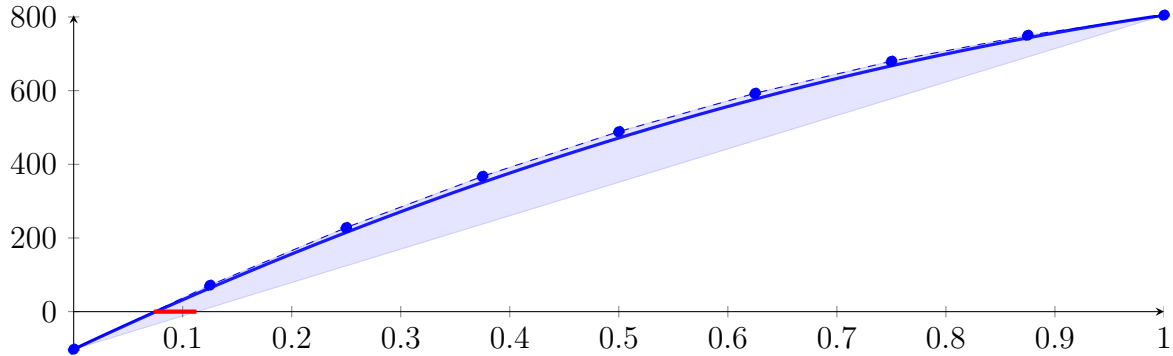
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

130.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

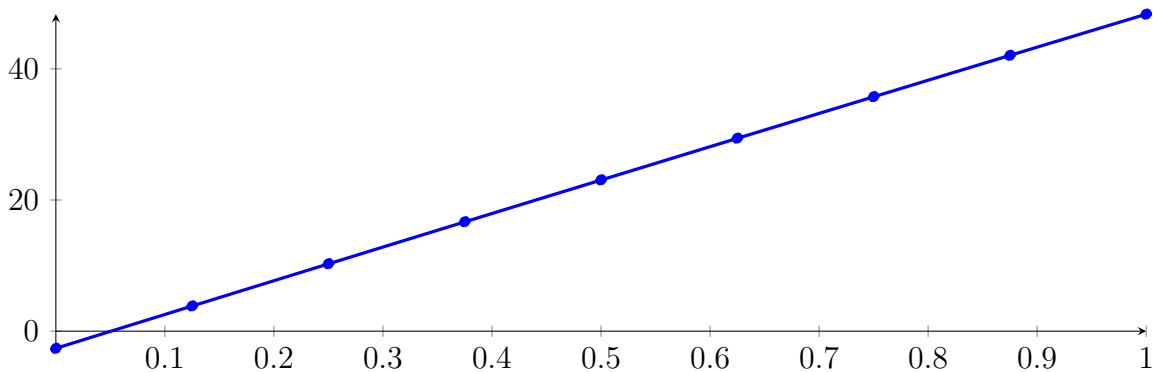
Longest intersection interval: 0.0391855

\implies Selective recursion: interval 1: [0.332635, 0.34642],

130.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.03577 \cdot 10^{-15} X^8 - 1.54633 \cdot 10^{-12} X^7 - 4.95836 \cdot 10^{-10} X^6 + 3.66751 \cdot 10^{-08} X^5 \\
 &\quad + 3.25466 \cdot 10^{-05} X^4 - 0.000586142 X^3 - 0.747315 X^2 + 51.7118 X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589

\implies Selective recursion: interval 1: [0.333333, 0.333343],

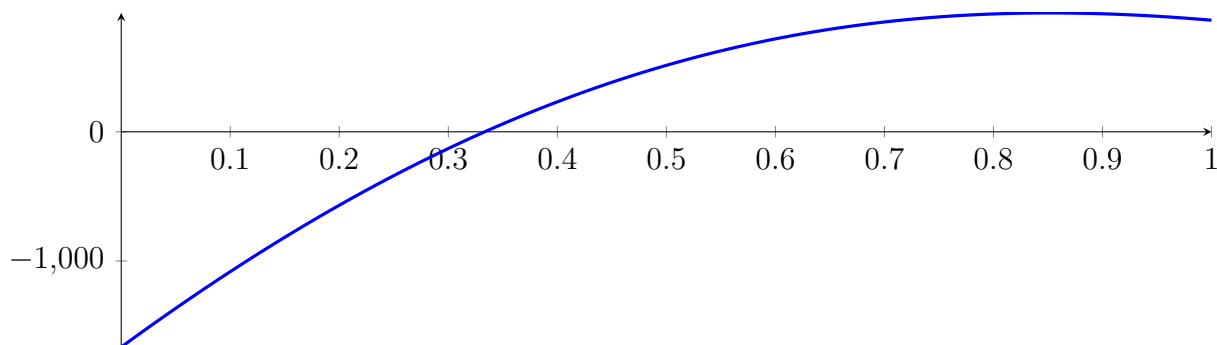
130.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Found root in interval [0.333333, 0.333343] at recursion depth 4!

130.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333343]$$

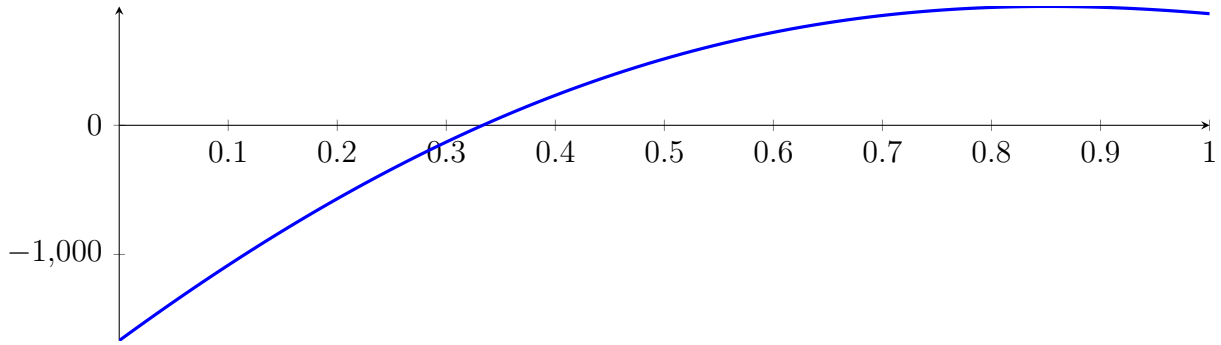
with precision $\varepsilon = 0.0001$.

131 Running QuadClip on f_8 with epsilon 4

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

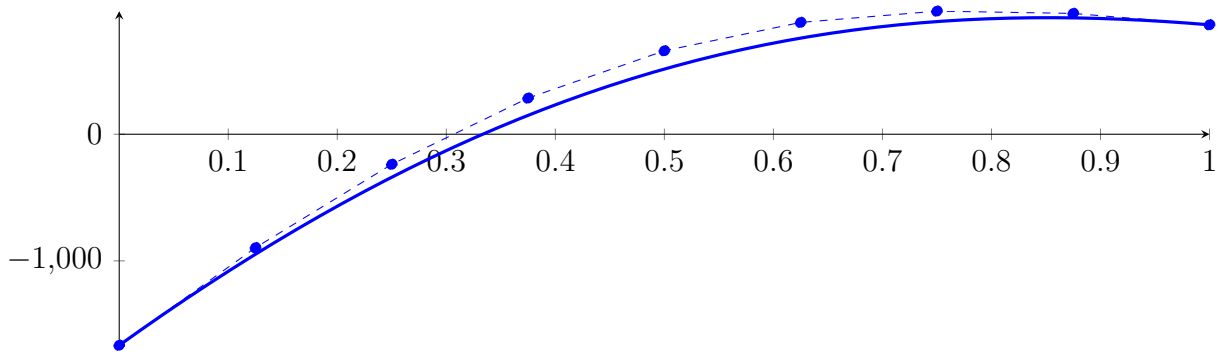
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



131.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

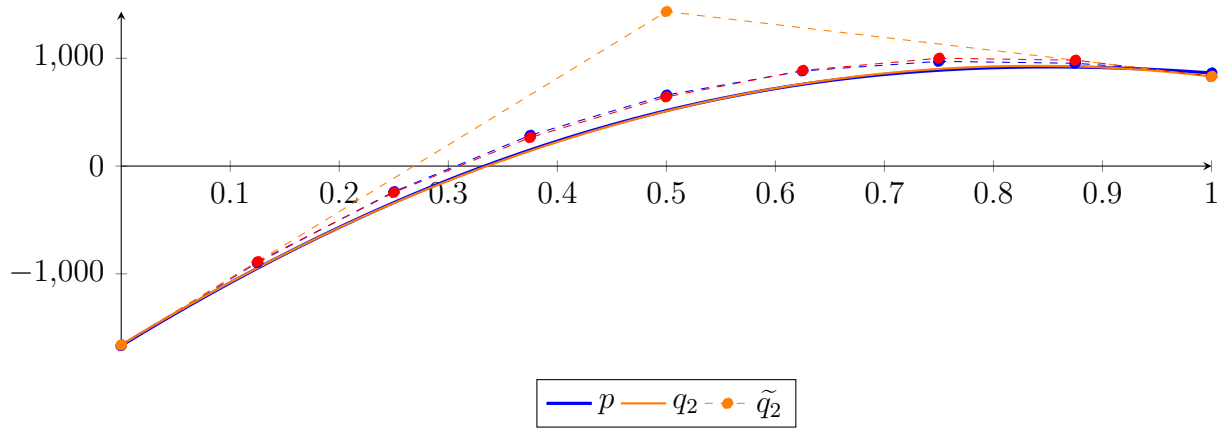
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -5.06304 \cdot 10^{-10}X^8 + 1.93305 \cdot 10^{-09}X^7 - 2.97598 \cdot 10^{-09}X^6 + 2.35116 \cdot 10^{-09}X^5 \\ &\quad - 9.91232 \cdot 10^{-10}X^4 + 2.00705 \cdot 10^{-10}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

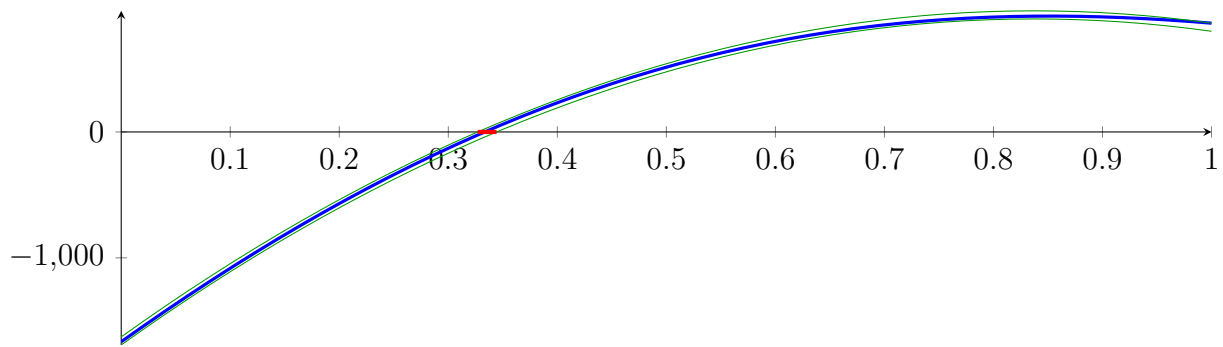
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

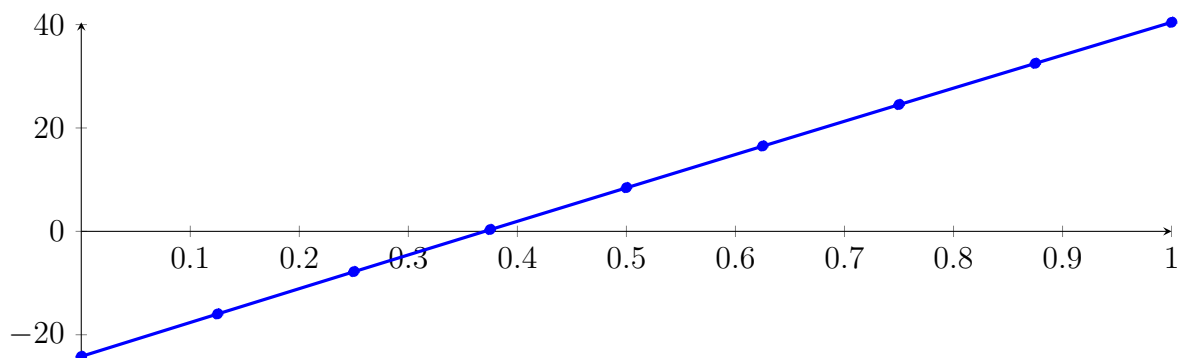
Longest intersection interval: 0.0173372

\implies Selective recursion: interval 1: $[0.326917, 0.344255]$,

131.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -8.80546 \cdot 10^{-15} X^8 - 7.66587 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

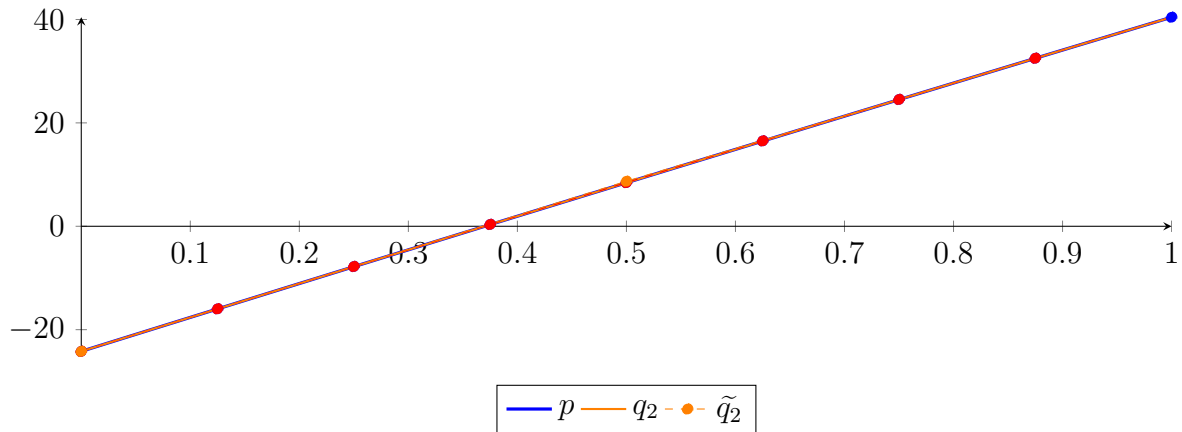
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = -2.59056 \cdot 10^{-11} X^8 + 1.00262 \cdot 10^{-10} X^7 - 1.57692 \cdot 10^{-10} X^6 + 1.29283 \cdot 10^{-10} X^5$$

$$- 5.86775 \cdot 10^{-11} X^4 + 1.42642 \cdot 10^{-11} X^3 - 1.18261 X^2 + 65.8162 X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

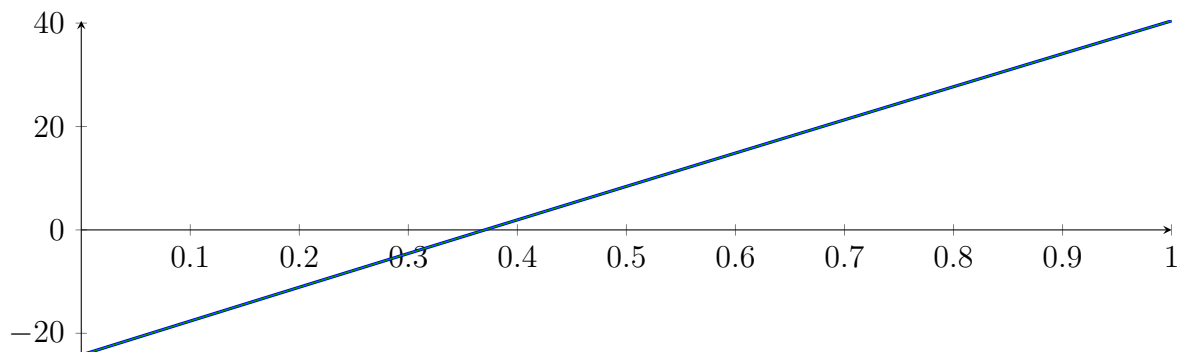
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

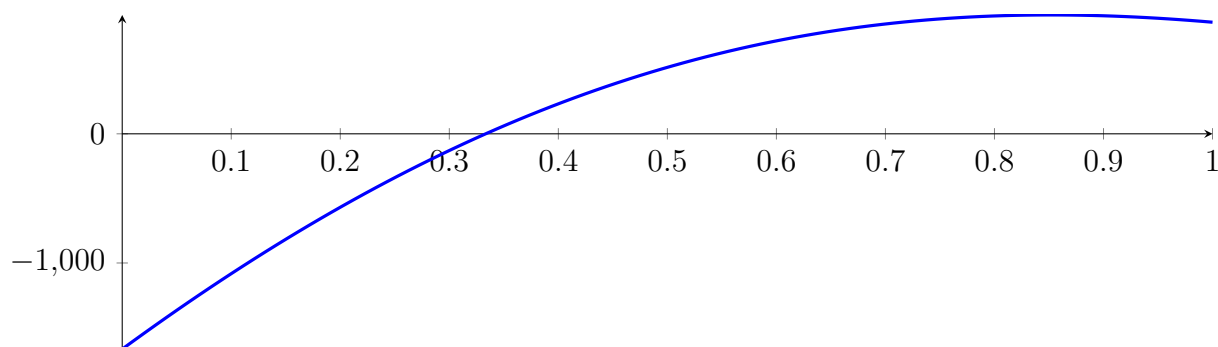
131.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

131.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

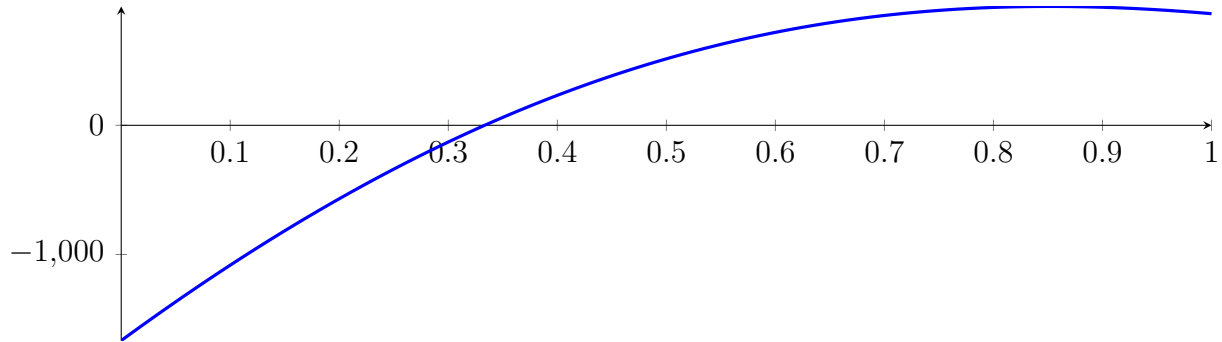
with precision $\varepsilon = 0.0001$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

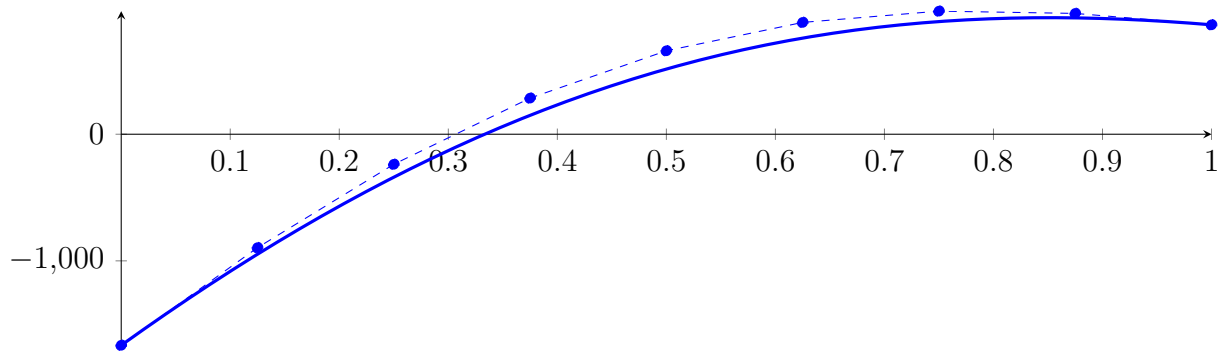
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



132.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

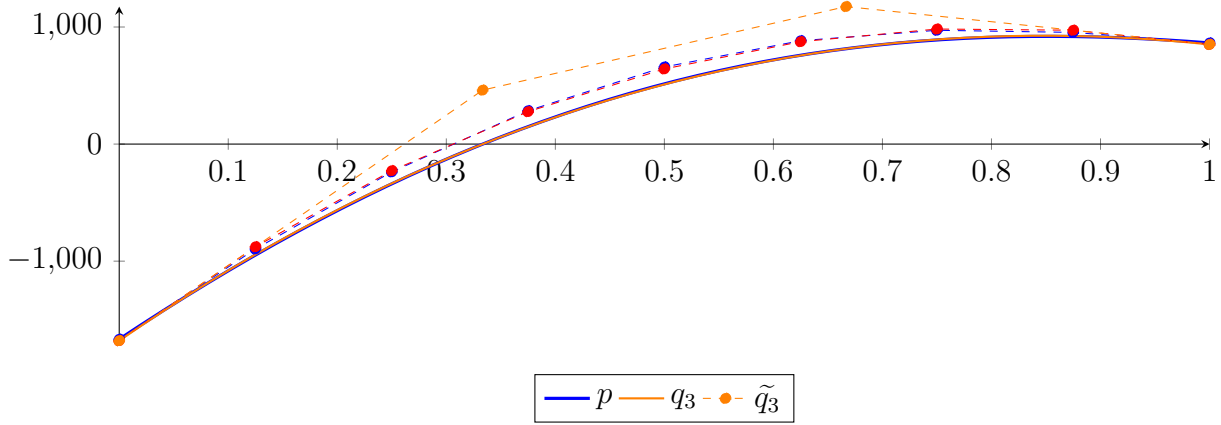
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.39389 \cdot 10^{-09}X^8 + 9.60737 \cdot 10^{-09}X^7 - 1.57477 \cdot 10^{-08}X^6 + 1.35479 \cdot 10^{-08}X^5 \\ &\quad - 6.53397 \cdot 10^{-09}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

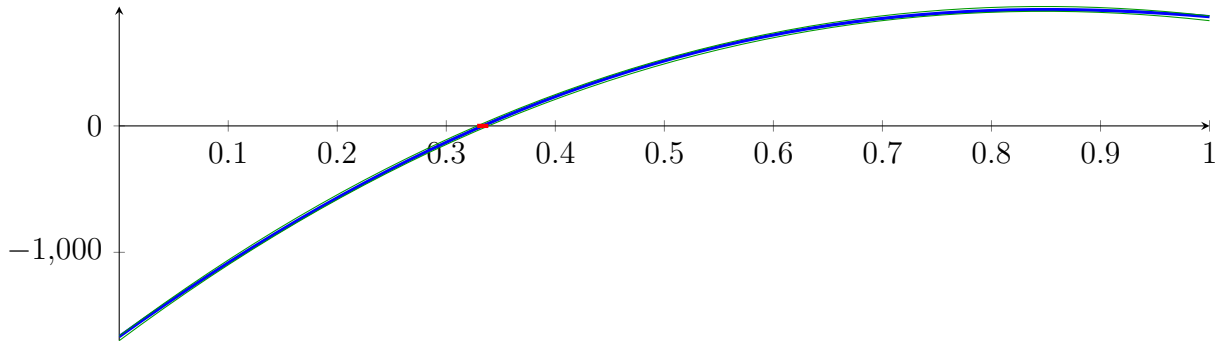
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

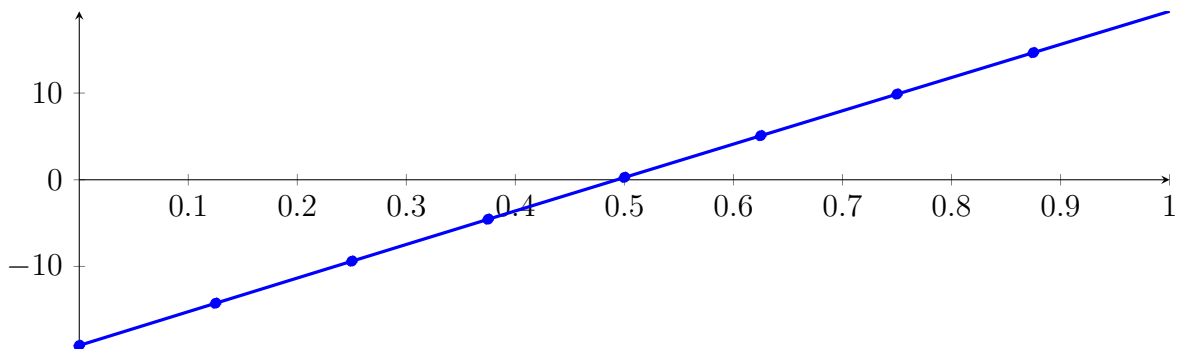
Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

132.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.66533 \cdot 10^{-16} X^8 - 1.99007 \cdot 10^{-13} X^7 - 8.53059 \cdot 10^{-11} X^6 + 8.7284 \cdot 10^{-09} X^5 \\ &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

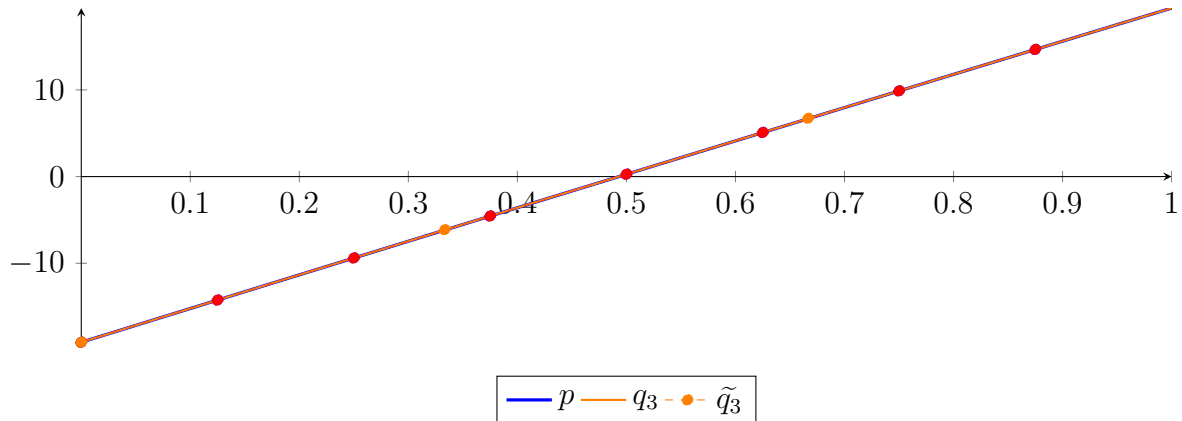
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = 2.82588 \cdot 10^{-12}X^8 - 1.08529 \cdot 10^{-11}X^7 + 1.68738 \cdot 10^{-11}X^6 - 1.34636 \cdot 10^{-11}X^5$$

$$+ 5.72618 \cdot 10^{-12}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m :

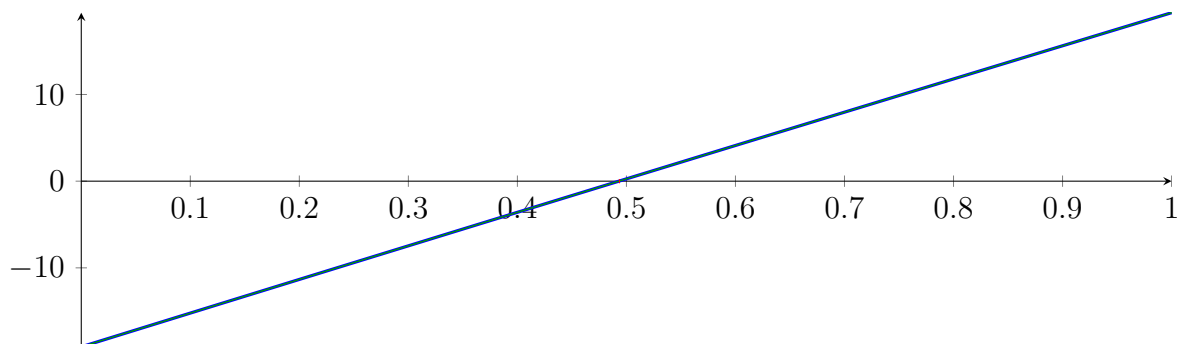
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \quad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.1252 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

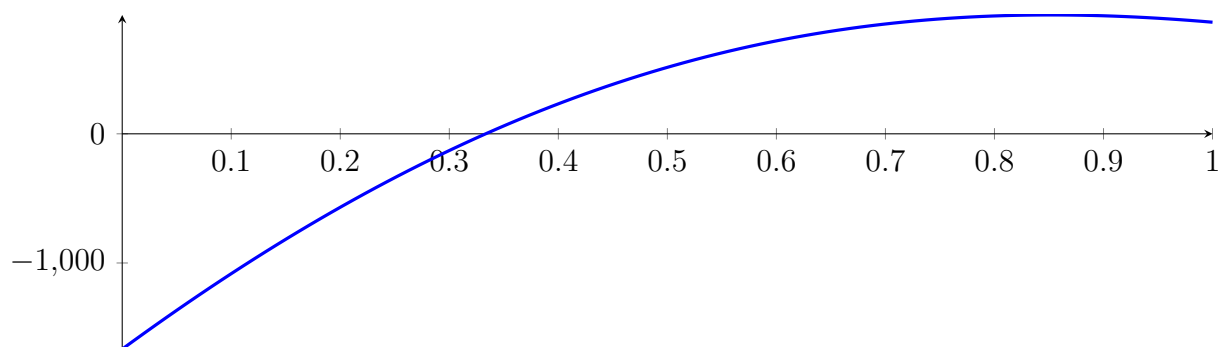
132.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

132.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

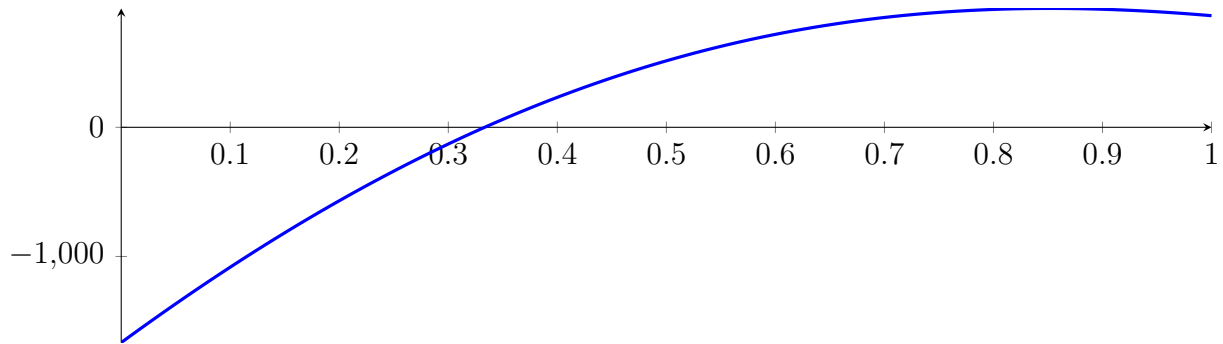
with precision $\varepsilon = 0.0001$.

133 Running BezClip on f_8 with epsilon 8

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

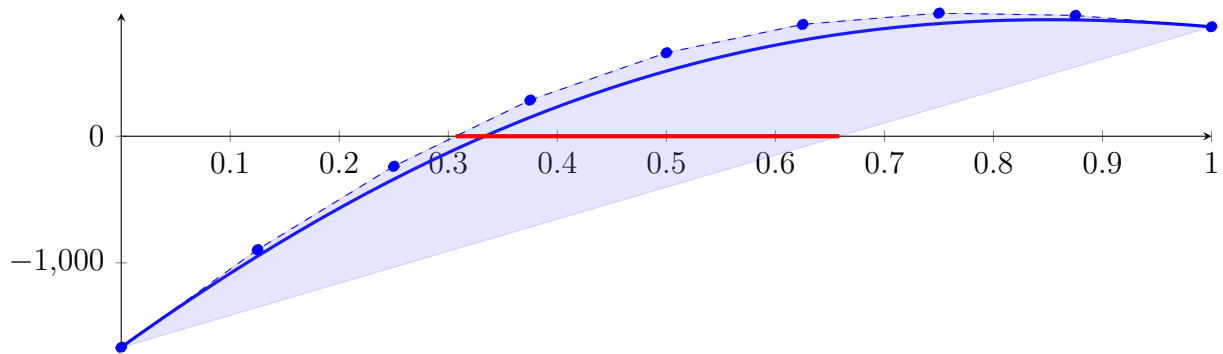
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



133.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

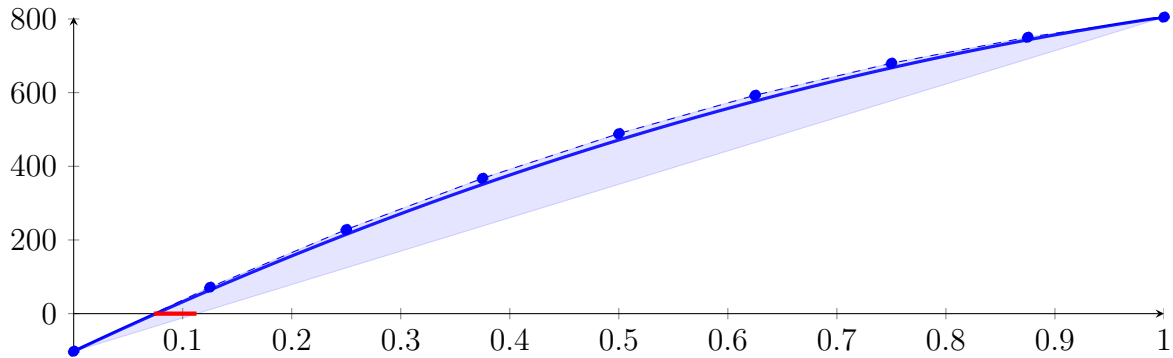
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

133.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

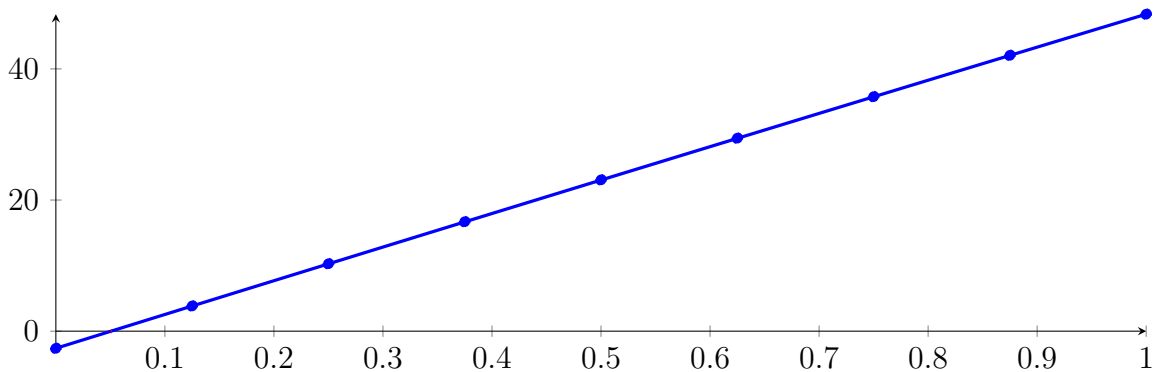
Longest intersection interval: 0.0391855

⇒ Selective recursion: interval 1: [0.332635, 0.34642],

133.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.03577 \cdot 10^{-15} X^8 - 1.54633 \cdot 10^{-12} X^7 - 4.95836 \cdot 10^{-10} X^6 + 3.66751 \cdot 10^{-08} X^5 \\
 &\quad + 3.25466 \cdot 10^{-05} X^4 - 0.000586142 X^3 - 0.747315 X^2 + 51.7118 X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

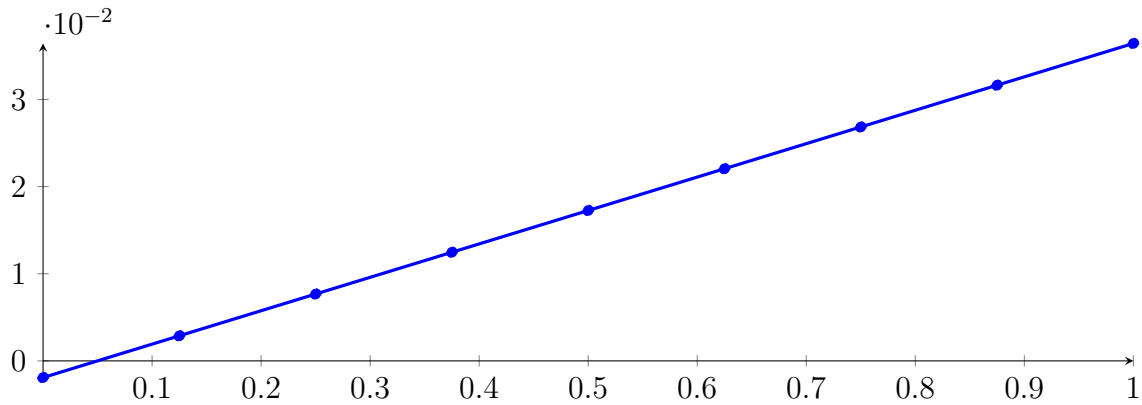
Longest intersection interval: 0.000742589

⇒ Selective recursion: interval 1: [0.333333, 0.333343],

133.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.28918 \cdot 10^{-18} X^8 - 1.32815 \cdot 10^{-18} X^7 - 4.50622 \cdot 10^{-18} X^6 + 2.22939 \cdot 10^{-18} X^5 \\
 &\quad + 9.48677 \cdot 10^{-18} X^4 - 2.37322 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\
 &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\
 &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\
 &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

Longest intersection interval: $5.36469 \cdot 10^{-07}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

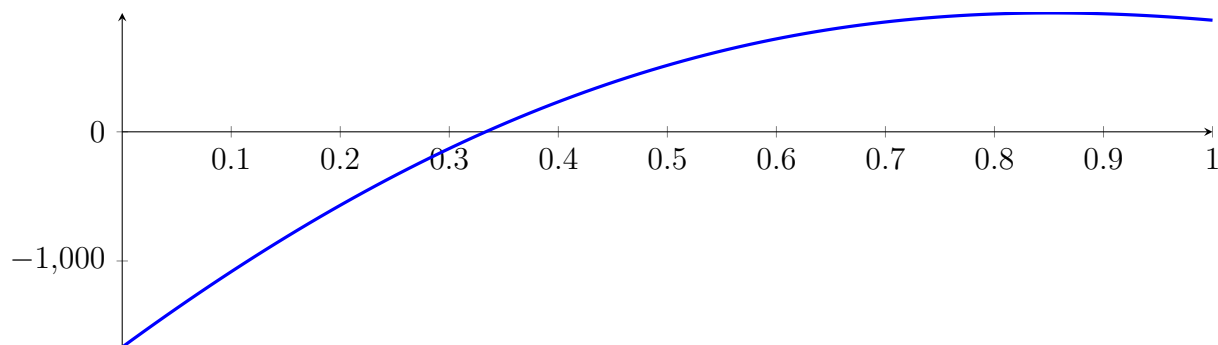
133.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

133.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

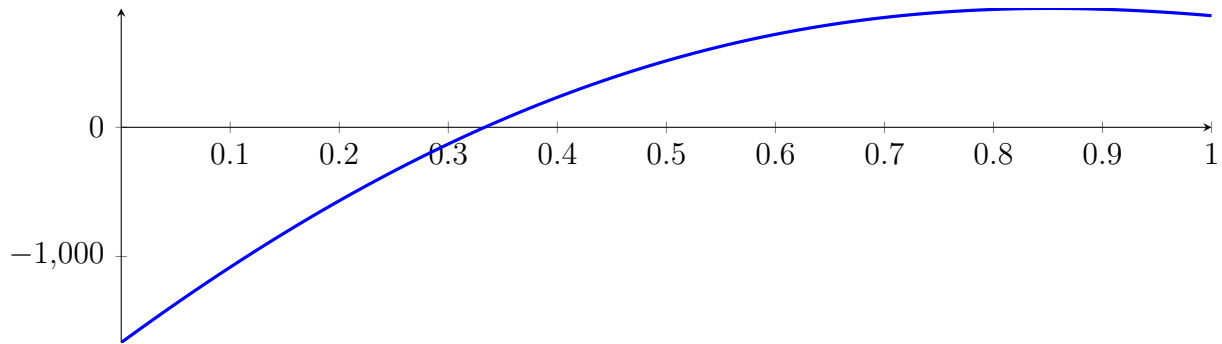
with precision $\varepsilon = 1 \cdot 10^{-08}$.

134 Running QuadClip on f_8 with epsilon 8

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

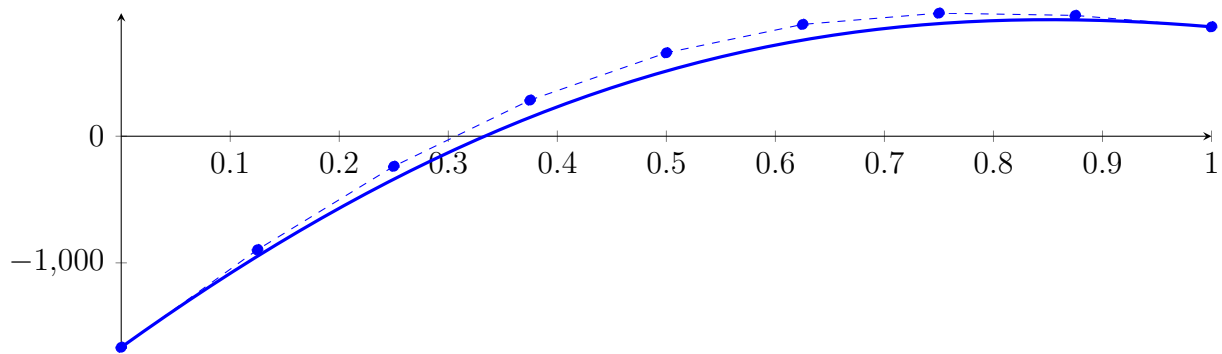
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



134.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

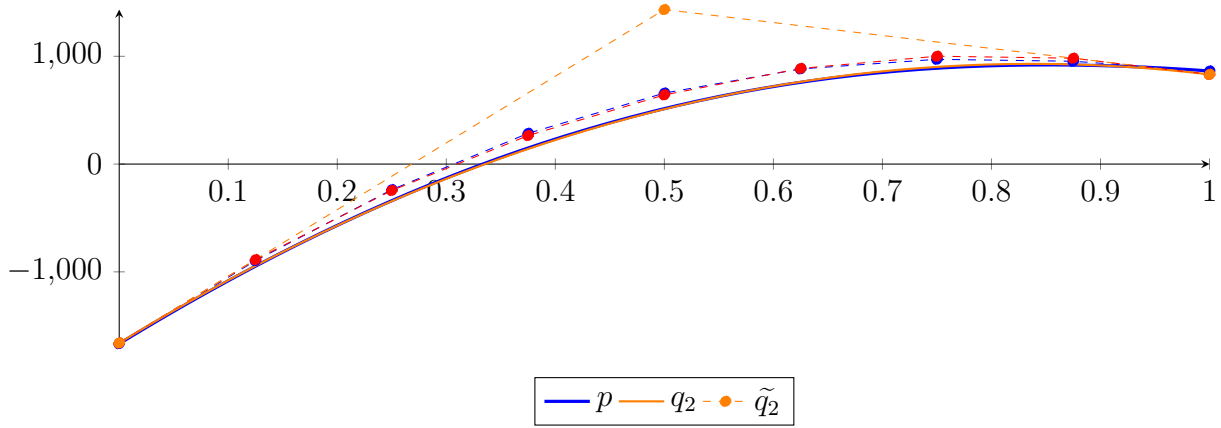
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -5.06304 \cdot 10^{-10}X^8 + 1.93305 \cdot 10^{-09}X^7 - 2.97598 \cdot 10^{-09}X^6 + 2.35116 \cdot 10^{-09}X^5 \\ &\quad - 9.91232 \cdot 10^{-10}X^4 + 2.00705 \cdot 10^{-10}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

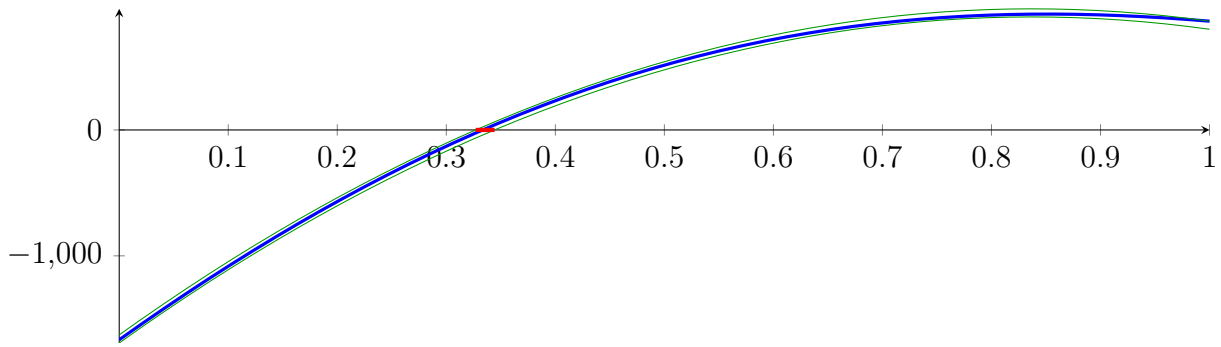
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

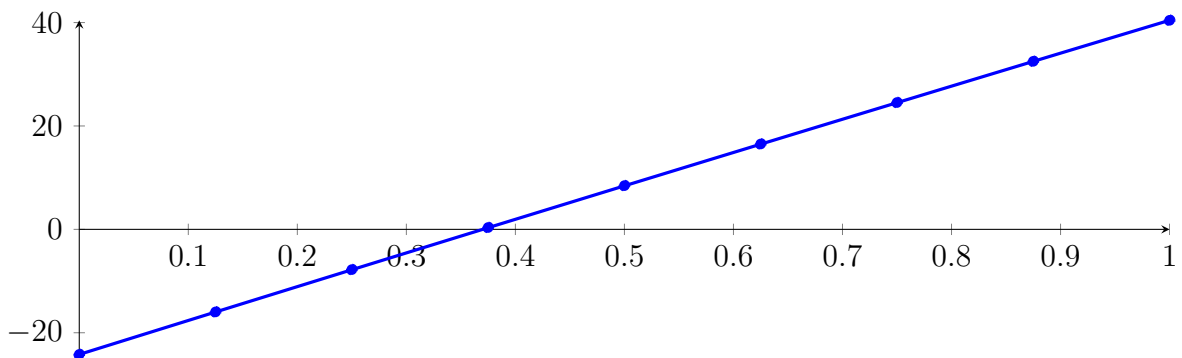
Longest intersection interval: 0.0173372

\implies Selective recursion: interval 1: $[0.326917, 0.344255]$,

134.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -8.80546 \cdot 10^{-15} X^8 - 7.66587 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

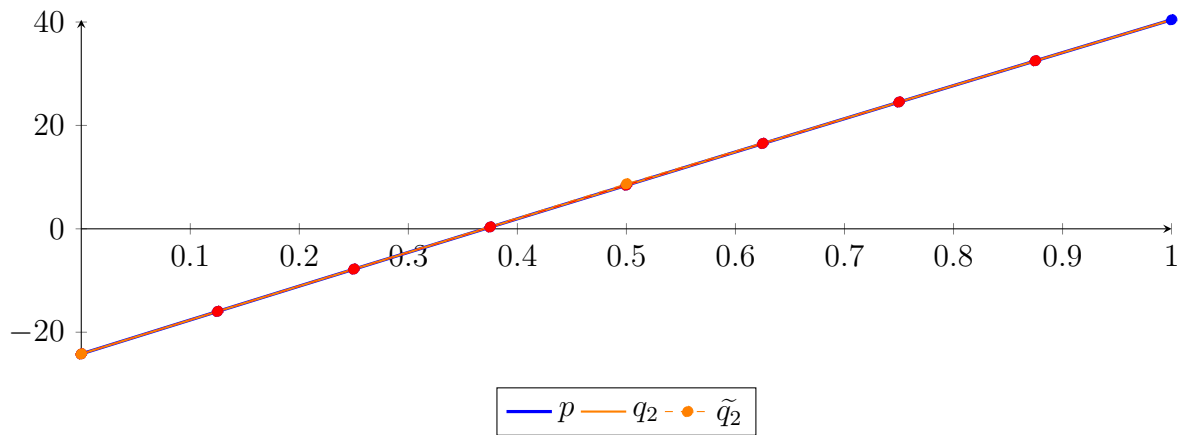
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = -2.59056 \cdot 10^{-11}X^8 + 1.00262 \cdot 10^{-10}X^7 - 1.57692 \cdot 10^{-10}X^6 + 1.29283 \cdot 10^{-10}X^5$$

$$- 5.86775 \cdot 10^{-11}X^4 + 1.42642 \cdot 10^{-11}X^3 - 1.18261X^2 + 65.8162X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

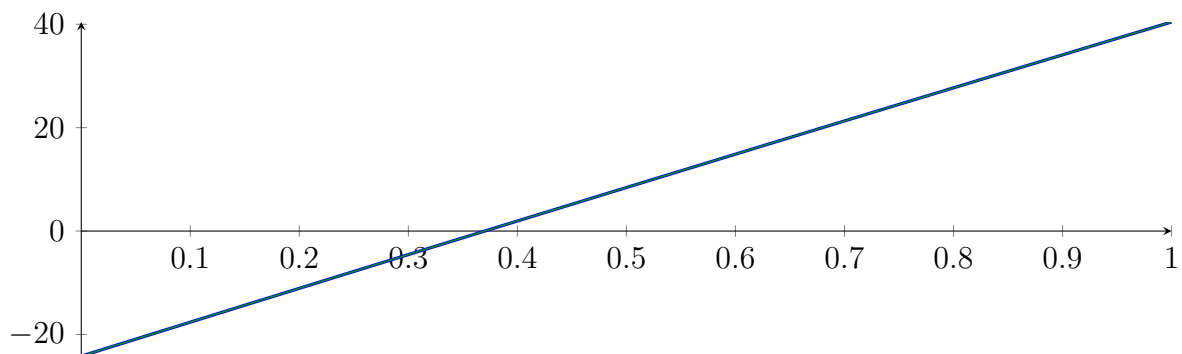
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

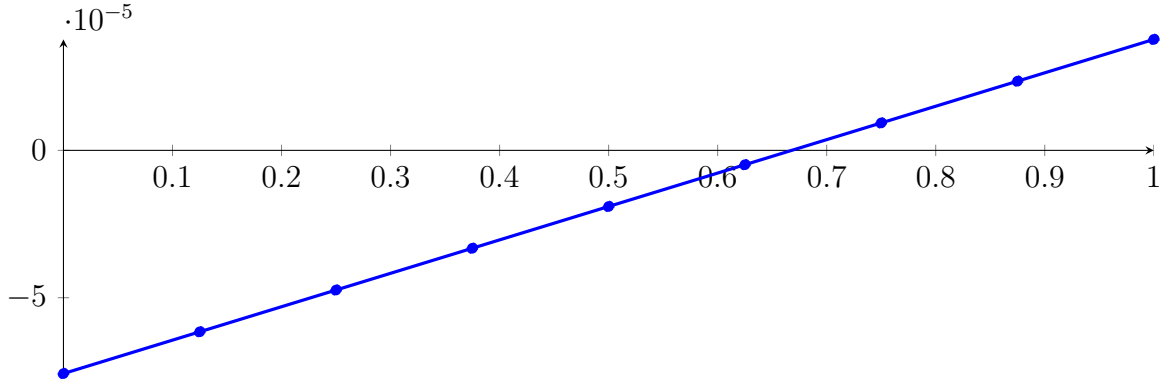
Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

134.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

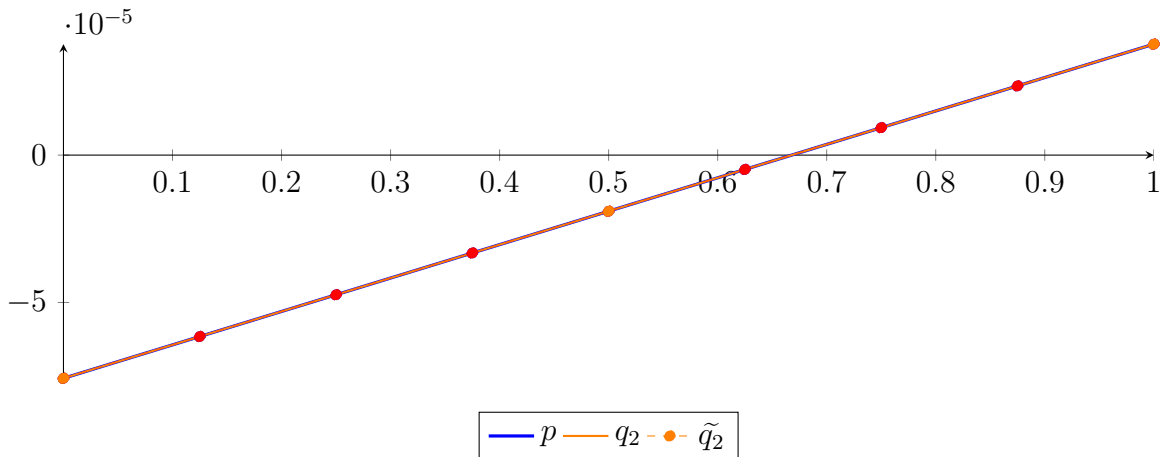
$$\begin{aligned}
 p &= 1.16467 \cdot 10^{-21} X^8 + 3.17637 \cdot 10^{-21} X^7 + 1.18585 \cdot 10^{-20} X^6 - 1.48231 \cdot 10^{-21} X^5 + 9.26442 \\
 &\quad \cdot 10^{-22} X^4 - 5.92923 \cdot 10^{-21} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) \\
 &\quad - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) \\
 &\quad + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 3.26671 \cdot 10^{-17} X^8 - 1.38104 \cdot 10^{-16} X^7 + 2.39221 \cdot 10^{-16} X^6 - 2.17429 \cdot 10^{-16} X^5 + 1.10046 \\
 &\quad \cdot 10^{-16} X^4 - 3.0162 \cdot 10^{-17} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \\
 &\quad \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.84643 \cdot 10^{-19}$.

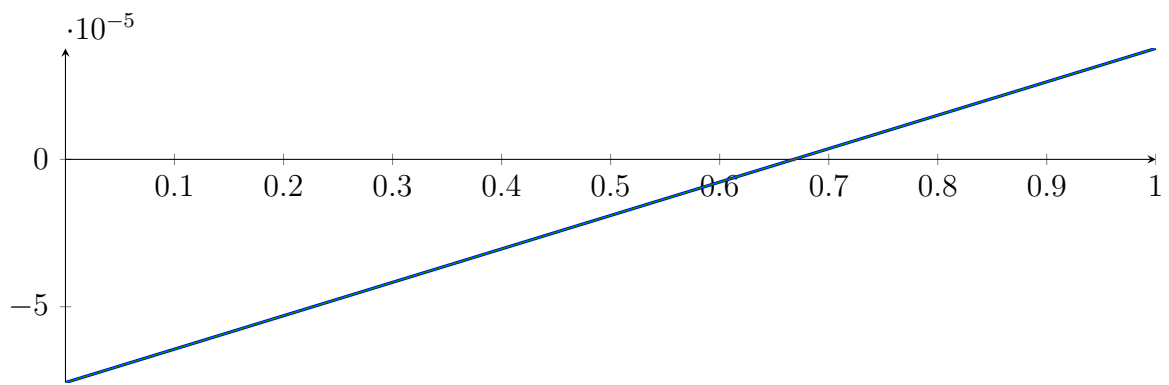
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 m &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.667955, 3.14644 \cdot 10^7\} \qquad N(m) = \{0.667955, 3.14644 \cdot 10^7\}$$

Intersection intervals:



[0.667955, 0.667955]

Longest intersection interval: $3.08439 \cdot 10^{-13}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

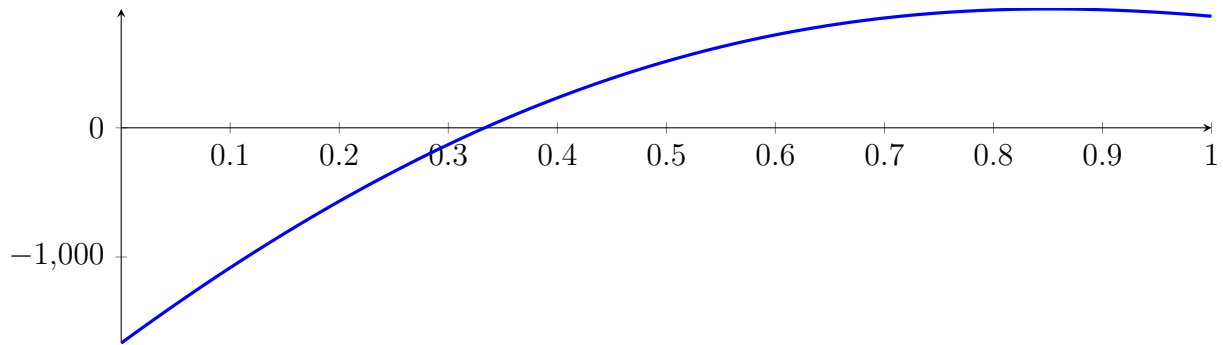
134.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

134.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

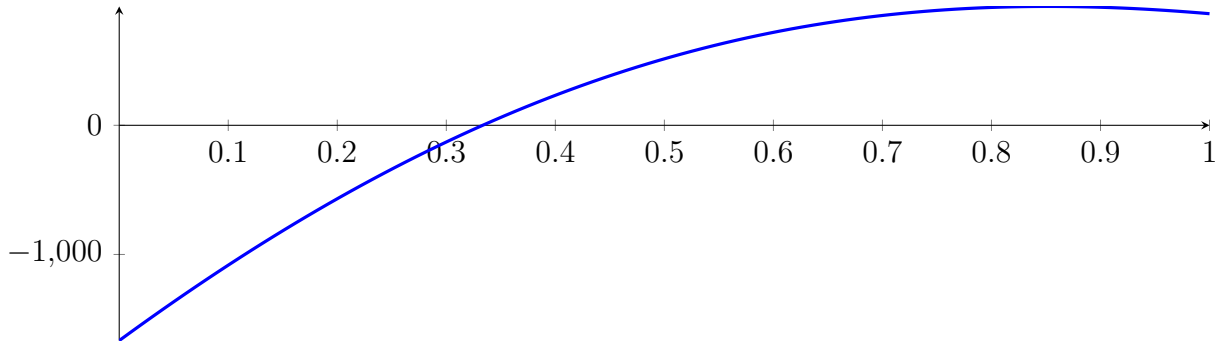
with precision $\varepsilon = 1 \cdot 10^{-08}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

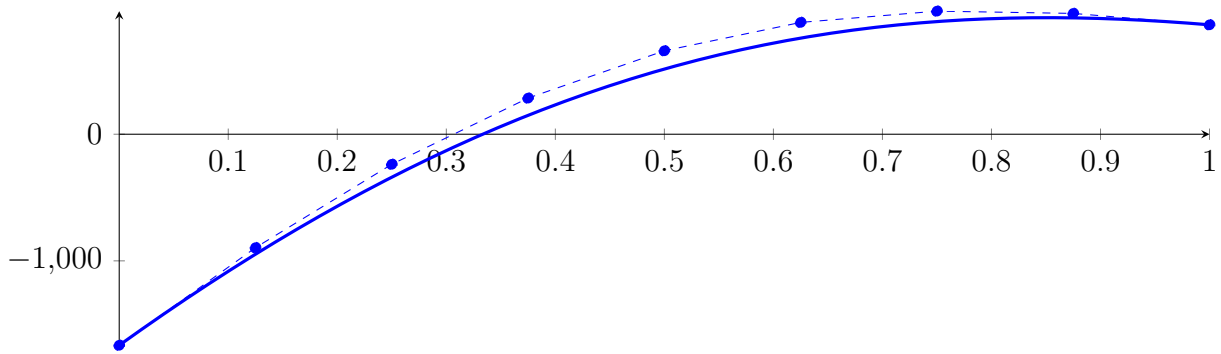
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



135.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

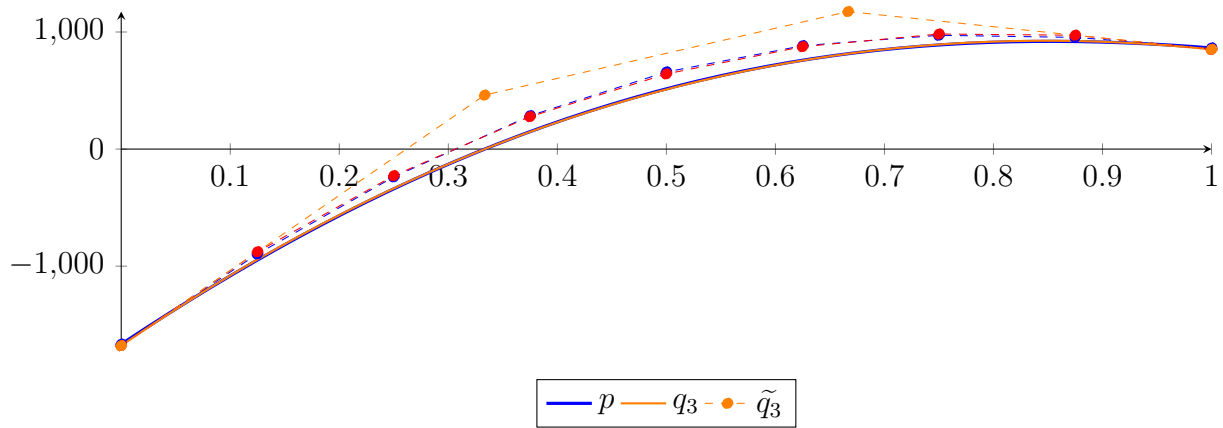
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.39389 \cdot 10^{-09}X^8 + 9.60737 \cdot 10^{-09}X^7 - 1.57477 \cdot 10^{-08}X^6 + 1.35479 \cdot 10^{-08}X^5 \\ &\quad - 6.53397 \cdot 10^{-09}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

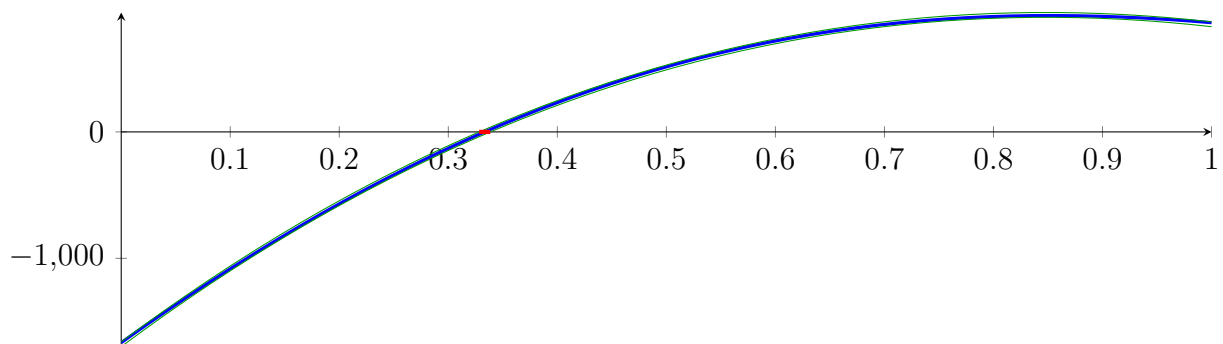
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

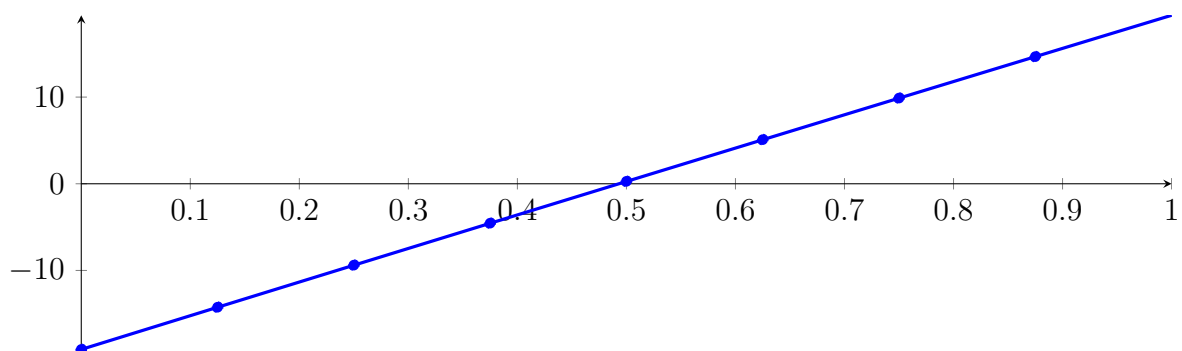
Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

135.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.66533 \cdot 10^{-16} X^8 - 1.99007 \cdot 10^{-13} X^7 - 8.53059 \cdot 10^{-11} X^6 + 8.7284 \cdot 10^{-09} X^5 \\
 &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\
 &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\
 &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

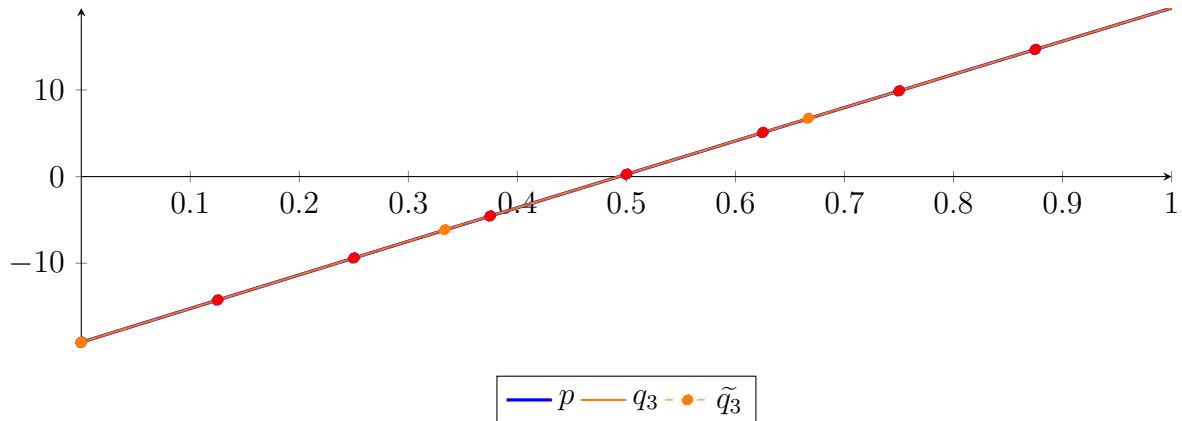
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = 2.82588 \cdot 10^{-12}X^8 - 1.08529 \cdot 10^{-11}X^7 + 1.68738 \cdot 10^{-11}X^6 - 1.34636 \cdot 10^{-11}X^5$$

$$+ 5.72618 \cdot 10^{-12}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m :

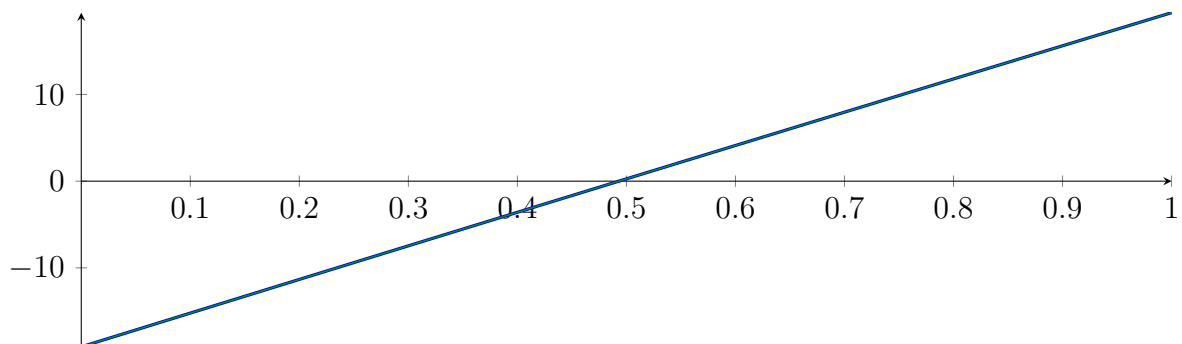
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \quad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.1252 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

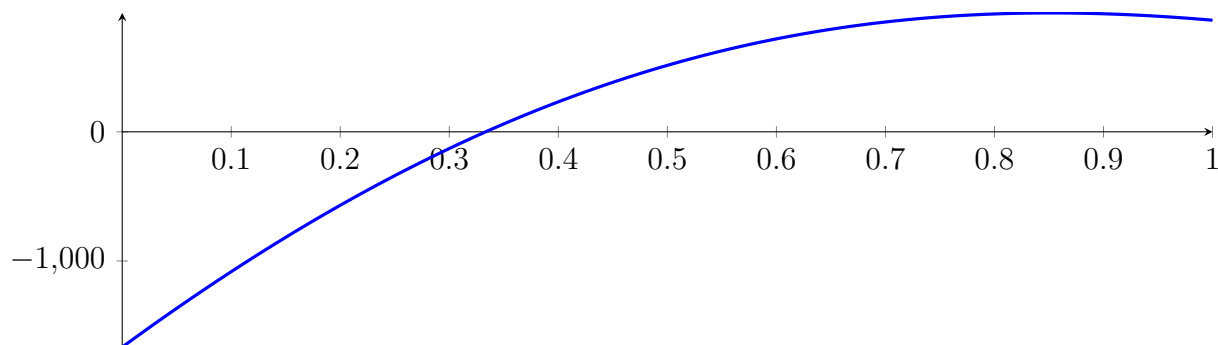
135.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

135.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

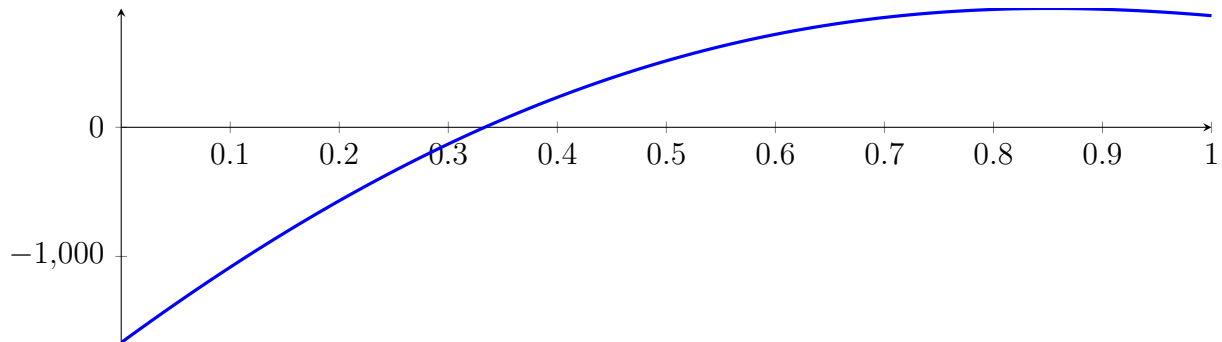
with precision $\varepsilon = 1 \cdot 10^{-08}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

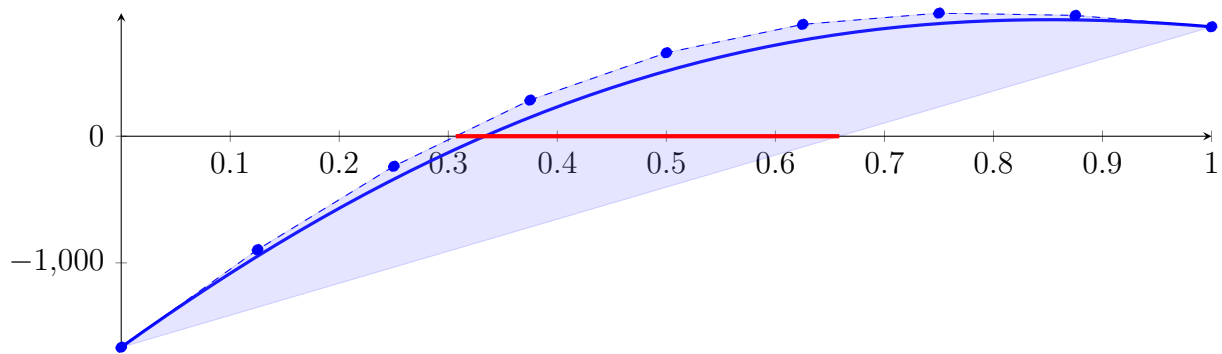
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



136.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

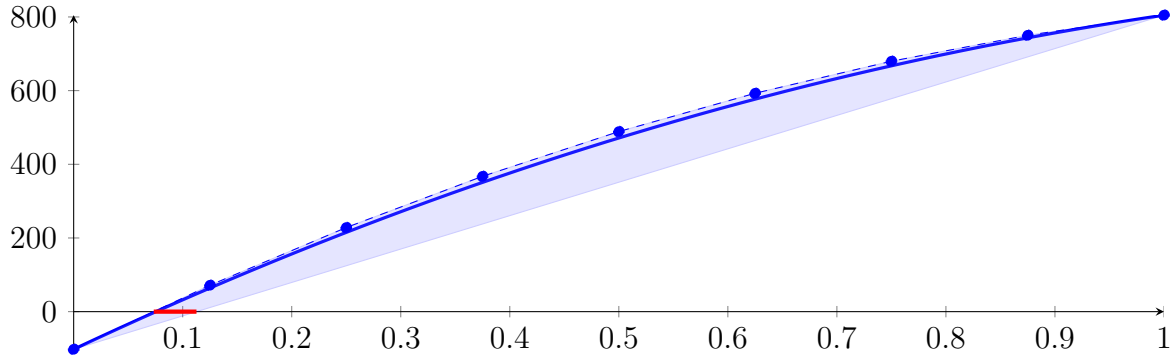
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

136.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

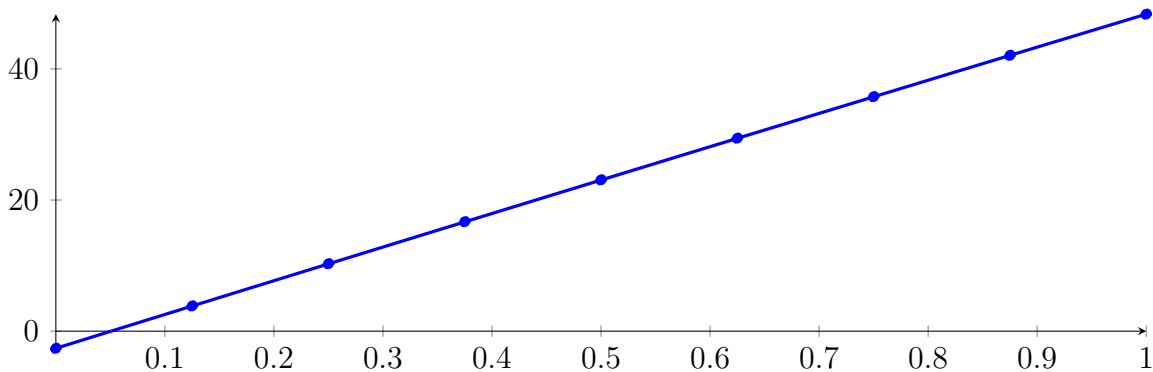
Longest intersection interval: 0.0391855

⇒ Selective recursion: interval 1: [0.332635, 0.34642],

136.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.03577 \cdot 10^{-15} X^8 - 1.54633 \cdot 10^{-12} X^7 - 4.95836 \cdot 10^{-10} X^6 + 3.66751 \cdot 10^{-08} X^5 \\
 &\quad + 3.25466 \cdot 10^{-05} X^4 - 0.000586142 X^3 - 0.747315 X^2 + 51.7118 X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

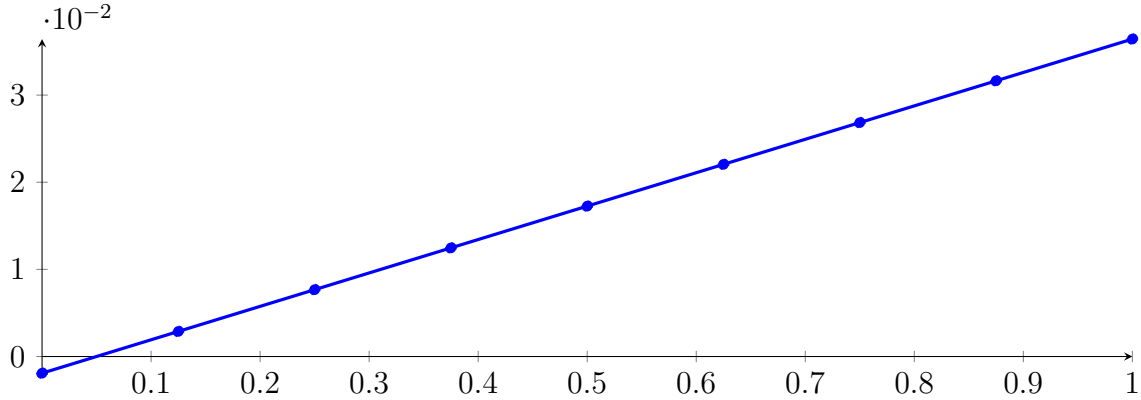
Longest intersection interval: 0.000742589

⇒ Selective recursion: interval 1: [0.333333, 0.333343],

136.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.28918 \cdot 10^{-18} X^8 - 1.32815 \cdot 10^{-18} X^7 - 4.50622 \cdot 10^{-18} X^6 + 2.22939 \cdot 10^{-18} X^5 \\
 &\quad + 9.48677 \cdot 10^{-18} X^4 - 2.37322 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\
 &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\
 &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\
 &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

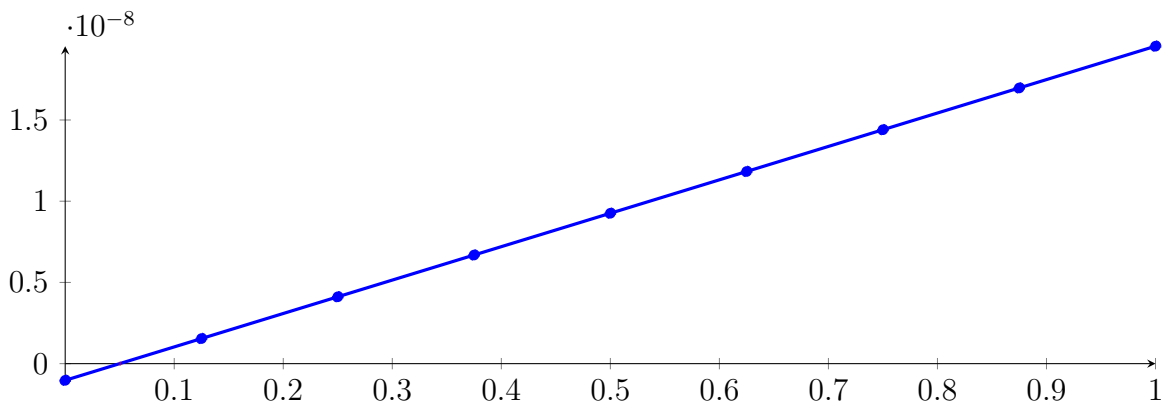
Longest intersection interval: $5.36469 \cdot 10^{-07}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

136.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -7.43978 \cdot 10^{-25} X^8 - 4.71751 \cdot 10^{-25} X^7 - 2.488 \cdot 10^{-24} X^6 + 1.04044 \cdot 10^{-24} X^5 \\
 &\quad - 2.26182 \cdot 10^{-25} X^4 - 1.18615 \cdot 10^{-19} X^2 + 2.05706 \cdot 10^{-08} X - 1.02667 \cdot 10^{-09} \\
 &= -1.02667 \cdot 10^{-09} B_{0,8}(X) + 1.54465 \cdot 10^{-09} B_{1,8}(X) + 4.11597 \cdot 10^{-09} B_{2,8}(X) \\
 &\quad + 6.68729 \cdot 10^{-09} B_{3,8}(X) + 9.25862 \cdot 10^{-09} B_{4,8}(X) + 1.18299 \cdot 10^{-08} B_{5,8}(X) \\
 &\quad + 1.44013 \cdot 10^{-08} B_{6,8}(X) + 1.69726 \cdot 10^{-08} B_{7,8}(X) + 1.95439 \cdot 10^{-08} B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87793 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

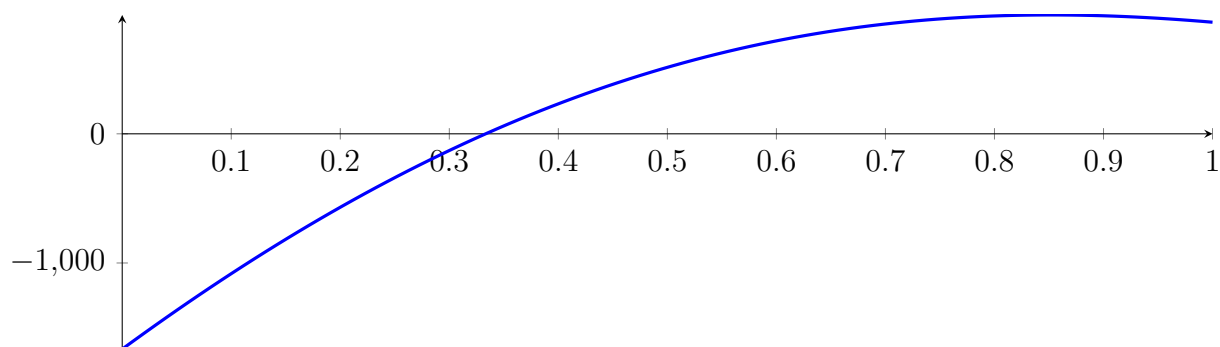
136.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

136.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

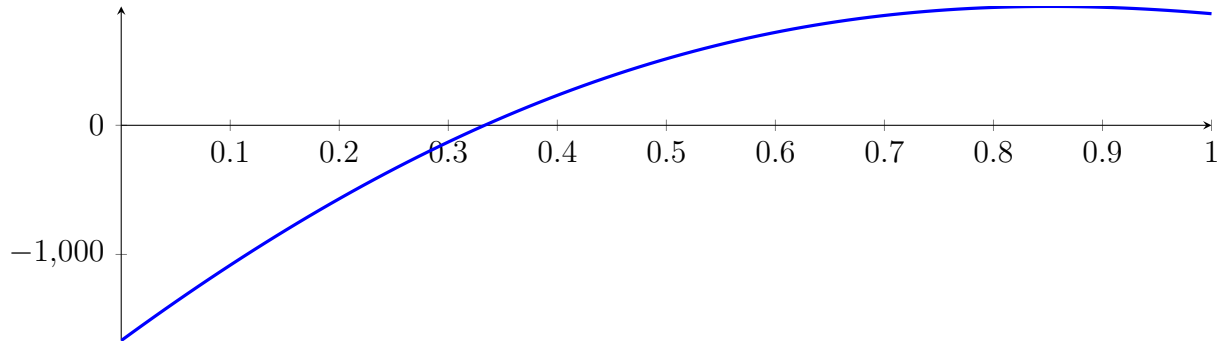
with precision $\varepsilon = 1 \cdot 10^{-16}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

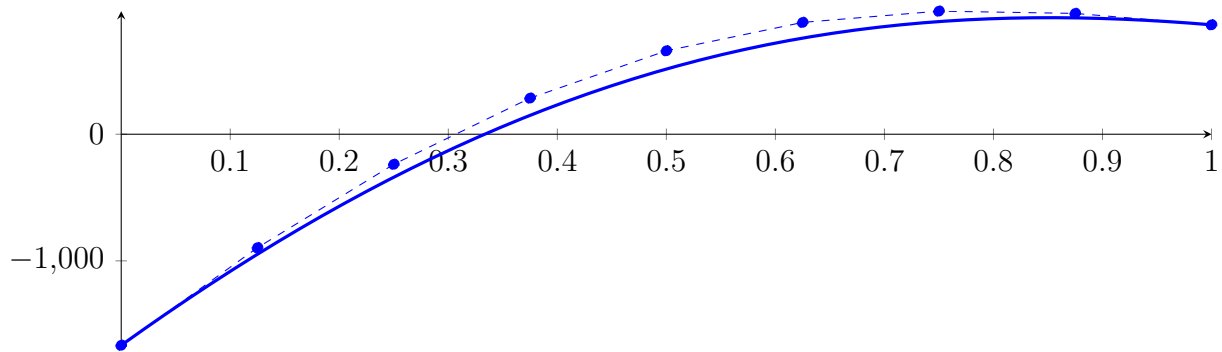
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



137.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

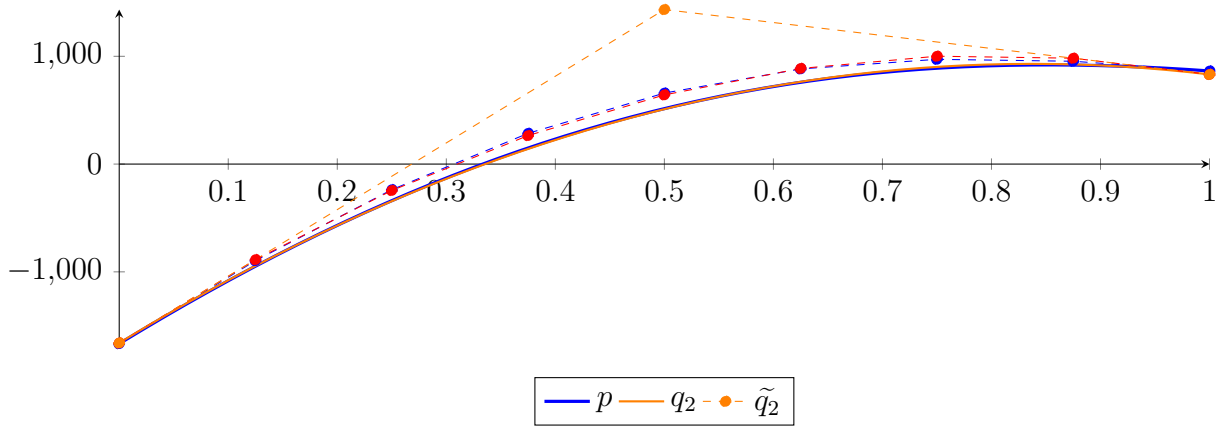
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -5.06304 \cdot 10^{-10}X^8 + 1.93305 \cdot 10^{-09}X^7 - 2.97598 \cdot 10^{-09}X^6 + 2.35116 \cdot 10^{-09}X^5 \\ &\quad - 9.91232 \cdot 10^{-10}X^4 + 2.00705 \cdot 10^{-10}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

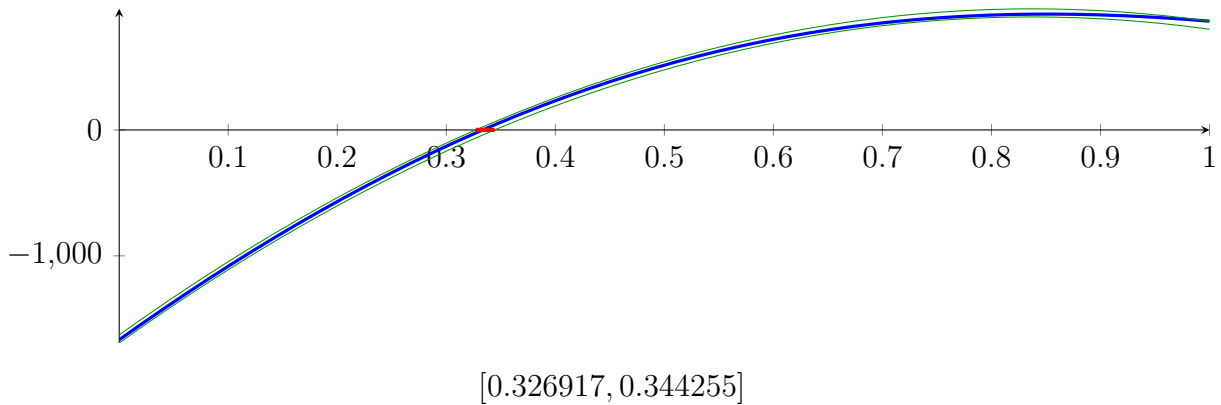
$$M = -3695.78X^2 + 6187.64X - 1627.86$$

$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\} \qquad N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



Longest intersection interval: 0.0173372

\implies Selective recursion: interval 1: [0.326917, 0.344255],

137.2 Recursion Branch 1 1 in Interval 1: [0.326917, 0.344255]

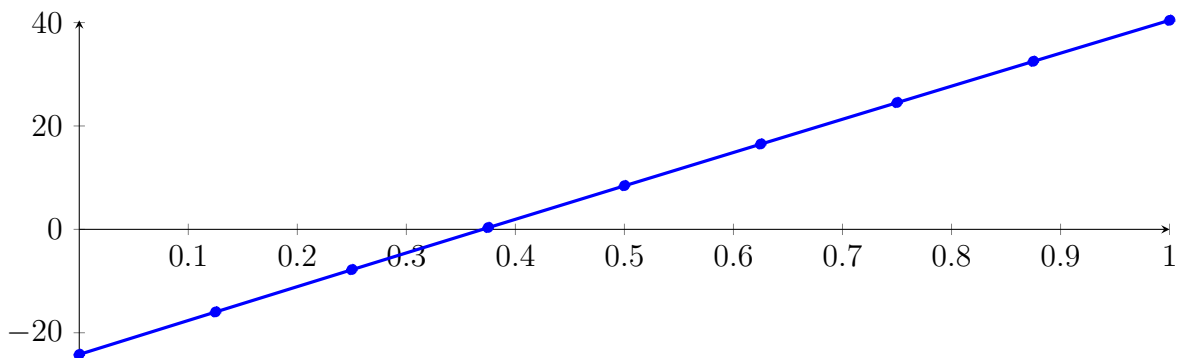
Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -8.80546 \cdot 10^{-15} X^8 - 7.66587 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5$$

$$+ 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945$$

$$= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X)$$

$$+ 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X)$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

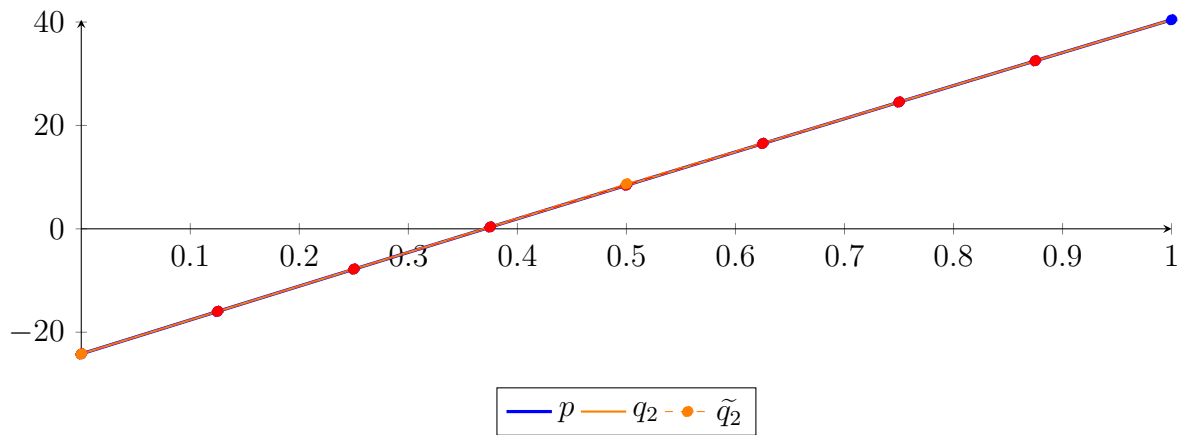
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = -2.59056 \cdot 10^{-11}X^8 + 1.00262 \cdot 10^{-10}X^7 - 1.57692 \cdot 10^{-10}X^6 + 1.29283 \cdot 10^{-10}X^5$$

$$- 5.86775 \cdot 10^{-11}X^4 + 1.42642 \cdot 10^{-11}X^3 - 1.18261X^2 + 65.8162X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

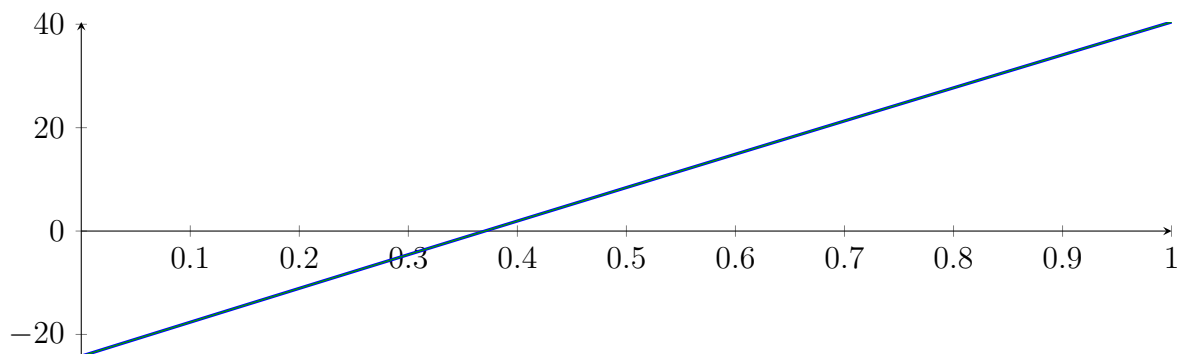
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

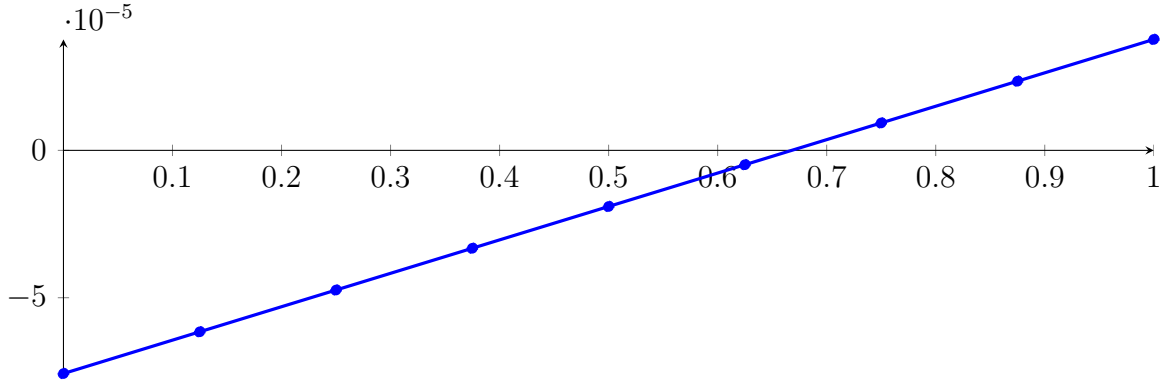
Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

137.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

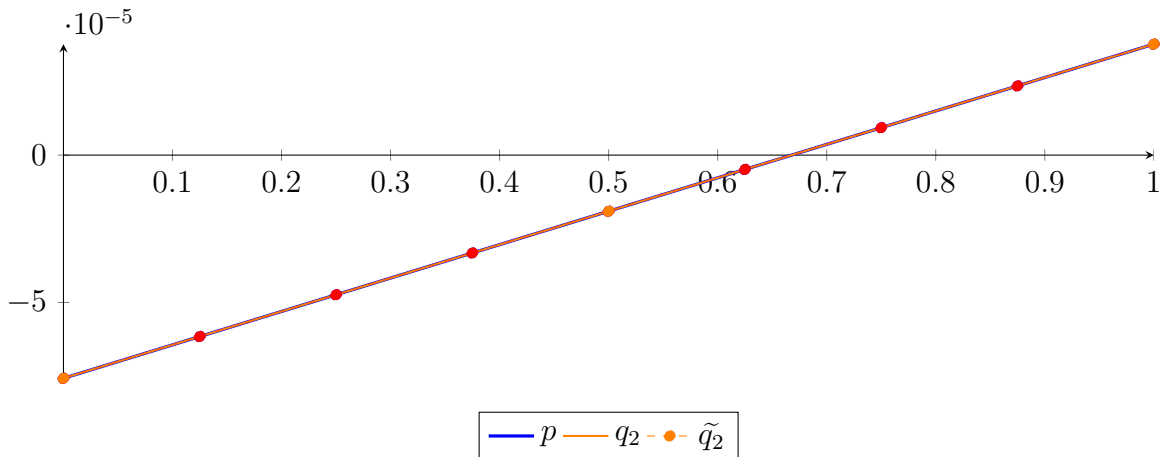
$$\begin{aligned}
 p &= 1.16467 \cdot 10^{-21} X^8 + 3.17637 \cdot 10^{-21} X^7 + 1.18585 \cdot 10^{-20} X^6 - 1.48231 \cdot 10^{-21} X^5 + 9.26442 \\
 &\quad \cdot 10^{-22} X^4 - 5.92923 \cdot 10^{-21} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) \\
 &\quad - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) \\
 &\quad + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 3.26671 \cdot 10^{-17} X^8 - 1.38104 \cdot 10^{-16} X^7 + 2.39221 \cdot 10^{-16} X^6 - 2.17429 \cdot 10^{-16} X^5 + 1.10046 \\
 &\quad \cdot 10^{-16} X^4 - 3.0162 \cdot 10^{-17} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \\
 &\quad \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.84643 \cdot 10^{-19}$.

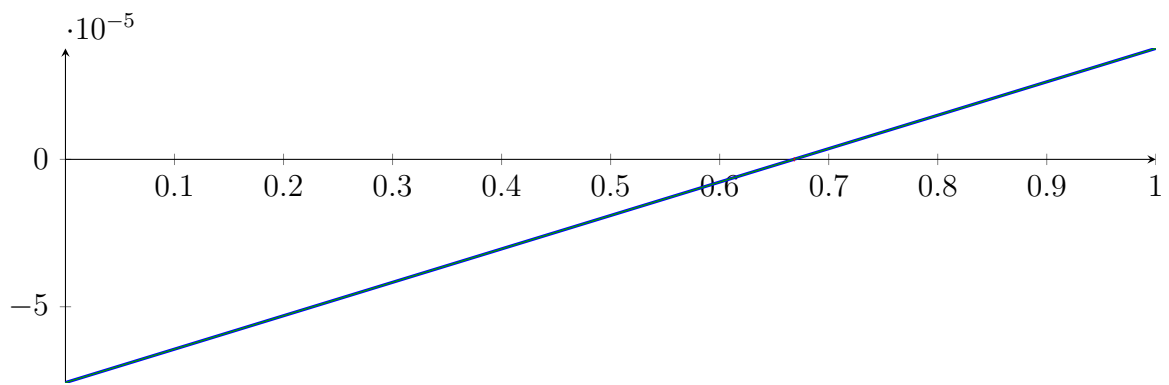
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 m &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.667955, 3.14644 \cdot 10^7\} \qquad N(m) = \{0.667955, 3.14644 \cdot 10^7\}$$

Intersection intervals:



[0.667955, 0.667955]

Longest intersection interval: $3.08439 \cdot 10^{-13}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

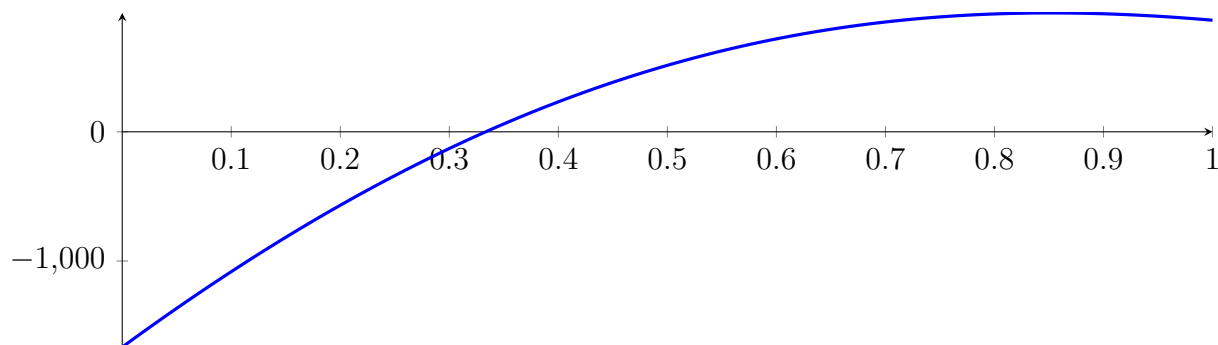
137.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

137.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

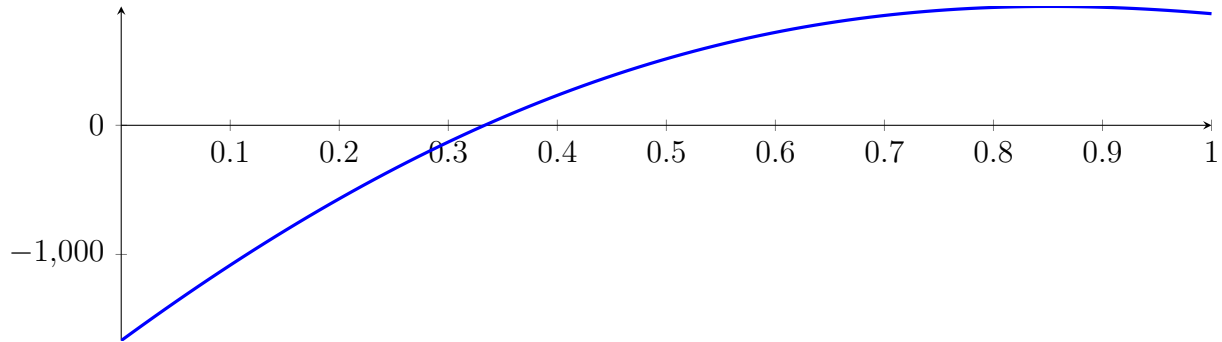
with precision $\varepsilon = 1 \cdot 10^{-16}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

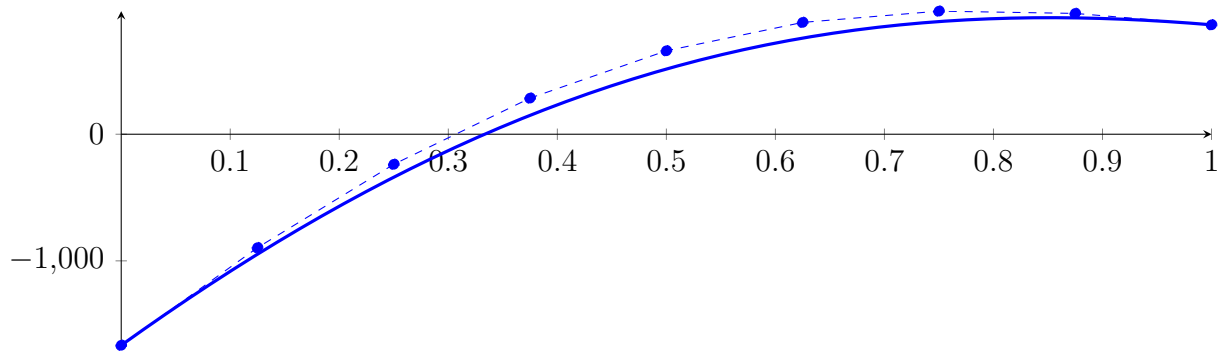
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



138.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

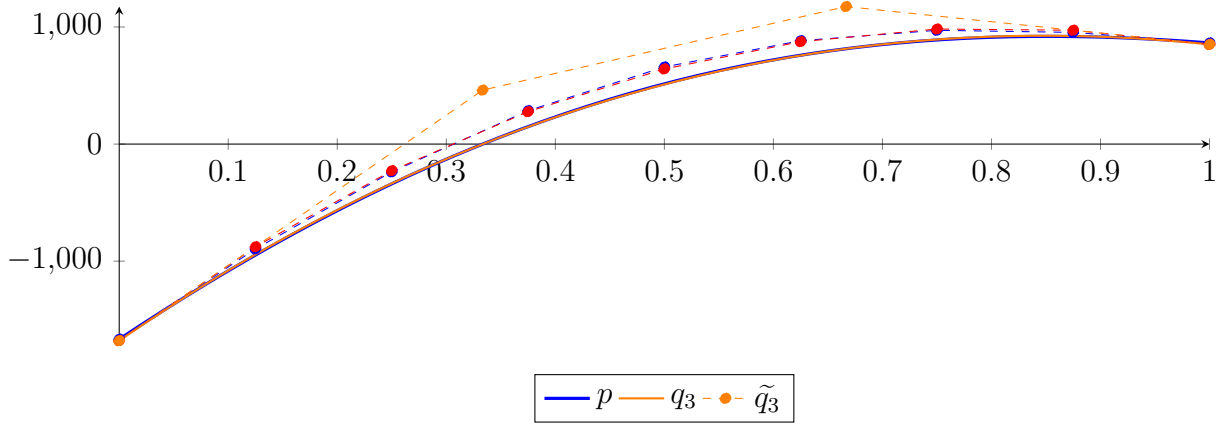
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.39389 \cdot 10^{-09}X^8 + 9.60737 \cdot 10^{-09}X^7 - 1.57477 \cdot 10^{-08}X^6 + 1.35479 \cdot 10^{-08}X^5 \\ &\quad - 6.53397 \cdot 10^{-09}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

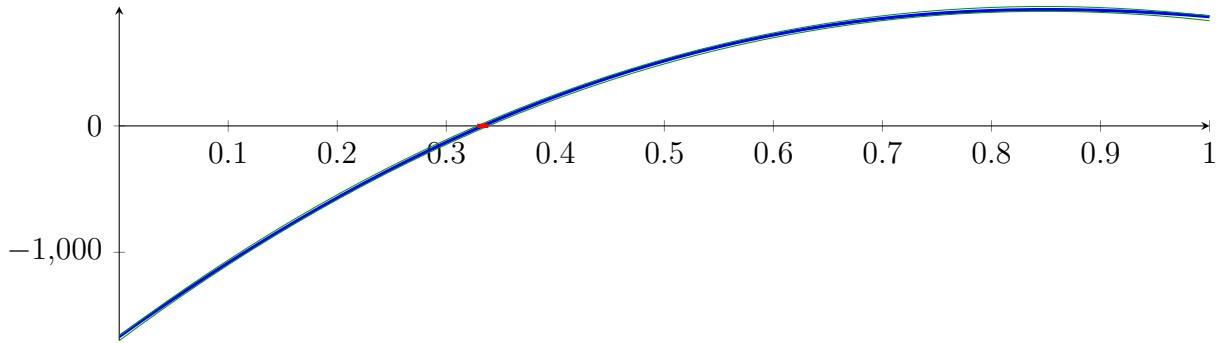
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

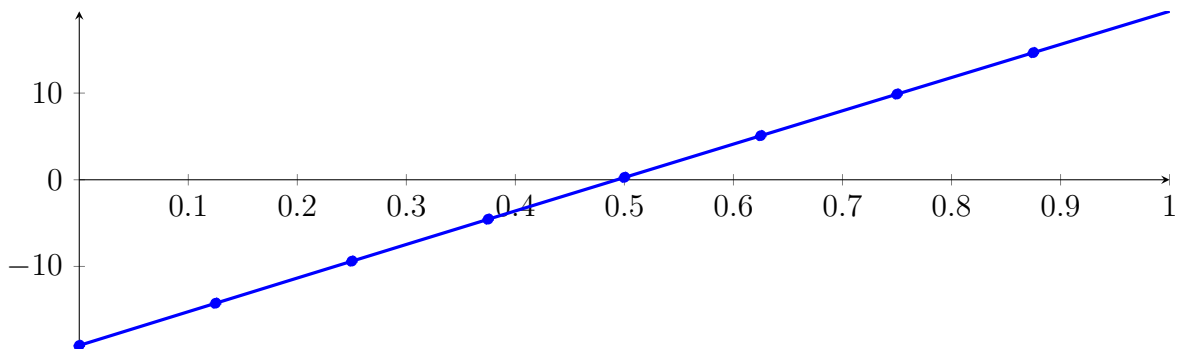
Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

138.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.66533 \cdot 10^{-16} X^8 - 1.99007 \cdot 10^{-13} X^7 - 8.53059 \cdot 10^{-11} X^6 + 8.7284 \cdot 10^{-09} X^5 \\
 &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\
 &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\
 &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

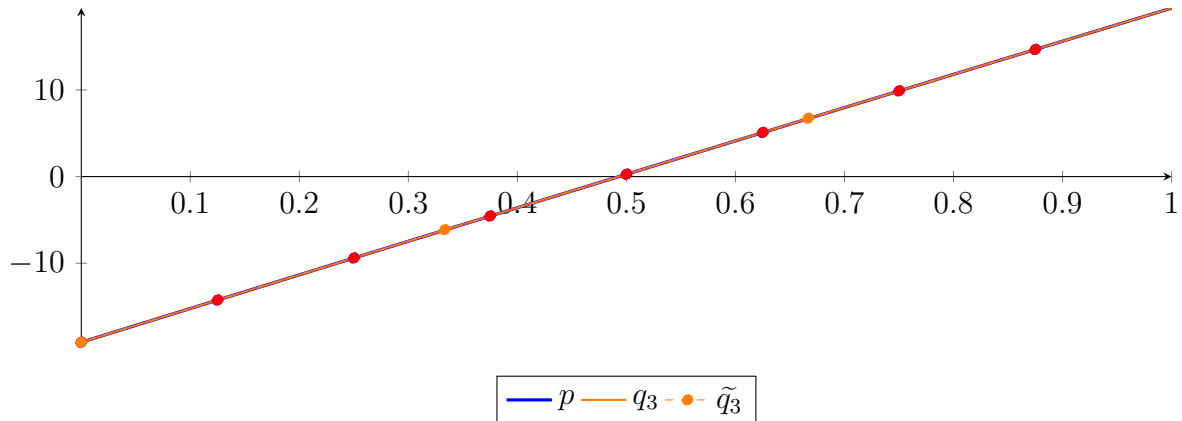
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = 2.82588 \cdot 10^{-12}X^8 - 1.08529 \cdot 10^{-11}X^7 + 1.68738 \cdot 10^{-11}X^6 - 1.34636 \cdot 10^{-11}X^5$$

$$+ 5.72618 \cdot 10^{-12}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m :

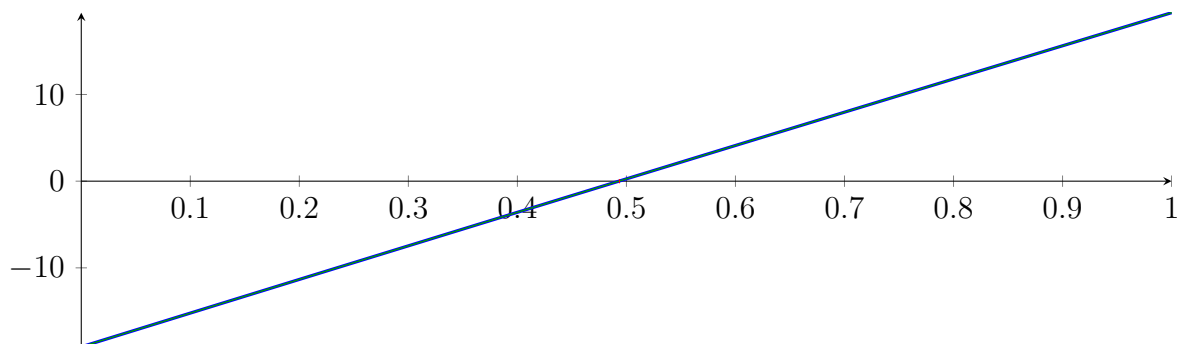
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \quad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

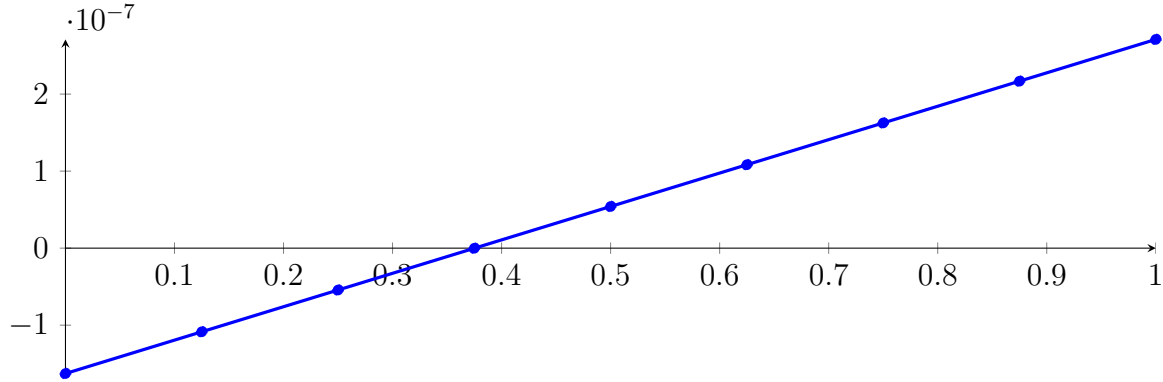
Longest intersection interval: $1.1252 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

138.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

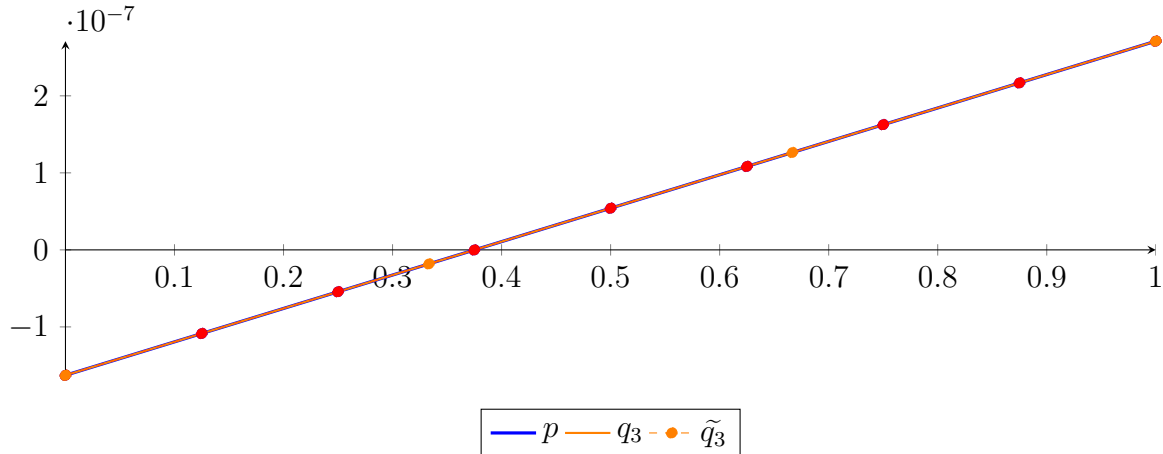
$$\begin{aligned}
 p &= -4.65289 \cdot 10^{-24} X^8 - 8.27181 \cdot 10^{-25} X^7 - 4.3427 \cdot 10^{-24} X^6 + 8.6854 \cdot 10^{-24} X^5 + 1.80946 \\
 &\quad \cdot 10^{-24} X^4 - 7.23783 \cdot 10^{-25} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,8}(X) - 1.08585 \cdot 10^{-07} B_{1,8}(X) - 5.43593 \cdot 10^{-08} B_{2,8}(X) \\
 &\quad - 1.33715 \cdot 10^{-10} B_{3,8}(X) + 5.40918 \cdot 10^{-08} B_{4,8}(X) + 1.08317 \cdot 10^{-07} B_{5,8}(X) \\
 &\quad + 1.62543 \cdot 10^{-07} B_{6,8}(X) + 2.16768 \cdot 10^{-07} B_{7,8}(X) + 2.70994 \cdot 10^{-07} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= 1.42947 \cdot 10^{-23} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,3} - 1.82089 \cdot 10^{-08} B_{1,3} + 1.26393 \cdot 10^{-07} B_{2,3} + 2.70994 \cdot 10^{-07} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= -2.97639 \cdot 10^{-20} X^8 + 1.26259 \cdot 10^{-19} X^7 - 2.19172 \cdot 10^{-19} X^6 + 2.00419 \cdot 10^{-19} X^5 - 1.02758 \\
 &\quad \cdot 10^{-19} X^4 + 2.83318 \cdot 10^{-20} X^3 - 5.27552 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,8} - 1.08585 \cdot 10^{-07} B_{1,8} - 5.43593 \cdot 10^{-08} B_{2,8} - 1.33715 \cdot 10^{-10} B_{3,8} + 5.40918 \\
 &\quad \cdot 10^{-08} B_{4,8} + 1.08317 \cdot 10^{-07} B_{5,8} + 1.62543 \cdot 10^{-07} B_{6,8} + 2.16768 \cdot 10^{-07} B_{7,8} + 2.70994 \cdot 10^{-07} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.97535 \cdot 10^{-22}$.

Bounding polynomials M and m :

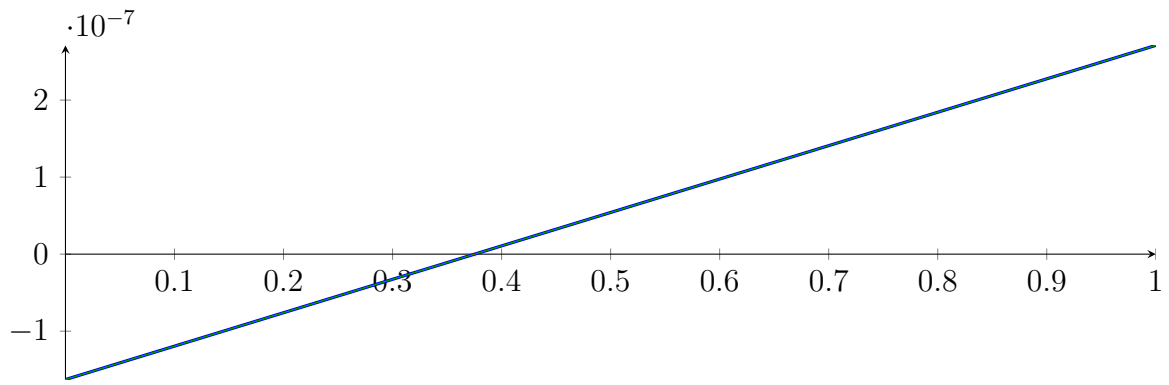
$$\begin{aligned}
 M &= 1.42689 \cdot 10^{-23} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 m &= 1.43206 \cdot 10^{-23} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.375308\}$$

$$N(m) = \{0.375308\}$$

Intersection intervals:



[0.375308, 0.375308]

Longest intersection interval: $1.36424 \cdot 10^{-12}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

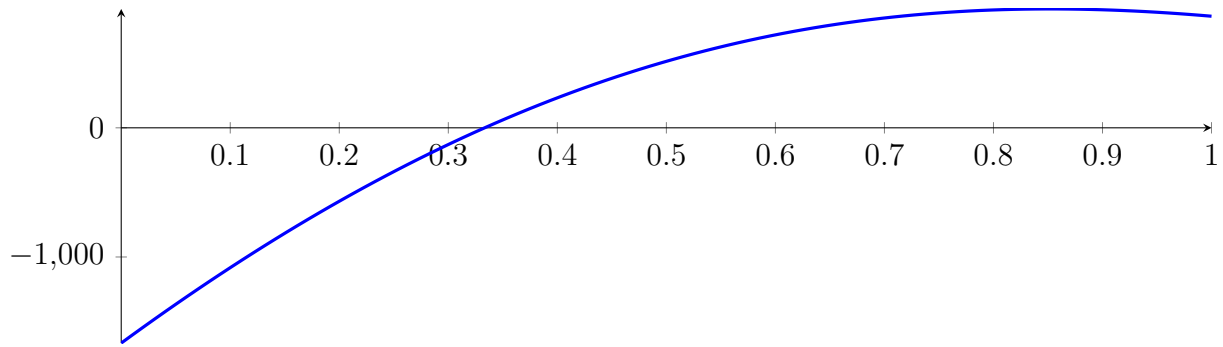
138.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

138.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

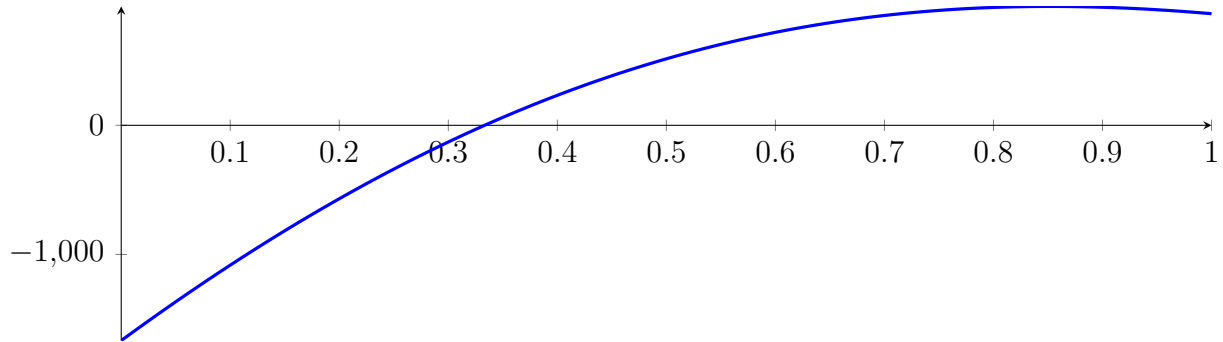
with precision $\varepsilon = 1 \cdot 10^{-16}$.

139 Running BezClip on f_8 with epsilon 32

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

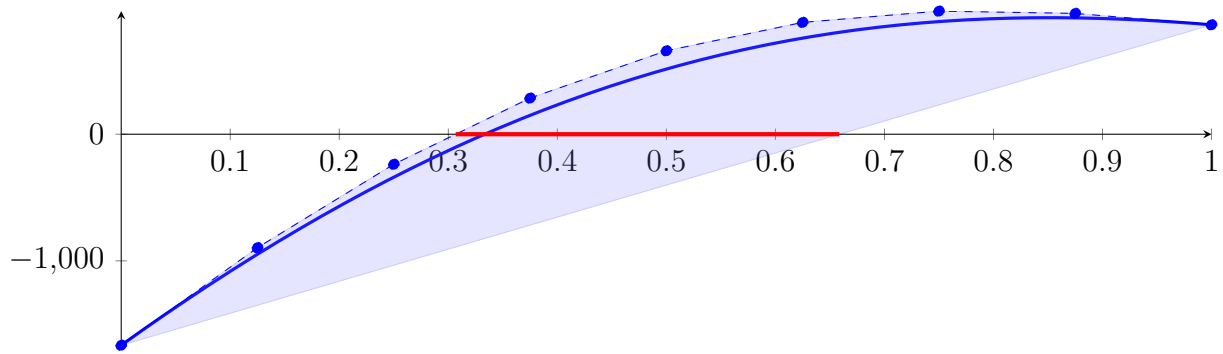
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



139.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

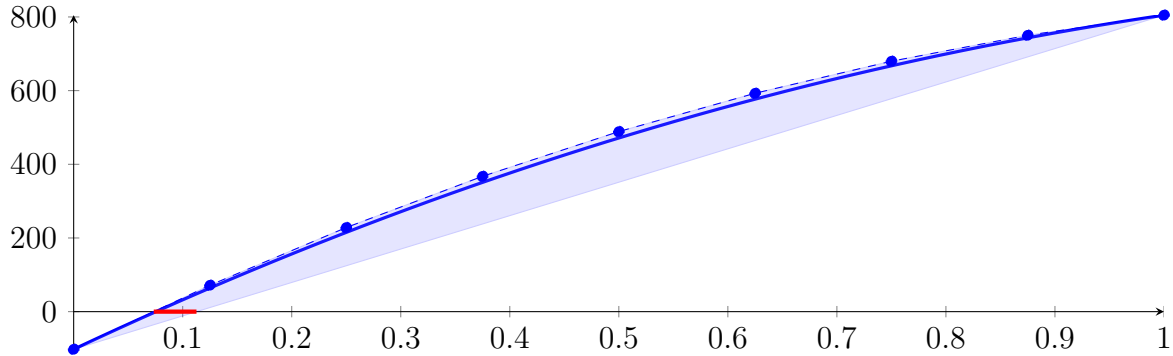
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

139.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

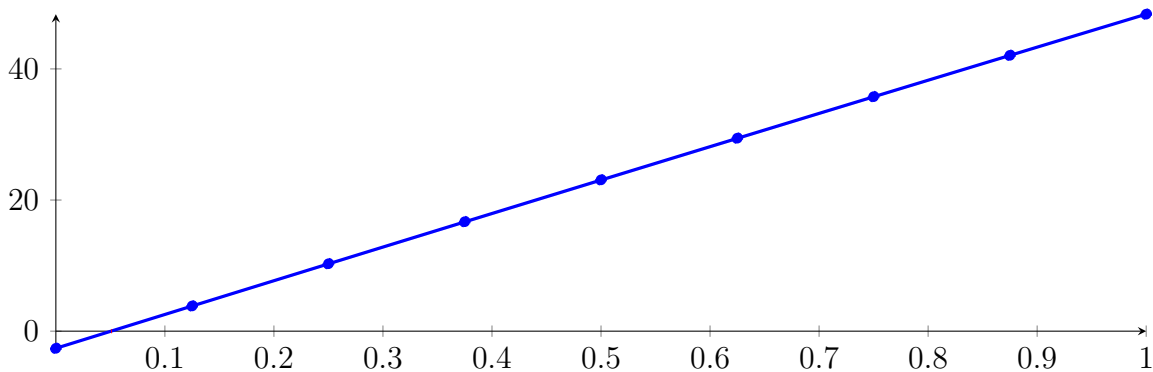
Longest intersection interval: 0.0391855

⇒ Selective recursion: interval 1: [0.332635, 0.34642],

139.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.03577 \cdot 10^{-15} X^8 - 1.54633 \cdot 10^{-12} X^7 - 4.95836 \cdot 10^{-10} X^6 + 3.66751 \cdot 10^{-08} X^5 \\
 &\quad + 3.25466 \cdot 10^{-05} X^4 - 0.000586142 X^3 - 0.747315 X^2 + 51.7118 X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

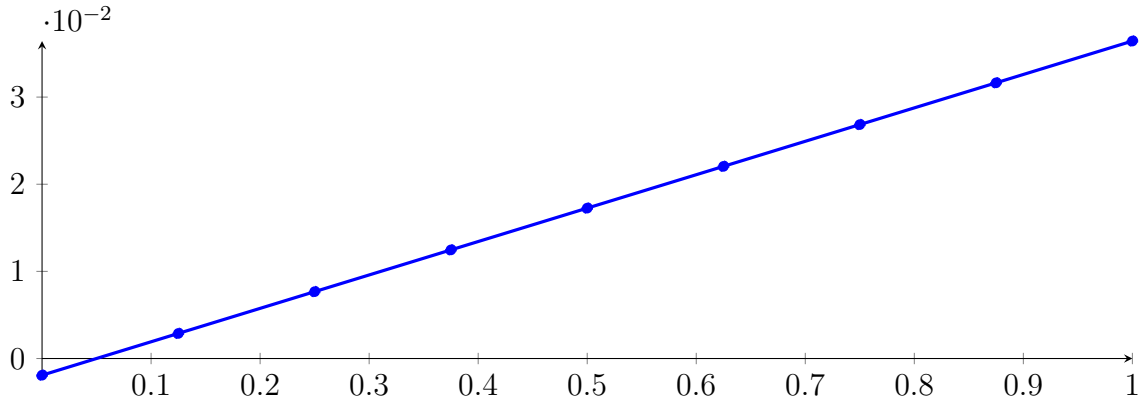
Longest intersection interval: 0.000742589

⇒ Selective recursion: interval 1: [0.333333, 0.333343],

139.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.28918 \cdot 10^{-18} X^8 - 1.32815 \cdot 10^{-18} X^7 - 4.50622 \cdot 10^{-18} X^6 + 2.22939 \cdot 10^{-18} X^5 \\
 &\quad + 9.48677 \cdot 10^{-18} X^4 - 2.37322 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\
 &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\
 &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\
 &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

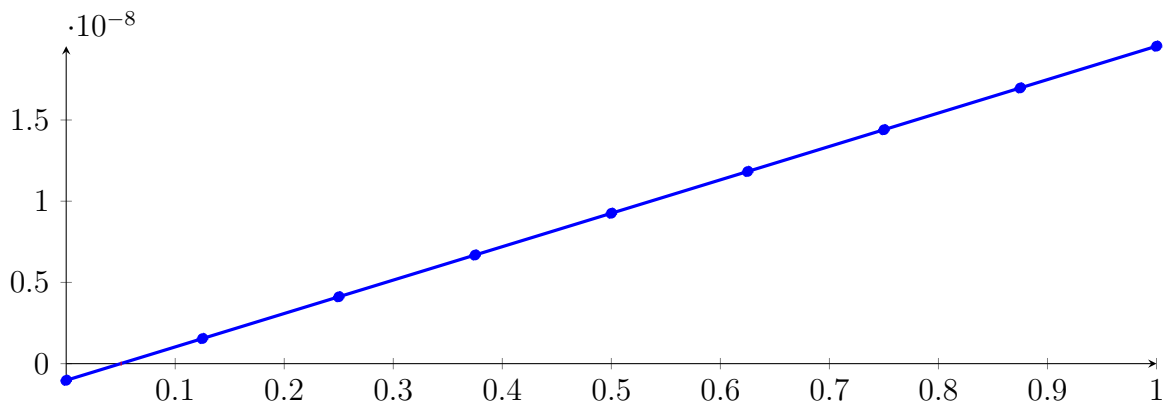
Longest intersection interval: $5.36469 \cdot 10^{-07}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

139.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -7.43978 \cdot 10^{-25} X^8 - 4.71751 \cdot 10^{-25} X^7 - 2.488 \cdot 10^{-24} X^6 + 1.04044 \cdot 10^{-24} X^5 \\
 &\quad - 2.26182 \cdot 10^{-25} X^4 - 1.18615 \cdot 10^{-19} X^2 + 2.05706 \cdot 10^{-08} X - 1.02667 \cdot 10^{-09} \\
 &= -1.02667 \cdot 10^{-09} B_{0,8}(X) + 1.54465 \cdot 10^{-09} B_{1,8}(X) + 4.11597 \cdot 10^{-09} B_{2,8}(X) \\
 &\quad + 6.68729 \cdot 10^{-09} B_{3,8}(X) + 9.25862 \cdot 10^{-09} B_{4,8}(X) + 1.18299 \cdot 10^{-08} B_{5,8}(X) \\
 &\quad + 1.44013 \cdot 10^{-08} B_{6,8}(X) + 1.69726 \cdot 10^{-08} B_{7,8}(X) + 1.95439 \cdot 10^{-08} B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87793 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

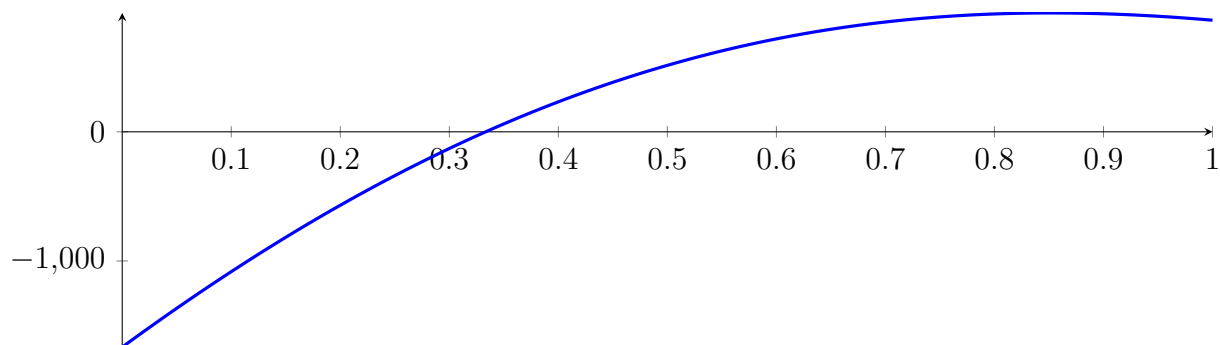
139.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

139.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

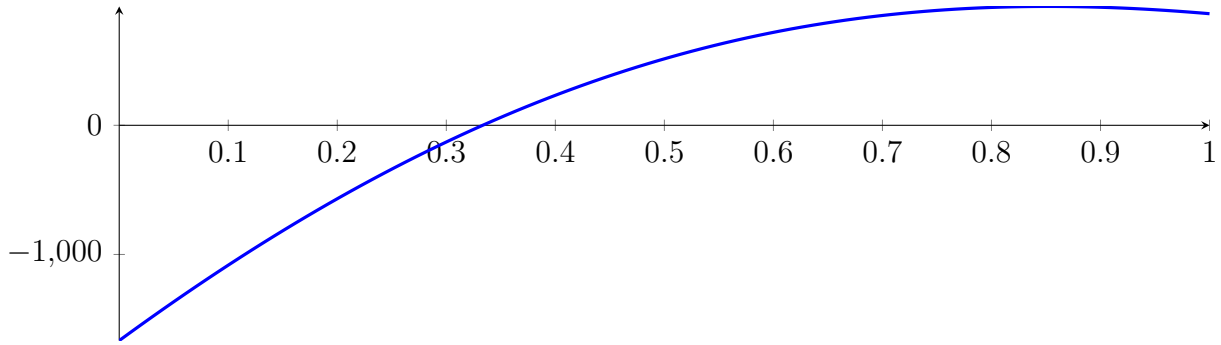
with precision $\varepsilon = 1 \cdot 10^{-32}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

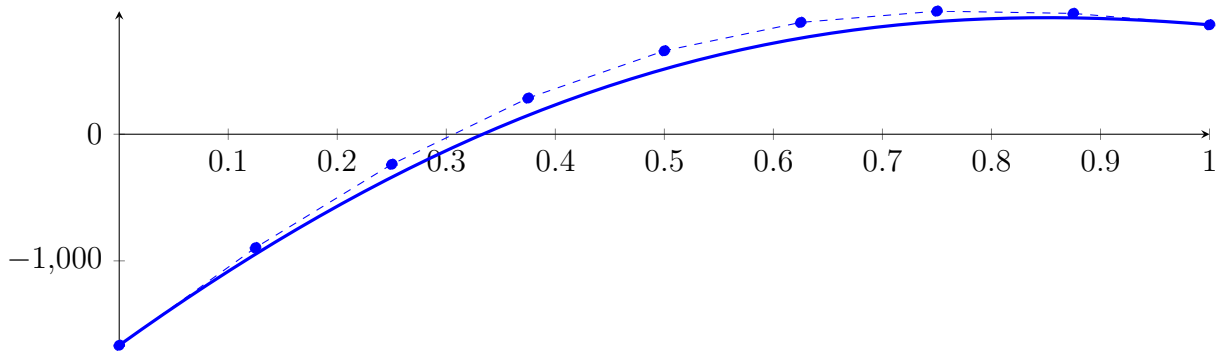
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



140.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

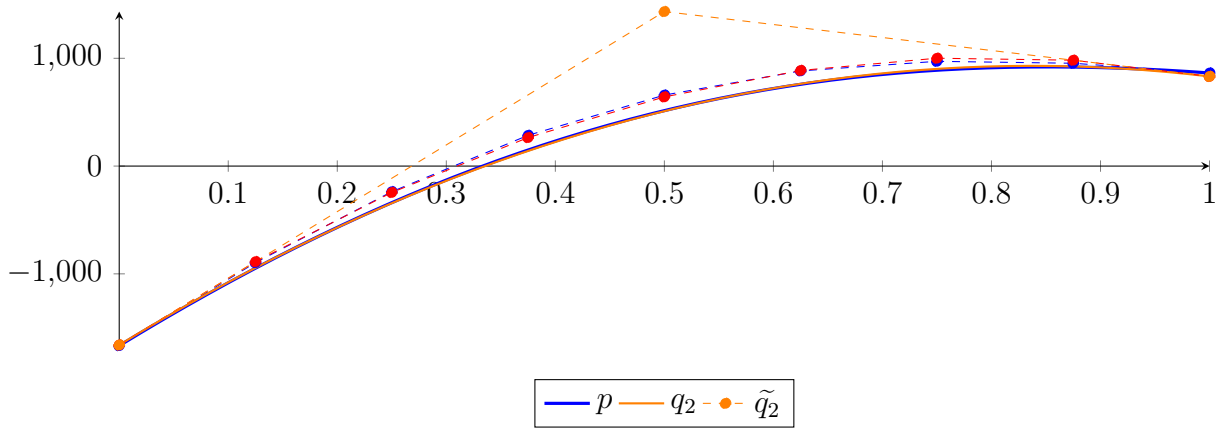
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -5.06304 \cdot 10^{-10}X^8 + 1.93305 \cdot 10^{-09}X^7 - 2.97598 \cdot 10^{-09}X^6 + 2.35116 \cdot 10^{-09}X^5 \\ &\quad - 9.91232 \cdot 10^{-10}X^4 + 2.00705 \cdot 10^{-10}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

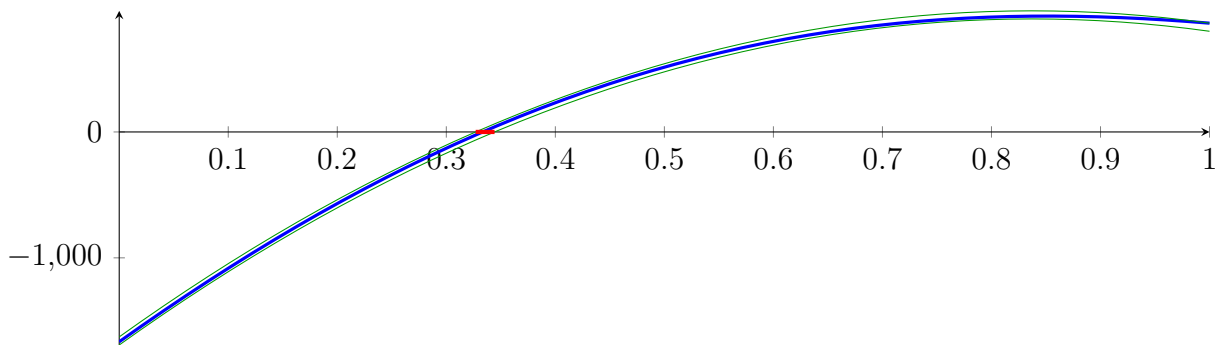
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

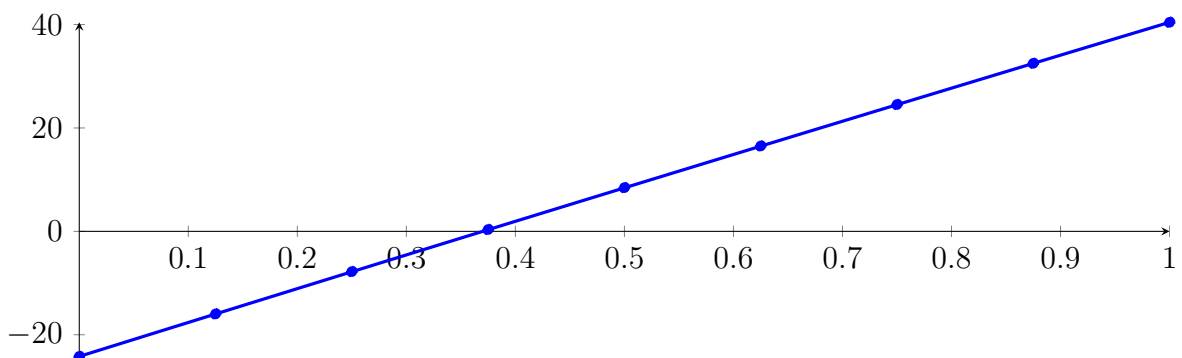
Longest intersection interval: 0.0173372

\implies Selective recursion: interval 1: $[0.326917, 0.344255]$,

140.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -8.80546 \cdot 10^{-15} X^8 - 7.66587 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

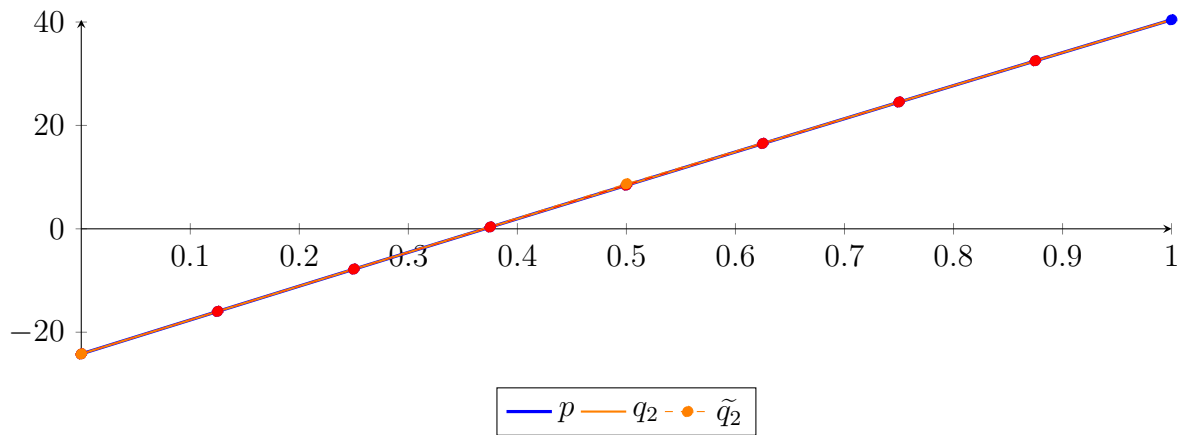
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = -2.59056 \cdot 10^{-11}X^8 + 1.00262 \cdot 10^{-10}X^7 - 1.57692 \cdot 10^{-10}X^6 + 1.29283 \cdot 10^{-10}X^5$$

$$- 5.86775 \cdot 10^{-11}X^4 + 1.42642 \cdot 10^{-11}X^3 - 1.18261X^2 + 65.8162X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

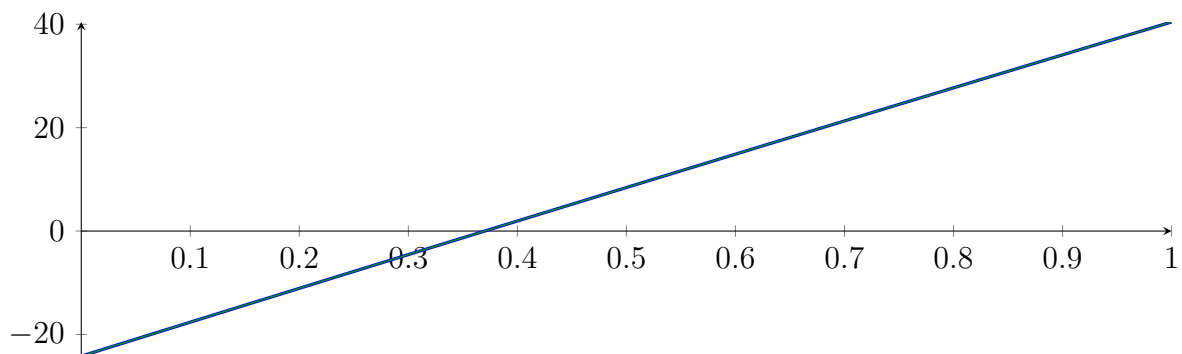
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

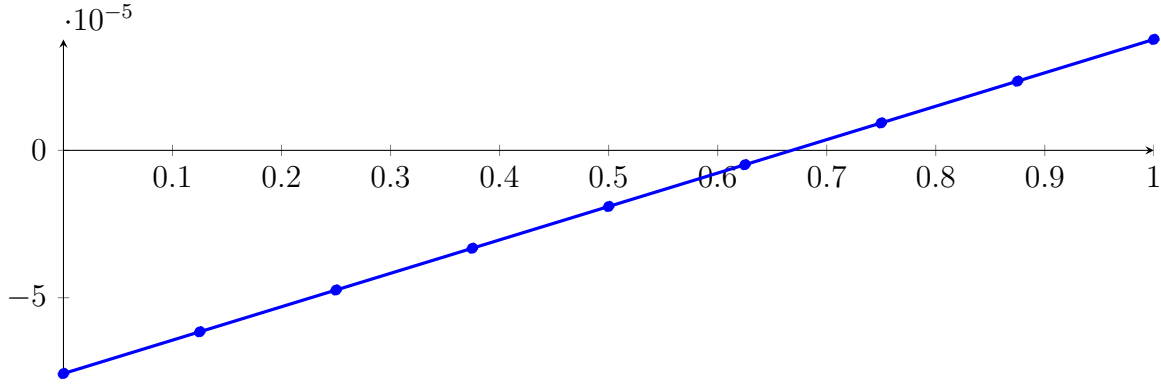
Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

140.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

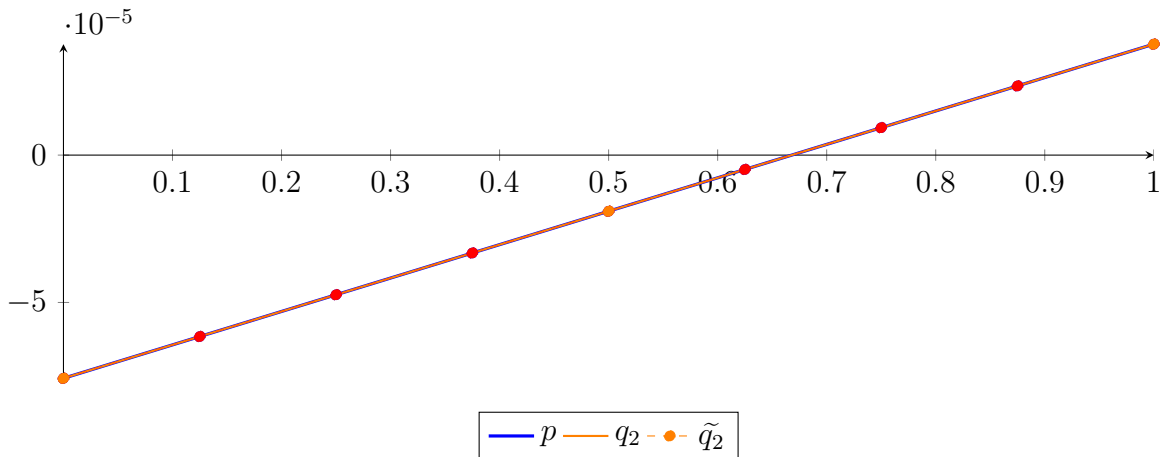
$$\begin{aligned}
 p &= 1.16467 \cdot 10^{-21} X^8 + 3.17637 \cdot 10^{-21} X^7 + 1.18585 \cdot 10^{-20} X^6 - 1.48231 \cdot 10^{-21} X^5 + 9.26442 \\
 &\quad \cdot 10^{-22} X^4 - 5.92923 \cdot 10^{-21} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) \\
 &\quad - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) \\
 &\quad + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 3.26671 \cdot 10^{-17} X^8 - 1.38104 \cdot 10^{-16} X^7 + 2.39221 \cdot 10^{-16} X^6 - 2.17429 \cdot 10^{-16} X^5 + 1.10046 \\
 &\quad \cdot 10^{-16} X^4 - 3.0162 \cdot 10^{-17} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \\
 &\quad \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.84643 \cdot 10^{-19}$.

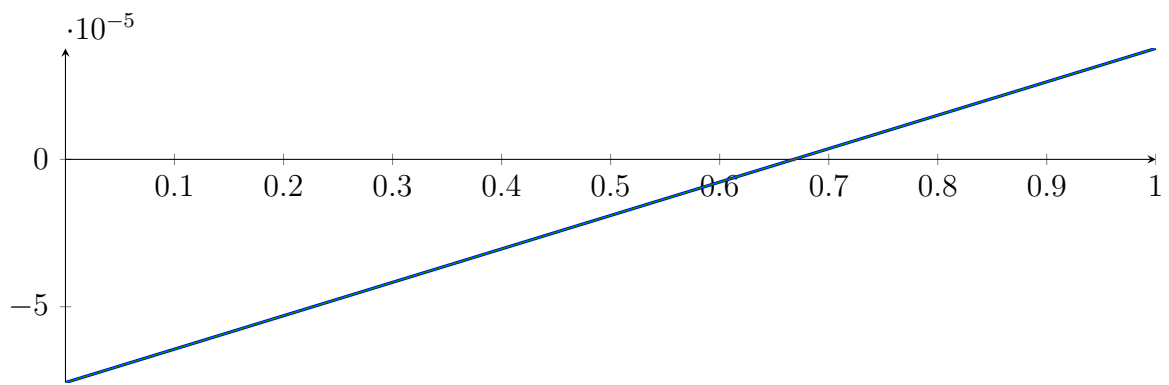
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 m &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.667955, 3.14644 \cdot 10^7\} \qquad N(m) = \{0.667955, 3.14644 \cdot 10^7\}$$

Intersection intervals:



[0.667955, 0.667955]

Longest intersection interval: $3.08439 \cdot 10^{-13}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

140.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

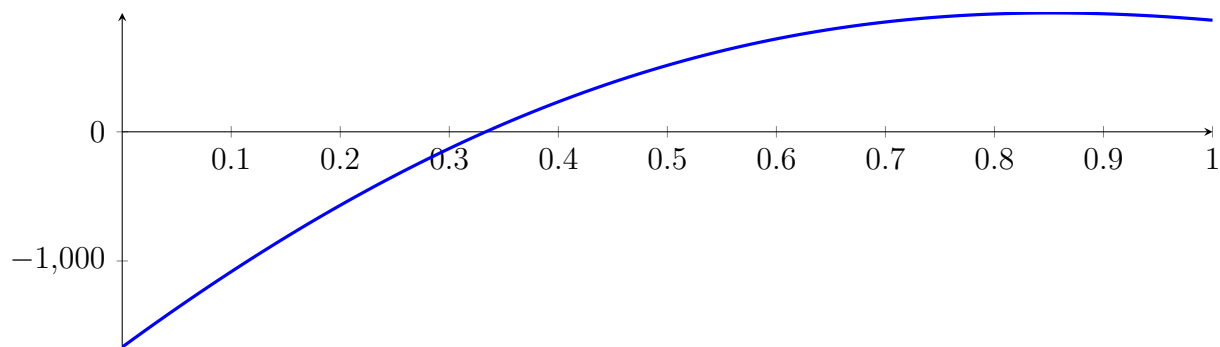
Reached interval [0.333333, 0.333333] **without sign change** at depth 4!

$p(0) = 2.85706e-18$ - $p(1) 3.78276e-17$

140.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

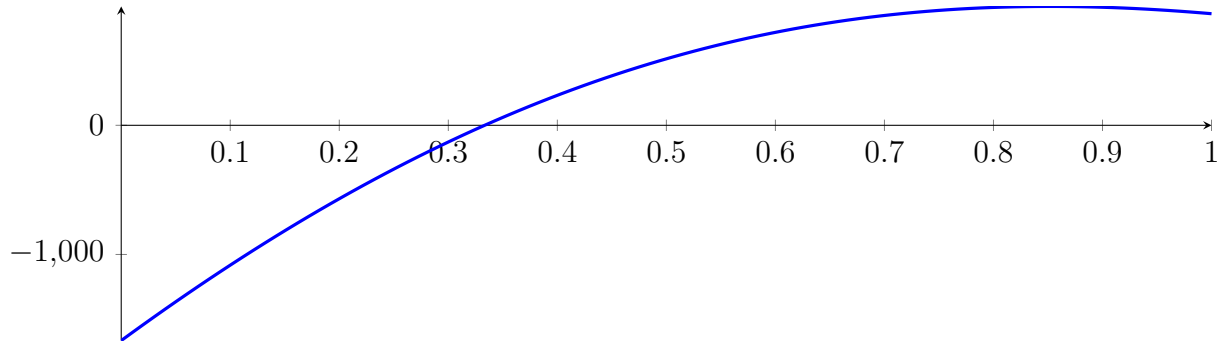
with precision $\varepsilon = 1 \cdot 10^{-32}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

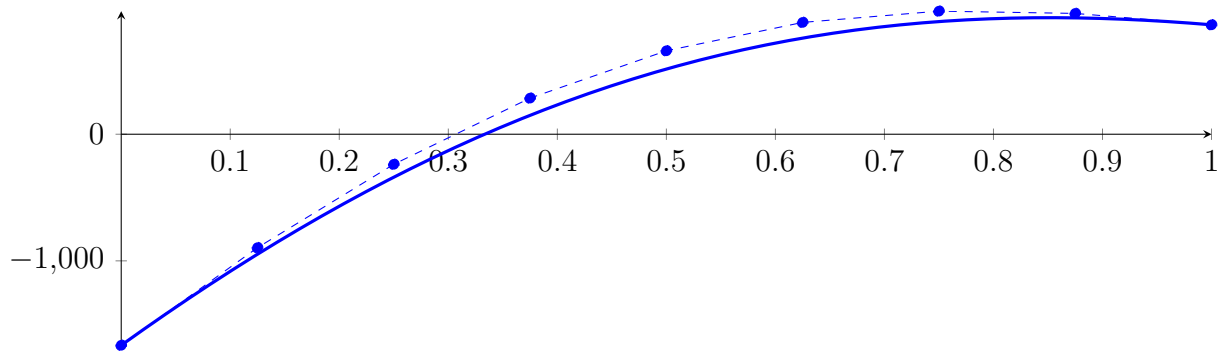
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



141.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

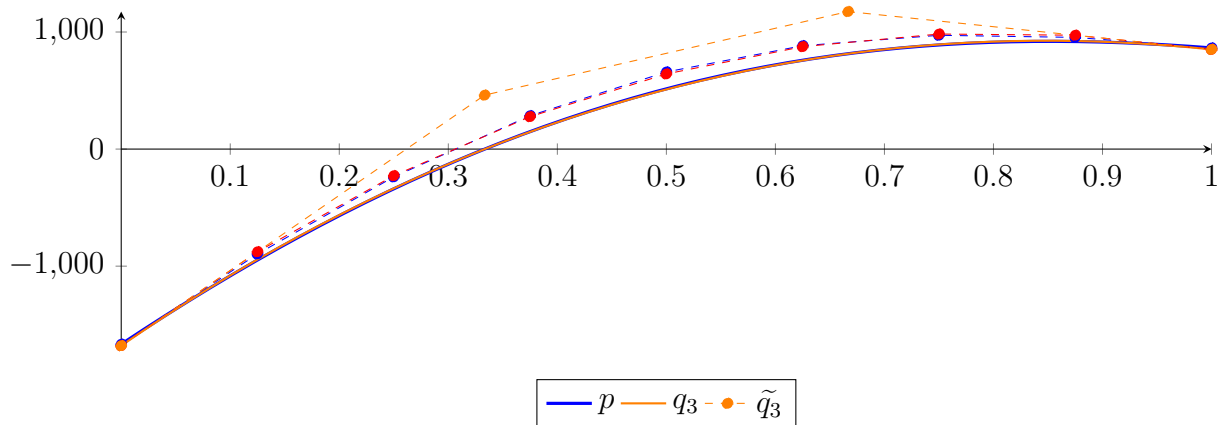
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.39389 \cdot 10^{-09}X^8 + 9.60737 \cdot 10^{-09}X^7 - 1.57477 \cdot 10^{-08}X^6 + 1.35479 \cdot 10^{-08}X^5 \\ &\quad - 6.53397 \cdot 10^{-09}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

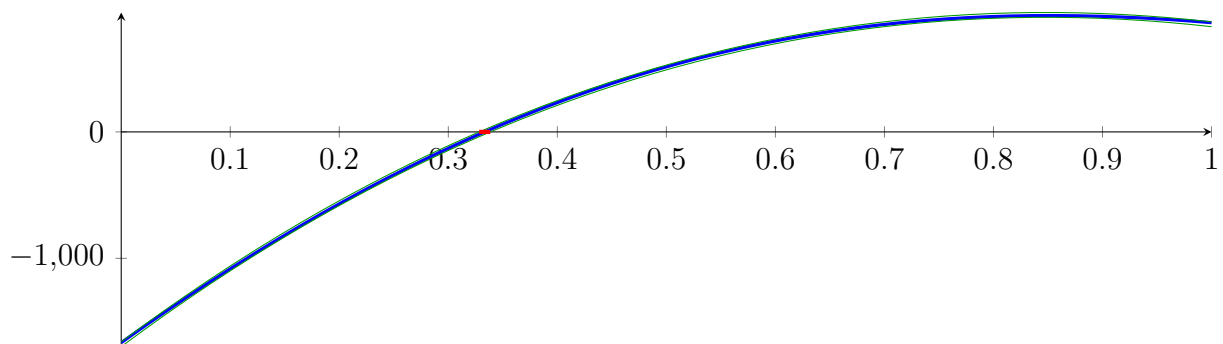
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

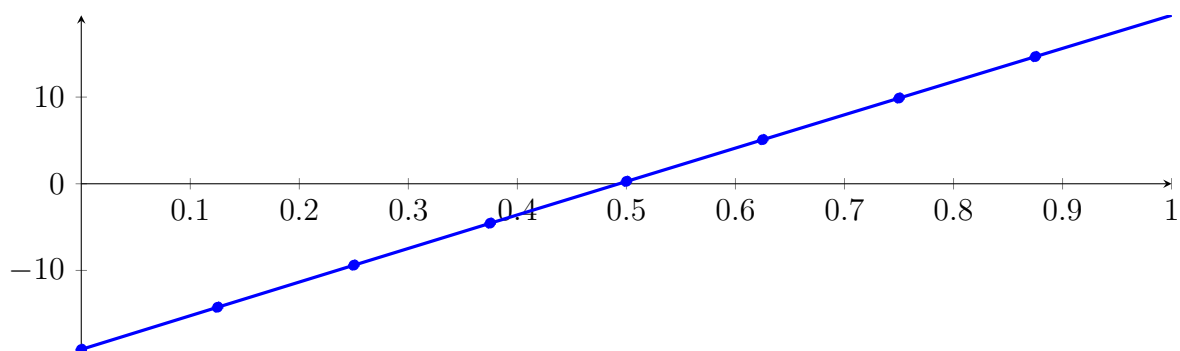
Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

141.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.66533 \cdot 10^{-16} X^8 - 1.99007 \cdot 10^{-13} X^7 - 8.53059 \cdot 10^{-11} X^6 + 8.7284 \cdot 10^{-09} X^5 \\ &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

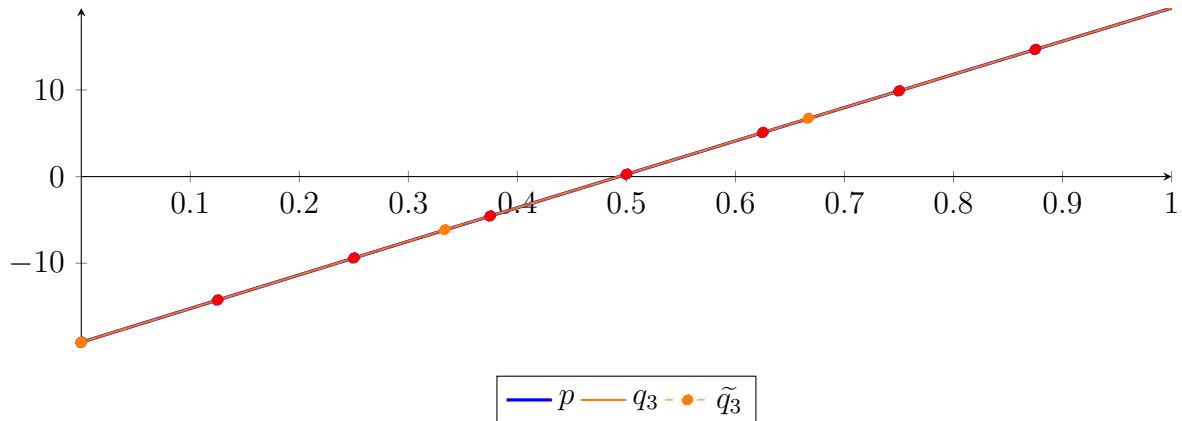
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = 2.82588 \cdot 10^{-12}X^8 - 1.08529 \cdot 10^{-11}X^7 + 1.68738 \cdot 10^{-11}X^6 - 1.34636 \cdot 10^{-11}X^5$$

$$+ 5.72618 \cdot 10^{-12}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m :

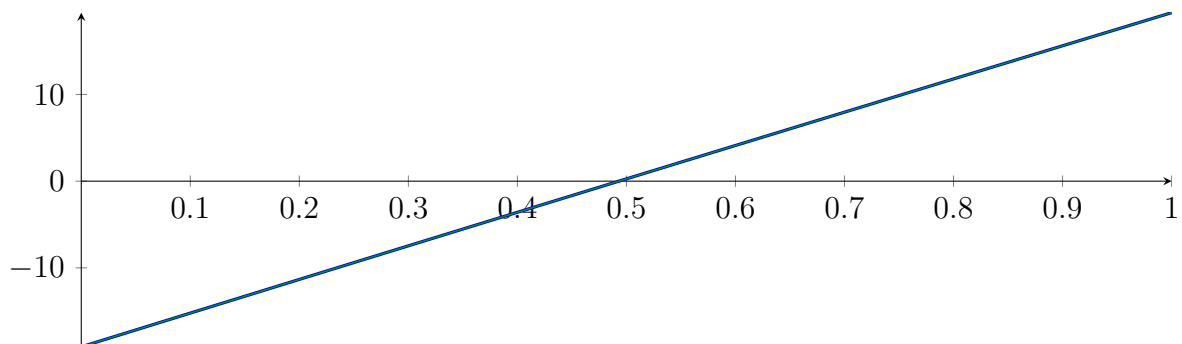
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \quad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

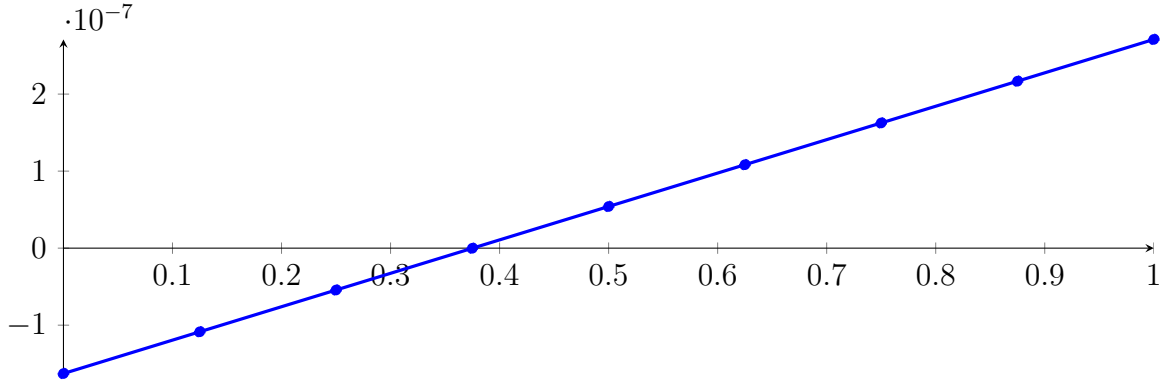
Longest intersection interval: $1.1252 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

141.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

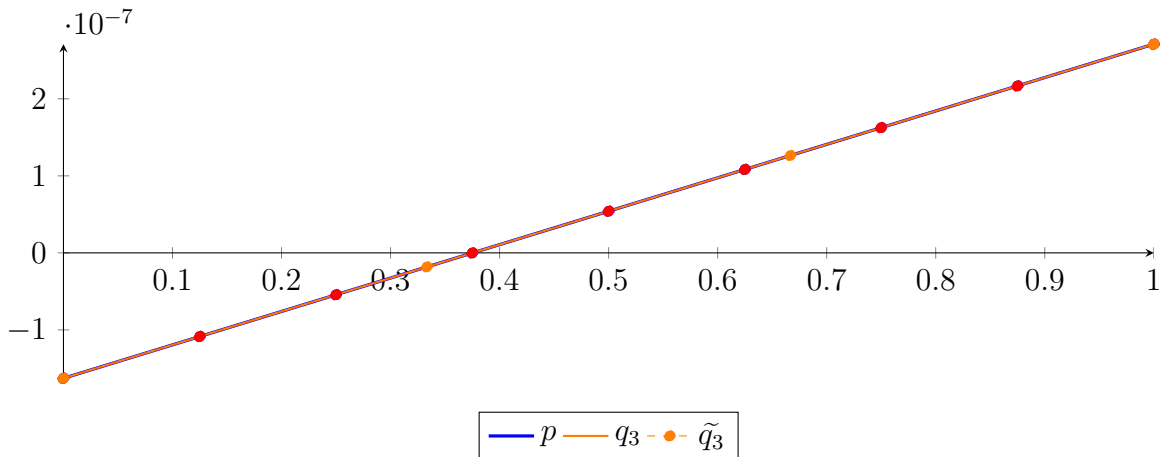
$$\begin{aligned}
 p &= -4.65289 \cdot 10^{-24} X^8 - 8.27181 \cdot 10^{-25} X^7 - 4.3427 \cdot 10^{-24} X^6 + 8.6854 \cdot 10^{-24} X^5 + 1.80946 \\
 &\quad \cdot 10^{-24} X^4 - 7.23783 \cdot 10^{-25} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,8}(X) - 1.08585 \cdot 10^{-07} B_{1,8}(X) - 5.43593 \cdot 10^{-08} B_{2,8}(X) \\
 &\quad - 1.33715 \cdot 10^{-10} B_{3,8}(X) + 5.40918 \cdot 10^{-08} B_{4,8}(X) + 1.08317 \cdot 10^{-07} B_{5,8}(X) \\
 &\quad + 1.62543 \cdot 10^{-07} B_{6,8}(X) + 2.16768 \cdot 10^{-07} B_{7,8}(X) + 2.70994 \cdot 10^{-07} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= 1.42947 \cdot 10^{-23} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,3} - 1.82089 \cdot 10^{-08} B_{1,3} + 1.26393 \cdot 10^{-07} B_{2,3} + 2.70994 \cdot 10^{-07} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= -2.97639 \cdot 10^{-20} X^8 + 1.26259 \cdot 10^{-19} X^7 - 2.19172 \cdot 10^{-19} X^6 + 2.00419 \cdot 10^{-19} X^5 - 1.02758 \\
 &\quad \cdot 10^{-19} X^4 + 2.83318 \cdot 10^{-20} X^3 - 5.27552 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,8} - 1.08585 \cdot 10^{-07} B_{1,8} - 5.43593 \cdot 10^{-08} B_{2,8} - 1.33715 \cdot 10^{-10} B_{3,8} + 5.40918 \\
 &\quad \cdot 10^{-08} B_{4,8} + 1.08317 \cdot 10^{-07} B_{5,8} + 1.62543 \cdot 10^{-07} B_{6,8} + 2.16768 \cdot 10^{-07} B_{7,8} + 2.70994 \cdot 10^{-07} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.97535 \cdot 10^{-22}$.

Bounding polynomials M and m :

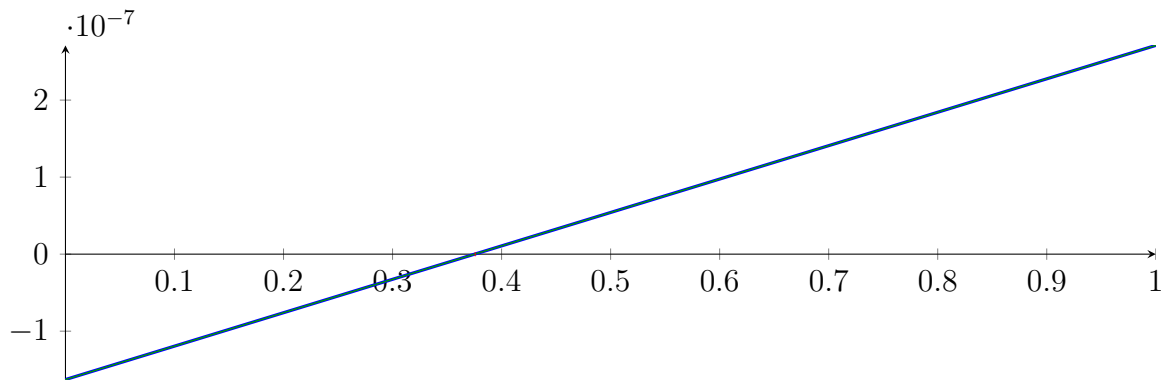
$$\begin{aligned}
 M &= 1.42689 \cdot 10^{-23} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 m &= 1.43206 \cdot 10^{-23} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.375308\}$$

$$N(m) = \{0.375308\}$$

Intersection intervals:



[0.375308, 0.375308]

Longest intersection interval: $1.36424 \cdot 10^{-12}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

141.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

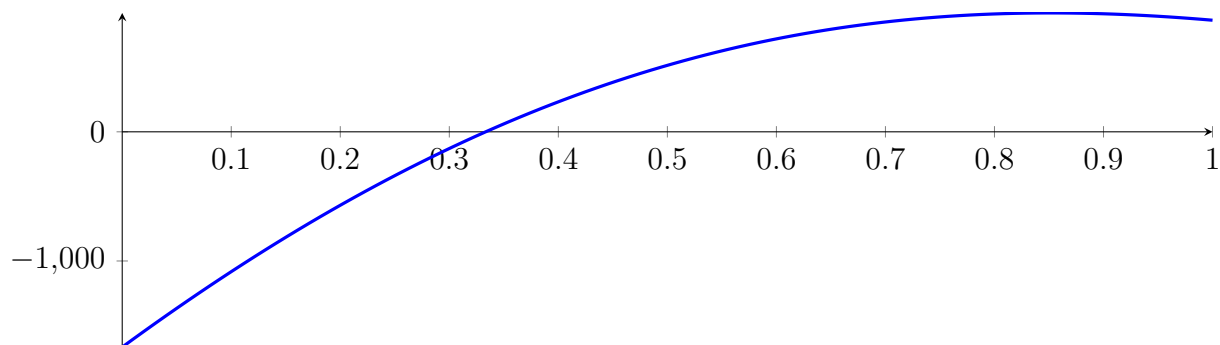
Reached interval [0.333333, 0.333333] **without sign change** at depth 4!

$p(0) = -1.10673e-18$ - $p(1) -5.14919e-19$

141.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

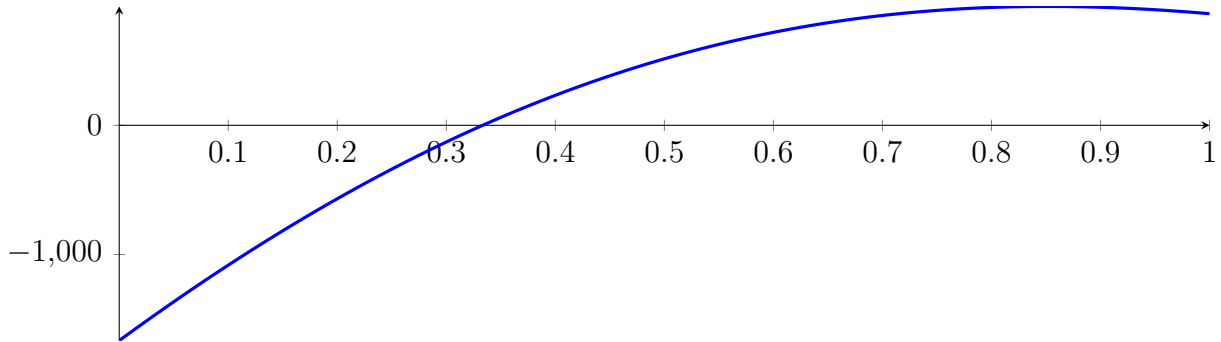
with precision $\varepsilon = 1 \cdot 10^{-32}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

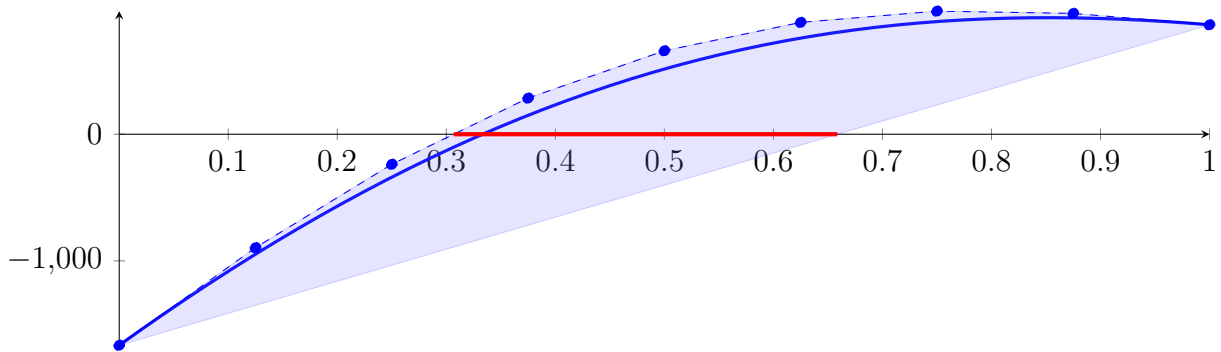
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



142.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

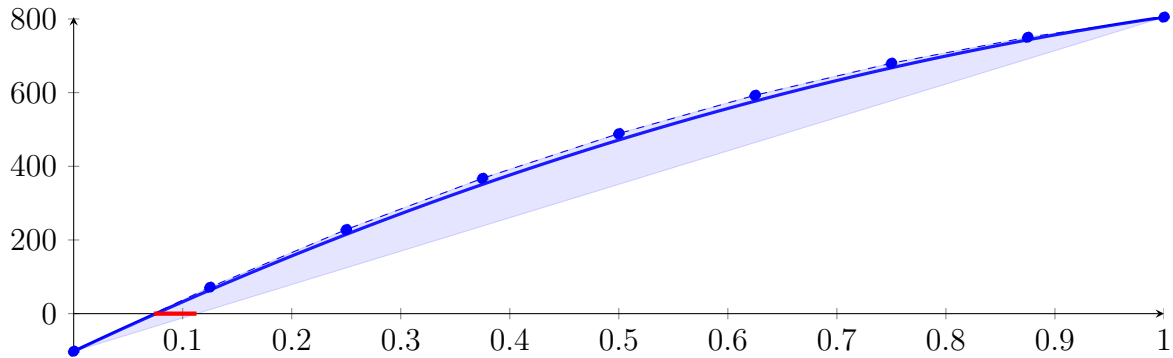
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

142.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

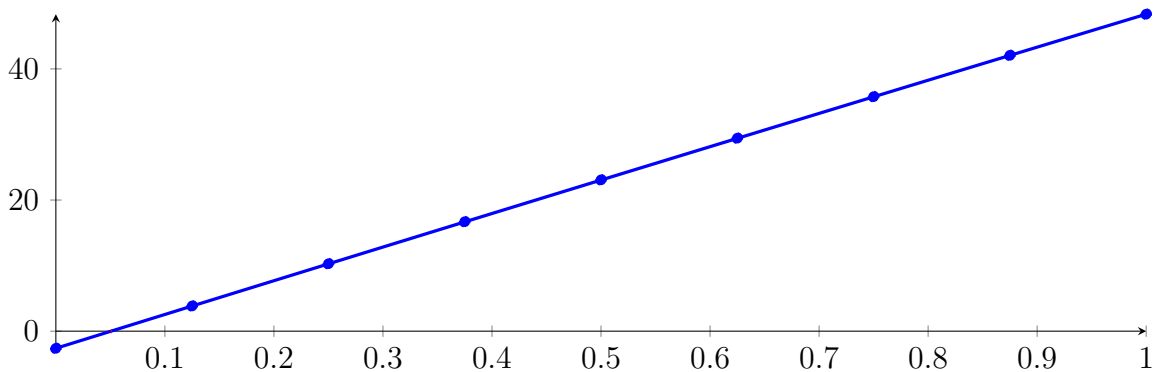
Longest intersection interval: 0.0391855

\implies Selective recursion: interval 1: [0.332635, 0.34642],

142.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.03577 \cdot 10^{-15} X^8 - 1.54633 \cdot 10^{-12} X^7 - 4.95836 \cdot 10^{-10} X^6 + 3.66751 \cdot 10^{-08} X^5 \\
 &\quad + 3.25466 \cdot 10^{-05} X^4 - 0.000586142 X^3 - 0.747315 X^2 + 51.7118 X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

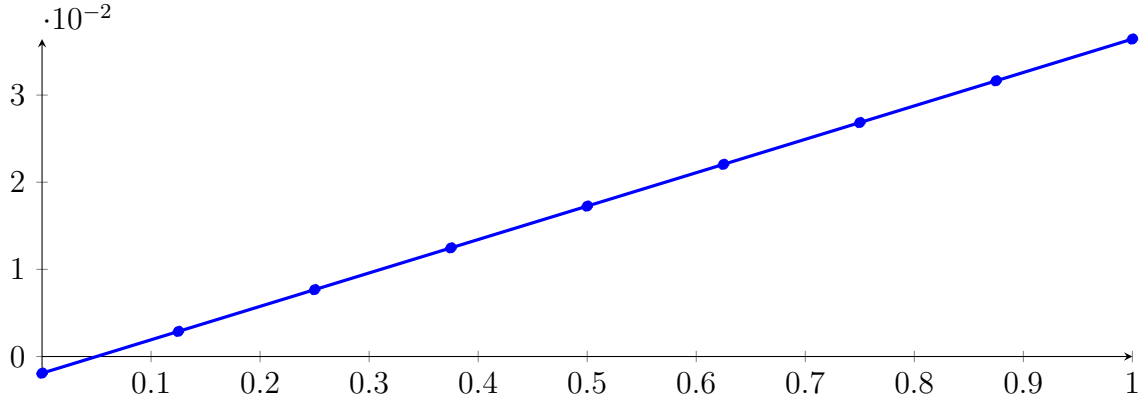
Longest intersection interval: 0.000742589

\implies Selective recursion: interval 1: [0.333333, 0.333343],

142.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.28918 \cdot 10^{-18} X^8 - 1.32815 \cdot 10^{-18} X^7 - 4.50622 \cdot 10^{-18} X^6 + 2.22939 \cdot 10^{-18} X^5 \\
 &\quad + 9.48677 \cdot 10^{-18} X^4 - 2.37322 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\
 &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\
 &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\
 &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

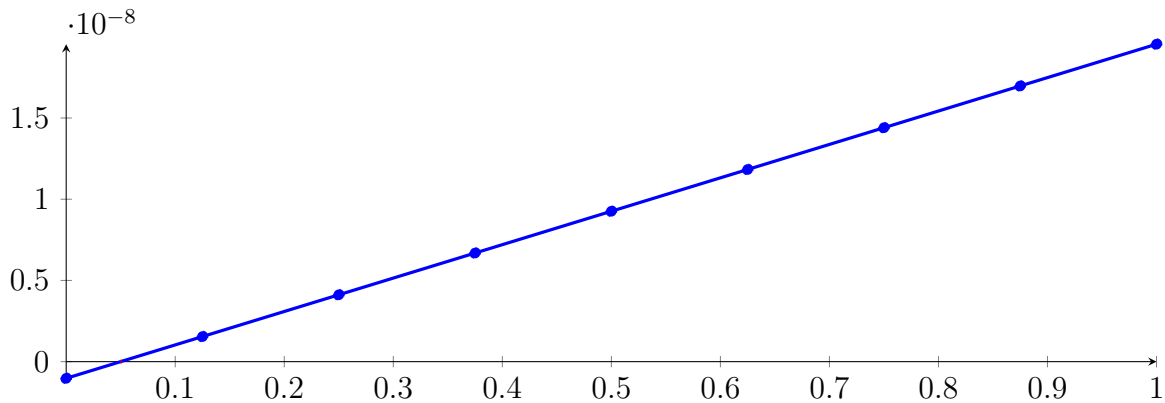
Longest intersection interval: $5.36469 \cdot 10^{-07}$

⇒ Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

142.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -7.43978 \cdot 10^{-25} X^8 - 4.71751 \cdot 10^{-25} X^7 - 2.488 \cdot 10^{-24} X^6 + 1.04044 \cdot 10^{-24} X^5 \\
 &\quad - 2.26182 \cdot 10^{-25} X^4 - 1.18615 \cdot 10^{-19} X^2 + 2.05706 \cdot 10^{-08} X - 1.02667 \cdot 10^{-09} \\
 &= -1.02667 \cdot 10^{-09} B_{0,8}(X) + 1.54465 \cdot 10^{-09} B_{1,8}(X) + 4.11597 \cdot 10^{-09} B_{2,8}(X) \\
 &\quad + 6.68729 \cdot 10^{-09} B_{3,8}(X) + 9.25862 \cdot 10^{-09} B_{4,8}(X) + 1.18299 \cdot 10^{-08} B_{5,8}(X) \\
 &\quad + 1.44013 \cdot 10^{-08} B_{6,8}(X) + 1.69726 \cdot 10^{-08} B_{7,8}(X) + 1.95439 \cdot 10^{-08} B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87793 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

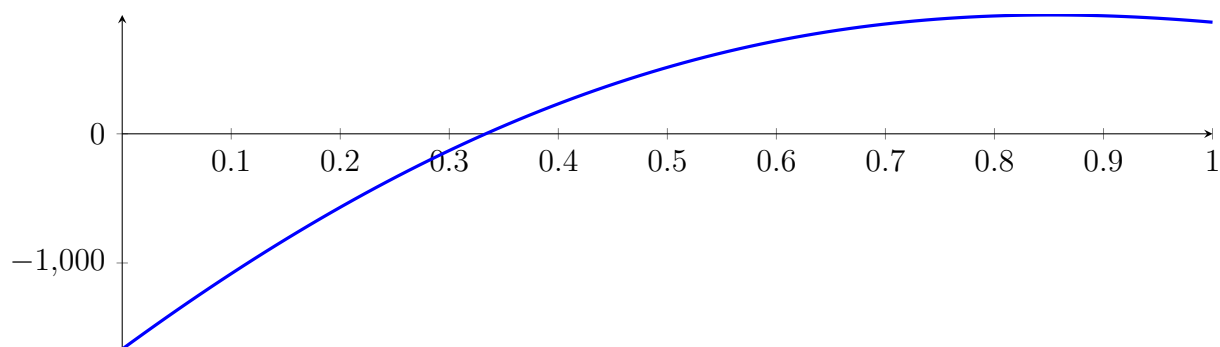
142.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

142.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

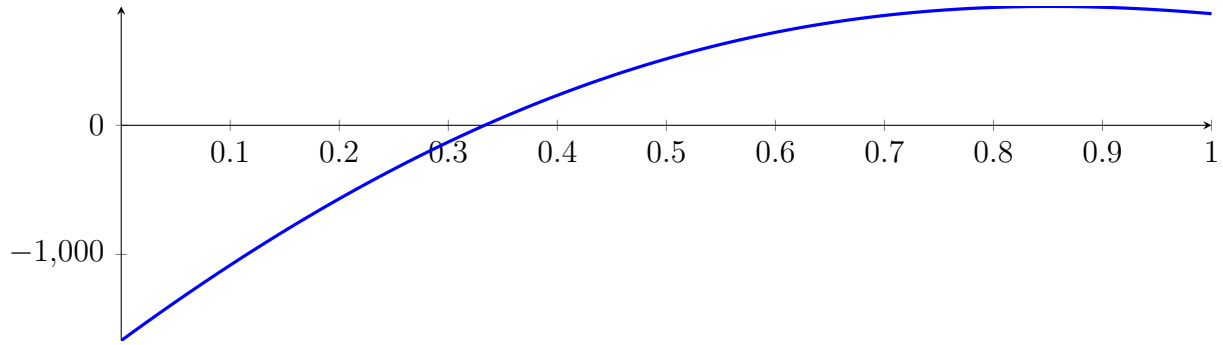
with precision $\varepsilon = 1 \cdot 10^{-64}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

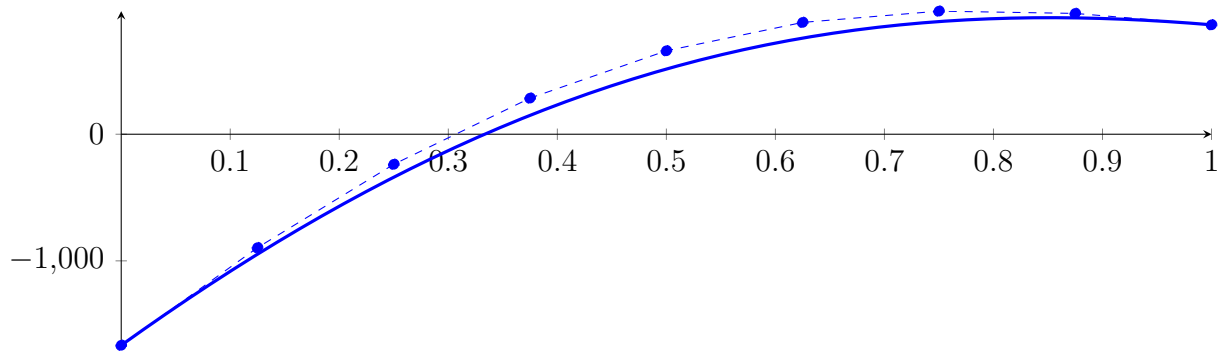
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



143.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

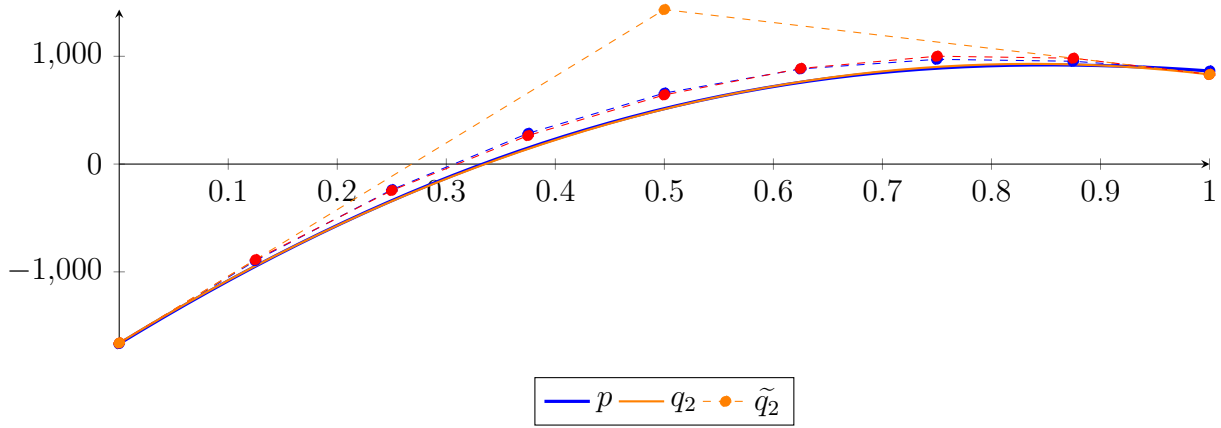
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -5.06304 \cdot 10^{-10}X^8 + 1.93305 \cdot 10^{-09}X^7 - 2.97598 \cdot 10^{-09}X^6 + 2.35116 \cdot 10^{-09}X^5 \\ &\quad - 9.91232 \cdot 10^{-10}X^4 + 2.00705 \cdot 10^{-10}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

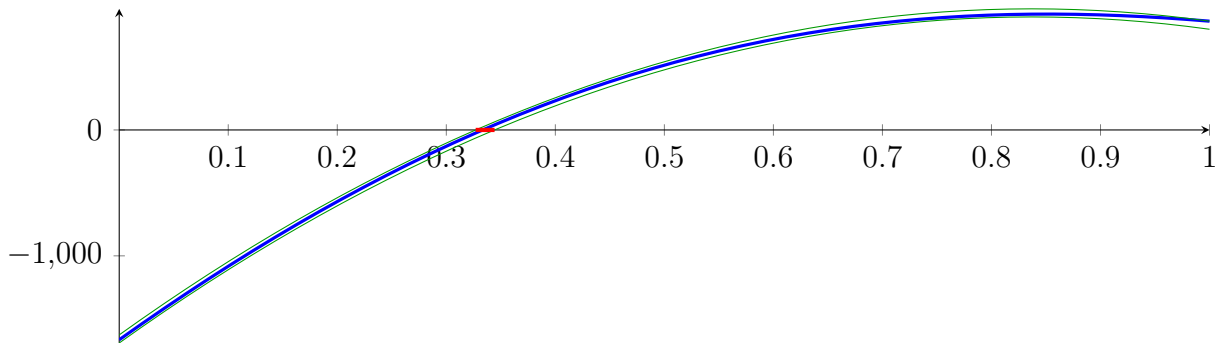
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

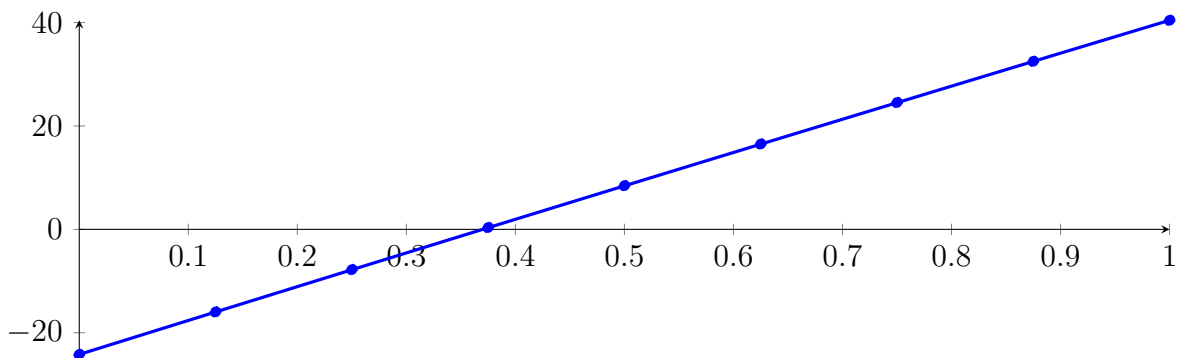
Longest intersection interval: 0.0173372

\implies Selective recursion: interval 1: $[0.326917, 0.344255]$,

143.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -8.80546 \cdot 10^{-15} X^8 - 7.66587 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

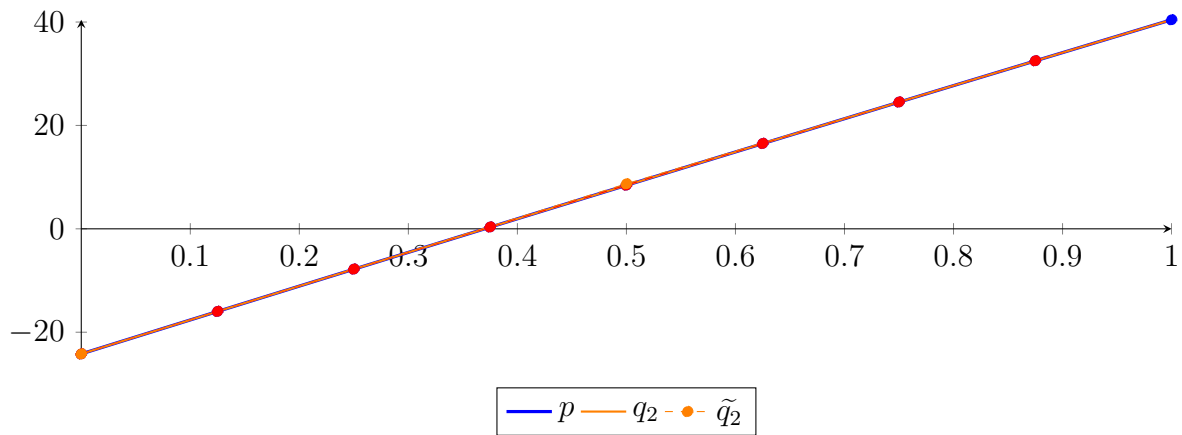
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = -2.59056 \cdot 10^{-11}X^8 + 1.00262 \cdot 10^{-10}X^7 - 1.57692 \cdot 10^{-10}X^6 + 1.29283 \cdot 10^{-10}X^5$$

$$- 5.86775 \cdot 10^{-11}X^4 + 1.42642 \cdot 10^{-11}X^3 - 1.18261X^2 + 65.8162X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

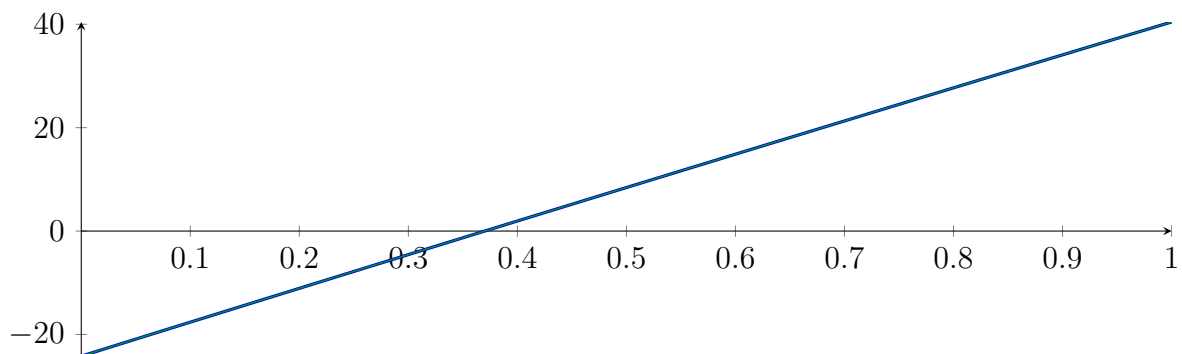
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

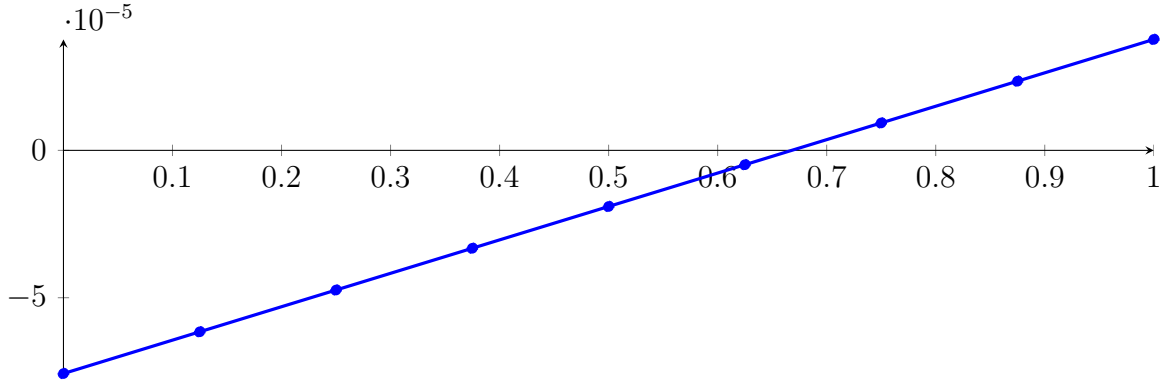
Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

143.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

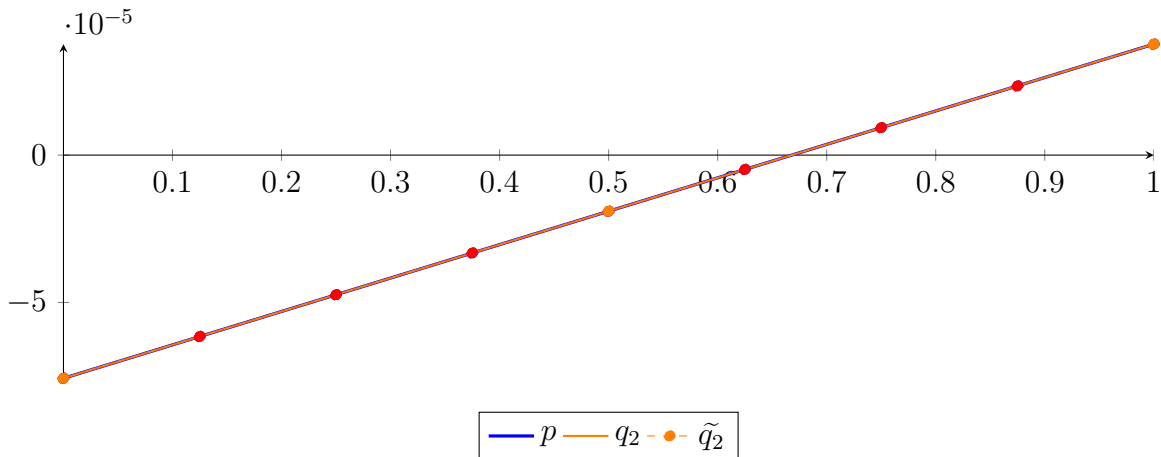
$$\begin{aligned}
 p &= 1.16467 \cdot 10^{-21} X^8 + 3.17637 \cdot 10^{-21} X^7 + 1.18585 \cdot 10^{-20} X^6 - 1.48231 \cdot 10^{-21} X^5 + 9.26442 \\
 &\quad \cdot 10^{-22} X^4 - 5.92923 \cdot 10^{-21} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) \\
 &\quad - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) \\
 &\quad + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 3.26671 \cdot 10^{-17} X^8 - 1.38104 \cdot 10^{-16} X^7 + 2.39221 \cdot 10^{-16} X^6 - 2.17429 \cdot 10^{-16} X^5 + 1.10046 \\
 &\quad \cdot 10^{-16} X^4 - 3.0162 \cdot 10^{-17} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \\
 &\quad \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.84643 \cdot 10^{-19}$.

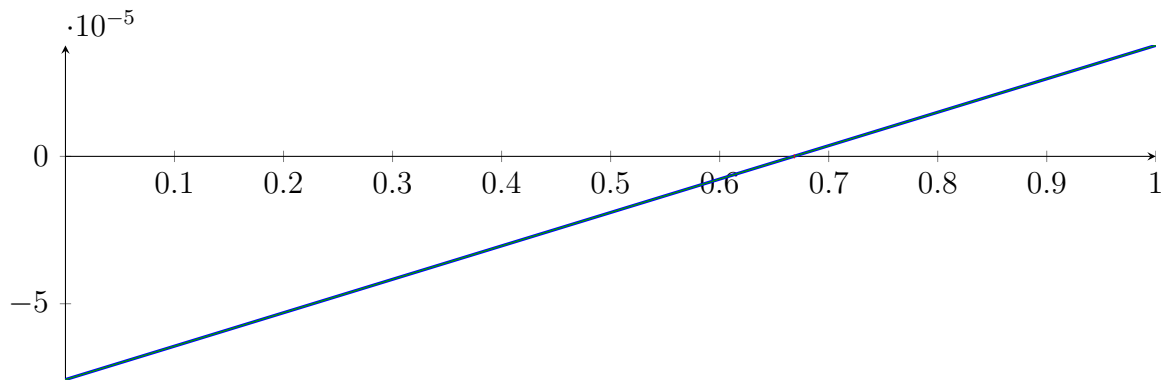
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 m &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.667955, 3.14644 \cdot 10^7\} \qquad N(m) = \{0.667955, 3.14644 \cdot 10^7\}$$

Intersection intervals:



[0.667955, 0.667955]

Longest intersection interval: $3.08439 \cdot 10^{-13}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

143.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

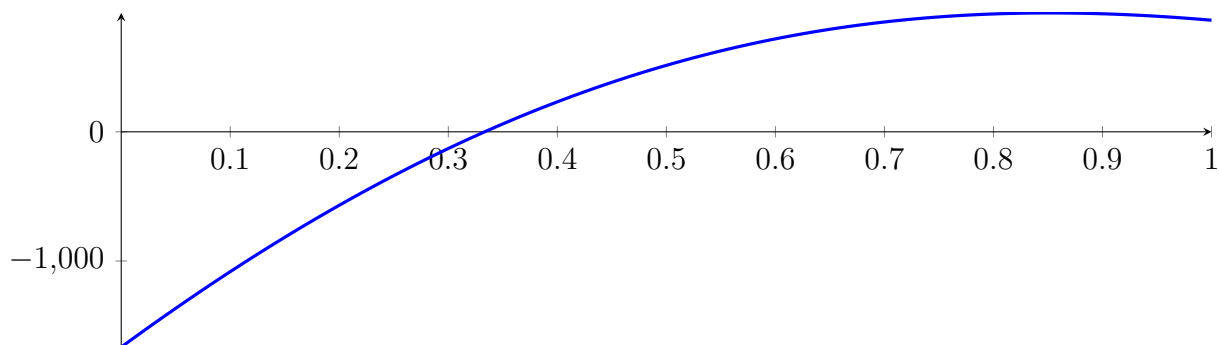
Reached interval [0.333333, 0.333333] **without sign change** at depth 4!

$p(0) = 2.85706e-18$ - $p(1) 3.78276e-17$

143.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

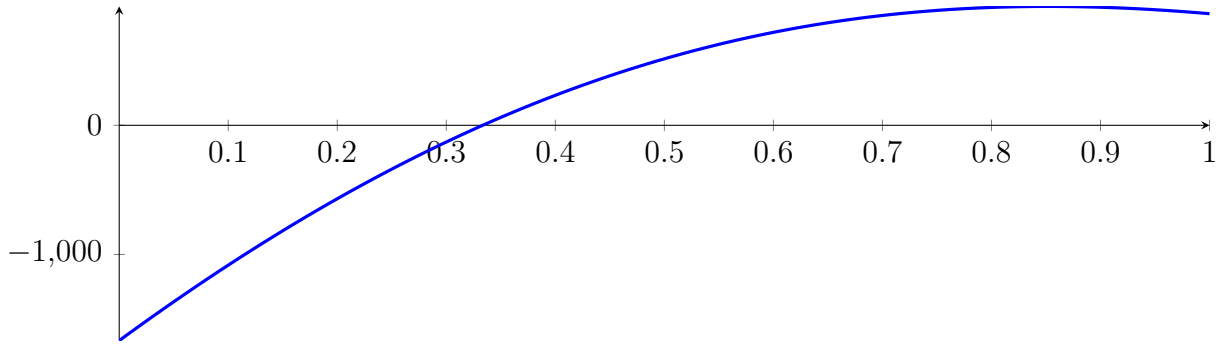
with precision $\varepsilon = 1 \cdot 10^{-64}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

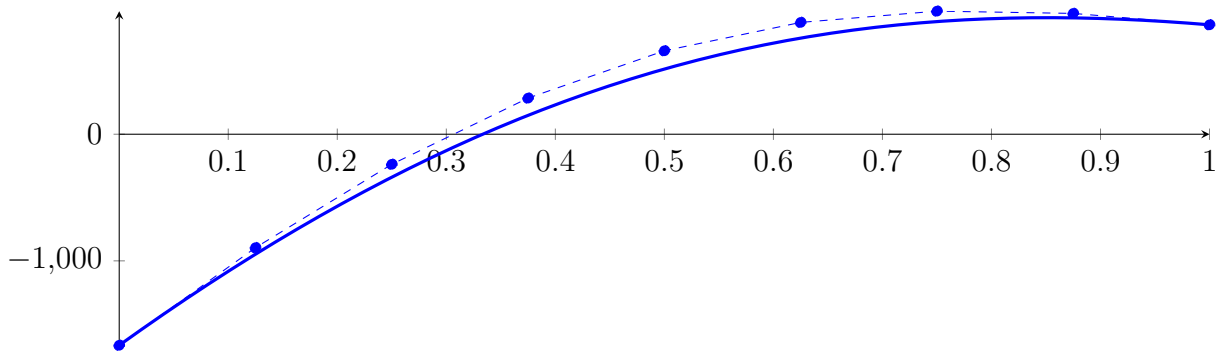
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



144.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

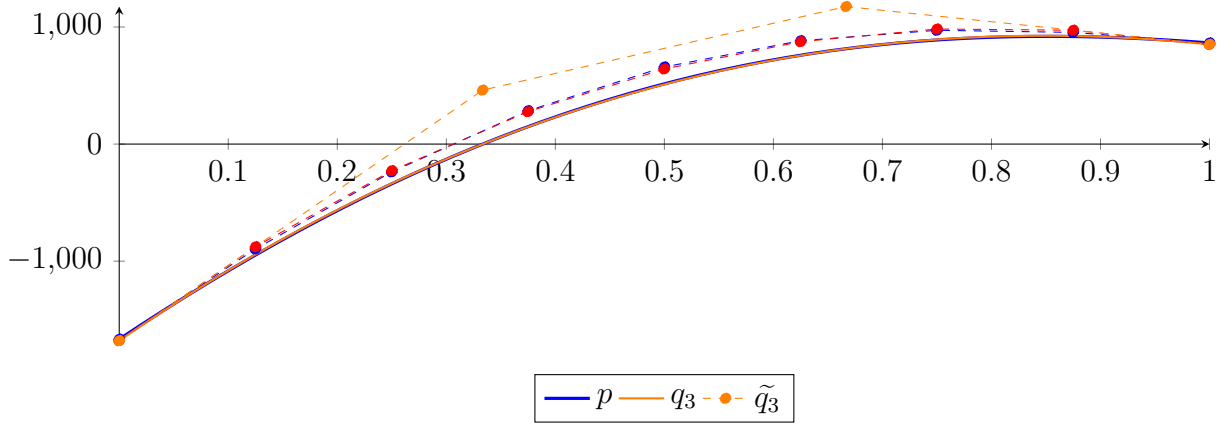
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.39389 \cdot 10^{-09}X^8 + 9.60737 \cdot 10^{-09}X^7 - 1.57477 \cdot 10^{-08}X^6 + 1.35479 \cdot 10^{-08}X^5 \\ &\quad - 6.53397 \cdot 10^{-09}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

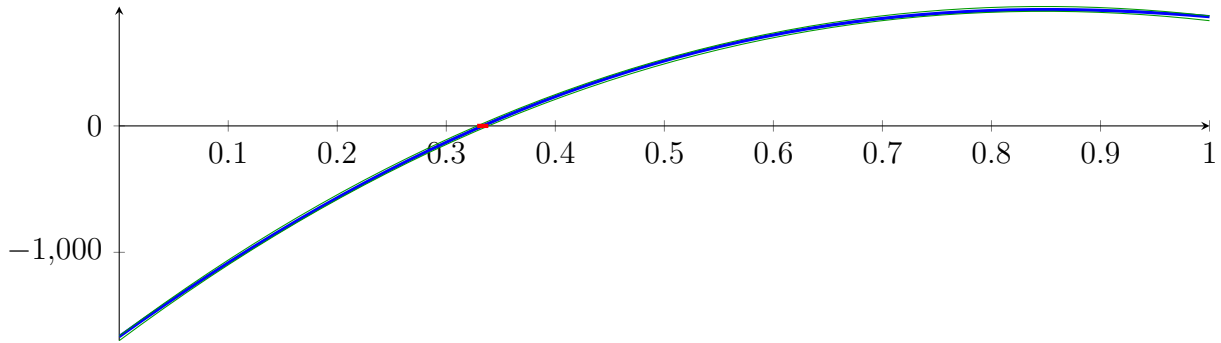
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

144.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

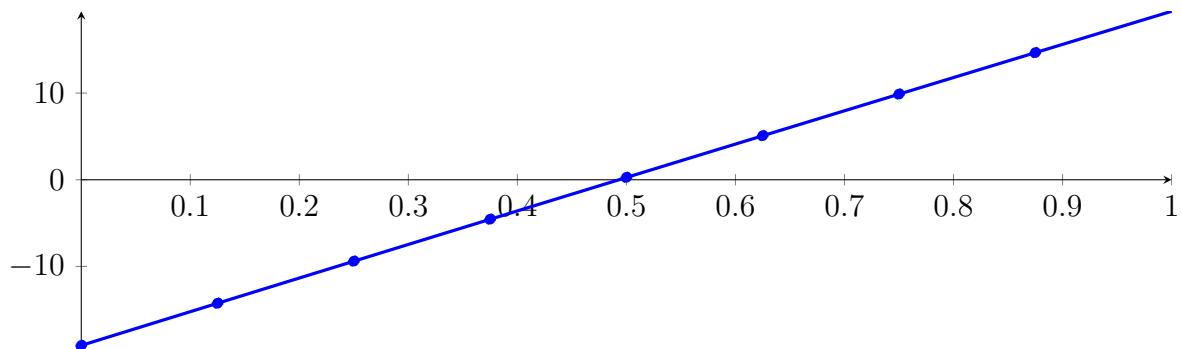
Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.66533 \cdot 10^{-16} X^8 - 1.99007 \cdot 10^{-13} X^7 - 8.53059 \cdot 10^{-11} X^6 + 8.7284 \cdot 10^{-09} X^5$$

$$+ 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124$$

$$= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X)$$

$$+ 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

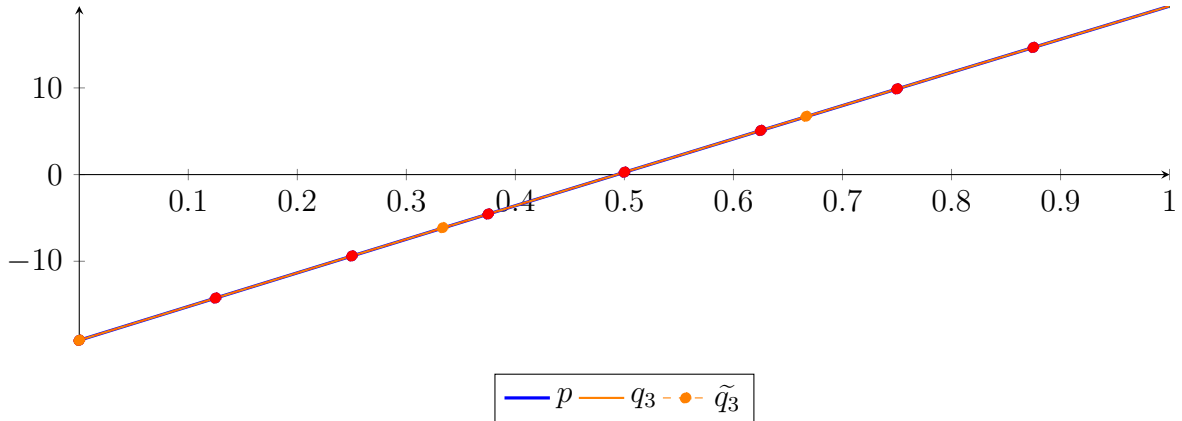
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = 2.82588 \cdot 10^{-12}X^8 - 1.08529 \cdot 10^{-11}X^7 + 1.68738 \cdot 10^{-11}X^6 - 1.34636 \cdot 10^{-11}X^5$$

$$+ 5.72618 \cdot 10^{-12}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m :

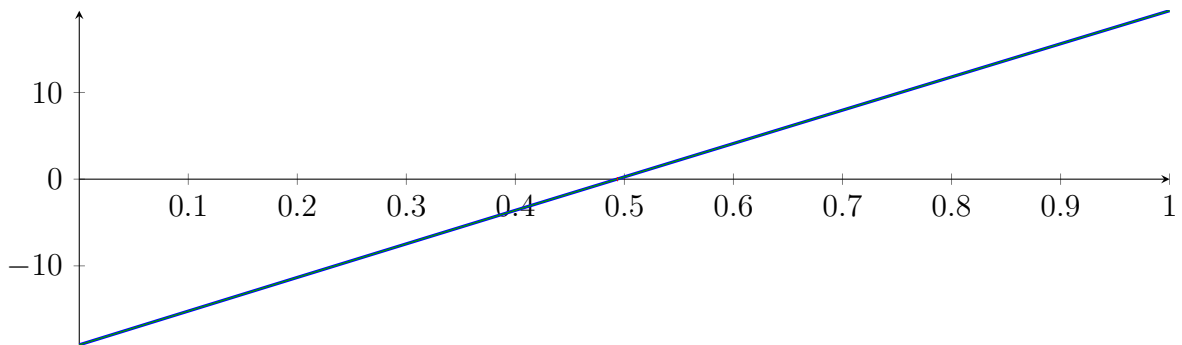
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \quad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

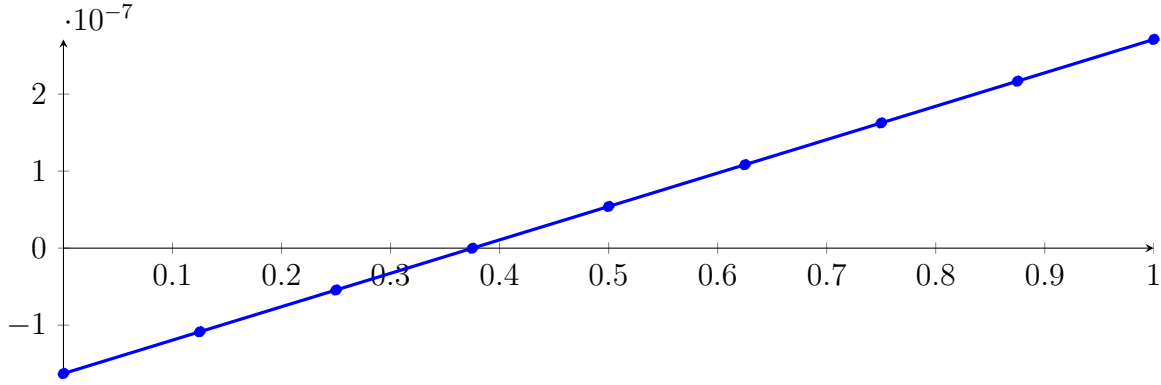
Longest intersection interval: $1.1252 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

144.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

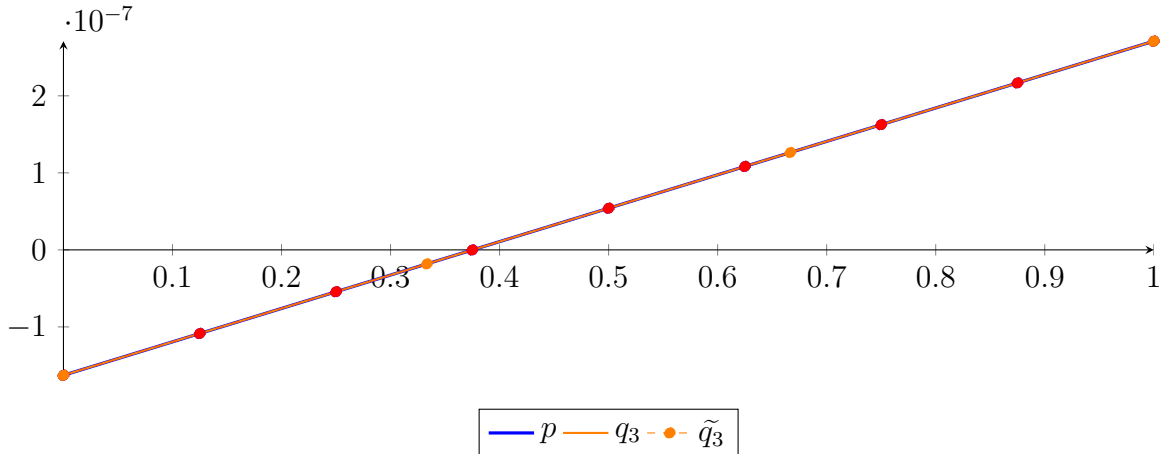
$$\begin{aligned}
 p &= -4.65289 \cdot 10^{-24} X^8 - 8.27181 \cdot 10^{-25} X^7 - 4.3427 \cdot 10^{-24} X^6 + 8.6854 \cdot 10^{-24} X^5 + 1.80946 \\
 &\quad \cdot 10^{-24} X^4 - 7.23783 \cdot 10^{-25} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,8}(X) - 1.08585 \cdot 10^{-07} B_{1,8}(X) - 5.43593 \cdot 10^{-08} B_{2,8}(X) \\
 &\quad - 1.33715 \cdot 10^{-10} B_{3,8}(X) + 5.40918 \cdot 10^{-08} B_{4,8}(X) + 1.08317 \cdot 10^{-07} B_{5,8}(X) \\
 &\quad + 1.62543 \cdot 10^{-07} B_{6,8}(X) + 2.16768 \cdot 10^{-07} B_{7,8}(X) + 2.70994 \cdot 10^{-07} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= 1.42947 \cdot 10^{-23} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,3} - 1.82089 \cdot 10^{-08} B_{1,3} + 1.26393 \cdot 10^{-07} B_{2,3} + 2.70994 \cdot 10^{-07} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= -2.97639 \cdot 10^{-20} X^8 + 1.26259 \cdot 10^{-19} X^7 - 2.19172 \cdot 10^{-19} X^6 + 2.00419 \cdot 10^{-19} X^5 - 1.02758 \\
 &\quad \cdot 10^{-19} X^4 + 2.83318 \cdot 10^{-20} X^3 - 5.27552 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,8} - 1.08585 \cdot 10^{-07} B_{1,8} - 5.43593 \cdot 10^{-08} B_{2,8} - 1.33715 \cdot 10^{-10} B_{3,8} + 5.40918 \\
 &\quad \cdot 10^{-08} B_{4,8} + 1.08317 \cdot 10^{-07} B_{5,8} + 1.62543 \cdot 10^{-07} B_{6,8} + 2.16768 \cdot 10^{-07} B_{7,8} + 2.70994 \cdot 10^{-07} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.97535 \cdot 10^{-22}$.

Bounding polynomials M and m :

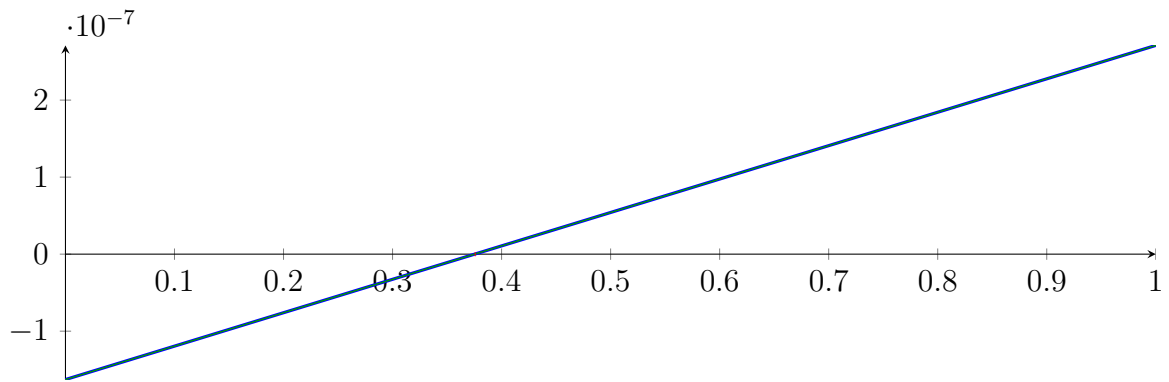
$$\begin{aligned}
 M &= 1.42689 \cdot 10^{-23} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 m &= 1.43206 \cdot 10^{-23} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.375308\}$$

$$N(m) = \{0.375308\}$$

Intersection intervals:



[0.375308, 0.375308]

Longest intersection interval: $1.36424 \cdot 10^{-12}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

144.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

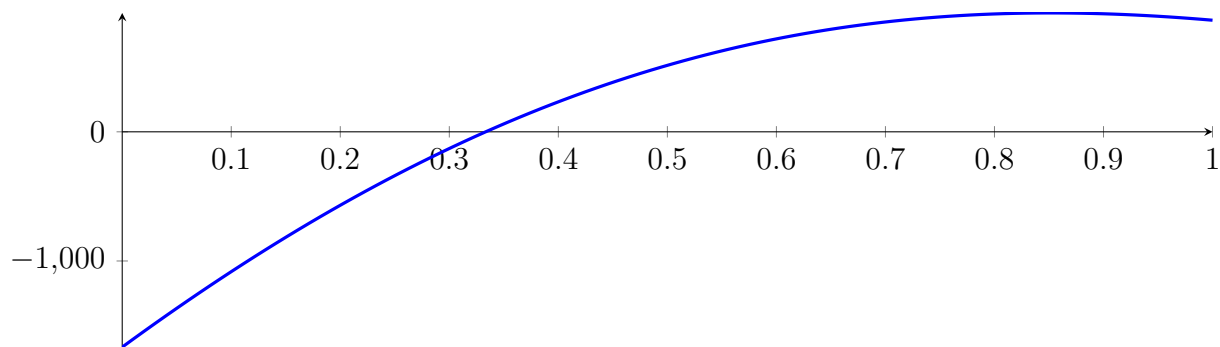
Reached interval [0.333333, 0.333333] **without sign change** at depth 4!

$p(0) = -1.10673e-18$ - $p(1) -5.14919e-19$

144.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

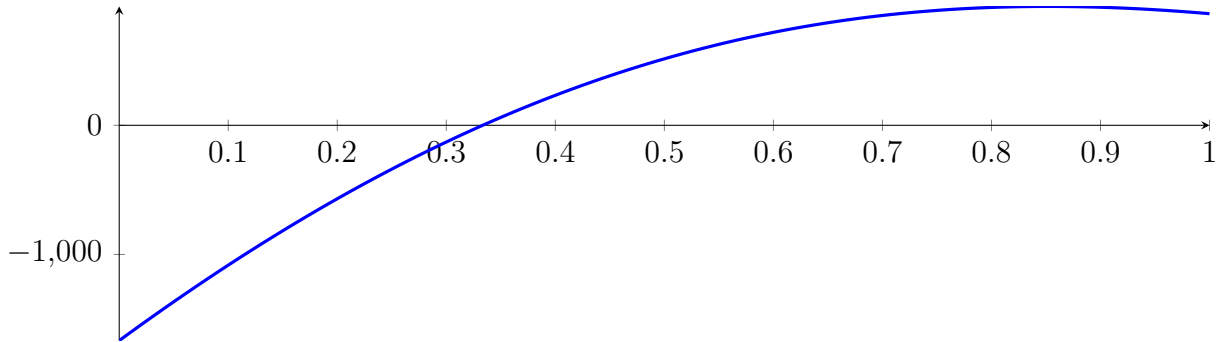
with precision $\varepsilon = 1 \cdot 10^{-64}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

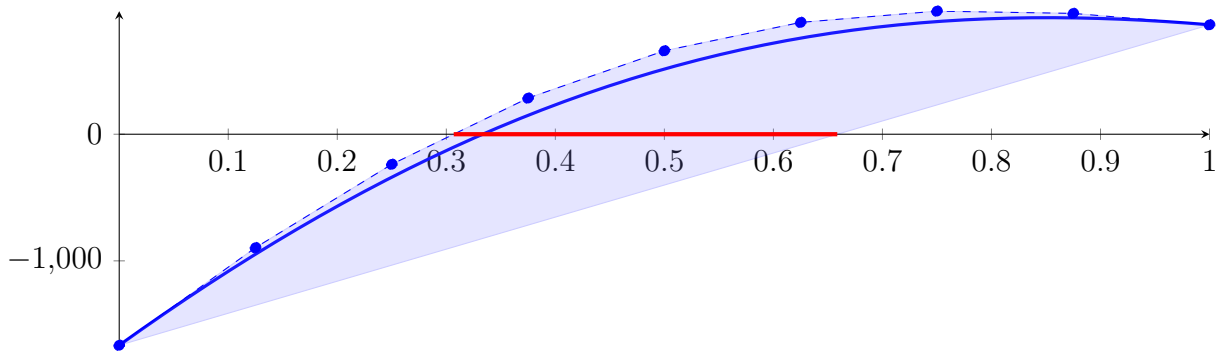
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



145.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

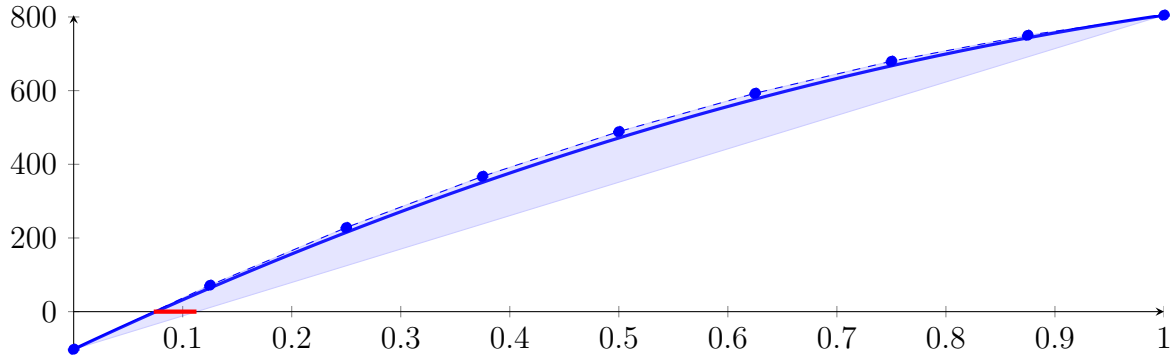
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

145.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

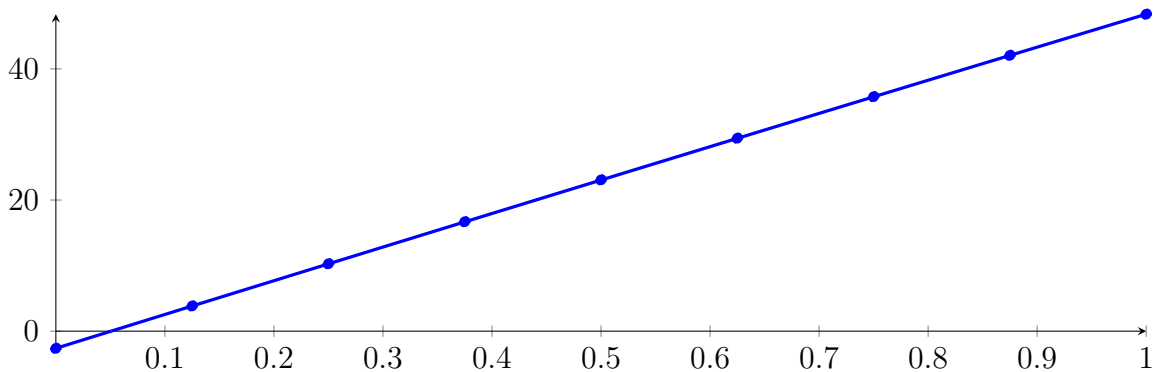
Longest intersection interval: 0.0391855

⇒ Selective recursion: interval 1: [0.332635, 0.34642],

145.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.03577 \cdot 10^{-15} X^8 - 1.54633 \cdot 10^{-12} X^7 - 4.95836 \cdot 10^{-10} X^6 + 3.66751 \cdot 10^{-08} X^5 \\
 &\quad + 3.25466 \cdot 10^{-05} X^4 - 0.000586142 X^3 - 0.747315 X^2 + 51.7118 X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

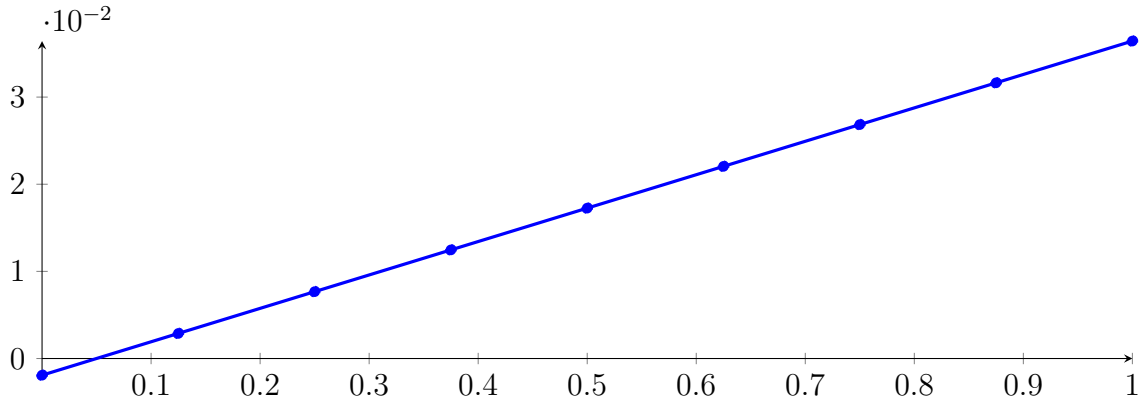
Longest intersection interval: 0.000742589

⇒ Selective recursion: interval 1: [0.333333, 0.333343],

145.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.28918 \cdot 10^{-18} X^8 - 1.32815 \cdot 10^{-18} X^7 - 4.50622 \cdot 10^{-18} X^6 + 2.22939 \cdot 10^{-18} X^5 \\
 &\quad + 9.48677 \cdot 10^{-18} X^4 - 2.37322 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\
 &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\
 &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\
 &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

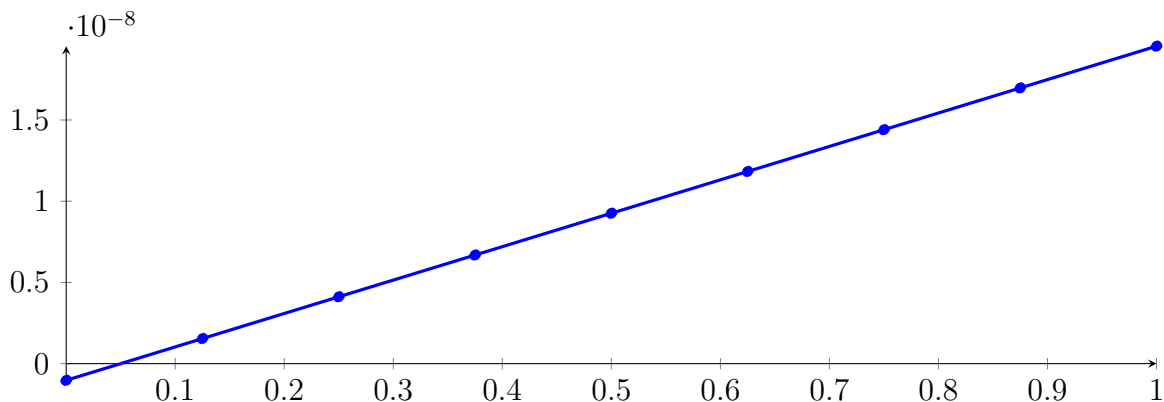
Longest intersection interval: $5.36469 \cdot 10^{-07}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

145.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -7.43978 \cdot 10^{-25} X^8 - 4.71751 \cdot 10^{-25} X^7 - 2.488 \cdot 10^{-24} X^6 + 1.04044 \cdot 10^{-24} X^5 \\
 &\quad - 2.26182 \cdot 10^{-25} X^4 - 1.18615 \cdot 10^{-19} X^2 + 2.05706 \cdot 10^{-08} X - 1.02667 \cdot 10^{-09} \\
 &= -1.02667 \cdot 10^{-09} B_{0,8}(X) + 1.54465 \cdot 10^{-09} B_{1,8}(X) + 4.11597 \cdot 10^{-09} B_{2,8}(X) \\
 &\quad + 6.68729 \cdot 10^{-09} B_{3,8}(X) + 9.25862 \cdot 10^{-09} B_{4,8}(X) + 1.18299 \cdot 10^{-08} B_{5,8}(X) \\
 &\quad + 1.44013 \cdot 10^{-08} B_{6,8}(X) + 1.69726 \cdot 10^{-08} B_{7,8}(X) + 1.95439 \cdot 10^{-08} B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87793 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

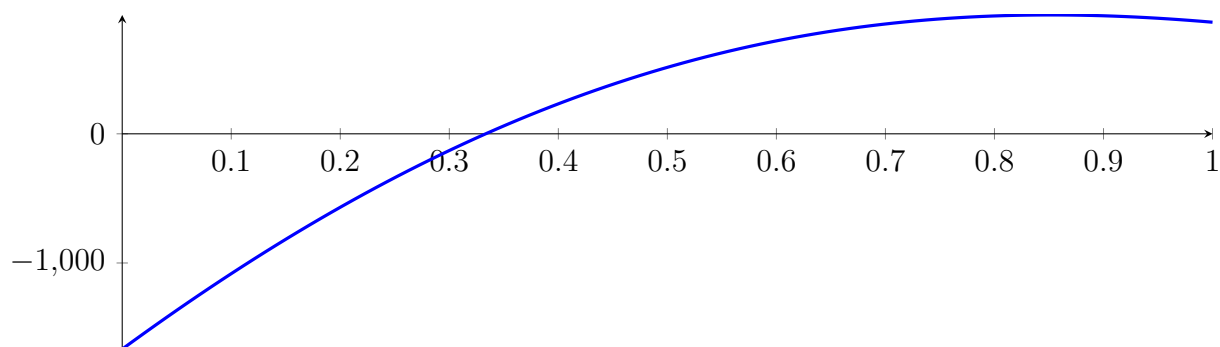
145.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

145.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

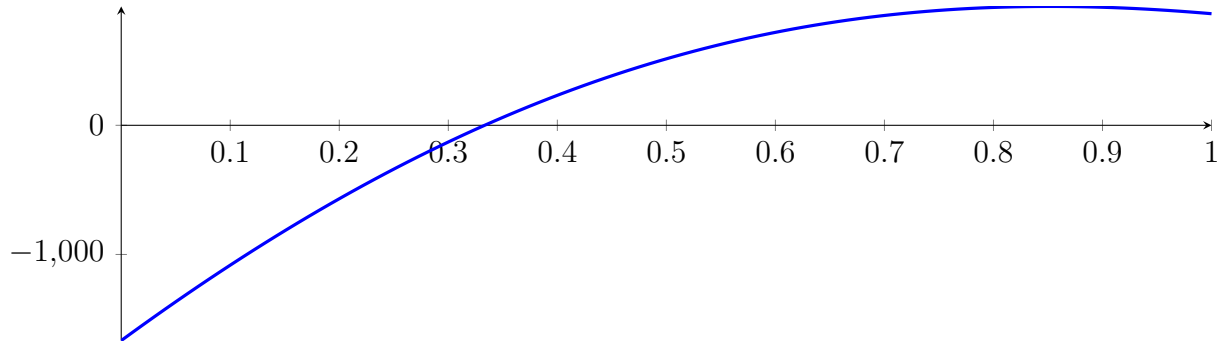
with precision $\varepsilon = 1 \cdot 10^{-128}$.

146 Running QuadClip on f_8 with epsilon 128

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

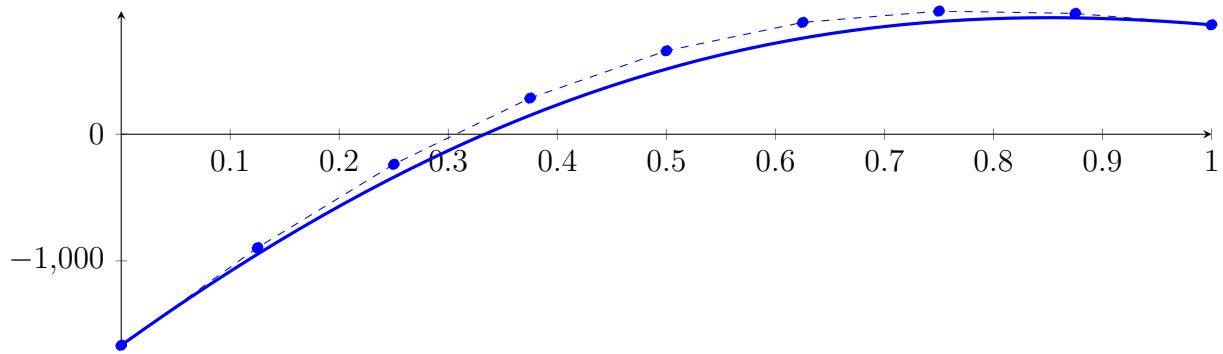
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



146.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

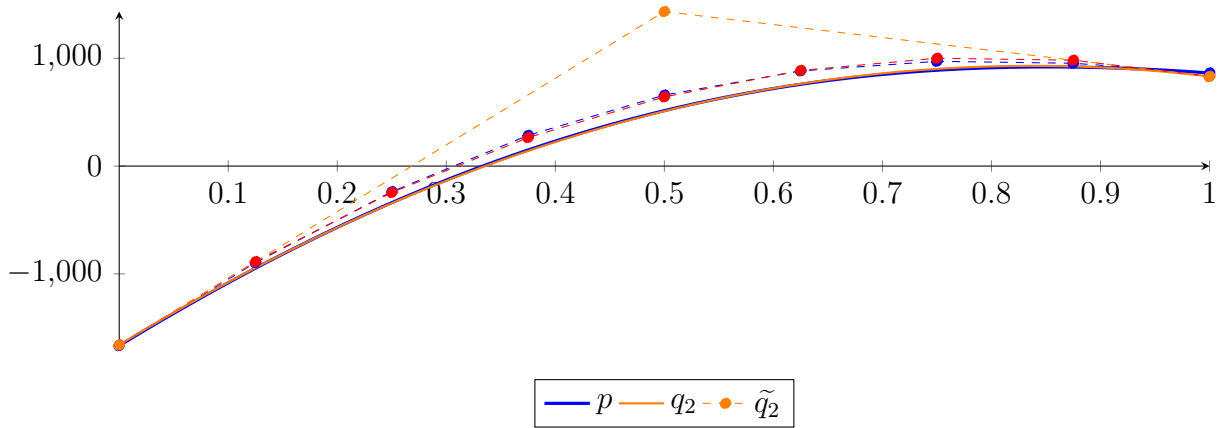
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -5.06304 \cdot 10^{-10}X^8 + 1.93305 \cdot 10^{-09}X^7 - 2.97598 \cdot 10^{-09}X^6 + 2.35116 \cdot 10^{-09}X^5 \\ &\quad - 9.91232 \cdot 10^{-10}X^4 + 2.00705 \cdot 10^{-10}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

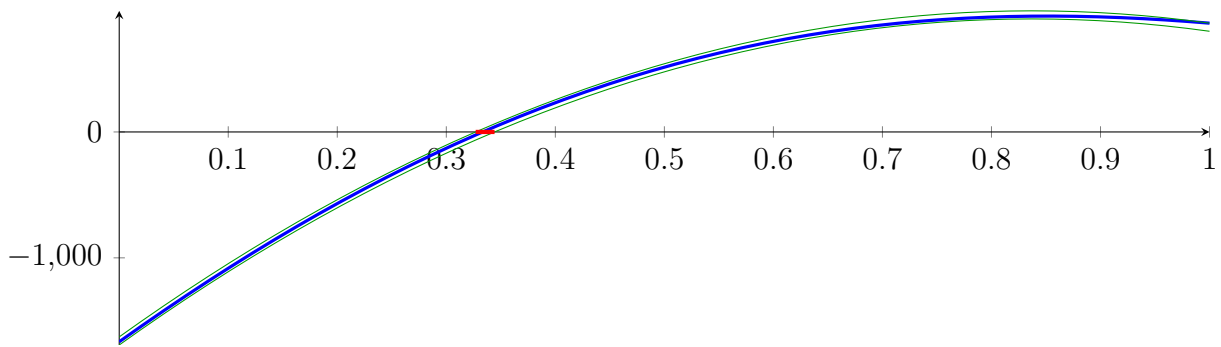
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

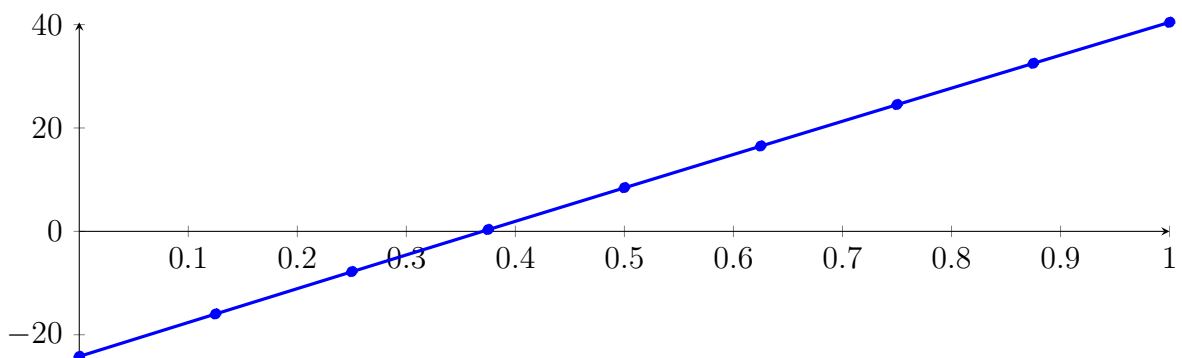
Longest intersection interval: 0.0173372

\implies Selective recursion: interval 1: $[0.326917, 0.344255]$,

146.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -8.80546 \cdot 10^{-15} X^8 - 7.66587 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

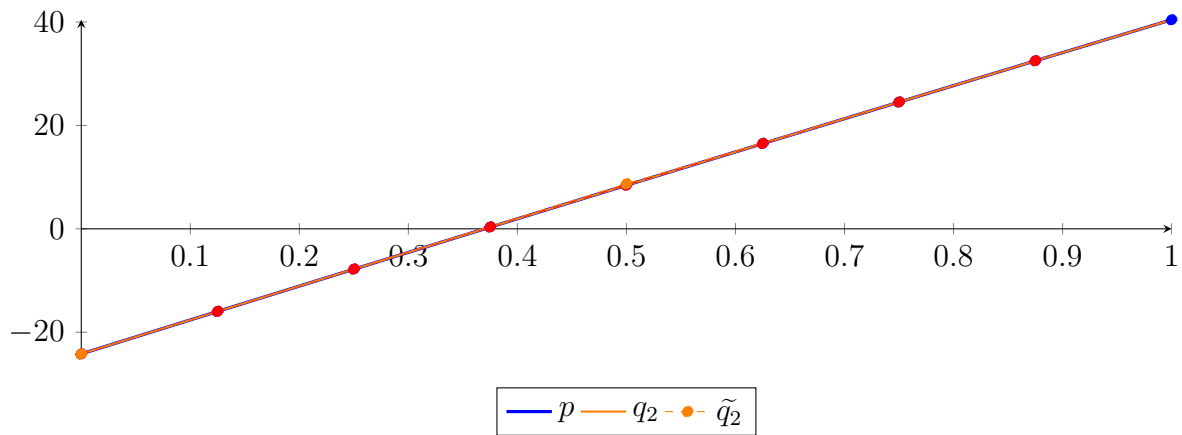
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = -2.59056 \cdot 10^{-11}X^8 + 1.00262 \cdot 10^{-10}X^7 - 1.57692 \cdot 10^{-10}X^6 + 1.29283 \cdot 10^{-10}X^5$$

$$- 5.86775 \cdot 10^{-11}X^4 + 1.42642 \cdot 10^{-11}X^3 - 1.18261X^2 + 65.8162X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

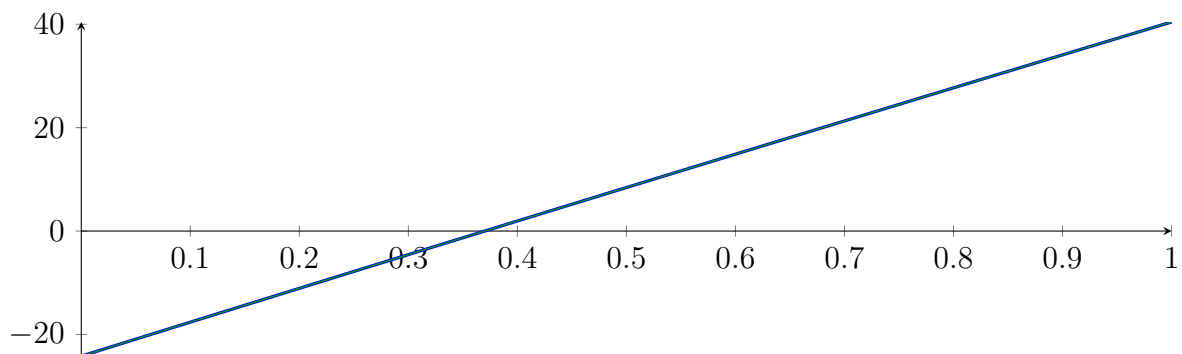
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

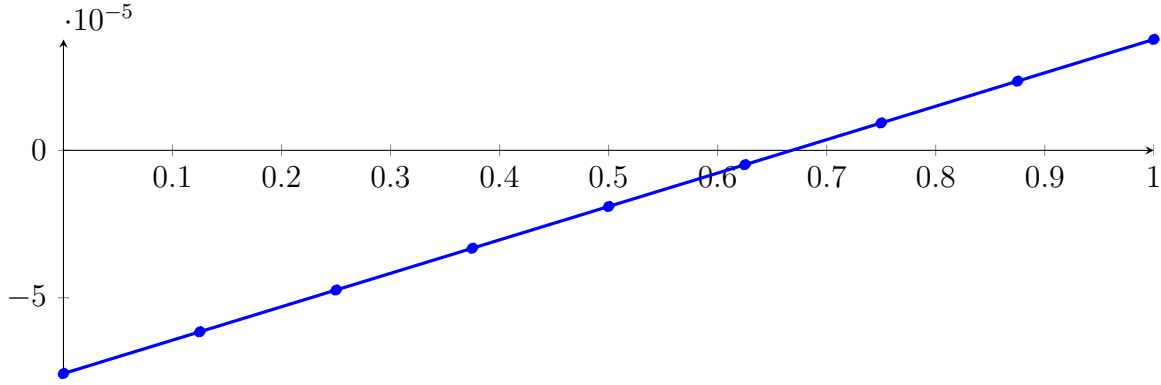
Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

146.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

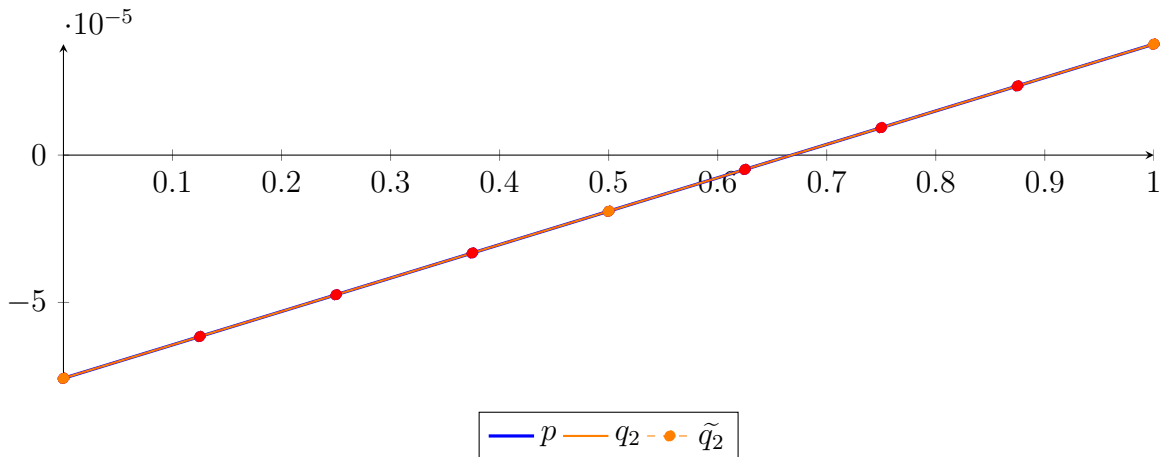
$$\begin{aligned}
 p &= 1.16467 \cdot 10^{-21} X^8 + 3.17637 \cdot 10^{-21} X^7 + 1.18585 \cdot 10^{-20} X^6 - 1.48231 \cdot 10^{-21} X^5 + 9.26442 \\
 &\quad \cdot 10^{-22} X^4 - 5.92923 \cdot 10^{-21} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) \\
 &\quad - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) \\
 &\quad + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 3.26671 \cdot 10^{-17} X^8 - 1.38104 \cdot 10^{-16} X^7 + 2.39221 \cdot 10^{-16} X^6 - 2.17429 \cdot 10^{-16} X^5 + 1.10046 \\
 &\quad \cdot 10^{-16} X^4 - 3.0162 \cdot 10^{-17} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \\
 &\quad \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.84643 \cdot 10^{-19}$.

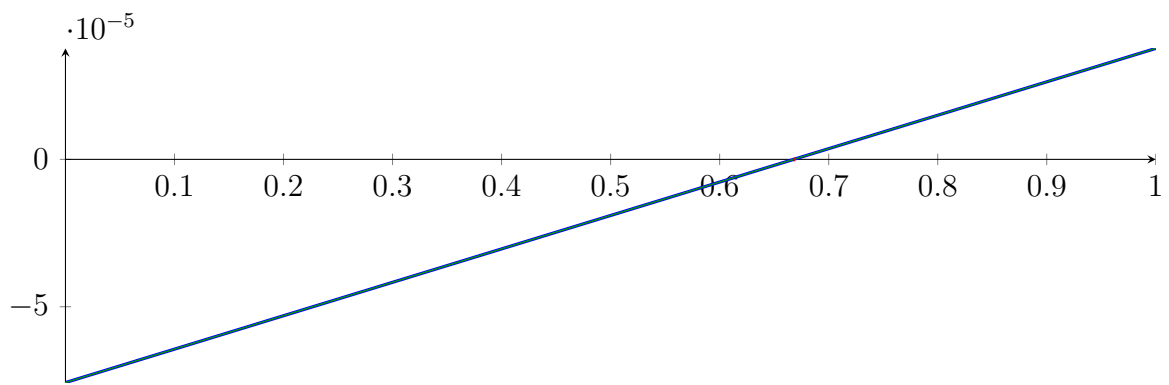
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 m &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.667955, 3.14644 \cdot 10^7\} \qquad N(m) = \{0.667955, 3.14644 \cdot 10^7\}$$

Intersection intervals:



[0.667955, 0.667955]

Longest intersection interval: $3.08439 \cdot 10^{-13}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

146.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

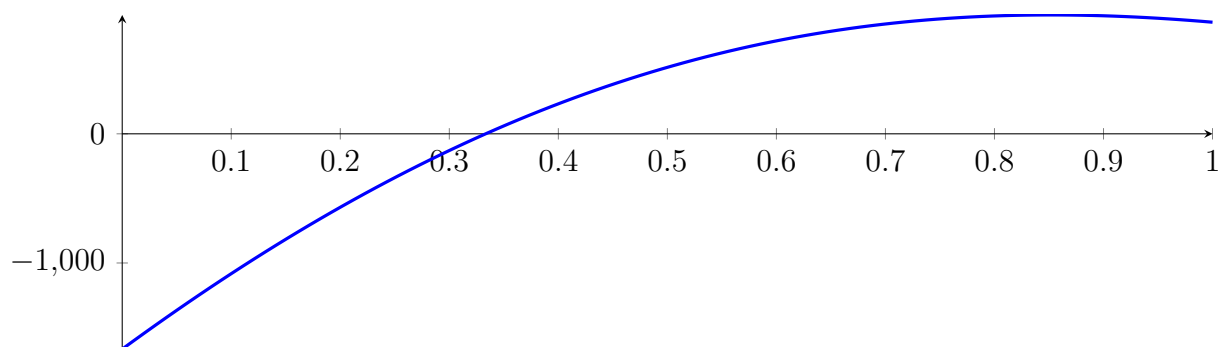
Reached interval [0.333333, 0.333333] **without sign change** at depth 4!

$p(0) = 2.85706e-18$ - $p(1) 3.78276e-17$

146.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

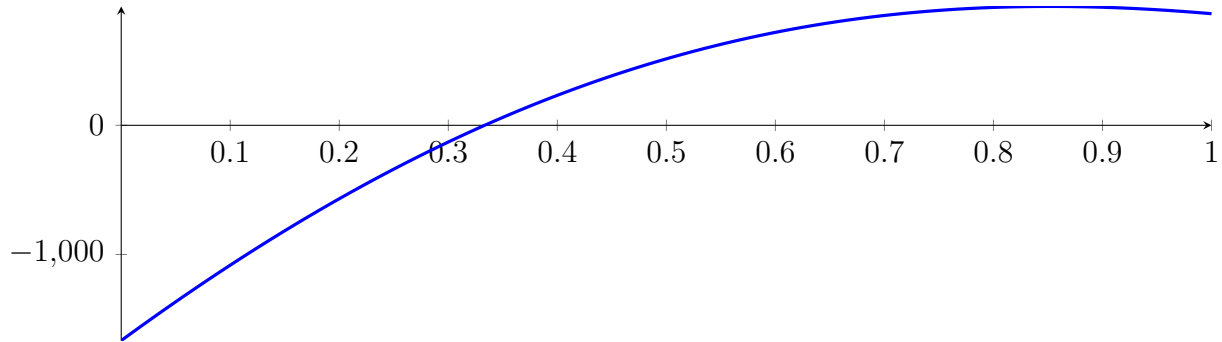
with precision $\varepsilon = 1 \cdot 10^{-128}$.

147 Running CubeClip on f_8 with epsilon 128

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

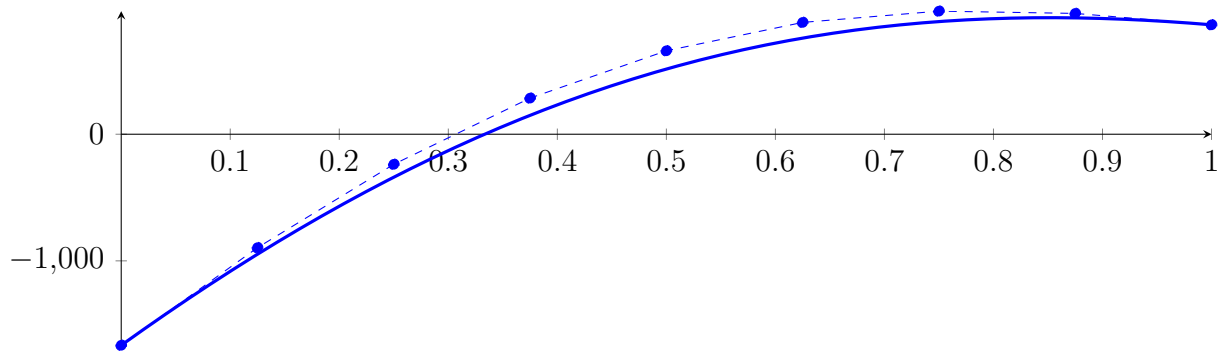
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



147.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

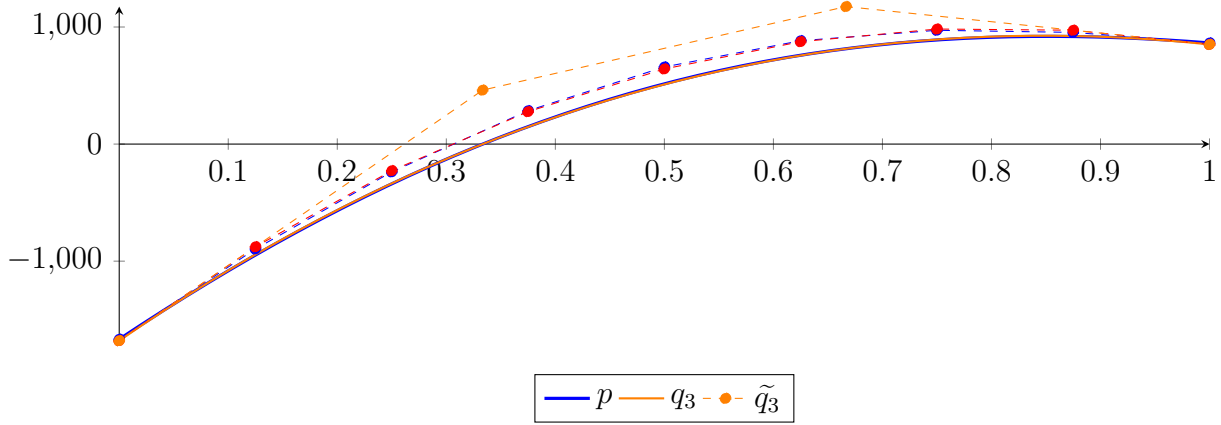
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.39389 \cdot 10^{-09}X^8 + 9.60737 \cdot 10^{-09}X^7 - 1.57477 \cdot 10^{-08}X^6 + 1.35479 \cdot 10^{-08}X^5 \\ &\quad - 6.53397 \cdot 10^{-09}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

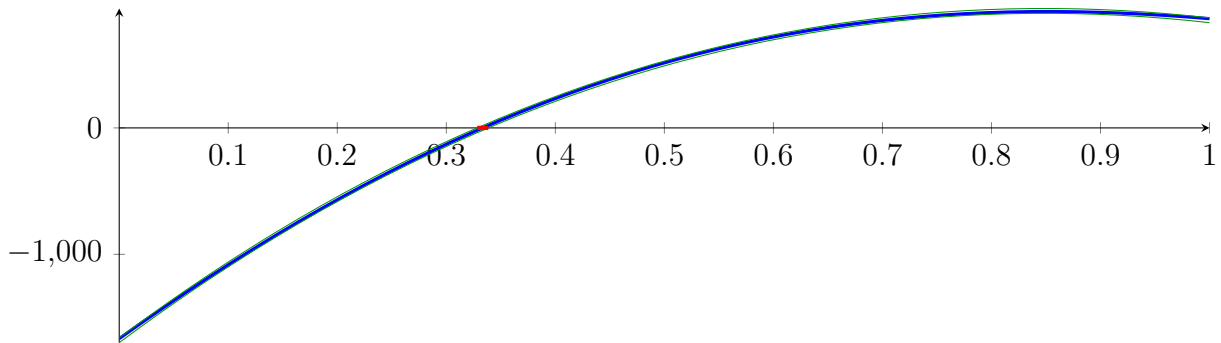
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

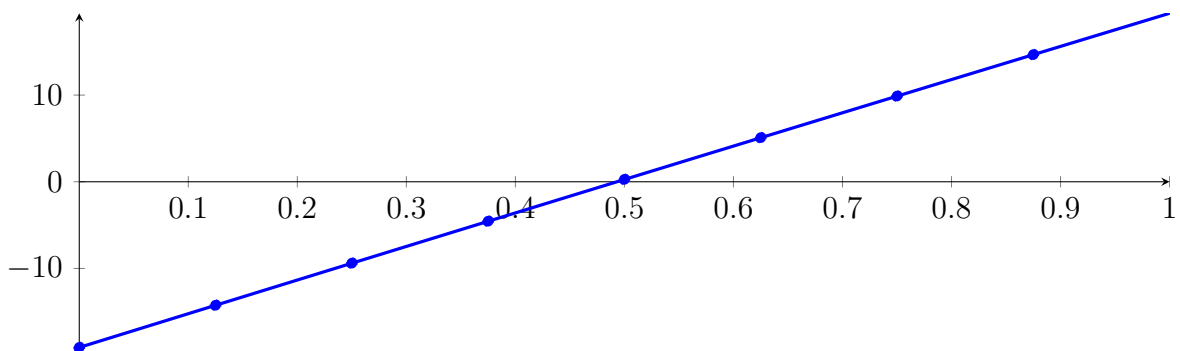
Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

147.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.66533 \cdot 10^{-16} X^8 - 1.99007 \cdot 10^{-13} X^7 - 8.53059 \cdot 10^{-11} X^6 + 8.7284 \cdot 10^{-09} X^5 \\
 &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\
 &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\
 &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

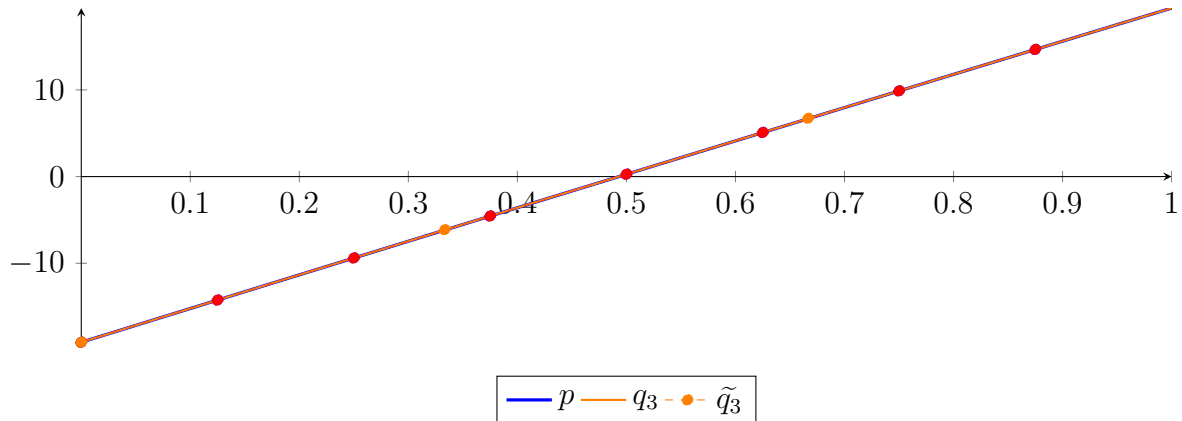
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = 2.82588 \cdot 10^{-12}X^8 - 1.08529 \cdot 10^{-11}X^7 + 1.68738 \cdot 10^{-11}X^6 - 1.34636 \cdot 10^{-11}X^5$$

$$+ 5.72618 \cdot 10^{-12}X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m :

$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

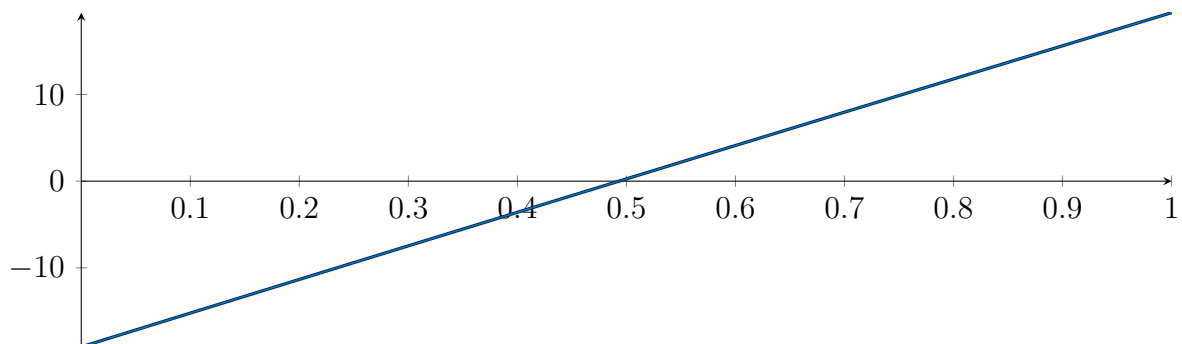
$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\}$$

$$N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

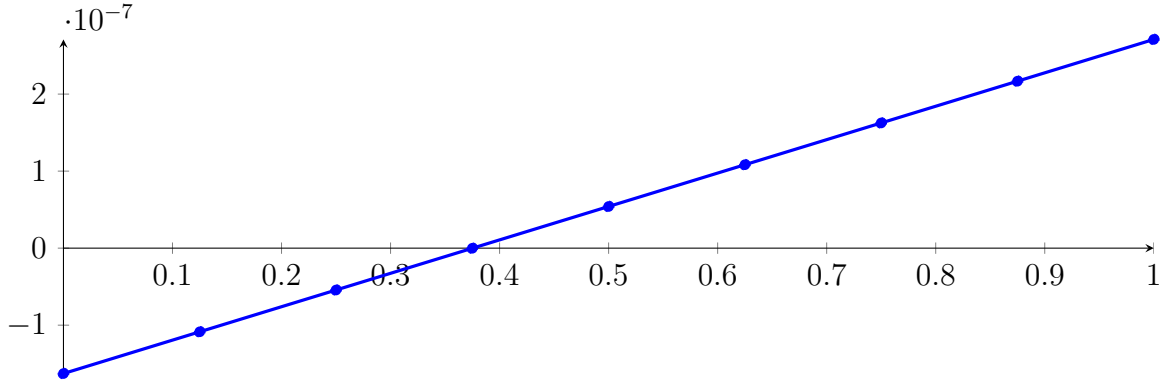
Longest intersection interval: $1.1252 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

147.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

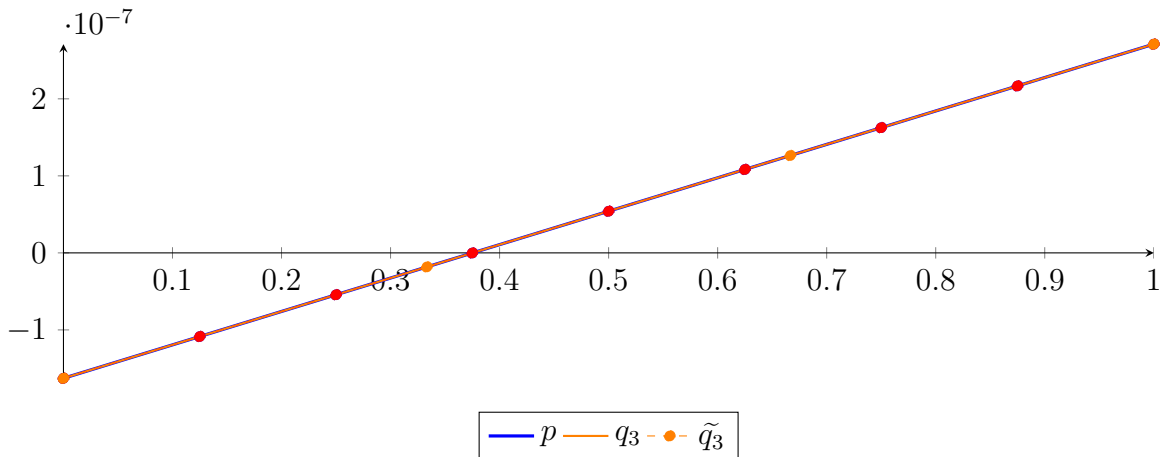
$$\begin{aligned}
 p &= -4.65289 \cdot 10^{-24} X^8 - 8.27181 \cdot 10^{-25} X^7 - 4.3427 \cdot 10^{-24} X^6 + 8.6854 \cdot 10^{-24} X^5 + 1.80946 \\
 &\quad \cdot 10^{-24} X^4 - 7.23783 \cdot 10^{-25} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,8}(X) - 1.08585 \cdot 10^{-07} B_{1,8}(X) - 5.43593 \cdot 10^{-08} B_{2,8}(X) \\
 &\quad - 1.33715 \cdot 10^{-10} B_{3,8}(X) + 5.40918 \cdot 10^{-08} B_{4,8}(X) + 1.08317 \cdot 10^{-07} B_{5,8}(X) \\
 &\quad + 1.62543 \cdot 10^{-07} B_{6,8}(X) + 2.16768 \cdot 10^{-07} B_{7,8}(X) + 2.70994 \cdot 10^{-07} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= 1.42947 \cdot 10^{-23} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,3} - 1.82089 \cdot 10^{-08} B_{1,3} + 1.26393 \cdot 10^{-07} B_{2,3} + 2.70994 \cdot 10^{-07} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= -2.97639 \cdot 10^{-20} X^8 + 1.26259 \cdot 10^{-19} X^7 - 2.19172 \cdot 10^{-19} X^6 + 2.00419 \cdot 10^{-19} X^5 - 1.02758 \\
 &\quad \cdot 10^{-19} X^4 + 2.83318 \cdot 10^{-20} X^3 - 5.27552 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,8} - 1.08585 \cdot 10^{-07} B_{1,8} - 5.43593 \cdot 10^{-08} B_{2,8} - 1.33715 \cdot 10^{-10} B_{3,8} + 5.40918 \\
 &\quad \cdot 10^{-08} B_{4,8} + 1.08317 \cdot 10^{-07} B_{5,8} + 1.62543 \cdot 10^{-07} B_{6,8} + 2.16768 \cdot 10^{-07} B_{7,8} + 2.70994 \cdot 10^{-07} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.97535 \cdot 10^{-22}$.

Bounding polynomials M and m :

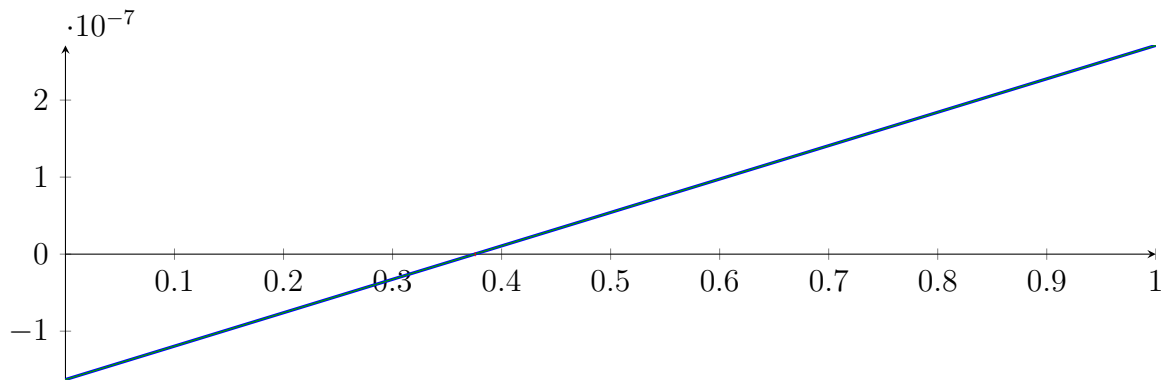
$$\begin{aligned}
 M &= 1.42689 \cdot 10^{-23} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 m &= 1.43206 \cdot 10^{-23} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.375308\}$$

$$N(m) = \{0.375308\}$$

Intersection intervals:



[0.375308, 0.375308]

Longest intersection interval: $1.36424 \cdot 10^{-12}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

147.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

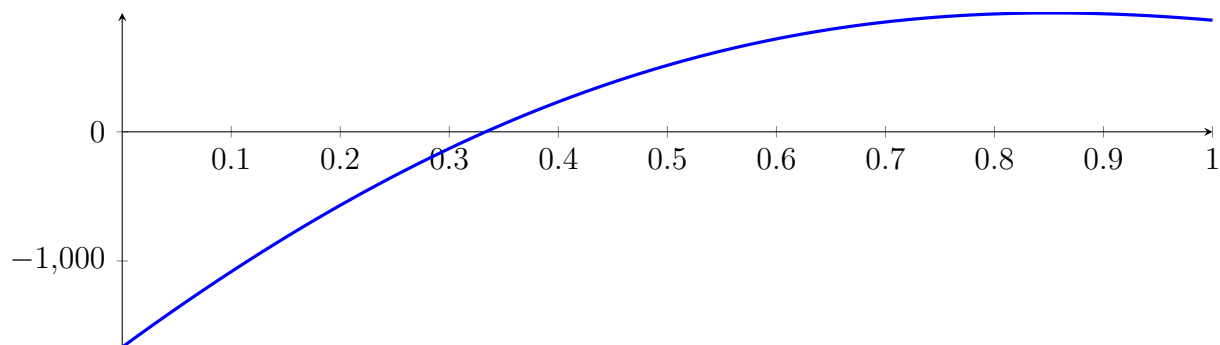
Reached interval [0.333333, 0.333333] **without sign change** at depth 4!

$p(0) = -1.10673e-18$ - $p(1) -5.14919e-19$

147.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

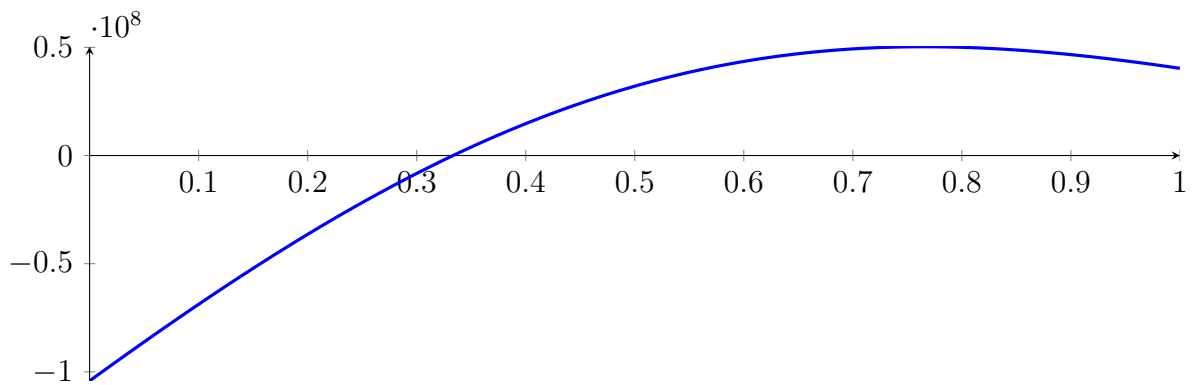
with precision $\varepsilon = 1 \cdot 10^{-128}$.

148 Running BezClip on f_{16} with epsilon 2

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

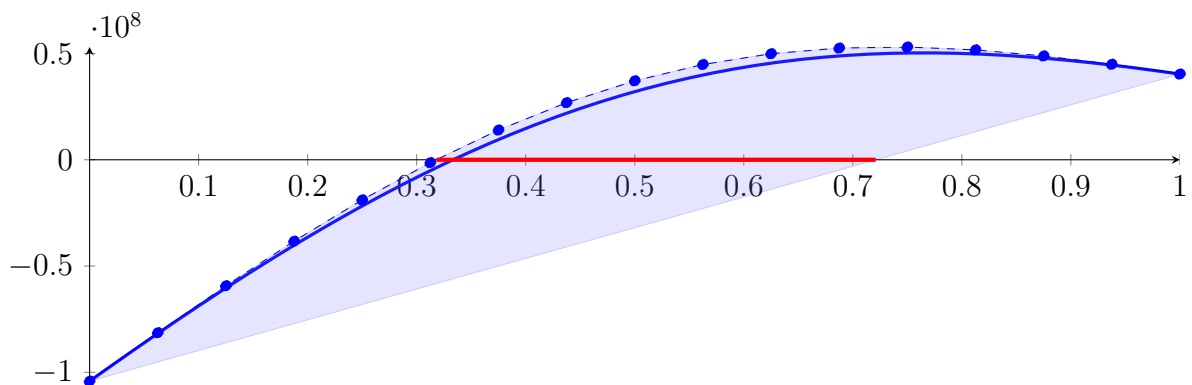
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



148.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

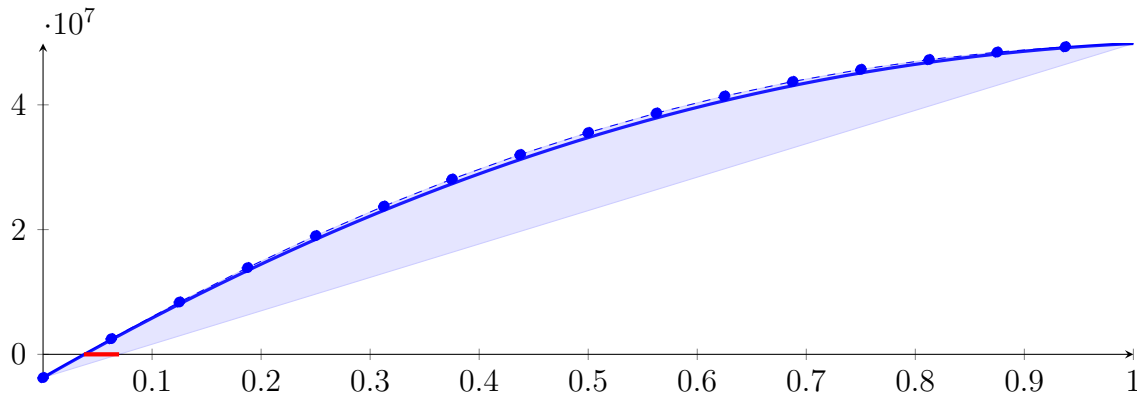
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

148.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.59825 \cdot 10^{-06} X^{16} - 5.93153 \cdot 10^{-05} X^{15} - 0.00248867 X^{14} - 0.0650056 X^{13} - 0.909142 X^{12} \\ &\quad - 5.03931 X^{11} + 36.4549 X^{10} + 692.921 X^9 + 1886.97 X^8 - 25792 X^7 - 149671 X^6 + 492605 X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

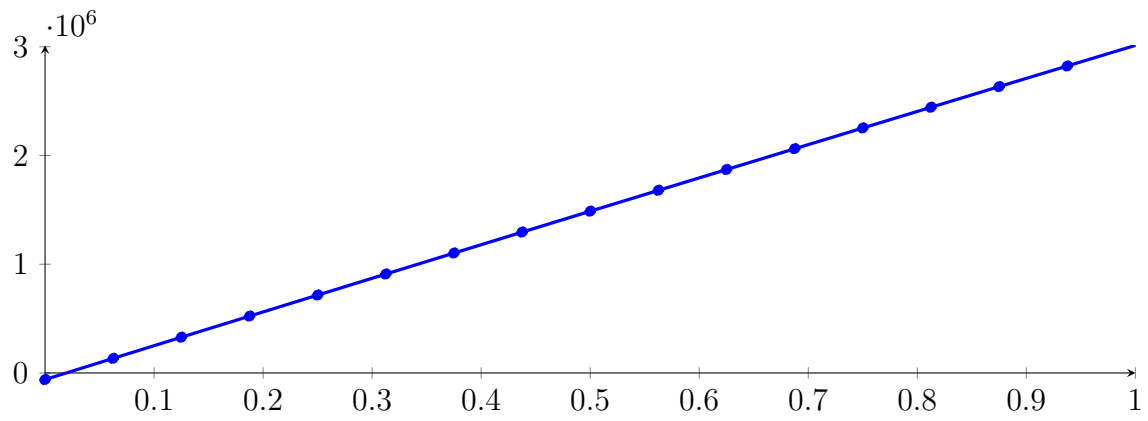
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

148.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 9.01396 \cdot 10^{-08} X^{16} - 2.65848 \cdot 10^{-07} X^{15} + 2.13948 \cdot 10^{-06} X^{14} - 1.33627 \cdot 10^{-06} X^{13} + 2.46973 \cdot 10^{-06} X^{12} \\ &\quad - 2.45524 \cdot 10^{-06} X^{11} + 5.50112 \cdot 10^{-07} X^{10} - 1.64198 \cdot 10^{-07} X^9 - 7.35598 \cdot 10^{-07} X^8 - 1.00892 \cdot 10^{-06} X^7 \\ &\quad - 0.000177509 X^6 + 0.0161038 X^5 + 4.36155 X^4 - 234.216 X^3 - 45622.2 X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554

\implies Selective recursion: interval 1: $[0.333333, 0.333337]$,

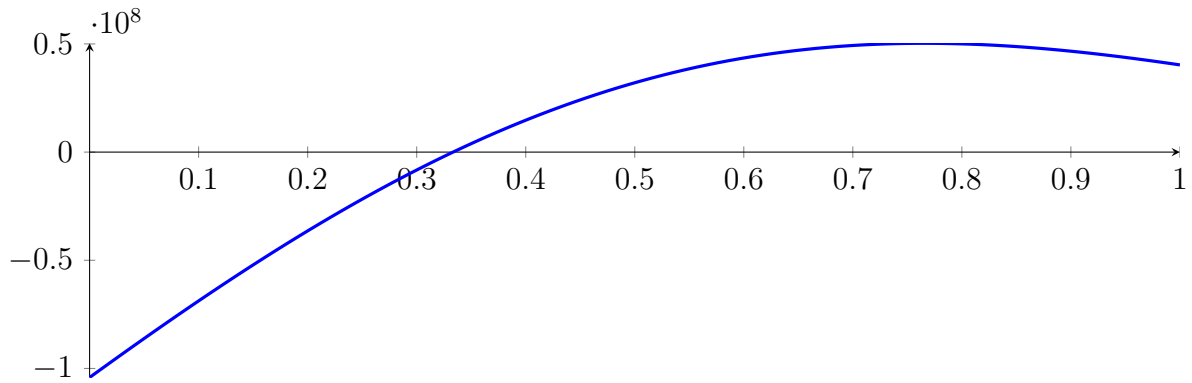
148.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Found root in interval $[0.333333, 0.333337]$ at recursion depth 4!

148.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333337]$$

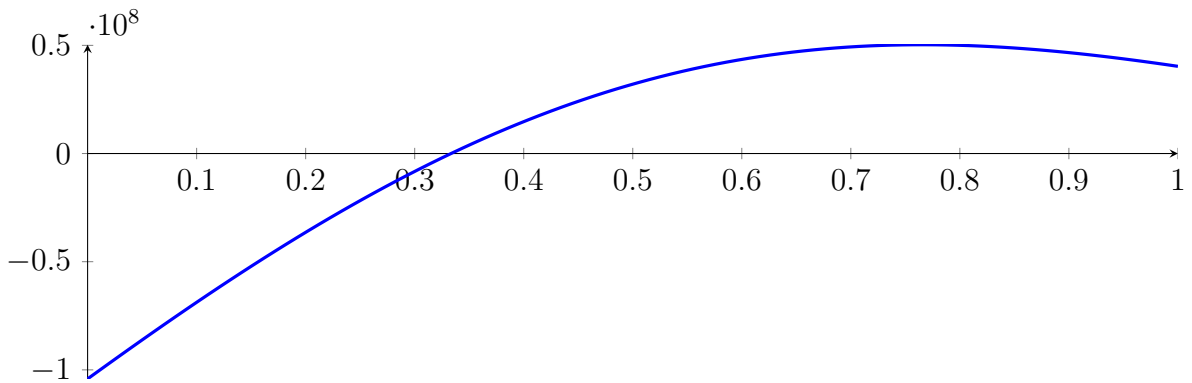
with precision $\varepsilon = 0.01$.

149 Running QuadClip on f_{16} with epsilon 2

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

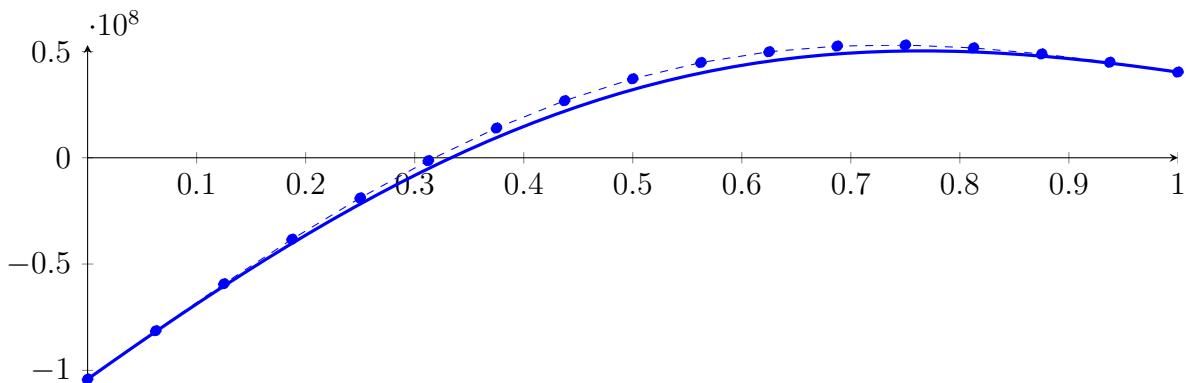
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



149.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = 6049.18X^{16} - 48305.2X^{15} + 174971X^{14} - 380294X^{13} + 552846X^{12} - 567203X^{11}$$

$$+ 422303X^{10} - 231038X^9 + 93003.6X^8 - 27320.1X^7 + 5752.57X^6 - 843.63X^5$$

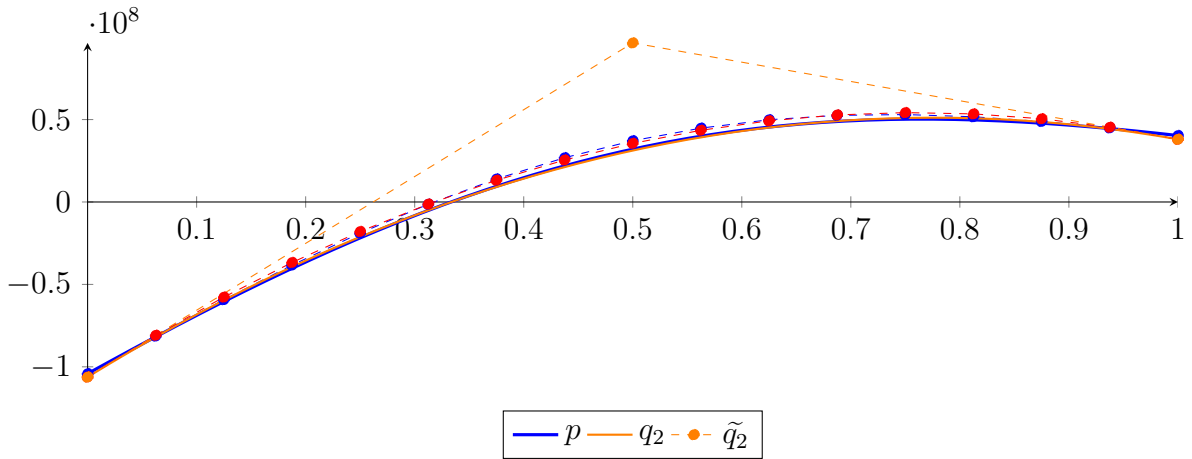
$$+ 82.5145X^4 - 5.01388X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

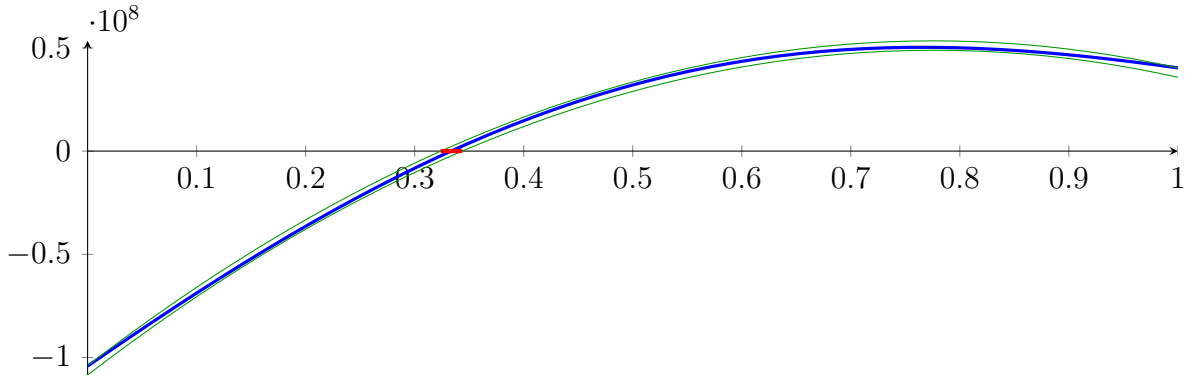
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

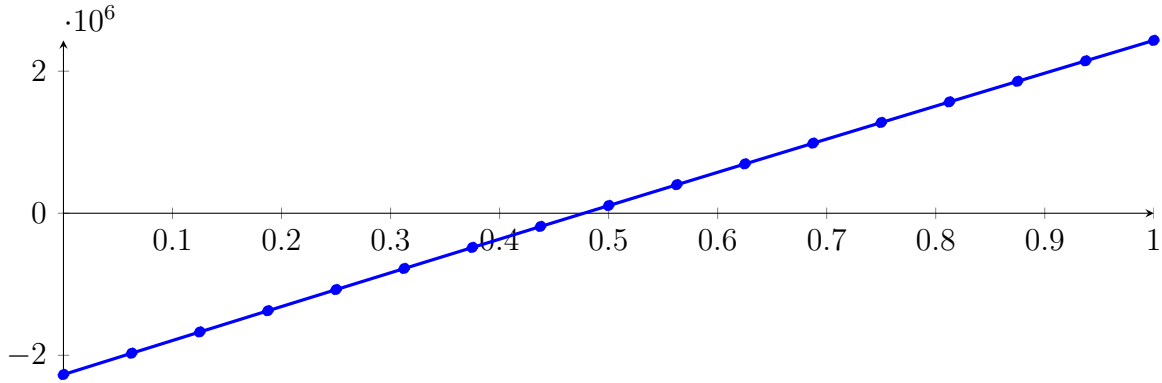
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

149.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

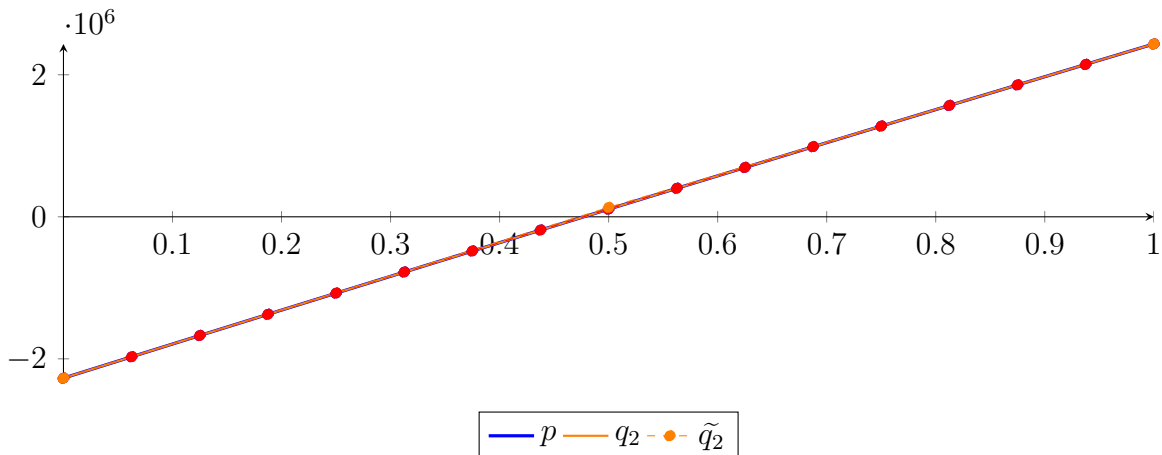
$$\begin{aligned}
 p &= -1.54841 \cdot 10^{-10} X^{16} - 1.66965 \cdot 10^{-07} X^{15} - 2.92739 \cdot 10^{-07} X^{14} - 1.77943 \cdot 10^{-06} X^{13} - 1.17235 \cdot 10^{-06} X^{12} \\
 &\quad - 2.42234 \cdot 10^{-06} X^{11} - 6.86445 \cdot 10^{-07} X^{10} - 1.39162 \cdot 10^{-06} X^9 + 1.07395 \cdot 10^{-06} X^8 - 1.67072 \cdot 10^{-05} X^7 \\
 &\quad - 0.00205879 X^6 + 0.132721 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 22.7036 X^{16} - 178.964 X^{15} + 638.976 X^{14} - 1366.64 X^{13} + 1951.1 X^{12} - 1960.92 X^{11} \\
 &\quad + 1425.35 X^{10} - 757.631 X^9 + 294.177 X^8 - 82.4368 X^7 + 16.2856 X^6 - 2.18949 X^5 \\
 &\quad + 0.191237 X^4 - 0.0101048 X^3 - 104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} \\
 &\quad + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

$$M = -104265 X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

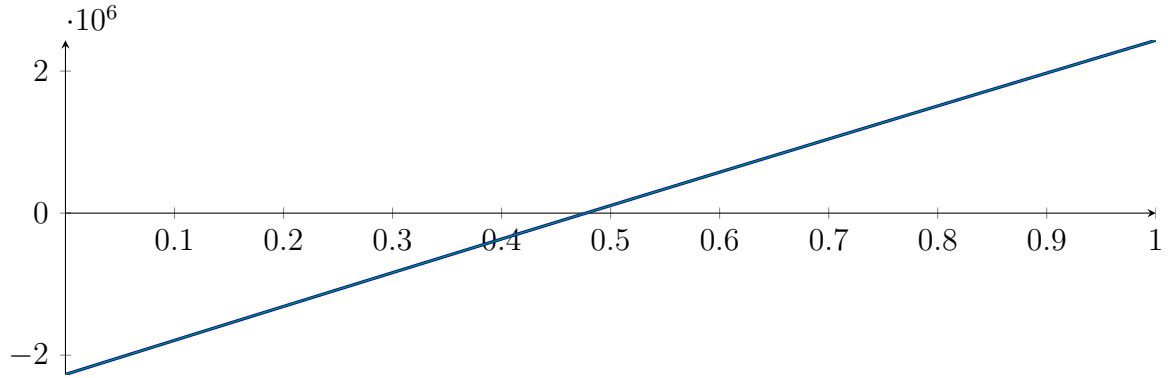
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\}$$

$$N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

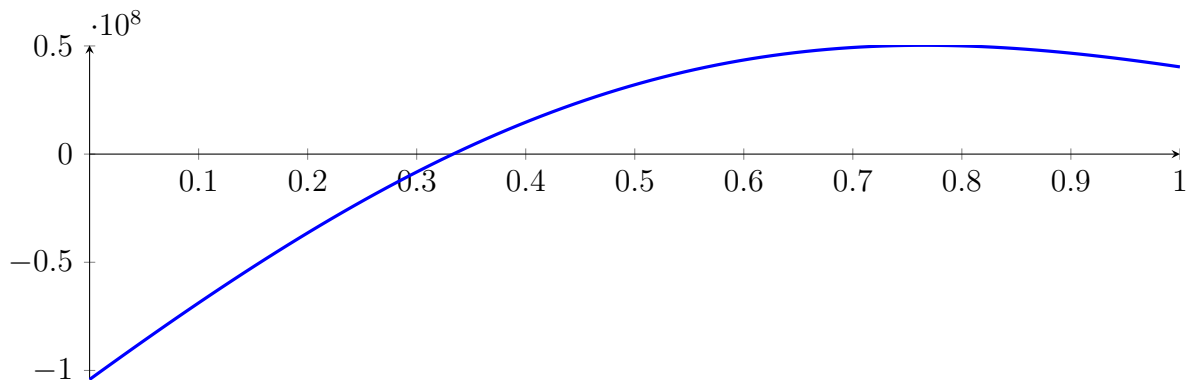
149.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

149.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

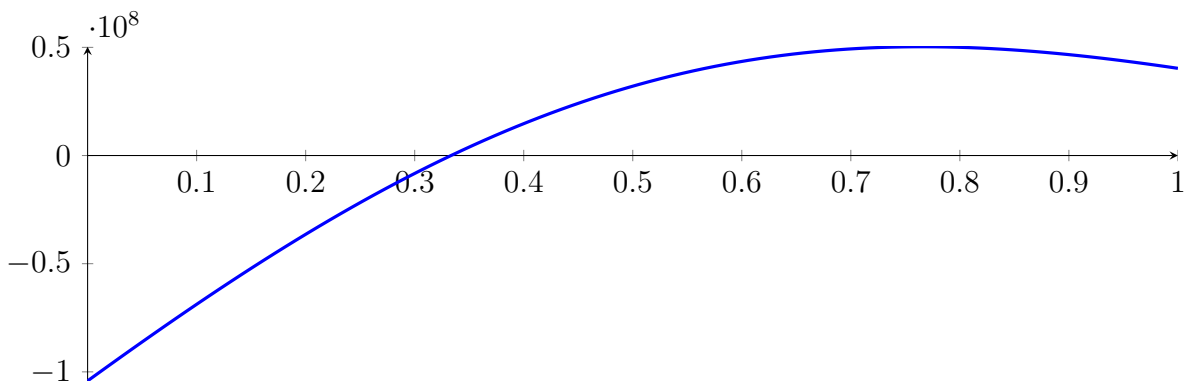
with precision $\varepsilon = 0.01$.

150 Running CubeClip on f_{16} with epsilon 2

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

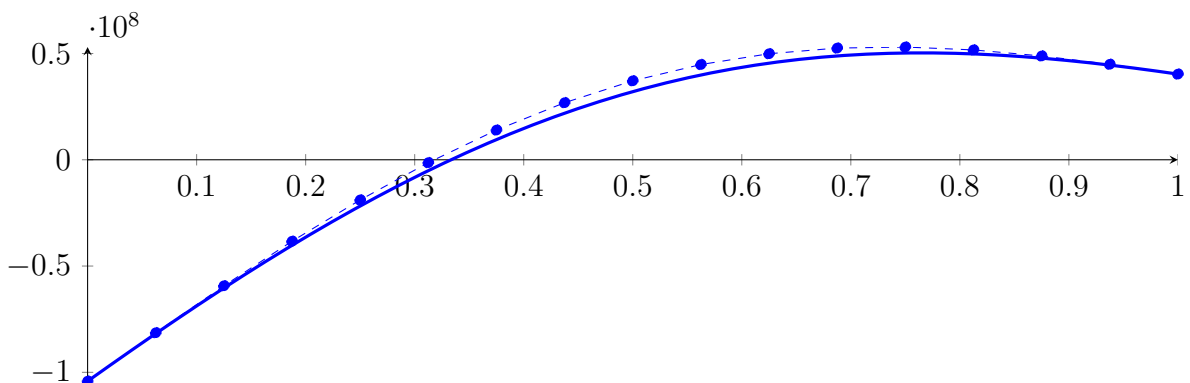
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



150.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 2461.93X^{16} - 19614.9X^{15} + 70879.5X^{14} - 153661X^{13} + 222746X^{12} - 227755X^{11}$$

$$+ 168826X^{10} - 91798.7X^9 + 36630.3X^8 - 10627.3X^7 + 2200.54X^6 - 316.059X^5$$

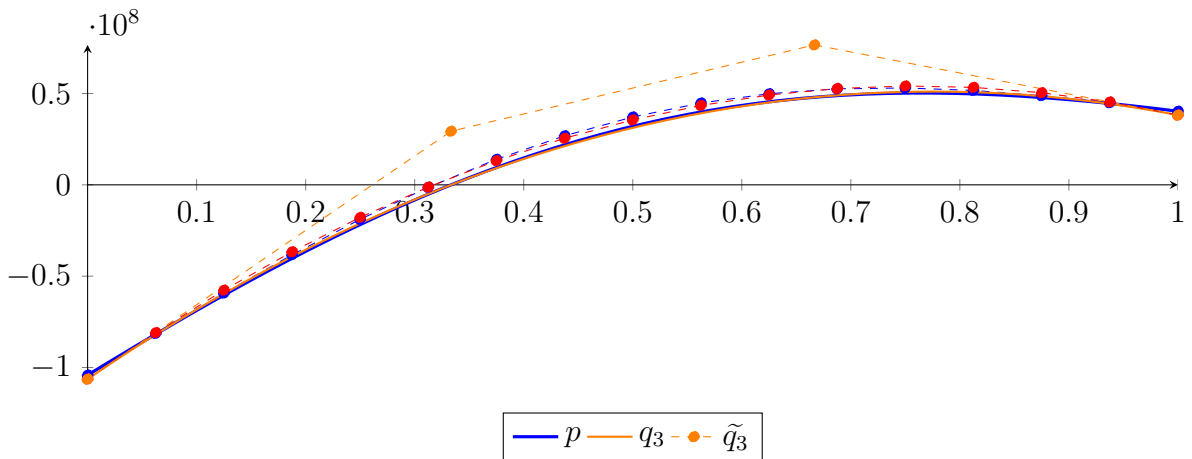
$$+ 30.1958X^4 + 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18551 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91815 \cdot 10^7 B_{10,16} + 5.27353 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

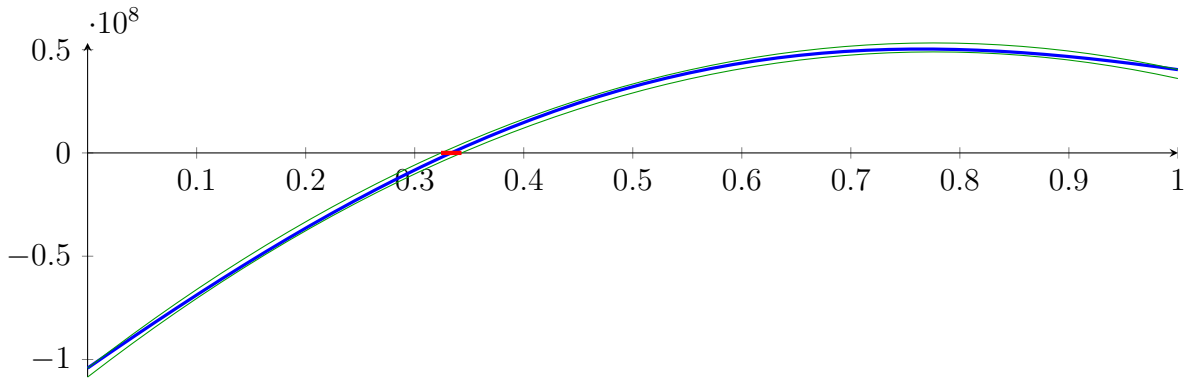
$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\} \quad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

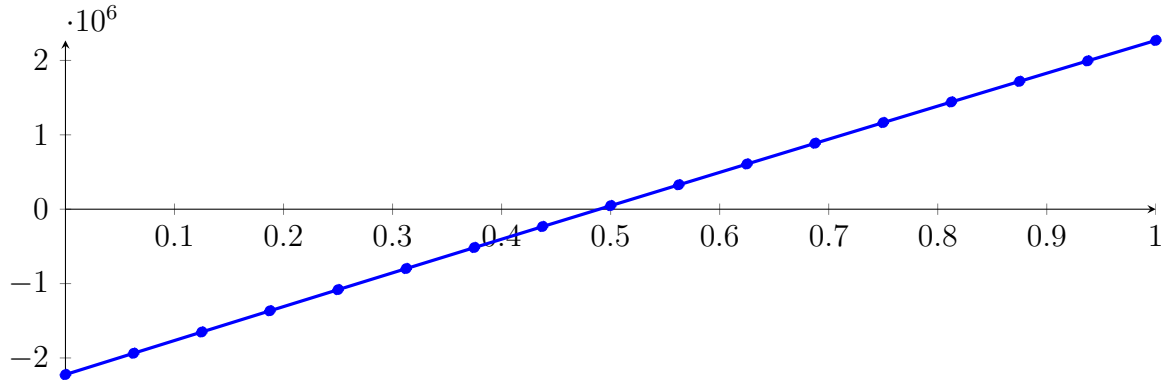
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

150.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

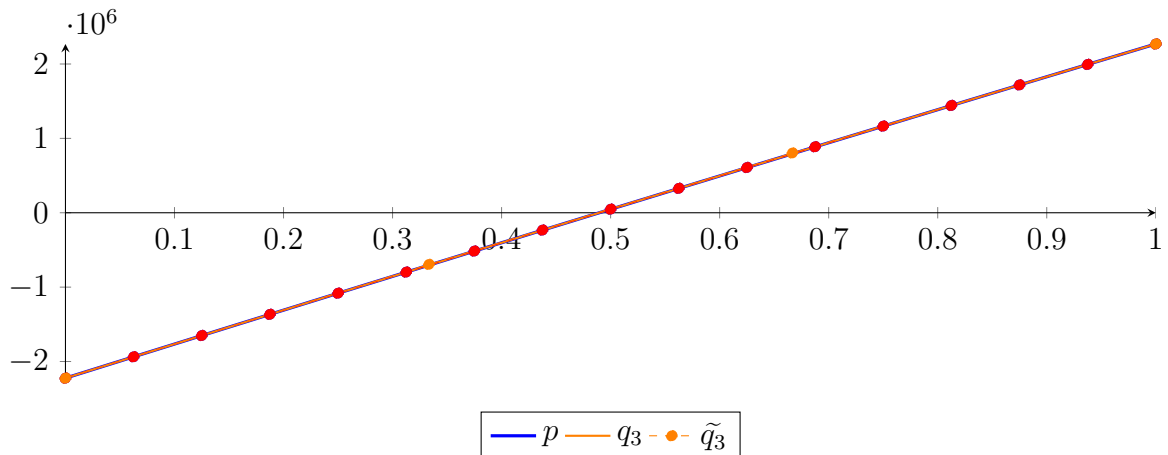
$$\begin{aligned}
 p &= -3.66617 \cdot 10^{-09} X^{16} - 1.53217 \cdot 10^{-07} X^{15} - 3.62234 \cdot 10^{-07} X^{14} - 1.65579 \cdot 10^{-06} X^{13} - 1.15373 \cdot 10^{-06} X^{12} \\
 &\quad - 2.3399 \cdot 10^{-06} X^{11} - 5.02543 \cdot 10^{-07} X^{10} - 1.38381 \cdot 10^{-06} X^9 + 1.1237 \cdot 10^{-06} X^8 - 1.19653 \cdot 10^{-05} X^7 \\
 &\quad - 0.00155608 X^6 + 0.10496 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 16.4956 X^{16} - 129.161 X^{15} + 457.83 X^{14} - 971.671 X^{13} + 1375.95 X^{12} - 1370.96 X^{11} \\
 &\quad + 987.265 X^{10} - 519.476 X^9 + 199.587 X^8 - 55.434 X^7 + 10.9237 X^6 - 1.48019 X^5 \\
 &\quad + 0.129516 X^4 - 700.679 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} \\
 &\quad + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m :

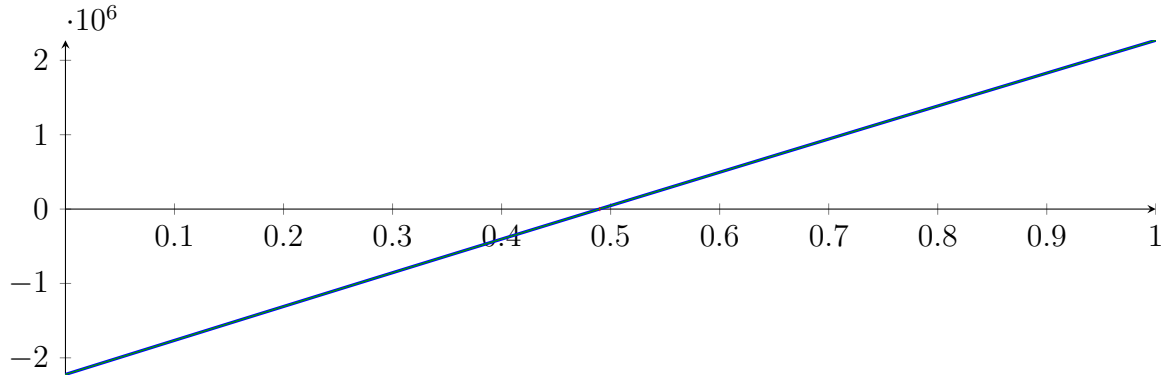
$$M = -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\} \quad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174 \cdot 10^{-07}$

\implies Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

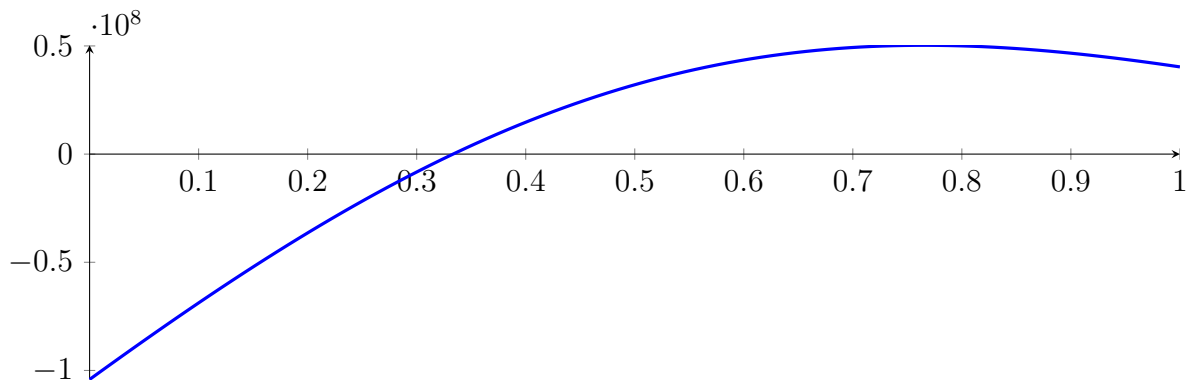
150.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 3!

150.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

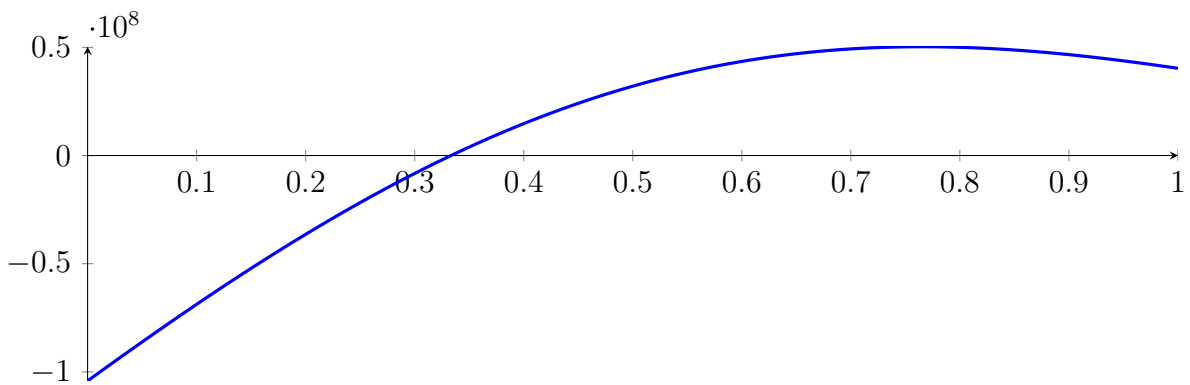
with precision $\varepsilon = 0.01$.

151 Running BezClip on f_{16} with epsilon 4

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

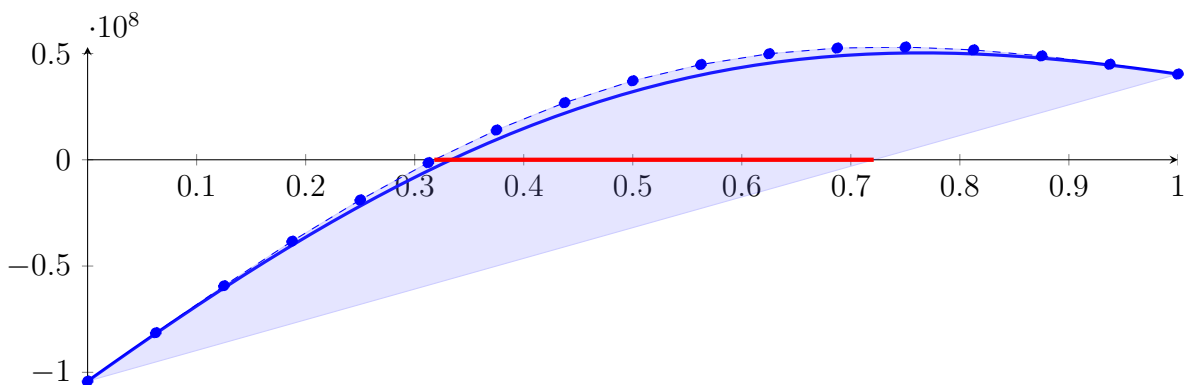
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



151.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

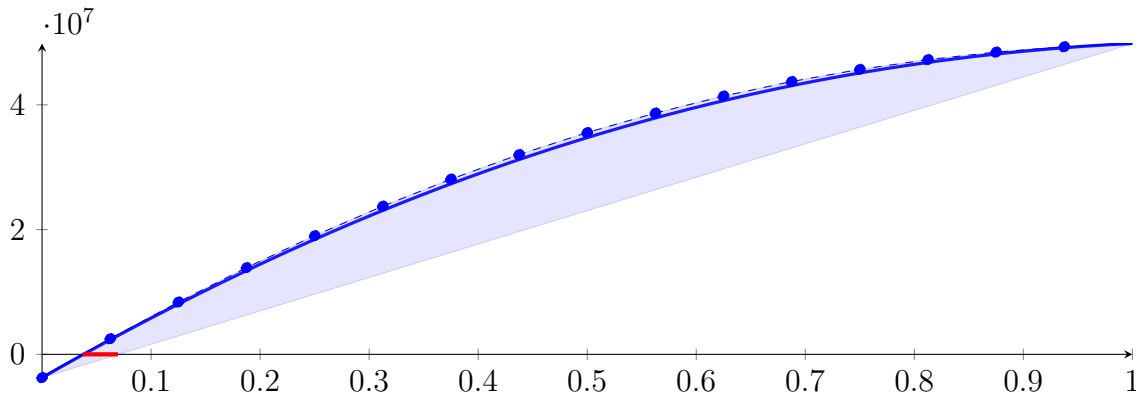
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

151.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.59825 \cdot 10^{-06} X^{16} - 5.93153 \cdot 10^{-05} X^{15} - 0.00248867 X^{14} - 0.0650056 X^{13} - 0.909142 X^{12} \\ &\quad - 5.03931 X^{11} + 36.4549 X^{10} + 692.921 X^9 + 1886.97 X^8 - 25792 X^7 - 149671 X^6 + 492605 X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

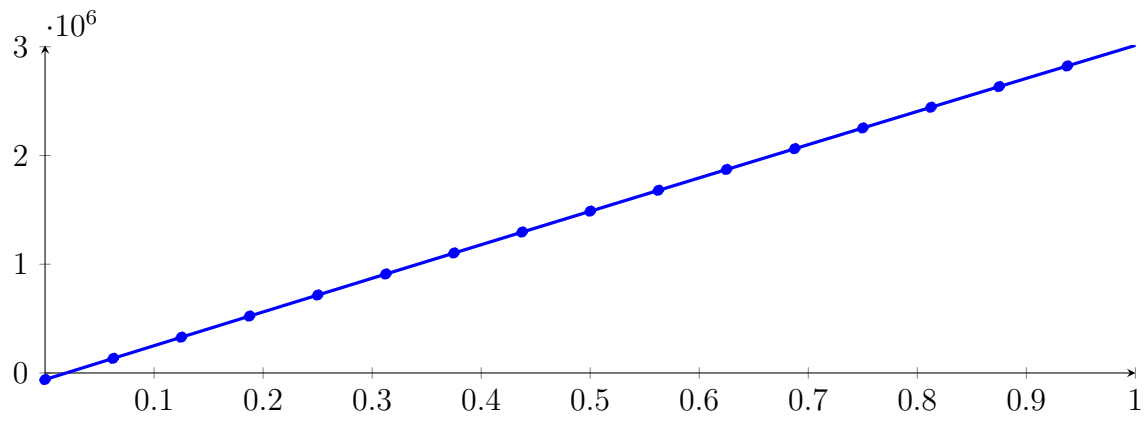
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

151.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 9.01396 \cdot 10^{-08} X^{16} - 2.65848 \cdot 10^{-07} X^{15} + 2.13948 \cdot 10^{-06} X^{14} - 1.33627 \cdot 10^{-06} X^{13} + 2.46973 \cdot 10^{-06} X^{12} \\ &\quad - 2.45524 \cdot 10^{-06} X^{11} + 5.50112 \cdot 10^{-07} X^{10} - 1.64198 \cdot 10^{-07} X^9 - 7.35598 \cdot 10^{-07} X^8 - 1.00892 \cdot 10^{-06} X^7 \\ &\quad - 0.000177509 X^6 + 0.0161038 X^5 + 4.36155 X^4 - 234.216 X^3 - 45622.2 X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554

\implies Selective recursion: interval 1: $[0.333333, 0.333337]$,

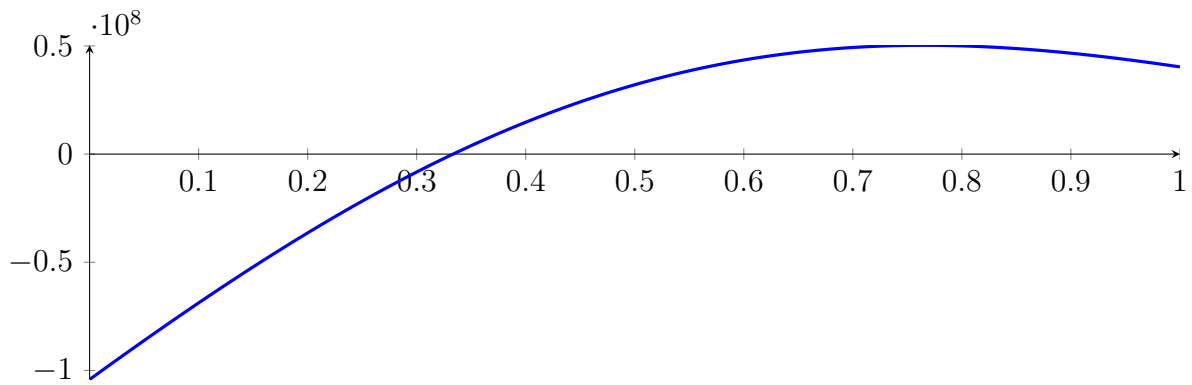
151.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Found root in interval $[0.333333, 0.333337]$ at recursion depth 4!

151.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333337]$$

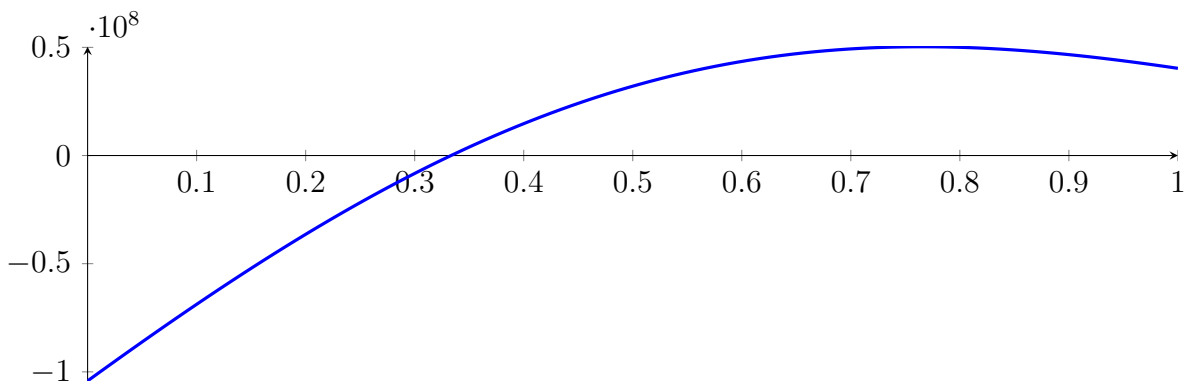
with precision $\varepsilon = 0.0001$.

152 Running QuadClip on f_{16} with epsilon 4

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

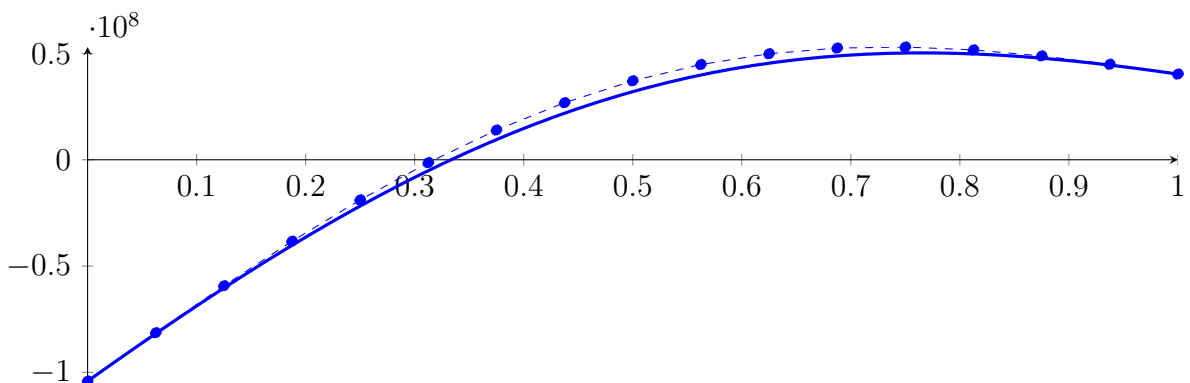
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



152.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = 6049.18X^{16} - 48305.2X^{15} + 174971X^{14} - 380294X^{13} + 552846X^{12} - 567203X^{11}$$

$$+ 422303X^{10} - 231038X^9 + 93003.6X^8 - 27320.1X^7 + 5752.57X^6 - 843.63X^5$$

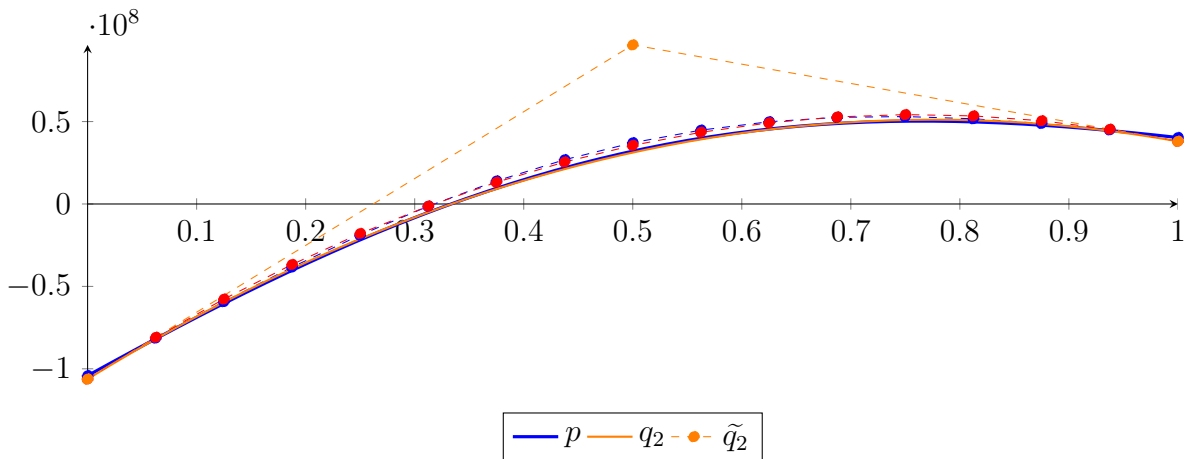
$$+ 82.5145X^4 - 5.01388X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

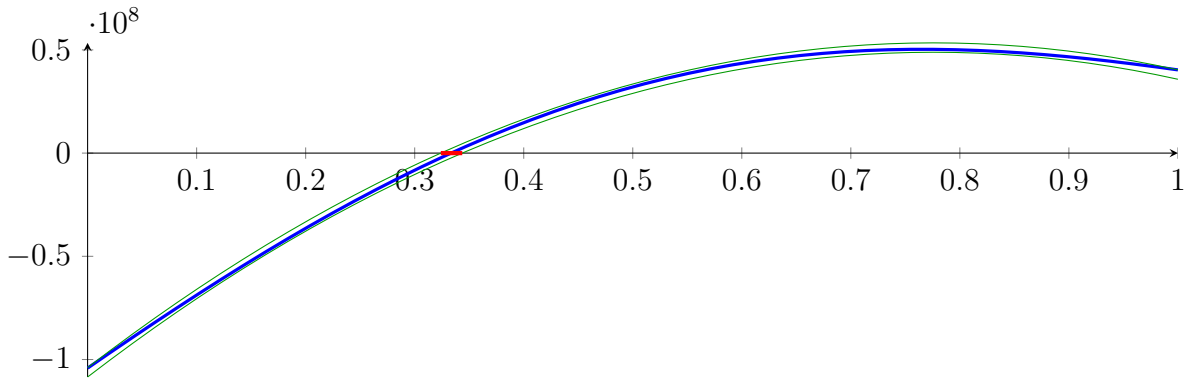
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

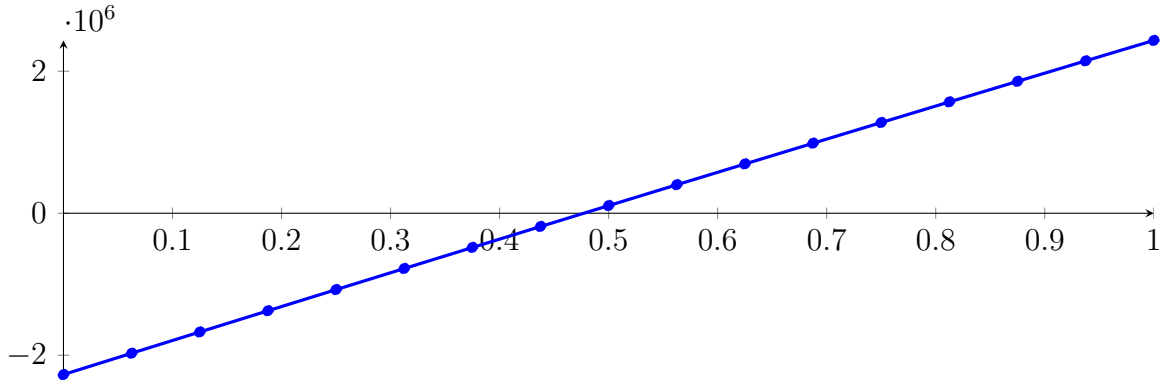
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

152.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

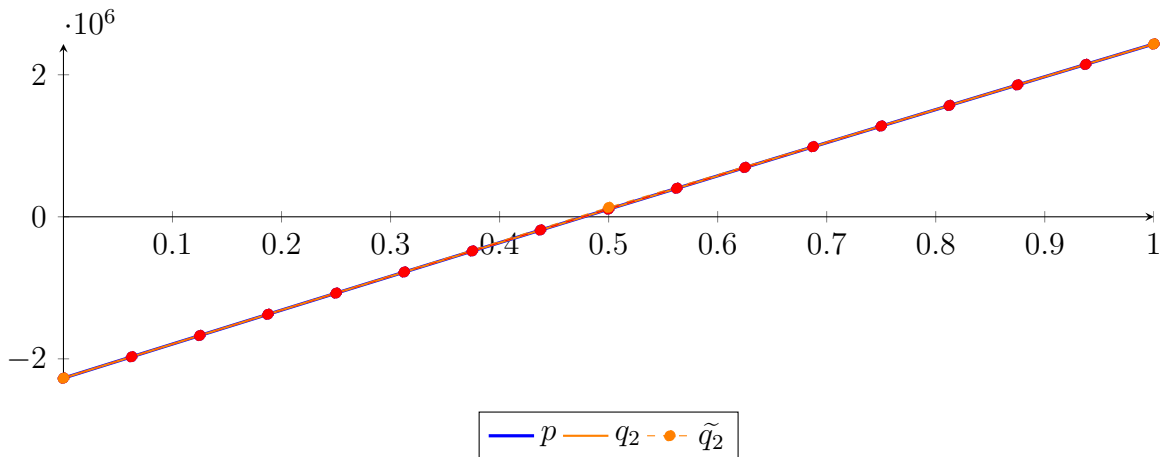
$$\begin{aligned}
 p &= -1.54841 \cdot 10^{-10} X^{16} - 1.66965 \cdot 10^{-07} X^{15} - 2.92739 \cdot 10^{-07} X^{14} - 1.77943 \cdot 10^{-06} X^{13} - 1.17235 \cdot 10^{-06} X^{12} \\
 &\quad - 2.42234 \cdot 10^{-06} X^{11} - 6.86445 \cdot 10^{-07} X^{10} - 1.39162 \cdot 10^{-06} X^9 + 1.07395 \cdot 10^{-06} X^8 - 1.67072 \cdot 10^{-05} X^7 \\
 &\quad - 0.00205879 X^6 + 0.132721 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 22.7036 X^{16} - 178.964 X^{15} + 638.976 X^{14} - 1366.64 X^{13} + 1951.1 X^{12} - 1960.92 X^{11} \\
 &\quad + 1425.35 X^{10} - 757.631 X^9 + 294.177 X^8 - 82.4368 X^7 + 16.2856 X^6 - 2.18949 X^5 \\
 &\quad + 0.191237 X^4 - 0.0101048 X^3 - 104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} \\
 &\quad + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

$$M = -104265 X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

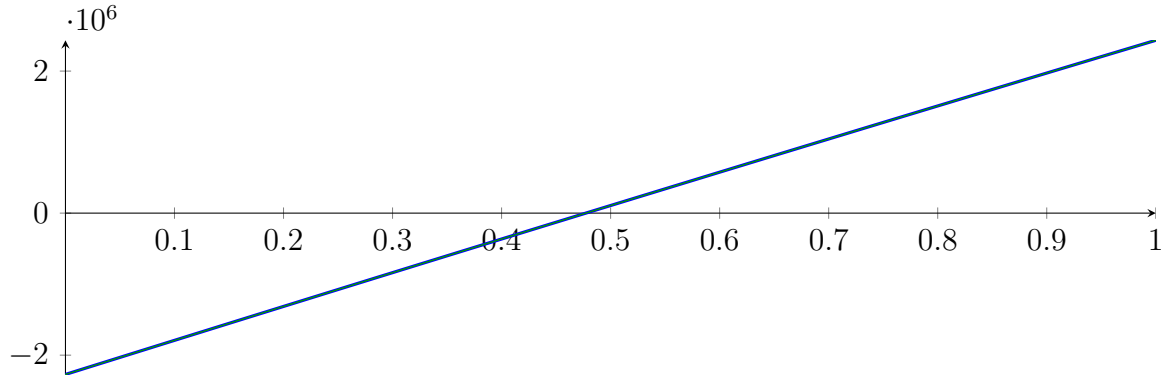
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\}$$

$$N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301 \cdot 10^{-05}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

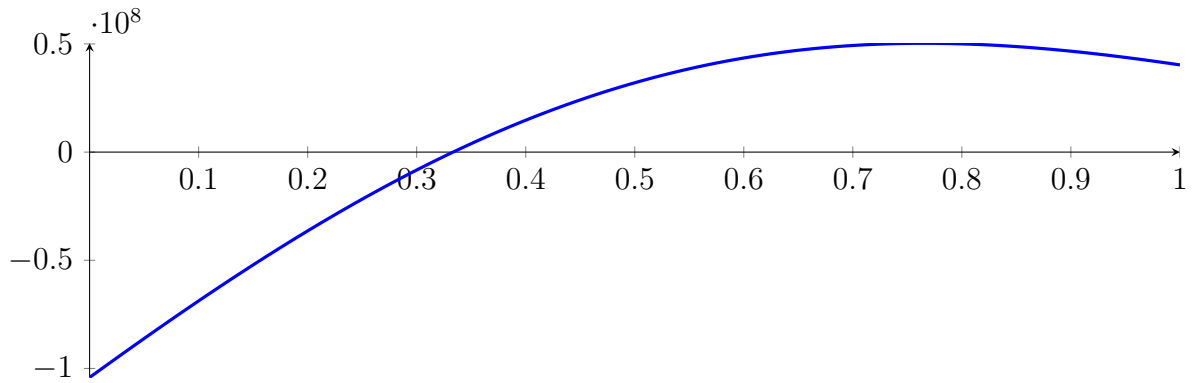
152.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

152.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

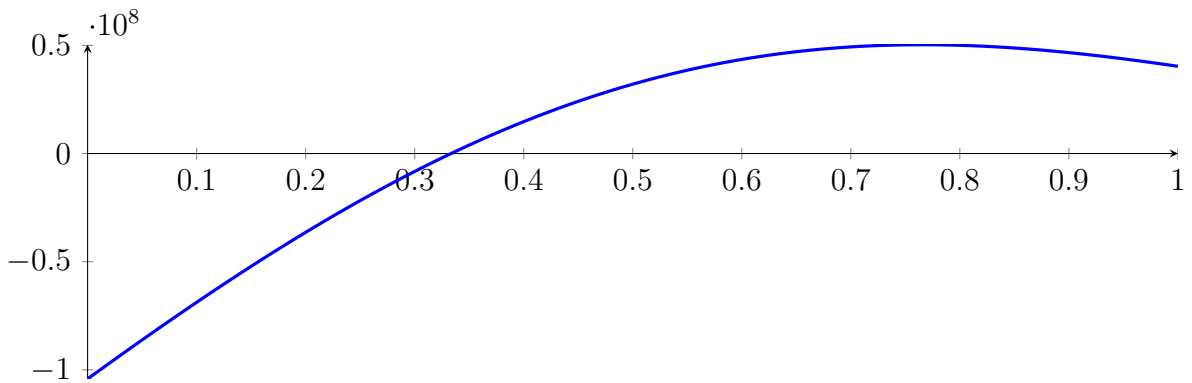
with precision $\varepsilon = 0.0001$.

153 Running CubeClip on f_{16} with epsilon 4

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

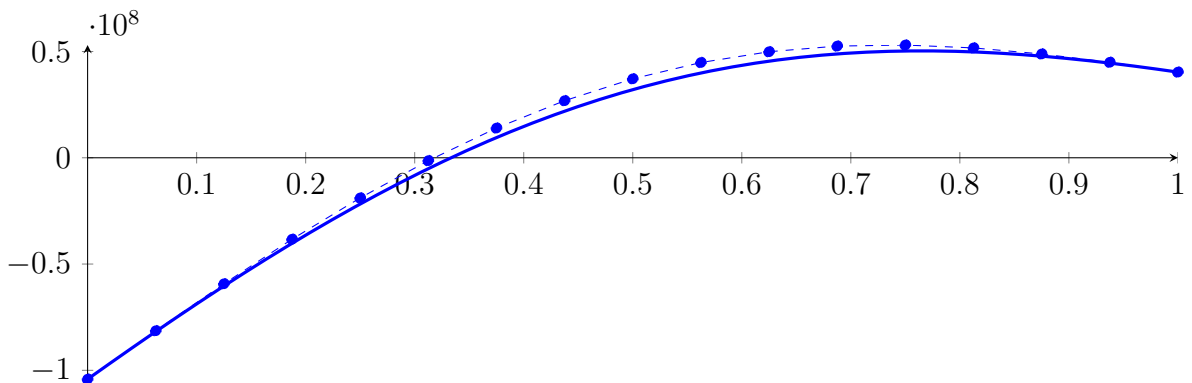
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



153.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 2461.93X^{16} - 19614.9X^{15} + 70879.5X^{14} - 153661X^{13} + 222746X^{12} - 227755X^{11}$$

$$+ 168826X^{10} - 91798.7X^9 + 36630.3X^8 - 10627.3X^7 + 2200.54X^6 - 316.059X^5$$

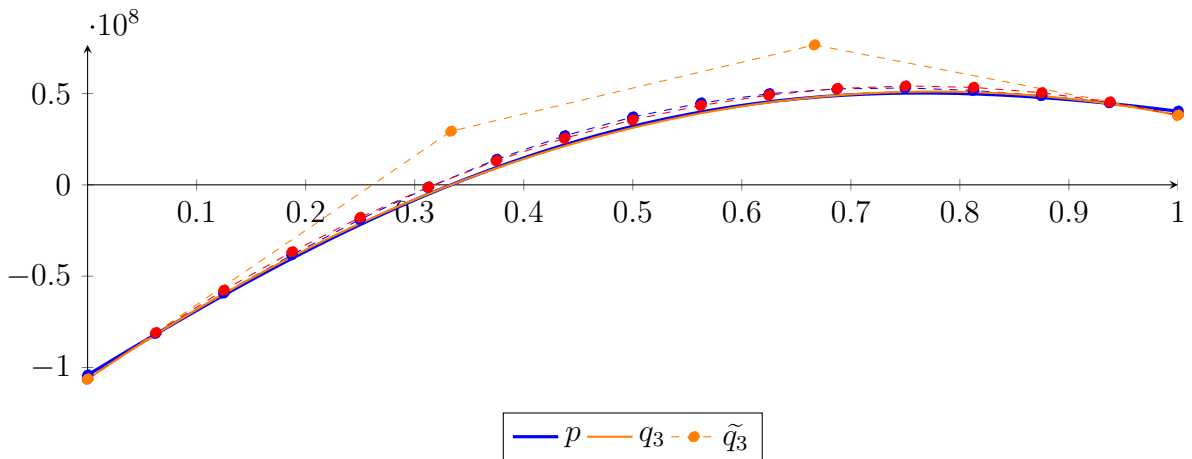
$$+ 30.1958X^4 + 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18551 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91815 \cdot 10^7 B_{10,16} + 5.27353 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

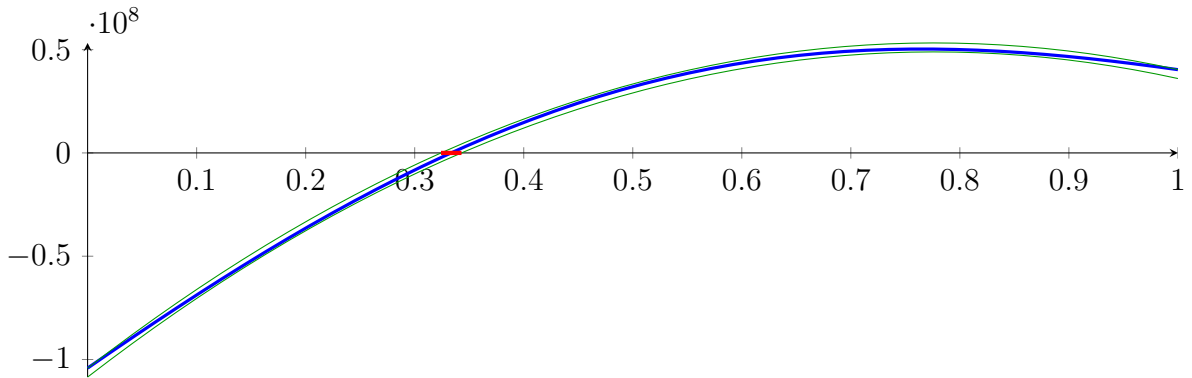
$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\}$$

$$N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

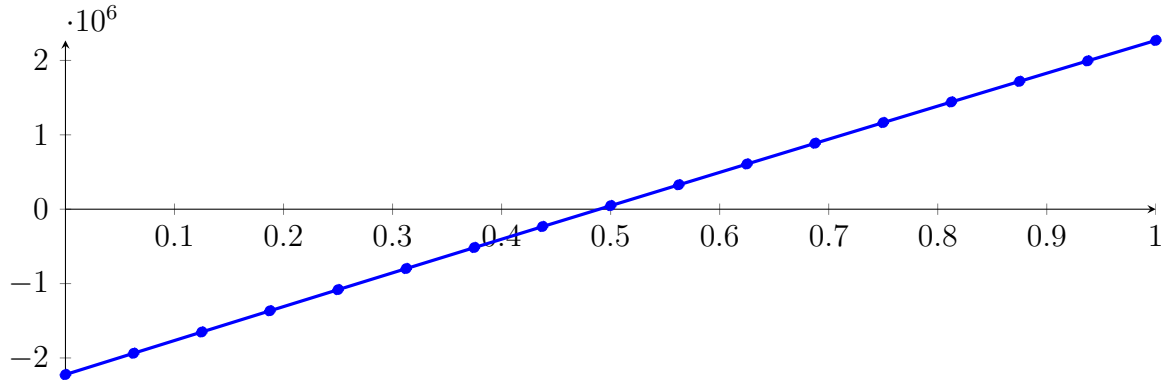
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

153.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

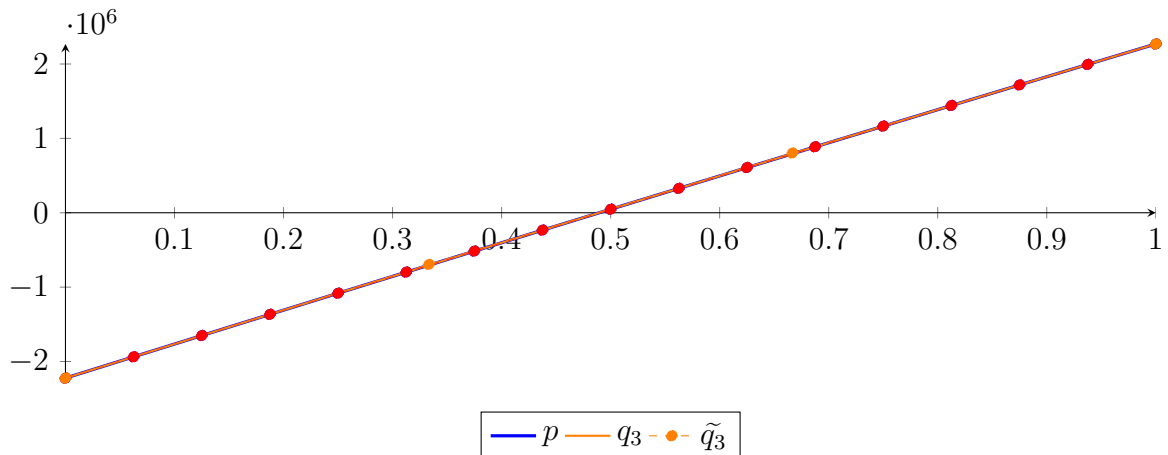
$$\begin{aligned}
 p &= -3.66617 \cdot 10^{-09} X^{16} - 1.53217 \cdot 10^{-07} X^{15} - 3.62234 \cdot 10^{-07} X^{14} - 1.65579 \cdot 10^{-06} X^{13} - 1.15373 \cdot 10^{-06} X^{12} \\
 &\quad - 2.3399 \cdot 10^{-06} X^{11} - 5.02543 \cdot 10^{-07} X^{10} - 1.38381 \cdot 10^{-06} X^9 + 1.1237 \cdot 10^{-06} X^8 - 1.19653 \cdot 10^{-05} X^7 \\
 &\quad - 0.00155608 X^6 + 0.10496 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 16.4956 X^{16} - 129.161 X^{15} + 457.83 X^{14} - 971.671 X^{13} + 1375.95 X^{12} - 1370.96 X^{11} \\
 &\quad + 987.265 X^{10} - 519.476 X^9 + 199.587 X^8 - 55.434 X^7 + 10.9237 X^6 - 1.48019 X^5 \\
 &\quad + 0.129516 X^4 - 700.679 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} \\
 &\quad + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m :

$$M = -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

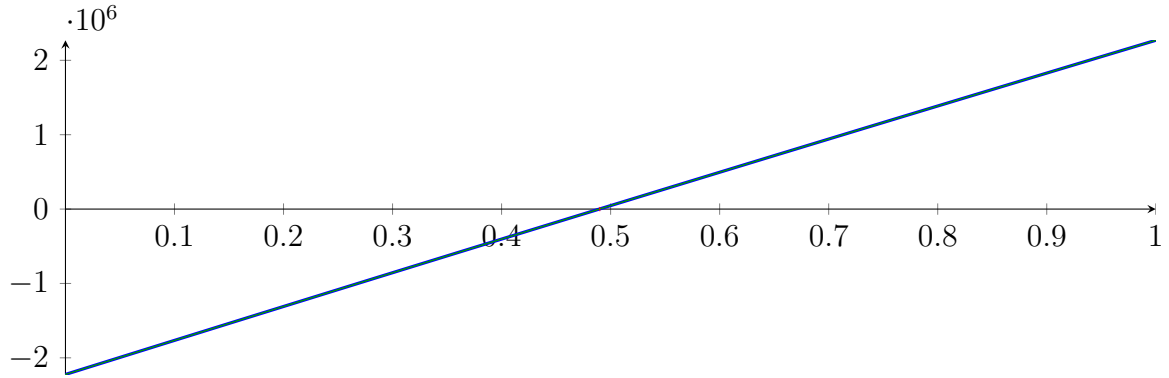
$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\}$$

$$N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174 \cdot 10^{-07}$

\implies Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

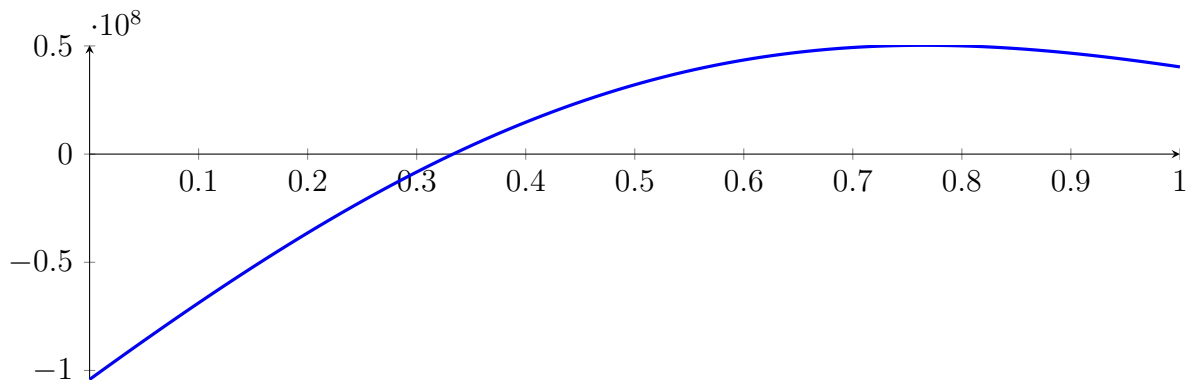
153.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 3!

153.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

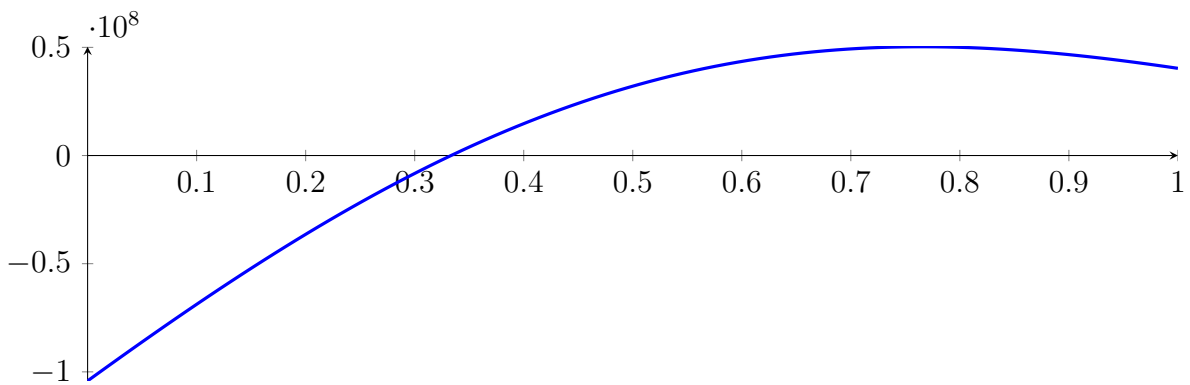
with precision $\varepsilon = 0.0001$.

154 Running BezClip on f_{16} with epsilon 8

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

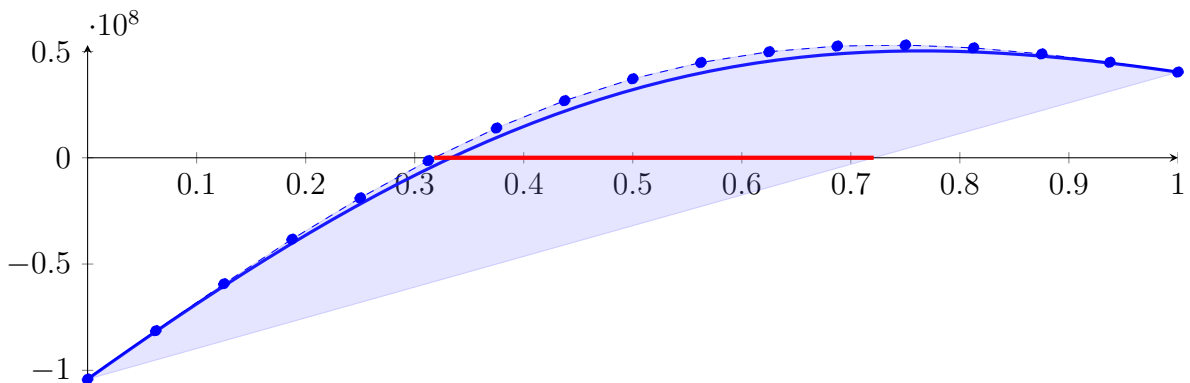
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



154.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

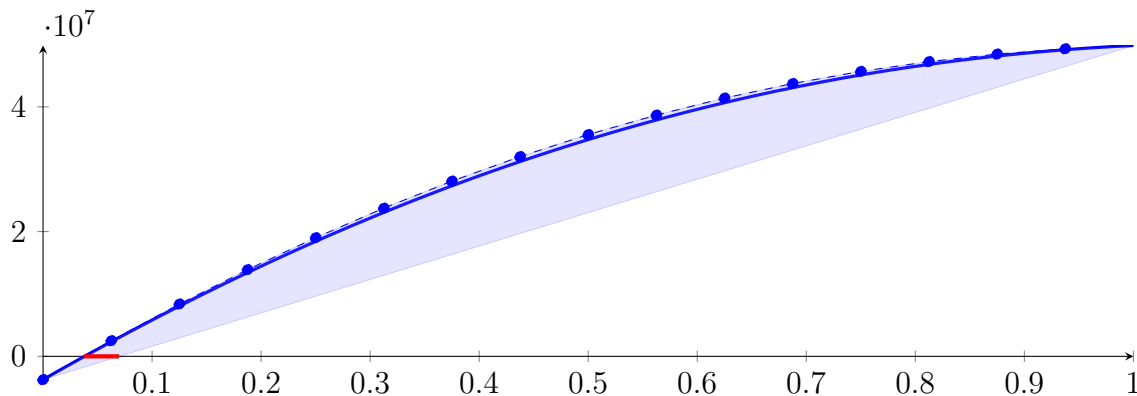
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

154.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.59825 \cdot 10^{-06} X^{16} - 5.93153 \cdot 10^{-05} X^{15} - 0.00248867 X^{14} - 0.0650056 X^{13} - 0.909142 X^{12} \\ &\quad - 5.03931 X^{11} + 36.4549 X^{10} + 692.921 X^9 + 1886.97 X^8 - 25792 X^7 - 149671 X^6 + 492605 X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

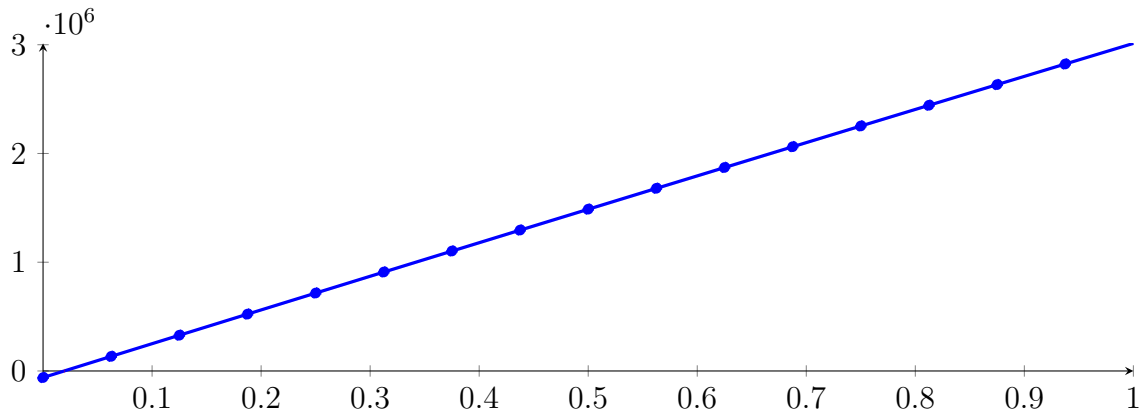
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

154.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 9.01396 \cdot 10^{-08} X^{16} - 2.65848 \cdot 10^{-07} X^{15} + 2.13948 \cdot 10^{-06} X^{14} - 1.33627 \cdot 10^{-06} X^{13} + 2.46973 \cdot 10^{-06} X^{12} \\ &\quad - 2.45524 \cdot 10^{-06} X^{11} + 5.50112 \cdot 10^{-07} X^{10} - 1.64198 \cdot 10^{-07} X^9 - 7.35598 \cdot 10^{-07} X^8 - 1.00892 \cdot 10^{-06} X^7 \\ &\quad - 0.000177509 X^6 + 0.0161038 X^5 + 4.36155 X^4 - 234.216 X^3 - 45622.2 X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

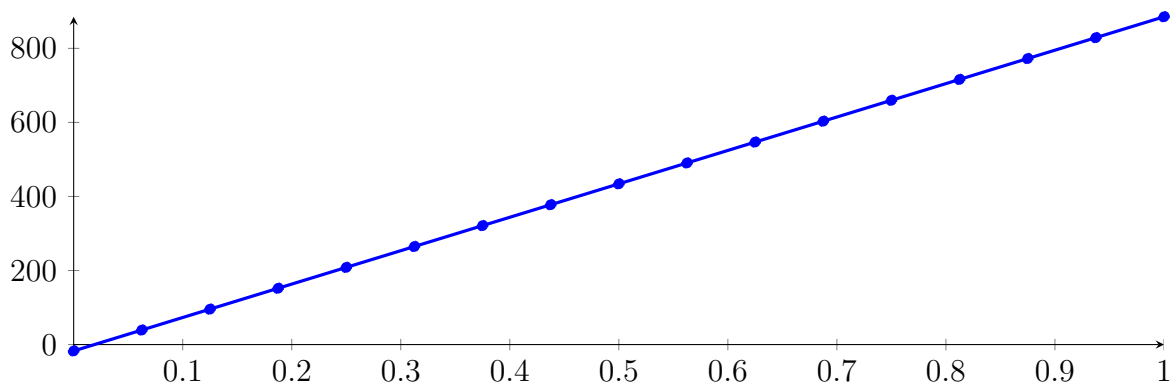
Longest intersection interval: 0.000289554

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333337]$,

154.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 2.55372 \cdot 10^{-11} X^{16} - 7.21263 \cdot 10^{-11} X^{15} + 6.24141 \cdot 10^{-10} X^{14} - 4.11162 \cdot 10^{-10} X^{13} \\
 &+ 6.82359 \cdot 10^{-10} X^{12} - 7.09475 \cdot 10^{-10} X^{11} + 9.71305 \cdot 10^{-11} X^{10} - 3.46101 \cdot 10^{-11} X^9 \\
 &- 2.13971 \cdot 10^{-10} X^8 - 1.46061 \cdot 10^{-11} X^7 - 1.63366 \cdot 10^{-11} X^6 + 1.87916 \cdot 10^{-12} X^5 \\
 &+ 2.52576 \cdot 10^{-14} X^4 - 5.67777 \cdot 10^{-09} X^3 - 0.00382618 X^2 + 902.448 X - 17.178 \\
 &= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
 &+ 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
 &+ 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
 &+ 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045 \cdot 10^{-08}$

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333333]$,

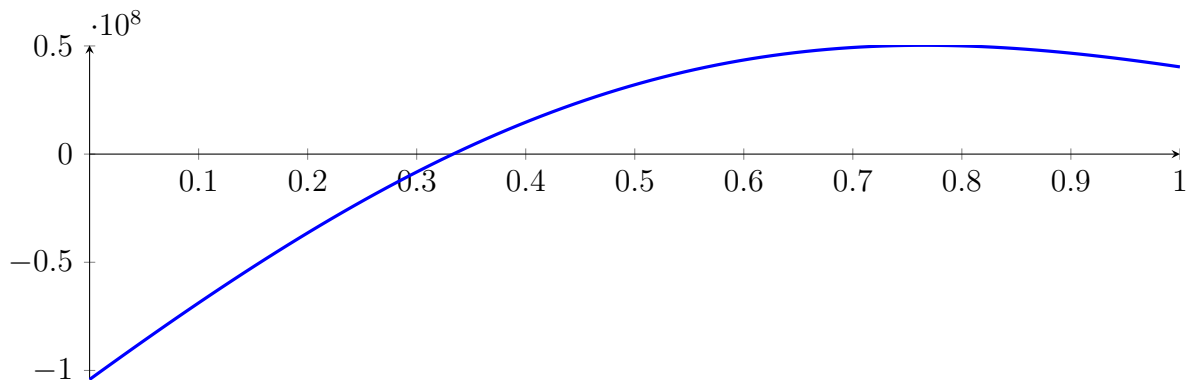
154.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

154.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

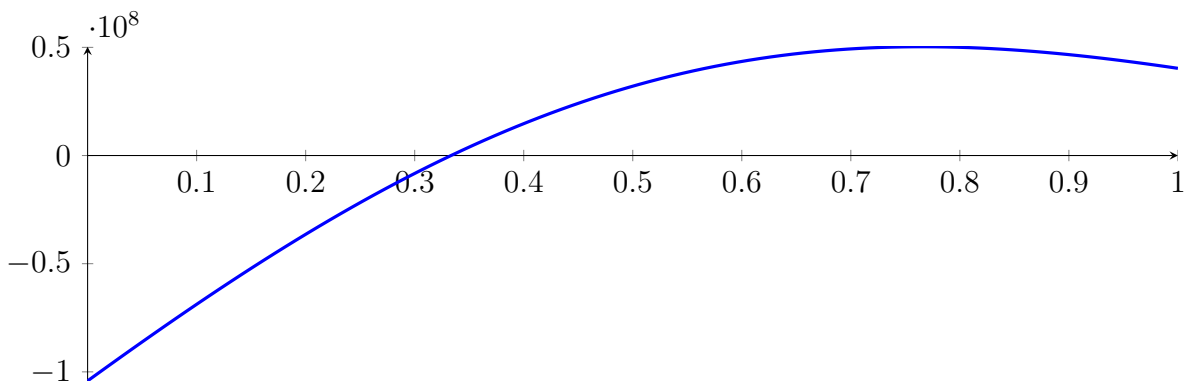
with precision $\varepsilon = 1 \cdot 10^{-08}$.

155 Running QuadClip on f_{16} with epsilon 8

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

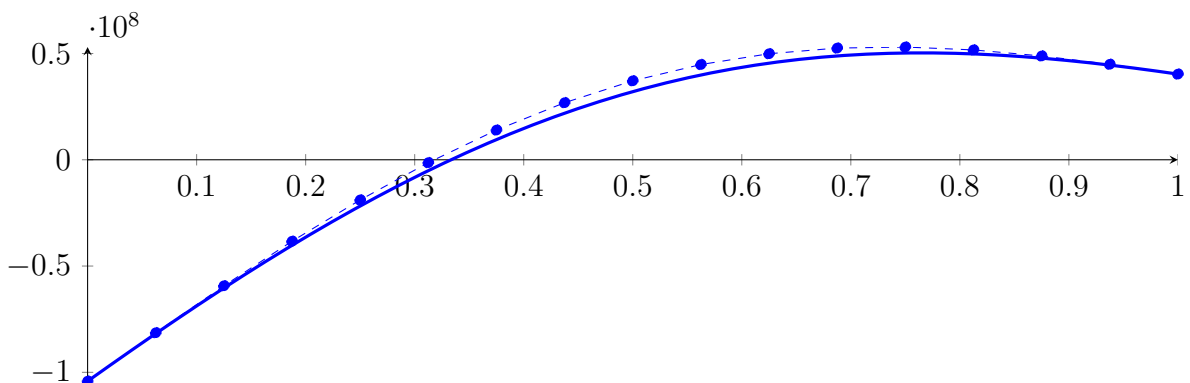
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



155.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = 6049.18X^{16} - 48305.2X^{15} + 174971X^{14} - 380294X^{13} + 552846X^{12} - 567203X^{11}$$

$$+ 422303X^{10} - 231038X^9 + 93003.6X^8 - 27320.1X^7 + 5752.57X^6 - 843.63X^5$$

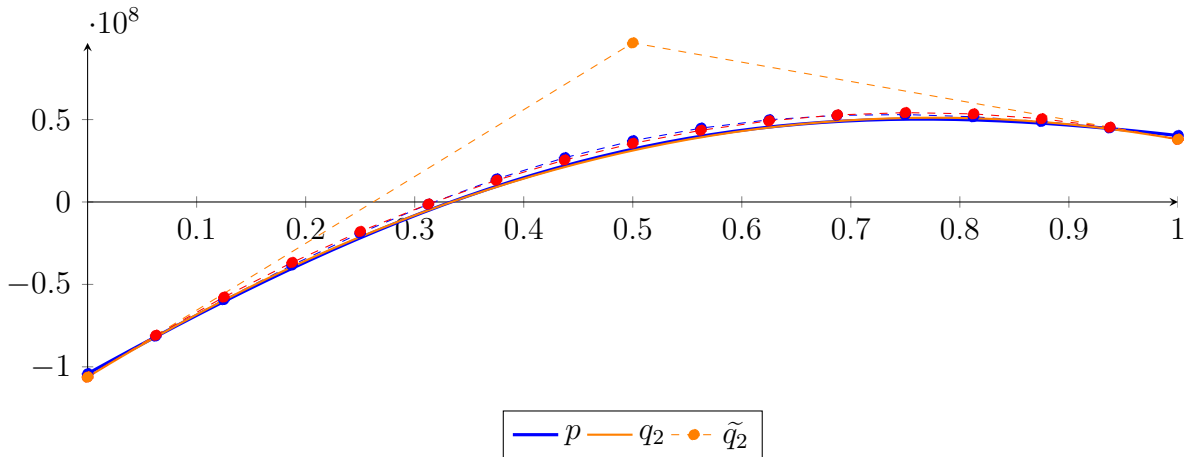
$$+ 82.5145X^4 - 5.01388X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

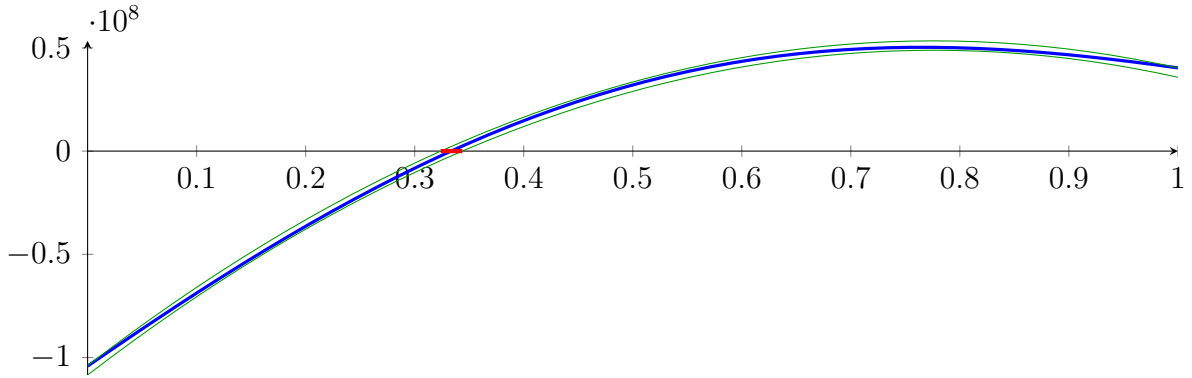
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\} \quad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

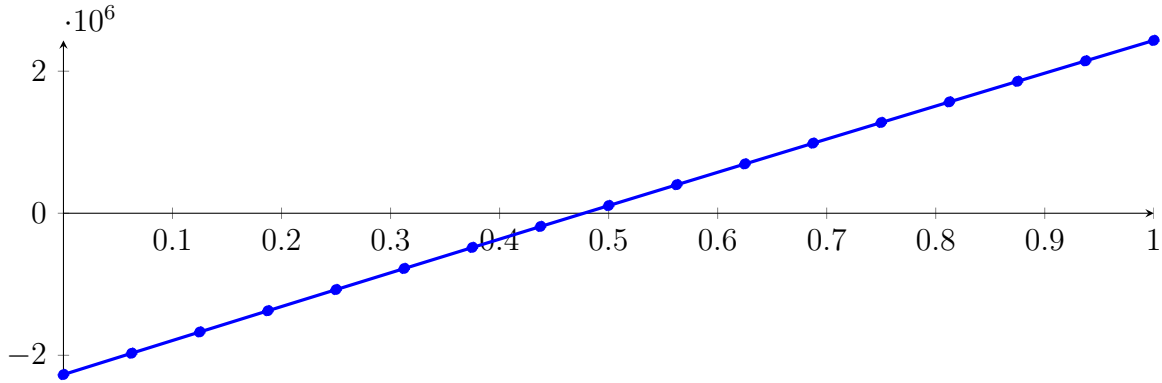
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

155.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

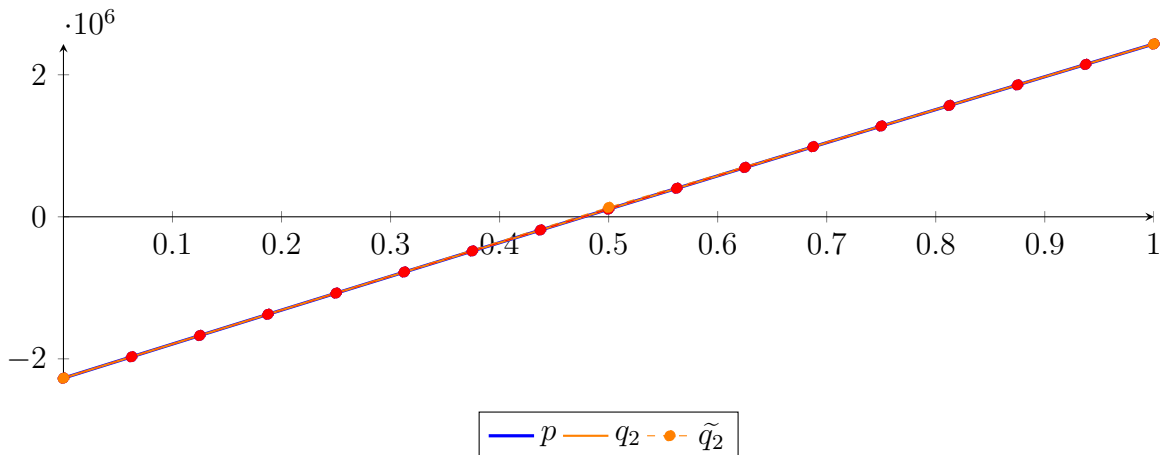
$$\begin{aligned}
 p &= -1.54841 \cdot 10^{-10} X^{16} - 1.66965 \cdot 10^{-07} X^{15} - 2.92739 \cdot 10^{-07} X^{14} - 1.77943 \cdot 10^{-06} X^{13} - 1.17235 \cdot 10^{-06} X^{12} \\
 &\quad - 2.42234 \cdot 10^{-06} X^{11} - 6.86445 \cdot 10^{-07} X^{10} - 1.39162 \cdot 10^{-06} X^9 + 1.07395 \cdot 10^{-06} X^8 - 1.67072 \cdot 10^{-05} X^7 \\
 &\quad - 0.00205879 X^6 + 0.132721 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 22.7036 X^{16} - 178.964 X^{15} + 638.976 X^{14} - 1366.64 X^{13} + 1951.1 X^{12} - 1960.92 X^{11} \\
 &\quad + 1425.35 X^{10} - 757.631 X^9 + 294.177 X^8 - 82.4368 X^7 + 16.2856 X^6 - 2.18949 X^5 \\
 &\quad + 0.191237 X^4 - 0.0101048 X^3 - 104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} \\
 &\quad + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

$$M = -104265 X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

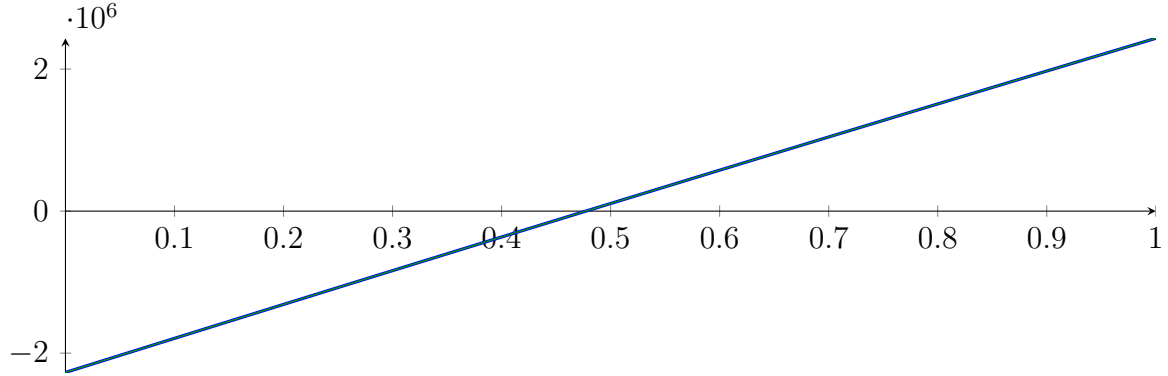
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\}$$

$$N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

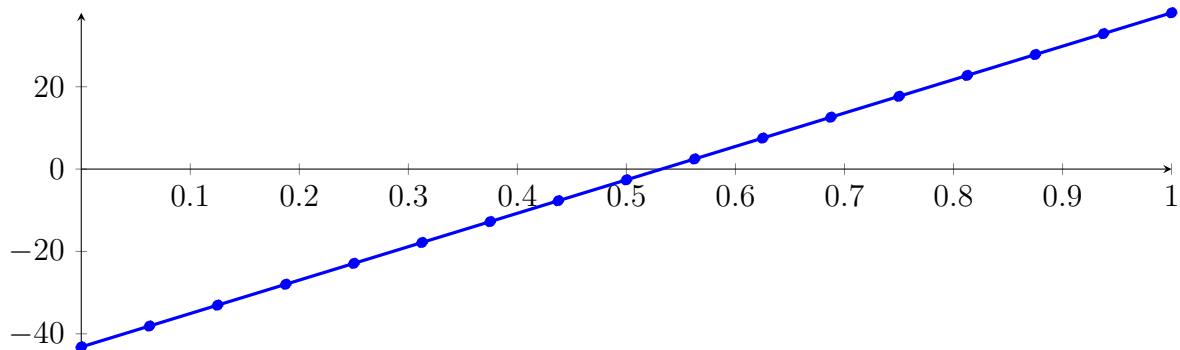
Longest intersection interval: $1.72301 \cdot 10^{-05}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

155.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

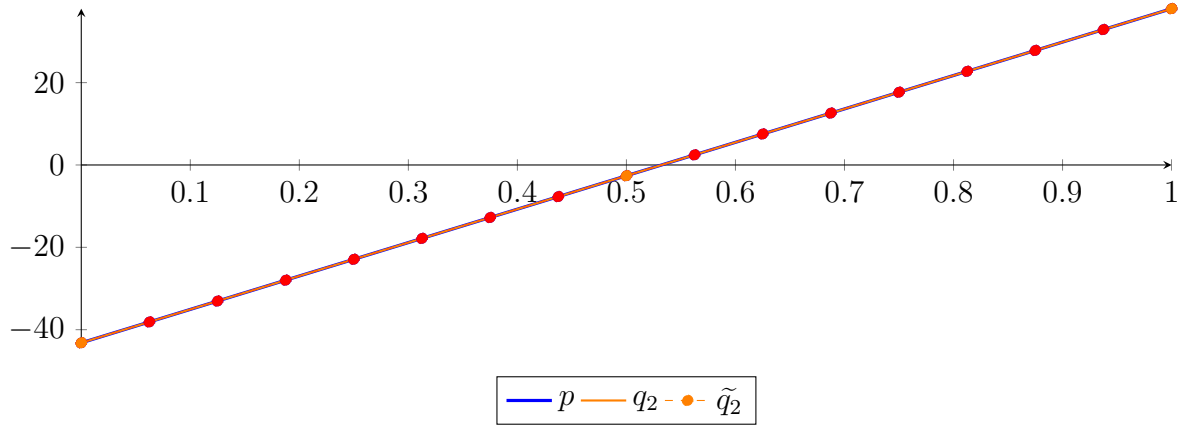
$$\begin{aligned} p &= -2.76723 \cdot 10^{-13} X^{16} - 2.40874 \cdot 10^{-12} X^{15} - 1.25233 \cdot 10^{-11} X^{14} - 3.02935 \cdot 10^{-11} X^{13} \\ &\quad - 3.05617 \cdot 10^{-11} X^{12} - 3.83107 \cdot 10^{-11} X^{11} - 1.26692 \cdot 10^{-11} X^{10} - 2.6672 \cdot 10^{-11} X^9 \\ &\quad + 2.0004 \cdot 10^{-11} X^8 + 4.12781 \cdot 10^{-12} X^7 + 2.44493 \cdot 10^{-12} X^6 - 1.21236 \cdot 10^{-13} X^5 \\ &\quad + 1.26288 \cdot 10^{-14} X^4 - 4.1267 \cdot 10^{-12} X^3 - 3.09388 \cdot 10^{-05} X^2 + 81.1506X - 43.1911 \\ &= -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\ &\quad - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X) \\ &\quad + 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X) \\ &\quad + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$\begin{aligned}
\tilde{q}_2 &= 5.96265 \cdot 10^{-05} X^{16} - 0.000436042 X^{15} + 0.00141812 X^{14} - 0.00269475 X^{13} \\
&+ 0.00329809 X^{12} - 0.00268757 X^{11} + 0.00143268 X^{10} - 0.000439599 X^9 \\
&+ 1.98418 \cdot 10^{-05} X^8 + 4.87608 \cdot 10^{-05} X^7 - 2.46333 \cdot 10^{-05} X^6 + 6.35808 \cdot 10^{-06} X^5 \\
&- 9.62755 \cdot 10^{-07} X^4 + 8.21372 \cdot 10^{-08} X^3 - 3.09429 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911 \\
&= -43.1911 B_{0,16} - 38.1192 B_{1,16} - 33.0473 B_{2,16} - 27.9754 B_{3,16} - 22.9035 B_{4,16} - 17.8316 B_{5,16} \\
&- 12.7597 B_{6,16} - 7.68778 B_{7,16} - 2.61587 B_{8,16} + 2.45604 B_{9,16} + 7.52795 B_{10,16} + 12.5999 B_{11,16} \\
&+ 17.6718 B_{12,16} + 22.7437 B_{13,16} + 27.8156 B_{14,16} + 32.8875 B_{15,16} + 37.9594 B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.5947 \cdot 10^{-09}$.

Bounding polynomials M and m :

$$M = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

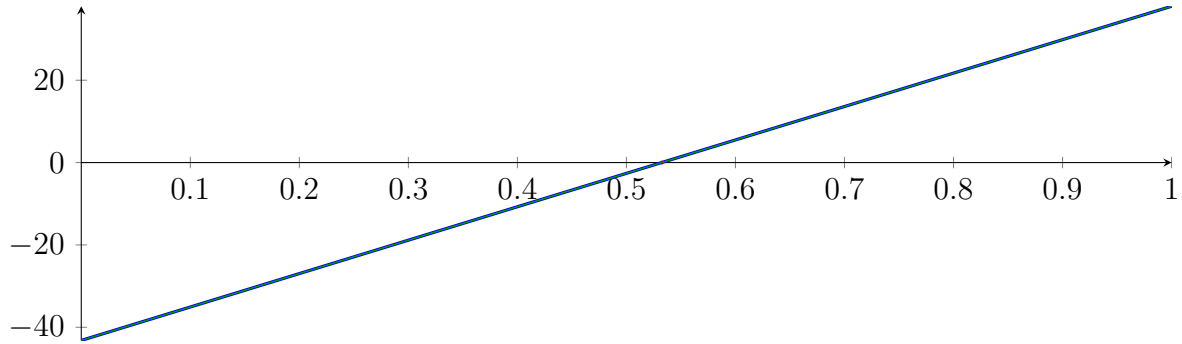
$$m = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

Root of M and m :

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\}$$

$$N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

Longest intersection interval: $3.93535 \cdot 10^{-11}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

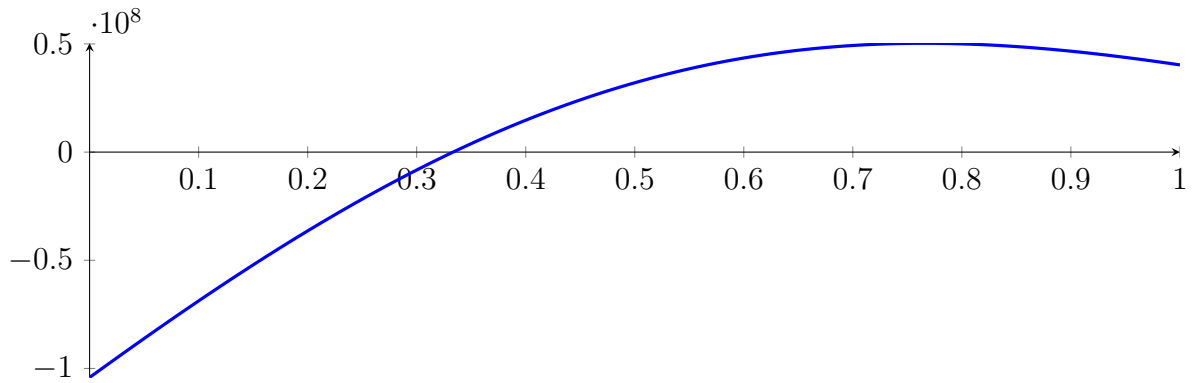
155.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

155.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

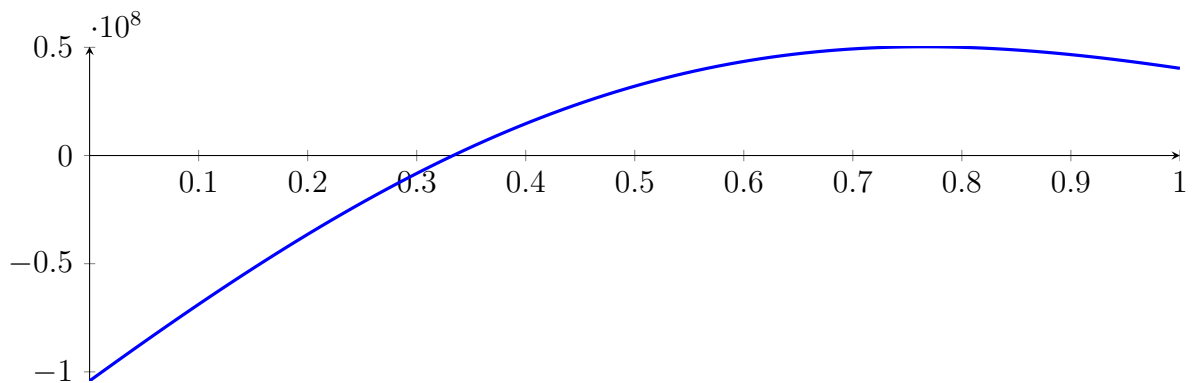
with precision $\varepsilon = 1 \cdot 10^{-08}$.

156 Running CubeClip on f_{16} with epsilon 8

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

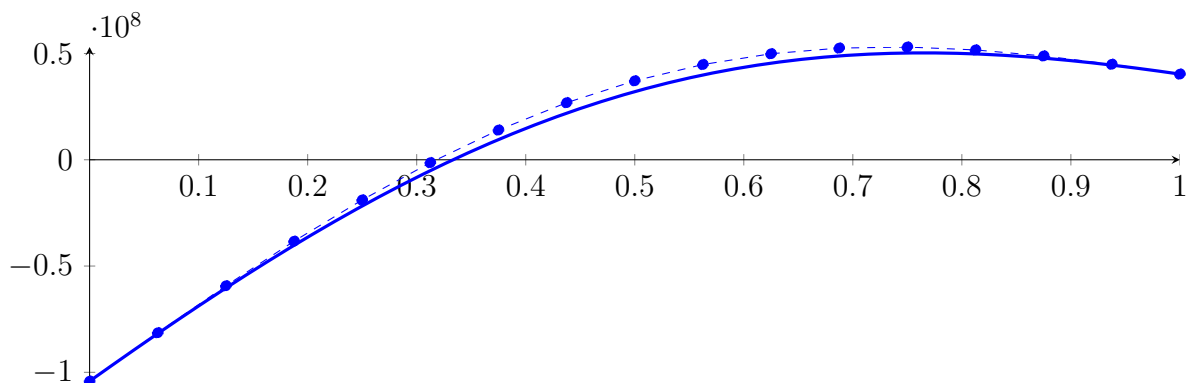
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



156.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 2461.93X^{16} - 19614.9X^{15} + 70879.5X^{14} - 153661X^{13} + 222746X^{12} - 227755X^{11}$$

$$+ 168826X^{10} - 91798.7X^9 + 36630.3X^8 - 10627.3X^7 + 2200.54X^6 - 316.059X^5$$

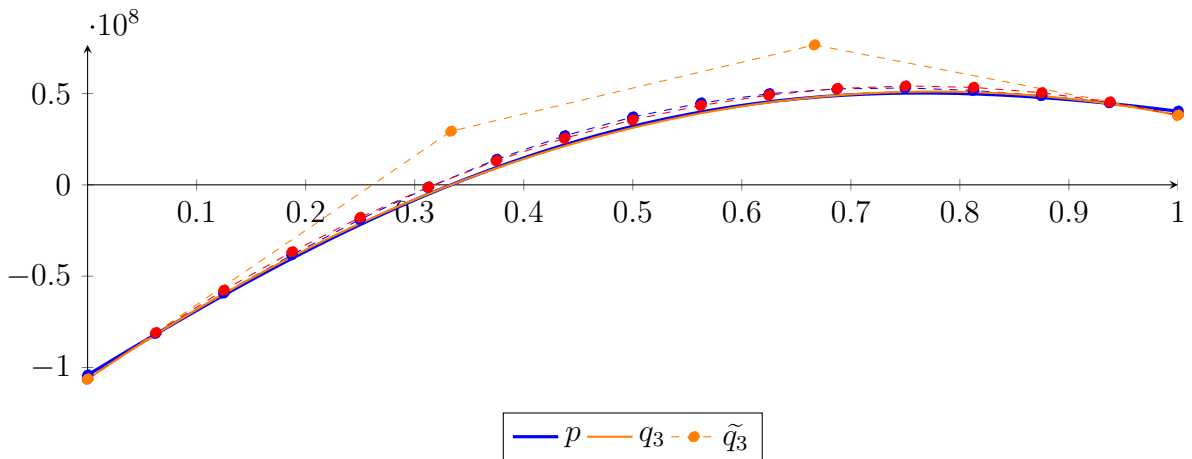
$$+ 30.1958X^4 + 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18551 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91815 \cdot 10^7 B_{10,16} + 5.27353 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

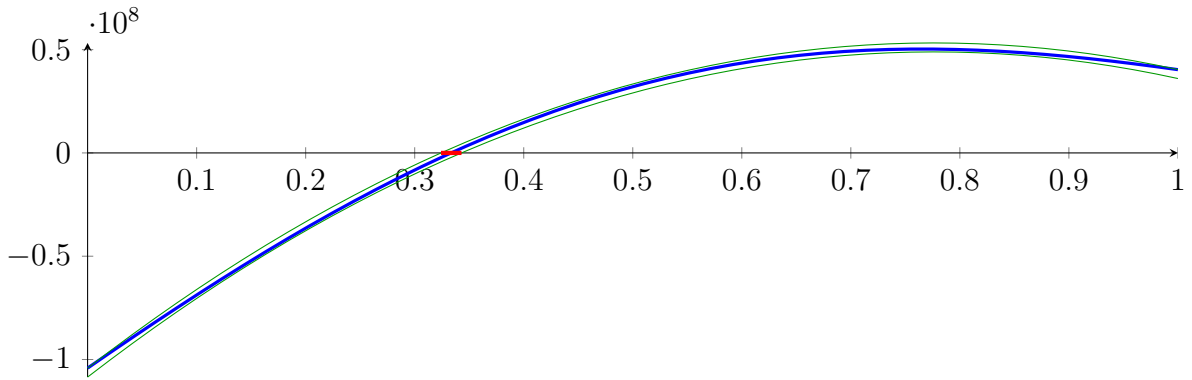
$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\}$$

$$N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

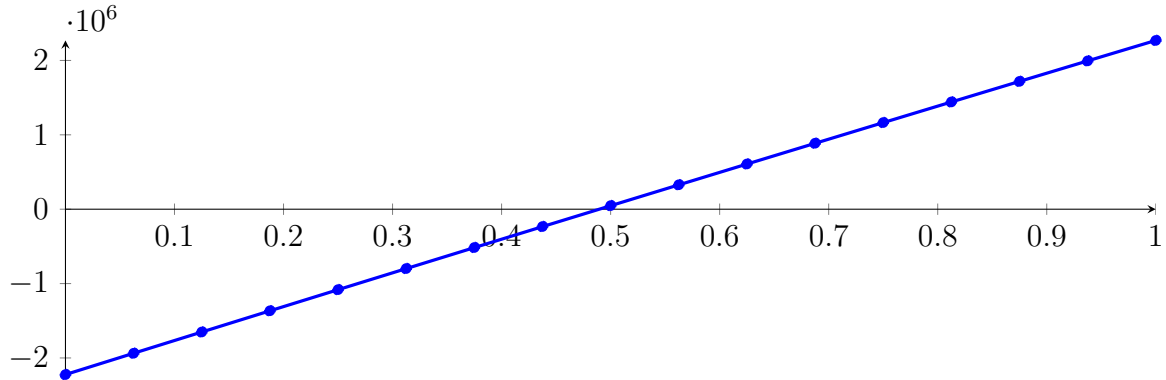
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

156.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

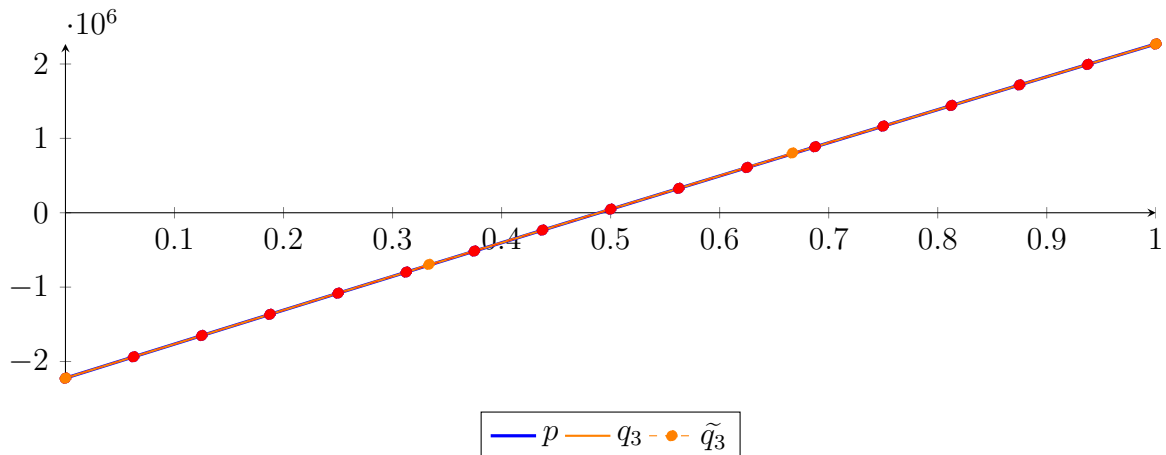
$$\begin{aligned}
 p &= -3.66617 \cdot 10^{-09} X^{16} - 1.53217 \cdot 10^{-07} X^{15} - 3.62234 \cdot 10^{-07} X^{14} - 1.65579 \cdot 10^{-06} X^{13} - 1.15373 \cdot 10^{-06} X^{12} \\
 &\quad - 2.3399 \cdot 10^{-06} X^{11} - 5.02543 \cdot 10^{-07} X^{10} - 1.38381 \cdot 10^{-06} X^9 + 1.1237 \cdot 10^{-06} X^8 - 1.19653 \cdot 10^{-05} X^7 \\
 &\quad - 0.00155608 X^6 + 0.10496 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 16.4956 X^{16} - 129.161 X^{15} + 457.83 X^{14} - 971.671 X^{13} + 1375.95 X^{12} - 1370.96 X^{11} \\
 &\quad + 987.265 X^{10} - 519.476 X^9 + 199.587 X^8 - 55.434 X^7 + 10.9237 X^6 - 1.48019 X^5 \\
 &\quad + 0.129516 X^4 - 700.679 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} \\
 &\quad + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m :

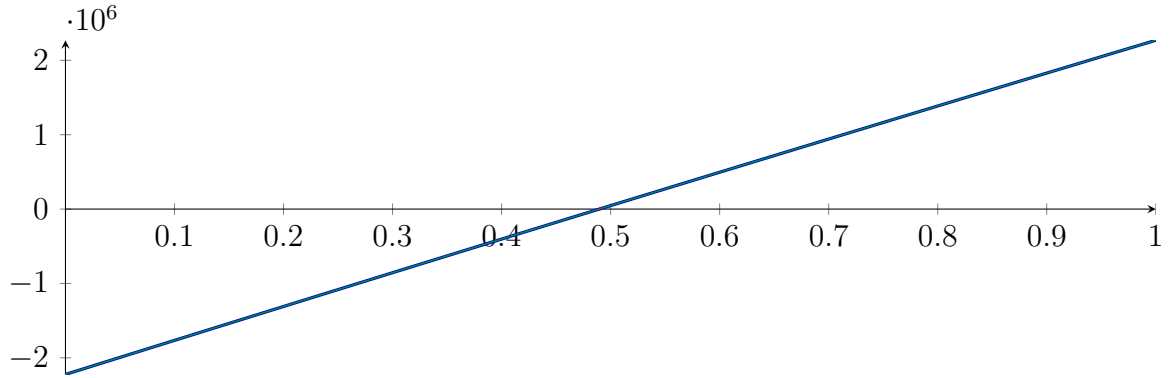
$$M = -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\} \quad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174 \cdot 10^{-07}$

\implies Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

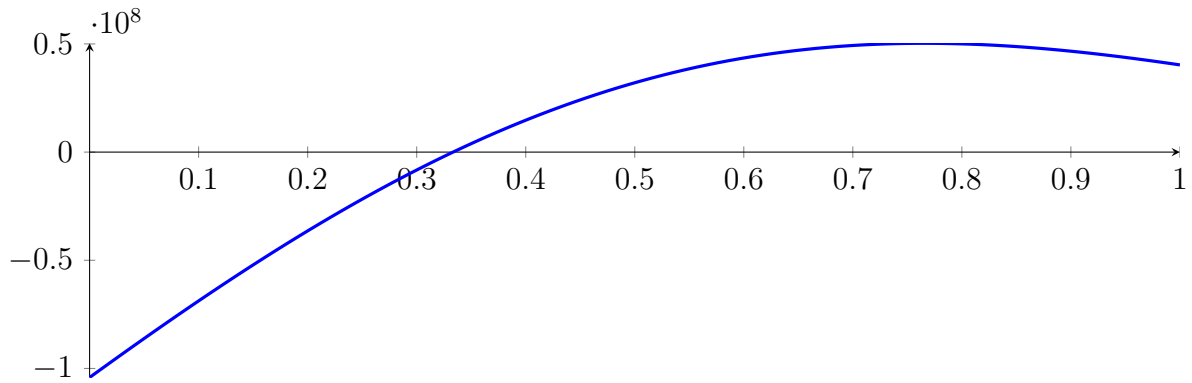
156.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 3!

156.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

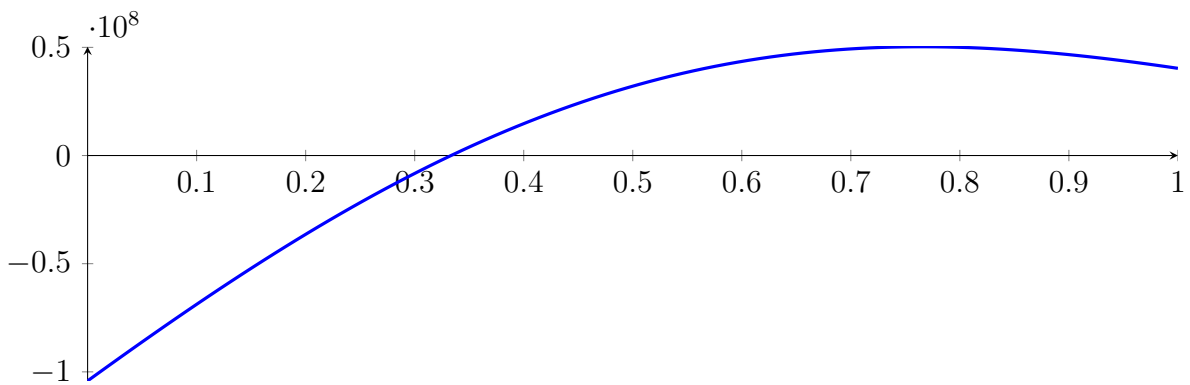
with precision $\varepsilon = 1 \cdot 10^{-08}$.

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$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

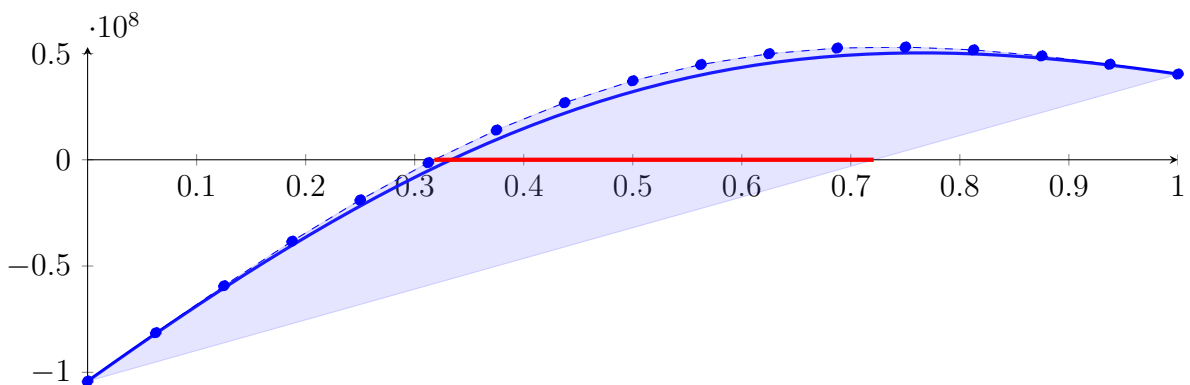
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



157.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

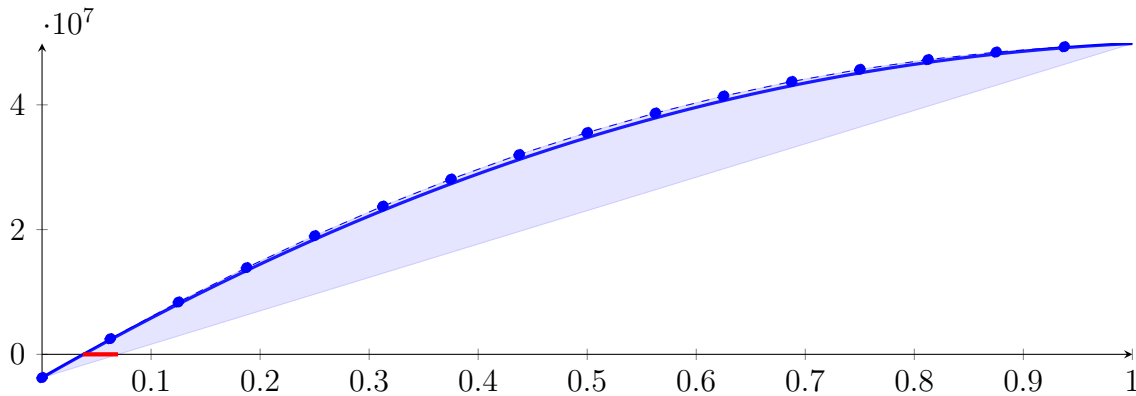
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

157.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.59825 \cdot 10^{-06} X^{16} - 5.93153 \cdot 10^{-05} X^{15} - 0.00248867 X^{14} - 0.0650056 X^{13} - 0.909142 X^{12} \\ &\quad - 5.03931 X^{11} + 36.4549 X^{10} + 692.921 X^9 + 1886.97 X^8 - 25792 X^7 - 149671 X^6 + 492605 X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

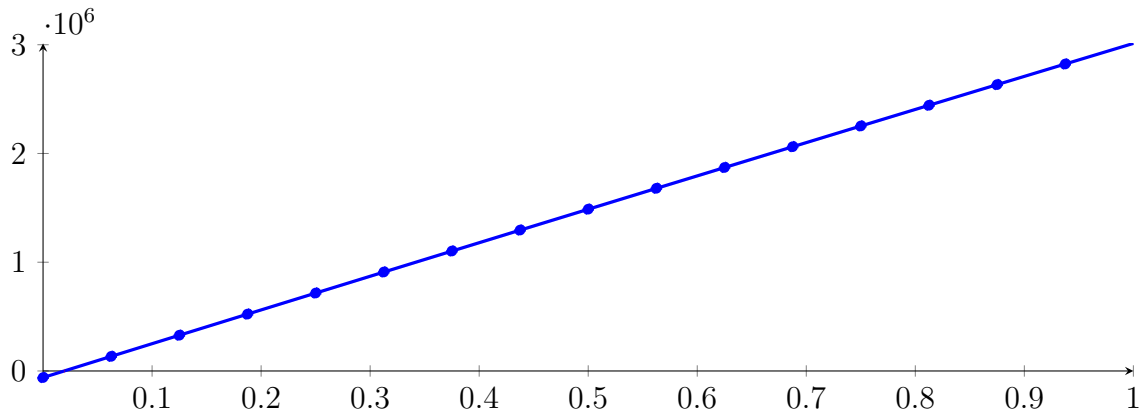
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

157.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 9.01396 \cdot 10^{-08} X^{16} - 2.65848 \cdot 10^{-07} X^{15} + 2.13948 \cdot 10^{-06} X^{14} - 1.33627 \cdot 10^{-06} X^{13} + 2.46973 \cdot 10^{-06} X^{12} \\ &\quad - 2.45524 \cdot 10^{-06} X^{11} + 5.50112 \cdot 10^{-07} X^{10} - 1.64198 \cdot 10^{-07} X^9 - 7.35598 \cdot 10^{-07} X^8 - 1.00892 \cdot 10^{-06} X^7 \\ &\quad - 0.000177509 X^6 + 0.0161038 X^5 + 4.36155 X^4 - 234.216 X^3 - 45622.2 X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

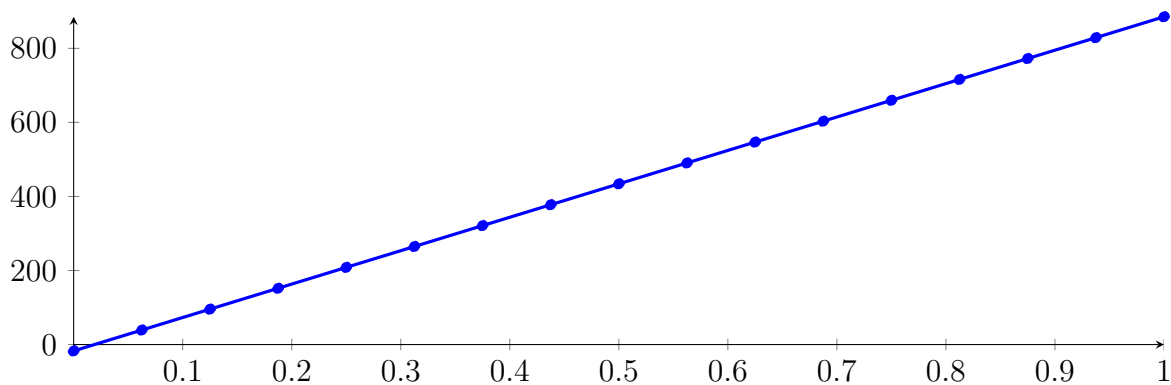
Longest intersection interval: 0.000289554

⇒ Selective recursion: interval 1: [0.333333, 0.333337],

157.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333337]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 2.55372 \cdot 10^{-11} X^{16} - 7.21263 \cdot 10^{-11} X^{15} + 6.24141 \cdot 10^{-10} X^{14} - 4.11162 \cdot 10^{-10} X^{13} \\
 &+ 6.82359 \cdot 10^{-10} X^{12} - 7.09475 \cdot 10^{-10} X^{11} + 9.71305 \cdot 10^{-11} X^{10} - 3.46101 \cdot 10^{-11} X^9 \\
 &- 2.13971 \cdot 10^{-10} X^8 - 1.46061 \cdot 10^{-11} X^7 - 1.63366 \cdot 10^{-11} X^6 + 1.87916 \cdot 10^{-12} X^5 \\
 &+ 2.52576 \cdot 10^{-14} X^4 - 5.67777 \cdot 10^{-09} X^3 - 0.00382618 X^2 + 902.448 X - 17.178 \\
 &= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
 &+ 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
 &+ 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
 &+ 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

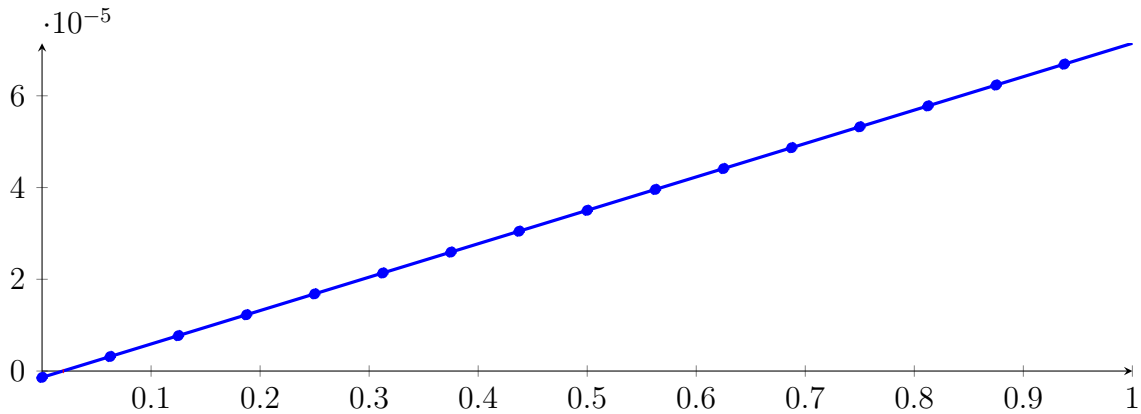
Longest intersection interval: $8.07045 \cdot 10^{-08}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

157.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 2.14261 \cdot 10^{-18} X^{16} - 6.28573 \cdot 10^{-18} X^{15} + 5.28612 \cdot 10^{-17} X^{14} - 3.67279 \cdot 10^{-17} X^{13} \\
 &+ 6.10136 \cdot 10^{-17} X^{12} - 6.60335 \cdot 10^{-17} X^{11} + 1.66661 \cdot 10^{-17} X^{10} - 8.36524 \cdot 10^{-18} X^9 \\
 &- 1.56919 \cdot 10^{-17} X^8 - 1.85474 \cdot 10^{-18} X^7 - 1.4308 \cdot 10^{-18} X^6 + 1.1562 \cdot 10^{-19} X^5 - 1.20437 \\
 &\cdot 10^{-20} X^4 - 4.63221 \cdot 10^{-22} X^3 - 2.49207 \cdot 10^{-17} X^2 + 7.28316 \cdot 10^{-05} X - 1.38634 \cdot 10^{-06} \\
 &= -1.38634 \cdot 10^{-06} B_{0,16}(X) + 3.16564 \cdot 10^{-06} B_{1,16}(X) + 7.71761 \cdot 10^{-06} B_{2,16}(X) + 1.22696 \\
 &\cdot 10^{-05} B_{3,16}(X) + 1.68216 \cdot 10^{-05} B_{4,16}(X) + 2.13735 \cdot 10^{-05} B_{5,16}(X) + 2.59255 \cdot 10^{-05} B_{6,16}(X) \\
 &+ 3.04775 \cdot 10^{-05} B_{7,16}(X) + 3.50295 \cdot 10^{-05} B_{8,16}(X) + 3.95814 \cdot 10^{-05} B_{9,16}(X) + 4.41334 \\
 &\cdot 10^{-05} B_{10,16}(X) + 4.86854 \cdot 10^{-05} B_{11,16}(X) + 5.32374 \cdot 10^{-05} B_{12,16}(X) + 5.77893 \cdot 10^{-05} B_{13,16}(X) \\
 &+ 6.23413 \cdot 10^{-05} B_{14,16}(X) + 6.68933 \cdot 10^{-05} B_{15,16}(X) + 7.14453 \cdot 10^{-05} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.51313 \cdot 10^{-15}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

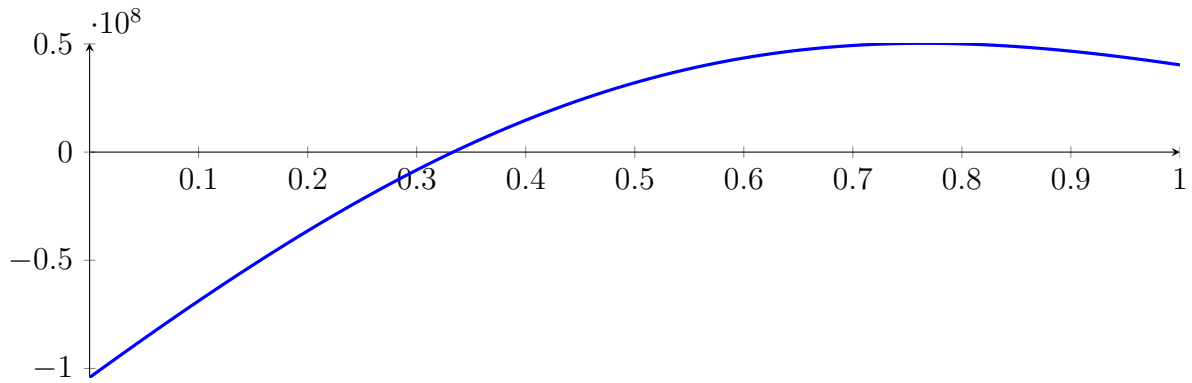
157.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

157.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

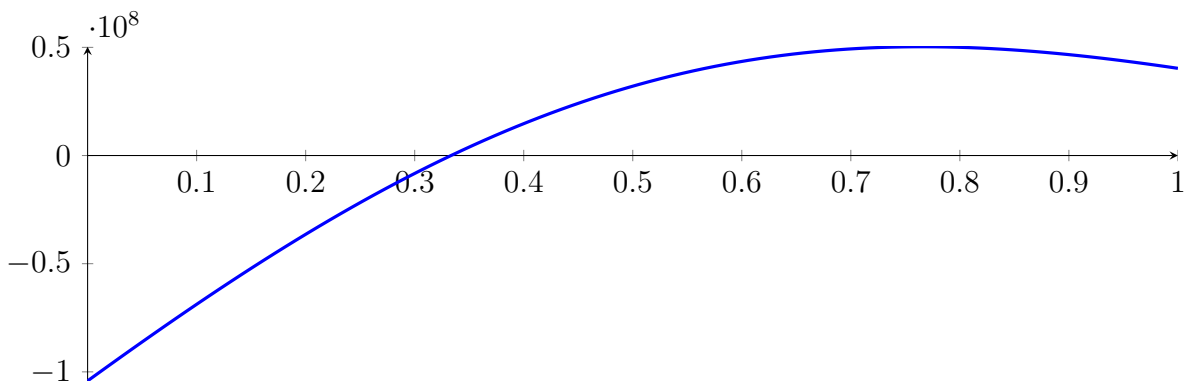
with precision $\varepsilon = 1 \cdot 10^{-16}$.

158 Running QuadClip on f_{16} with epsilon 16

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

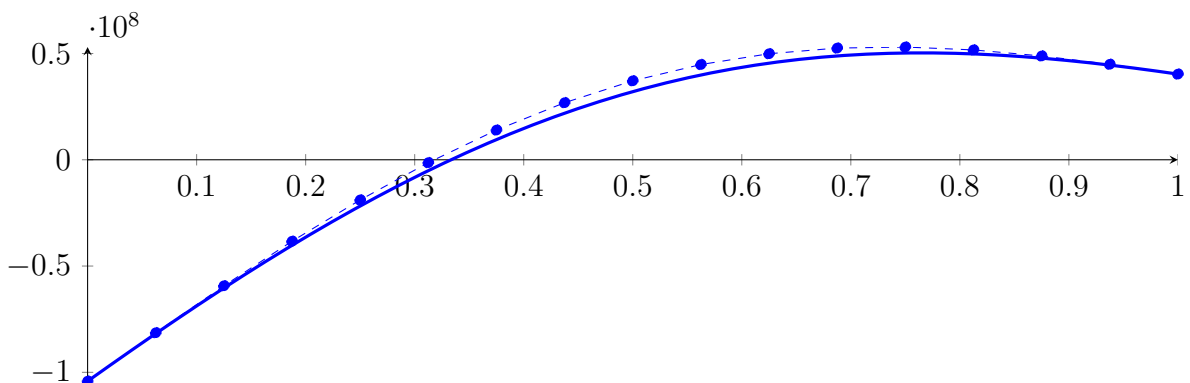
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



158.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = 6049.18X^{16} - 48305.2X^{15} + 174971X^{14} - 380294X^{13} + 552846X^{12} - 567203X^{11}$$

$$+ 422303X^{10} - 231038X^9 + 93003.6X^8 - 27320.1X^7 + 5752.57X^6 - 843.63X^5$$

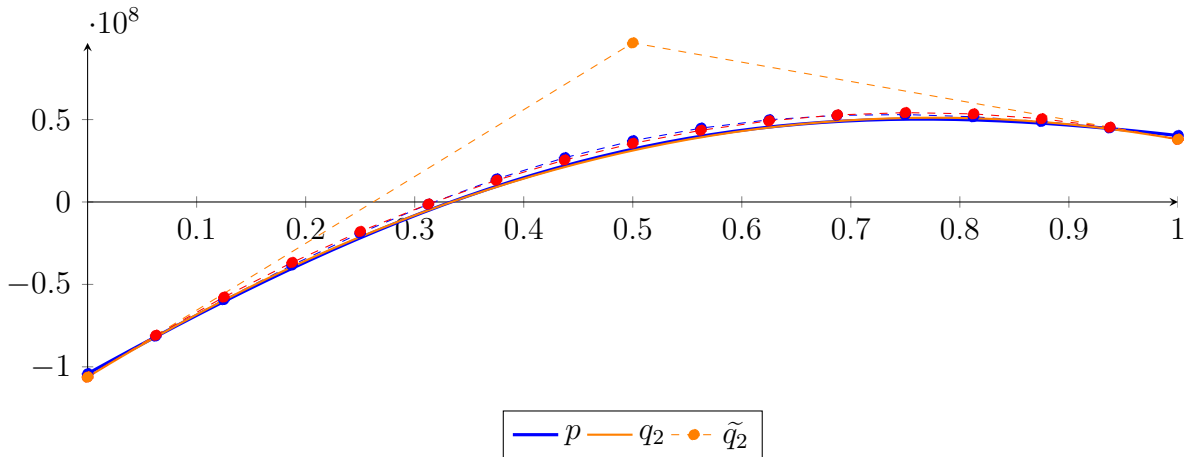
$$+ 82.5145X^4 - 5.01388X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

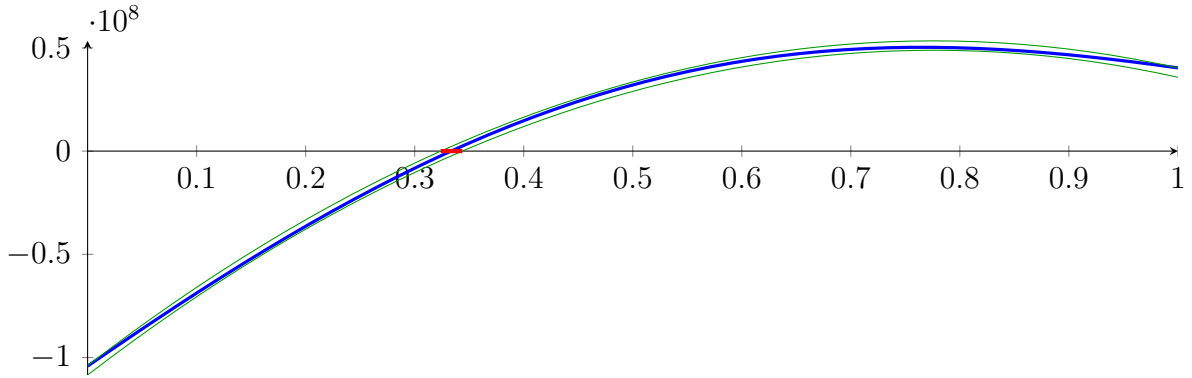
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

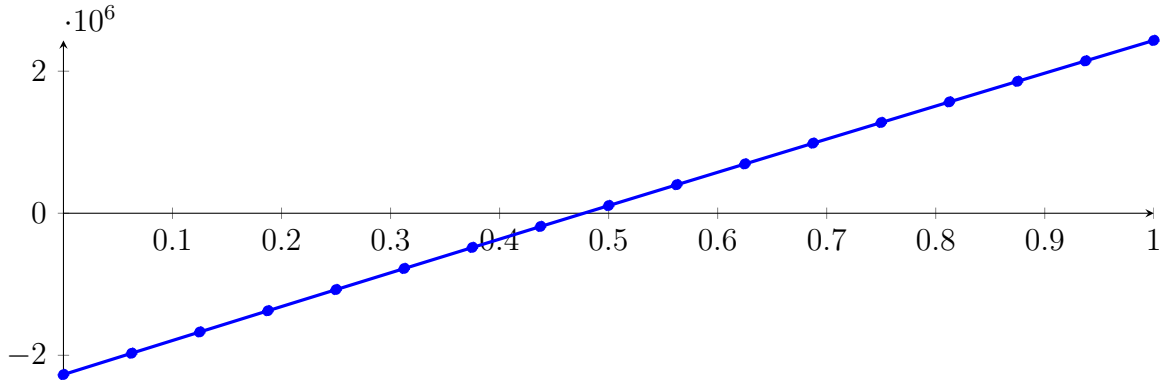
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

158.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

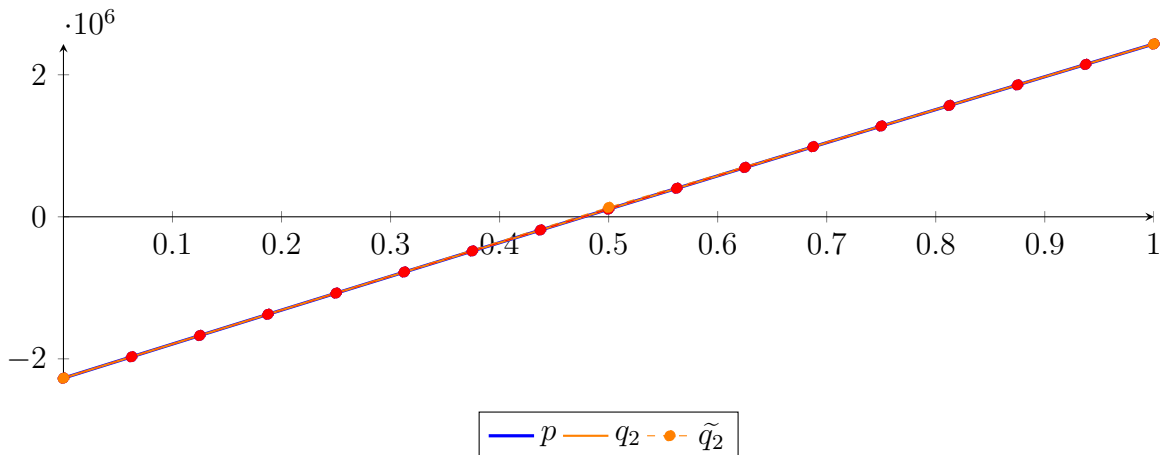
$$\begin{aligned}
 p &= -1.54841 \cdot 10^{-10} X^{16} - 1.66965 \cdot 10^{-07} X^{15} - 2.92739 \cdot 10^{-07} X^{14} - 1.77943 \cdot 10^{-06} X^{13} - 1.17235 \cdot 10^{-06} X^{12} \\
 &\quad - 2.42234 \cdot 10^{-06} X^{11} - 6.86445 \cdot 10^{-07} X^{10} - 1.39162 \cdot 10^{-06} X^9 + 1.07395 \cdot 10^{-06} X^8 - 1.67072 \cdot 10^{-05} X^7 \\
 &\quad - 0.00205879 X^6 + 0.132721 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 22.7036 X^{16} - 178.964 X^{15} + 638.976 X^{14} - 1366.64 X^{13} + 1951.1 X^{12} - 1960.92 X^{11} \\
 &\quad + 1425.35 X^{10} - 757.631 X^9 + 294.177 X^8 - 82.4368 X^7 + 16.2856 X^6 - 2.18949 X^5 \\
 &\quad + 0.191237 X^4 - 0.0101048 X^3 - 104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} \\
 &\quad + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

$$M = -104265 X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

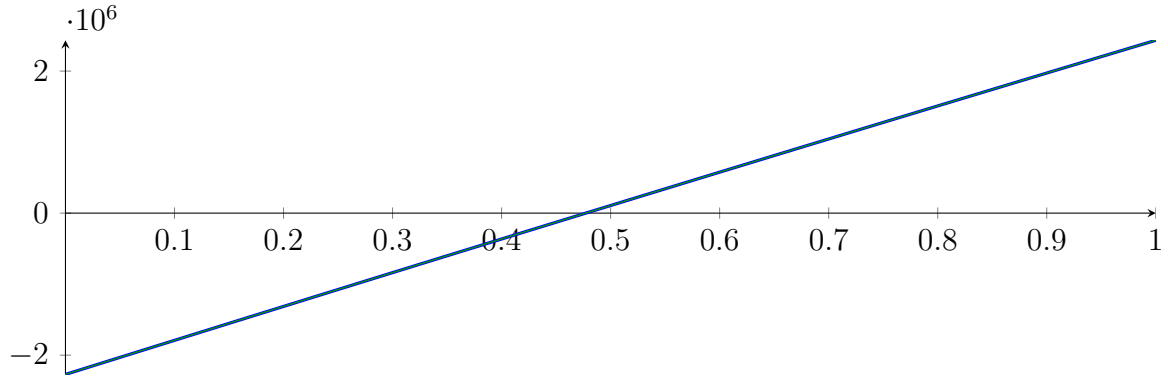
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\}$$

$$N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

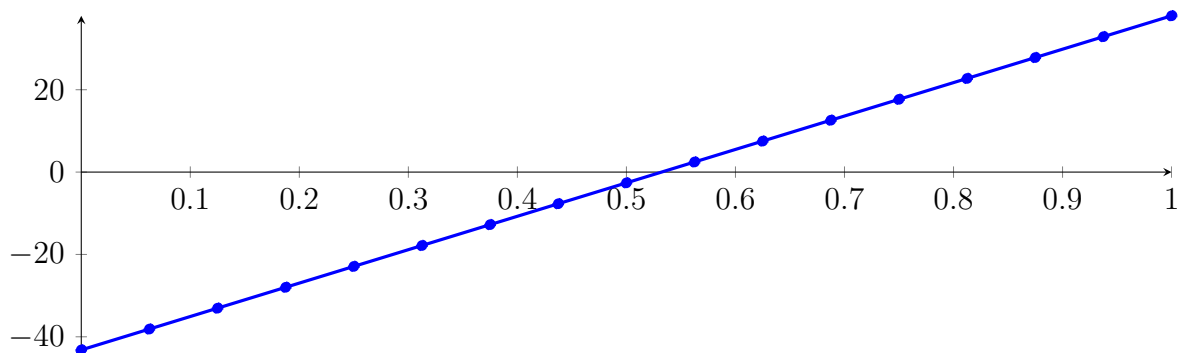
Longest intersection interval: $1.72301 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

158.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

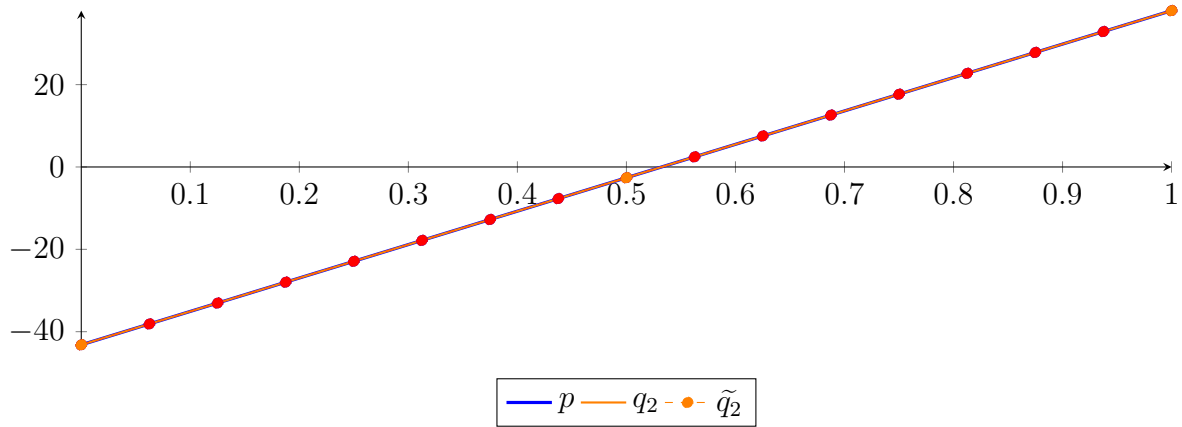
$$\begin{aligned} p &= -2.76723 \cdot 10^{-13} X^{16} - 2.40874 \cdot 10^{-12} X^{15} - 1.25233 \cdot 10^{-11} X^{14} - 3.02935 \cdot 10^{-11} X^{13} \\ &\quad - 3.05617 \cdot 10^{-11} X^{12} - 3.83107 \cdot 10^{-11} X^{11} - 1.26692 \cdot 10^{-11} X^{10} - 2.6672 \cdot 10^{-11} X^9 \\ &\quad + 2.0004 \cdot 10^{-11} X^8 + 4.12781 \cdot 10^{-12} X^7 + 2.44493 \cdot 10^{-12} X^6 - 1.21236 \cdot 10^{-13} X^5 \\ &\quad + 1.26288 \cdot 10^{-14} X^4 - 4.1267 \cdot 10^{-12} X^3 - 3.09388 \cdot 10^{-05} X^2 + 81.1506X - 43.1911 \\ &= -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\ &\quad - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X) \\ &\quad + 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X) \\ &\quad + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$\begin{aligned}
\tilde{q}_2 &= 5.96265 \cdot 10^{-05} X^{16} - 0.000436042 X^{15} + 0.00141812 X^{14} - 0.00269475 X^{13} \\
&+ 0.00329809 X^{12} - 0.00268757 X^{11} + 0.00143268 X^{10} - 0.000439599 X^9 \\
&+ 1.98418 \cdot 10^{-05} X^8 + 4.87608 \cdot 10^{-05} X^7 - 2.46333 \cdot 10^{-05} X^6 + 6.35808 \cdot 10^{-06} X^5 \\
&- 9.62755 \cdot 10^{-07} X^4 + 8.21372 \cdot 10^{-08} X^3 - 3.09429 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911 \\
&= -43.1911 B_{0,16} - 38.1192 B_{1,16} - 33.0473 B_{2,16} - 27.9754 B_{3,16} - 22.9035 B_{4,16} - 17.8316 B_{5,16} \\
&- 12.7597 B_{6,16} - 7.68778 B_{7,16} - 2.61587 B_{8,16} + 2.45604 B_{9,16} + 7.52795 B_{10,16} + 12.5999 B_{11,16} \\
&+ 17.6718 B_{12,16} + 22.7437 B_{13,16} + 27.8156 B_{14,16} + 32.8875 B_{15,16} + 37.9594 B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.5947 \cdot 10^{-09}$.

Bounding polynomials M and m :

$$M = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

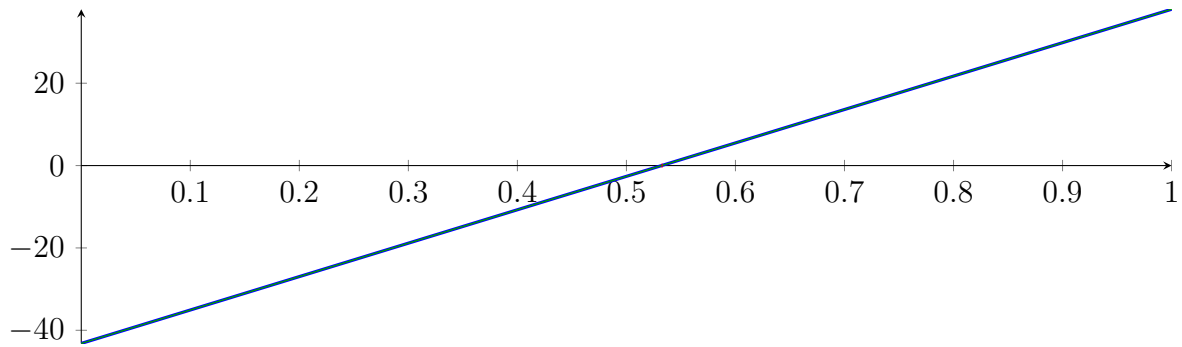
$$m = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

Root of M and m :

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\}$$

$$N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

Longest intersection interval: $3.93535 \cdot 10^{-11}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

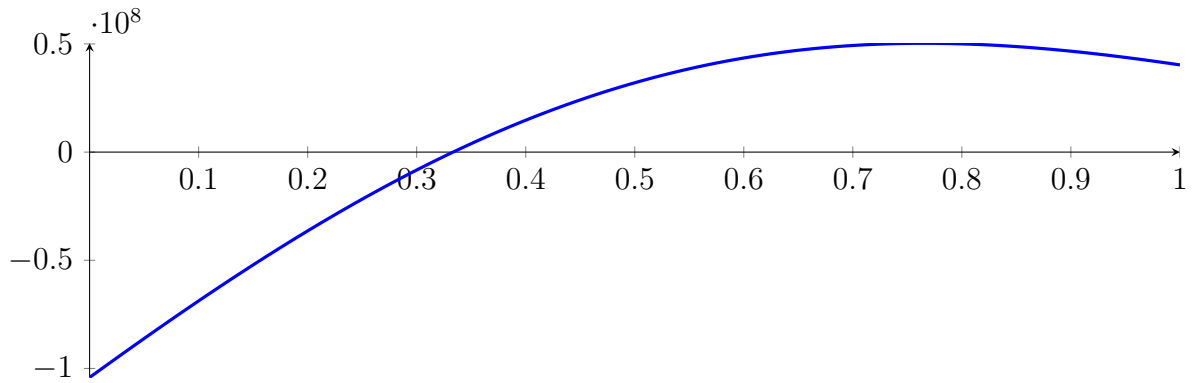
158.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

158.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

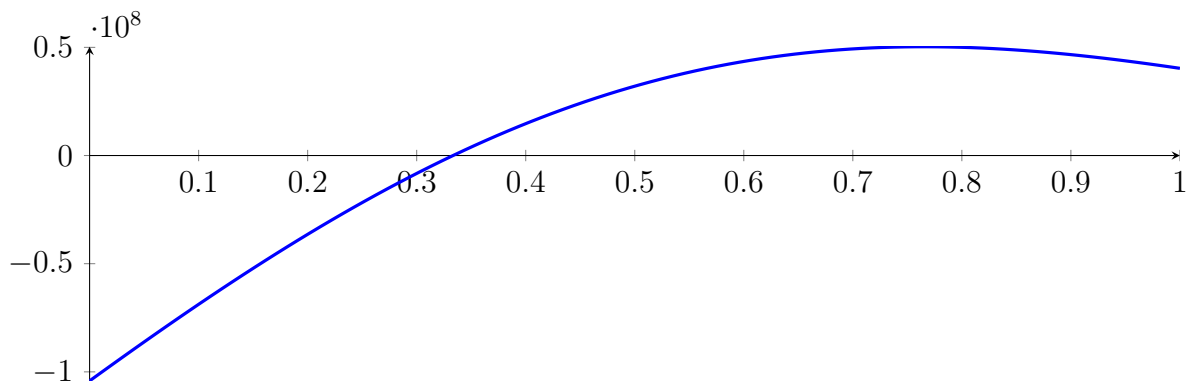
with precision $\varepsilon = 1 \cdot 10^{-16}$.

159 Running CubeClip on f_{16} with epsilon 16

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

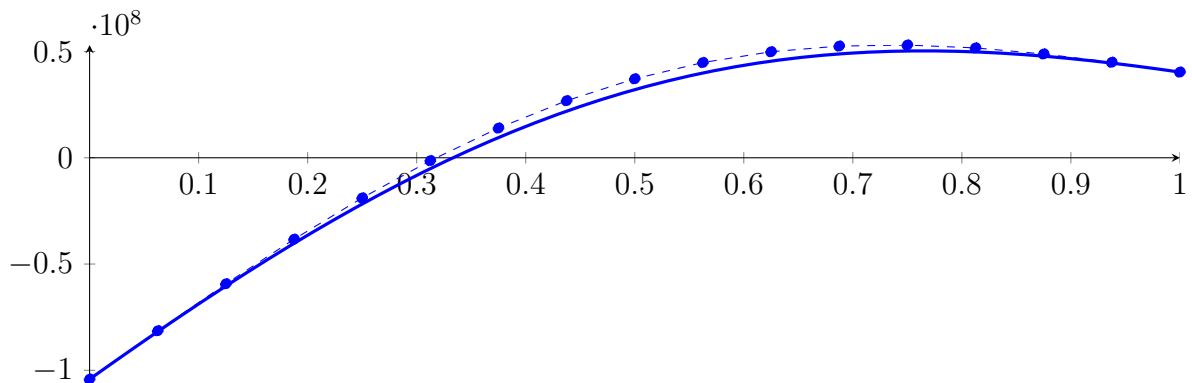
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



159.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 2461.93X^{16} - 19614.9X^{15} + 70879.5X^{14} - 153661X^{13} + 222746X^{12} - 227755X^{11}$$

$$+ 168826X^{10} - 91798.7X^9 + 36630.3X^8 - 10627.3X^7 + 2200.54X^6 - 316.059X^5$$

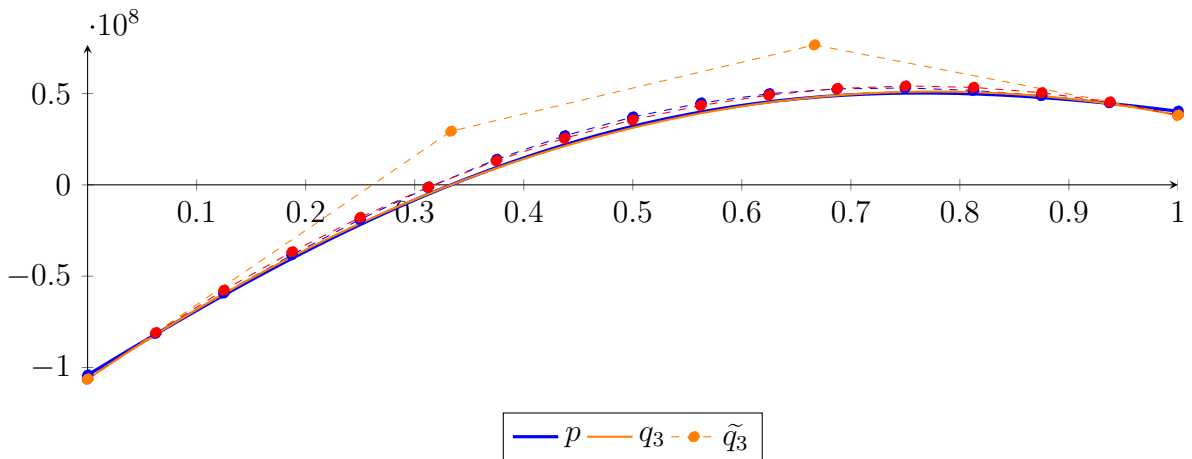
$$+ 30.1958X^4 + 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18551 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91815 \cdot 10^7 B_{10,16} + 5.27353 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

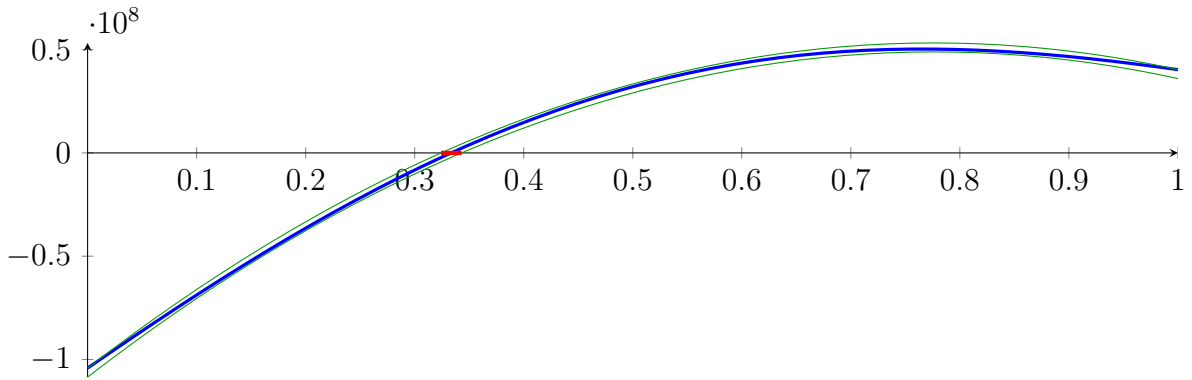
$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\} \quad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

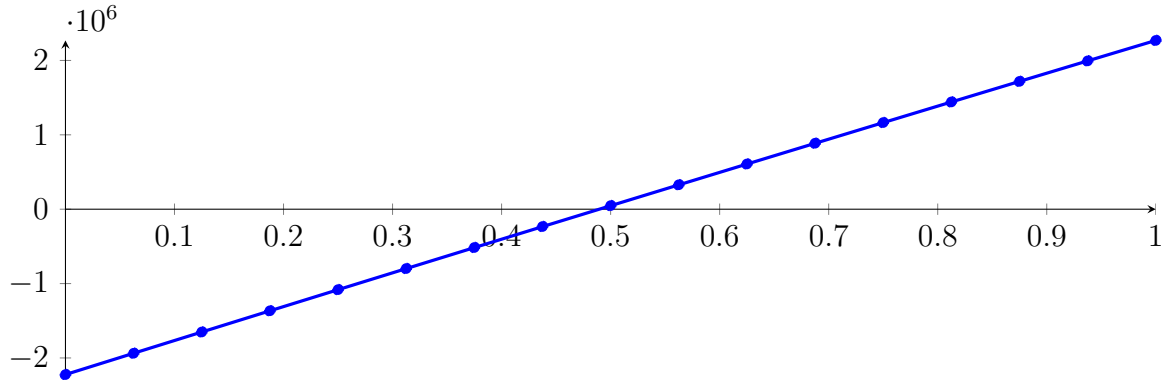
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

159.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

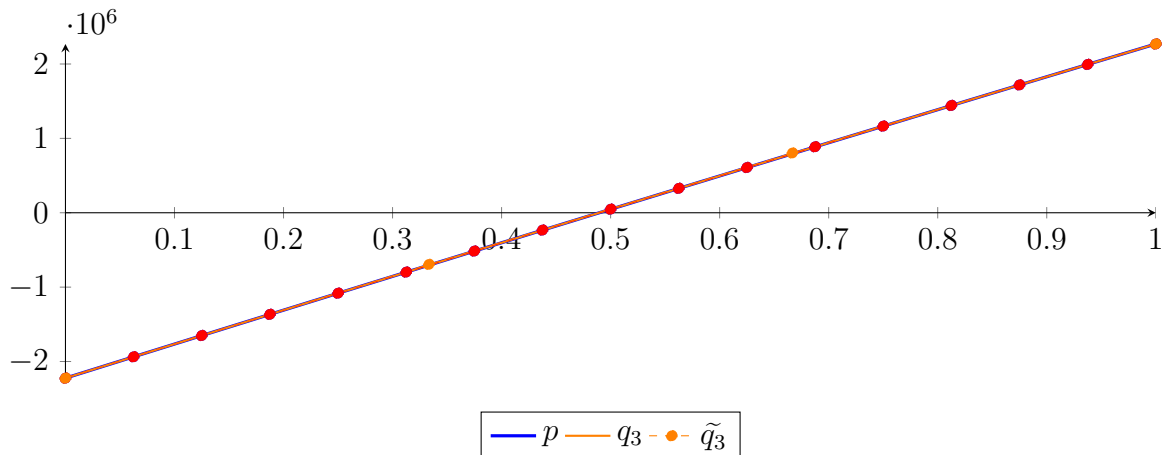
$$\begin{aligned}
 p &= -3.66617 \cdot 10^{-09} X^{16} - 1.53217 \cdot 10^{-07} X^{15} - 3.62234 \cdot 10^{-07} X^{14} - 1.65579 \cdot 10^{-06} X^{13} - 1.15373 \cdot 10^{-06} X^{12} \\
 &\quad - 2.3399 \cdot 10^{-06} X^{11} - 5.02543 \cdot 10^{-07} X^{10} - 1.38381 \cdot 10^{-06} X^9 + 1.1237 \cdot 10^{-06} X^8 - 1.19653 \cdot 10^{-05} X^7 \\
 &\quad - 0.00155608 X^6 + 0.10496 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 16.4956 X^{16} - 129.161 X^{15} + 457.83 X^{14} - 971.671 X^{13} + 1375.95 X^{12} - 1370.96 X^{11} \\
 &\quad + 987.265 X^{10} - 519.476 X^9 + 199.587 X^8 - 55.434 X^7 + 10.9237 X^6 - 1.48019 X^5 \\
 &\quad + 0.129516 X^4 - 700.679 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} \\
 &\quad + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m :

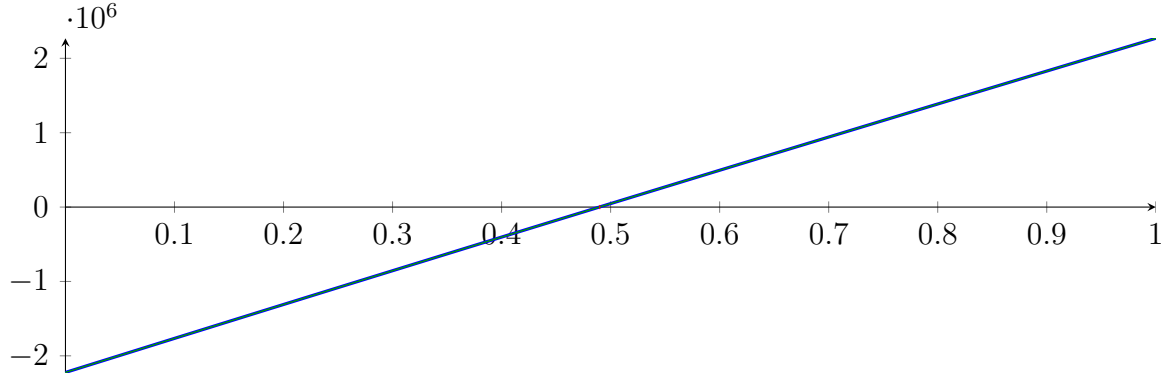
$$M = -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\} \quad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

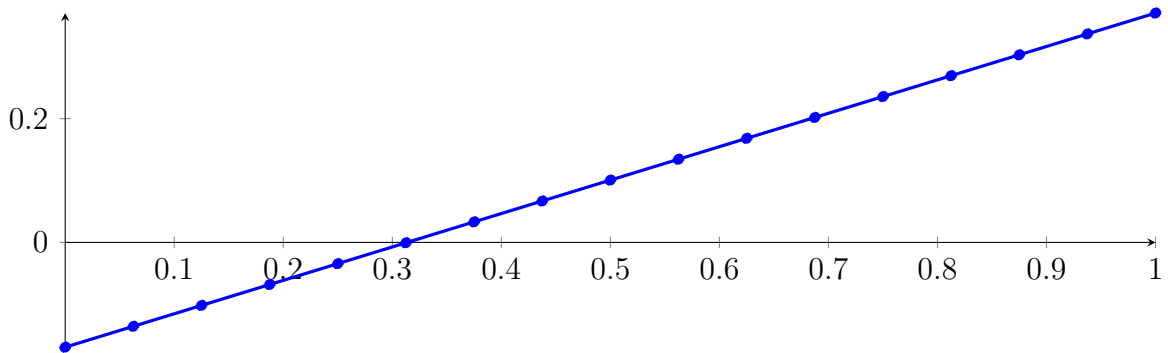
Longest intersection interval: $1.20174 \cdot 10^{-07}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

159.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

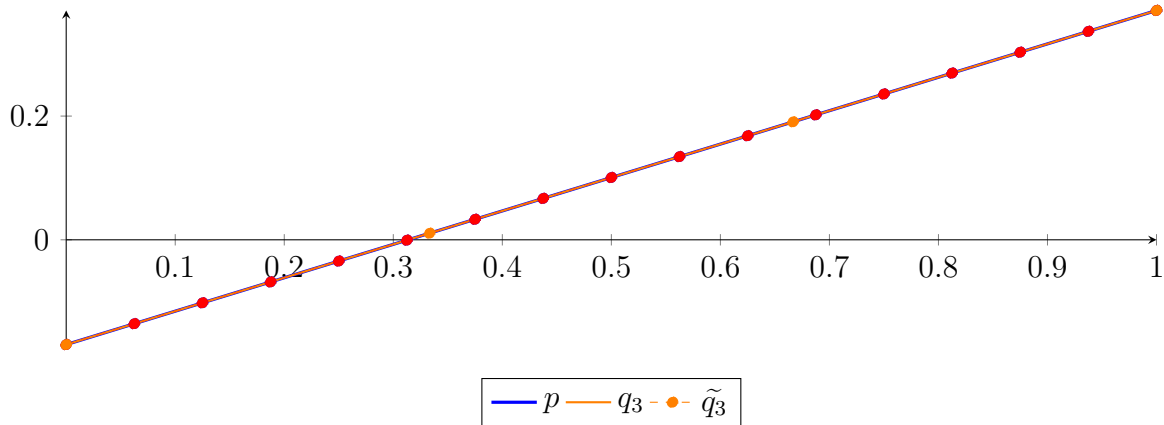
$$\begin{aligned} p &= 5.55524 \cdot 10^{-15} X^{16} - 2.94313 \cdot 10^{-14} X^{15} + 1.19384 \cdot 10^{-13} X^{14} - 2.17482 \cdot 10^{-13} X^{13} + 7.26155 \cdot 10^{-14} X^{12} \\ &\quad - 3.44766 \cdot 10^{-13} X^{11} - 3.47292 \cdot 10^{-15} X^{10} - 1.1287 \cdot 10^{-13} X^9 + 2.93027 \cdot 10^{-14} X^8 + 8.06213 \cdot 10^{-15} X^7 \\ &\quad + 5.64349 \cdot 10^{-15} X^6 + 4.93312 \cdot 10^{-17} X^4 + 1.51788 \cdot 10^{-17} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396 \\ &= -0.169396 B_{0,16}(X) - 0.135637 B_{1,16}(X) - 0.101877 B_{2,16}(X) - 0.068118 B_{3,16}(X) \\ &\quad - 0.0343587 B_{4,16}(X) - 0.000599476 B_{5,16}(X) + 0.0331598 B_{6,16}(X) \\ &\quad + 0.0669191 B_{7,16}(X) + 0.100678 B_{8,16}(X) + 0.134438 B_{9,16}(X) + 0.168197 B_{10,16}(X) \\ &\quad + 0.201956 B_{11,16}(X) + 0.235715 B_{12,16}(X) + 0.269475 B_{13,16}(X) \\ &\quad + 0.303234 B_{14,16}(X) + 0.336993 B_{15,16}(X) + 0.370752 B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07065 \cdot 10^{-17} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396 \\ &= -0.169396 B_{0,3} + 0.0106536 B_{1,3} + 0.190703 B_{2,3} + 0.370752 B_{3,3} \end{aligned}$$

$$\begin{aligned}
\tilde{q}_3 &= 8.59095 \cdot 10^{-06} X^{16} - 6.82648 \cdot 10^{-05} X^{15} + 0.000245968 X^{14} - 0.000531568 X^{13} \\
&+ 0.000767923 X^{12} - 0.000782231 X^{11} + 0.0005774 X^{10} - 0.000312464 X^9 \\
&+ 0.000123994 X^8 - 3.57388 \cdot 10^{-05} X^7 + 7.34249 \cdot 10^{-06} X^6 - 1.04474 \cdot 10^{-06} X^5 \\
&+ 9.86739 \cdot 10^{-08} X^4 - 5.7553 \cdot 10^{-09} X^3 - 1.19186 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396 \\
&= -0.169396 B_{0,16} - 0.135637 B_{1,16} - 0.101877 B_{2,16} - 0.068118 B_{3,16} - 0.0343587 B_{4,16} \\
&- 0.000599476 B_{5,16} + 0.0331598 B_{6,16} + 0.0669191 B_{7,16} + 0.100678 B_{8,16} \\
&+ 0.134438 B_{9,16} + 0.168197 B_{10,16} + 0.201956 B_{11,16} + 0.235715 B_{12,16} \\
&+ 0.269475 B_{13,16} + 0.303234 B_{14,16} + 0.336993 B_{15,16} + 0.370752 B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.81206 \cdot 10^{-10}$.

Bounding polynomials M and m :

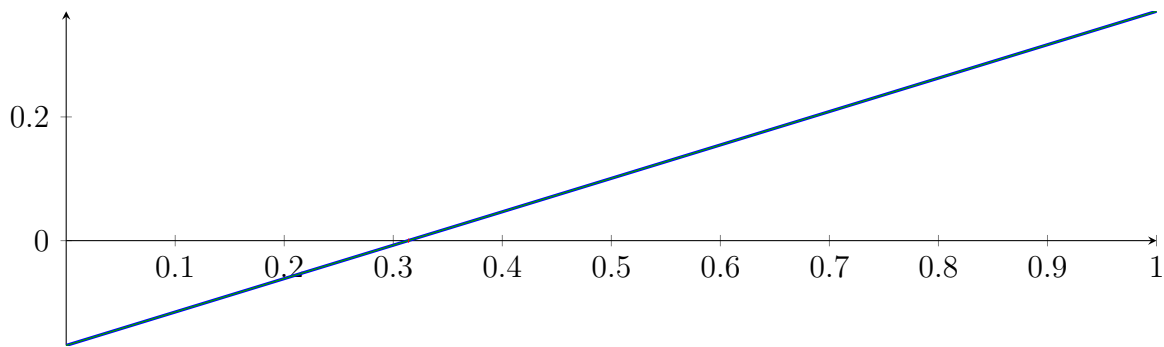
$$M = -1.07065 \cdot 10^{-17} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

$$m = -1.07065 \cdot 10^{-17} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

Root of M and m :

$$N(M) = \{-2.97569 \cdot 10^8, 0.31361, 1.69542 \cdot 10^8\} \quad N(m) = \{-2.97569 \cdot 10^8, 0.31361, 1.69542 \cdot 10^8\}$$

Intersection intervals:



$$[0.31361, 0.31361]$$

Longest intersection interval: $7.85803 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

159.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

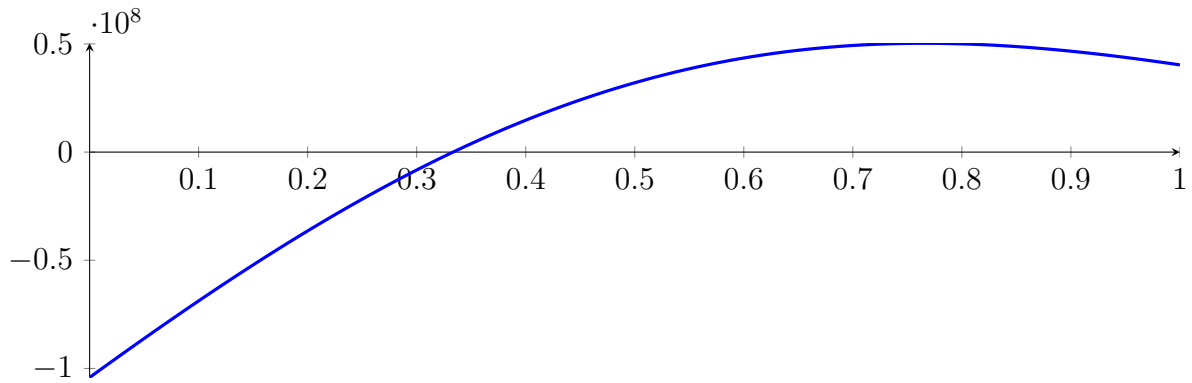
Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 4!

$$p(0) = -2.39831e-08 - p(1) -2.35587e-08$$

159.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

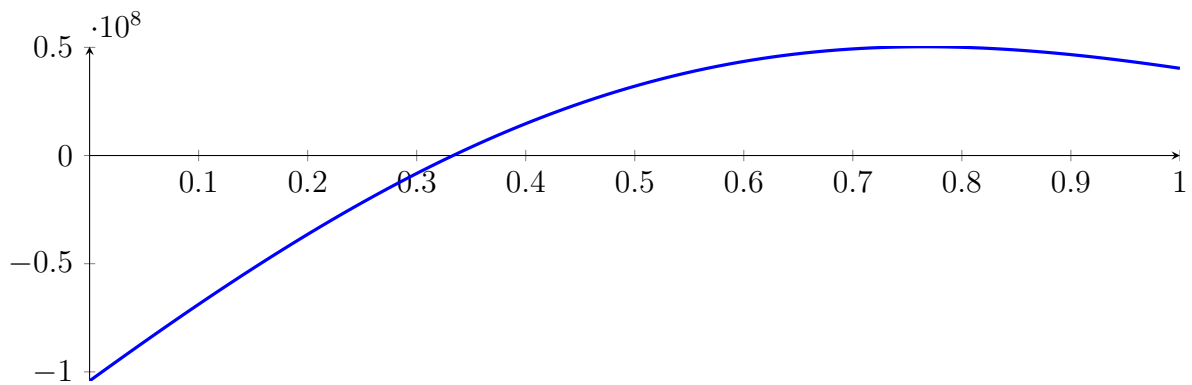
with precision $\varepsilon = 1 \cdot 10^{-16}$.

160 Running BezClip on f_{16} with epsilon 32

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

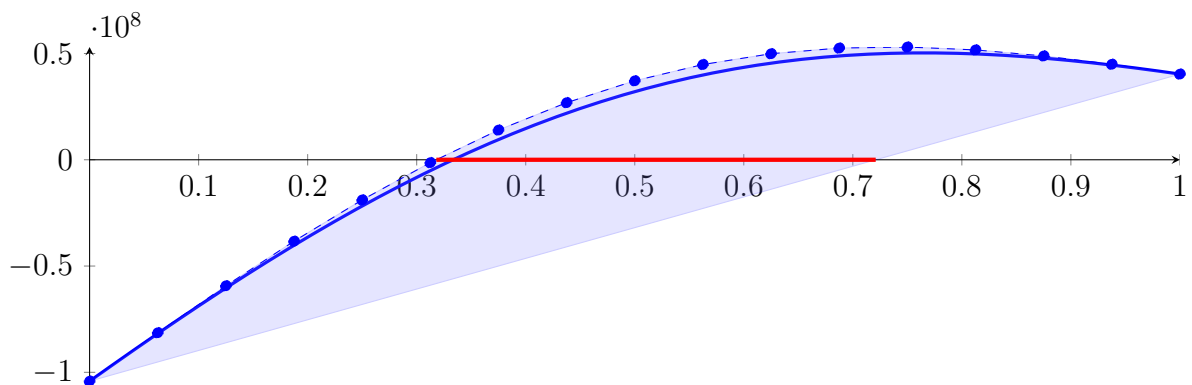
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



160.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

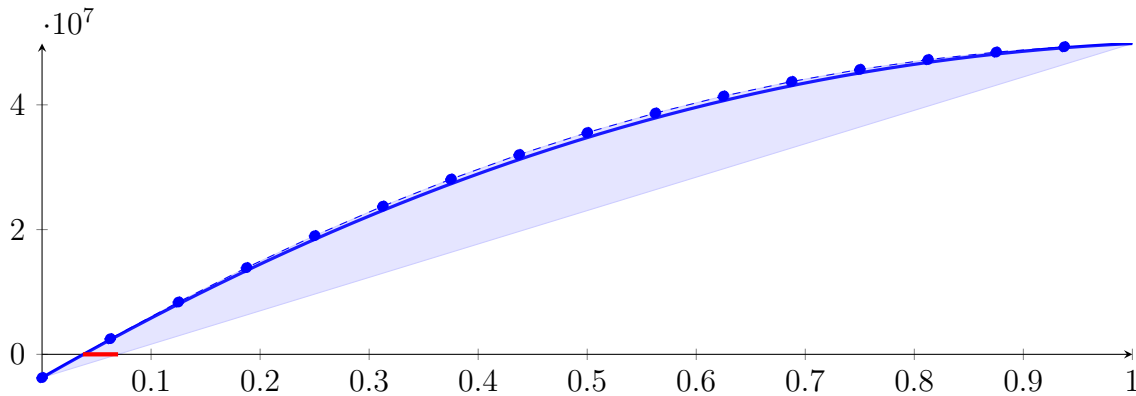
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

160.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.59825 \cdot 10^{-06} X^{16} - 5.93153 \cdot 10^{-05} X^{15} - 0.00248867 X^{14} - 0.0650056 X^{13} - 0.909142 X^{12} \\ &\quad - 5.03931 X^{11} + 36.4549 X^{10} + 692.921 X^9 + 1886.97 X^8 - 25792 X^7 - 149671 X^6 + 492605 X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

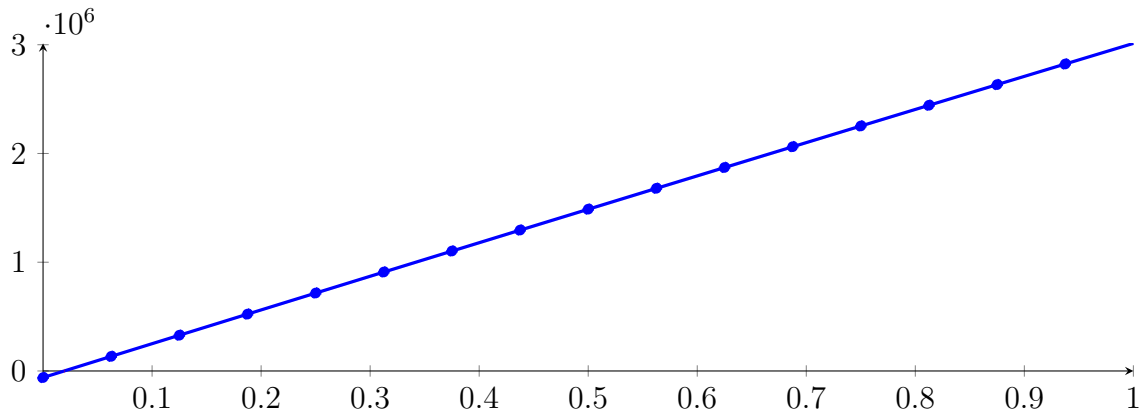
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

160.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 9.01396 \cdot 10^{-08} X^{16} - 2.65848 \cdot 10^{-07} X^{15} + 2.13948 \cdot 10^{-06} X^{14} - 1.33627 \cdot 10^{-06} X^{13} + 2.46973 \cdot 10^{-06} X^{12} \\ &\quad - 2.45524 \cdot 10^{-06} X^{11} + 5.50112 \cdot 10^{-07} X^{10} - 1.64198 \cdot 10^{-07} X^9 - 7.35598 \cdot 10^{-07} X^8 - 1.00892 \cdot 10^{-06} X^7 \\ &\quad - 0.000177509 X^6 + 0.0161038 X^5 + 4.36155 X^4 - 234.216 X^3 - 45622.2 X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

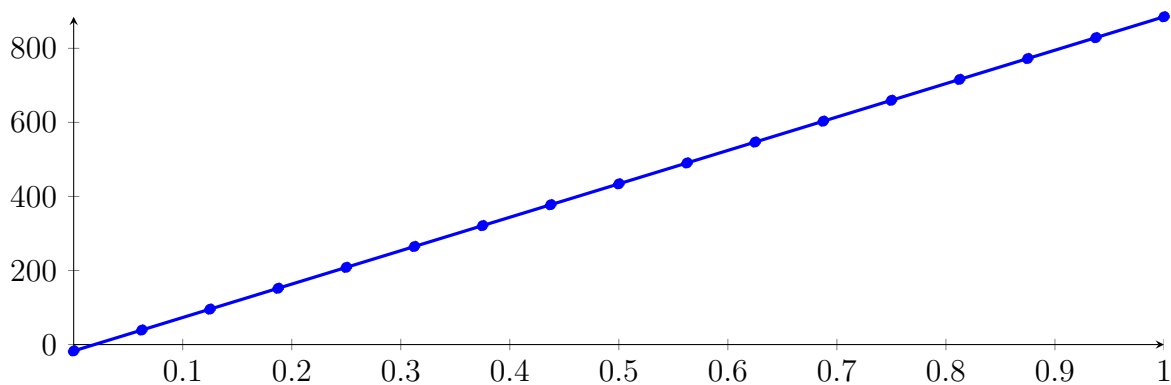
Longest intersection interval: 0.000289554

⇒ Selective recursion: interval 1: [0.333333, 0.333337],

160.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333337]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 2.55372 \cdot 10^{-11} X^{16} - 7.21263 \cdot 10^{-11} X^{15} + 6.24141 \cdot 10^{-10} X^{14} - 4.11162 \cdot 10^{-10} X^{13} \\
 &+ 6.82359 \cdot 10^{-10} X^{12} - 7.09475 \cdot 10^{-10} X^{11} + 9.71305 \cdot 10^{-11} X^{10} - 3.46101 \cdot 10^{-11} X^9 \\
 &- 2.13971 \cdot 10^{-10} X^8 - 1.46061 \cdot 10^{-11} X^7 - 1.63366 \cdot 10^{-11} X^6 + 1.87916 \cdot 10^{-12} X^5 \\
 &+ 2.52576 \cdot 10^{-14} X^4 - 5.67777 \cdot 10^{-09} X^3 - 0.00382618 X^2 + 902.448 X - 17.178 \\
 &= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
 &+ 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
 &+ 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
 &+ 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

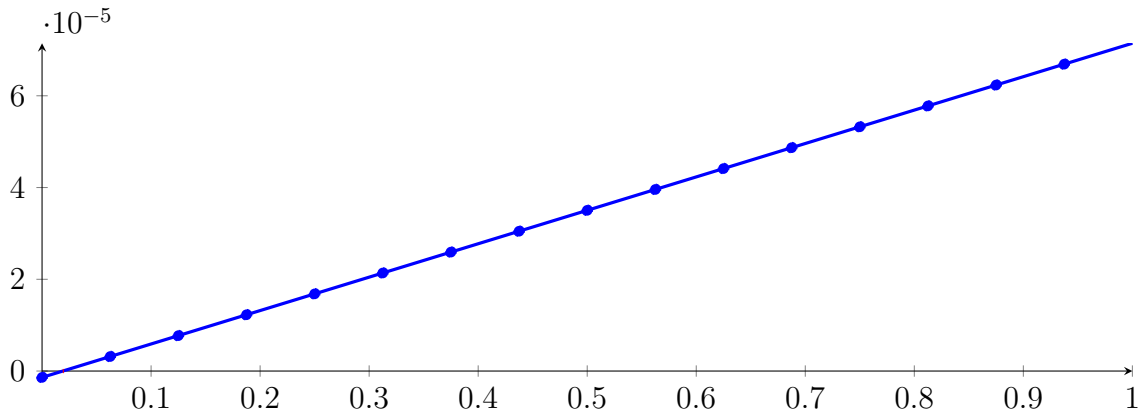
Longest intersection interval: $8.07045 \cdot 10^{-08}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

160.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 2.14261 \cdot 10^{-18} X^{16} - 6.28573 \cdot 10^{-18} X^{15} + 5.28612 \cdot 10^{-17} X^{14} - 3.67279 \cdot 10^{-17} X^{13} \\
 &\quad + 6.10136 \cdot 10^{-17} X^{12} - 6.60335 \cdot 10^{-17} X^{11} + 1.66661 \cdot 10^{-17} X^{10} - 8.36524 \cdot 10^{-18} X^9 \\
 &\quad - 1.56919 \cdot 10^{-17} X^8 - 1.85474 \cdot 10^{-18} X^7 - 1.4308 \cdot 10^{-18} X^6 + 1.1562 \cdot 10^{-19} X^5 - 1.20437 \\
 &\quad \cdot 10^{-20} X^4 - 4.63221 \cdot 10^{-22} X^3 - 2.49207 \cdot 10^{-17} X^2 + 7.28316 \cdot 10^{-05} X - 1.38634 \cdot 10^{-06} \\
 &= -1.38634 \cdot 10^{-06} B_{0,16}(X) + 3.16564 \cdot 10^{-06} B_{1,16}(X) + 7.71761 \cdot 10^{-06} B_{2,16}(X) + 1.22696 \\
 &\quad \cdot 10^{-05} B_{3,16}(X) + 1.68216 \cdot 10^{-05} B_{4,16}(X) + 2.13735 \cdot 10^{-05} B_{5,16}(X) + 2.59255 \cdot 10^{-05} B_{6,16}(X) \\
 &\quad + 3.04775 \cdot 10^{-05} B_{7,16}(X) + 3.50295 \cdot 10^{-05} B_{8,16}(X) + 3.95814 \cdot 10^{-05} B_{9,16}(X) + 4.41334 \\
 &\quad \cdot 10^{-05} B_{10,16}(X) + 4.86854 \cdot 10^{-05} B_{11,16}(X) + 5.32374 \cdot 10^{-05} B_{12,16}(X) + 5.77893 \cdot 10^{-05} B_{13,16}(X) \\
 &\quad + 6.23413 \cdot 10^{-05} B_{14,16}(X) + 6.68933 \cdot 10^{-05} B_{15,16}(X) + 7.14453 \cdot 10^{-05} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.51313 \cdot 10^{-15}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

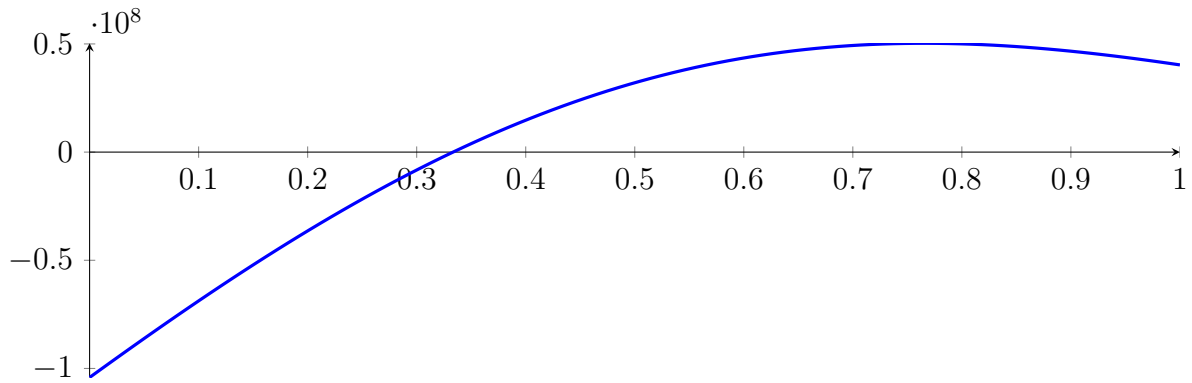
160.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

160.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

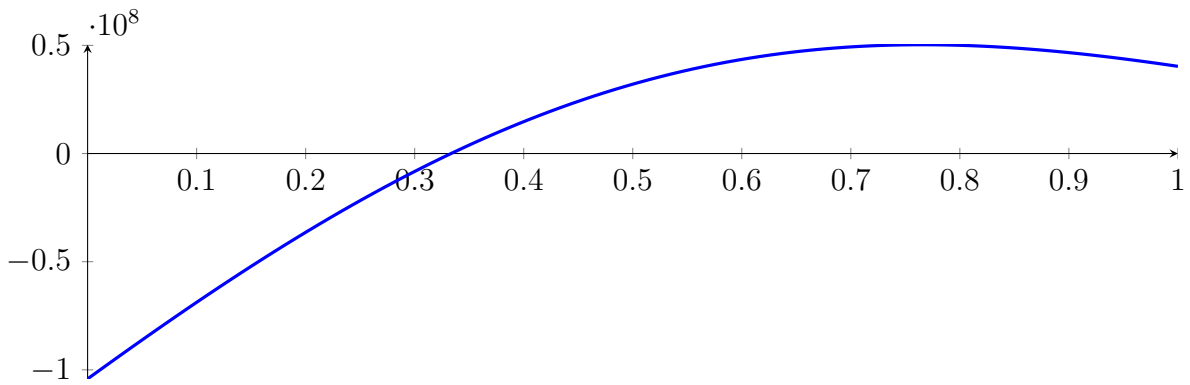
with precision $\varepsilon = 1 \cdot 10^{-32}$.

161 Running QuadClip on f_{16} with epsilon 32

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

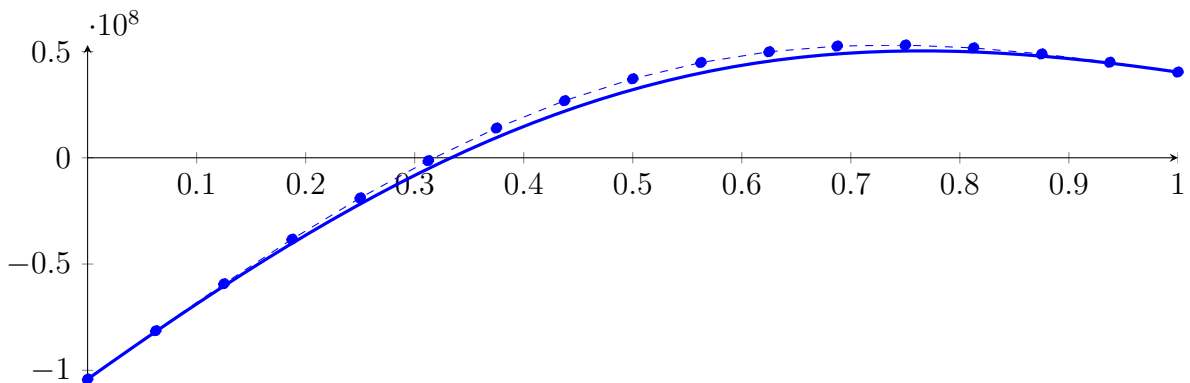
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



161.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = 6049.18X^{16} - 48305.2X^{15} + 174971X^{14} - 380294X^{13} + 552846X^{12} - 567203X^{11}$$

$$+ 422303X^{10} - 231038X^9 + 93003.6X^8 - 27320.1X^7 + 5752.57X^6 - 843.63X^5$$

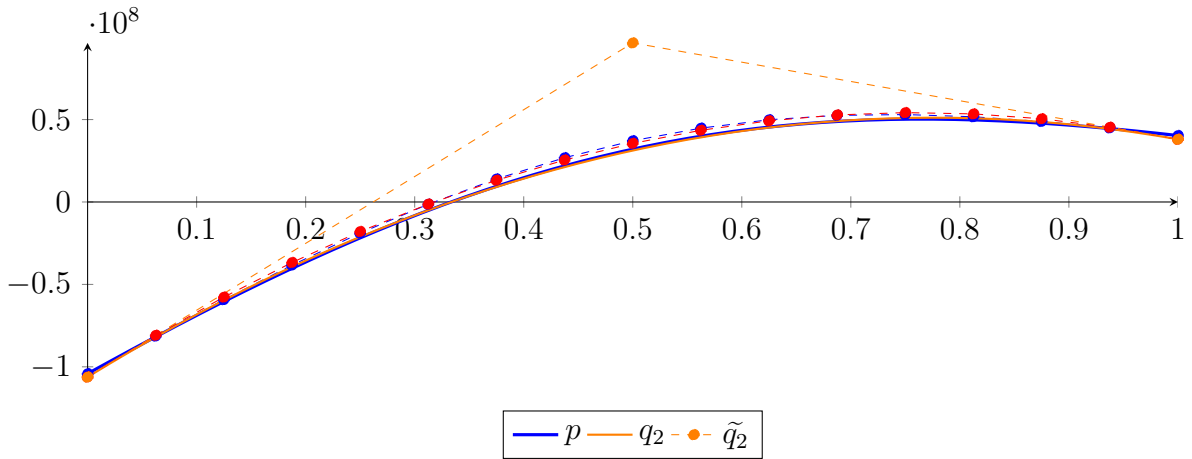
$$+ 82.5145X^4 - 5.01388X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

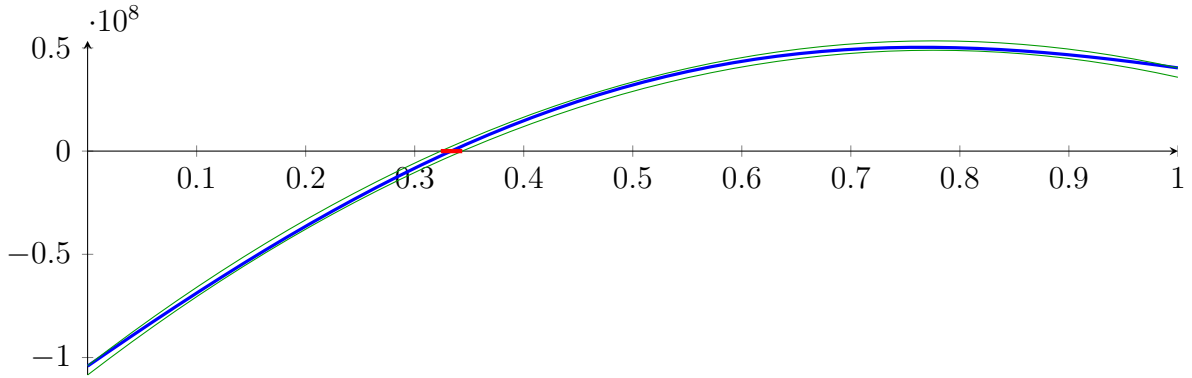
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

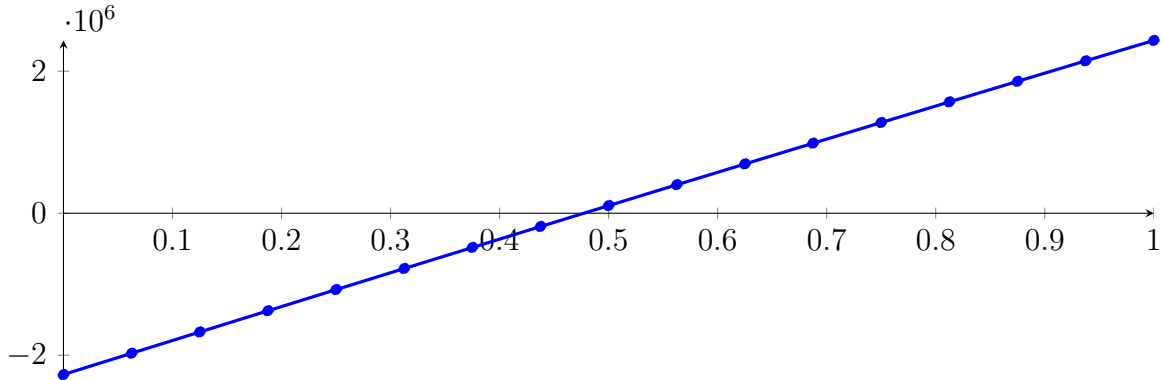
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

161.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

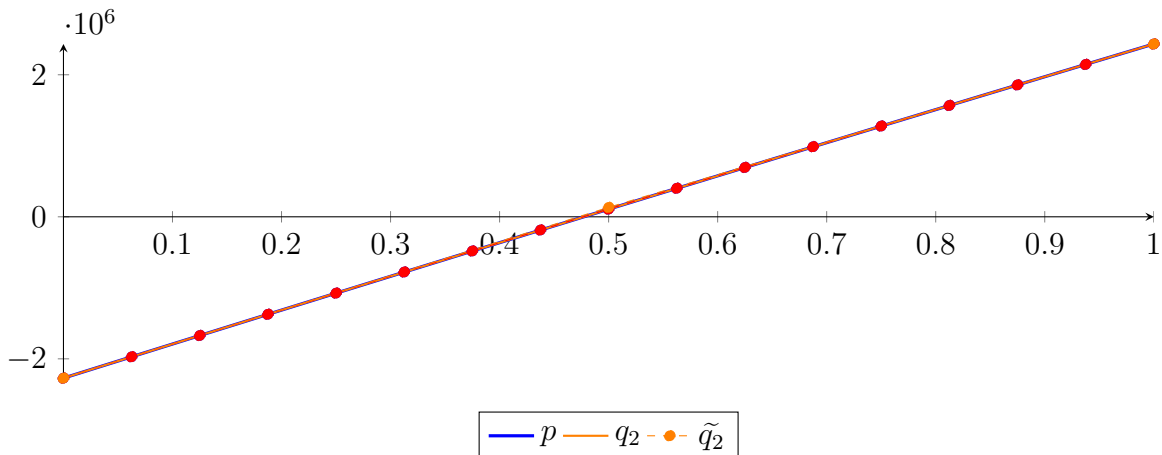
$$\begin{aligned}
 p &= -1.54841 \cdot 10^{-10} X^{16} - 1.66965 \cdot 10^{-07} X^{15} - 2.92739 \cdot 10^{-07} X^{14} - 1.77943 \cdot 10^{-06} X^{13} - 1.17235 \cdot 10^{-06} X^{12} \\
 &\quad - 2.42234 \cdot 10^{-06} X^{11} - 6.86445 \cdot 10^{-07} X^{10} - 1.39162 \cdot 10^{-06} X^9 + 1.07395 \cdot 10^{-06} X^8 - 1.67072 \cdot 10^{-05} X^7 \\
 &\quad - 0.00205879 X^6 + 0.132721 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 22.7036 X^{16} - 178.964 X^{15} + 638.976 X^{14} - 1366.64 X^{13} + 1951.1 X^{12} - 1960.92 X^{11} \\
 &\quad + 1425.35 X^{10} - 757.631 X^9 + 294.177 X^8 - 82.4368 X^7 + 16.2856 X^6 - 2.18949 X^5 \\
 &\quad + 0.191237 X^4 - 0.0101048 X^3 - 104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} \\
 &\quad + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

$$M = -104265 X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

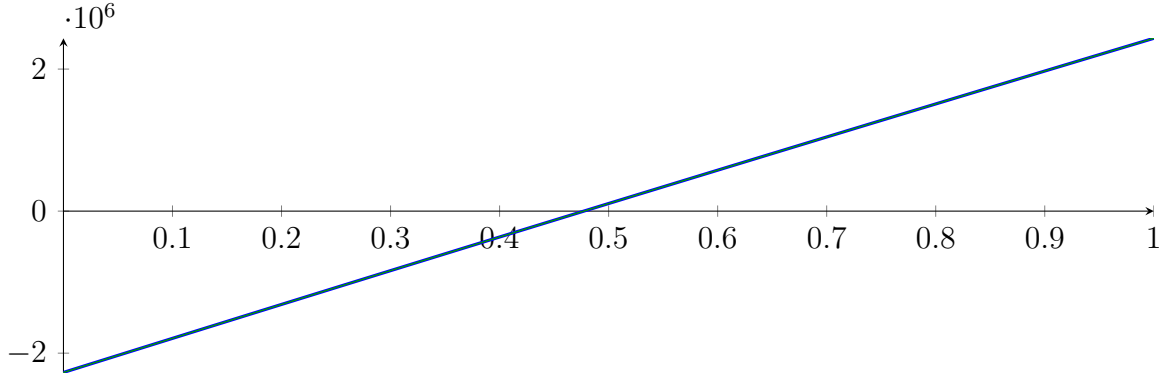
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\}$$

$$N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

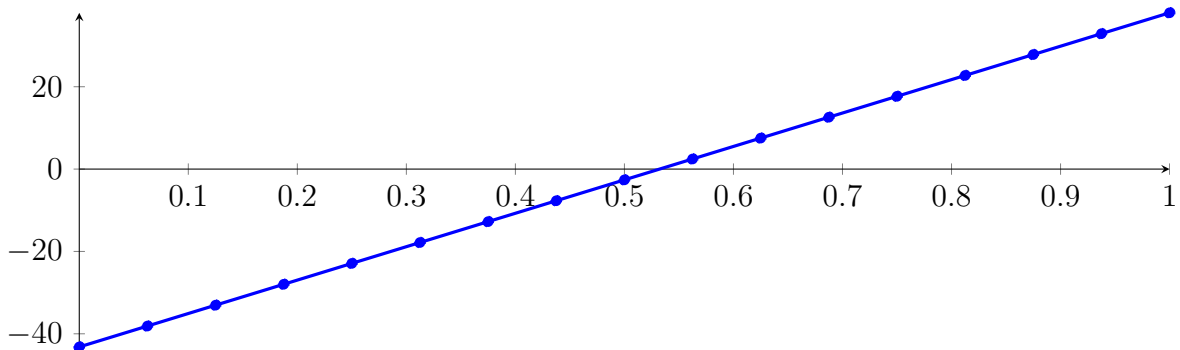
Longest intersection interval: $1.72301 \cdot 10^{-05}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

161.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

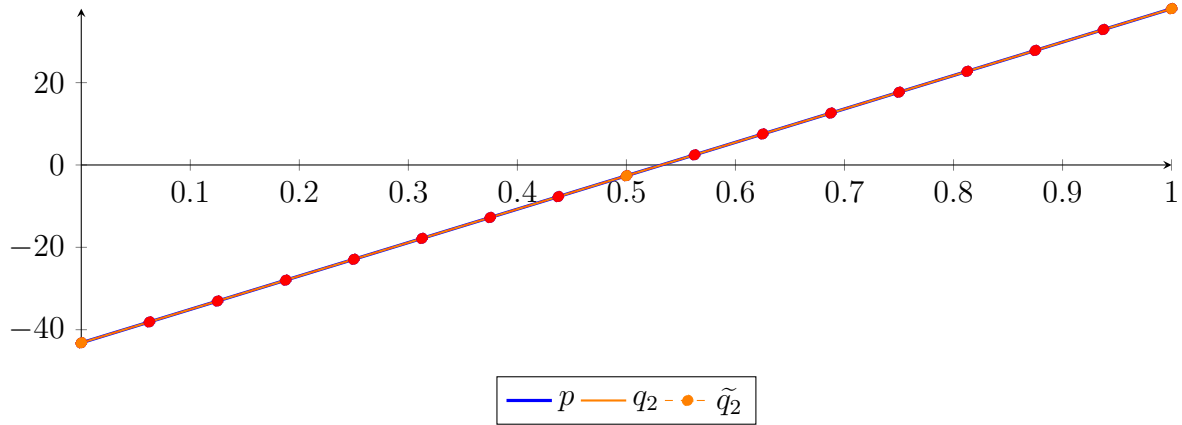
$$\begin{aligned} p &= -2.76723 \cdot 10^{-13} X^{16} - 2.40874 \cdot 10^{-12} X^{15} - 1.25233 \cdot 10^{-11} X^{14} - 3.02935 \cdot 10^{-11} X^{13} \\ &\quad - 3.05617 \cdot 10^{-11} X^{12} - 3.83107 \cdot 10^{-11} X^{11} - 1.26692 \cdot 10^{-11} X^{10} - 2.6672 \cdot 10^{-11} X^9 \\ &\quad + 2.0004 \cdot 10^{-11} X^8 + 4.12781 \cdot 10^{-12} X^7 + 2.44493 \cdot 10^{-12} X^6 - 1.21236 \cdot 10^{-13} X^5 \\ &\quad + 1.26288 \cdot 10^{-14} X^4 - 4.1267 \cdot 10^{-12} X^3 - 3.09388 \cdot 10^{-05} X^2 + 81.1506X - 43.1911 \\ &= -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\ &\quad - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X) \\ &\quad + 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X) \\ &\quad + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$\begin{aligned}
\tilde{q}_2 &= 5.96265 \cdot 10^{-05} X^{16} - 0.000436042 X^{15} + 0.00141812 X^{14} - 0.00269475 X^{13} \\
&+ 0.00329809 X^{12} - 0.00268757 X^{11} + 0.00143268 X^{10} - 0.000439599 X^9 \\
&+ 1.98418 \cdot 10^{-05} X^8 + 4.87608 \cdot 10^{-05} X^7 - 2.46333 \cdot 10^{-05} X^6 + 6.35808 \cdot 10^{-06} X^5 \\
&- 9.62755 \cdot 10^{-07} X^4 + 8.21372 \cdot 10^{-08} X^3 - 3.09429 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911 \\
&= -43.1911 B_{0,16} - 38.1192 B_{1,16} - 33.0473 B_{2,16} - 27.9754 B_{3,16} - 22.9035 B_{4,16} - 17.8316 B_{5,16} \\
&- 12.7597 B_{6,16} - 7.68778 B_{7,16} - 2.61587 B_{8,16} + 2.45604 B_{9,16} + 7.52795 B_{10,16} + 12.5999 B_{11,16} \\
&+ 17.6718 B_{12,16} + 22.7437 B_{13,16} + 27.8156 B_{14,16} + 32.8875 B_{15,16} + 37.9594 B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.5947 \cdot 10^{-09}$.

Bounding polynomials M and m :

$$M = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

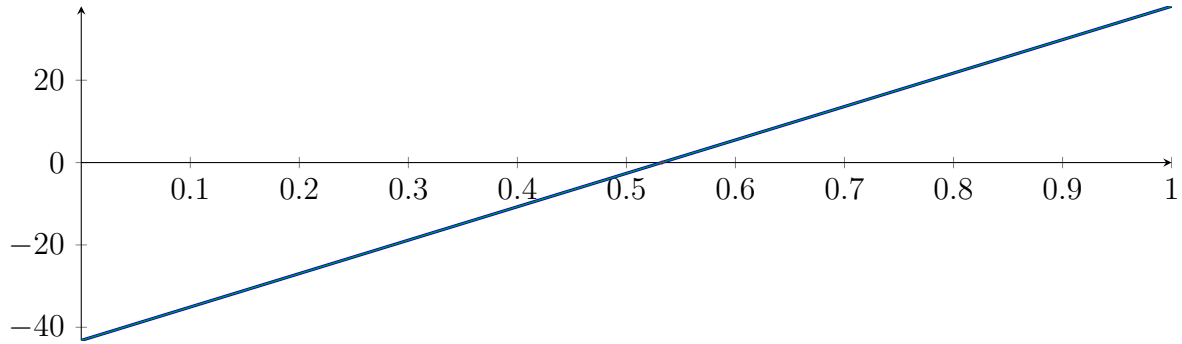
$$m = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

Root of M and m :

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\}$$

$$N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

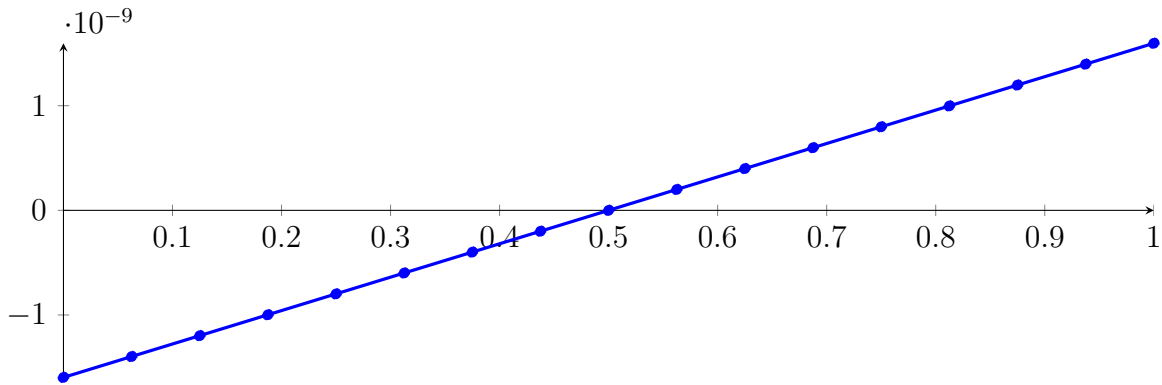
Longest intersection interval: $3.93535 \cdot 10^{-11}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

161.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

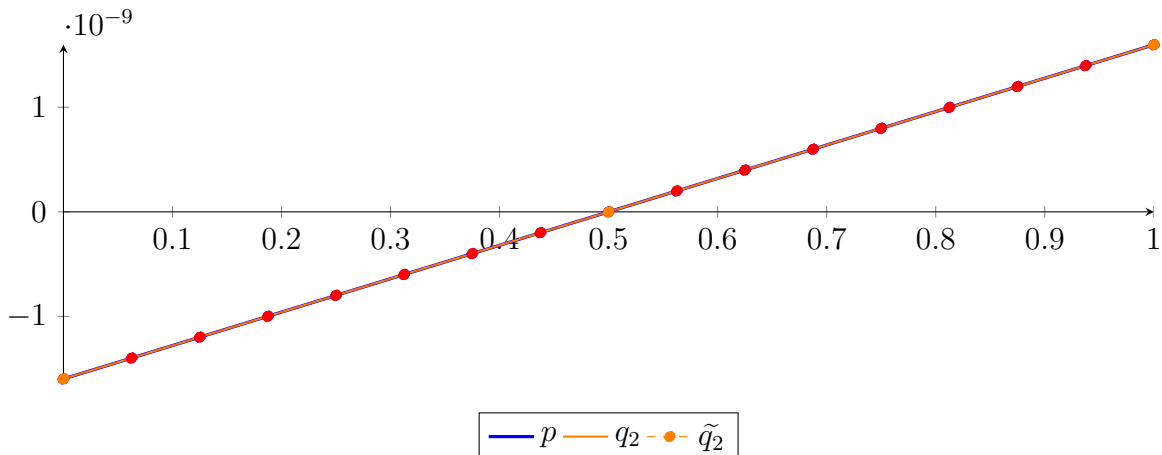
$$\begin{aligned}
 p &= -4.89361 \cdot 10^{-24} X^{16} - 1.05466 \cdot 10^{-22} X^{15} - 3.09805 \cdot 10^{-22} X^{14} - 1.16981 \cdot 10^{-21} X^{13} \\
 &\quad - 9.76202 \cdot 10^{-22} X^{12} - 1.69365 \cdot 10^{-21} X^{11} - 5.95131 \cdot 10^{-22} X^{10} - 1.05349 \cdot 10^{-21} X^9 \\
 &\quad + 7.5893 \cdot 10^{-22} X^8 + 1.47859 \cdot 10^{-22} X^7 + 9.70322 \cdot 10^{-23} X^6 - 7.05688 \cdot 10^{-24} X^5 \\
 &\quad + 1.47018 \cdot 10^{-24} X^4 - 4.84676 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09} \\
 &= -1.59674 \cdot 10^{-09} B_{0,16}(X) - 1.39715 \cdot 10^{-09} B_{1,16}(X) - 1.19755 \cdot 10^{-09} B_{2,16}(X) - 9.97951 \\
 &\quad \cdot 10^{-10} B_{3,16}(X) - 7.98353 \cdot 10^{-10} B_{4,16}(X) - 5.98756 \cdot 10^{-10} B_{5,16}(X) - 3.99159 \cdot 10^{-10} B_{6,16}(X) \\
 &\quad - 1.99561 \cdot 10^{-10} B_{7,16}(X) + 3.6039 \cdot 10^{-14} B_{8,16}(X) + 1.99633 \cdot 10^{-10} B_{9,16}(X) + 3.99231 \\
 &\quad \cdot 10^{-10} B_{10,16}(X) + 5.98828 \cdot 10^{-10} B_{11,16}(X) + 7.98425 \cdot 10^{-10} B_{12,16}(X) + 9.98023 \cdot 10^{-10} B_{13,16}(X) \\
 &\quad + 1.19762 \cdot 10^{-09} B_{14,16}(X) + 1.39722 \cdot 10^{-09} B_{15,16}(X) + 1.59681 \cdot 10^{-09} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -4.83666 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09} \\
 &= -1.59674 \cdot 10^{-09} B_{0,2} + 3.6039 \cdot 10^{-14} B_{1,2} + 1.59681 \cdot 10^{-09} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 9.45798 \cdot 10^{-15} X^{16} - 7.39324 \cdot 10^{-14} X^{15} + 2.61437 \cdot 10^{-13} X^{14} - 5.52989 \cdot 10^{-13} X^{13} \\
 &\quad + 7.79462 \cdot 10^{-13} X^{12} - 7.71893 \cdot 10^{-13} X^{11} + 5.51461 \cdot 10^{-13} X^{10} - 2.8712 \cdot 10^{-13} X^9 \\
 &\quad + 1.08634 \cdot 10^{-13} X^8 - 2.94042 \cdot 10^{-14} X^7 + 5.52081 \cdot 10^{-15} X^6 - 6.84058 \cdot 10^{-16} X^5 + 5.20623 \\
 &\quad \cdot 10^{-17} X^4 - 2.16513 \cdot 10^{-18} X^3 + 1.74369 \cdot 10^{-20} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09} \\
 &= -1.59674 \cdot 10^{-09} B_{0,16} - 1.39715 \cdot 10^{-09} B_{1,16} - 1.19755 \cdot 10^{-09} B_{2,16} - 9.97951 \cdot 10^{-10} B_{3,16} - 7.98353 \\
 &\quad \cdot 10^{-10} B_{4,16} - 5.98756 \cdot 10^{-10} B_{5,16} - 3.99159 \cdot 10^{-10} B_{6,16} - 1.99561 \cdot 10^{-10} B_{7,16} + 3.60393 \cdot 10^{-14} B_{8,16} \\
 &\quad + 1.99633 \cdot 10^{-10} B_{9,16} + 3.99231 \cdot 10^{-10} B_{10,16} + 5.98828 \cdot 10^{-10} B_{11,16} + 7.98425 \cdot 10^{-10} B_{12,16} \\
 &\quad + 9.98023 \cdot 10^{-10} B_{13,16} + 1.19762 \cdot 10^{-09} B_{14,16} + 1.39722 \cdot 10^{-09} B_{15,16} + 1.59681 \cdot 10^{-09} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.02367 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -4.82657 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$

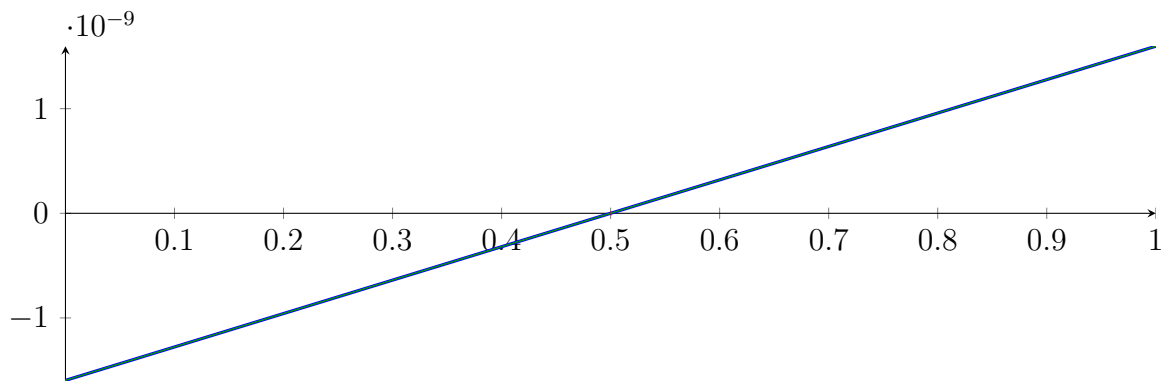
$$m = -4.84676 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$

Root of M and m :

$$N(M) = \{0.5, 6.61662 \cdot 10^{16}\}$$

$$N(m) = \{0.5, 6.58905 \cdot 10^{16}\}$$

Intersection intervals:



$$[0.5, 0.5]$$

Longest intersection interval: 0

⇒ Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

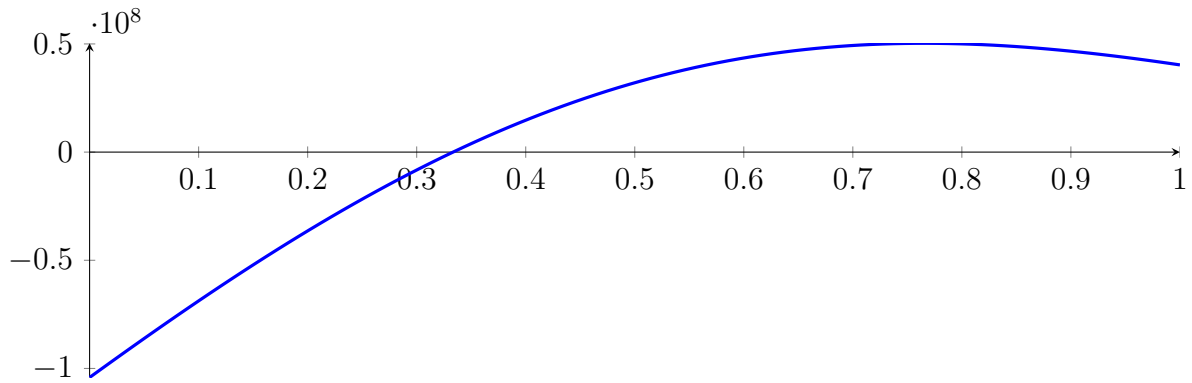
161.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

161.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

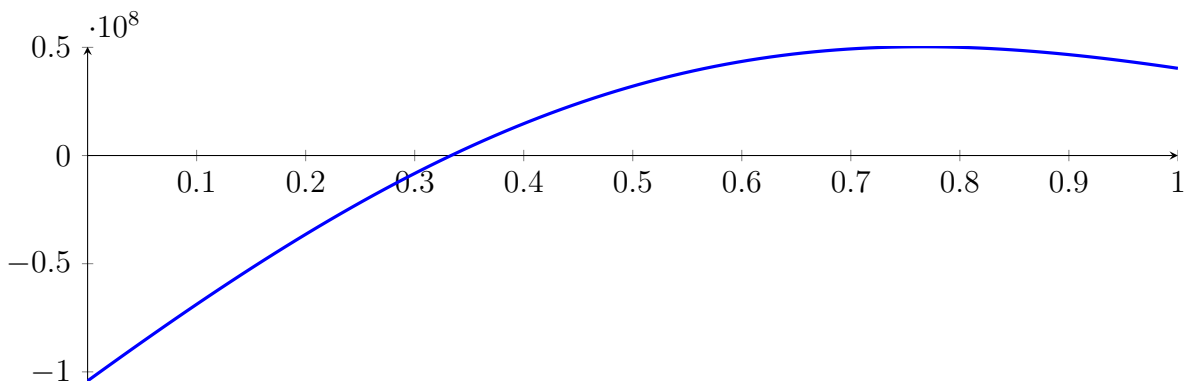
with precision $\varepsilon = 1 \cdot 10^{-32}$.

162 Running CubeClip on f_{16} with epsilon 32

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

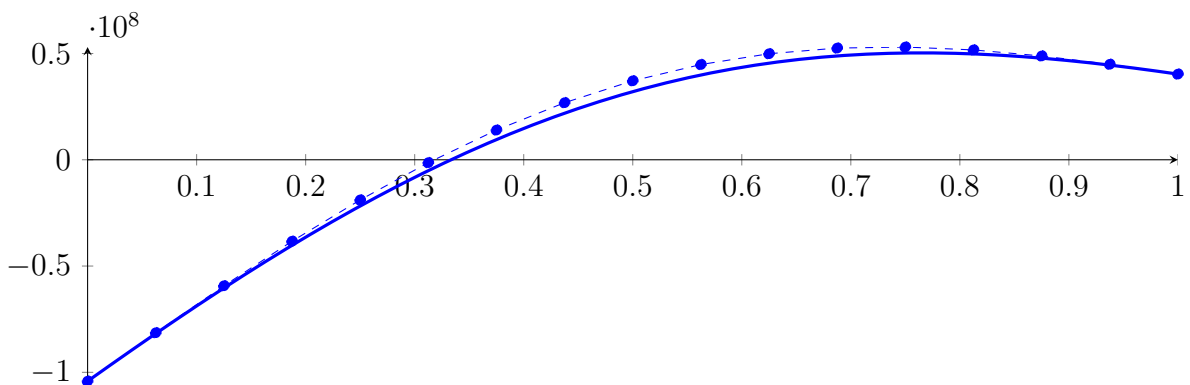
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



162.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 2461.93X^{16} - 19614.9X^{15} + 70879.5X^{14} - 153661X^{13} + 222746X^{12} - 227755X^{11}$$

$$+ 168826X^{10} - 91798.7X^9 + 36630.3X^8 - 10627.3X^7 + 2200.54X^6 - 316.059X^5$$

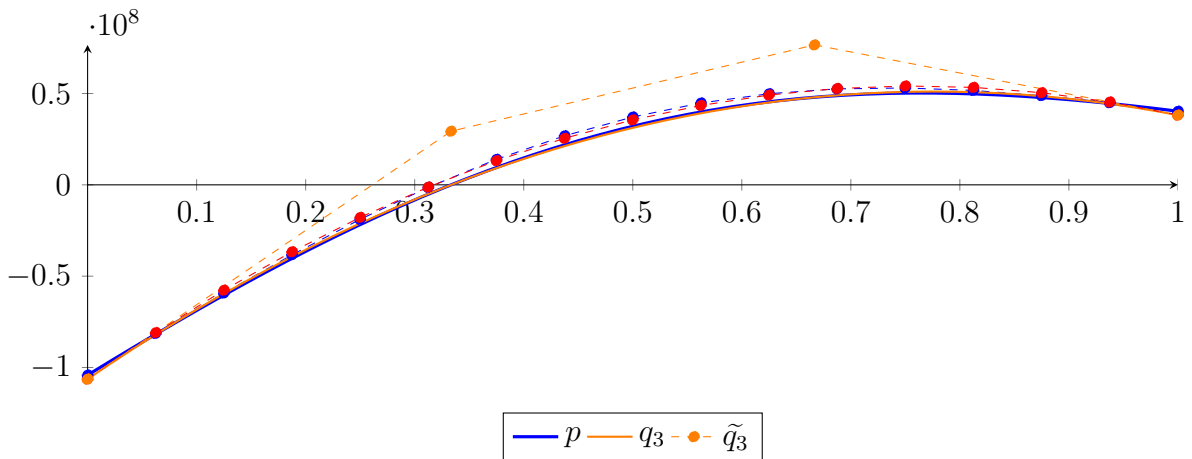
$$+ 30.1958X^4 + 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18551 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91815 \cdot 10^7 B_{10,16} + 5.27353 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

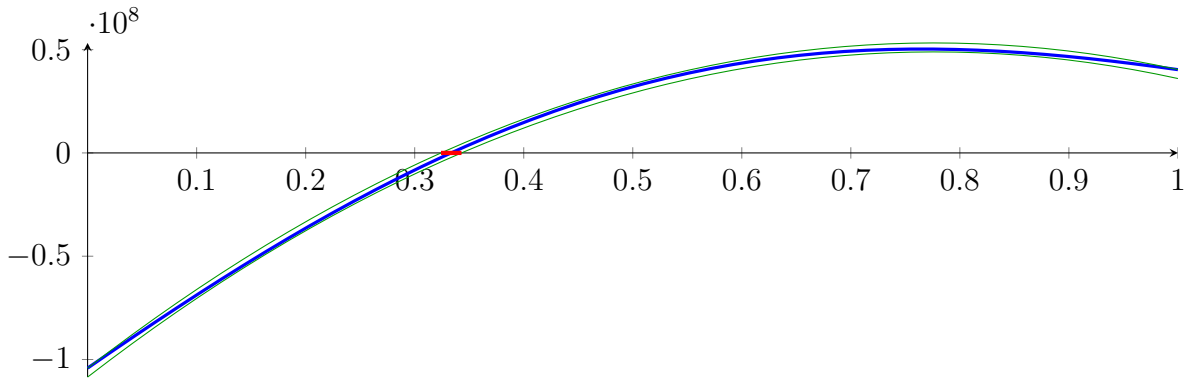
$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\}$$

$$N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

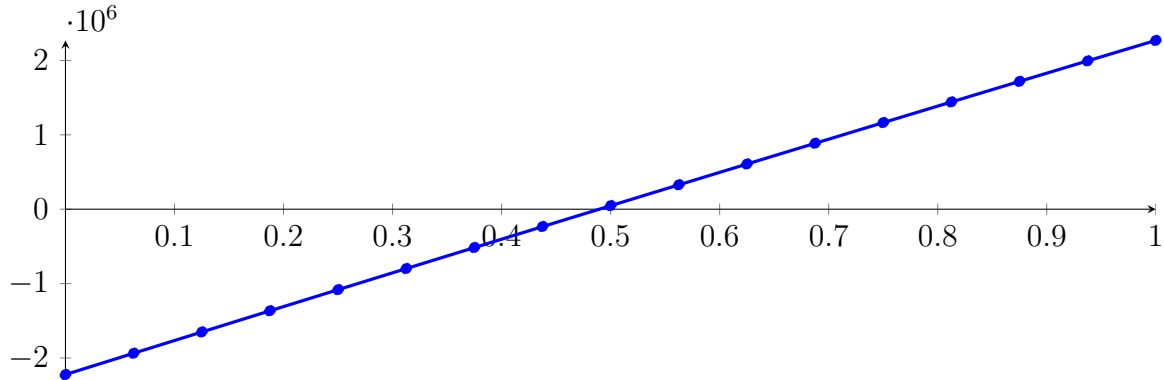
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

162.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

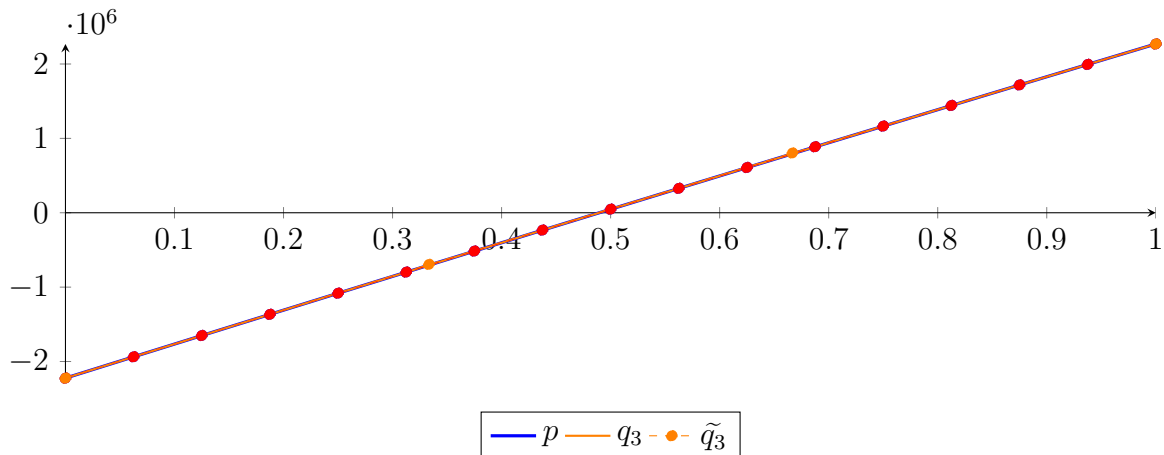
$$\begin{aligned}
 p &= -3.66617 \cdot 10^{-09} X^{16} - 1.53217 \cdot 10^{-07} X^{15} - 3.62234 \cdot 10^{-07} X^{14} - 1.65579 \cdot 10^{-06} X^{13} - 1.15373 \cdot 10^{-06} X^{12} \\
 &\quad - 2.3399 \cdot 10^{-06} X^{11} - 5.02543 \cdot 10^{-07} X^{10} - 1.38381 \cdot 10^{-06} X^9 + 1.1237 \cdot 10^{-06} X^8 - 1.19653 \cdot 10^{-05} X^7 \\
 &\quad - 0.00155608 X^6 + 0.10496 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 16.4956 X^{16} - 129.161 X^{15} + 457.83 X^{14} - 971.671 X^{13} + 1375.95 X^{12} - 1370.96 X^{11} \\
 &\quad + 987.265 X^{10} - 519.476 X^9 + 199.587 X^8 - 55.434 X^7 + 10.9237 X^6 - 1.48019 X^5 \\
 &\quad + 0.129516 X^4 - 700.679 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} \\
 &\quad + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m :

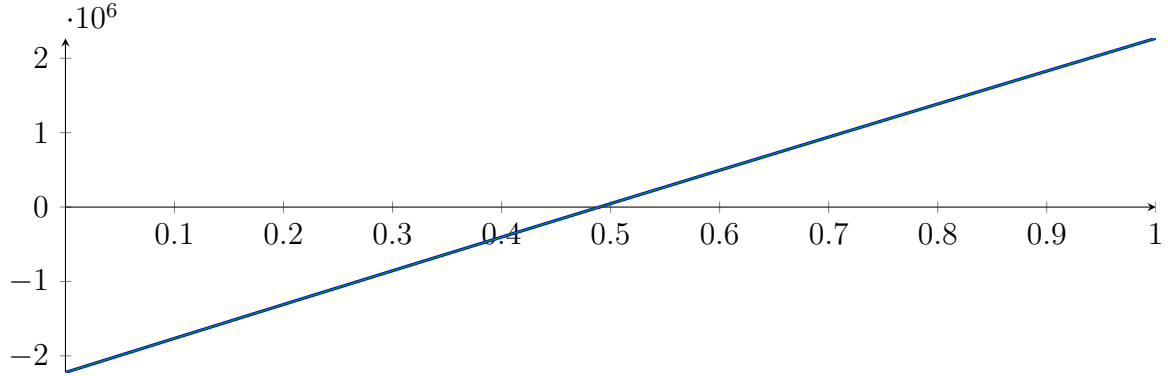
$$M = -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\} \quad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

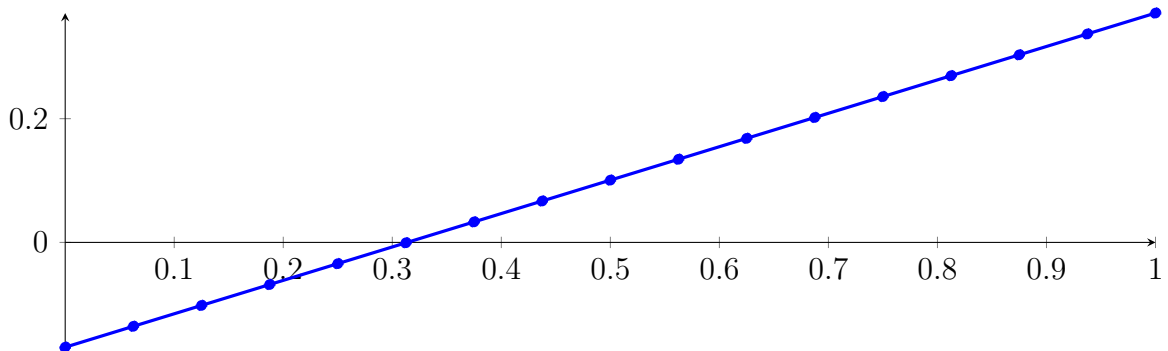
Longest intersection interval: $1.20174 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

162.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

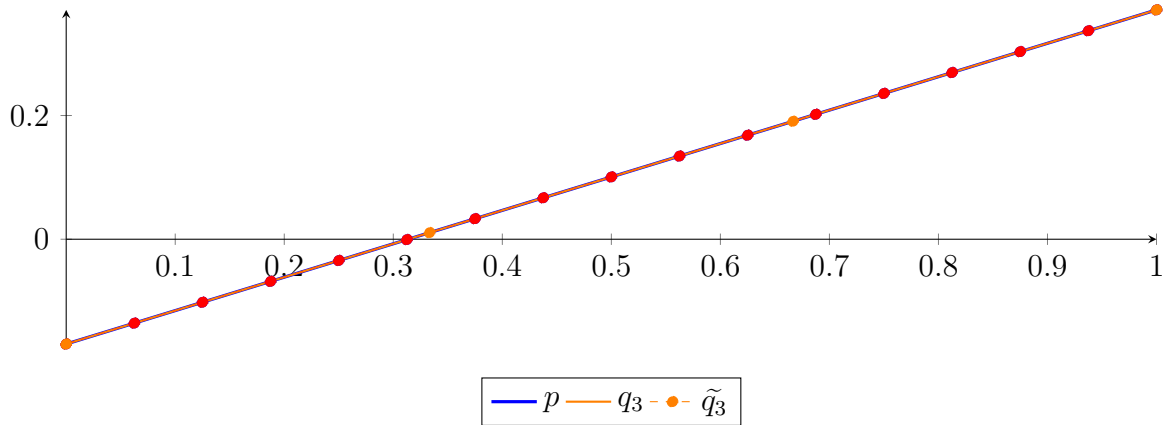
$$\begin{aligned} p &= 5.55524 \cdot 10^{-15} X^{16} - 2.94313 \cdot 10^{-14} X^{15} + 1.19384 \cdot 10^{-13} X^{14} - 2.17482 \cdot 10^{-13} X^{13} + 7.26155 \cdot 10^{-14} X^{12} \\ &\quad - 3.44766 \cdot 10^{-13} X^{11} - 3.47292 \cdot 10^{-15} X^{10} - 1.1287 \cdot 10^{-13} X^9 + 2.93027 \cdot 10^{-14} X^8 + 8.06213 \cdot 10^{-15} X^7 \\ &\quad + 5.64349 \cdot 10^{-15} X^6 + 4.93312 \cdot 10^{-17} X^4 + 1.51788 \cdot 10^{-17} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396 \\ &= -0.169396 B_{0,16}(X) - 0.135637 B_{1,16}(X) - 0.101877 B_{2,16}(X) - 0.068118 B_{3,16}(X) \\ &\quad - 0.0343587 B_{4,16}(X) - 0.000599476 B_{5,16}(X) + 0.0331598 B_{6,16}(X) \\ &\quad + 0.0669191 B_{7,16}(X) + 0.100678 B_{8,16}(X) + 0.134438 B_{9,16}(X) + 0.168197 B_{10,16}(X) \\ &\quad + 0.201956 B_{11,16}(X) + 0.235715 B_{12,16}(X) + 0.269475 B_{13,16}(X) \\ &\quad + 0.303234 B_{14,16}(X) + 0.336993 B_{15,16}(X) + 0.370752 B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07065 \cdot 10^{-17} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396 \\ &= -0.169396 B_{0,3} + 0.0106536 B_{1,3} + 0.190703 B_{2,3} + 0.370752 B_{3,3} \end{aligned}$$

$$\begin{aligned}
\tilde{q}_3 &= 8.59095 \cdot 10^{-06} X^{16} - 6.82648 \cdot 10^{-05} X^{15} + 0.000245968 X^{14} - 0.000531568 X^{13} \\
&+ 0.000767923 X^{12} - 0.000782231 X^{11} + 0.0005774 X^{10} - 0.000312464 X^9 \\
&+ 0.000123994 X^8 - 3.57388 \cdot 10^{-05} X^7 + 7.34249 \cdot 10^{-06} X^6 - 1.04474 \cdot 10^{-06} X^5 \\
&+ 9.86739 \cdot 10^{-08} X^4 - 5.7553 \cdot 10^{-09} X^3 - 1.19186 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396 \\
&= -0.169396 B_{0,16} - 0.135637 B_{1,16} - 0.101877 B_{2,16} - 0.068118 B_{3,16} - 0.0343587 B_{4,16} \\
&- 0.000599476 B_{5,16} + 0.0331598 B_{6,16} + 0.0669191 B_{7,16} + 0.100678 B_{8,16} \\
&+ 0.134438 B_{9,16} + 0.168197 B_{10,16} + 0.201956 B_{11,16} + 0.235715 B_{12,16} \\
&+ 0.269475 B_{13,16} + 0.303234 B_{14,16} + 0.336993 B_{15,16} + 0.370752 B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.81206 \cdot 10^{-10}$.

Bounding polynomials M and m :

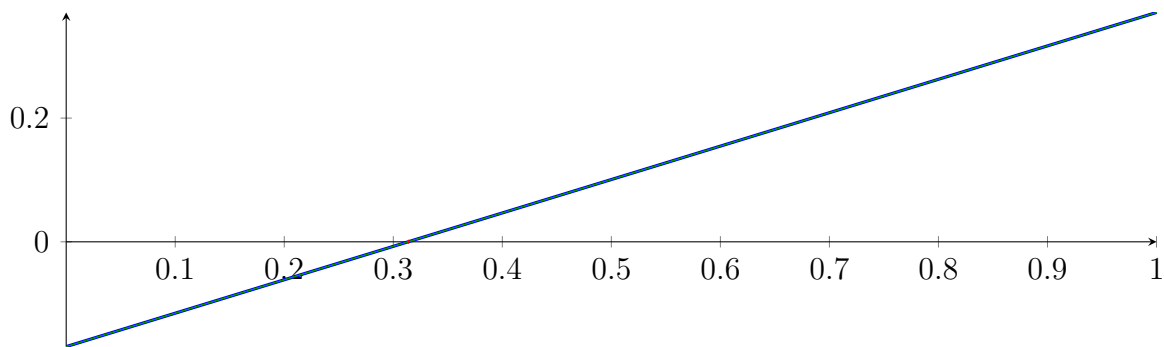
$$M = -1.07065 \cdot 10^{-17} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

$$m = -1.07065 \cdot 10^{-17} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

Root of M and m :

$$N(M) = \{-2.97569 \cdot 10^8, 0.31361, 1.69542 \cdot 10^8\} \quad N(m) = \{-2.97569 \cdot 10^8, 0.31361, 1.69542 \cdot 10^8\}$$

Intersection intervals:



$$[0.31361, 0.31361]$$

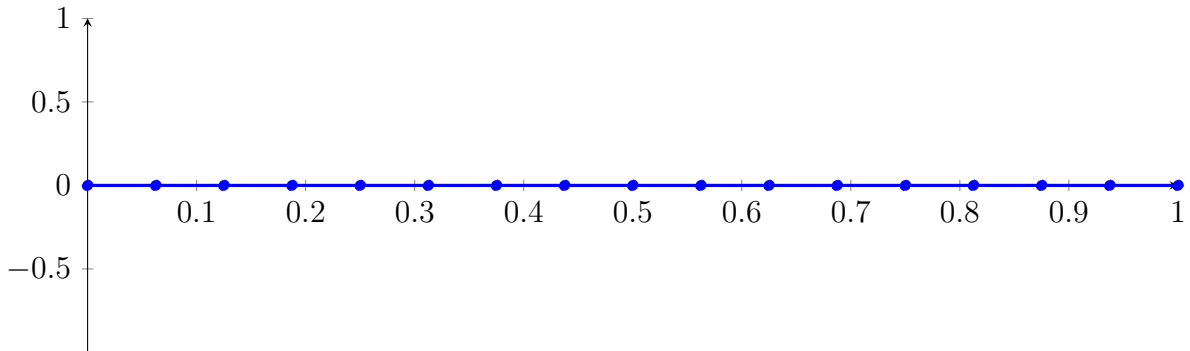
Longest intersection interval: $7.85803 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

162.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

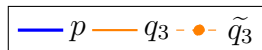
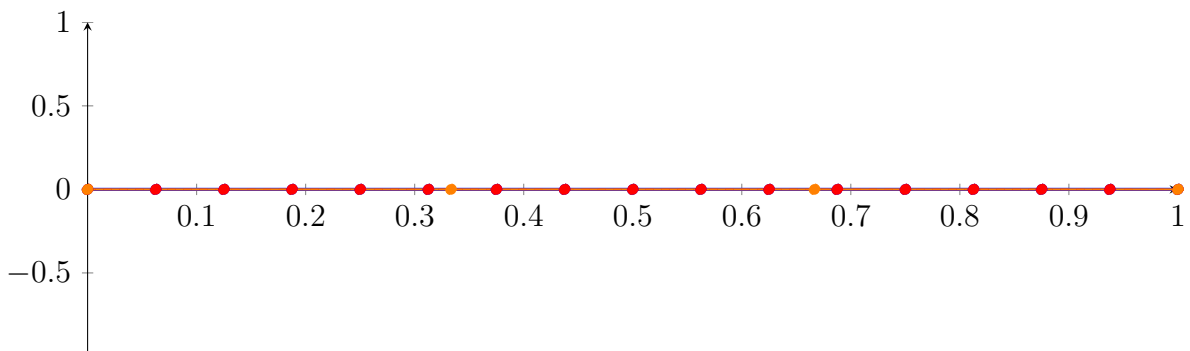
$$\begin{aligned}
 p &= -1.51576 \cdot 10^{-21} X^{16} + 2.62009 \cdot 10^{-21} X^{15} - 3.98039 \cdot 10^{-20} X^{14} + 3.2136 \cdot 10^{-21} X^{13} - 5.16564 \cdot 10^{-20} X^{12} \\
 &\quad + 1.52429 \cdot 10^{-20} X^{11} - 1.44901 \cdot 10^{-20} X^{10} - 1.40466 \cdot 10^{-20} X^9 + 2.34541 \cdot 10^{-20} X^8 + 3.25289 \cdot 10^{-21} X^7 \\
 &\quad + 2.38052 \cdot 10^{-21} X^6 - 2.2582 \cdot 10^{-22} X^5 + 3.52844 \cdot 10^{-23} X^4 + 4.2445 \cdot 10^{-10} X - 2.39831 \cdot 10^{-08} \\
 &= -2.39831 \cdot 10^{-08} B_{0,16}(X) - 2.39566 \cdot 10^{-08} B_{1,16}(X) - 2.39301 \cdot 10^{-08} B_{2,16}(X) - 2.39036 \\
 &\quad \cdot 10^{-08} B_{3,16}(X) - 2.3877 \cdot 10^{-08} B_{4,16}(X) - 2.38505 \cdot 10^{-08} B_{5,16}(X) - 2.3824 \cdot 10^{-08} B_{6,16}(X) \\
 &\quad - 2.37974 \cdot 10^{-08} B_{7,16}(X) - 2.37709 \cdot 10^{-08} B_{8,16}(X) - 2.37444 \cdot 10^{-08} B_{9,16}(X) - 2.37179 \\
 &\quad \cdot 10^{-08} B_{10,16}(X) - 2.36913 \cdot 10^{-08} B_{11,16}(X) - 2.36648 \cdot 10^{-08} B_{12,16}(X) - 2.36383 \cdot 10^{-08} B_{13,16}(X) \\
 &\quad - 2.36118 \cdot 10^{-08} B_{14,16}(X) - 2.35852 \cdot 10^{-08} B_{15,16}(X) - 2.35587 \cdot 10^{-08} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -3.5866 \cdot 10^{-25} X^3 + 8.33643 \cdot 10^{-25} X^2 + 4.2445 \cdot 10^{-10} X - 2.39831 \cdot 10^{-08} \\
 &= -2.39831 \cdot 10^{-08} B_{0,3} - 2.38417 \cdot 10^{-08} B_{1,3} - 2.37002 \cdot 10^{-08} B_{2,3} - 2.35587 \cdot 10^{-08} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= -1.64958 \cdot 10^{-12} X^{16} + 1.3166 \cdot 10^{-11} X^{15} - 4.76688 \cdot 10^{-11} X^{14} + 1.03558 \cdot 10^{-10} X^{13} \\
 &\quad - 1.50448 \cdot 10^{-10} X^{12} + 1.54183 \cdot 10^{-10} X^{11} - 1.1456 \cdot 10^{-10} X^{10} + 6.24452 \cdot 10^{-11} X^9 \\
 &\quad - 2.49793 \cdot 10^{-11} X^8 + 7.26358 \cdot 10^{-12} X^7 - 1.50649 \cdot 10^{-12} X^6 + 2.16616 \cdot 10^{-13} X^5 - 2.07725 \\
 &\quad \cdot 10^{-14} X^4 + 1.24748 \cdot 10^{-15} X^3 - 4.0727 \cdot 10^{-17} X^2 + 4.2445 \cdot 10^{-10} X - 2.39831 \cdot 10^{-08} \\
 &= -2.39831 \cdot 10^{-08} B_{0,16} - 2.39566 \cdot 10^{-08} B_{1,16} - 2.39301 \cdot 10^{-08} B_{2,16} - 2.39036 \cdot 10^{-08} B_{3,16} - 2.3877 \\
 &\quad \cdot 10^{-08} B_{4,16} - 2.38505 \cdot 10^{-08} B_{5,16} - 2.3824 \cdot 10^{-08} B_{6,16} - 2.37974 \cdot 10^{-08} B_{7,16} - 2.37709 \cdot 10^{-08} B_{8,16} \\
 &\quad - 2.37444 \cdot 10^{-08} B_{9,16} - 2.37179 \cdot 10^{-08} B_{10,16} - 2.36913 \cdot 10^{-08} B_{11,16} - 2.36648 \cdot 10^{-08} B_{12,16} \\
 &\quad - 2.36383 \cdot 10^{-08} B_{13,16} - 2.36118 \cdot 10^{-08} B_{14,16} - 2.35852 \cdot 10^{-08} B_{15,16} - 2.35587 \cdot 10^{-08} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.51589 \cdot 10^{-17}$.

Bounding polynomials M and m :

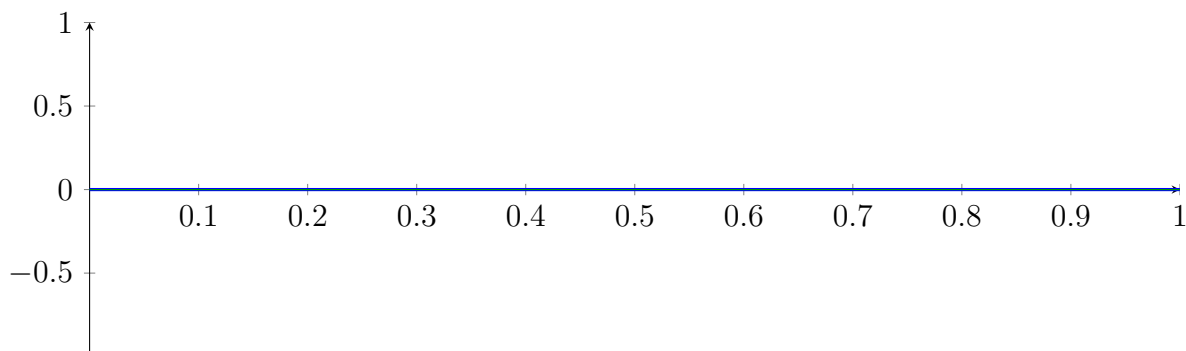
$$M = -3.5866 \cdot 10^{-25} X^3 + 8.33643 \cdot 10^{-25} X^2 + 4.2445 \cdot 10^{-10} X - 2.39831 \cdot 10^{-08}$$

$$m = -3.5866 \cdot 10^{-25} X^3 + 8.33643 \cdot 10^{-25} X^2 + 4.2445 \cdot 10^{-10} X - 2.39831 \cdot 10^{-08}$$

Root of M and m :

$$N(M) = \{-3.44011 \cdot 10^7, 56.504, 3.4401 \cdot 10^7\} \quad N(m) = \{-3.44011 \cdot 10^7, 56.504, 3.4401 \cdot 10^7\}$$

Intersection intervals:

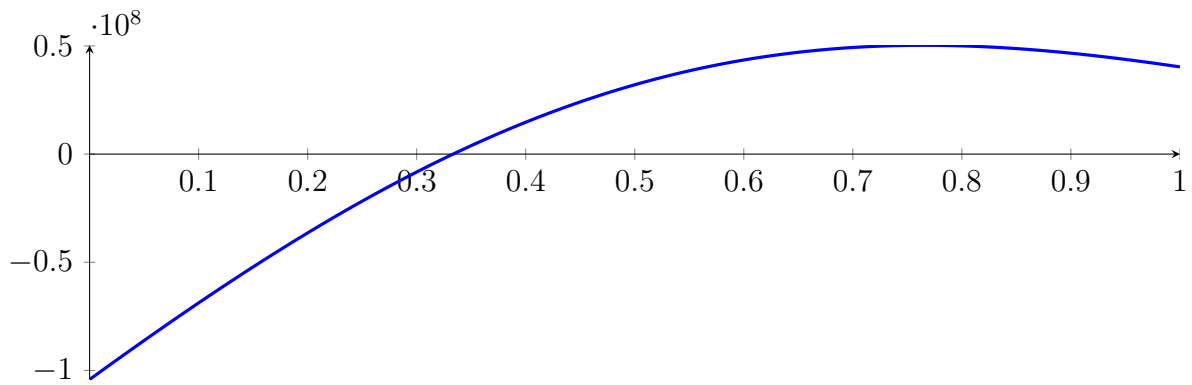


No intersection intervals with the x axis.

162.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

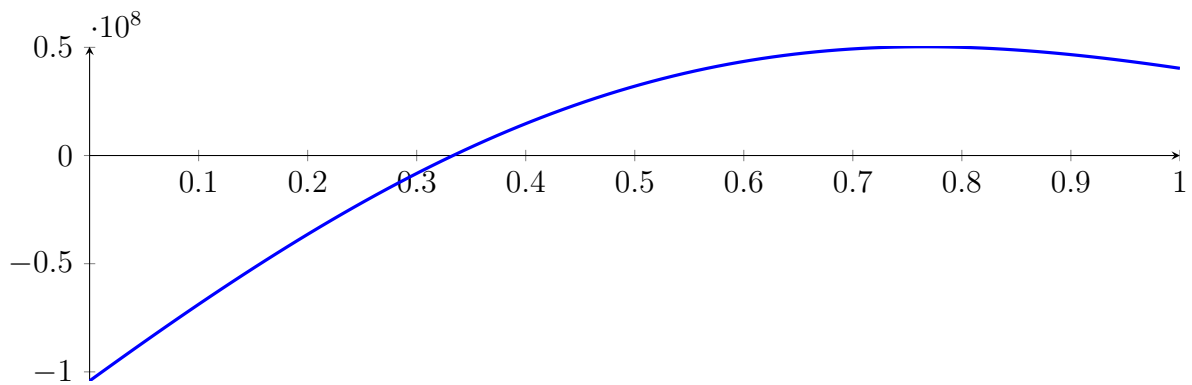
with precision $\varepsilon = 1 \cdot 10^{-32}$.

163 Running BezClip on f_{16} with epsilon 64

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

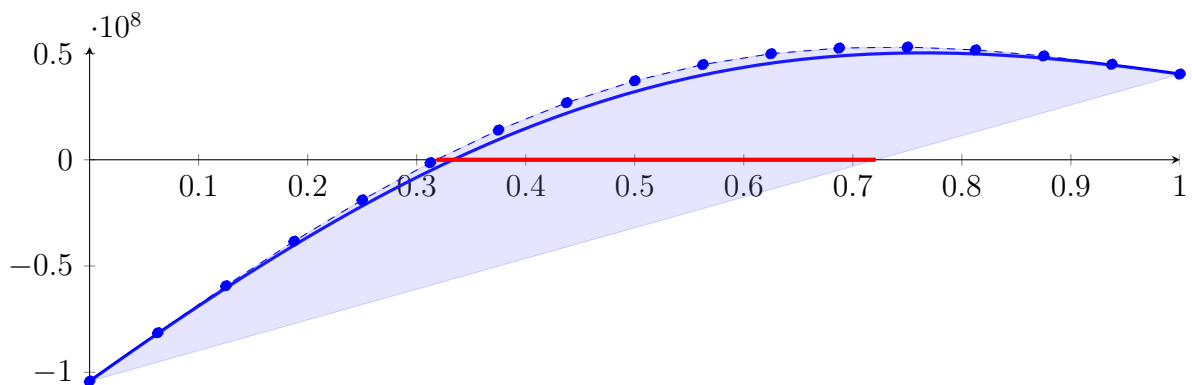
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



163.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

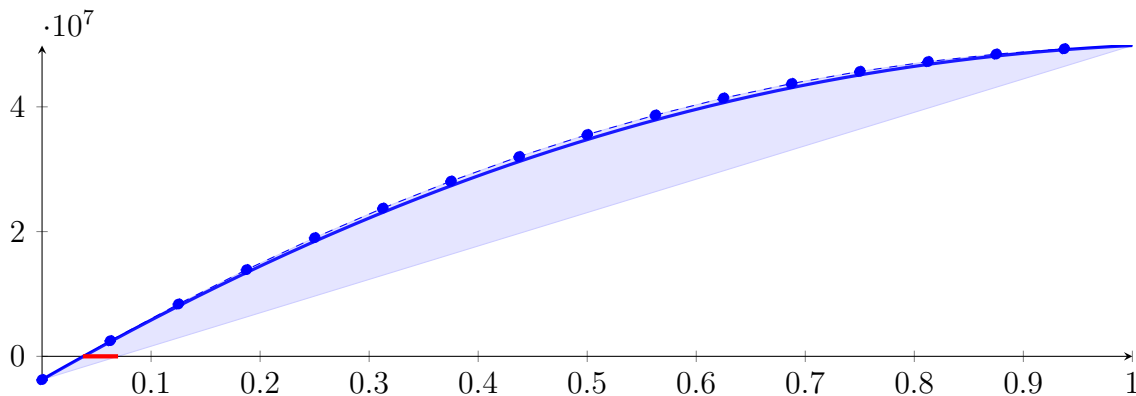
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

163.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.59825 \cdot 10^{-06} X^{16} - 5.93153 \cdot 10^{-05} X^{15} - 0.00248867 X^{14} - 0.0650056 X^{13} - 0.909142 X^{12} \\ &\quad - 5.03931 X^{11} + 36.4549 X^{10} + 692.921 X^9 + 1886.97 X^8 - 25792 X^7 - 149671 X^6 + 492605 X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

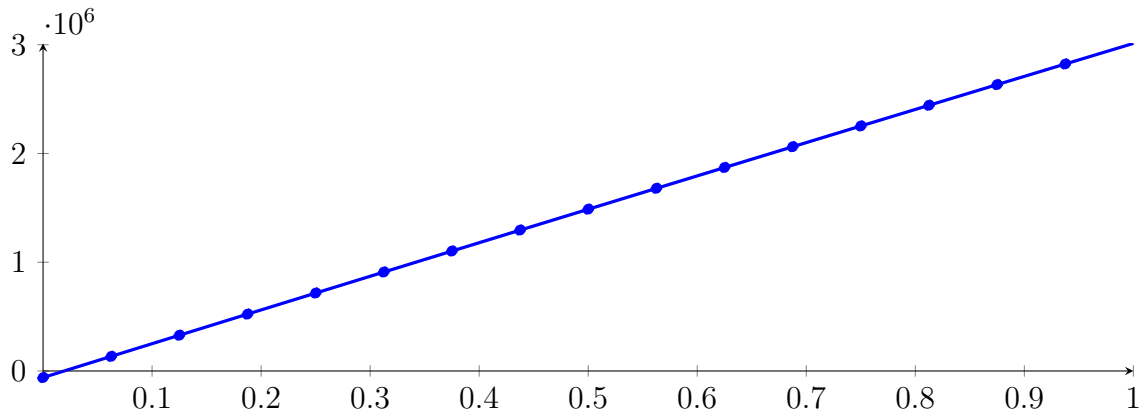
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

163.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 9.01396 \cdot 10^{-08} X^{16} - 2.65848 \cdot 10^{-07} X^{15} + 2.13948 \cdot 10^{-06} X^{14} - 1.33627 \cdot 10^{-06} X^{13} + 2.46973 \cdot 10^{-06} X^{12} \\ &\quad - 2.45524 \cdot 10^{-06} X^{11} + 5.50112 \cdot 10^{-07} X^{10} - 1.64198 \cdot 10^{-07} X^9 - 7.35598 \cdot 10^{-07} X^8 - 1.00892 \cdot 10^{-06} X^7 \\ &\quad - 0.000177509 X^6 + 0.0161038 X^5 + 4.36155 X^4 - 234.216 X^3 - 45622.2 X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

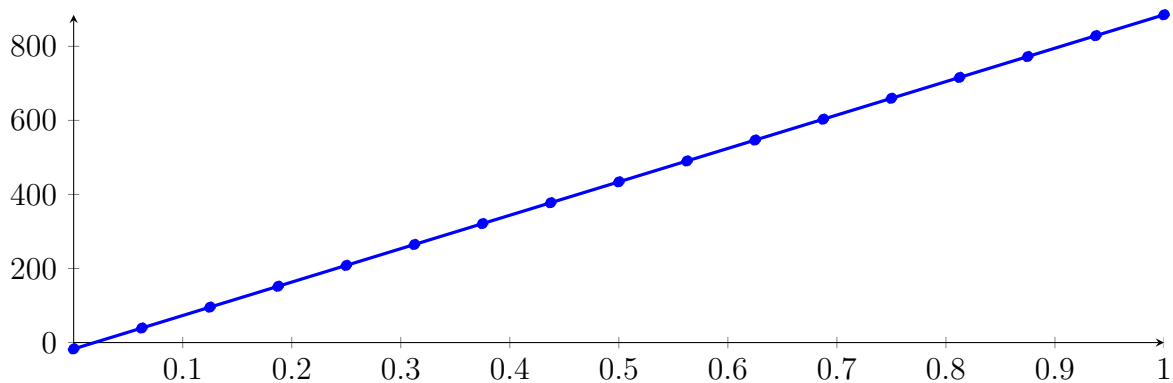
Longest intersection interval: 0.000289554

⇒ Selective recursion: interval 1: $[0.333333, 0.333337]$,

163.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 2.55372 \cdot 10^{-11} X^{16} - 7.21263 \cdot 10^{-11} X^{15} + 6.24141 \cdot 10^{-10} X^{14} - 4.11162 \cdot 10^{-10} X^{13} \\
 &+ 6.82359 \cdot 10^{-10} X^{12} - 7.09475 \cdot 10^{-10} X^{11} + 9.71305 \cdot 10^{-11} X^{10} - 3.46101 \cdot 10^{-11} X^9 \\
 &- 2.13971 \cdot 10^{-10} X^8 - 1.46061 \cdot 10^{-11} X^7 - 1.63366 \cdot 10^{-11} X^6 + 1.87916 \cdot 10^{-12} X^5 \\
 &+ 2.52576 \cdot 10^{-14} X^4 - 5.67777 \cdot 10^{-09} X^3 - 0.00382618 X^2 + 902.448 X - 17.178 \\
 &= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
 &+ 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
 &+ 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
 &+ 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

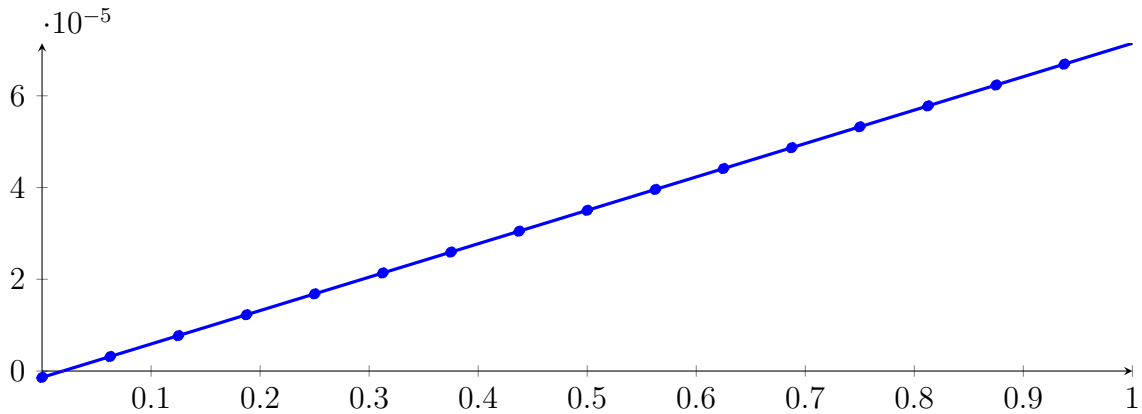
Longest intersection interval: $8.07045 \cdot 10^{-08}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

163.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 2.14261 \cdot 10^{-18} X^{16} - 6.28573 \cdot 10^{-18} X^{15} + 5.28612 \cdot 10^{-17} X^{14} - 3.67279 \cdot 10^{-17} X^{13} \\
 &+ 6.10136 \cdot 10^{-17} X^{12} - 6.60335 \cdot 10^{-17} X^{11} + 1.66661 \cdot 10^{-17} X^{10} - 8.36524 \cdot 10^{-18} X^9 \\
 &- 1.56919 \cdot 10^{-17} X^8 - 1.85474 \cdot 10^{-18} X^7 - 1.4308 \cdot 10^{-18} X^6 + 1.1562 \cdot 10^{-19} X^5 - 1.20437 \\
 &\cdot 10^{-20} X^4 - 4.63221 \cdot 10^{-22} X^3 - 2.49207 \cdot 10^{-17} X^2 + 7.28316 \cdot 10^{-05} X - 1.38634 \cdot 10^{-06} \\
 &= -1.38634 \cdot 10^{-06} B_{0,16}(X) + 3.16564 \cdot 10^{-06} B_{1,16}(X) + 7.71761 \cdot 10^{-06} B_{2,16}(X) + 1.22696 \\
 &\cdot 10^{-05} B_{3,16}(X) + 1.68216 \cdot 10^{-05} B_{4,16}(X) + 2.13735 \cdot 10^{-05} B_{5,16}(X) + 2.59255 \cdot 10^{-05} B_{6,16}(X) \\
 &+ 3.04775 \cdot 10^{-05} B_{7,16}(X) + 3.50295 \cdot 10^{-05} B_{8,16}(X) + 3.95814 \cdot 10^{-05} B_{9,16}(X) + 4.41334 \\
 &\cdot 10^{-05} B_{10,16}(X) + 4.86854 \cdot 10^{-05} B_{11,16}(X) + 5.32374 \cdot 10^{-05} B_{12,16}(X) + 5.77893 \cdot 10^{-05} B_{13,16}(X) \\
 &+ 6.23413 \cdot 10^{-05} B_{14,16}(X) + 6.68933 \cdot 10^{-05} B_{15,16}(X) + 7.14453 \cdot 10^{-05} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.51313 \cdot 10^{-15}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

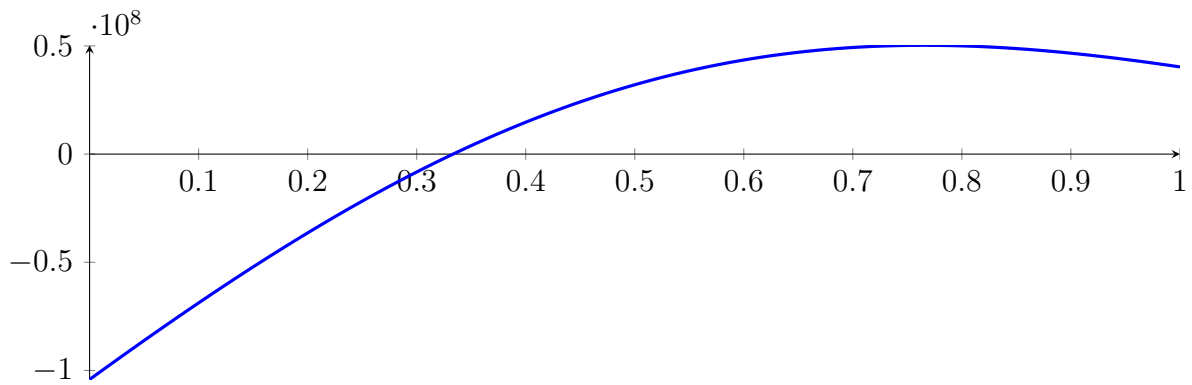
163.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

163.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

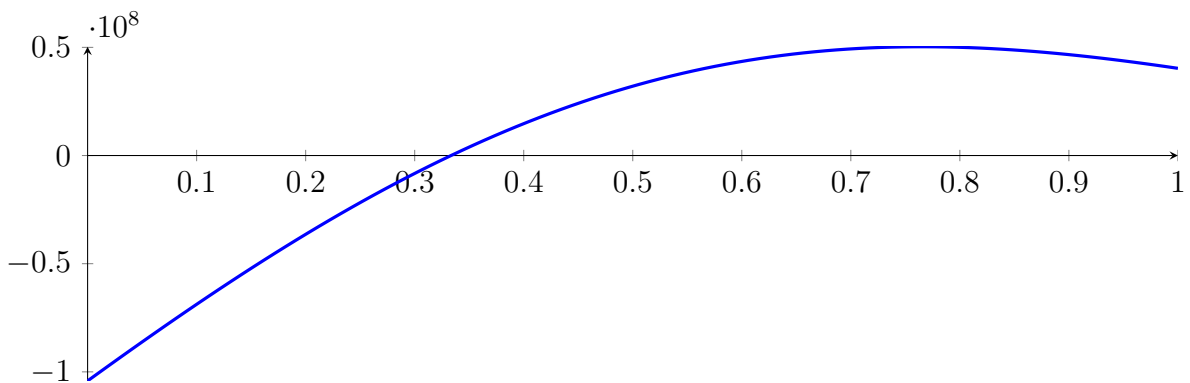
with precision $\varepsilon = 1 \cdot 10^{-64}$.

164 Running QuadClip on f_{16} with epsilon 64

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

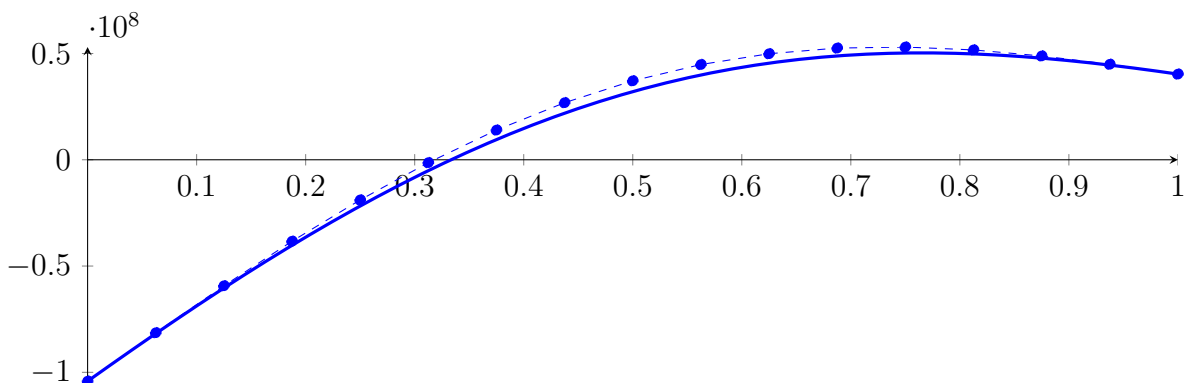
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



164.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = 6049.18X^{16} - 48305.2X^{15} + 174971X^{14} - 380294X^{13} + 552846X^{12} - 567203X^{11}$$

$$+ 422303X^{10} - 231038X^9 + 93003.6X^8 - 27320.1X^7 + 5752.57X^6 - 843.63X^5$$

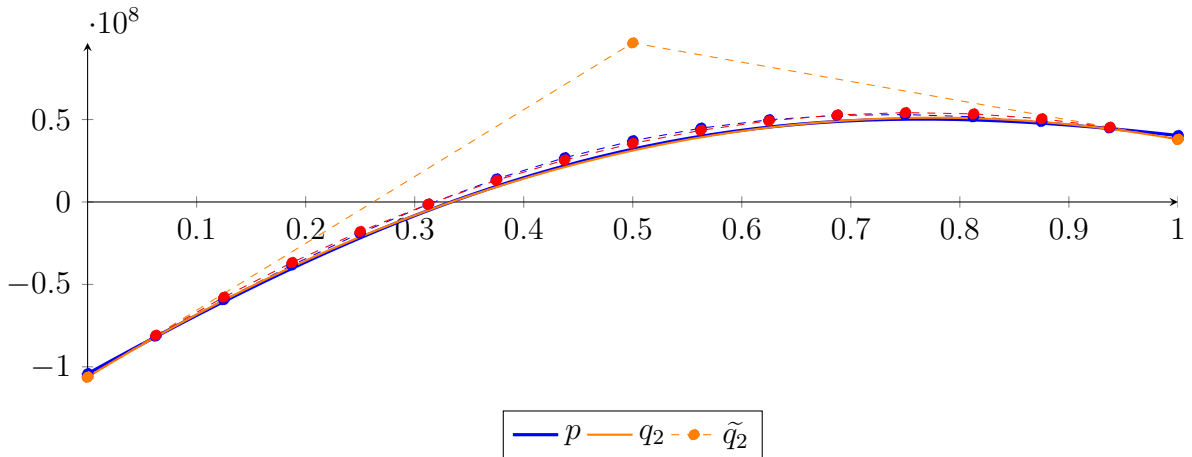
$$+ 82.5145X^4 - 5.01388X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

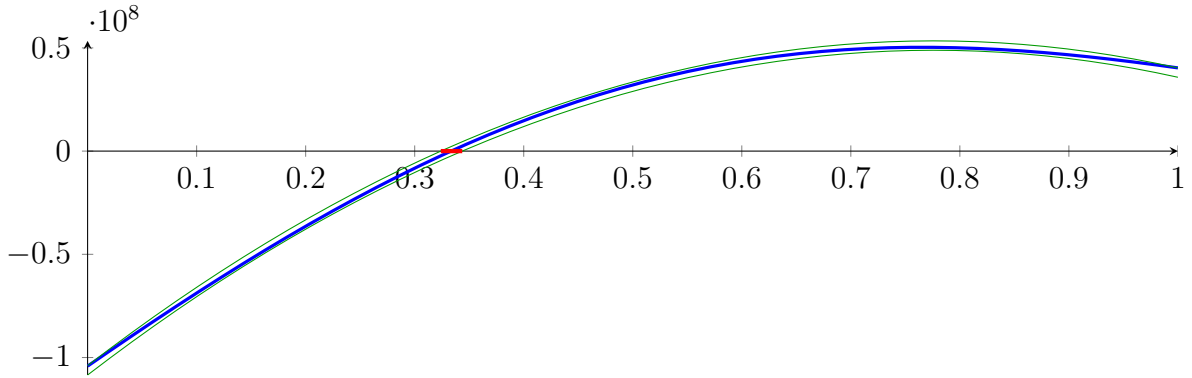
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

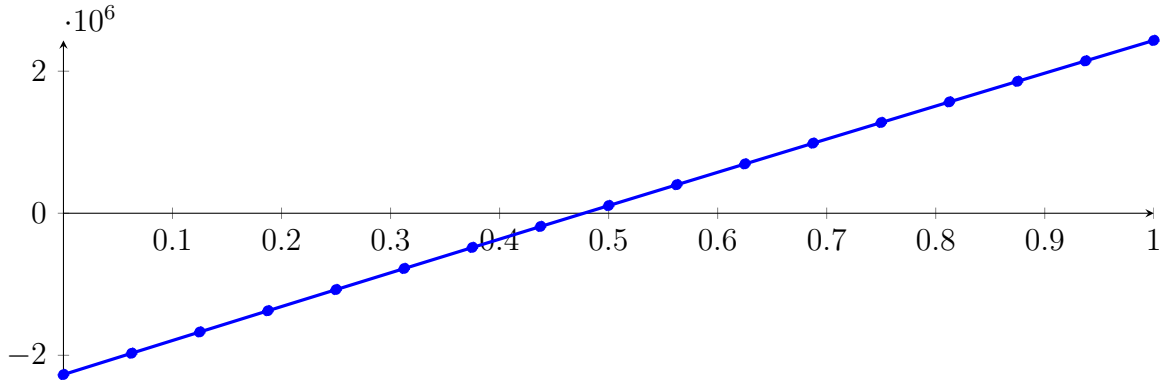
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

164.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

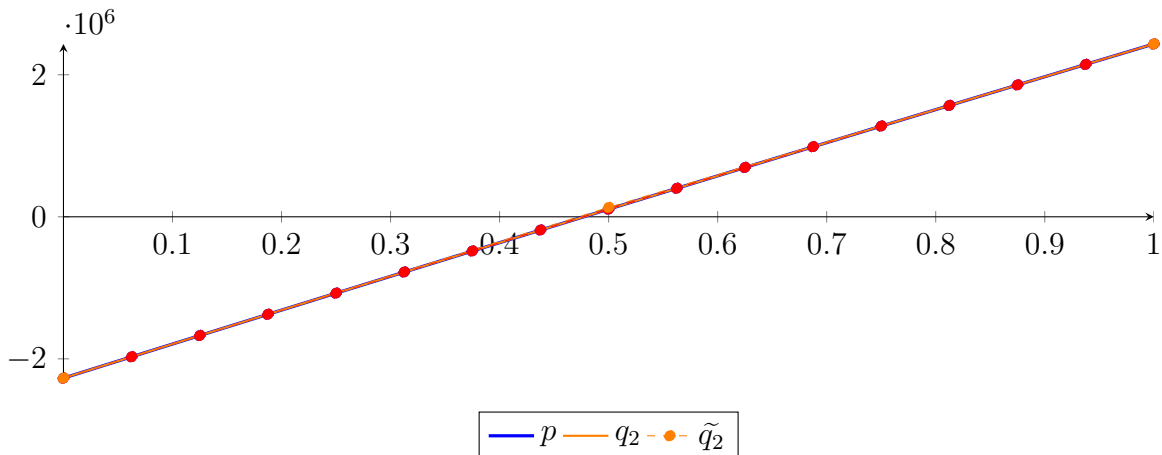
$$\begin{aligned}
 p &= -1.54841 \cdot 10^{-10} X^{16} - 1.66965 \cdot 10^{-07} X^{15} - 2.92739 \cdot 10^{-07} X^{14} - 1.77943 \cdot 10^{-06} X^{13} - 1.17235 \cdot 10^{-06} X^{12} \\
 &\quad - 2.42234 \cdot 10^{-06} X^{11} - 6.86445 \cdot 10^{-07} X^{10} - 1.39162 \cdot 10^{-06} X^9 + 1.07395 \cdot 10^{-06} X^8 - 1.67072 \cdot 10^{-05} X^7 \\
 &\quad - 0.00205879 X^6 + 0.132721 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 22.7036 X^{16} - 178.964 X^{15} + 638.976 X^{14} - 1366.64 X^{13} + 1951.1 X^{12} - 1960.92 X^{11} \\
 &\quad + 1425.35 X^{10} - 757.631 X^9 + 294.177 X^8 - 82.4368 X^7 + 16.2856 X^6 - 2.18949 X^5 \\
 &\quad + 0.191237 X^4 - 0.0101048 X^3 - 104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} \\
 &\quad + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

$$M = -104265 X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

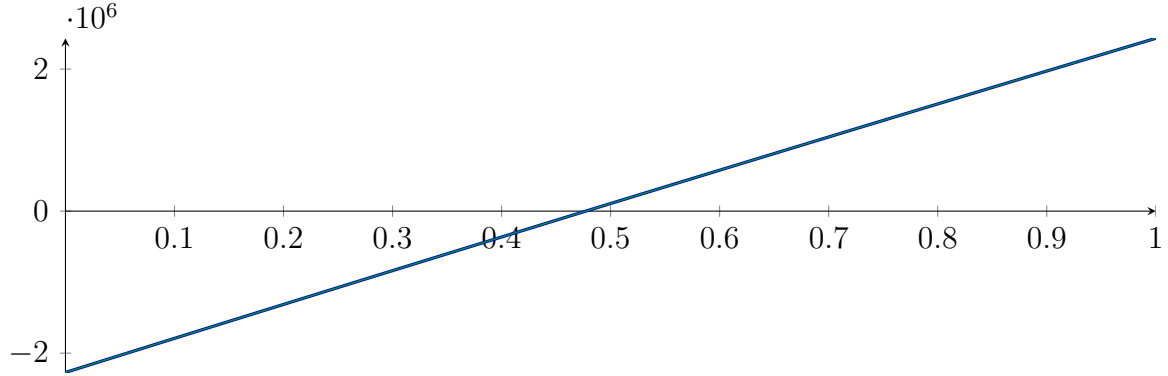
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\}$$

$$N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

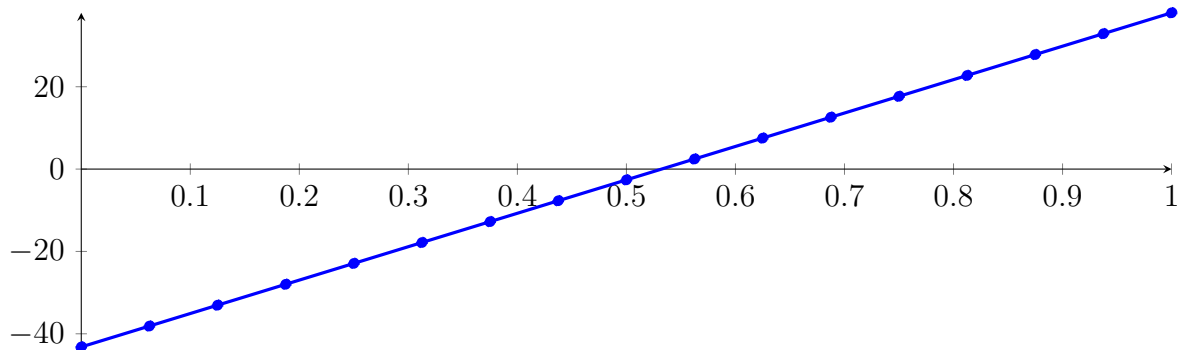
Longest intersection interval: $1.72301 \cdot 10^{-05}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

164.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

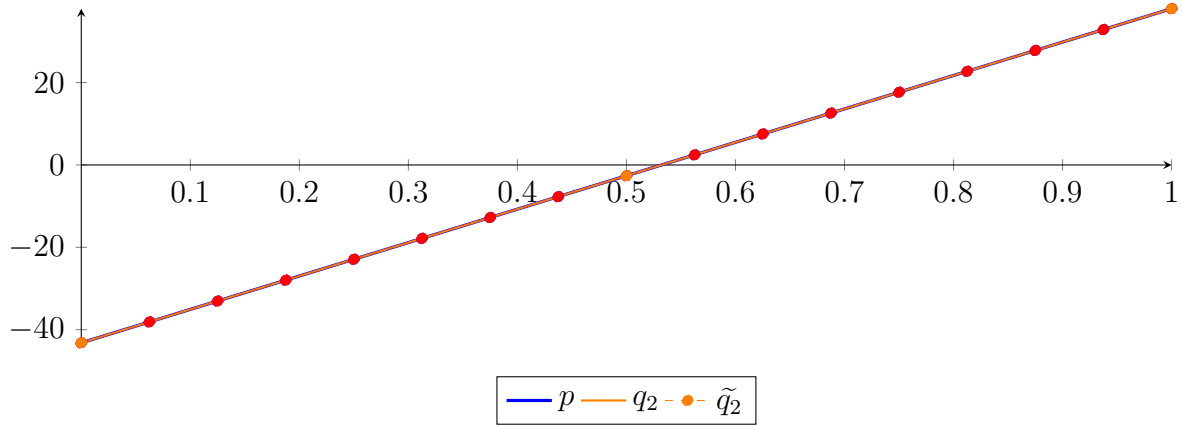
$$\begin{aligned} p &= -2.76723 \cdot 10^{-13} X^{16} - 2.40874 \cdot 10^{-12} X^{15} - 1.25233 \cdot 10^{-11} X^{14} - 3.02935 \cdot 10^{-11} X^{13} \\ &\quad - 3.05617 \cdot 10^{-11} X^{12} - 3.83107 \cdot 10^{-11} X^{11} - 1.26692 \cdot 10^{-11} X^{10} - 2.6672 \cdot 10^{-11} X^9 \\ &\quad + 2.0004 \cdot 10^{-11} X^8 + 4.12781 \cdot 10^{-12} X^7 + 2.44493 \cdot 10^{-12} X^6 - 1.21236 \cdot 10^{-13} X^5 \\ &\quad + 1.26288 \cdot 10^{-14} X^4 - 4.1267 \cdot 10^{-12} X^3 - 3.09388 \cdot 10^{-05} X^2 + 81.1506X - 43.1911 \\ &= -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\ &\quad - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X) \\ &\quad + 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X) \\ &\quad + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$\begin{aligned}
\tilde{q}_2 &= 5.96265 \cdot 10^{-05} X^{16} - 0.000436042 X^{15} + 0.00141812 X^{14} - 0.00269475 X^{13} \\
&+ 0.00329809 X^{12} - 0.00268757 X^{11} + 0.00143268 X^{10} - 0.000439599 X^9 \\
&+ 1.98418 \cdot 10^{-05} X^8 + 4.87608 \cdot 10^{-05} X^7 - 2.46333 \cdot 10^{-05} X^6 + 6.35808 \cdot 10^{-06} X^5 \\
&- 9.62755 \cdot 10^{-07} X^4 + 8.21372 \cdot 10^{-08} X^3 - 3.09429 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911 \\
&= -43.1911 B_{0,16} - 38.1192 B_{1,16} - 33.0473 B_{2,16} - 27.9754 B_{3,16} - 22.9035 B_{4,16} - 17.8316 B_{5,16} \\
&- 12.7597 B_{6,16} - 7.68778 B_{7,16} - 2.61587 B_{8,16} + 2.45604 B_{9,16} + 7.52795 B_{10,16} + 12.5999 B_{11,16} \\
&+ 17.6718 B_{12,16} + 22.7437 B_{13,16} + 27.8156 B_{14,16} + 32.8875 B_{15,16} + 37.9594 B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.5947 \cdot 10^{-09}$.

Bounding polynomials M and m :

$$M = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

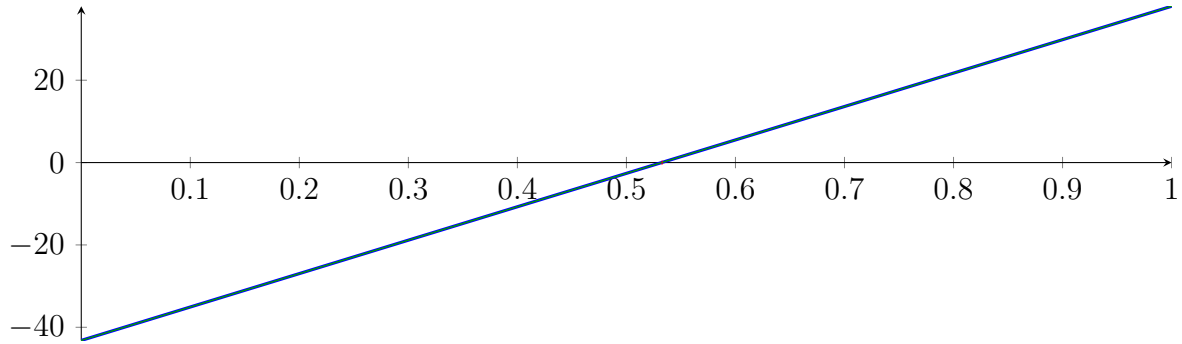
$$m = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

Root of M and m :

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\}$$

$$N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

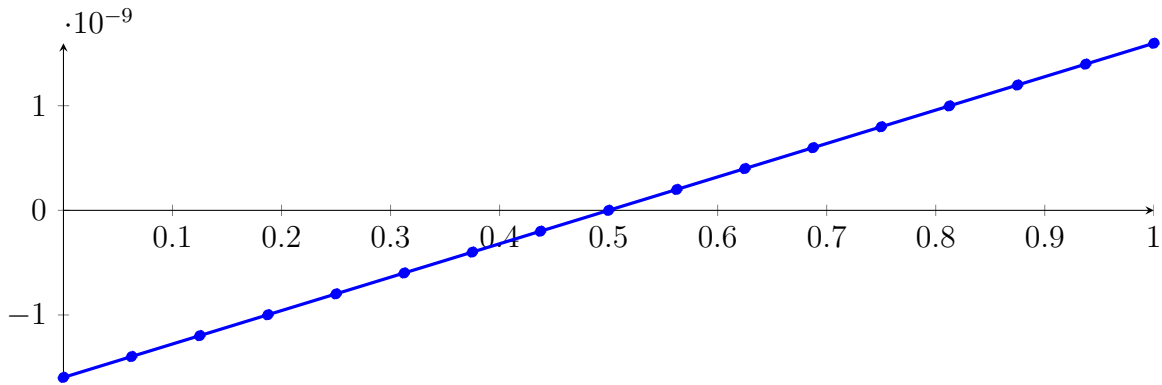
Longest intersection interval: $3.93535 \cdot 10^{-11}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

164.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

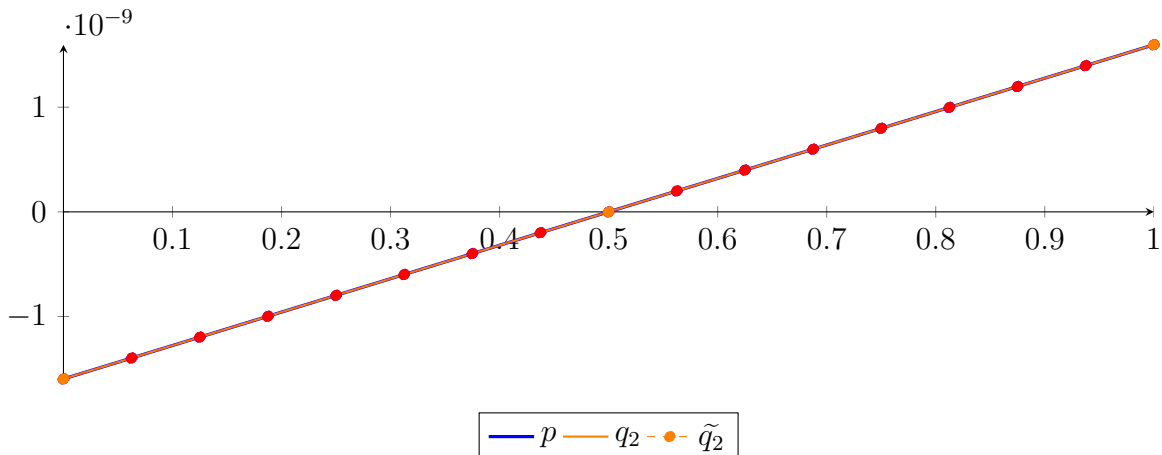
$$\begin{aligned}
 p &= -4.89361 \cdot 10^{-24} X^{16} - 1.05466 \cdot 10^{-22} X^{15} - 3.09805 \cdot 10^{-22} X^{14} - 1.16981 \cdot 10^{-21} X^{13} \\
 &\quad - 9.76202 \cdot 10^{-22} X^{12} - 1.69365 \cdot 10^{-21} X^{11} - 5.95131 \cdot 10^{-22} X^{10} - 1.05349 \cdot 10^{-21} X^9 \\
 &\quad + 7.5893 \cdot 10^{-22} X^8 + 1.47859 \cdot 10^{-22} X^7 + 9.70322 \cdot 10^{-23} X^6 - 7.05688 \cdot 10^{-24} X^5 \\
 &\quad + 1.47018 \cdot 10^{-24} X^4 - 4.84676 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09} \\
 &= -1.59674 \cdot 10^{-09} B_{0,16}(X) - 1.39715 \cdot 10^{-09} B_{1,16}(X) - 1.19755 \cdot 10^{-09} B_{2,16}(X) - 9.97951 \\
 &\quad \cdot 10^{-10} B_{3,16}(X) - 7.98353 \cdot 10^{-10} B_{4,16}(X) - 5.98756 \cdot 10^{-10} B_{5,16}(X) - 3.99159 \cdot 10^{-10} B_{6,16}(X) \\
 &\quad - 1.99561 \cdot 10^{-10} B_{7,16}(X) + 3.6039 \cdot 10^{-14} B_{8,16}(X) + 1.99633 \cdot 10^{-10} B_{9,16}(X) + 3.99231 \\
 &\quad \cdot 10^{-10} B_{10,16}(X) + 5.98828 \cdot 10^{-10} B_{11,16}(X) + 7.98425 \cdot 10^{-10} B_{12,16}(X) + 9.98023 \cdot 10^{-10} B_{13,16}(X) \\
 &\quad + 1.19762 \cdot 10^{-09} B_{14,16}(X) + 1.39722 \cdot 10^{-09} B_{15,16}(X) + 1.59681 \cdot 10^{-09} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -4.83666 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09} \\
 &= -1.59674 \cdot 10^{-09} B_{0,2} + 3.6039 \cdot 10^{-14} B_{1,2} + 1.59681 \cdot 10^{-09} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 9.45798 \cdot 10^{-15} X^{16} - 7.39324 \cdot 10^{-14} X^{15} + 2.61437 \cdot 10^{-13} X^{14} - 5.52989 \cdot 10^{-13} X^{13} \\
 &\quad + 7.79462 \cdot 10^{-13} X^{12} - 7.71893 \cdot 10^{-13} X^{11} + 5.51461 \cdot 10^{-13} X^{10} - 2.8712 \cdot 10^{-13} X^9 \\
 &\quad + 1.08634 \cdot 10^{-13} X^8 - 2.94042 \cdot 10^{-14} X^7 + 5.52081 \cdot 10^{-15} X^6 - 6.84058 \cdot 10^{-16} X^5 + 5.20623 \\
 &\quad \cdot 10^{-17} X^4 - 2.16513 \cdot 10^{-18} X^3 + 1.74369 \cdot 10^{-20} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09} \\
 &= -1.59674 \cdot 10^{-09} B_{0,16} - 1.39715 \cdot 10^{-09} B_{1,16} - 1.19755 \cdot 10^{-09} B_{2,16} - 9.97951 \cdot 10^{-10} B_{3,16} - 7.98353 \\
 &\quad \cdot 10^{-10} B_{4,16} - 5.98756 \cdot 10^{-10} B_{5,16} - 3.99159 \cdot 10^{-10} B_{6,16} - 1.99561 \cdot 10^{-10} B_{7,16} + 3.60393 \cdot 10^{-14} B_{8,16} \\
 &\quad + 1.99633 \cdot 10^{-10} B_{9,16} + 3.99231 \cdot 10^{-10} B_{10,16} + 5.98828 \cdot 10^{-10} B_{11,16} + 7.98425 \cdot 10^{-10} B_{12,16} \\
 &\quad + 9.98023 \cdot 10^{-10} B_{13,16} + 1.19762 \cdot 10^{-09} B_{14,16} + 1.39722 \cdot 10^{-09} B_{15,16} + 1.59681 \cdot 10^{-09} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.02367 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -4.82657 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$

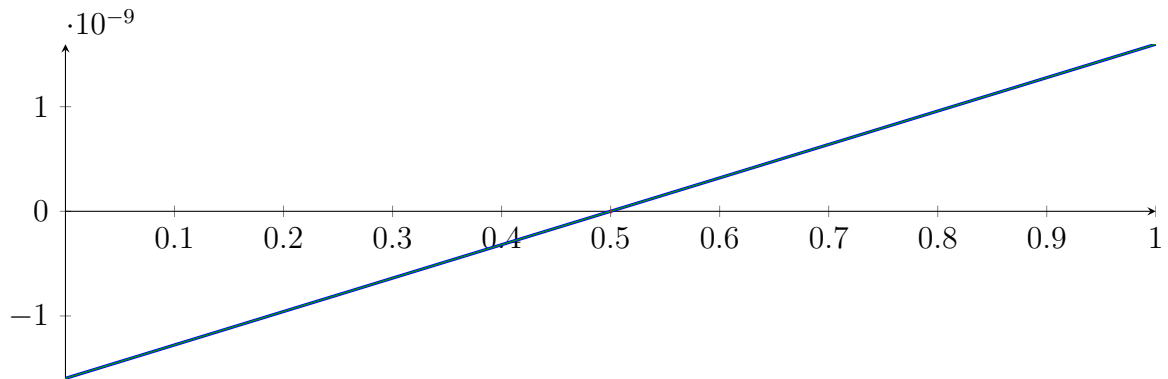
$$m = -4.84676 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$

Root of M and m :

$$N(M) = \{0.5, 6.61662 \cdot 10^{16}\}$$

$$N(m) = \{0.5, 6.58905 \cdot 10^{16}\}$$

Intersection intervals:



[0.5, 0.5]

Longest intersection interval: 0

⇒ Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

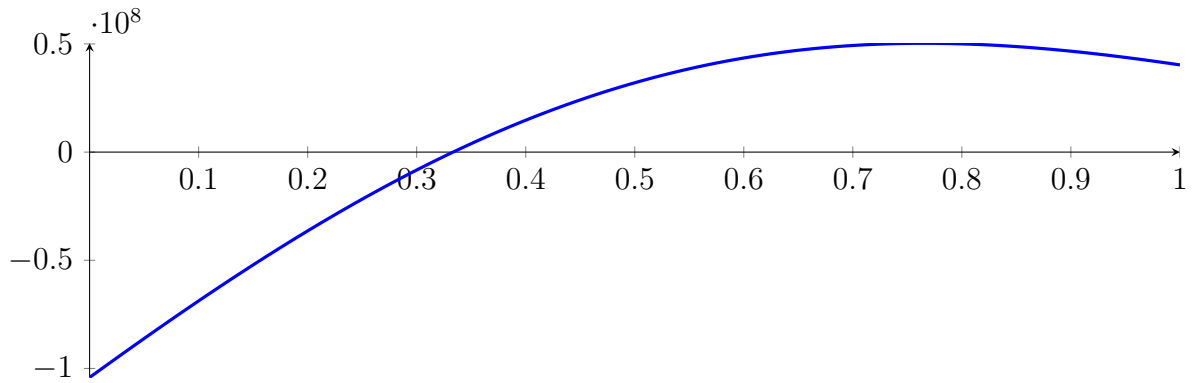
164.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

164.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

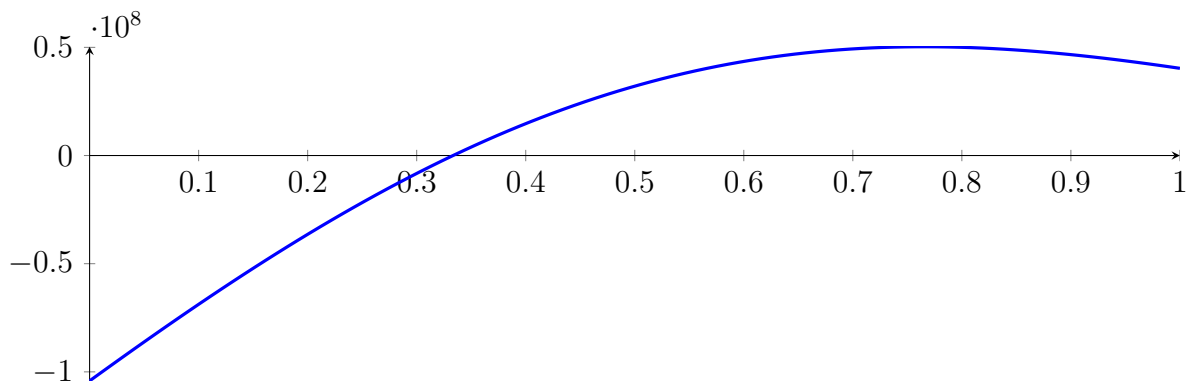
with precision $\varepsilon = 1 \cdot 10^{-64}$.

165 Running CubeClip on f_{16} with epsilon 64

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

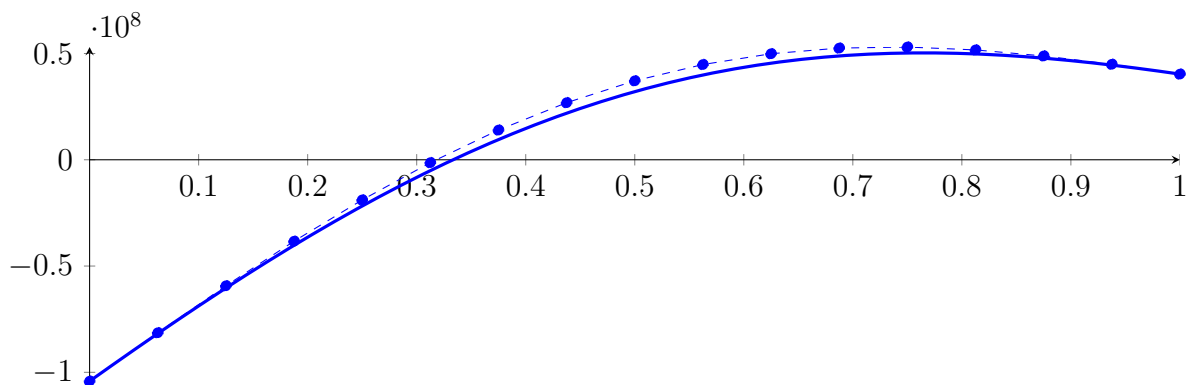
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



165.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 2461.93X^{16} - 19614.9X^{15} + 70879.5X^{14} - 153661X^{13} + 222746X^{12} - 227755X^{11}$$

$$+ 168826X^{10} - 91798.7X^9 + 36630.3X^8 - 10627.3X^7 + 2200.54X^6 - 316.059X^5$$

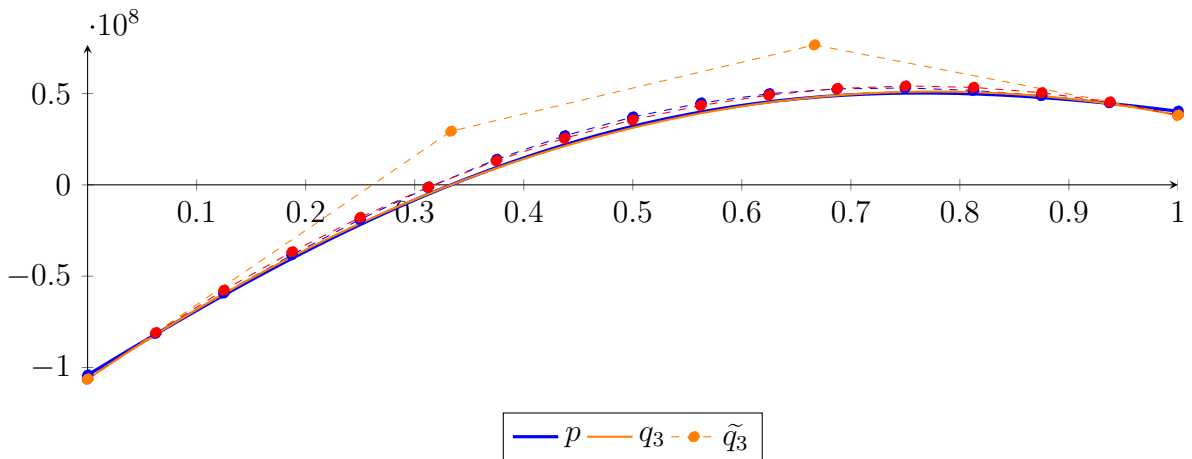
$$+ 30.1958X^4 + 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18551 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91815 \cdot 10^7 B_{10,16} + 5.27353 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

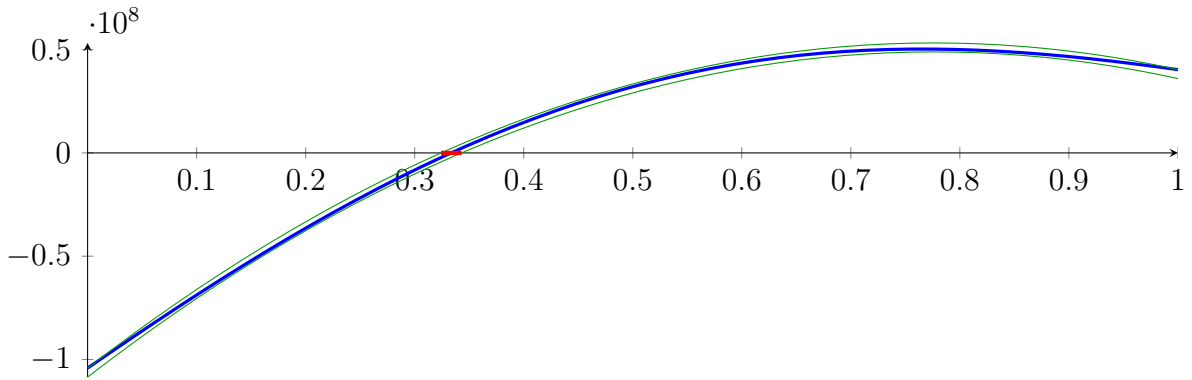
$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\} \quad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

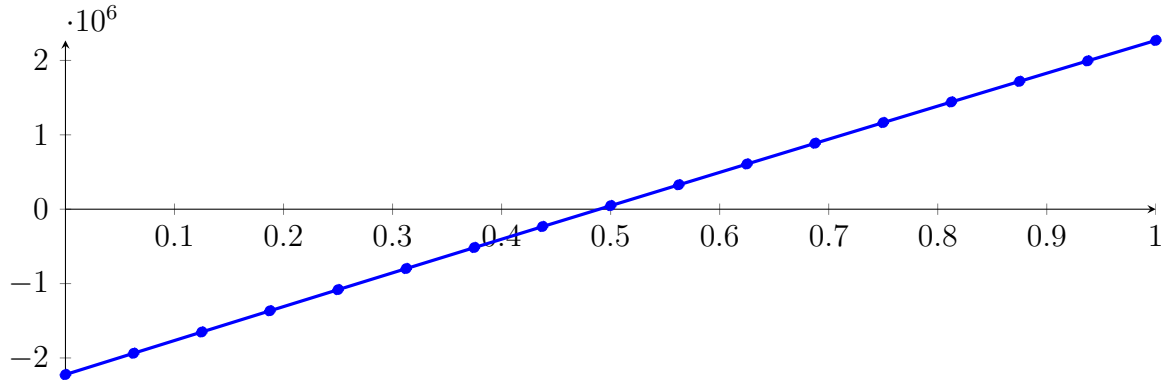
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

165.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

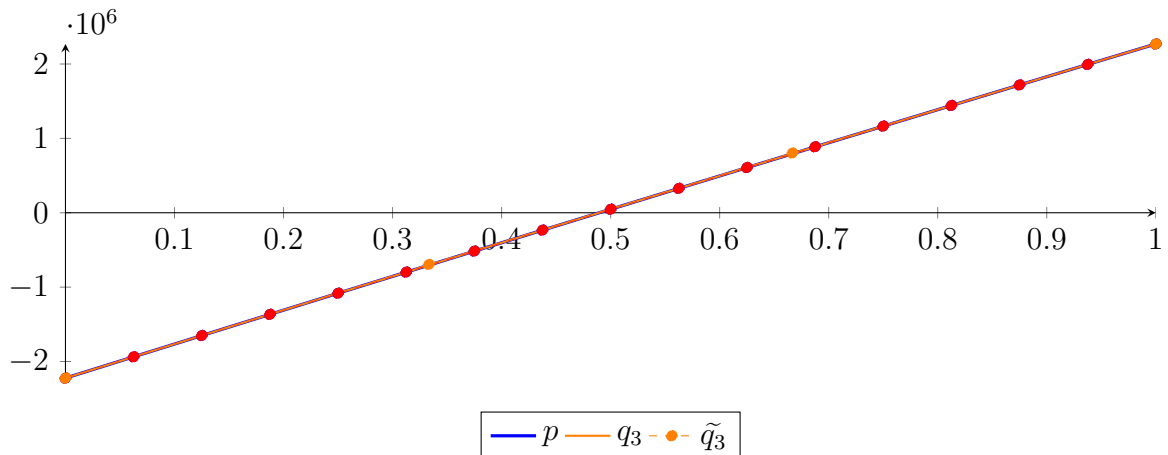
$$\begin{aligned}
 p &= -3.66617 \cdot 10^{-09} X^{16} - 1.53217 \cdot 10^{-07} X^{15} - 3.62234 \cdot 10^{-07} X^{14} - 1.65579 \cdot 10^{-06} X^{13} - 1.15373 \cdot 10^{-06} X^{12} \\
 &\quad - 2.3399 \cdot 10^{-06} X^{11} - 5.02543 \cdot 10^{-07} X^{10} - 1.38381 \cdot 10^{-06} X^9 + 1.1237 \cdot 10^{-06} X^8 - 1.19653 \cdot 10^{-05} X^7 \\
 &\quad - 0.00155608 X^6 + 0.10496 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 16.4956 X^{16} - 129.161 X^{15} + 457.83 X^{14} - 971.671 X^{13} + 1375.95 X^{12} - 1370.96 X^{11} \\
 &\quad + 987.265 X^{10} - 519.476 X^9 + 199.587 X^8 - 55.434 X^7 + 10.9237 X^6 - 1.48019 X^5 \\
 &\quad + 0.129516 X^4 - 700.679 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} \\
 &\quad + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m :

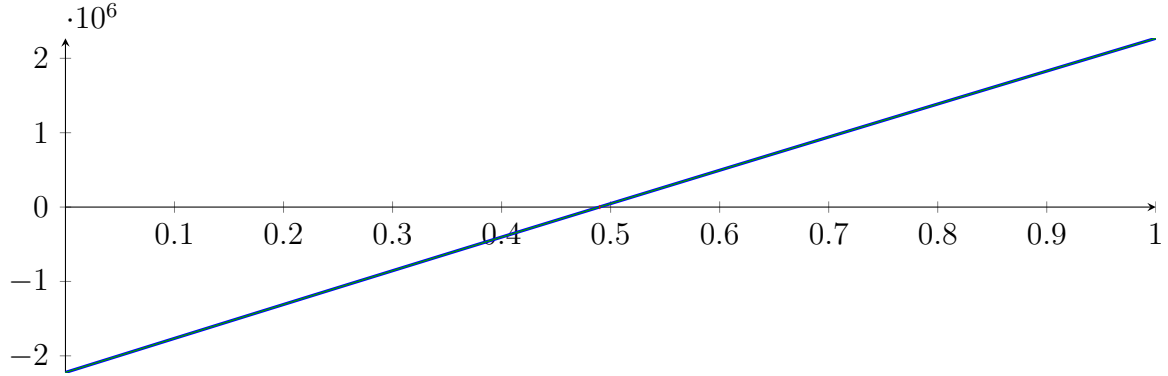
$$M = -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\} \quad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

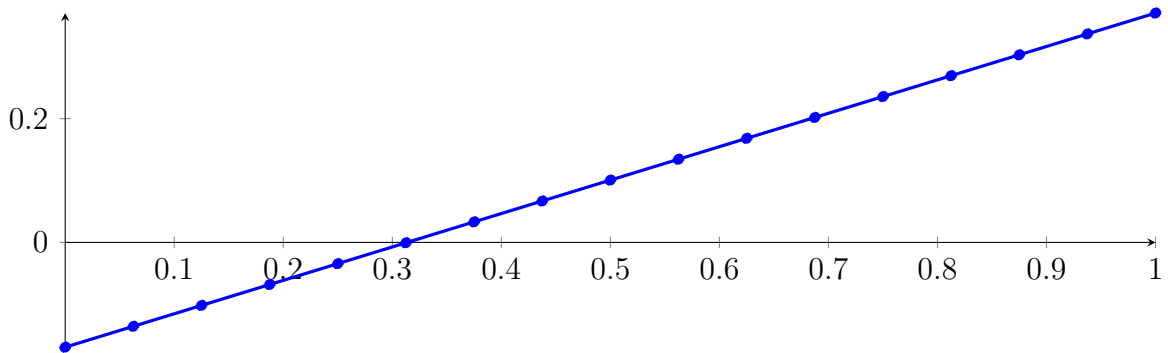
Longest intersection interval: $1.20174 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

165.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

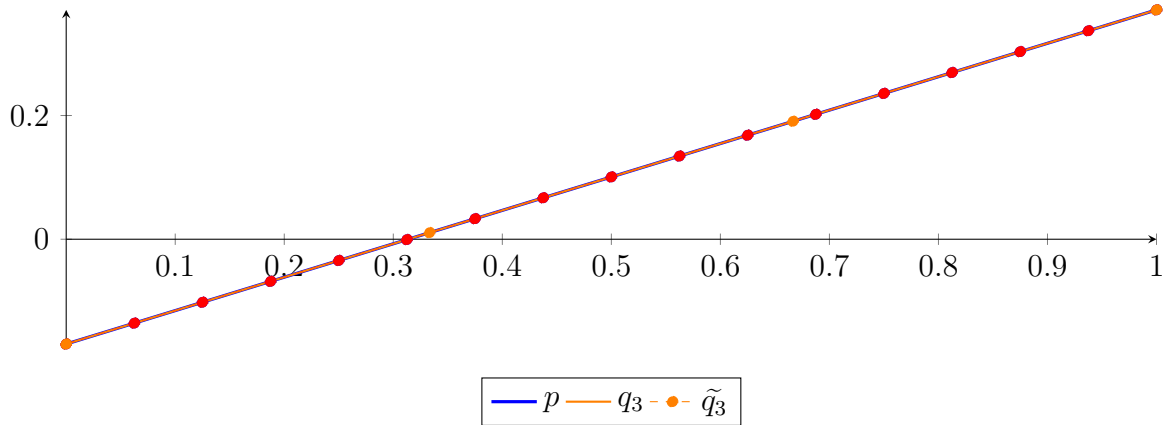
$$\begin{aligned} p &= 5.55524 \cdot 10^{-15} X^{16} - 2.94313 \cdot 10^{-14} X^{15} + 1.19384 \cdot 10^{-13} X^{14} - 2.17482 \cdot 10^{-13} X^{13} + 7.26155 \cdot 10^{-14} X^{12} \\ &\quad - 3.44766 \cdot 10^{-13} X^{11} - 3.47292 \cdot 10^{-15} X^{10} - 1.1287 \cdot 10^{-13} X^9 + 2.93027 \cdot 10^{-14} X^8 + 8.06213 \cdot 10^{-15} X^7 \\ &\quad + 5.64349 \cdot 10^{-15} X^6 + 4.93312 \cdot 10^{-17} X^4 + 1.51788 \cdot 10^{-17} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396 \\ &= -0.169396 B_{0,16}(X) - 0.135637 B_{1,16}(X) - 0.101877 B_{2,16}(X) - 0.068118 B_{3,16}(X) \\ &\quad - 0.0343587 B_{4,16}(X) - 0.000599476 B_{5,16}(X) + 0.0331598 B_{6,16}(X) \\ &\quad + 0.0669191 B_{7,16}(X) + 0.100678 B_{8,16}(X) + 0.134438 B_{9,16}(X) + 0.168197 B_{10,16}(X) \\ &\quad + 0.201956 B_{11,16}(X) + 0.235715 B_{12,16}(X) + 0.269475 B_{13,16}(X) \\ &\quad + 0.303234 B_{14,16}(X) + 0.336993 B_{15,16}(X) + 0.370752 B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07065 \cdot 10^{-17} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396 \\ &= -0.169396 B_{0,3} + 0.0106536 B_{1,3} + 0.190703 B_{2,3} + 0.370752 B_{3,3} \end{aligned}$$

$$\begin{aligned}
\tilde{q}_3 &= 8.59095 \cdot 10^{-06} X^{16} - 6.82648 \cdot 10^{-05} X^{15} + 0.000245968 X^{14} - 0.000531568 X^{13} \\
&+ 0.000767923 X^{12} - 0.000782231 X^{11} + 0.0005774 X^{10} - 0.000312464 X^9 \\
&+ 0.000123994 X^8 - 3.57388 \cdot 10^{-05} X^7 + 7.34249 \cdot 10^{-06} X^6 - 1.04474 \cdot 10^{-06} X^5 \\
&+ 9.86739 \cdot 10^{-08} X^4 - 5.7553 \cdot 10^{-09} X^3 - 1.19186 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396 \\
&= -0.169396 B_{0,16} - 0.135637 B_{1,16} - 0.101877 B_{2,16} - 0.068118 B_{3,16} - 0.0343587 B_{4,16} \\
&- 0.000599476 B_{5,16} + 0.0331598 B_{6,16} + 0.0669191 B_{7,16} + 0.100678 B_{8,16} \\
&+ 0.134438 B_{9,16} + 0.168197 B_{10,16} + 0.201956 B_{11,16} + 0.235715 B_{12,16} \\
&+ 0.269475 B_{13,16} + 0.303234 B_{14,16} + 0.336993 B_{15,16} + 0.370752 B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.81206 \cdot 10^{-10}$.

Bounding polynomials M and m :

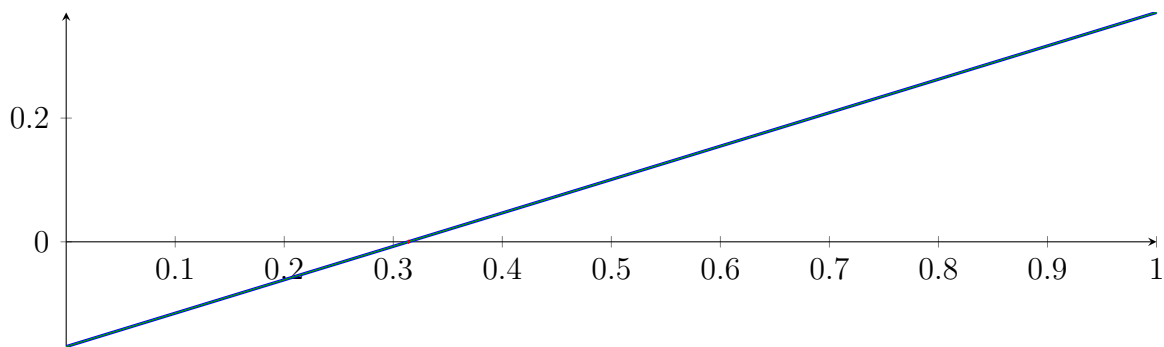
$$M = -1.07065 \cdot 10^{-17} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

$$m = -1.07065 \cdot 10^{-17} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

Root of M and m :

$$N(M) = \{-2.97569 \cdot 10^8, 0.31361, 1.69542 \cdot 10^8\} \quad N(m) = \{-2.97569 \cdot 10^8, 0.31361, 1.69542 \cdot 10^8\}$$

Intersection intervals:



$$[0.31361, 0.31361]$$

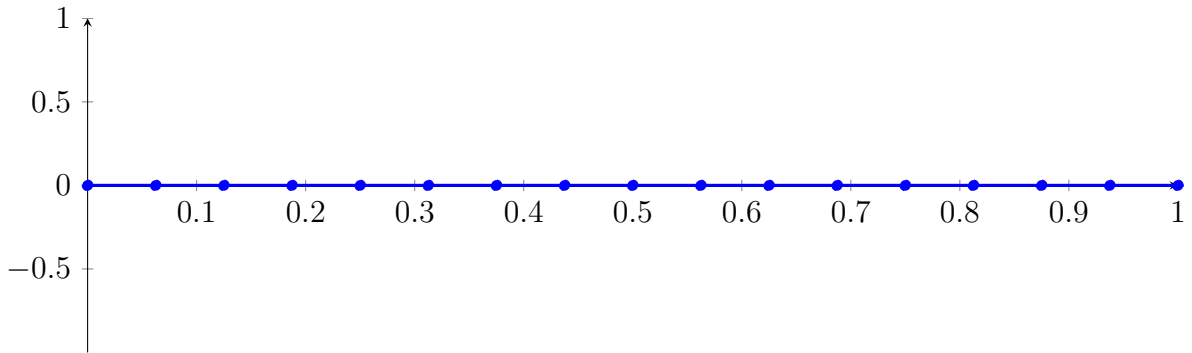
Longest intersection interval: $7.85803 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

165.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

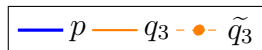
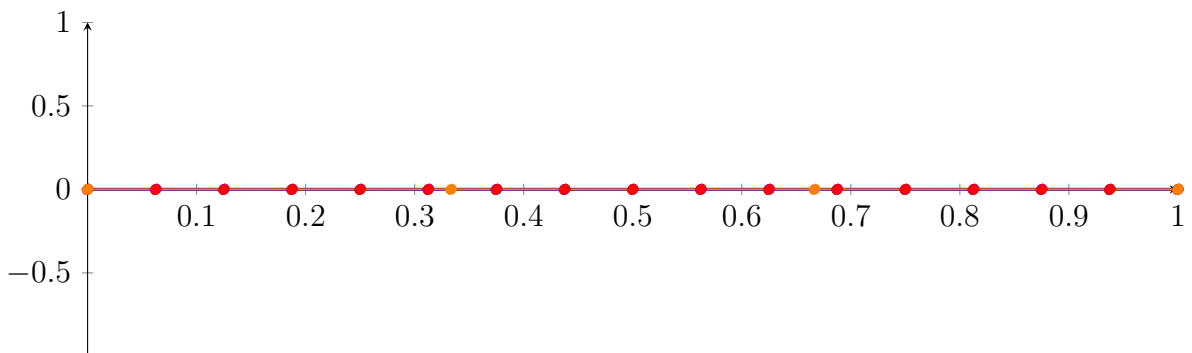
$$\begin{aligned}
 p &= -1.51576 \cdot 10^{-21} X^{16} + 2.62009 \cdot 10^{-21} X^{15} - 3.98039 \cdot 10^{-20} X^{14} + 3.2136 \cdot 10^{-21} X^{13} - 5.16564 \cdot 10^{-20} X^{12} \\
 &\quad + 1.52429 \cdot 10^{-20} X^{11} - 1.44901 \cdot 10^{-20} X^{10} - 1.40466 \cdot 10^{-20} X^9 + 2.34541 \cdot 10^{-20} X^8 + 3.25289 \cdot 10^{-21} X^7 \\
 &\quad + 2.38052 \cdot 10^{-21} X^6 - 2.2582 \cdot 10^{-22} X^5 + 3.52844 \cdot 10^{-23} X^4 + 4.2445 \cdot 10^{-10} X - 2.39831 \cdot 10^{-08} \\
 &= -2.39831 \cdot 10^{-08} B_{0,16}(X) - 2.39566 \cdot 10^{-08} B_{1,16}(X) - 2.39301 \cdot 10^{-08} B_{2,16}(X) - 2.39036 \\
 &\quad \cdot 10^{-08} B_{3,16}(X) - 2.3877 \cdot 10^{-08} B_{4,16}(X) - 2.38505 \cdot 10^{-08} B_{5,16}(X) - 2.3824 \cdot 10^{-08} B_{6,16}(X) \\
 &\quad - 2.37974 \cdot 10^{-08} B_{7,16}(X) - 2.37709 \cdot 10^{-08} B_{8,16}(X) - 2.37444 \cdot 10^{-08} B_{9,16}(X) - 2.37179 \\
 &\quad \cdot 10^{-08} B_{10,16}(X) - 2.36913 \cdot 10^{-08} B_{11,16}(X) - 2.36648 \cdot 10^{-08} B_{12,16}(X) - 2.36383 \cdot 10^{-08} B_{13,16}(X) \\
 &\quad - 2.36118 \cdot 10^{-08} B_{14,16}(X) - 2.35852 \cdot 10^{-08} B_{15,16}(X) - 2.35587 \cdot 10^{-08} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -3.5866 \cdot 10^{-25} X^3 + 8.33643 \cdot 10^{-25} X^2 + 4.2445 \cdot 10^{-10} X - 2.39831 \cdot 10^{-08} \\
 &= -2.39831 \cdot 10^{-08} B_{0,3} - 2.38417 \cdot 10^{-08} B_{1,3} - 2.37002 \cdot 10^{-08} B_{2,3} - 2.35587 \cdot 10^{-08} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= -1.64958 \cdot 10^{-12} X^{16} + 1.3166 \cdot 10^{-11} X^{15} - 4.76688 \cdot 10^{-11} X^{14} + 1.03558 \cdot 10^{-10} X^{13} \\
 &\quad - 1.50448 \cdot 10^{-10} X^{12} + 1.54183 \cdot 10^{-10} X^{11} - 1.1456 \cdot 10^{-10} X^{10} + 6.24452 \cdot 10^{-11} X^9 \\
 &\quad - 2.49793 \cdot 10^{-11} X^8 + 7.26358 \cdot 10^{-12} X^7 - 1.50649 \cdot 10^{-12} X^6 + 2.16616 \cdot 10^{-13} X^5 - 2.07725 \\
 &\quad \cdot 10^{-14} X^4 + 1.24748 \cdot 10^{-15} X^3 - 4.0727 \cdot 10^{-17} X^2 + 4.2445 \cdot 10^{-10} X - 2.39831 \cdot 10^{-08} \\
 &= -2.39831 \cdot 10^{-08} B_{0,16} - 2.39566 \cdot 10^{-08} B_{1,16} - 2.39301 \cdot 10^{-08} B_{2,16} - 2.39036 \cdot 10^{-08} B_{3,16} - 2.3877 \\
 &\quad \cdot 10^{-08} B_{4,16} - 2.38505 \cdot 10^{-08} B_{5,16} - 2.3824 \cdot 10^{-08} B_{6,16} - 2.37974 \cdot 10^{-08} B_{7,16} - 2.37709 \cdot 10^{-08} B_{8,16} \\
 &\quad - 2.37444 \cdot 10^{-08} B_{9,16} - 2.37179 \cdot 10^{-08} B_{10,16} - 2.36913 \cdot 10^{-08} B_{11,16} - 2.36648 \cdot 10^{-08} B_{12,16} \\
 &\quad - 2.36383 \cdot 10^{-08} B_{13,16} - 2.36118 \cdot 10^{-08} B_{14,16} - 2.35852 \cdot 10^{-08} B_{15,16} - 2.35587 \cdot 10^{-08} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.51589 \cdot 10^{-17}$.

Bounding polynomials M and m :

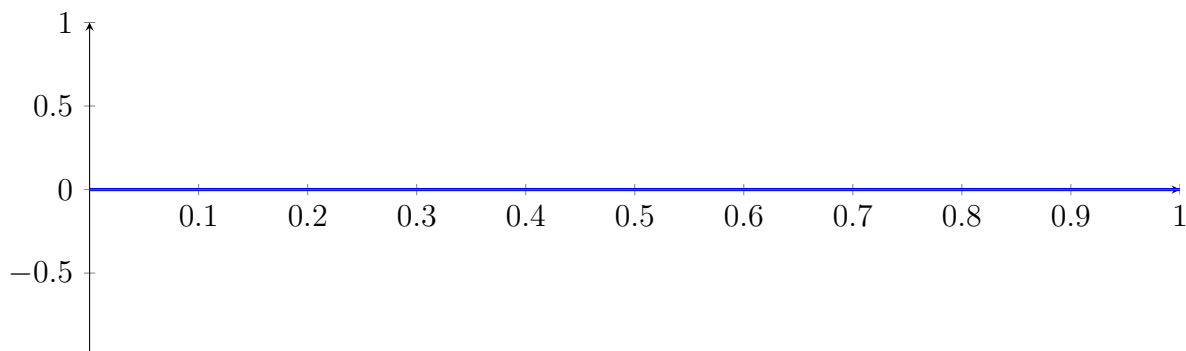
$$M = -3.5866 \cdot 10^{-25} X^3 + 8.33643 \cdot 10^{-25} X^2 + 4.2445 \cdot 10^{-10} X - 2.39831 \cdot 10^{-08}$$

$$m = -3.5866 \cdot 10^{-25} X^3 + 8.33643 \cdot 10^{-25} X^2 + 4.2445 \cdot 10^{-10} X - 2.39831 \cdot 10^{-08}$$

Root of M and m :

$$N(M) = \{-3.44011 \cdot 10^7, 56.504, 3.4401 \cdot 10^7\} \quad N(m) = \{-3.44011 \cdot 10^7, 56.504, 3.4401 \cdot 10^7\}$$

Intersection intervals:

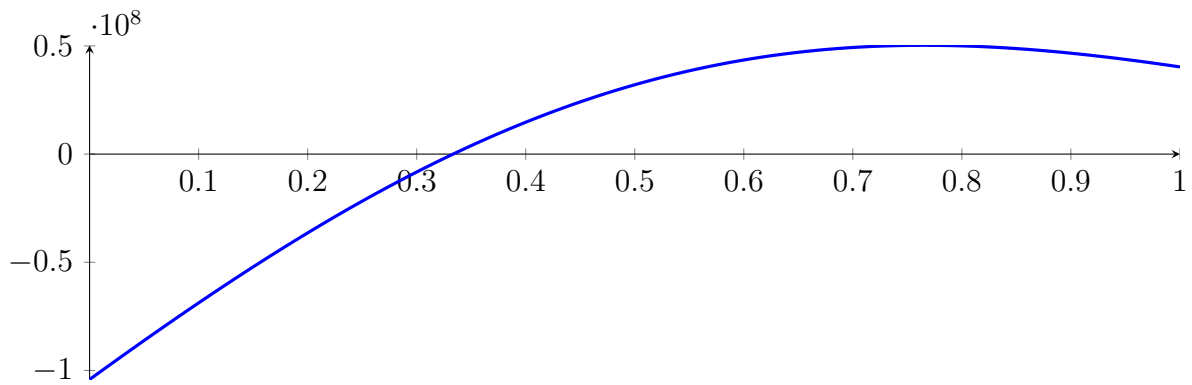


No intersection intervals with the x axis.

165.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

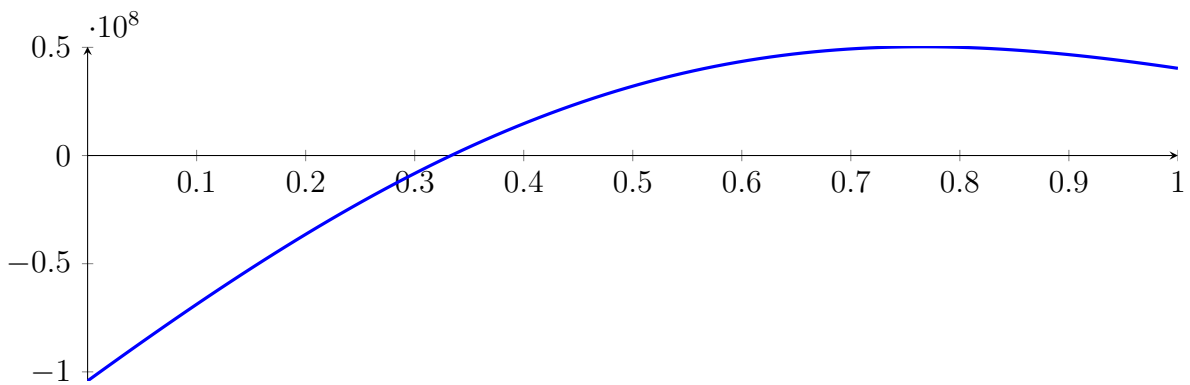
with precision $\varepsilon = 1 \cdot 10^{-64}$.

166 Running BezClip on f_{16} with epsilon 128

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

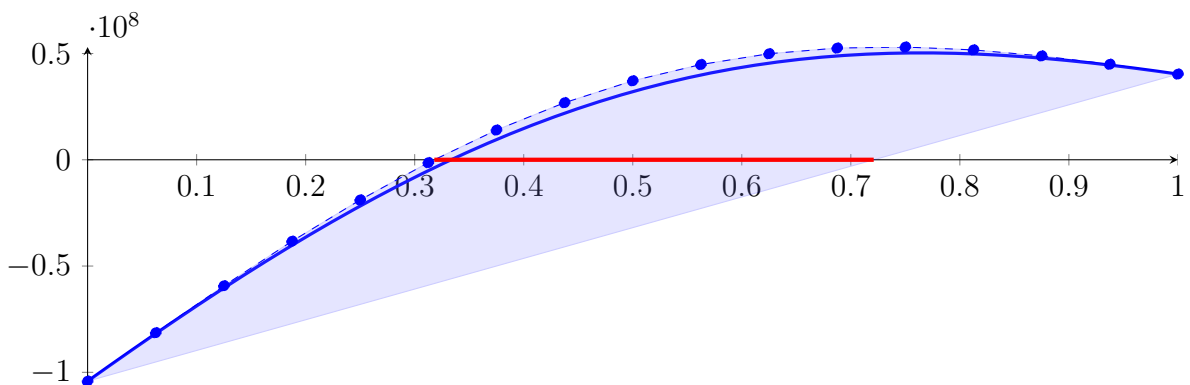
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



166.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

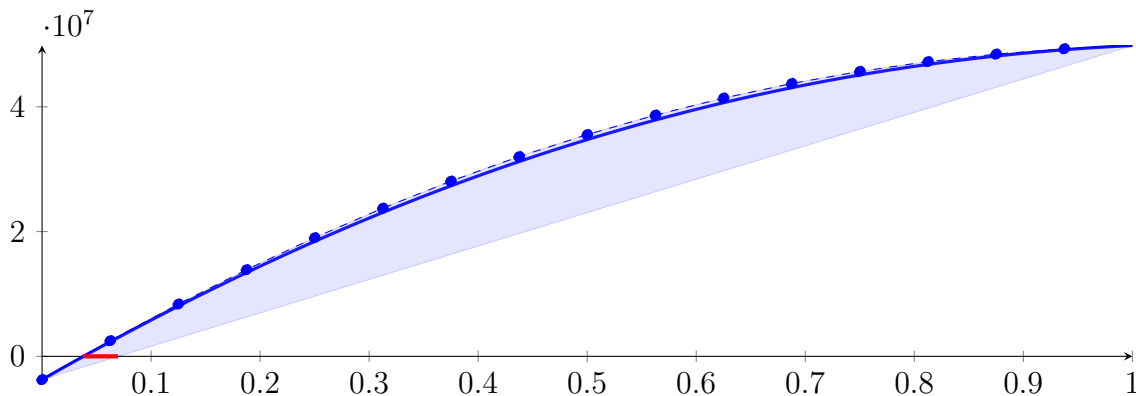
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

166.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.59825 \cdot 10^{-06} X^{16} - 5.93153 \cdot 10^{-05} X^{15} - 0.00248867 X^{14} - 0.0650056 X^{13} - 0.909142 X^{12} \\ &\quad - 5.03931 X^{11} + 36.4549 X^{10} + 692.921 X^9 + 1886.97 X^8 - 25792 X^7 - 149671 X^6 + 492605 X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

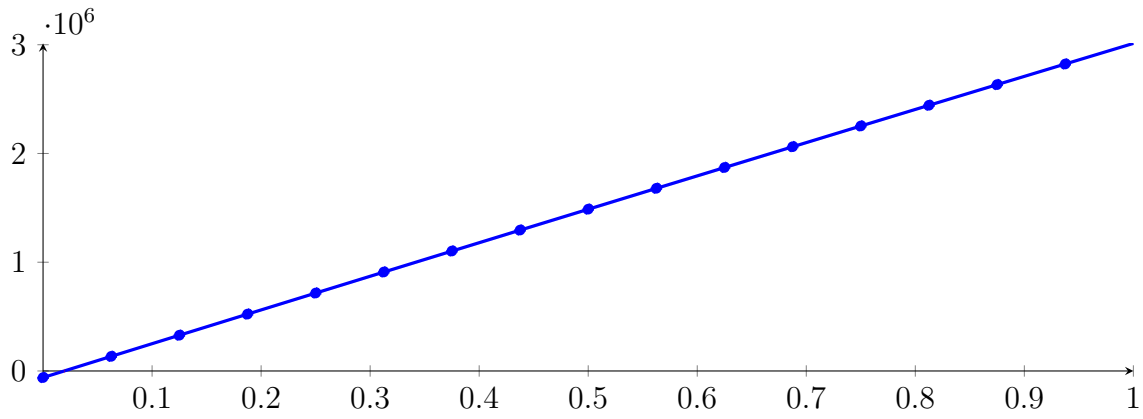
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

166.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 9.01396 \cdot 10^{-08} X^{16} - 2.65848 \cdot 10^{-07} X^{15} + 2.13948 \cdot 10^{-06} X^{14} - 1.33627 \cdot 10^{-06} X^{13} + 2.46973 \cdot 10^{-06} X^{12} \\ &\quad - 2.45524 \cdot 10^{-06} X^{11} + 5.50112 \cdot 10^{-07} X^{10} - 1.64198 \cdot 10^{-07} X^9 - 7.35598 \cdot 10^{-07} X^8 - 1.00892 \cdot 10^{-06} X^7 \\ &\quad - 0.000177509 X^6 + 0.0161038 X^5 + 4.36155 X^4 - 234.216 X^3 - 45622.2 X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

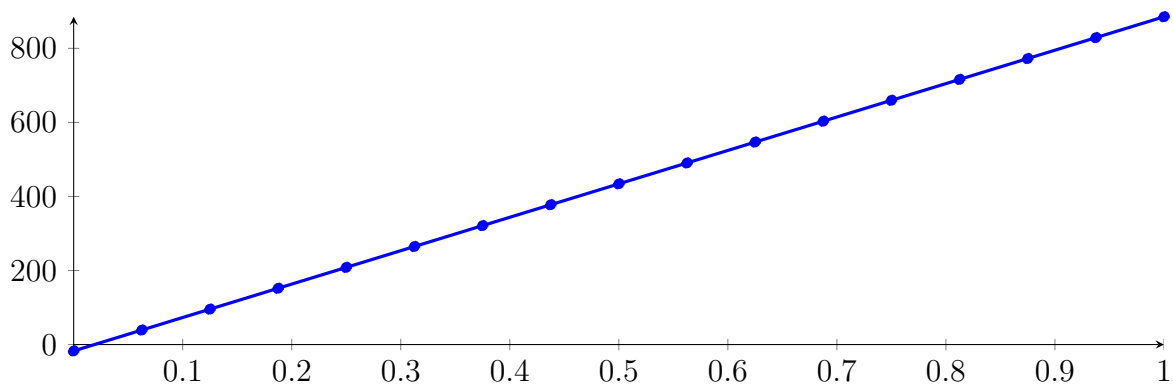
Longest intersection interval: 0.000289554

⇒ Selective recursion: interval 1: $[0.333333, 0.333337]$,

166.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 2.55372 \cdot 10^{-11} X^{16} - 7.21263 \cdot 10^{-11} X^{15} + 6.24141 \cdot 10^{-10} X^{14} - 4.11162 \cdot 10^{-10} X^{13} \\
 &+ 6.82359 \cdot 10^{-10} X^{12} - 7.09475 \cdot 10^{-10} X^{11} + 9.71305 \cdot 10^{-11} X^{10} - 3.46101 \cdot 10^{-11} X^9 \\
 &- 2.13971 \cdot 10^{-10} X^8 - 1.46061 \cdot 10^{-11} X^7 - 1.63366 \cdot 10^{-11} X^6 + 1.87916 \cdot 10^{-12} X^5 \\
 &+ 2.52576 \cdot 10^{-14} X^4 - 5.67777 \cdot 10^{-09} X^3 - 0.00382618 X^2 + 902.448 X - 17.178 \\
 &= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
 &+ 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
 &+ 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
 &+ 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

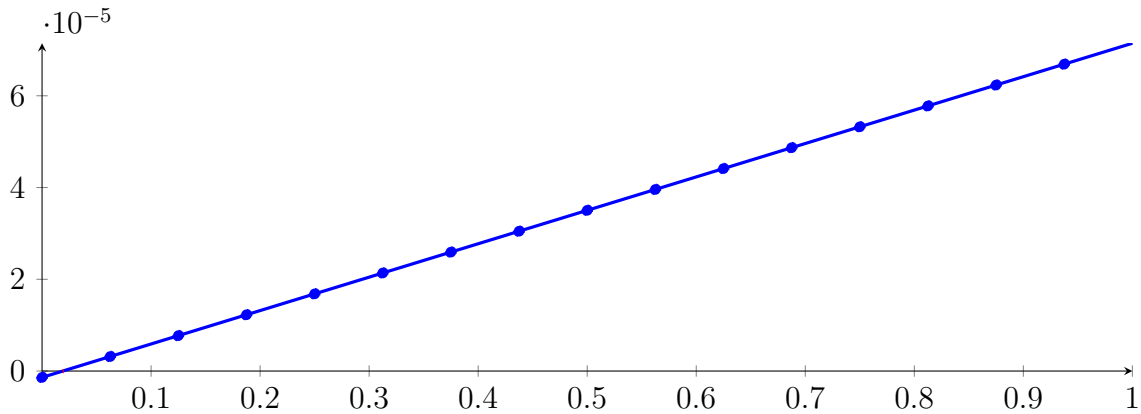
Longest intersection interval: $8.07045 \cdot 10^{-08}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

166.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 2.14261 \cdot 10^{-18} X^{16} - 6.28573 \cdot 10^{-18} X^{15} + 5.28612 \cdot 10^{-17} X^{14} - 3.67279 \cdot 10^{-17} X^{13} \\
 &\quad + 6.10136 \cdot 10^{-17} X^{12} - 6.60335 \cdot 10^{-17} X^{11} + 1.66661 \cdot 10^{-17} X^{10} - 8.36524 \cdot 10^{-18} X^9 \\
 &\quad - 1.56919 \cdot 10^{-17} X^8 - 1.85474 \cdot 10^{-18} X^7 - 1.4308 \cdot 10^{-18} X^6 + 1.1562 \cdot 10^{-19} X^5 - 1.20437 \\
 &\quad \cdot 10^{-20} X^4 - 4.63221 \cdot 10^{-22} X^3 - 2.49207 \cdot 10^{-17} X^2 + 7.28316 \cdot 10^{-05} X - 1.38634 \cdot 10^{-06} \\
 &= -1.38634 \cdot 10^{-06} B_{0,16}(X) + 3.16564 \cdot 10^{-06} B_{1,16}(X) + 7.71761 \cdot 10^{-06} B_{2,16}(X) + 1.22696 \\
 &\quad \cdot 10^{-05} B_{3,16}(X) + 1.68216 \cdot 10^{-05} B_{4,16}(X) + 2.13735 \cdot 10^{-05} B_{5,16}(X) + 2.59255 \cdot 10^{-05} B_{6,16}(X) \\
 &\quad + 3.04775 \cdot 10^{-05} B_{7,16}(X) + 3.50295 \cdot 10^{-05} B_{8,16}(X) + 3.95814 \cdot 10^{-05} B_{9,16}(X) + 4.41334 \\
 &\quad \cdot 10^{-05} B_{10,16}(X) + 4.86854 \cdot 10^{-05} B_{11,16}(X) + 5.32374 \cdot 10^{-05} B_{12,16}(X) + 5.77893 \cdot 10^{-05} B_{13,16}(X) \\
 &\quad + 6.23413 \cdot 10^{-05} B_{14,16}(X) + 6.68933 \cdot 10^{-05} B_{15,16}(X) + 7.14453 \cdot 10^{-05} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.51313 \cdot 10^{-15}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

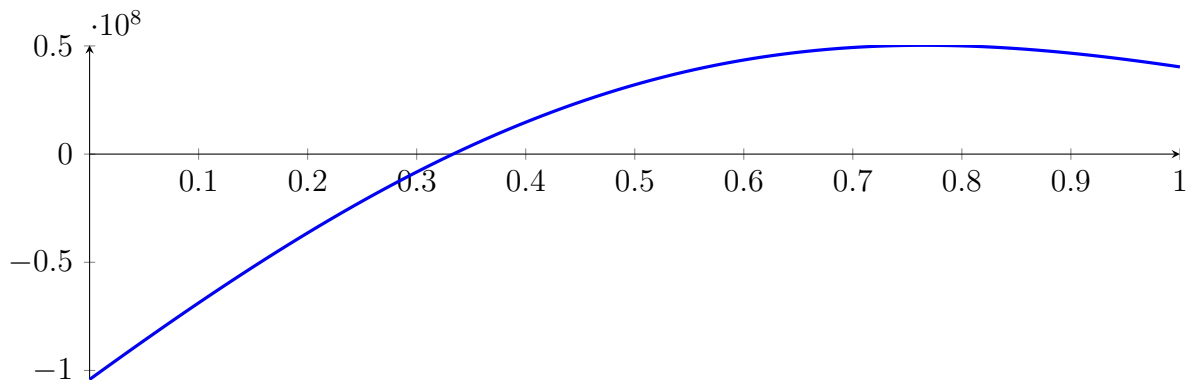
166.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

166.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

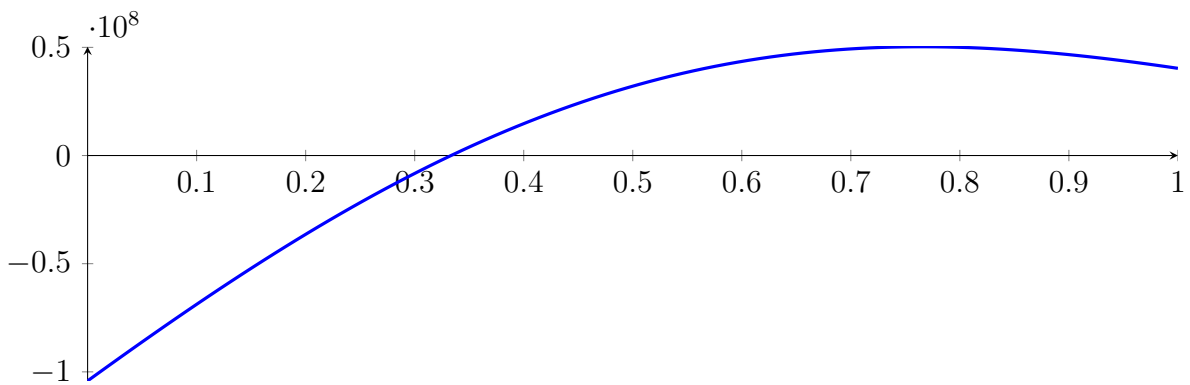
with precision $\varepsilon = 1 \cdot 10^{-128}$.

167 Running QuadClip on f_{16} with epsilon 128

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

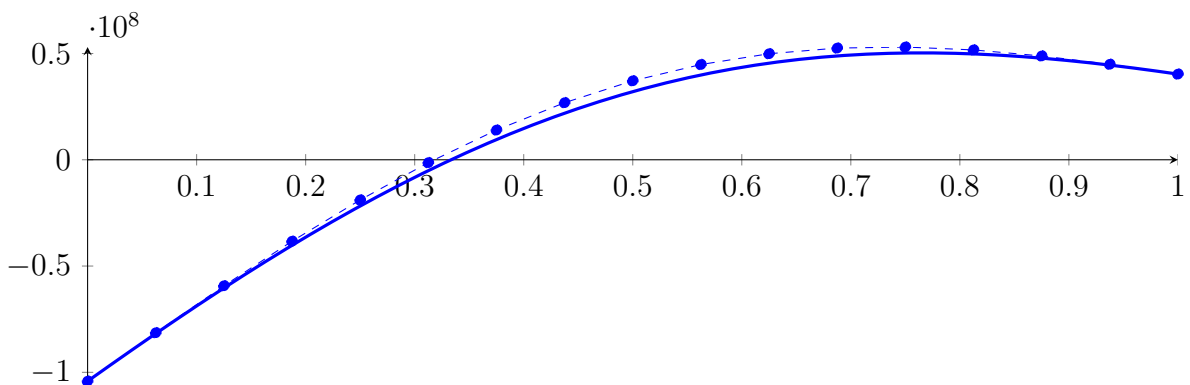
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



167.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = 6049.18X^{16} - 48305.2X^{15} + 174971X^{14} - 380294X^{13} + 552846X^{12} - 567203X^{11}$$

$$+ 422303X^{10} - 231038X^9 + 93003.6X^8 - 27320.1X^7 + 5752.57X^6 - 843.63X^5$$

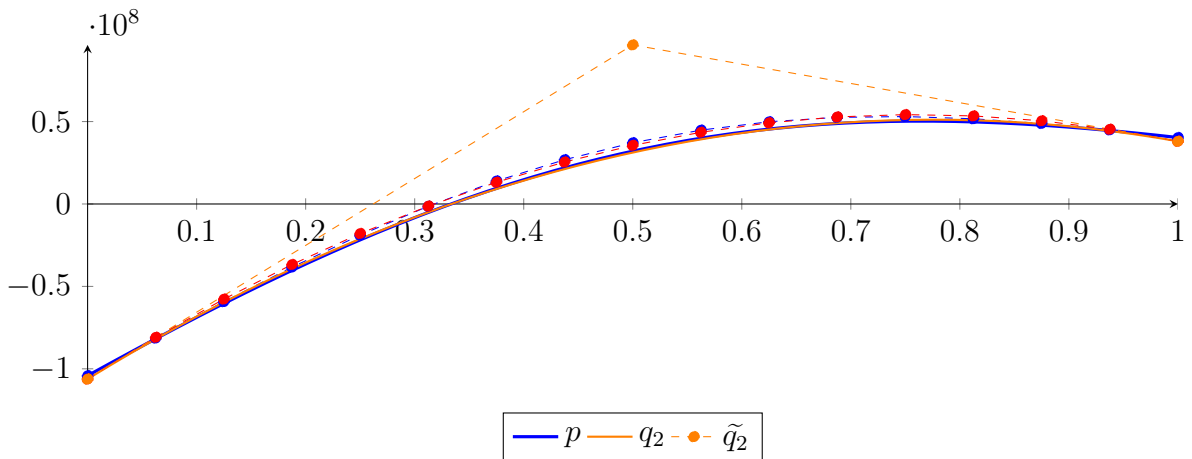
$$+ 82.5145X^4 - 5.01388X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

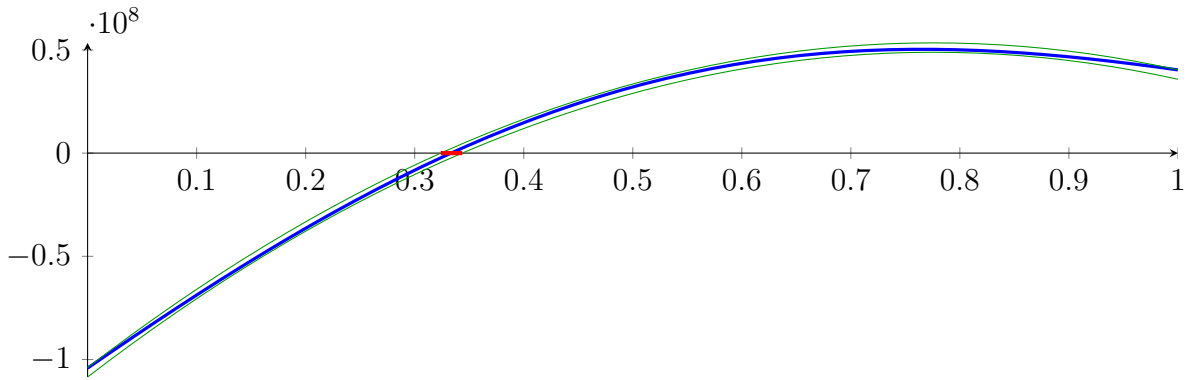
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

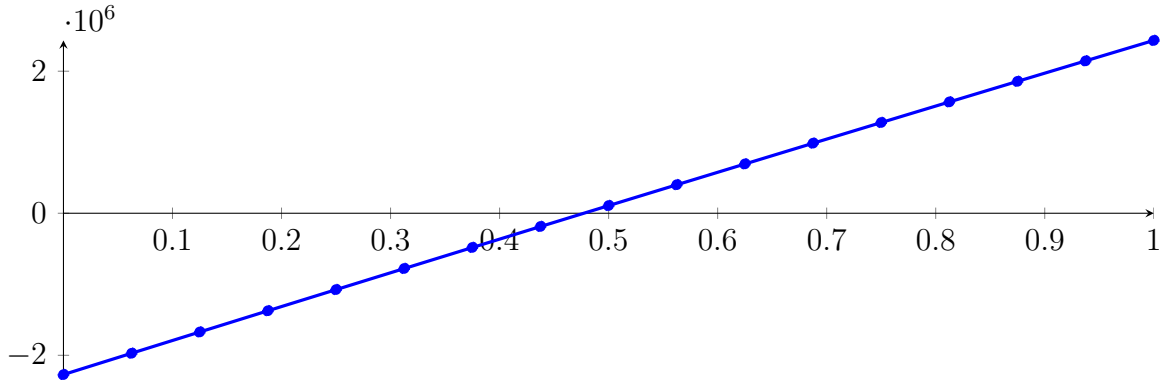
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

167.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

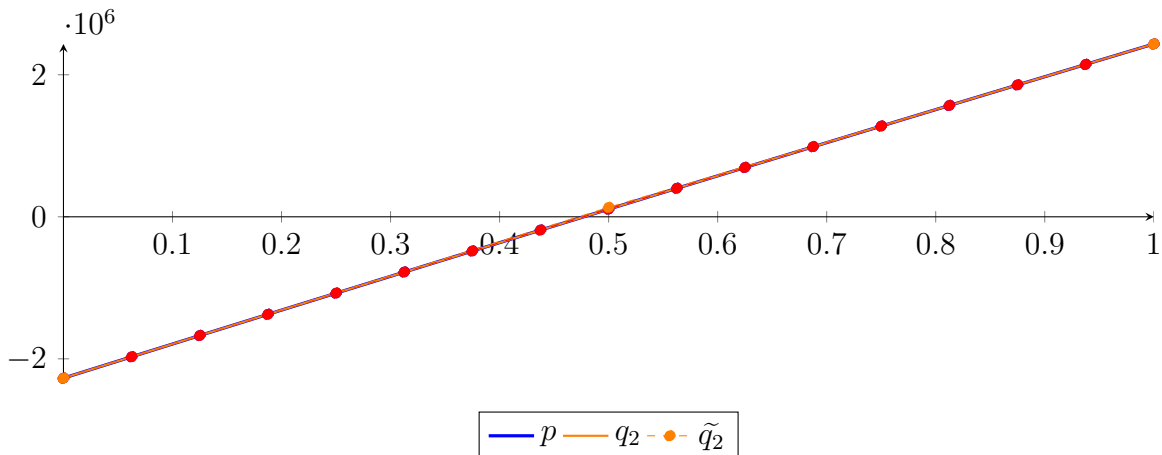
$$\begin{aligned}
 p &= -1.54841 \cdot 10^{-10} X^{16} - 1.66965 \cdot 10^{-07} X^{15} - 2.92739 \cdot 10^{-07} X^{14} - 1.77943 \cdot 10^{-06} X^{13} - 1.17235 \cdot 10^{-06} X^{12} \\
 &\quad - 2.42234 \cdot 10^{-06} X^{11} - 6.86445 \cdot 10^{-07} X^{10} - 1.39162 \cdot 10^{-06} X^9 + 1.07395 \cdot 10^{-06} X^8 - 1.67072 \cdot 10^{-05} X^7 \\
 &\quad - 0.00205879 X^6 + 0.132721 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 22.7036 X^{16} - 178.964 X^{15} + 638.976 X^{14} - 1366.64 X^{13} + 1951.1 X^{12} - 1960.92 X^{11} \\
 &\quad + 1425.35 X^{10} - 757.631 X^9 + 294.177 X^8 - 82.4368 X^7 + 16.2856 X^6 - 2.18949 X^5 \\
 &\quad + 0.191237 X^4 - 0.0101048 X^3 - 104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} \\
 &\quad + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

$$M = -104265 X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

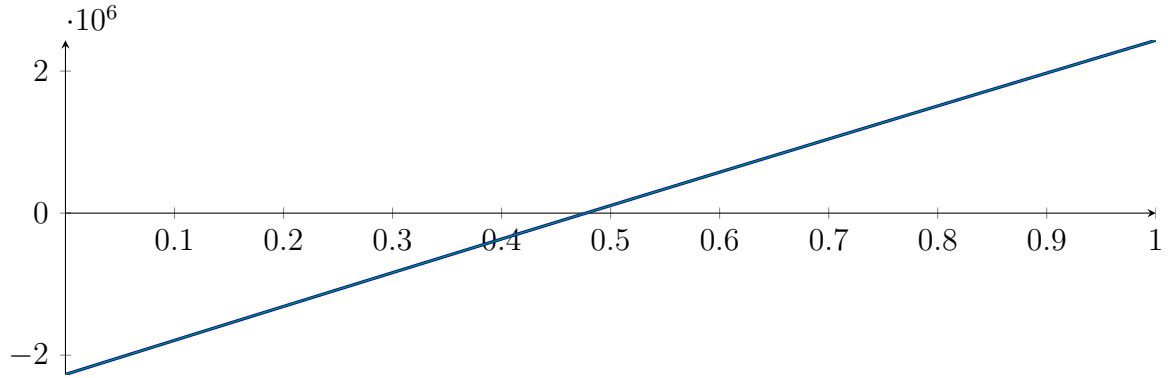
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\}$$

$$N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

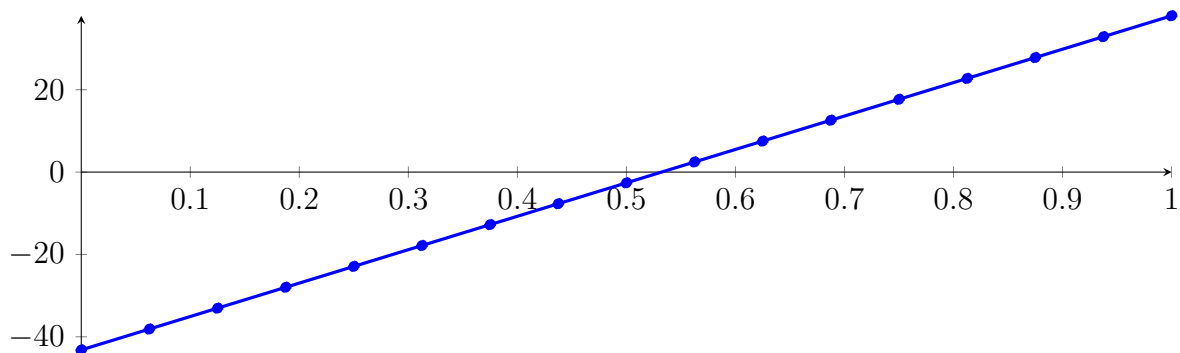
Longest intersection interval: $1.72301 \cdot 10^{-05}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

167.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

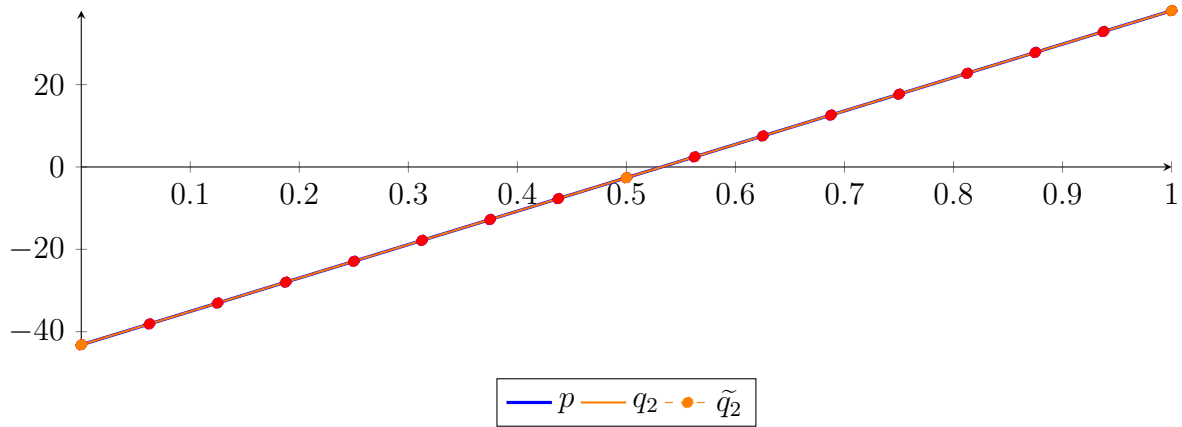
$$\begin{aligned} p &= -2.76723 \cdot 10^{-13} X^{16} - 2.40874 \cdot 10^{-12} X^{15} - 1.25233 \cdot 10^{-11} X^{14} - 3.02935 \cdot 10^{-11} X^{13} \\ &\quad - 3.05617 \cdot 10^{-11} X^{12} - 3.83107 \cdot 10^{-11} X^{11} - 1.26692 \cdot 10^{-11} X^{10} - 2.6672 \cdot 10^{-11} X^9 \\ &\quad + 2.0004 \cdot 10^{-11} X^8 + 4.12781 \cdot 10^{-12} X^7 + 2.44493 \cdot 10^{-12} X^6 - 1.21236 \cdot 10^{-13} X^5 \\ &\quad + 1.26288 \cdot 10^{-14} X^4 - 4.1267 \cdot 10^{-12} X^3 - 3.09388 \cdot 10^{-05} X^2 + 81.1506X - 43.1911 \\ &= -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\ &\quad - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X) \\ &\quad + 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X) \\ &\quad + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$\begin{aligned}
\tilde{q}_2 &= 5.96265 \cdot 10^{-05} X^{16} - 0.000436042 X^{15} + 0.00141812 X^{14} - 0.00269475 X^{13} \\
&+ 0.00329809 X^{12} - 0.00268757 X^{11} + 0.00143268 X^{10} - 0.000439599 X^9 \\
&+ 1.98418 \cdot 10^{-05} X^8 + 4.87608 \cdot 10^{-05} X^7 - 2.46333 \cdot 10^{-05} X^6 + 6.35808 \cdot 10^{-06} X^5 \\
&- 9.62755 \cdot 10^{-07} X^4 + 8.21372 \cdot 10^{-08} X^3 - 3.09429 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911 \\
&= -43.1911 B_{0,16} - 38.1192 B_{1,16} - 33.0473 B_{2,16} - 27.9754 B_{3,16} - 22.9035 B_{4,16} - 17.8316 B_{5,16} \\
&- 12.7597 B_{6,16} - 7.68778 B_{7,16} - 2.61587 B_{8,16} + 2.45604 B_{9,16} + 7.52795 B_{10,16} + 12.5999 B_{11,16} \\
&+ 17.6718 B_{12,16} + 22.7437 B_{13,16} + 27.8156 B_{14,16} + 32.8875 B_{15,16} + 37.9594 B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.5947 \cdot 10^{-09}$.

Bounding polynomials M and m :

$$M = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

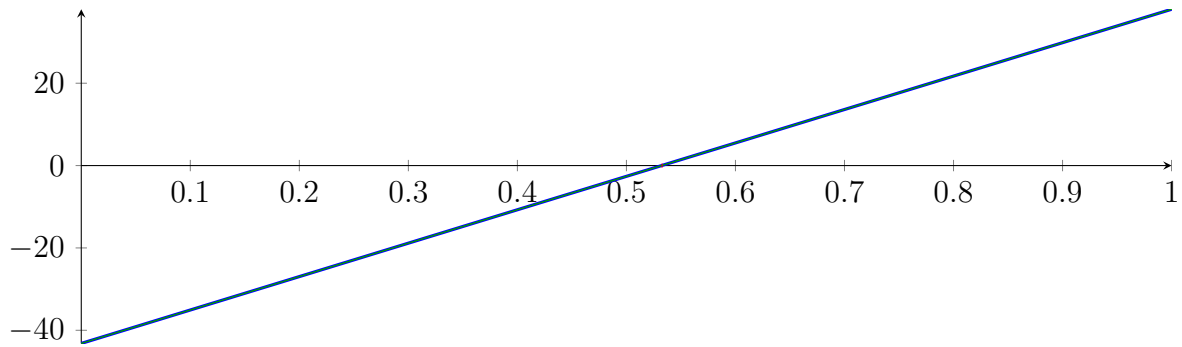
$$m = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

Root of M and m :

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\}$$

$$N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

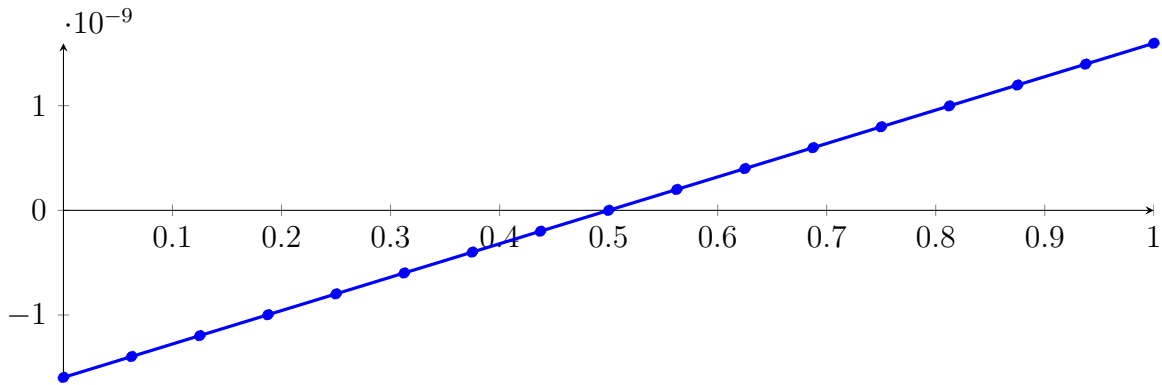
Longest intersection interval: $3.93535 \cdot 10^{-11}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

167.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

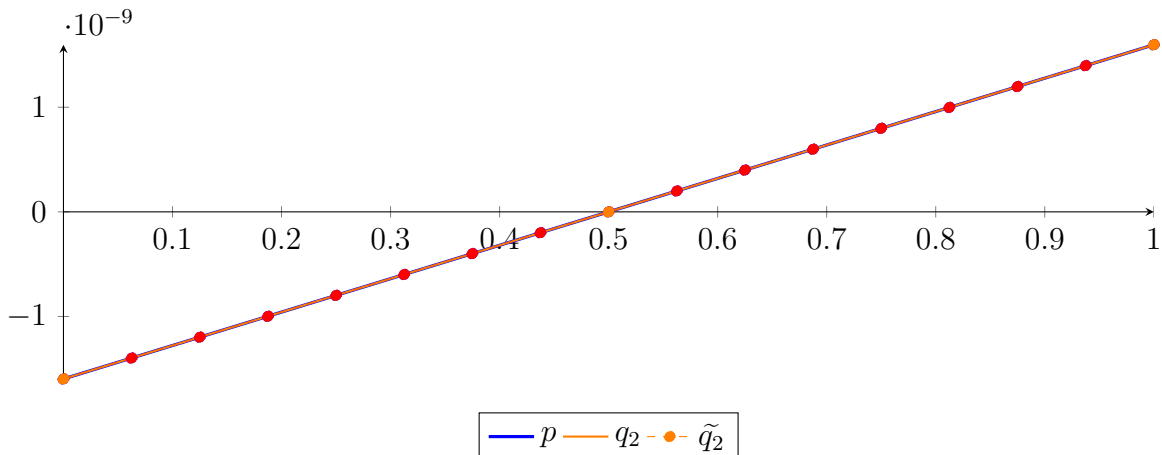
$$\begin{aligned}
 p &= -4.89361 \cdot 10^{-24} X^{16} - 1.05466 \cdot 10^{-22} X^{15} - 3.09805 \cdot 10^{-22} X^{14} - 1.16981 \cdot 10^{-21} X^{13} \\
 &\quad - 9.76202 \cdot 10^{-22} X^{12} - 1.69365 \cdot 10^{-21} X^{11} - 5.95131 \cdot 10^{-22} X^{10} - 1.05349 \cdot 10^{-21} X^9 \\
 &\quad + 7.5893 \cdot 10^{-22} X^8 + 1.47859 \cdot 10^{-22} X^7 + 9.70322 \cdot 10^{-23} X^6 - 7.05688 \cdot 10^{-24} X^5 \\
 &\quad + 1.47018 \cdot 10^{-24} X^4 - 4.84676 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09} \\
 &= -1.59674 \cdot 10^{-09} B_{0,16}(X) - 1.39715 \cdot 10^{-09} B_{1,16}(X) - 1.19755 \cdot 10^{-09} B_{2,16}(X) - 9.97951 \\
 &\quad \cdot 10^{-10} B_{3,16}(X) - 7.98353 \cdot 10^{-10} B_{4,16}(X) - 5.98756 \cdot 10^{-10} B_{5,16}(X) - 3.99159 \cdot 10^{-10} B_{6,16}(X) \\
 &\quad - 1.99561 \cdot 10^{-10} B_{7,16}(X) + 3.6039 \cdot 10^{-14} B_{8,16}(X) + 1.99633 \cdot 10^{-10} B_{9,16}(X) + 3.99231 \\
 &\quad \cdot 10^{-10} B_{10,16}(X) + 5.98828 \cdot 10^{-10} B_{11,16}(X) + 7.98425 \cdot 10^{-10} B_{12,16}(X) + 9.98023 \cdot 10^{-10} B_{13,16}(X) \\
 &\quad + 1.19762 \cdot 10^{-09} B_{14,16}(X) + 1.39722 \cdot 10^{-09} B_{15,16}(X) + 1.59681 \cdot 10^{-09} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -4.83666 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09} \\
 &= -1.59674 \cdot 10^{-09} B_{0,2} + 3.6039 \cdot 10^{-14} B_{1,2} + 1.59681 \cdot 10^{-09} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 9.45798 \cdot 10^{-15} X^{16} - 7.39324 \cdot 10^{-14} X^{15} + 2.61437 \cdot 10^{-13} X^{14} - 5.52989 \cdot 10^{-13} X^{13} \\
 &\quad + 7.79462 \cdot 10^{-13} X^{12} - 7.71893 \cdot 10^{-13} X^{11} + 5.51461 \cdot 10^{-13} X^{10} - 2.8712 \cdot 10^{-13} X^9 \\
 &\quad + 1.08634 \cdot 10^{-13} X^8 - 2.94042 \cdot 10^{-14} X^7 + 5.52081 \cdot 10^{-15} X^6 - 6.84058 \cdot 10^{-16} X^5 + 5.20623 \\
 &\quad \cdot 10^{-17} X^4 - 2.16513 \cdot 10^{-18} X^3 + 1.74369 \cdot 10^{-20} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09} \\
 &= -1.59674 \cdot 10^{-09} B_{0,16} - 1.39715 \cdot 10^{-09} B_{1,16} - 1.19755 \cdot 10^{-09} B_{2,16} - 9.97951 \cdot 10^{-10} B_{3,16} - 7.98353 \\
 &\quad \cdot 10^{-10} B_{4,16} - 5.98756 \cdot 10^{-10} B_{5,16} - 3.99159 \cdot 10^{-10} B_{6,16} - 1.99561 \cdot 10^{-10} B_{7,16} + 3.60393 \cdot 10^{-14} B_{8,16} \\
 &\quad + 1.99633 \cdot 10^{-10} B_{9,16} + 3.99231 \cdot 10^{-10} B_{10,16} + 5.98828 \cdot 10^{-10} B_{11,16} + 7.98425 \cdot 10^{-10} B_{12,16} \\
 &\quad + 9.98023 \cdot 10^{-10} B_{13,16} + 1.19762 \cdot 10^{-09} B_{14,16} + 1.39722 \cdot 10^{-09} B_{15,16} + 1.59681 \cdot 10^{-09} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.02367 \cdot 10^{-19}$.

Bounding polynomials M and m :

$$M = -4.82657 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$

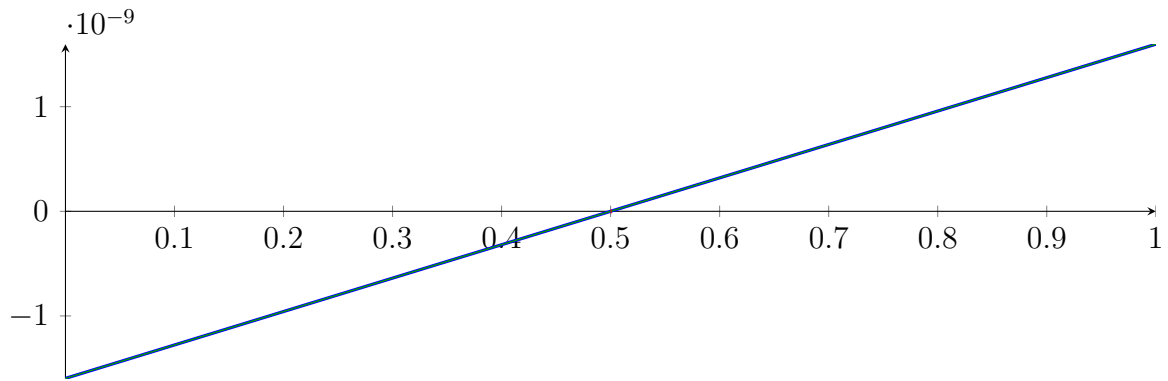
$$m = -4.84676 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$

Root of M and m :

$$N(M) = \{0.5, 6.61662 \cdot 10^{16}\}$$

$$N(m) = \{0.5, 6.58905 \cdot 10^{16}\}$$

Intersection intervals:



[0.5, 0.5]

Longest intersection interval: 0

⇒ Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

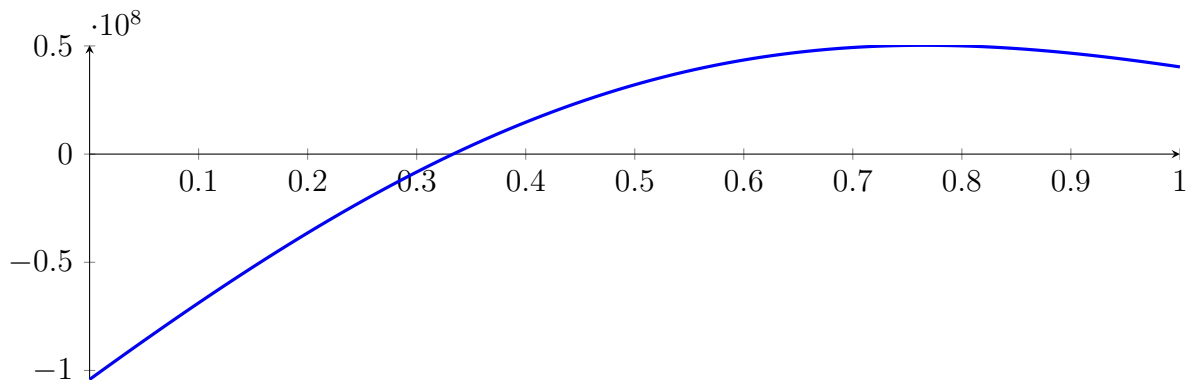
167.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

167.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

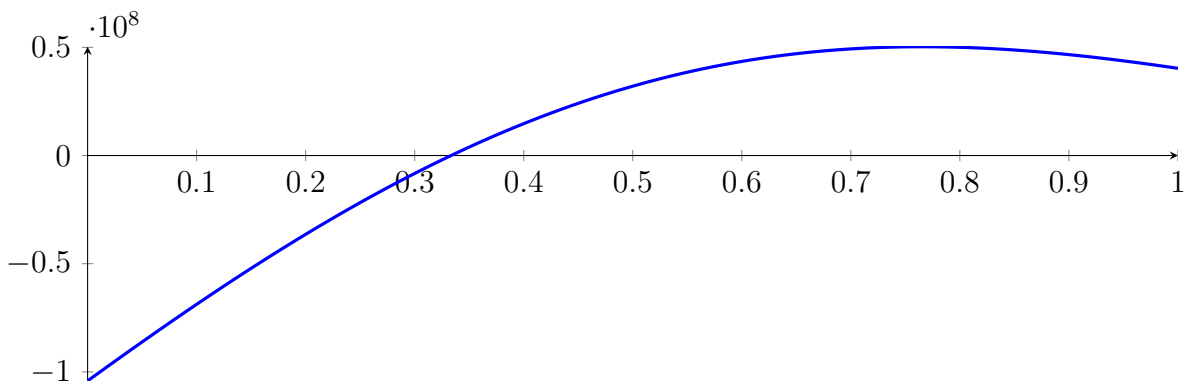
with precision $\varepsilon = 1 \cdot 10^{-128}$.

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$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

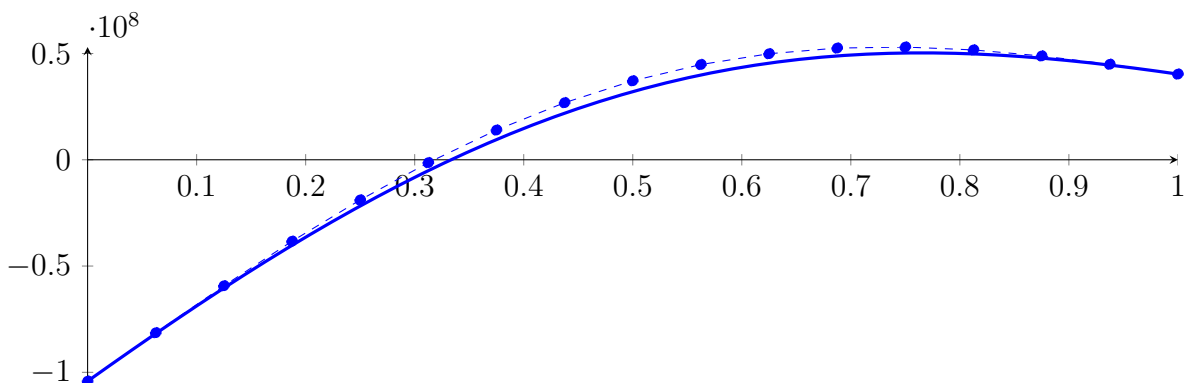
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



168.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\
 & + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \\
 & \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 2461.93X^{16} - 19614.9X^{15} + 70879.5X^{14} - 153661X^{13} + 222746X^{12} - 227755X^{11}$$

$$+ 168826X^{10} - 91798.7X^9 + 36630.3X^8 - 10627.3X^7 + 2200.54X^6 - 316.059X^5$$

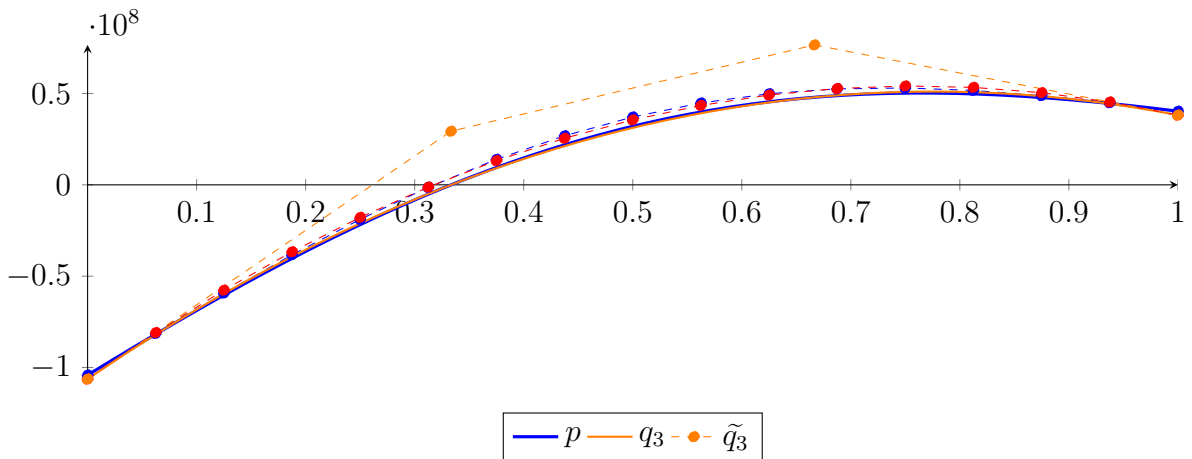
$$+ 30.1958X^4 + 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18551 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91815 \cdot 10^7 B_{10,16} + 5.27353 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

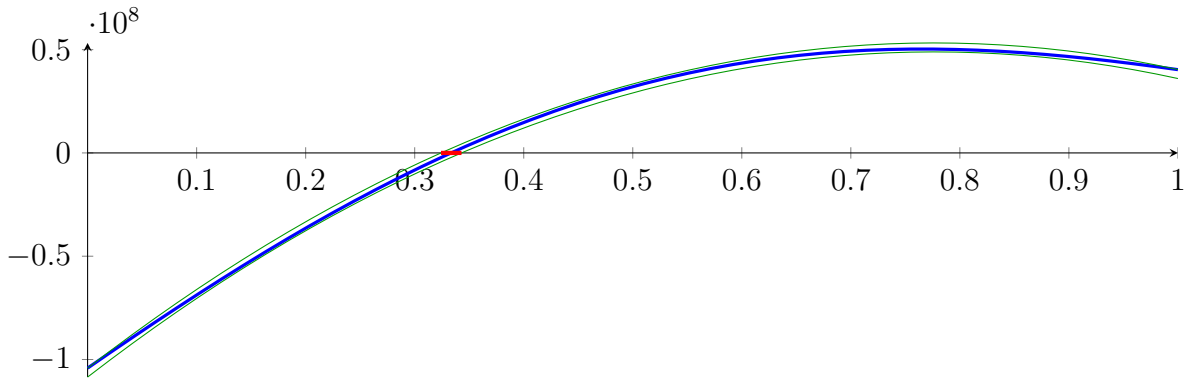
$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\} \quad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

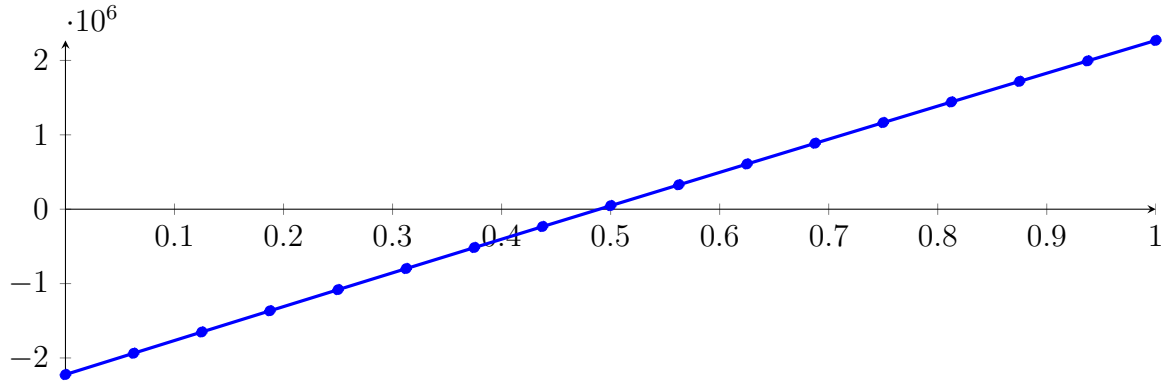
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

168.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

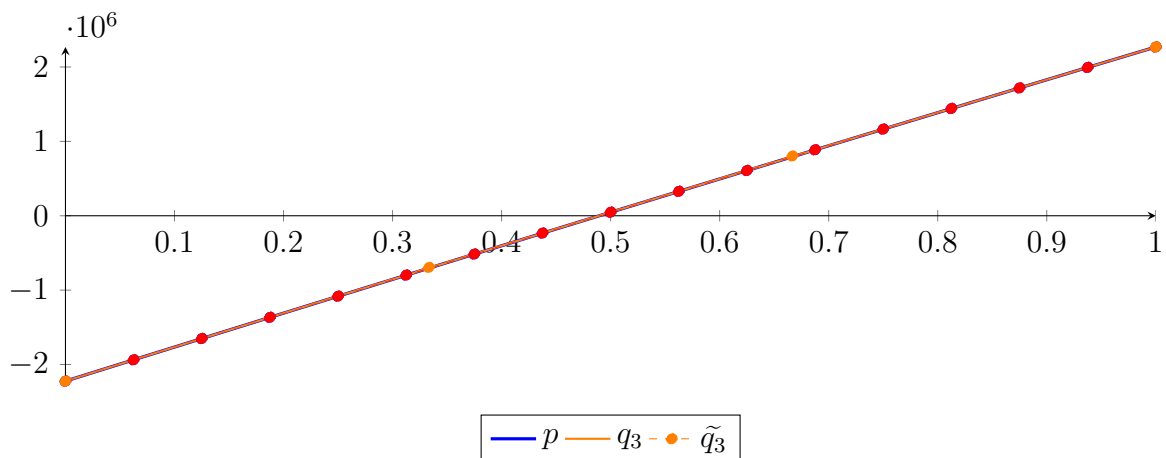
$$\begin{aligned}
 p &= -3.66617 \cdot 10^{-09} X^{16} - 1.53217 \cdot 10^{-07} X^{15} - 3.62234 \cdot 10^{-07} X^{14} - 1.65579 \cdot 10^{-06} X^{13} - 1.15373 \cdot 10^{-06} X^{12} \\
 &\quad - 2.3399 \cdot 10^{-06} X^{11} - 5.02543 \cdot 10^{-07} X^{10} - 1.38381 \cdot 10^{-06} X^9 + 1.1237 \cdot 10^{-06} X^8 - 1.19653 \cdot 10^{-05} X^7 \\
 &\quad - 0.00155608 X^6 + 0.10496 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 16.4956 X^{16} - 129.161 X^{15} + 457.83 X^{14} - 971.671 X^{13} + 1375.95 X^{12} - 1370.96 X^{11} \\
 &\quad + 987.265 X^{10} - 519.476 X^9 + 199.587 X^8 - 55.434 X^7 + 10.9237 X^6 - 1.48019 X^5 \\
 &\quad + 0.129516 X^4 - 700.679 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} \\
 &\quad + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m :

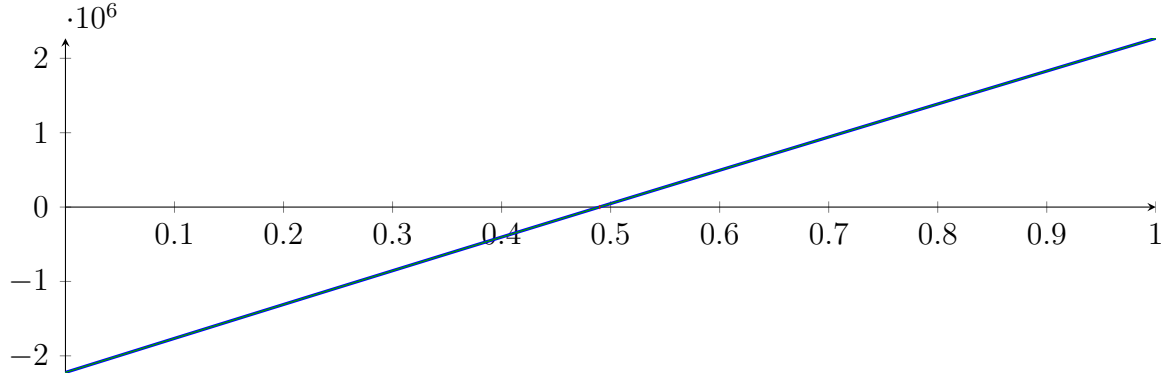
$$M = -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\} \quad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

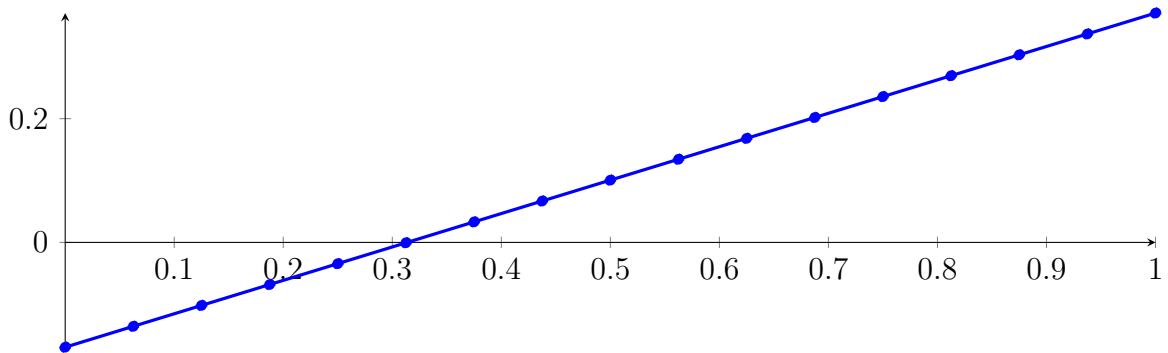
Longest intersection interval: $1.20174 \cdot 10^{-07}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

168.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

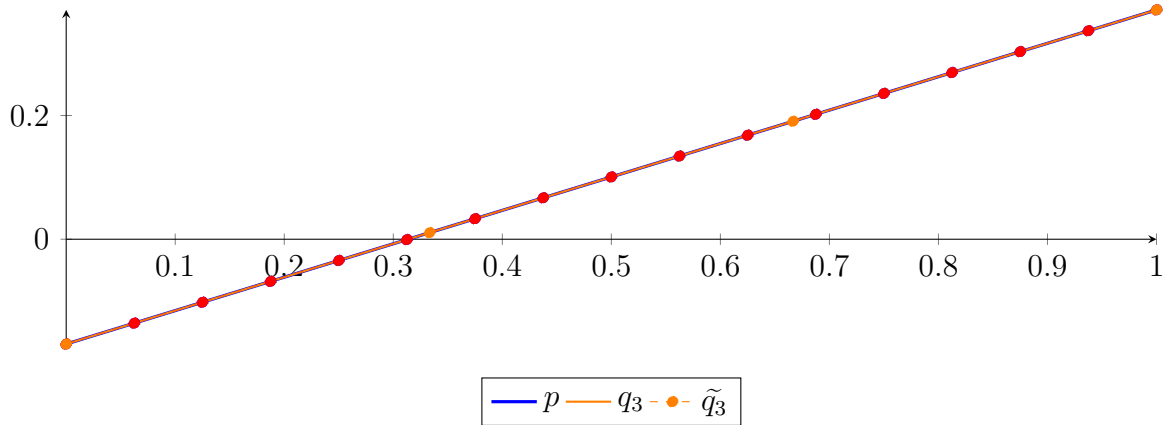
$$\begin{aligned} p &= 5.55524 \cdot 10^{-15} X^{16} - 2.94313 \cdot 10^{-14} X^{15} + 1.19384 \cdot 10^{-13} X^{14} - 2.17482 \cdot 10^{-13} X^{13} + 7.26155 \cdot 10^{-14} X^{12} \\ &\quad - 3.44766 \cdot 10^{-13} X^{11} - 3.47292 \cdot 10^{-15} X^{10} - 1.1287 \cdot 10^{-13} X^9 + 2.93027 \cdot 10^{-14} X^8 + 8.06213 \cdot 10^{-15} X^7 \\ &\quad + 5.64349 \cdot 10^{-15} X^6 + 4.93312 \cdot 10^{-17} X^4 + 1.51788 \cdot 10^{-17} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396 \\ &= -0.169396 B_{0,16}(X) - 0.135637 B_{1,16}(X) - 0.101877 B_{2,16}(X) - 0.068118 B_{3,16}(X) \\ &\quad - 0.0343587 B_{4,16}(X) - 0.000599476 B_{5,16}(X) + 0.0331598 B_{6,16}(X) \\ &\quad + 0.0669191 B_{7,16}(X) + 0.100678 B_{8,16}(X) + 0.134438 B_{9,16}(X) + 0.168197 B_{10,16}(X) \\ &\quad + 0.201956 B_{11,16}(X) + 0.235715 B_{12,16}(X) + 0.269475 B_{13,16}(X) \\ &\quad + 0.303234 B_{14,16}(X) + 0.336993 B_{15,16}(X) + 0.370752 B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07065 \cdot 10^{-17} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396 \\ &= -0.169396 B_{0,3} + 0.0106536 B_{1,3} + 0.190703 B_{2,3} + 0.370752 B_{3,3} \end{aligned}$$

$$\begin{aligned}
\tilde{q}_3 &= 8.59095 \cdot 10^{-06} X^{16} - 6.82648 \cdot 10^{-05} X^{15} + 0.000245968 X^{14} - 0.000531568 X^{13} \\
&+ 0.000767923 X^{12} - 0.000782231 X^{11} + 0.0005774 X^{10} - 0.000312464 X^9 \\
&+ 0.000123994 X^8 - 3.57388 \cdot 10^{-05} X^7 + 7.34249 \cdot 10^{-06} X^6 - 1.04474 \cdot 10^{-06} X^5 \\
&+ 9.86739 \cdot 10^{-08} X^4 - 5.7553 \cdot 10^{-09} X^3 - 1.19186 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396 \\
&= -0.169396 B_{0,16} - 0.135637 B_{1,16} - 0.101877 B_{2,16} - 0.068118 B_{3,16} - 0.0343587 B_{4,16} \\
&- 0.000599476 B_{5,16} + 0.0331598 B_{6,16} + 0.0669191 B_{7,16} + 0.100678 B_{8,16} \\
&+ 0.134438 B_{9,16} + 0.168197 B_{10,16} + 0.201956 B_{11,16} + 0.235715 B_{12,16} \\
&+ 0.269475 B_{13,16} + 0.303234 B_{14,16} + 0.336993 B_{15,16} + 0.370752 B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.81206 \cdot 10^{-10}$.

Bounding polynomials M and m :

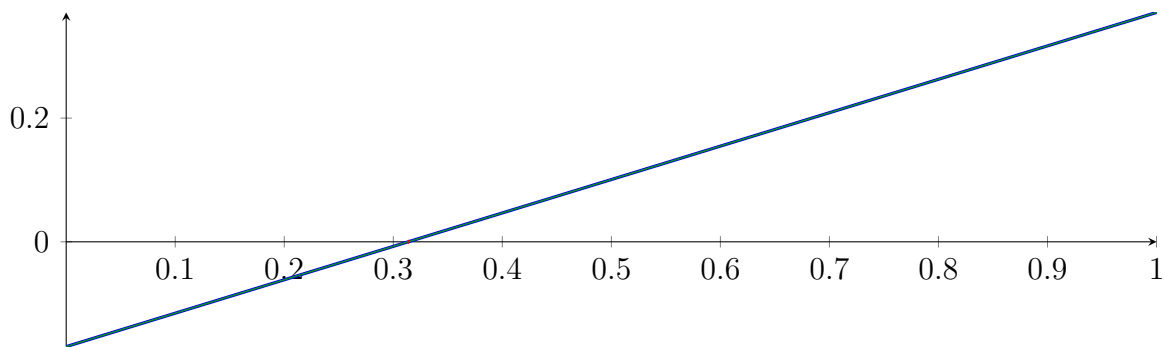
$$M = -1.07065 \cdot 10^{-17} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

$$m = -1.07065 \cdot 10^{-17} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

Root of M and m :

$$N(M) = \{-2.97569 \cdot 10^8, 0.31361, 1.69542 \cdot 10^8\} \quad N(m) = \{-2.97569 \cdot 10^8, 0.31361, 1.69542 \cdot 10^8\}$$

Intersection intervals:



$$[0.31361, 0.31361]$$

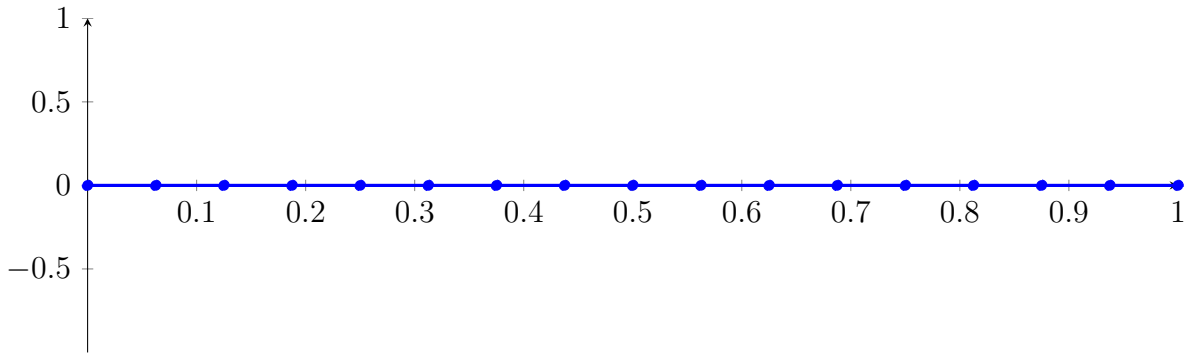
Longest intersection interval: $7.85803 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

168.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

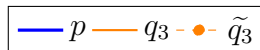
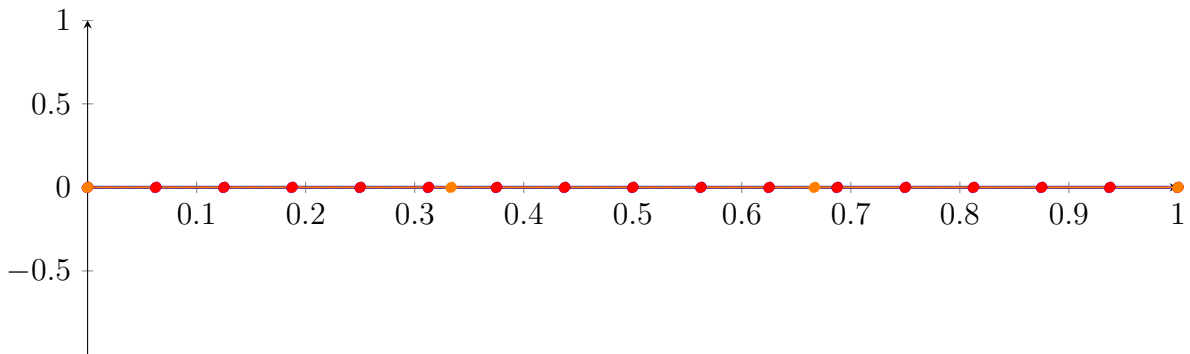
$$\begin{aligned}
 p &= -1.51576 \cdot 10^{-21} X^{16} + 2.62009 \cdot 10^{-21} X^{15} - 3.98039 \cdot 10^{-20} X^{14} + 3.2136 \cdot 10^{-21} X^{13} - 5.16564 \cdot 10^{-20} X^{12} \\
 &\quad + 1.52429 \cdot 10^{-20} X^{11} - 1.44901 \cdot 10^{-20} X^{10} - 1.40466 \cdot 10^{-20} X^9 + 2.34541 \cdot 10^{-20} X^8 + 3.25289 \cdot 10^{-21} X^7 \\
 &\quad + 2.38052 \cdot 10^{-21} X^6 - 2.2582 \cdot 10^{-22} X^5 + 3.52844 \cdot 10^{-23} X^4 + 4.2445 \cdot 10^{-10} X - 2.39831 \cdot 10^{-08} \\
 &= -2.39831 \cdot 10^{-08} B_{0,16}(X) - 2.39566 \cdot 10^{-08} B_{1,16}(X) - 2.39301 \cdot 10^{-08} B_{2,16}(X) - 2.39036 \\
 &\quad \cdot 10^{-08} B_{3,16}(X) - 2.3877 \cdot 10^{-08} B_{4,16}(X) - 2.38505 \cdot 10^{-08} B_{5,16}(X) - 2.3824 \cdot 10^{-08} B_{6,16}(X) \\
 &\quad - 2.37974 \cdot 10^{-08} B_{7,16}(X) - 2.37709 \cdot 10^{-08} B_{8,16}(X) - 2.37444 \cdot 10^{-08} B_{9,16}(X) - 2.37179 \\
 &\quad \cdot 10^{-08} B_{10,16}(X) - 2.36913 \cdot 10^{-08} B_{11,16}(X) - 2.36648 \cdot 10^{-08} B_{12,16}(X) - 2.36383 \cdot 10^{-08} B_{13,16}(X) \\
 &\quad - 2.36118 \cdot 10^{-08} B_{14,16}(X) - 2.35852 \cdot 10^{-08} B_{15,16}(X) - 2.35587 \cdot 10^{-08} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -3.5866 \cdot 10^{-25} X^3 + 8.33643 \cdot 10^{-25} X^2 + 4.2445 \cdot 10^{-10} X - 2.39831 \cdot 10^{-08} \\
 &= -2.39831 \cdot 10^{-08} B_{0,3} - 2.38417 \cdot 10^{-08} B_{1,3} - 2.37002 \cdot 10^{-08} B_{2,3} - 2.35587 \cdot 10^{-08} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= -1.64958 \cdot 10^{-12} X^{16} + 1.3166 \cdot 10^{-11} X^{15} - 4.76688 \cdot 10^{-11} X^{14} + 1.03558 \cdot 10^{-10} X^{13} \\
 &\quad - 1.50448 \cdot 10^{-10} X^{12} + 1.54183 \cdot 10^{-10} X^{11} - 1.1456 \cdot 10^{-10} X^{10} + 6.24452 \cdot 10^{-11} X^9 \\
 &\quad - 2.49793 \cdot 10^{-11} X^8 + 7.26358 \cdot 10^{-12} X^7 - 1.50649 \cdot 10^{-12} X^6 + 2.16616 \cdot 10^{-13} X^5 - 2.07725 \\
 &\quad \cdot 10^{-14} X^4 + 1.24748 \cdot 10^{-15} X^3 - 4.0727 \cdot 10^{-17} X^2 + 4.2445 \cdot 10^{-10} X - 2.39831 \cdot 10^{-08} \\
 &= -2.39831 \cdot 10^{-08} B_{0,16} - 2.39566 \cdot 10^{-08} B_{1,16} - 2.39301 \cdot 10^{-08} B_{2,16} - 2.39036 \cdot 10^{-08} B_{3,16} - 2.3877 \\
 &\quad \cdot 10^{-08} B_{4,16} - 2.38505 \cdot 10^{-08} B_{5,16} - 2.3824 \cdot 10^{-08} B_{6,16} - 2.37974 \cdot 10^{-08} B_{7,16} - 2.37709 \cdot 10^{-08} B_{8,16} \\
 &\quad - 2.37444 \cdot 10^{-08} B_{9,16} - 2.37179 \cdot 10^{-08} B_{10,16} - 2.36913 \cdot 10^{-08} B_{11,16} - 2.36648 \cdot 10^{-08} B_{12,16} \\
 &\quad - 2.36383 \cdot 10^{-08} B_{13,16} - 2.36118 \cdot 10^{-08} B_{14,16} - 2.35852 \cdot 10^{-08} B_{15,16} - 2.35587 \cdot 10^{-08} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.51589 \cdot 10^{-17}$.

Bounding polynomials M and m :

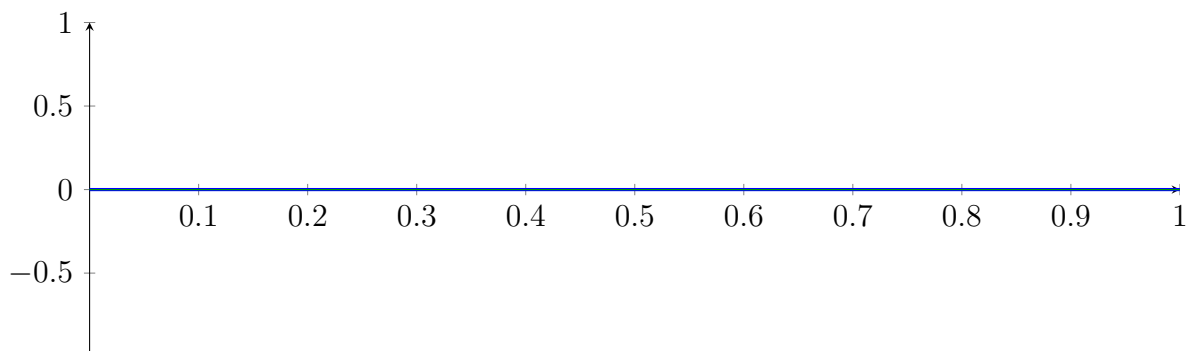
$$M = -3.5866 \cdot 10^{-25} X^3 + 8.33643 \cdot 10^{-25} X^2 + 4.2445 \cdot 10^{-10} X - 2.39831 \cdot 10^{-08}$$

$$m = -3.5866 \cdot 10^{-25} X^3 + 8.33643 \cdot 10^{-25} X^2 + 4.2445 \cdot 10^{-10} X - 2.39831 \cdot 10^{-08}$$

Root of M and m :

$$N(M) = \{-3.44011 \cdot 10^7, 56.504, 3.4401 \cdot 10^7\} \quad N(m) = \{-3.44011 \cdot 10^7, 56.504, 3.4401 \cdot 10^7\}$$

Intersection intervals:

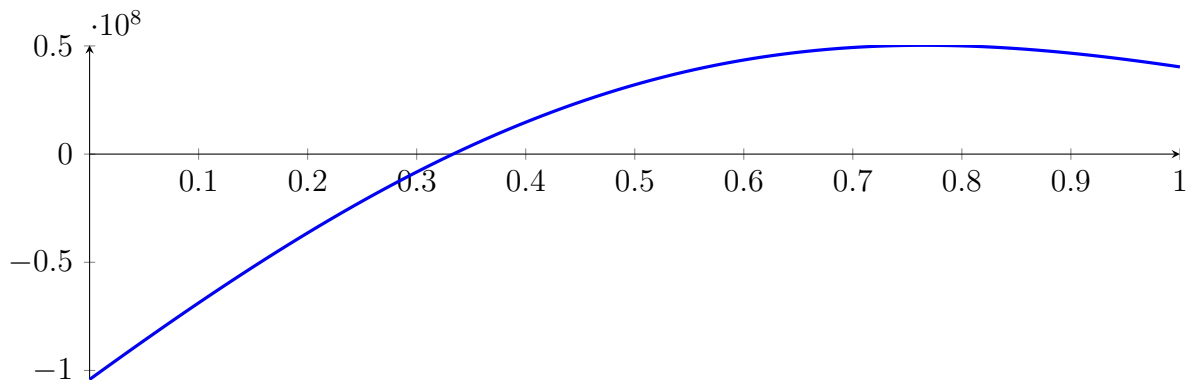


No intersection intervals with the x axis.

168.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

with precision $\varepsilon = 1 \cdot 10^{-128}$.

Part III

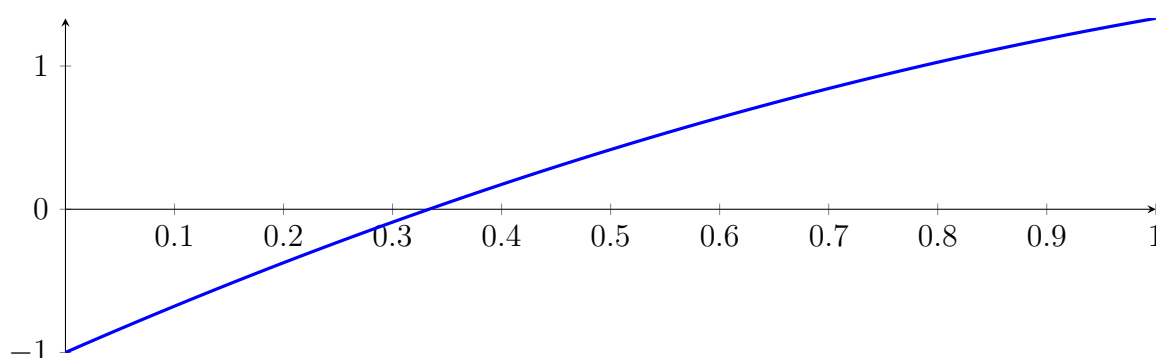
Numeric = MpfrFloat with precision 1024

169 Running BezClip on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

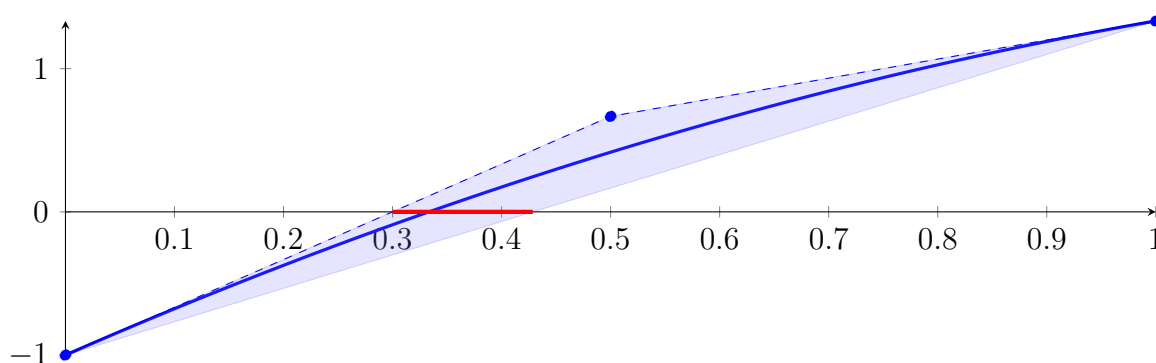
$$p = -1X^2 + 3.33333X - 1$$



169.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

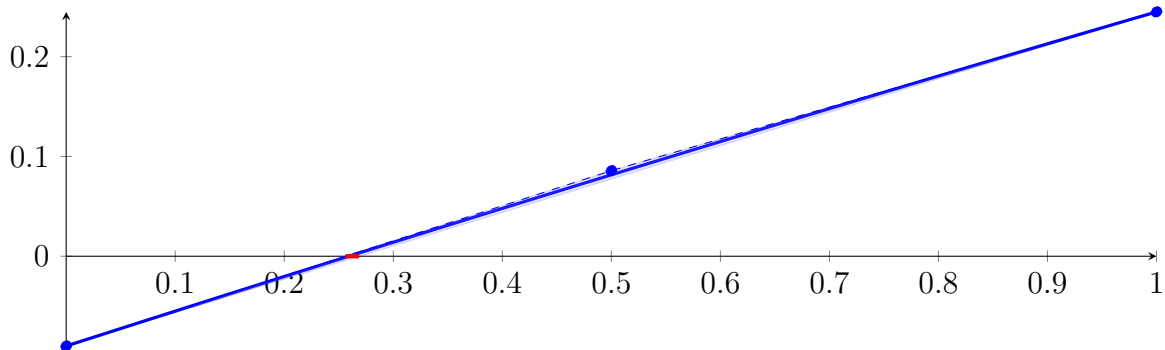
Longest intersection interval: 0.128571

\Rightarrow Selective recursion: interval 1: $[0.3, 0.428571]$,

169.2 Recursion Branch 1 1 in Interval 1: [0.3, 0.428571]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

Longest intersection interval: 0.012641

\implies Selective recursion: interval 1: [0.332927, 0.334552],

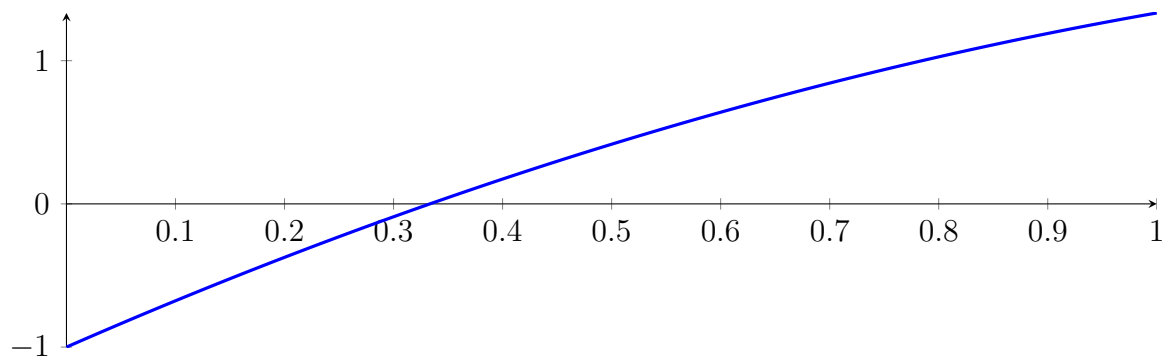
169.3 Recursion Branch 1 1 1 in Interval 1: [0.332927, 0.334552]

Found root in interval [0.332927, 0.334552] at recursion depth 3!

169.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.332927, 0.334552]$$

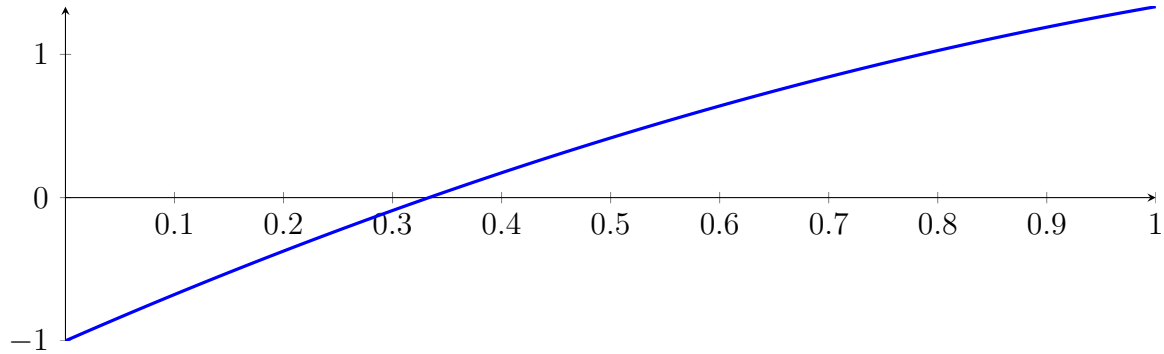
with precision $\varepsilon = 0.01$.

170 Running QuadClip on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

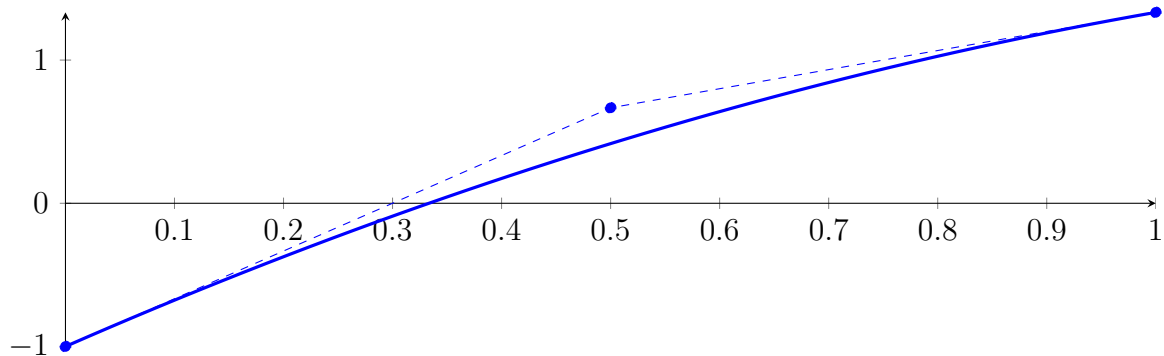
$$p = -1X^2 + 3.33333X - 1$$



170.1 Recursion Branch 1 for Input Interval $[0, 1]$

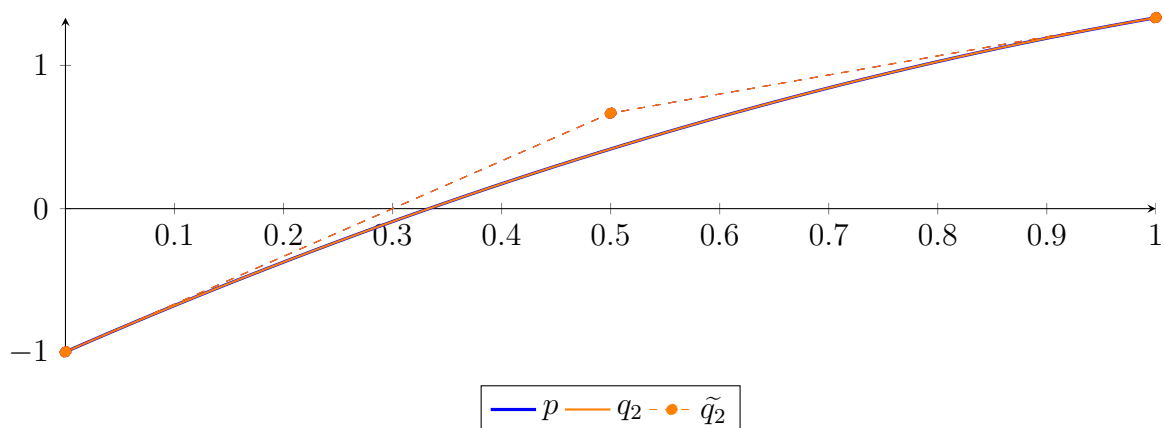
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.22507 \cdot 10^{-308}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

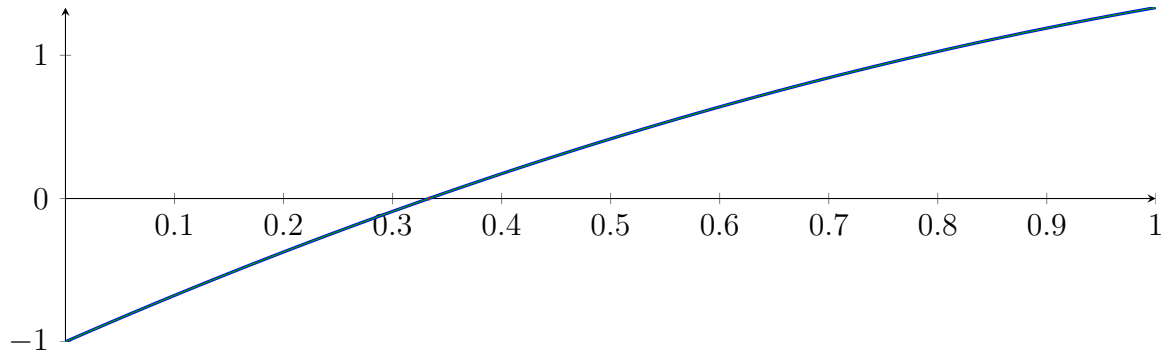
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $1.11254 \cdot 10^{-308}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

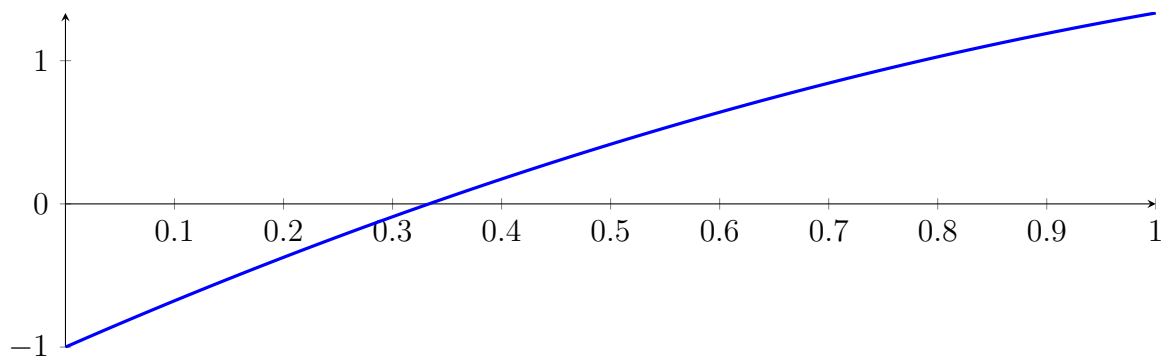
170.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

170.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

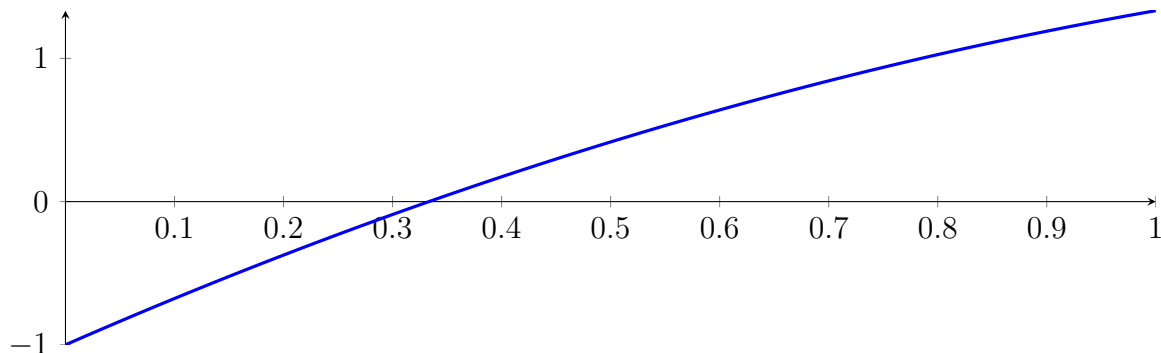
with precision $\varepsilon = 0.01$.

171 Running CubeClip on f_2 with epsilon 2

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

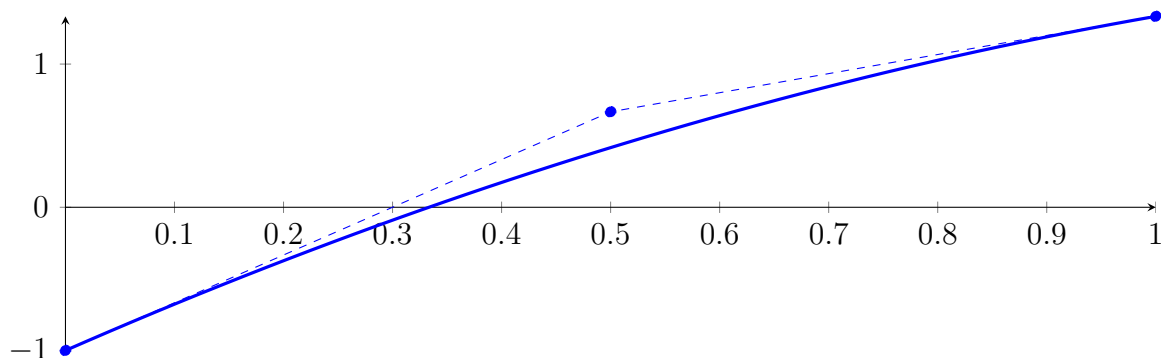
$$p = -1X^2 + 3.33333X - 1$$



171.1 Recursion Branch 1 for Input Interval $[0, 1]$

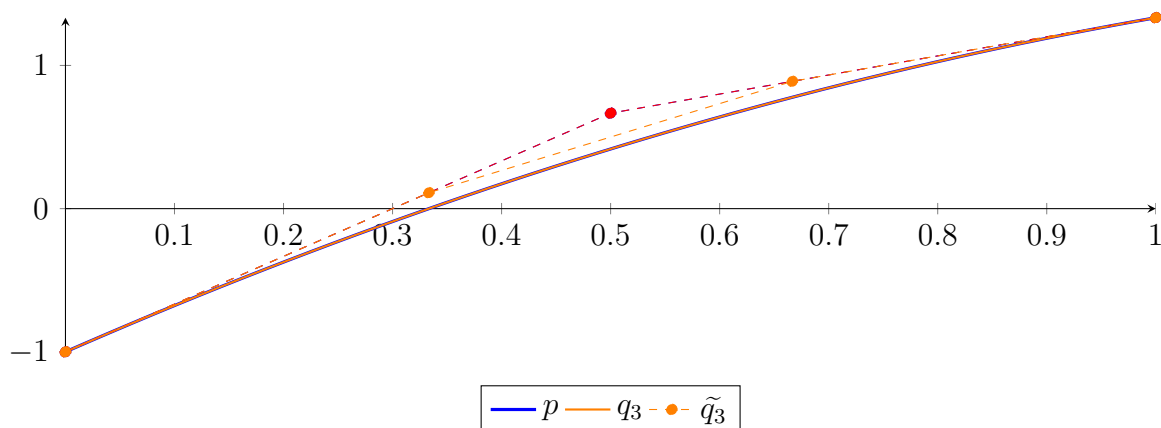
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.66881 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.00128 \cdot 10^{-307}$.

Bounding polynomials M and m :

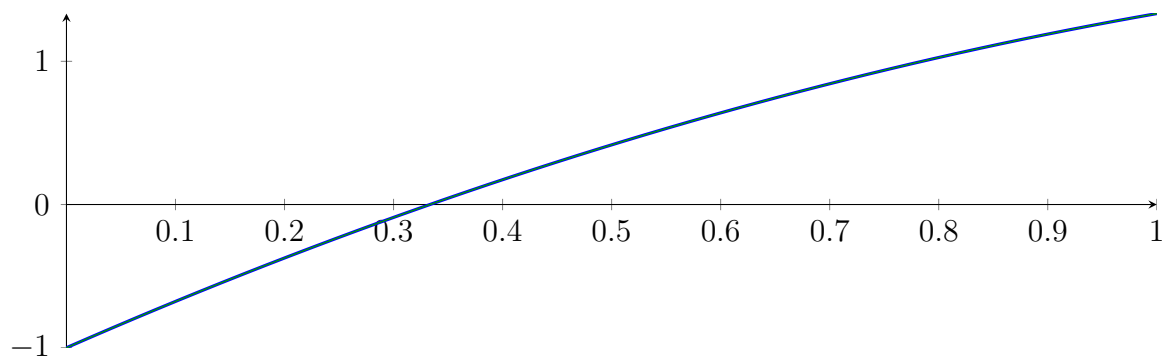
$$M = -2.78134 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1$$

$$m = -2.78134 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-3.59539 \cdot 10^{307}, -3.38949 \cdot 10^{291}, 1.03213 \cdot 10^{153}\} \quad N(m) = \{-3.59539 \cdot 10^{307}, -3.38949 \cdot 10^{291}, 1.75\}$$

Intersection intervals:

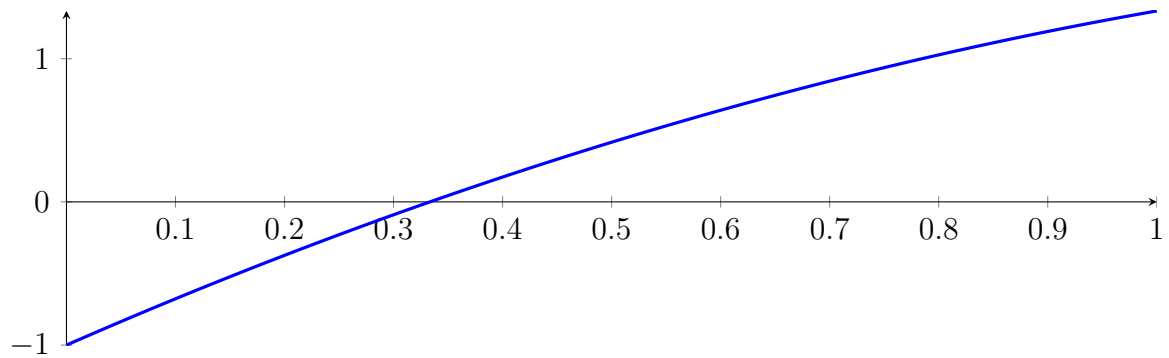


No intersection intervals with the x axis.

171.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

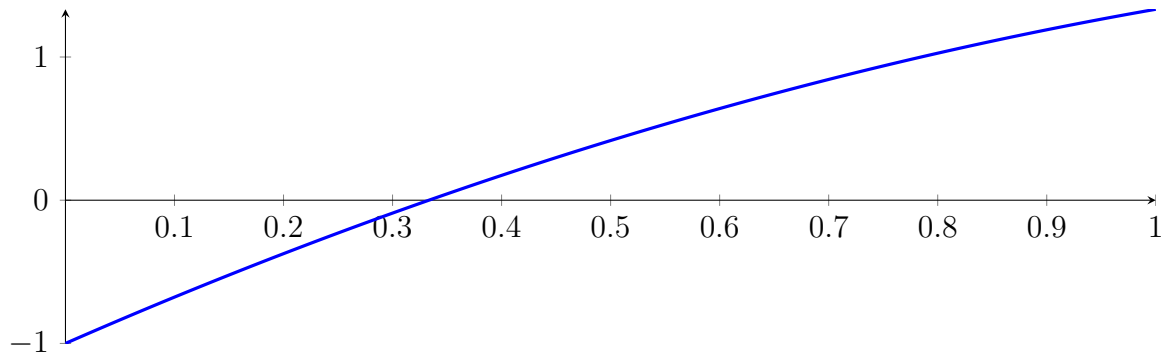
with precision $\varepsilon = 0.01$.

172 Running BezClip on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

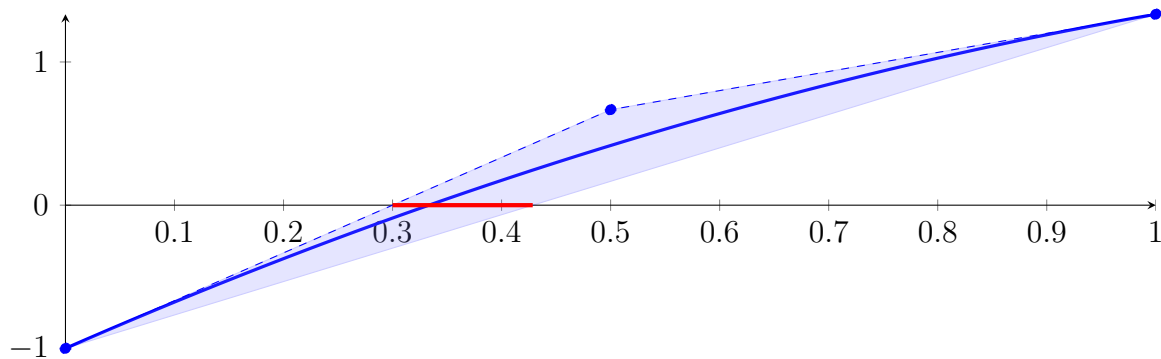
$$p = -1X^2 + 3.33333X - 1$$



172.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

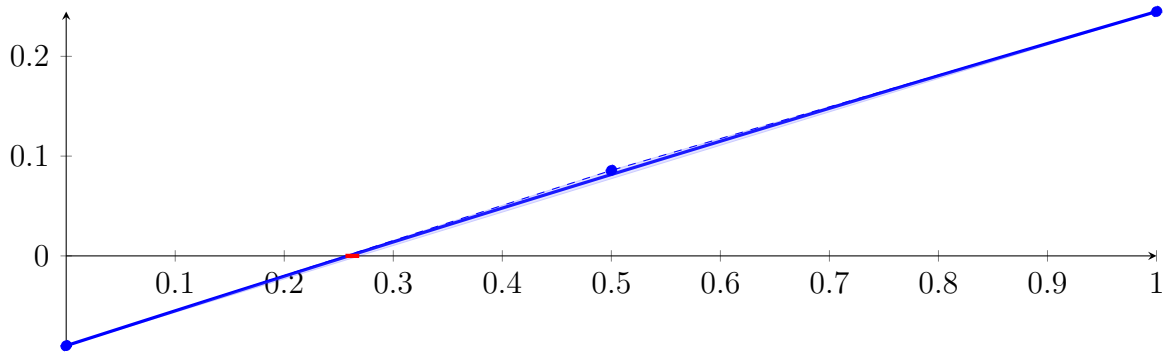
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

172.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

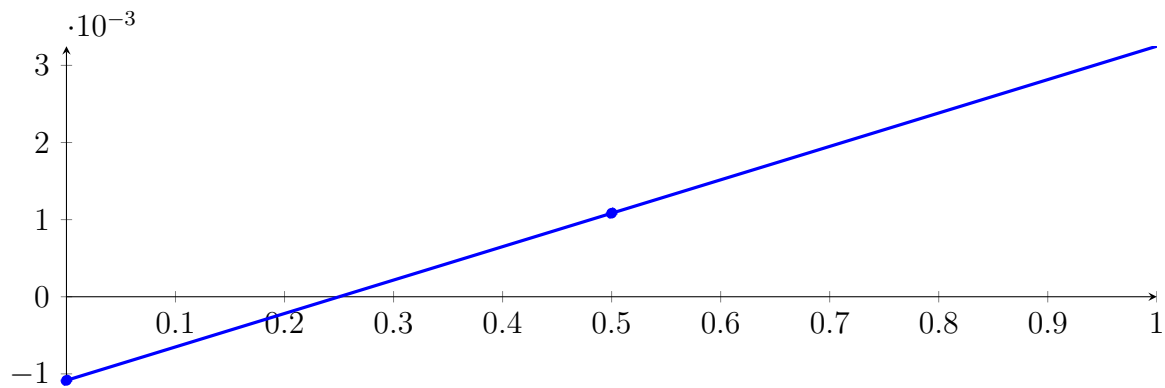
Longest intersection interval: 0.012641

⇒ Selective recursion: interval 1: $[0.332927, 0.334552]$,

172.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

Longest intersection interval: 0.000152462

⇒ Selective recursion: interval 1: $[0.333333, 0.333334]$,

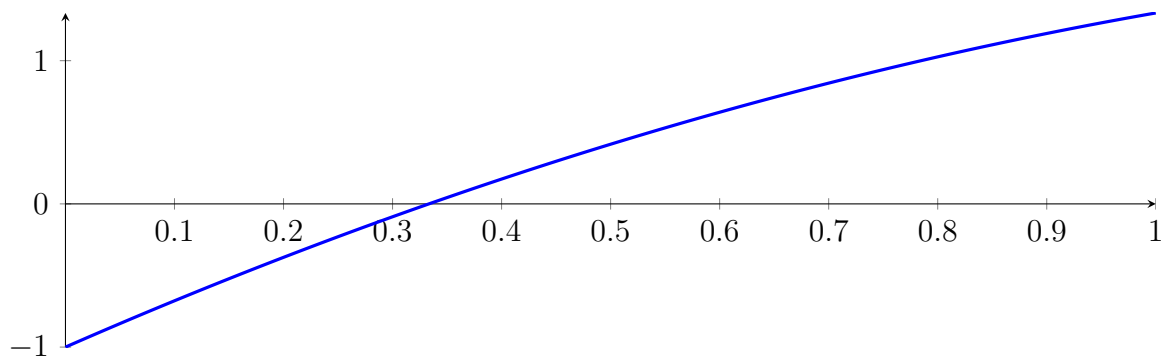
172.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333334]$

Found root in interval $[0.333333, 0.333334]$ at recursion depth 4!

172.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333334]$$

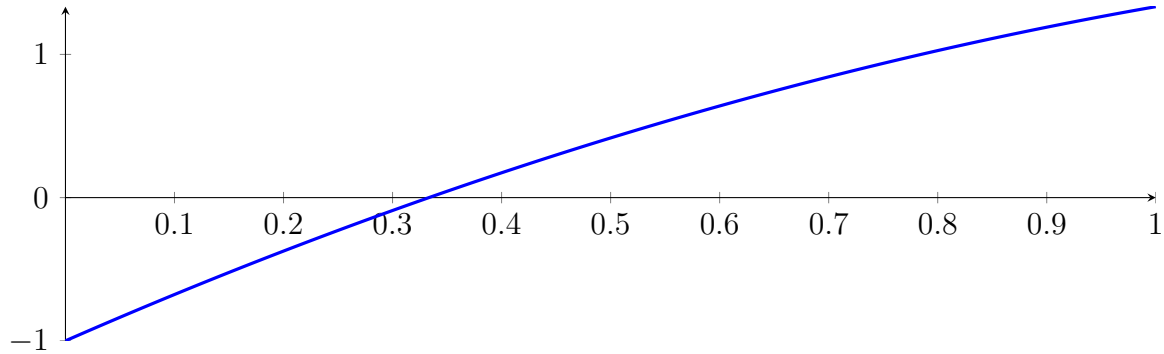
with precision $\varepsilon = 0.0001$.

173 Running QuadClip on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

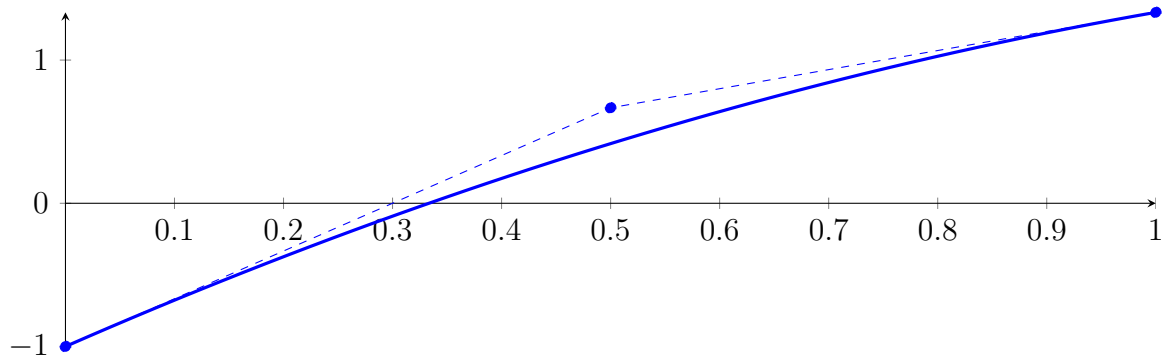
$$p = -1X^2 + 3.33333X - 1$$



173.1 Recursion Branch 1 for Input Interval $[0, 1]$

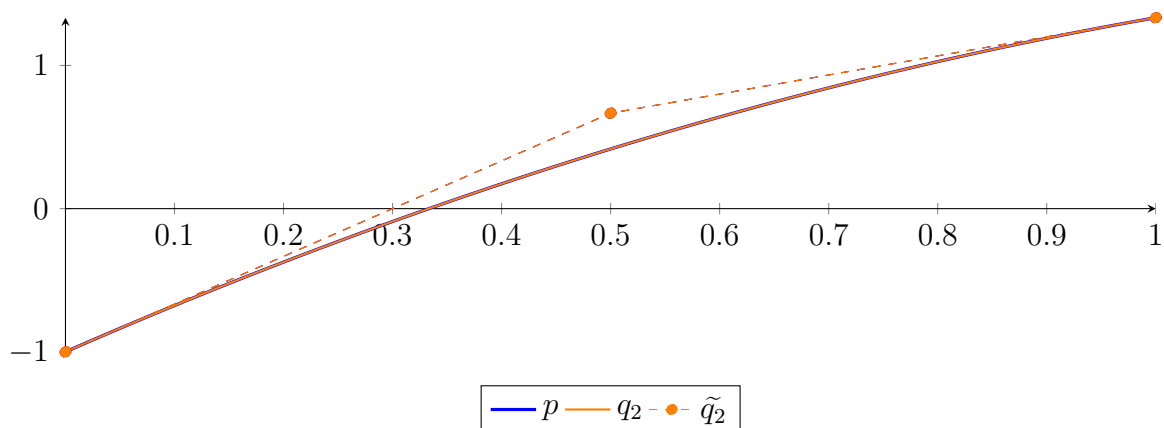
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.22507 \cdot 10^{-308}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

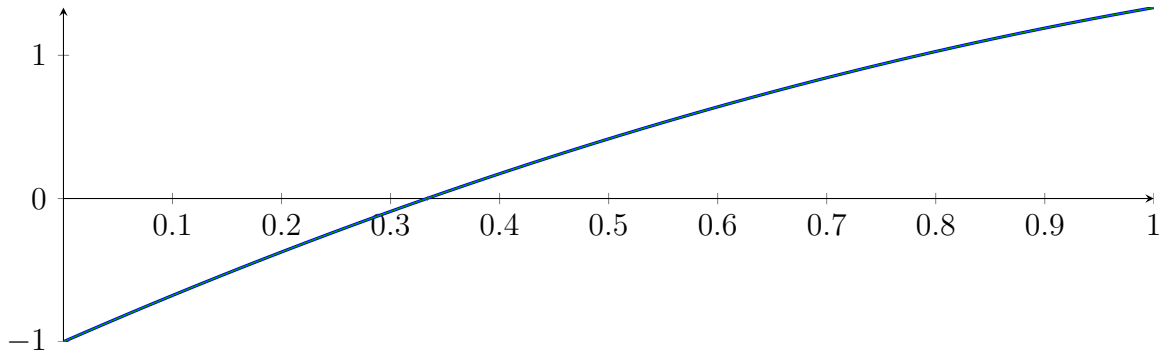
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $1.11254 \cdot 10^{-308}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

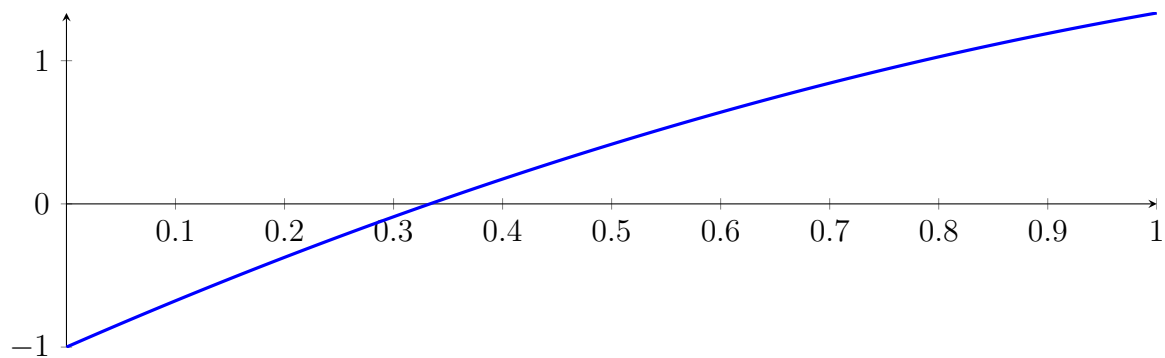
173.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

173.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

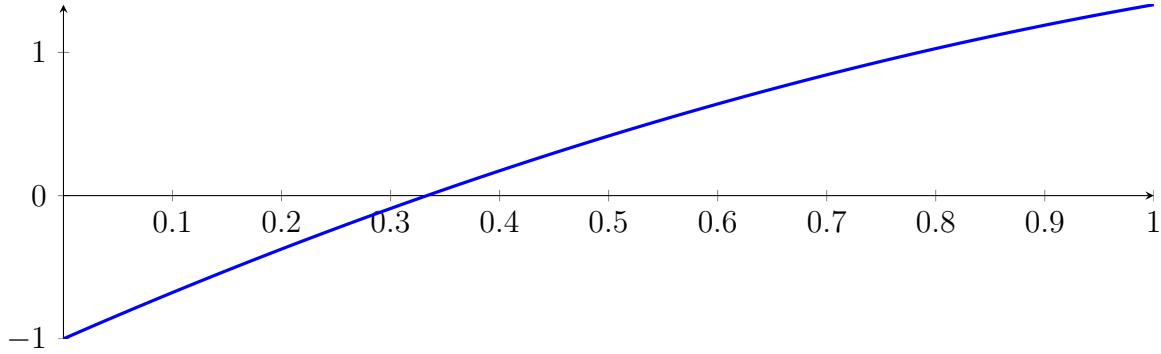
with precision $\varepsilon = 0.0001$.

174 Running CubeClip on f_2 with epsilon 4

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

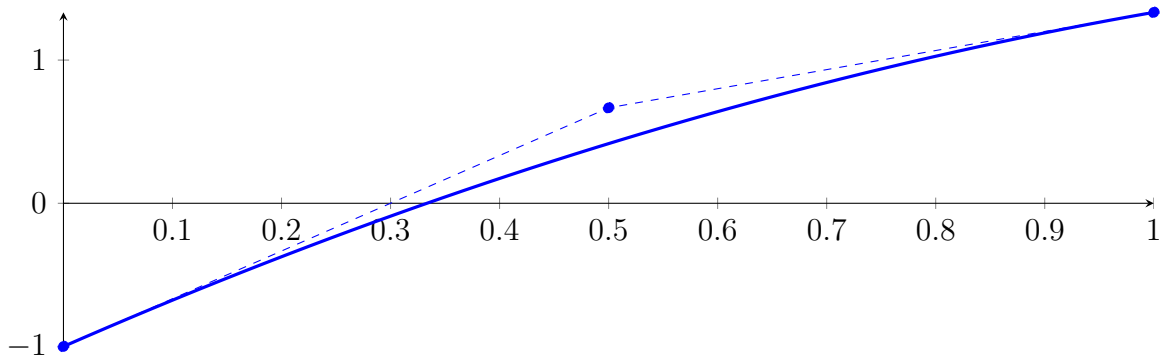
$$p = -1X^2 + 3.33333X - 1$$



174.1 Recursion Branch 1 for Input Interval $[0, 1]$

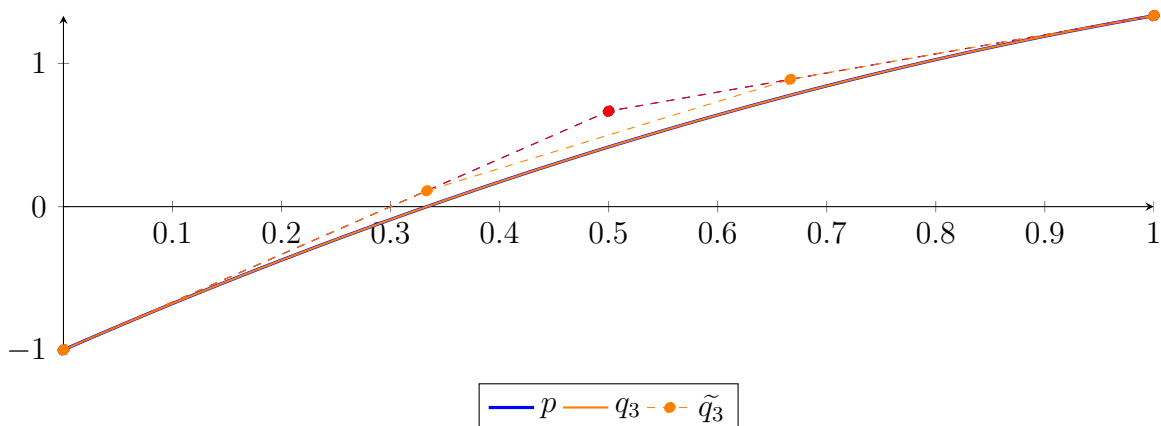
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.66881 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.00128 \cdot 10^{-307}$.

Bounding polynomials M and m :

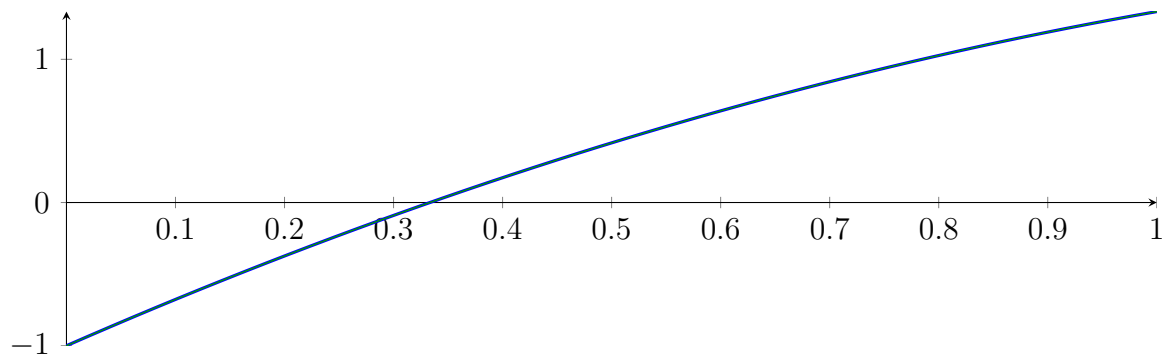
$$M = -2.78134 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1$$

$$m = -2.78134 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-3.59539 \cdot 10^{307}, -3.38949 \cdot 10^{291}, 1.03213 \cdot 10^{153}\} \quad N(m) = \{-3.59539 \cdot 10^{307}, -3.38949 \cdot 10^{291}, 1.75\}$$

Intersection intervals:

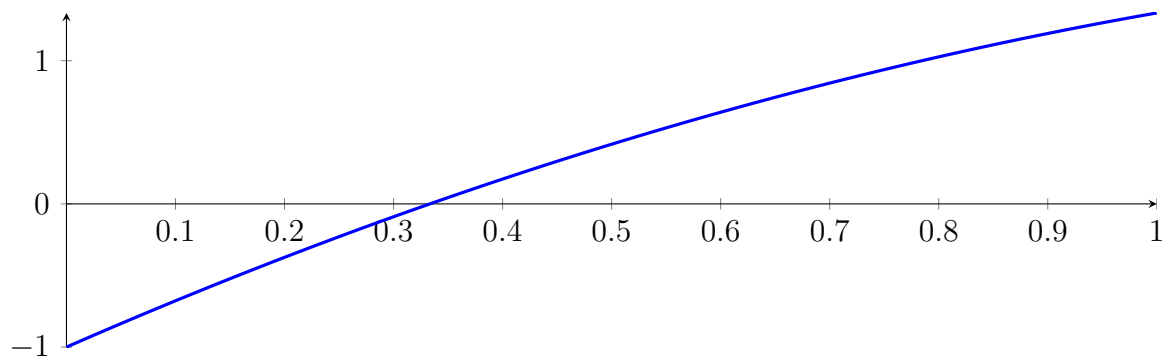


No intersection intervals with the x axis.

174.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

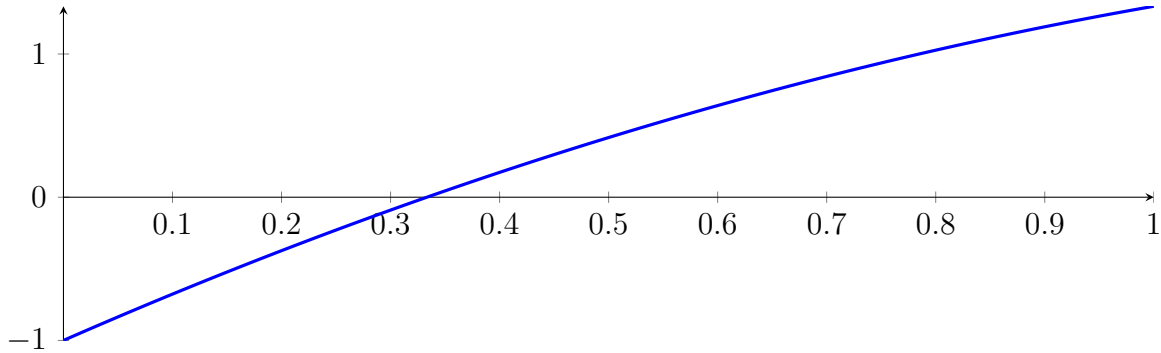
with precision $\varepsilon = 0.0001$.

175 Running BezClip on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

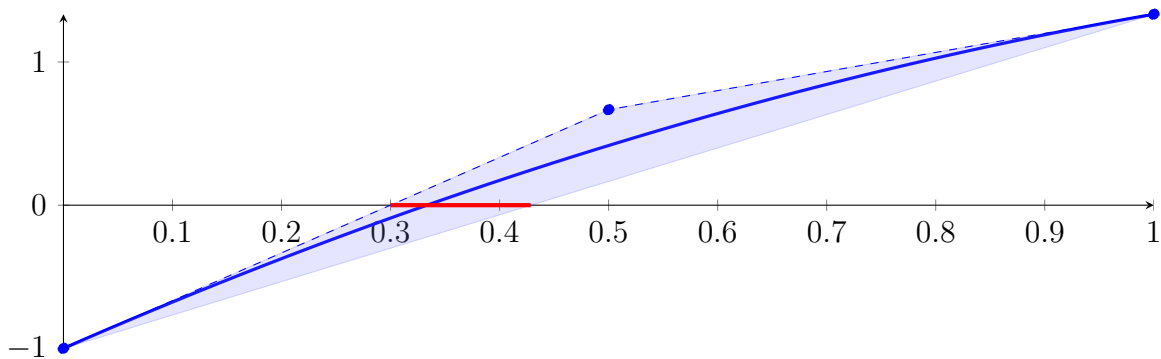
$$p = -1X^2 + 3.33333X - 1$$



175.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

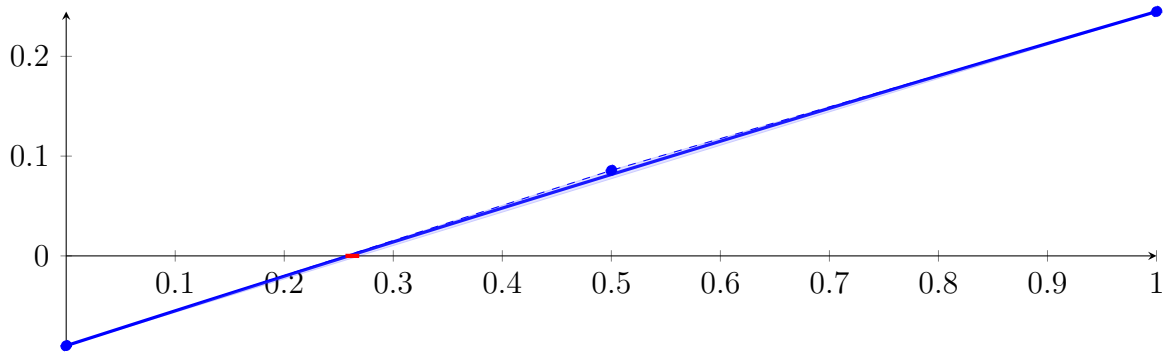
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

175.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

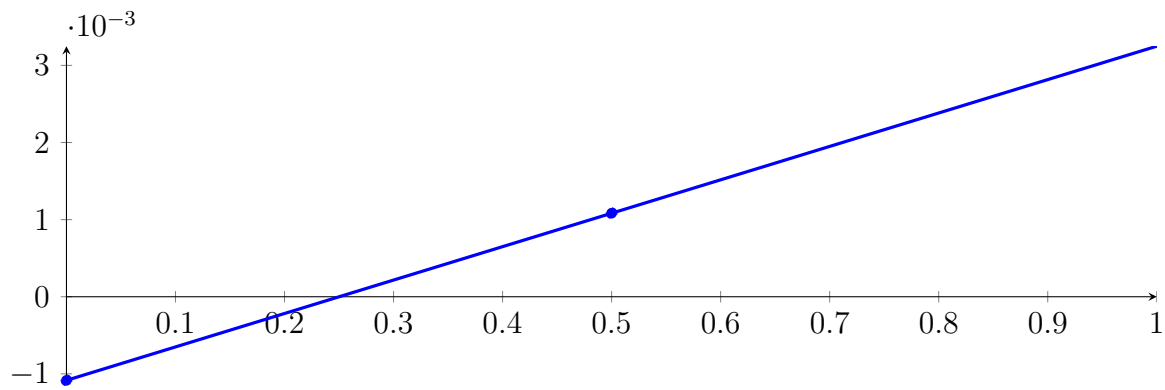
Longest intersection interval: 0.012641

\Rightarrow Selective recursion: interval 1: $[0.332927, 0.334552]$,

175.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

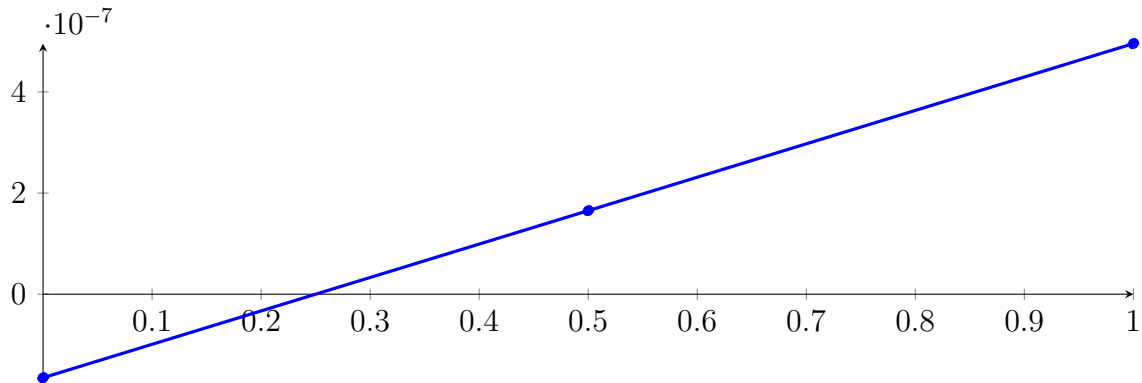
Longest intersection interval: 0.000152462

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333334]$,

175.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07} \\ &= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $2.32306 \cdot 10^{-08}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

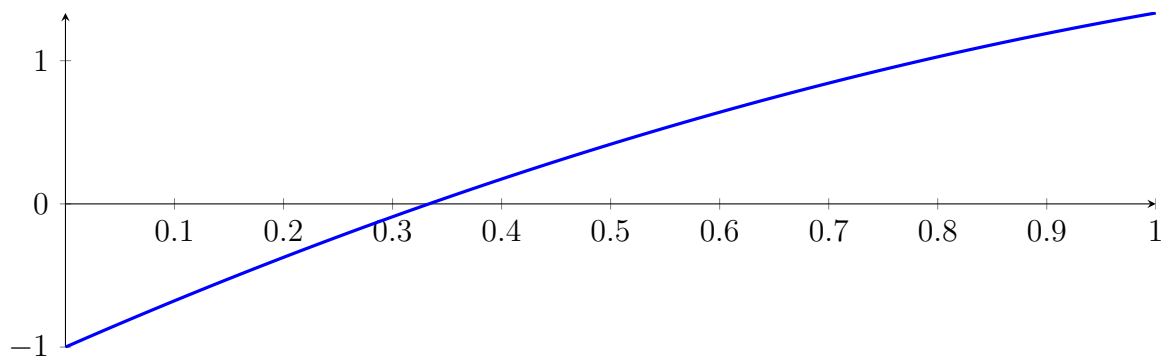
175.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

175.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

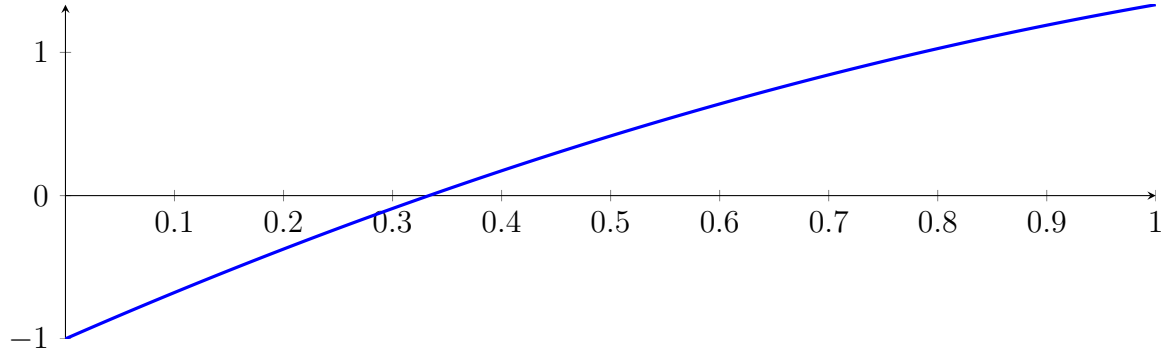
with precision $\varepsilon = 1 \cdot 10^{-08}$.

176 Running QuadClip on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

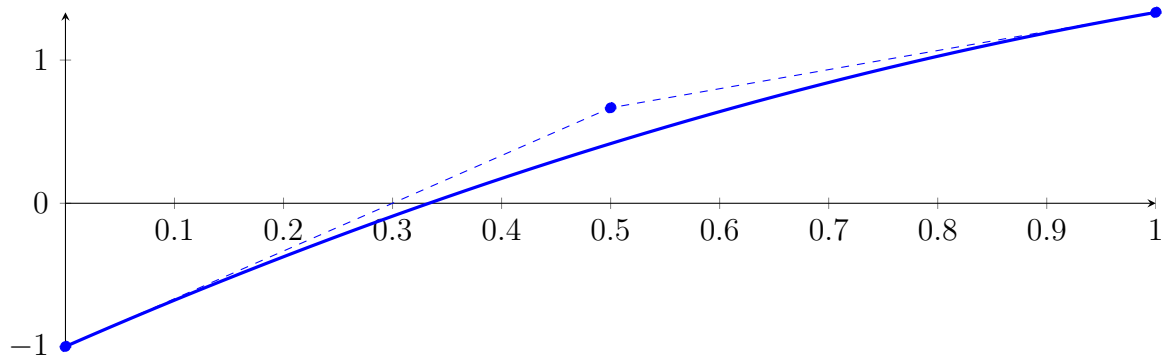
$$p = -1X^2 + 3.33333X - 1$$



176.1 Recursion Branch 1 for Input Interval $[0, 1]$

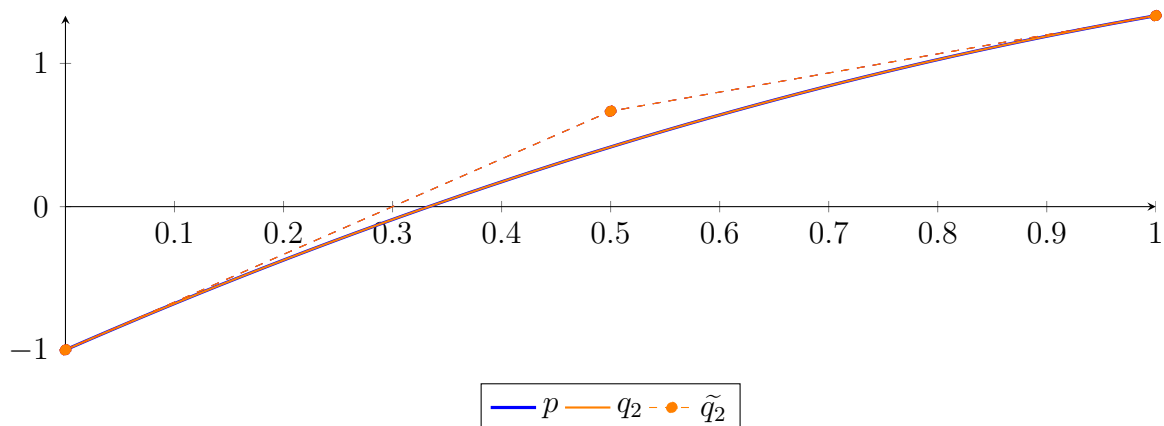
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.22507 \cdot 10^{-308}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

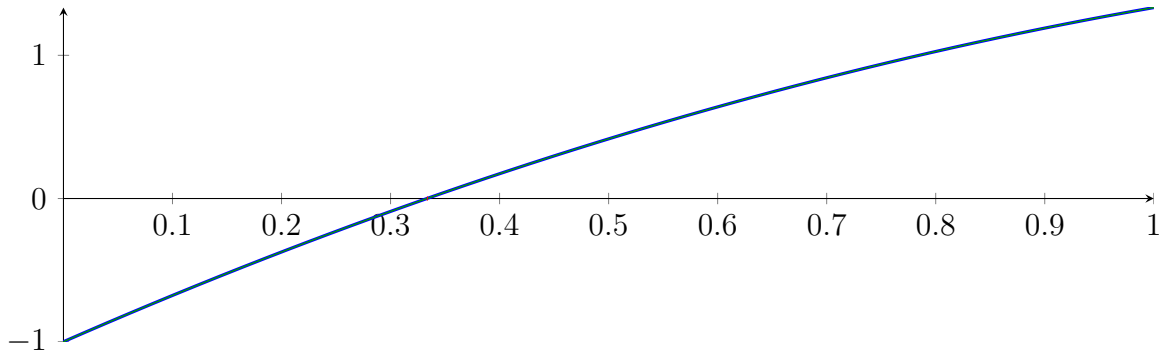
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $1.11254 \cdot 10^{-308}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

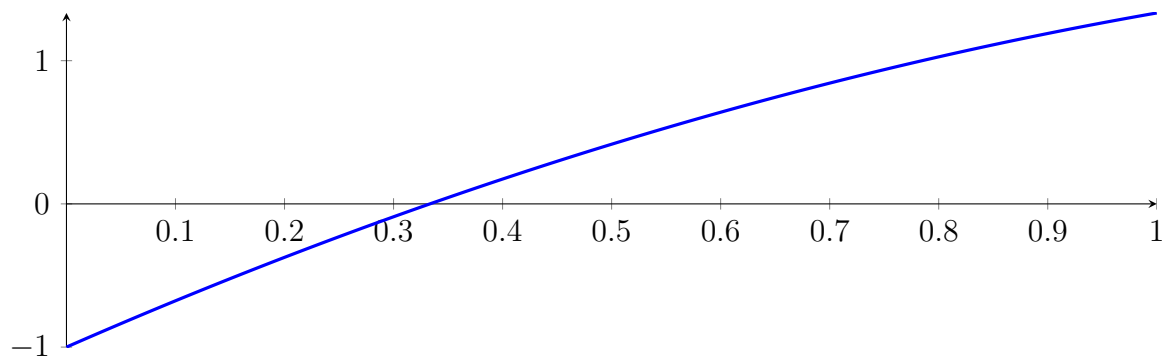
176.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

176.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

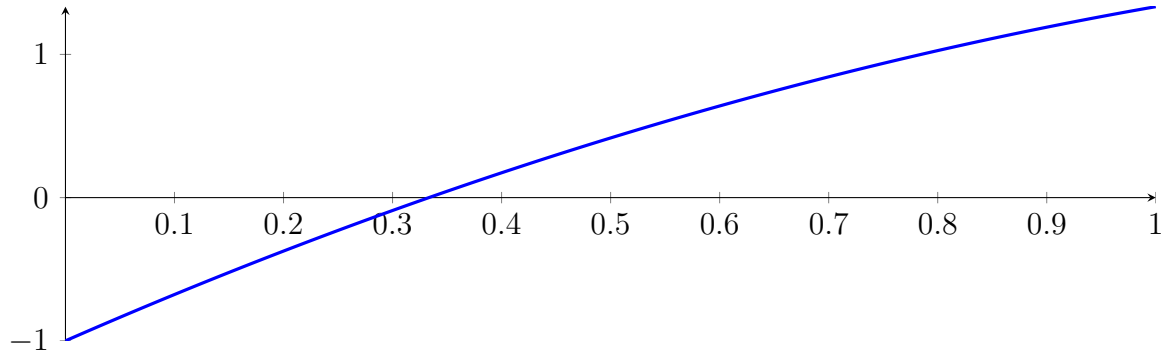
with precision $\varepsilon = 1 \cdot 10^{-08}$.

177 Running CubeClip on f_2 with epsilon 8

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

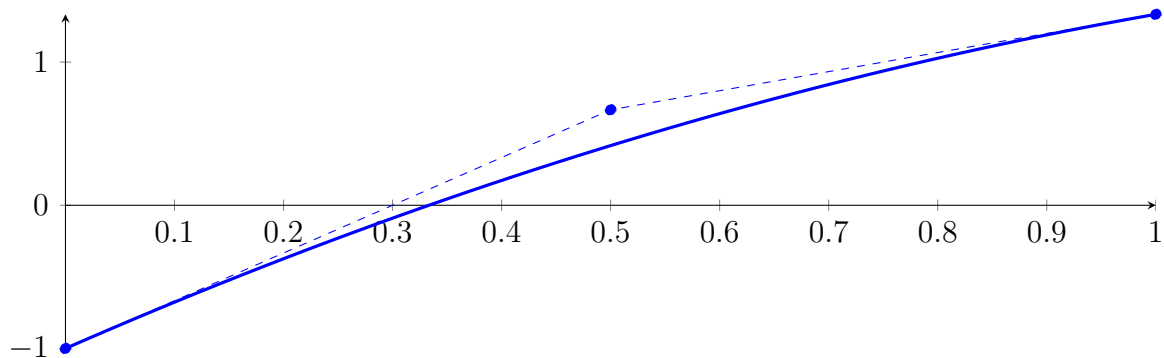
$$p = -1X^2 + 3.33333X - 1$$



177.1 Recursion Branch 1 for Input Interval $[0, 1]$

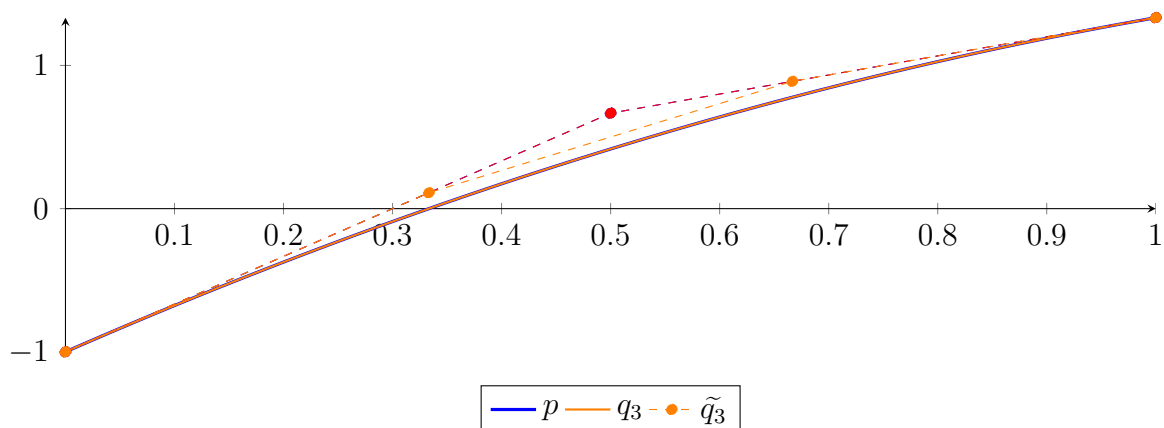
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.66881 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.00128 \cdot 10^{-307}$.

Bounding polynomials M and m :

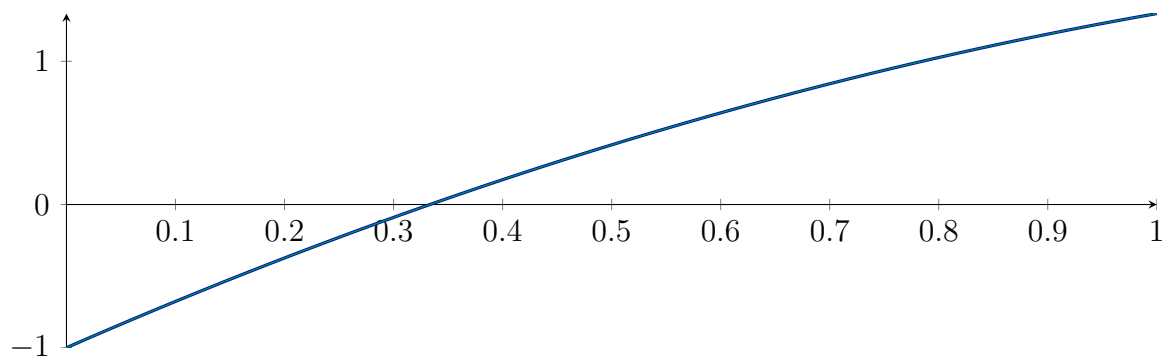
$$M = -2.78134 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1$$

$$m = -2.78134 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-3.59539 \cdot 10^{307}, -3.38949 \cdot 10^{291}, 1.03213 \cdot 10^{153}\} \quad N(m) = \{-3.59539 \cdot 10^{307}, -3.38949 \cdot 10^{291}, 1.75\}$$

Intersection intervals:

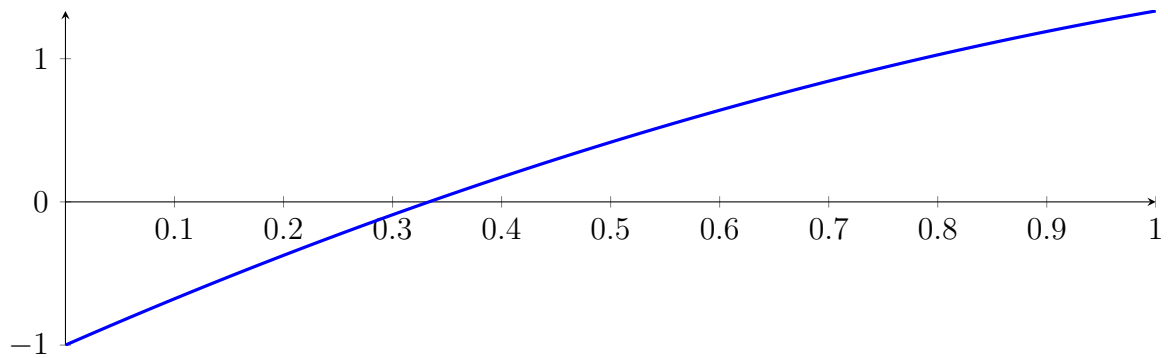


No intersection intervals with the x axis.

177.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

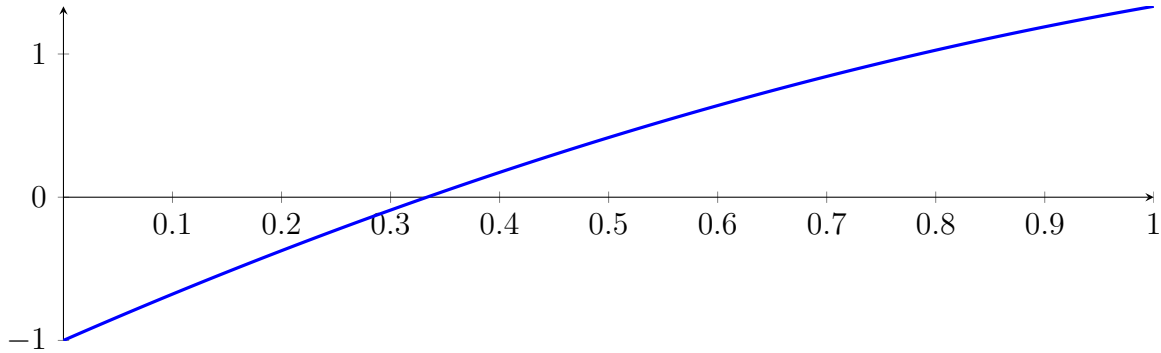
with precision $\varepsilon = 1 \cdot 10^{-08}$.

178 Running BezClip on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

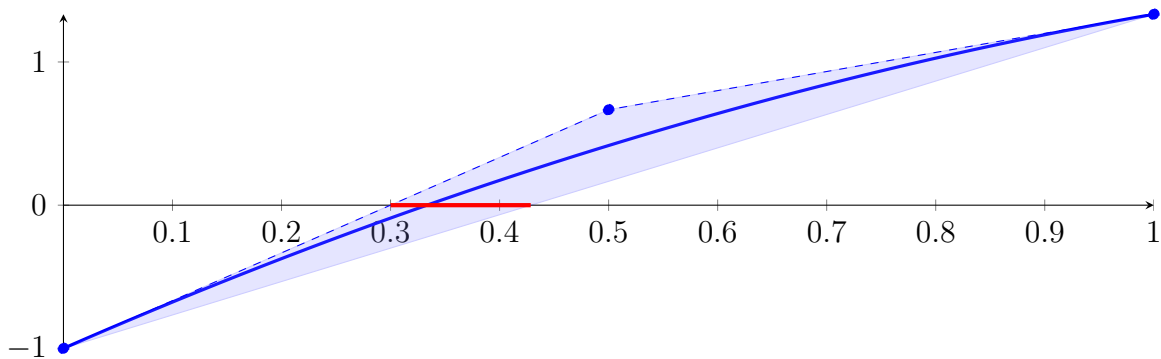
$$p = -1X^2 + 3.33333X - 1$$



178.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

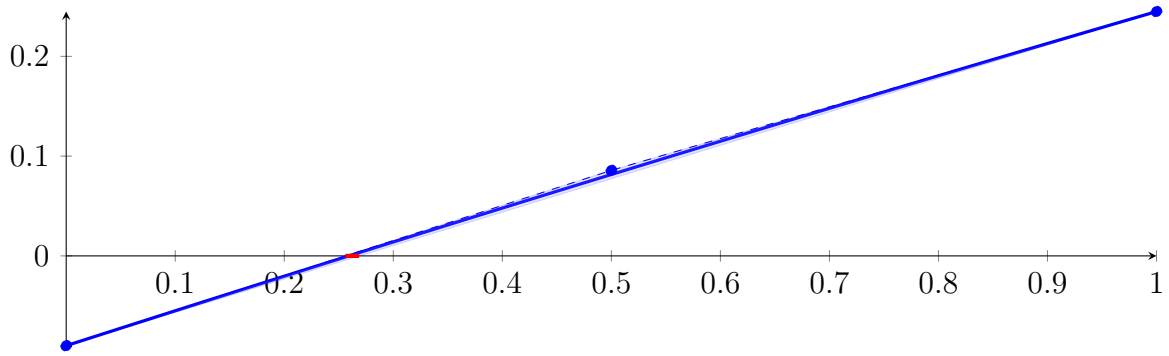
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

178.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

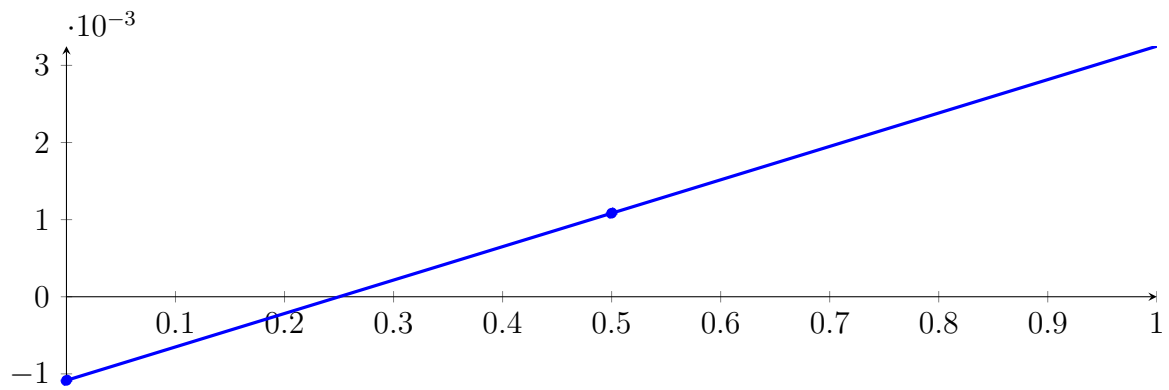
Longest intersection interval: 0.012641

\Rightarrow Selective recursion: interval 1: $[0.332927, 0.334552]$,

178.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

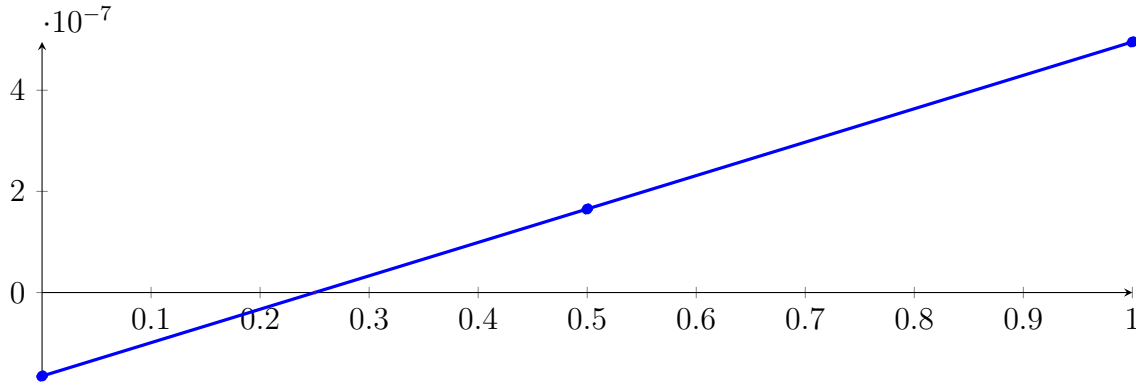
Longest intersection interval: 0.000152462

\Rightarrow Selective recursion: interval 1: $[0.333333, 0.333334]$,

178.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07} \\
 &= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:
 $\{0.25, 0.25\}$

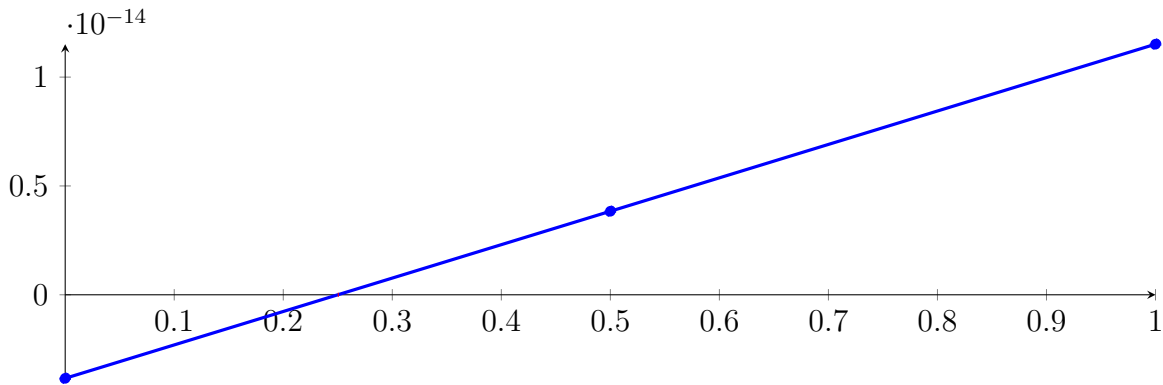
Intersection intervals with the x axis:
 $[0.25, 0.25]$

Longest intersection interval: $2.32306 \cdot 10^{-08}$
 \implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

178.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.31358 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15} \\
 &= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:
 $\{0.25, 0.25\}$

Intersection intervals with the x axis:
 $[0.25, 0.25]$

Longest intersection interval: $5.3966 \cdot 10^{-16}$
 \implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

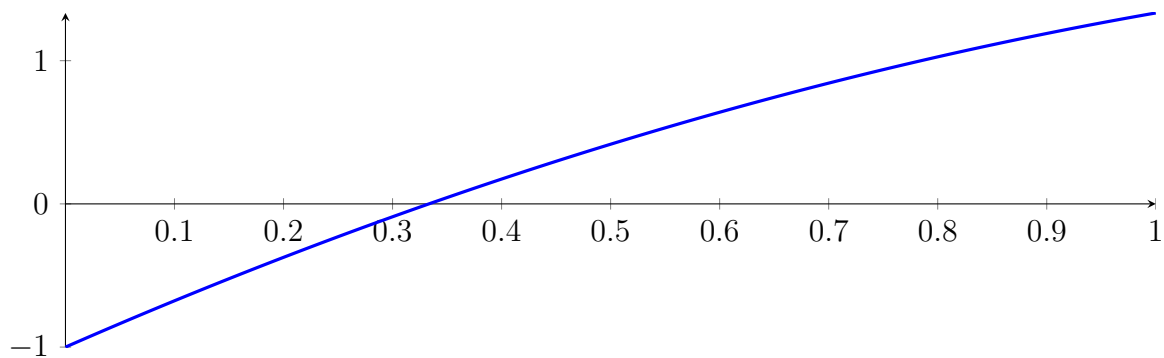
178.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

178.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

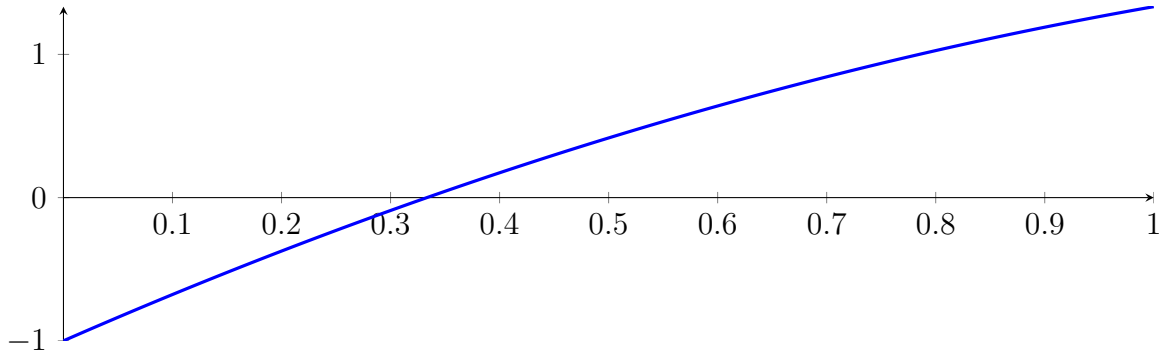
with precision $\varepsilon = 1 \cdot 10^{-16}$.

179 Running QuadClip on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

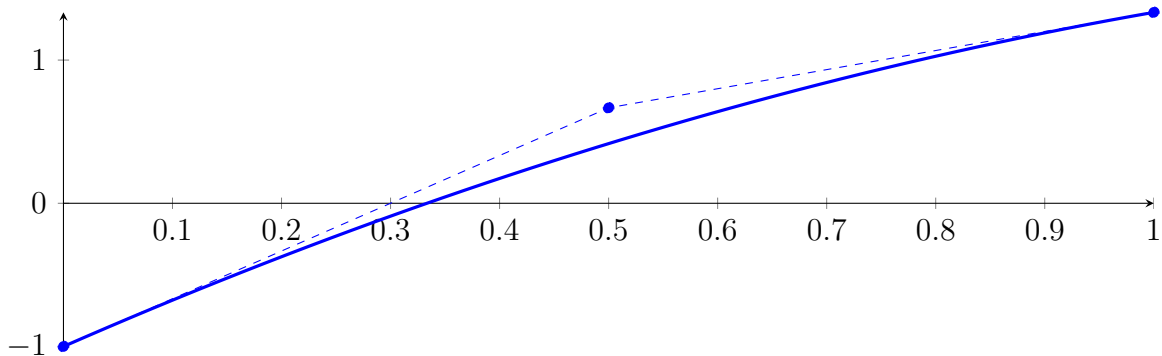
$$p = -1X^2 + 3.33333X - 1$$



179.1 Recursion Branch 1 for Input Interval $[0, 1]$

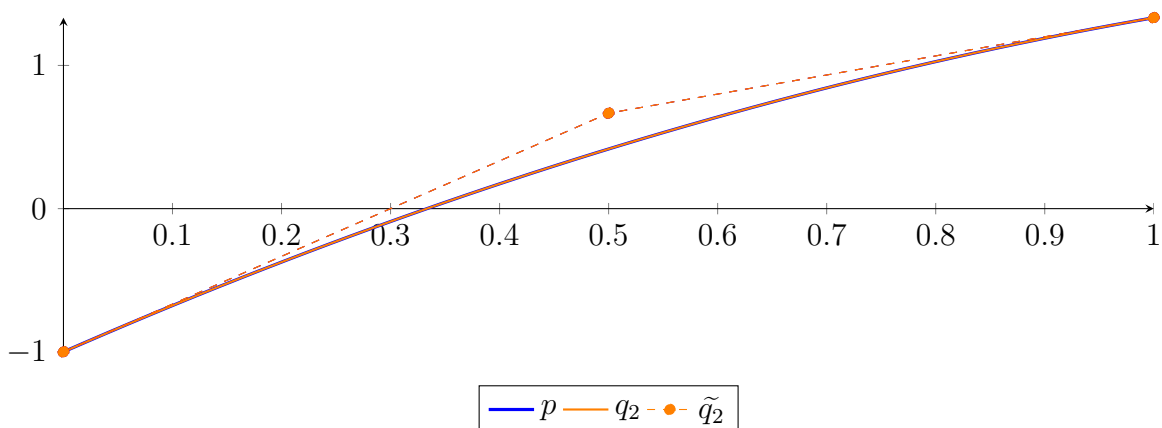
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.22507 \cdot 10^{-308}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

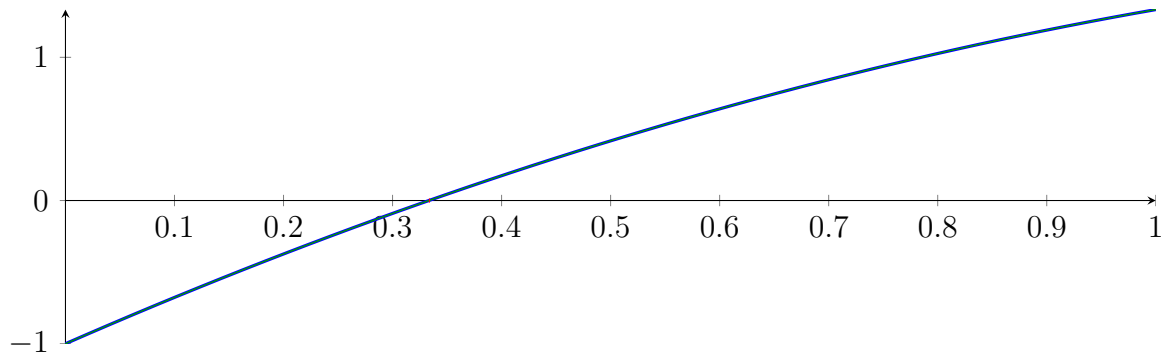
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $1.11254 \cdot 10^{-308}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

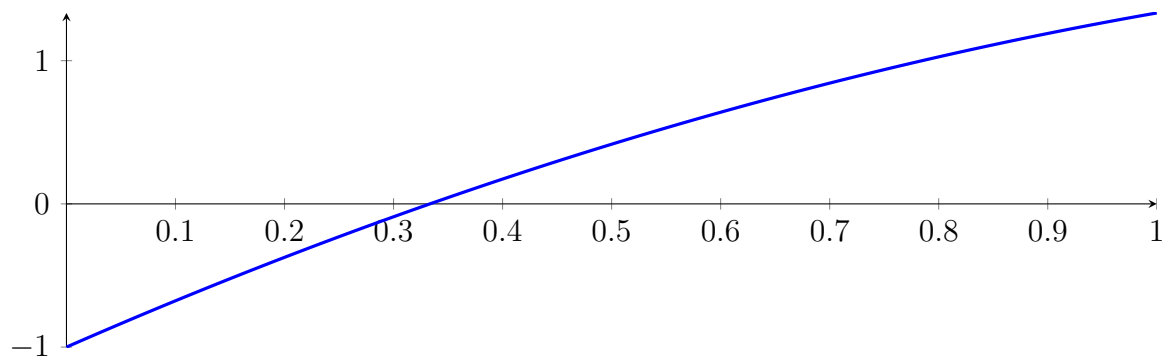
179.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

179.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

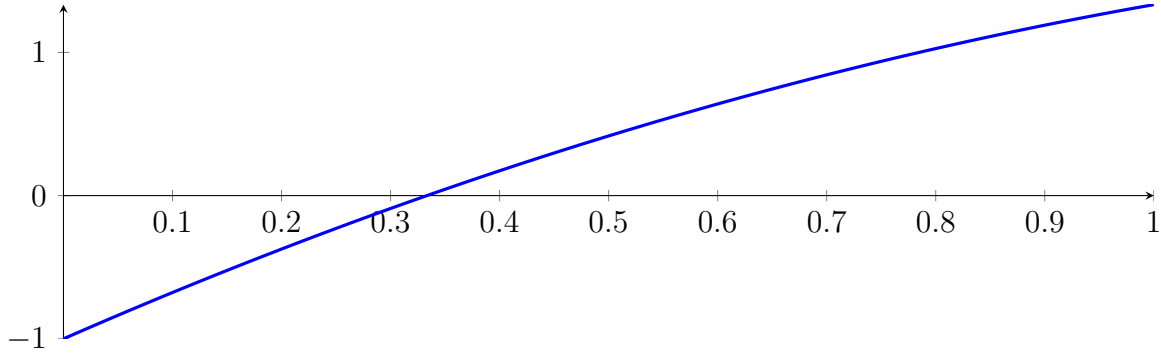
with precision $\varepsilon = 1 \cdot 10^{-16}$.

180 Running CubeClip on f_2 with epsilon 16

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

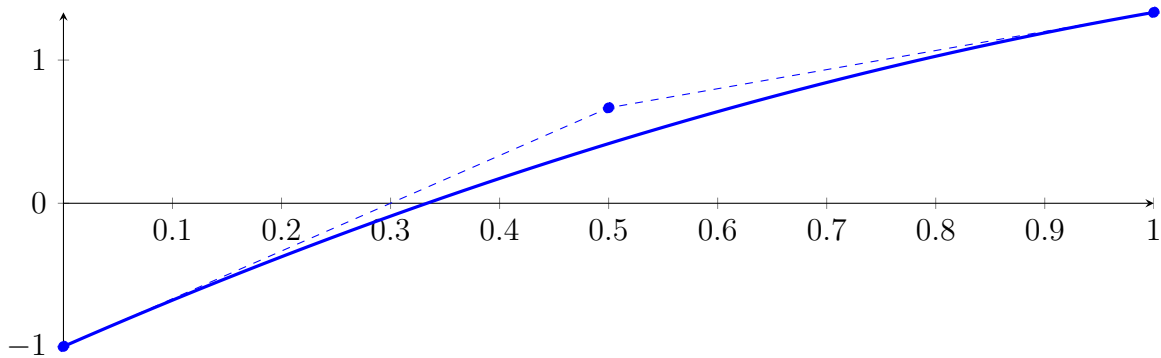
$$p = -1X^2 + 3.33333X - 1$$



180.1 Recursion Branch 1 for Input Interval $[0, 1]$

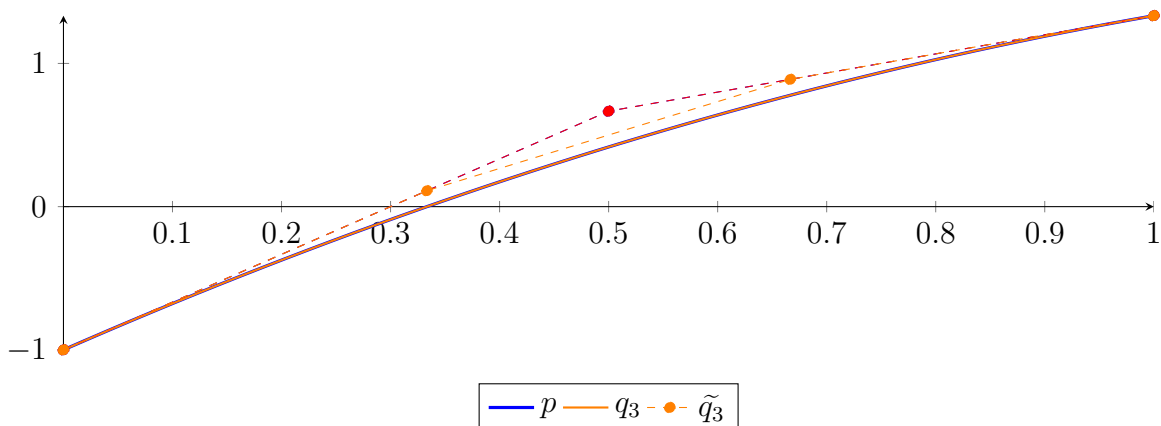
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.66881 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.00128 \cdot 10^{-307}$.

Bounding polynomials M and m :

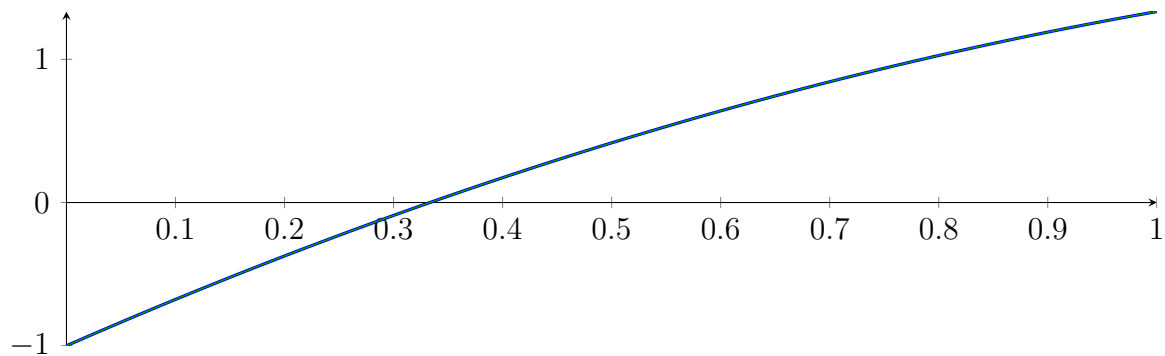
$$M = -2.78134 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1$$

$$m = -2.78134 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-3.59539 \cdot 10^{307}, -3.38949 \cdot 10^{291}, 1.03213 \cdot 10^{153}\} \quad N(m) = \{-3.59539 \cdot 10^{307}, -3.38949 \cdot 10^{291}, 1.75\}$$

Intersection intervals:

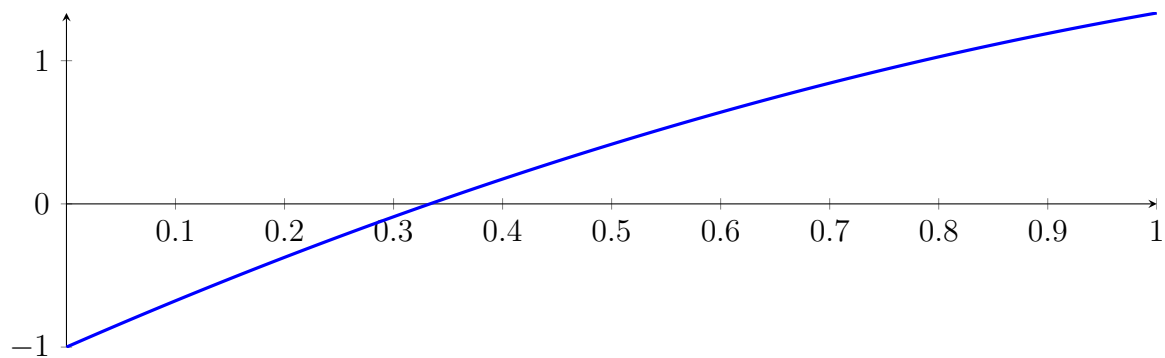


No intersection intervals with the x axis.

180.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

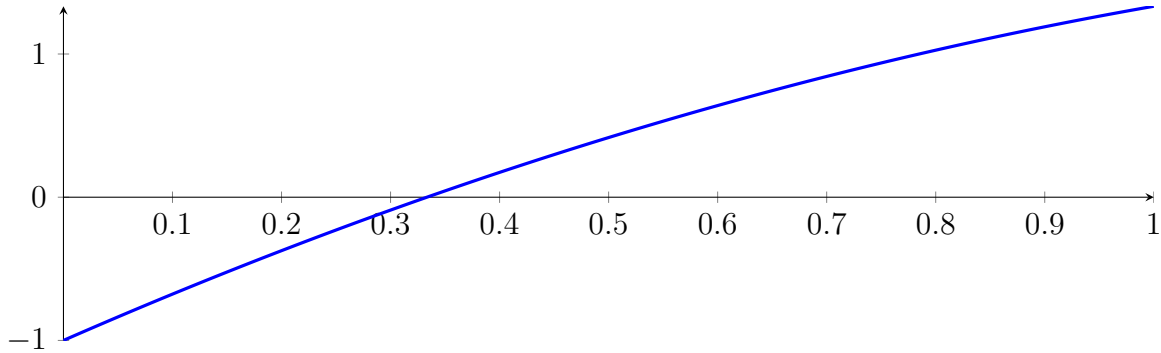
with precision $\varepsilon = 1 \cdot 10^{-16}$.

181 Running BezClip on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

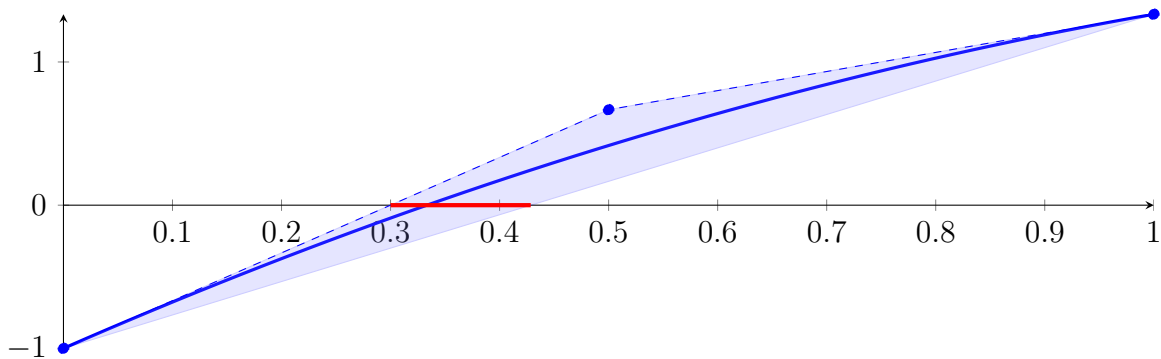
$$p = -1X^2 + 3.33333X - 1$$



181.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

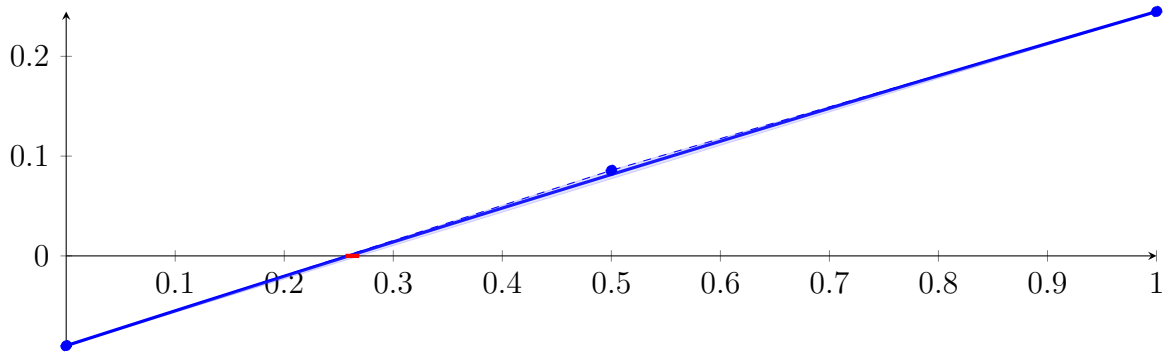
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

181.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

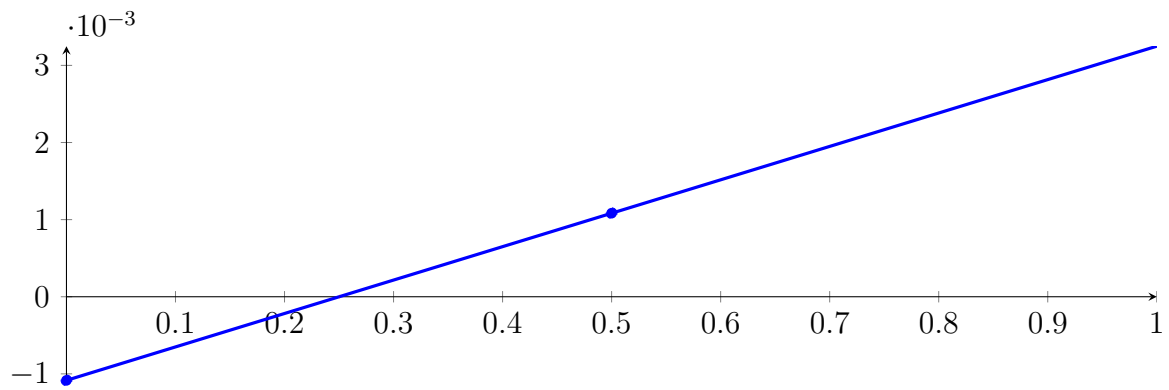
Longest intersection interval: 0.012641

⇒ Selective recursion: interval 1: $[0.332927, 0.334552]$,

181.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

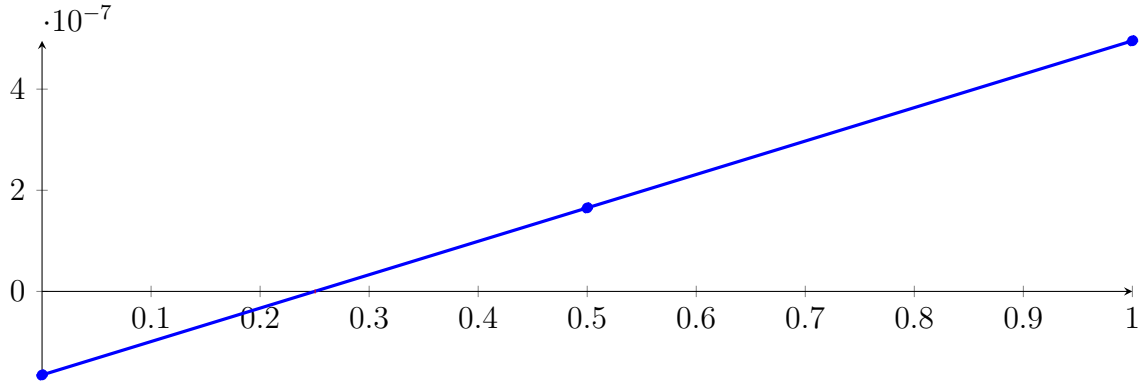
Longest intersection interval: 0.000152462

⇒ Selective recursion: interval 1: $[0.333333, 0.333334]$,

181.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07} \\
 &= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

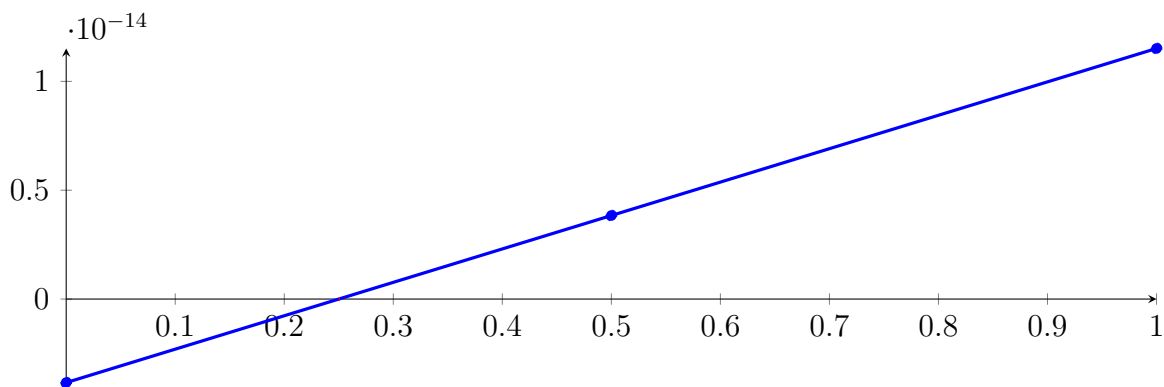
Longest intersection interval: $2.32306 \cdot 10^{-08}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

181.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.31358 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15} \\
 &= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

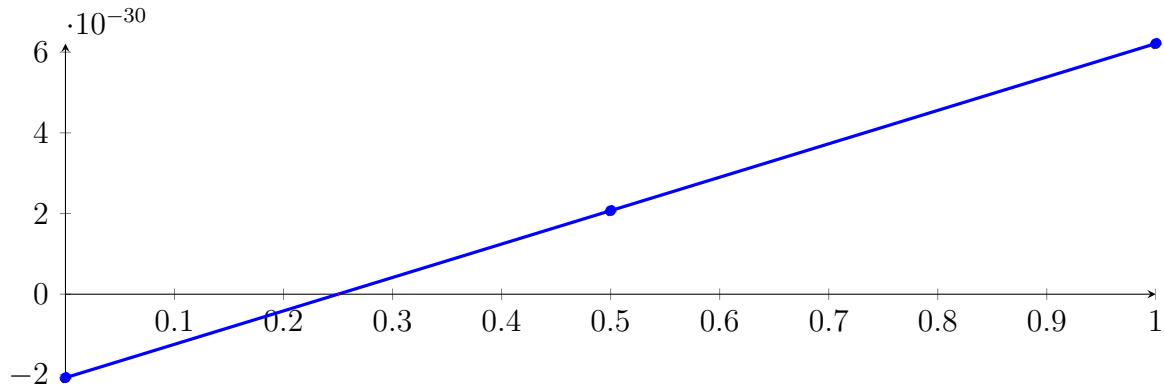
Longest intersection interval: $5.3966 \cdot 10^{-16}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

181.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.65021 \cdot 10^{-60} X^2 + 8.28394 \cdot 10^{-30} X - 2.07099 \cdot 10^{-30} \\ &= -2.07099 \cdot 10^{-30} B_{0,2}(X) + 2.07099 \cdot 10^{-30} B_{1,2}(X) + 6.21296 \cdot 10^{-30} B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $2.91232 \cdot 10^{-31}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

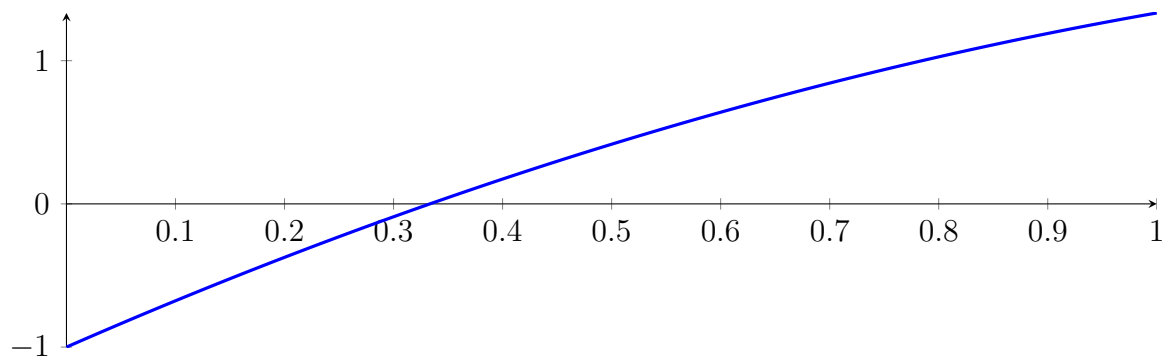
181.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 7!

181.8 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

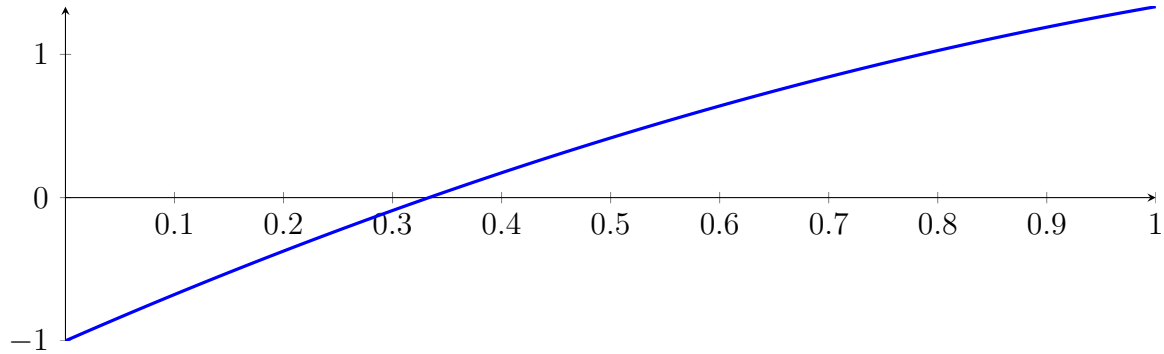
with precision $\varepsilon = 1 \cdot 10^{-32}$.

182 Running QuadClip on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

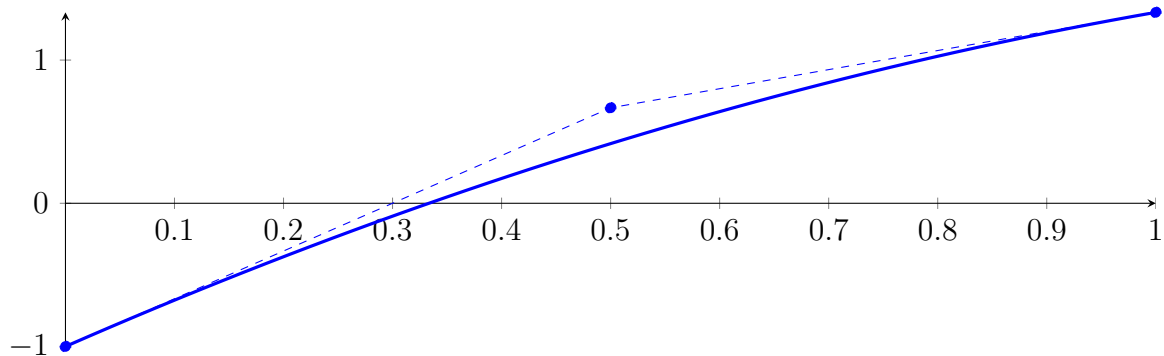
$$p = -1X^2 + 3.33333X - 1$$



182.1 Recursion Branch 1 for Input Interval $[0, 1]$

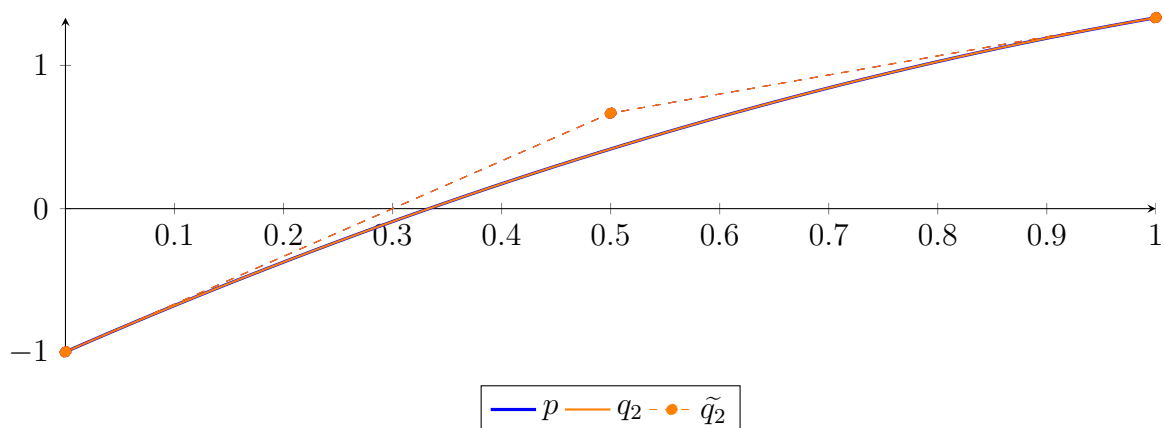
Normalized monomial and Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.22507 \cdot 10^{-308}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

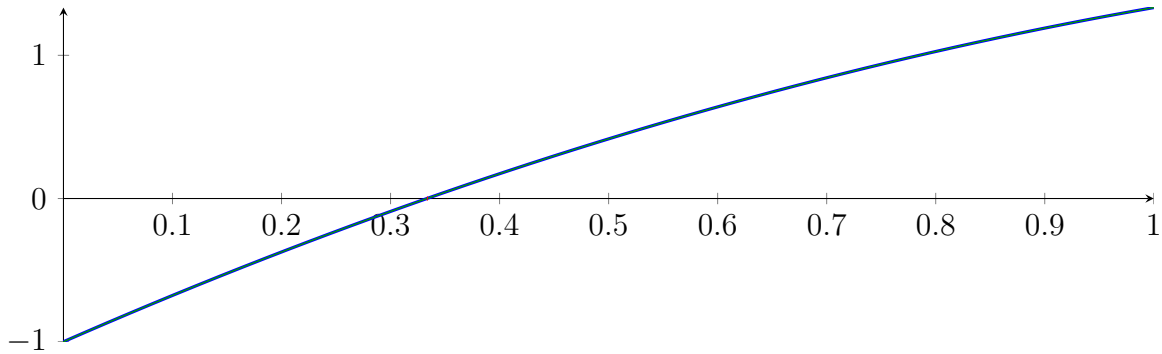
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $1.11254 \cdot 10^{-308}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

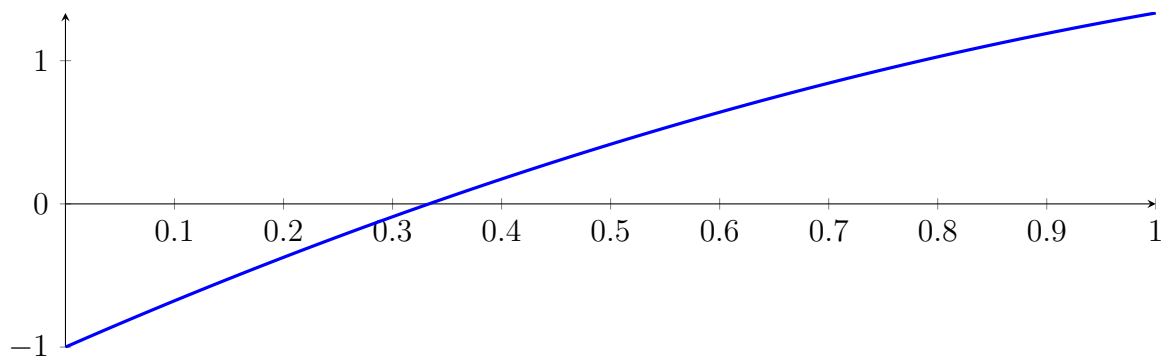
182.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

182.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

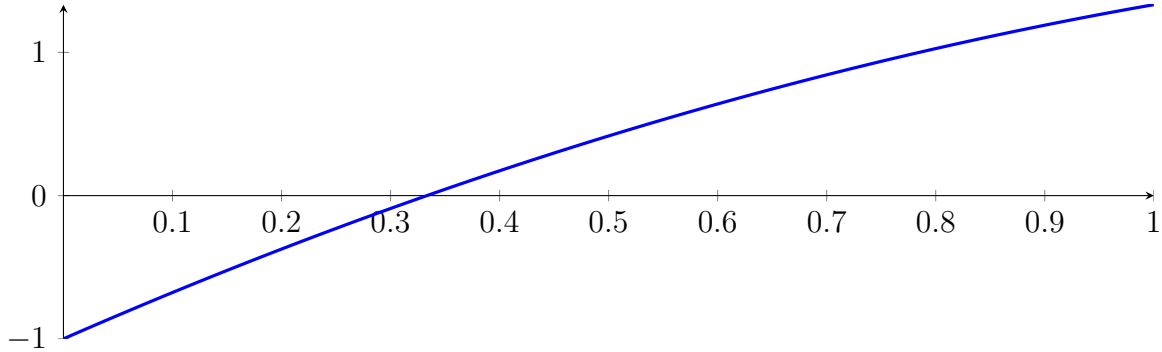
with precision $\varepsilon = 1 \cdot 10^{-32}$.

183 Running CubeClip on f_2 with epsilon 32

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

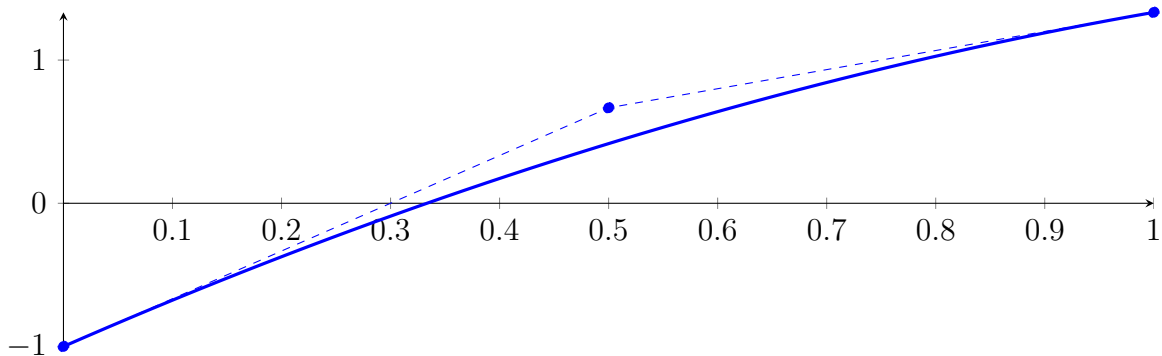
$$p = -1X^2 + 3.33333X - 1$$



183.1 Recursion Branch 1 for Input Interval $[0, 1]$

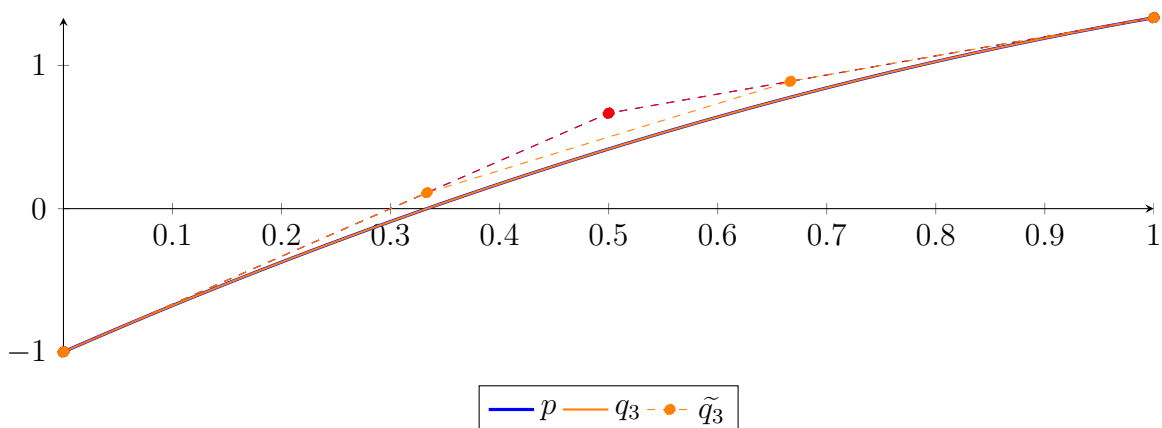
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.66881 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.00128 \cdot 10^{-307}$.

Bounding polynomials M and m :

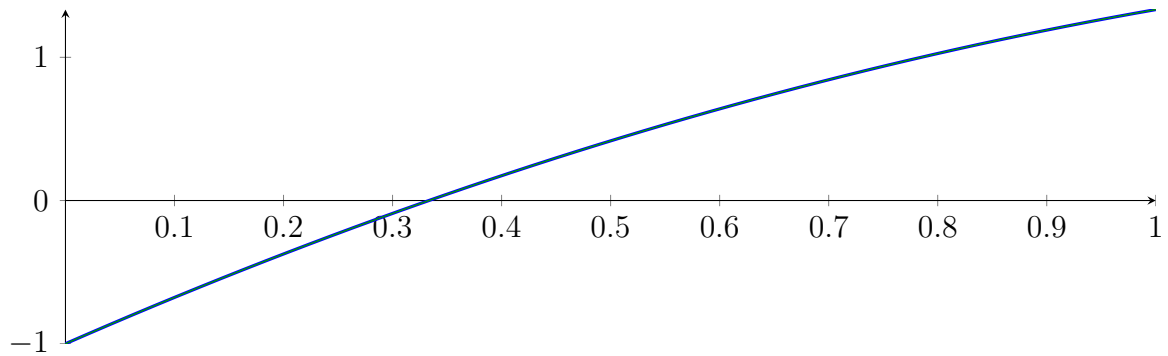
$$M = -2.78134 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1$$

$$m = -2.78134 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-3.59539 \cdot 10^{307}, -3.38949 \cdot 10^{291}, 1.03213 \cdot 10^{153}\} \quad N(m) = \{-3.59539 \cdot 10^{307}, -3.38949 \cdot 10^{291}, 1.75\}$$

Intersection intervals:

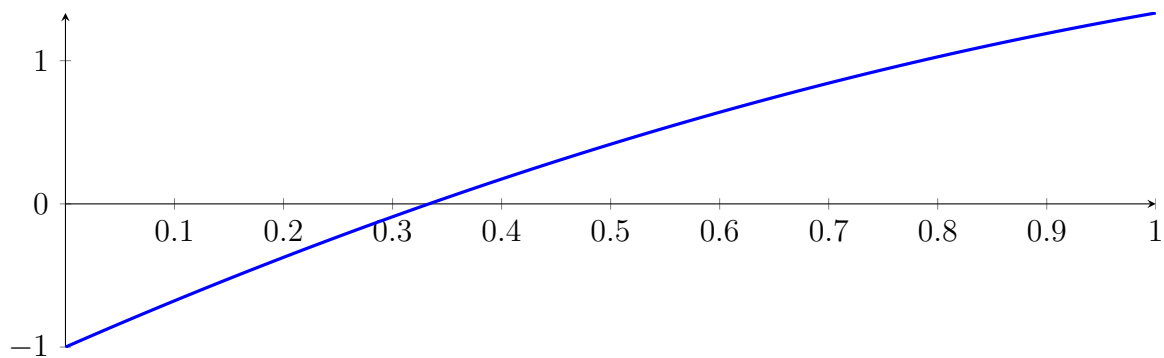


No intersection intervals with the x axis.

183.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

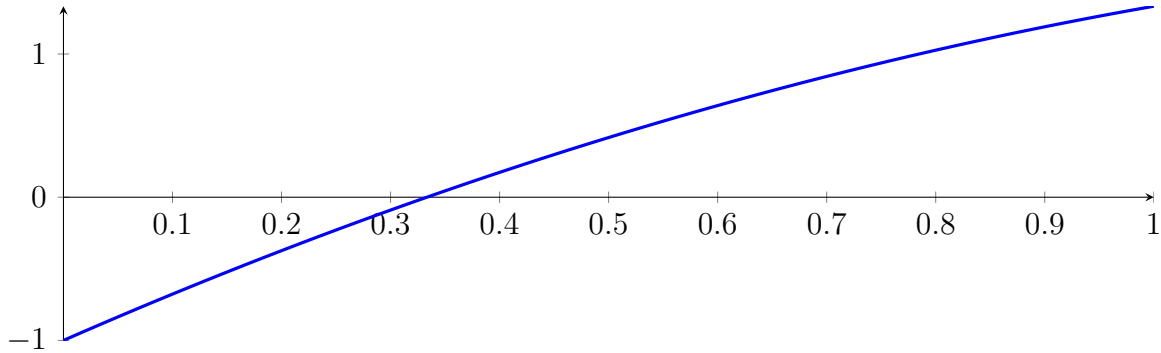
with precision $\varepsilon = 1 \cdot 10^{-32}$.

184 Running BezClip on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

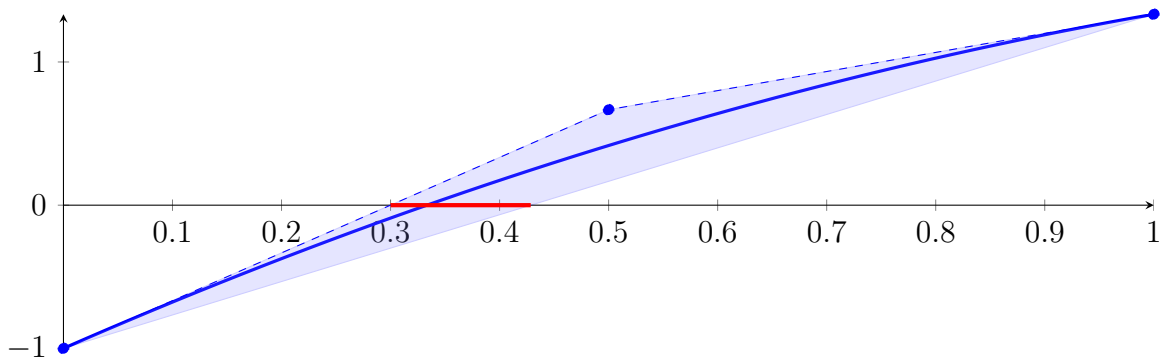
$$p = -1X^2 + 3.33333X - 1$$



184.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

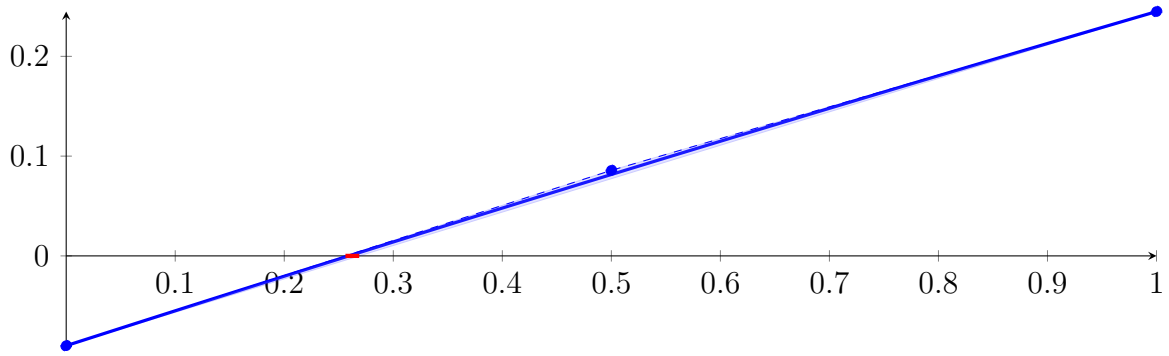
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

184.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

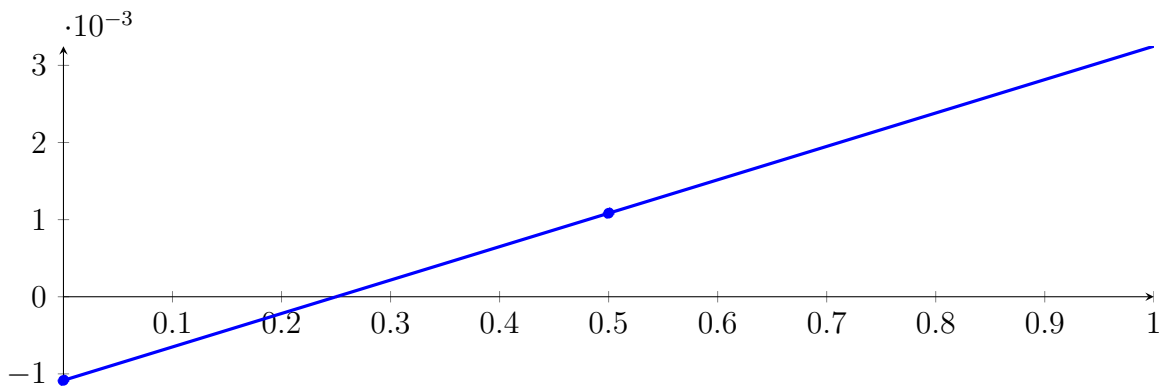
Longest intersection interval: 0.012641

⇒ Selective recursion: interval 1: $[0.332927, 0.334552]$,

184.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

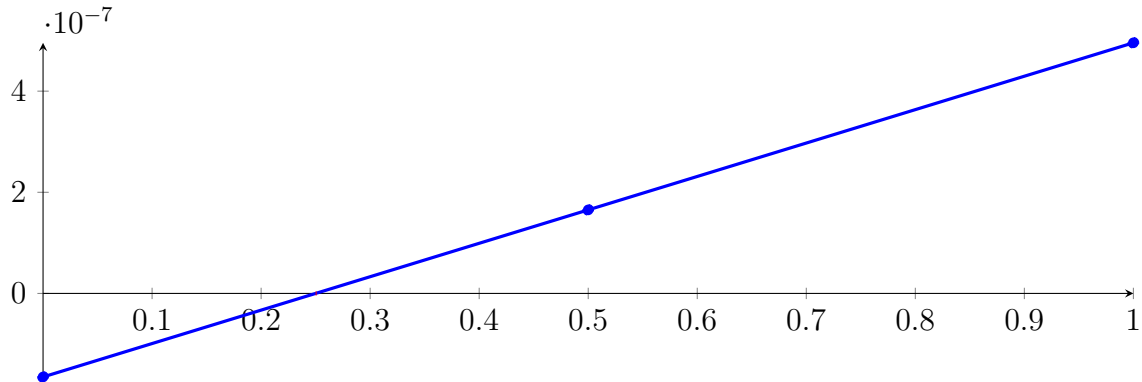
Longest intersection interval: 0.000152462

⇒ Selective recursion: interval 1: $[0.333333, 0.333334]$,

184.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07} \\
 &= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

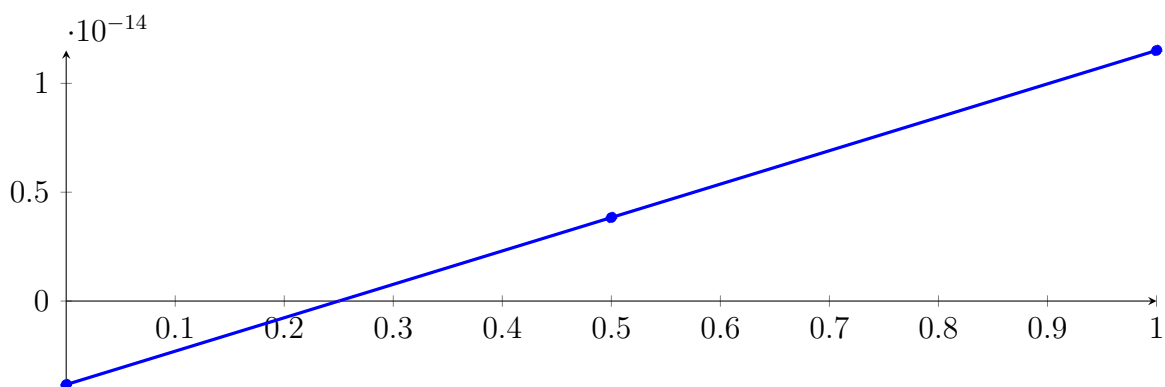
Longest intersection interval: $2.32306 \cdot 10^{-08}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

184.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.31358 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15} \\
 &= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

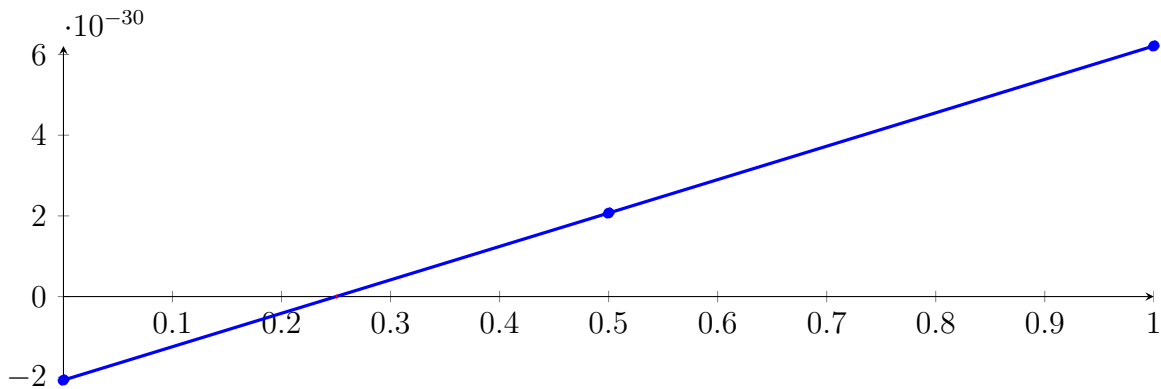
Longest intersection interval: $5.3966 \cdot 10^{-16}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

184.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -9.65021 \cdot 10^{-60} X^2 + 8.28394 \cdot 10^{-30} X - 2.07099 \cdot 10^{-30} \\
 &= -2.07099 \cdot 10^{-30} B_{0,2}(X) + 2.07099 \cdot 10^{-30} B_{1,2}(X) + 6.21296 \cdot 10^{-30} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

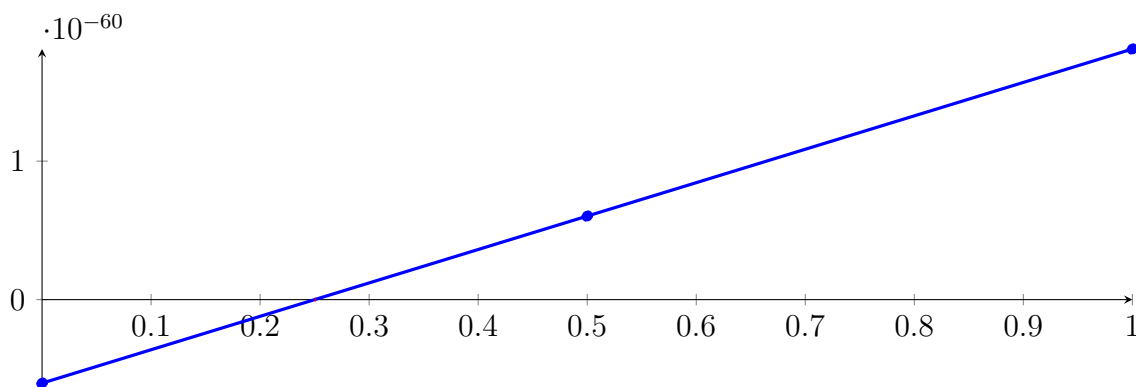
Longest intersection interval: $2.91232 \cdot 10^{-31}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

184.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -8.18495 \cdot 10^{-121} X^2 + 2.41255 \cdot 10^{-60} X - 6.03138 \cdot 10^{-61} \\
 &= -6.03138 \cdot 10^{-61} B_{0,2}(X) + 6.03138 \cdot 10^{-61} B_{1,2}(X) + 1.80941 \cdot 10^{-60} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $8.48163 \cdot 10^{-62}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

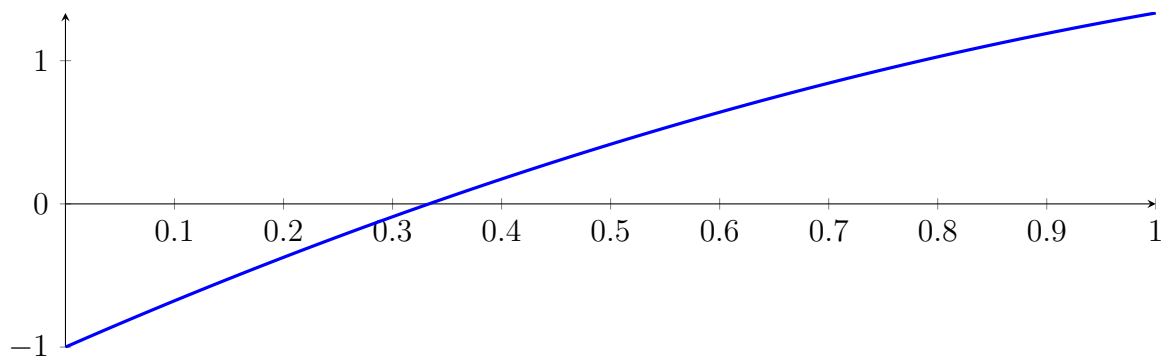
184.8 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 8!

184.9 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

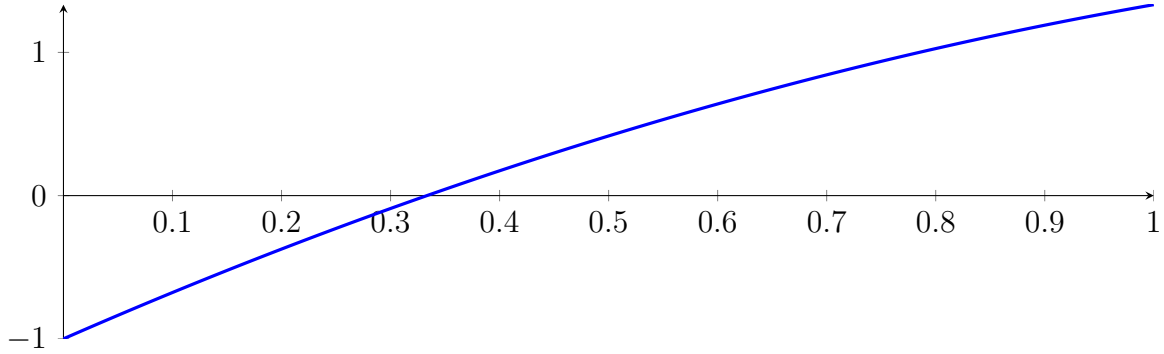
with precision $\varepsilon = 1 \cdot 10^{-64}$.

185 Running QuadClip on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

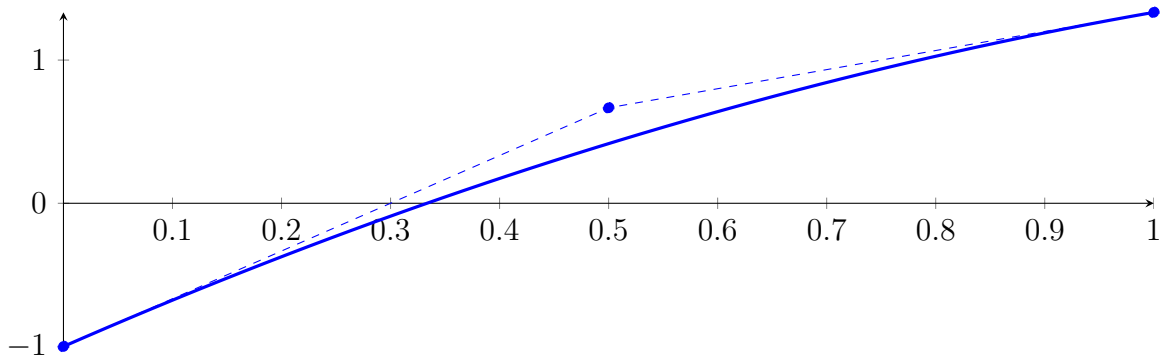
$$p = -1X^2 + 3.33333X - 1$$



185.1 Recursion Branch 1 for Input Interval $[0, 1]$

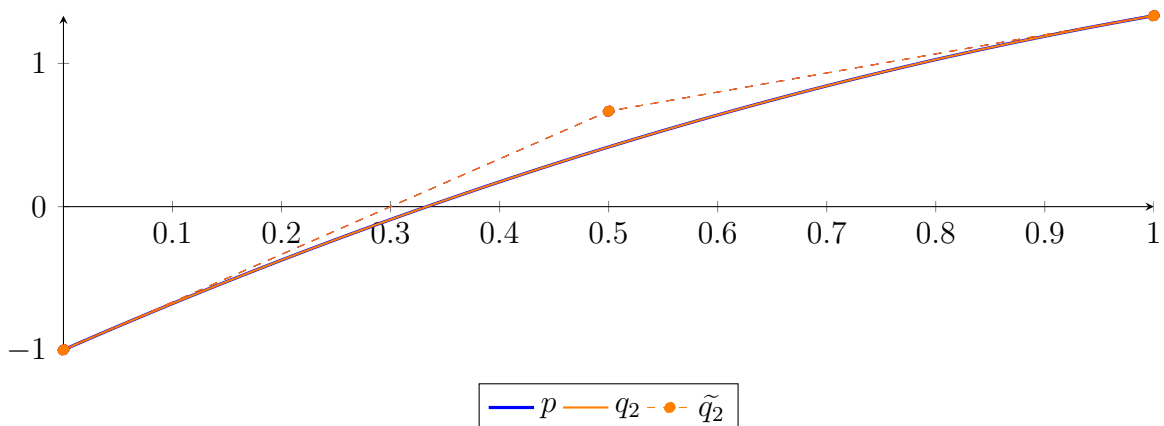
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.22507 \cdot 10^{-308}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

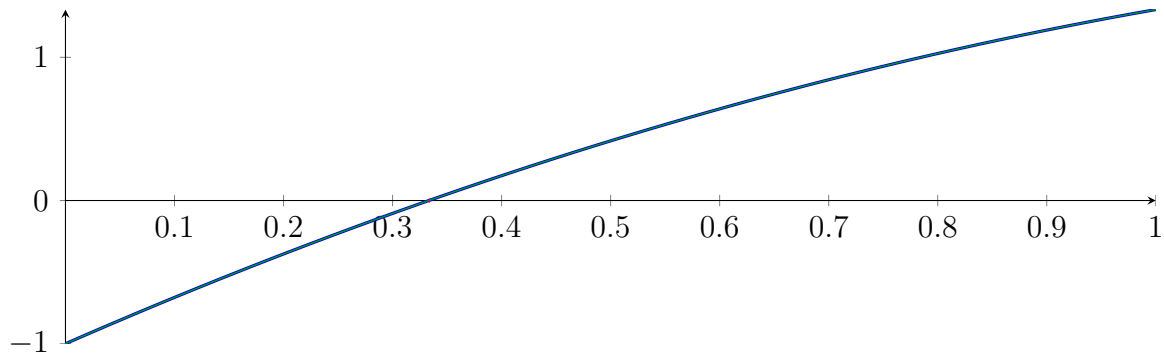
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $1.11254 \cdot 10^{-308}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

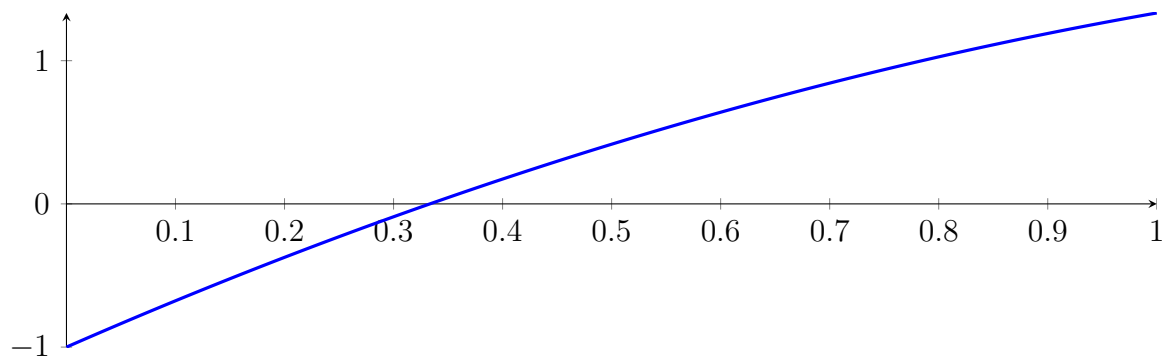
185.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

185.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

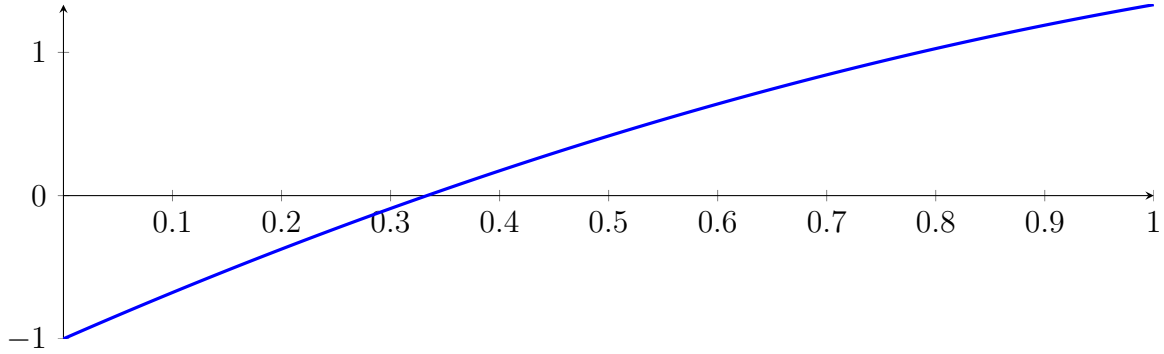
with precision $\varepsilon = 1 \cdot 10^{-64}$.

186 Running CubeClip on f_2 with epsilon 64

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

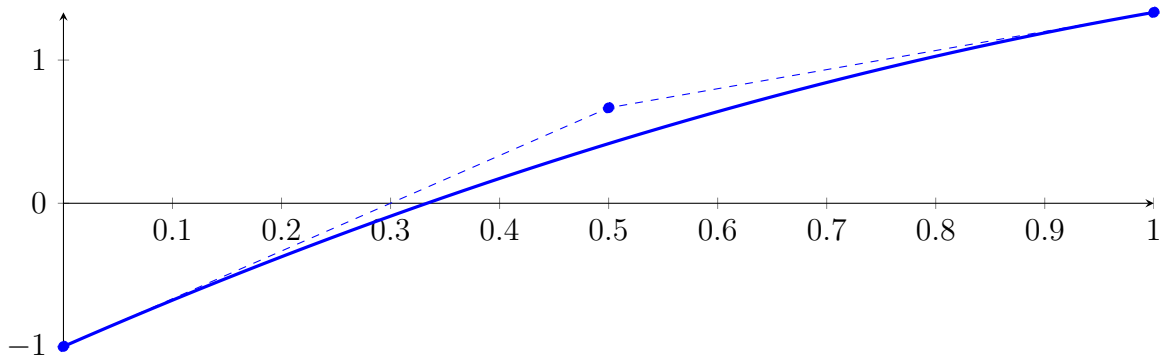
$$p = -1X^2 + 3.33333X - 1$$



186.1 Recursion Branch 1 for Input Interval $[0, 1]$

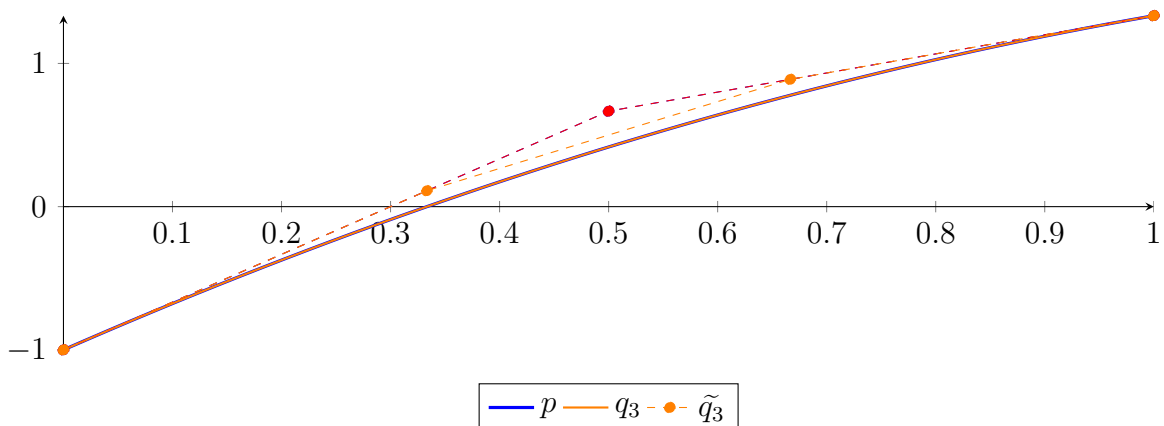
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.66881 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.00128 \cdot 10^{-307}$.

Bounding polynomials M and m :

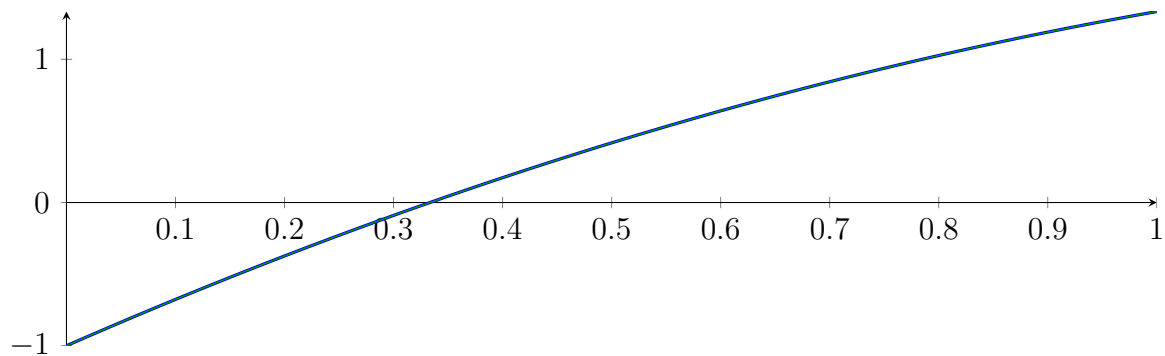
$$M = -2.78134 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1$$

$$m = -2.78134 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-3.59539 \cdot 10^{307}, -3.38949 \cdot 10^{291}, 1.03213 \cdot 10^{153}\} \quad N(m) = \{-3.59539 \cdot 10^{307}, -3.38949 \cdot 10^{291}, 1.75\}$$

Intersection intervals:

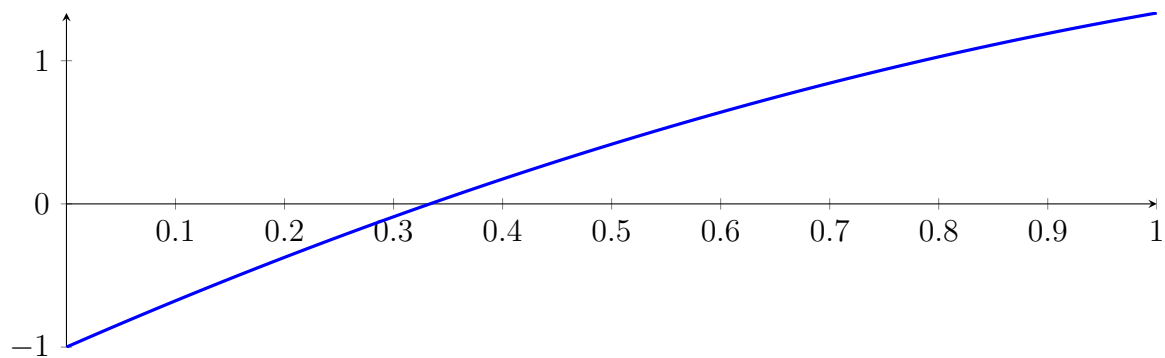


No intersection intervals with the x axis.

186.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

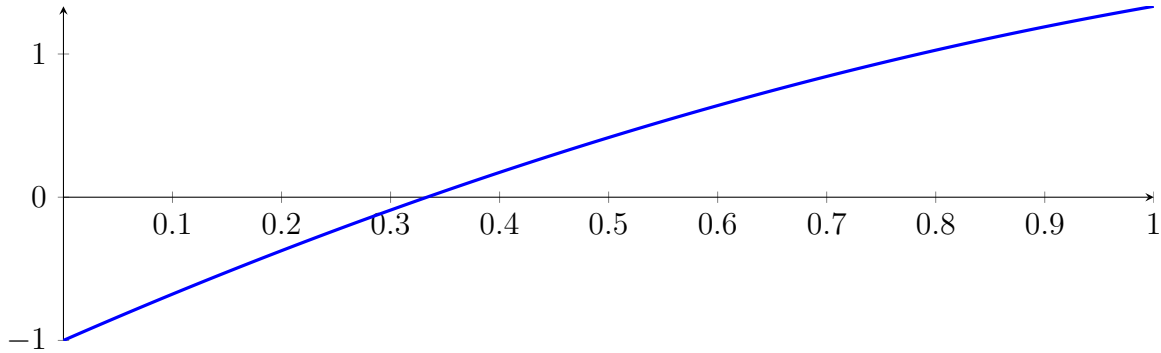
with precision $\varepsilon = 1 \cdot 10^{-64}$.

187 Running BezClip on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called BezClip with input polynomial on interval $[0, 1]$:

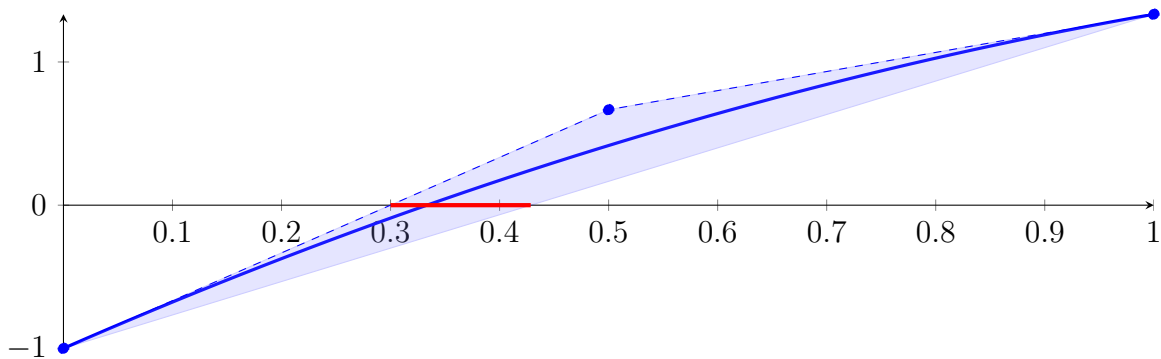
$$p = -1X^2 + 3.33333X - 1$$



187.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.3, 0.428571\}$$

Intersection intervals with the x axis:

$$[0.3, 0.428571]$$

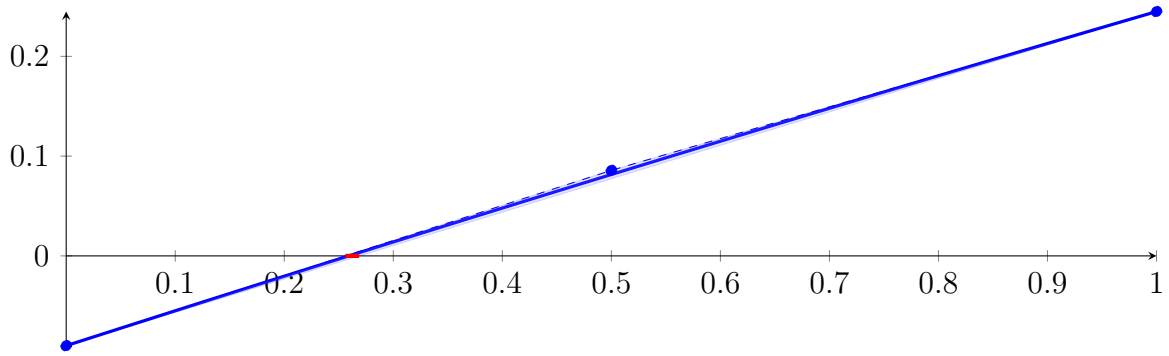
Longest intersection interval: 0.128571

\implies Selective recursion: interval 1: $[0.3, 0.428571]$,

187.2 Recursion Branch 1 1 in Interval 1: $[0.3, 0.428571]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -0.0165306X^2 + 0.351429X - 0.09 \\ &= -0.09B_{0,2}(X) + 0.0857143B_{1,2}(X) + 0.244898B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.256098, 0.268739\}$$

Intersection intervals with the x axis:

$$[0.256098, 0.268739]$$

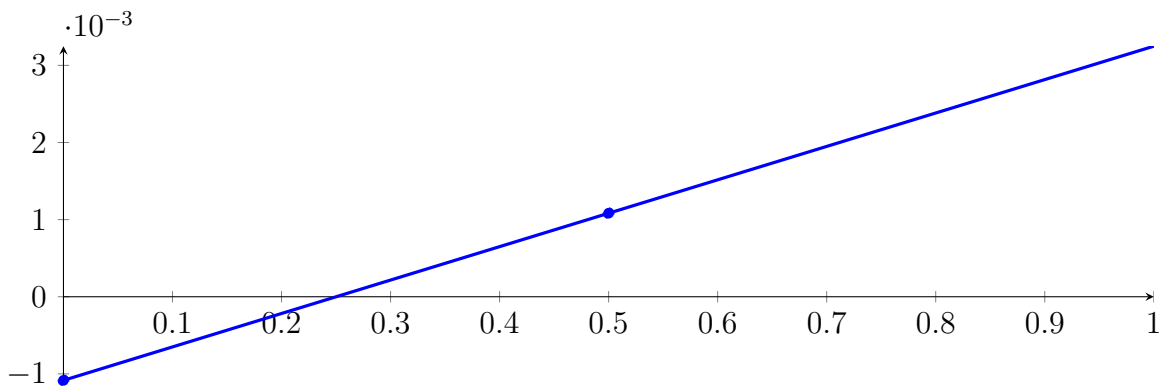
Longest intersection interval: 0.012641

\implies Selective recursion: interval 1: $[0.332927, 0.334552]$,

187.3 Recursion Branch 1 1 1 in Interval 1: $[0.332927, 0.334552]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -2.64151 \cdot 10^{-06} X^2 + 0.00433538 X - 0.00108418 \\ &= -0.00108418 B_{0,2}(X) + 0.00108352 B_{1,2}(X) + 0.00324857 B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.250076, 0.250229\}$$

Intersection intervals with the x axis:

$$[0.250076, 0.250229]$$

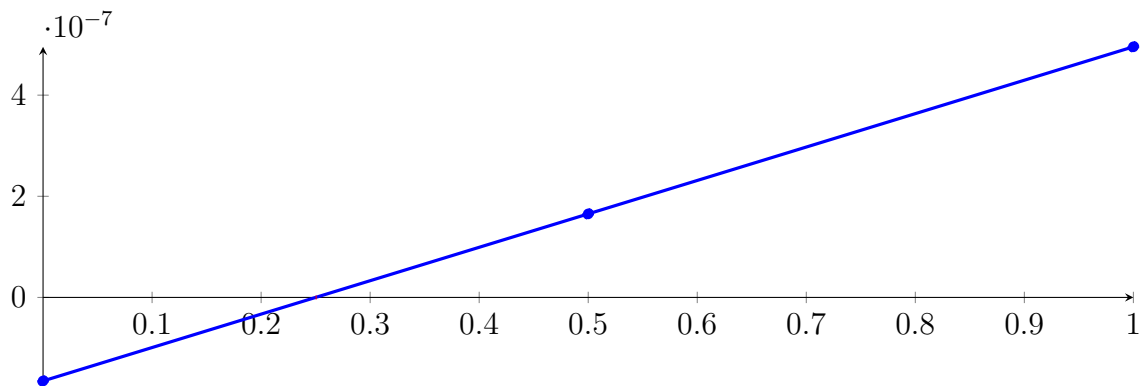
Longest intersection interval: 0.000152462

\implies Selective recursion: interval 1: $[0.333333, 0.333334]$,

187.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333334]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -6.14013 \cdot 10^{-14} X^2 + 6.60781 \cdot 10^{-07} X - 1.65195 \cdot 10^{-07} \\
 &= -1.65195 \cdot 10^{-07} B_{0,2}(X) + 1.65195 \cdot 10^{-07} B_{1,2}(X) + 4.95585 \cdot 10^{-07} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

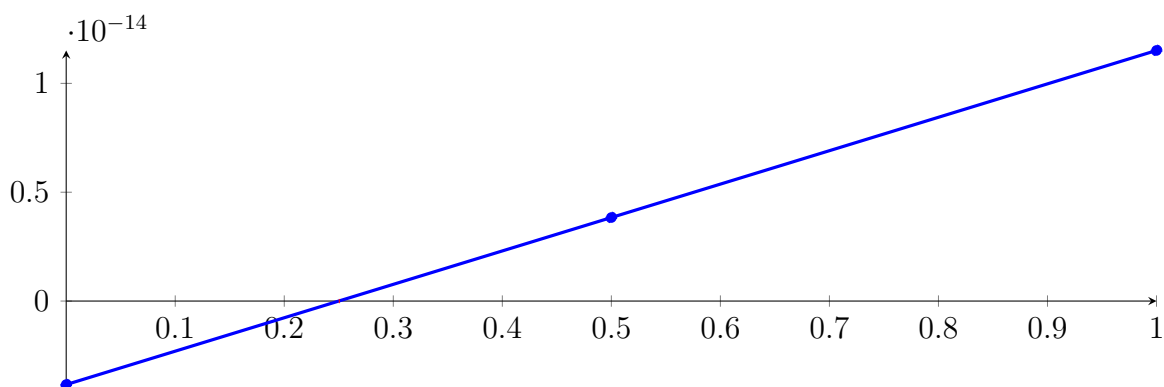
Longest intersection interval: $2.32306 \cdot 10^{-08}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

187.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.31358 \cdot 10^{-29} X^2 + 1.53503 \cdot 10^{-14} X - 3.83758 \cdot 10^{-15} \\
 &= -3.83758 \cdot 10^{-15} B_{0,2}(X) + 3.83758 \cdot 10^{-15} B_{1,2}(X) + 1.15127 \cdot 10^{-14} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

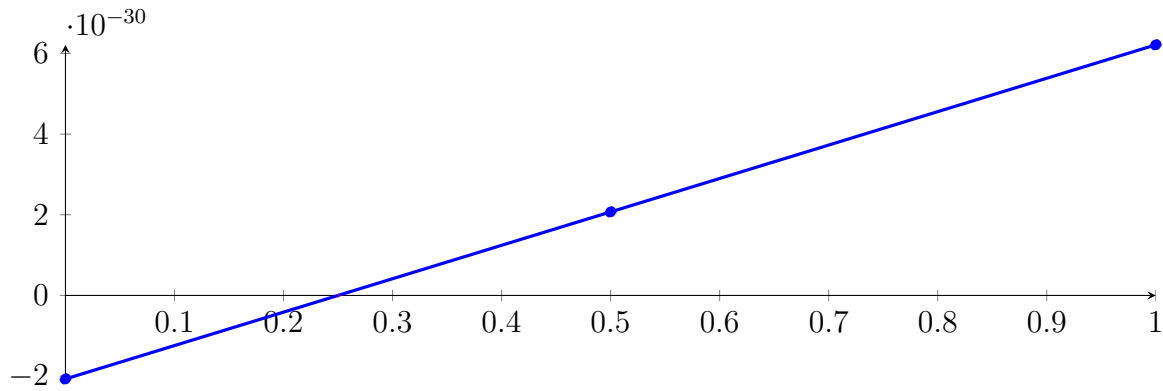
Longest intersection interval: $5.3966 \cdot 10^{-16}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

187.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -9.65021 \cdot 10^{-60} X^2 + 8.28394 \cdot 10^{-30} X - 2.07099 \cdot 10^{-30} \\
 &= -2.07099 \cdot 10^{-30} B_{0,2}(X) + 2.07099 \cdot 10^{-30} B_{1,2}(X) + 6.21296 \cdot 10^{-30} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

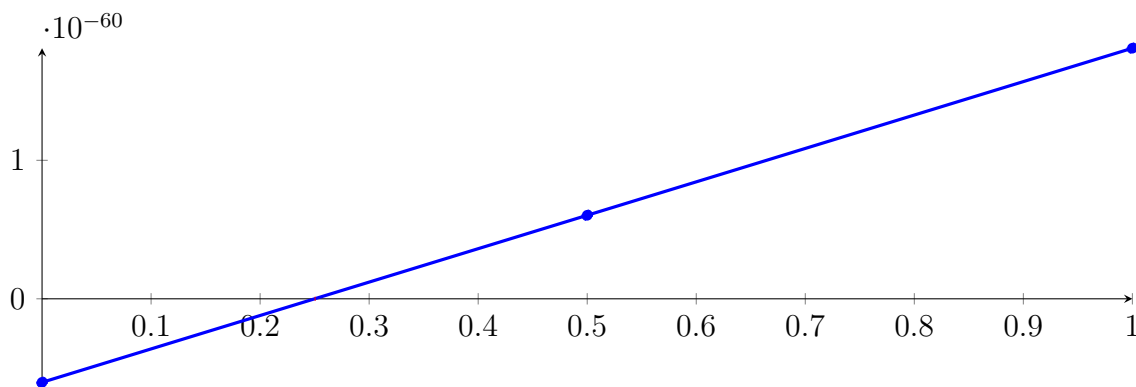
Longest intersection interval: $2.91232 \cdot 10^{-31}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

187.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -8.18495 \cdot 10^{-121} X^2 + 2.41255 \cdot 10^{-60} X - 6.03138 \cdot 10^{-61} \\
 &= -6.03138 \cdot 10^{-61} B_{0,2}(X) + 6.03138 \cdot 10^{-61} B_{1,2}(X) + 1.80941 \cdot 10^{-60} B_{2,2}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

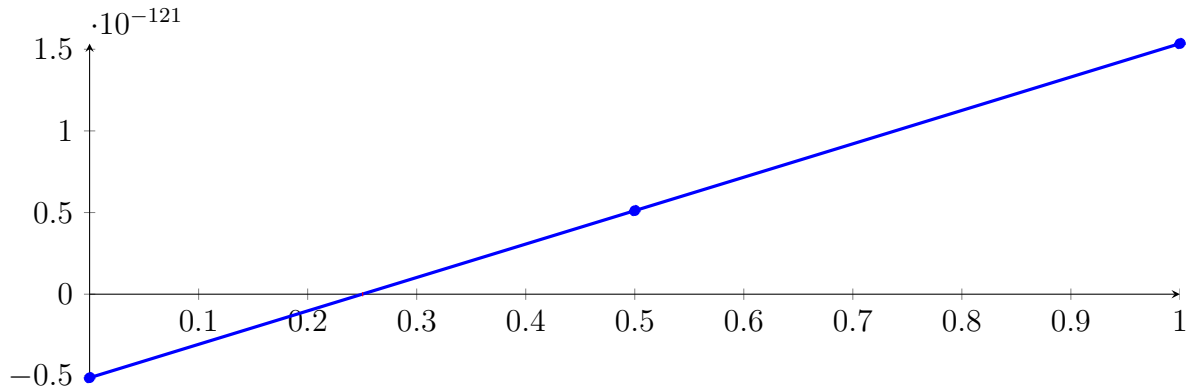
Longest intersection interval: $8.48163 \cdot 10^{-62}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

187.8 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.8881 \cdot 10^{-243} X^2 + 2.04624 \cdot 10^{-121} X - 5.1156 \cdot 10^{-122} \\ &= -5.1156 \cdot 10^{-122} B_{0,2}(X) + 5.1156 \cdot 10^{-122} B_{1,2}(X) + 1.53468 \cdot 10^{-121} B_{2,2}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.25, 0.25\}$$

Intersection intervals with the x axis:

$$[0.25, 0.25]$$

Longest intersection interval: $7.19381 \cdot 10^{-123}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

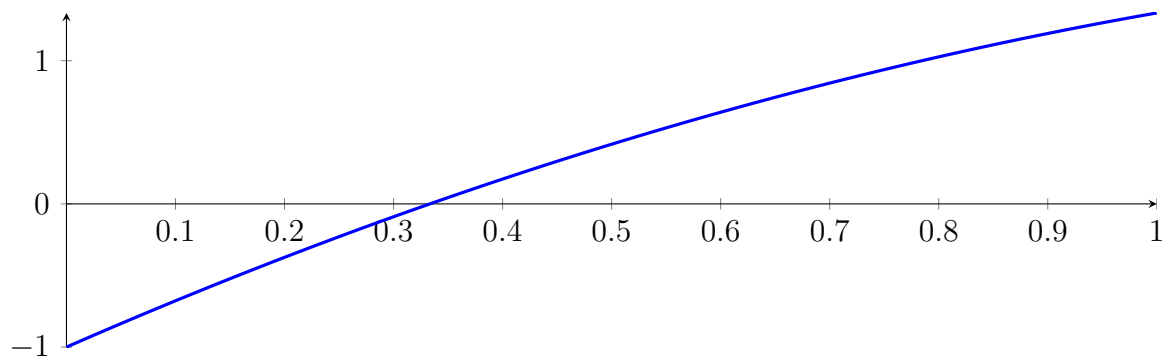
187.9 Recursion Branch 1 1 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 9!

187.10 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

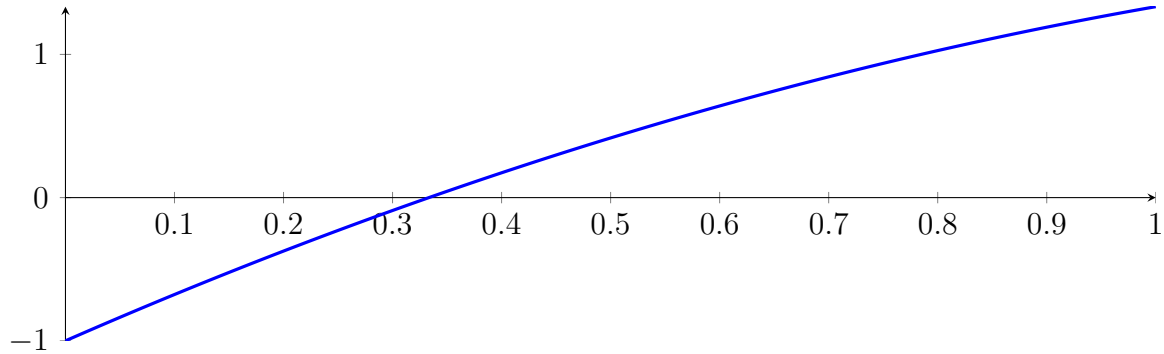
with precision $\varepsilon = 1 \cdot 10^{-128}$.

188 Running QuadClip on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called QuadClip with input polynomial on interval $[0, 1]$:

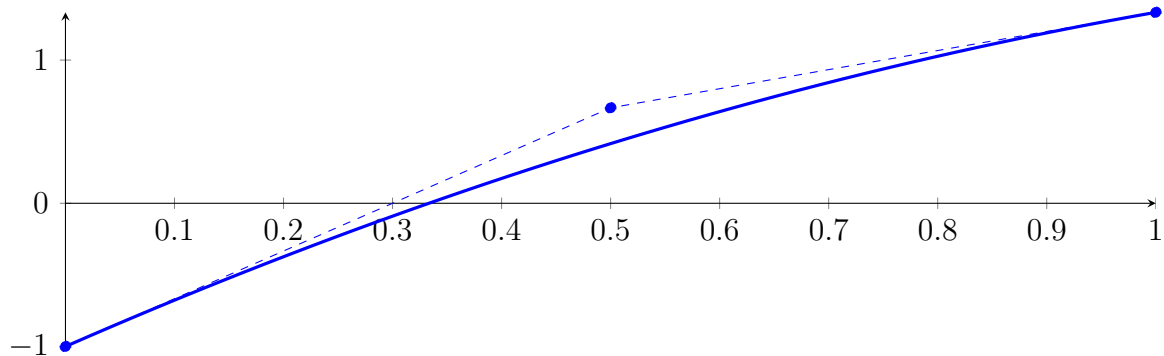
$$p = -1X^2 + 3.33333X - 1$$



188.1 Recursion Branch 1 for Input Interval $[0, 1]$

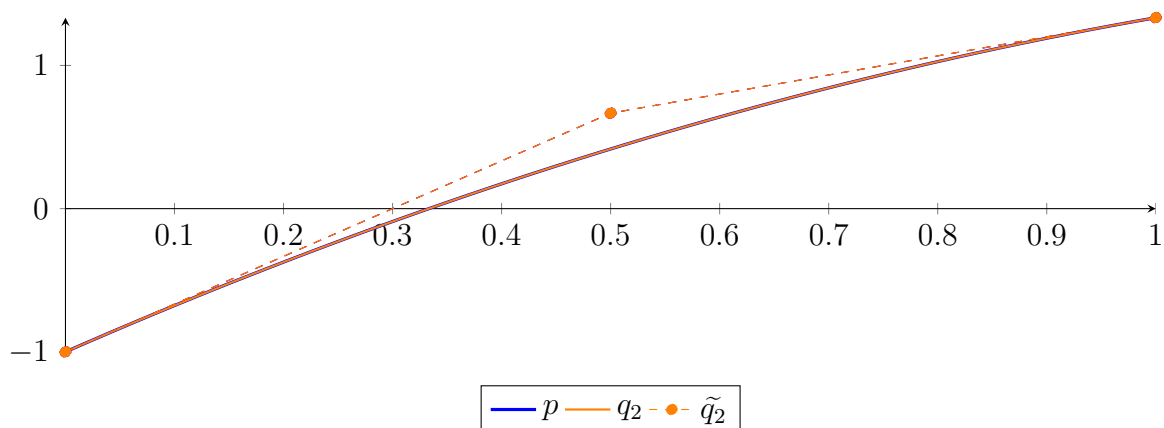
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \\ \tilde{q}_2 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.22507 \cdot 10^{-308}$.

Bounding polynomials M and m :

$$M = -1X^2 + 3.33333X - 1$$

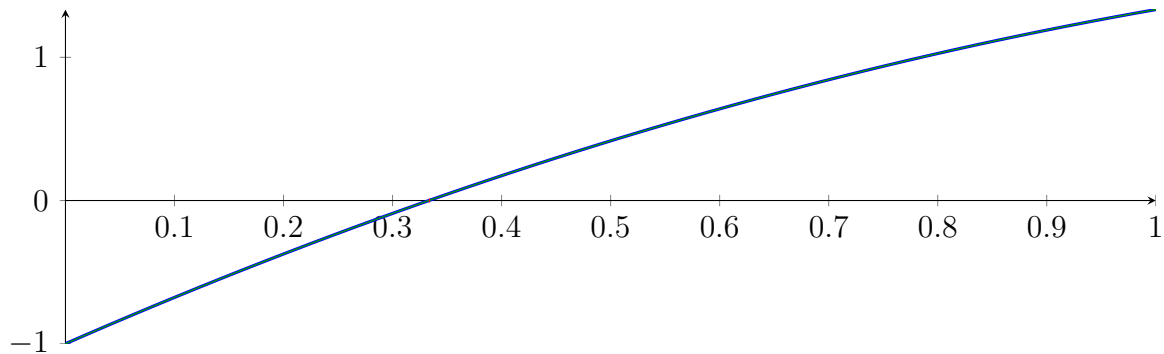
$$m = -1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{0.333333, 3\}$$

$$N(m) = \{0.333333, 3\}$$

Intersection intervals:



$$[0.333333, 0.333333]$$

Longest intersection interval: $1.11254 \cdot 10^{-308}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

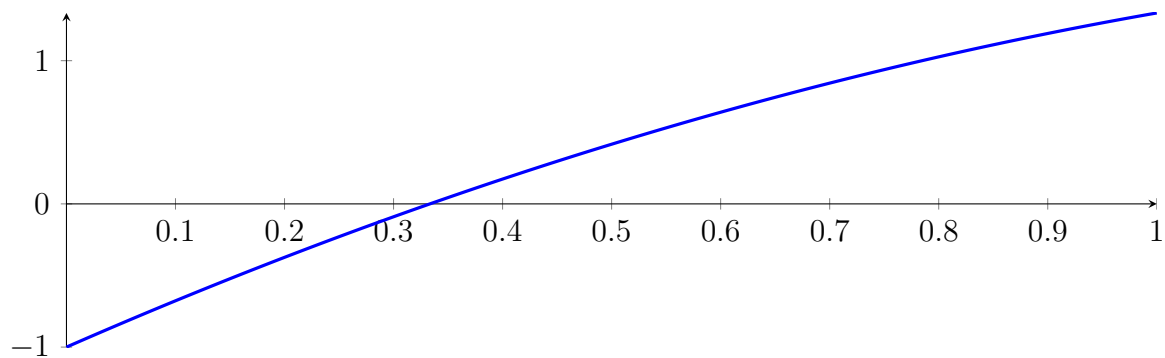
188.2 Recursion Branch 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 2!

188.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

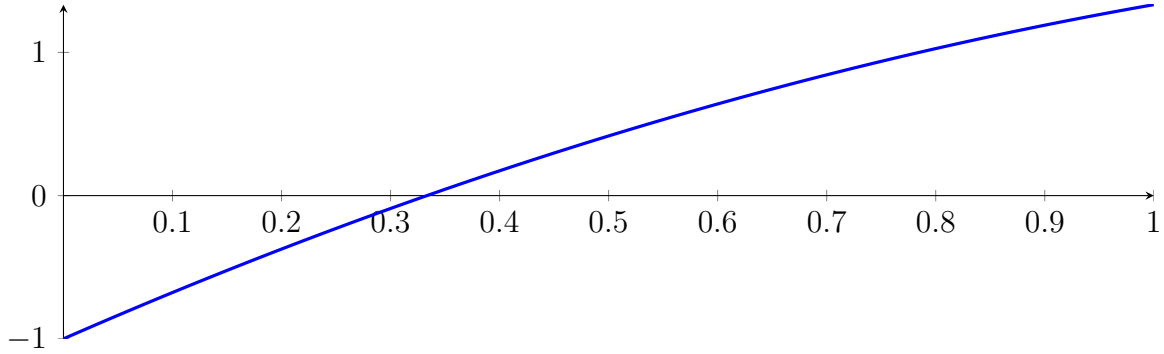
with precision $\varepsilon = 1 \cdot 10^{-128}$.

189 Running CubeClip on f_2 with epsilon 128

$$-1X^2 + 3.33333X - 1$$

Called CubeClip with input polynomial on interval $[0, 1]$:

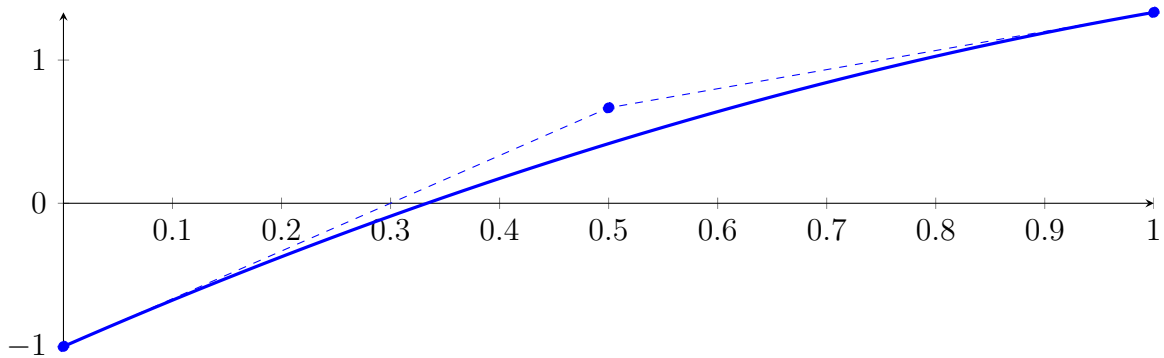
$$p = -1X^2 + 3.33333X - 1$$



189.1 Recursion Branch 1 for Input Interval $[0, 1]$

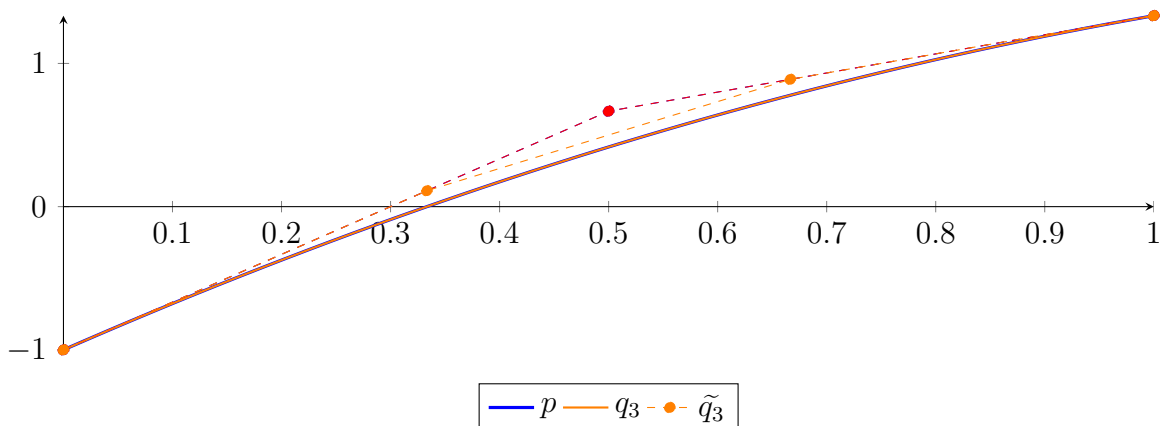
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2}(X) + 0.666667B_{1,2}(X) + 1.33333B_{2,2}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.66881 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1 \\ &= -1B_{0,3} + 0.111111B_{1,3} + 0.888889B_{2,3} + 1.33333B_{3,3} \\ \tilde{q}_3 &= -1X^2 + 3.33333X - 1 \\ &= -1B_{0,2} + 0.666667B_{1,2} + 1.33333B_{2,2} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.00128 \cdot 10^{-307}$.

Bounding polynomials M and m :

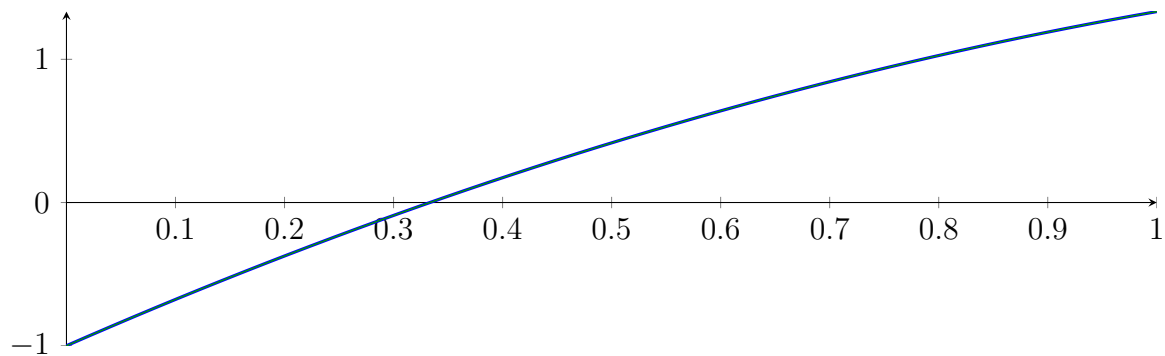
$$M = -2.78134 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1$$

$$m = -2.78134 \cdot 10^{-308} X^3 - 1X^2 + 3.33333X - 1$$

Root of M and m :

$$N(M) = \{-3.59539 \cdot 10^{307}, -3.38949 \cdot 10^{291}, 1.03213 \cdot 10^{153}\} \quad N(m) = \{-3.59539 \cdot 10^{307}, -3.38949 \cdot 10^{291}, 1.75\}$$

Intersection intervals:

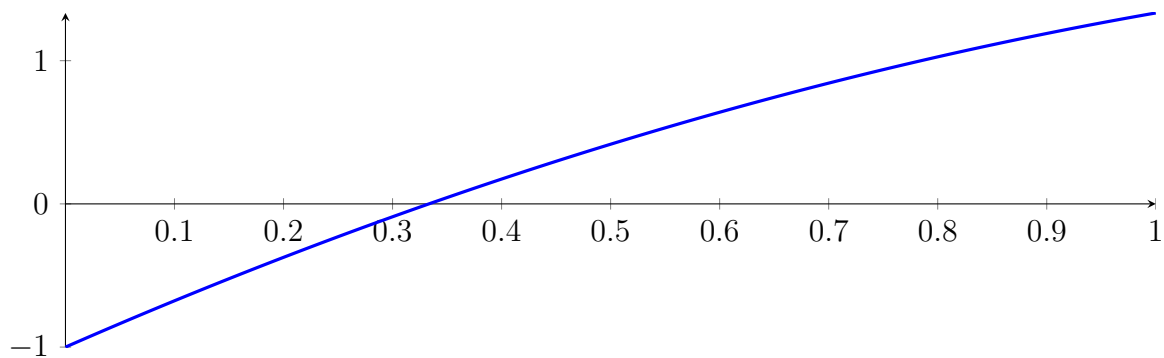


No intersection intervals with the x axis.

189.2 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^2 + 3.33333X - 1$$



Result: Root Intervals

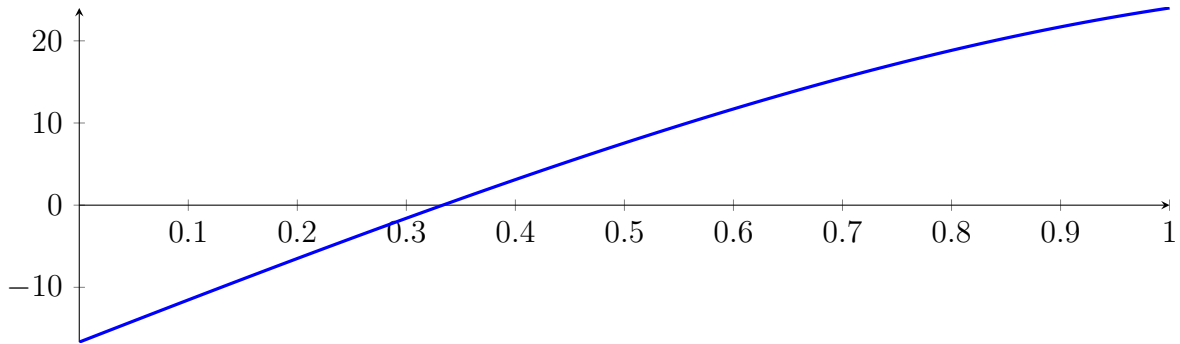
with precision $\varepsilon = 1 \cdot 10^{-128}$.

190 Running BezClip on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

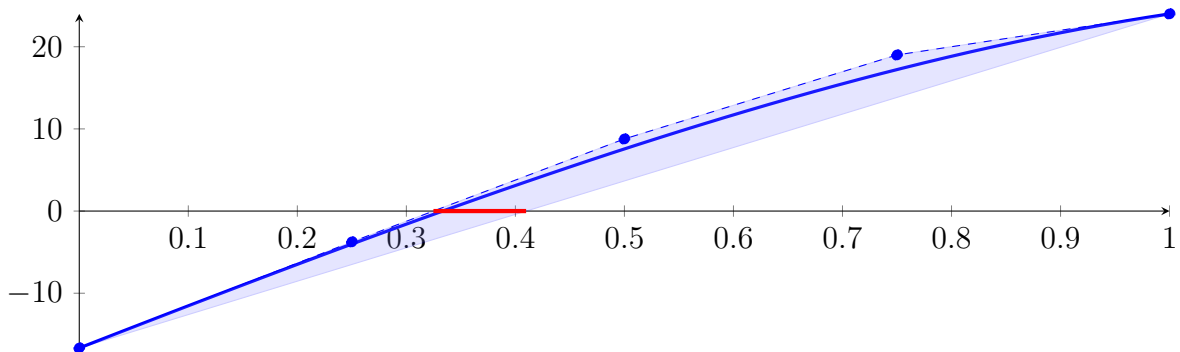
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



190.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

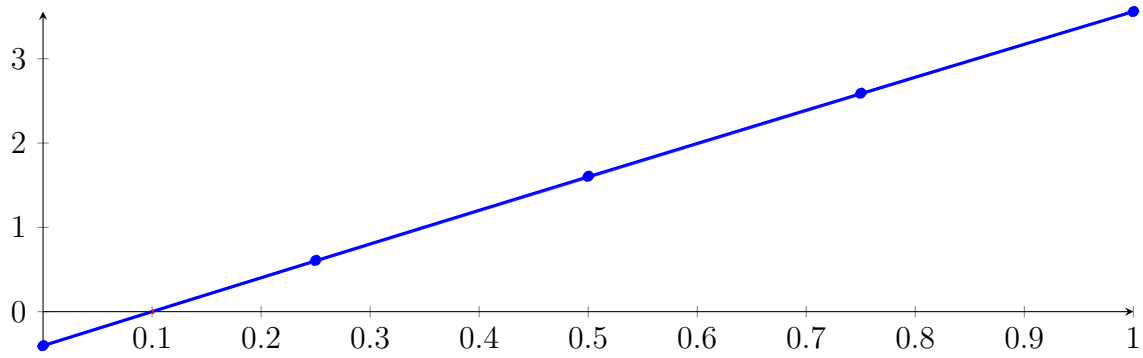
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

190.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

Longest intersection interval: 0.00203877

\implies Selective recursion: interval 1: $[0.333317, 0.333491]$,

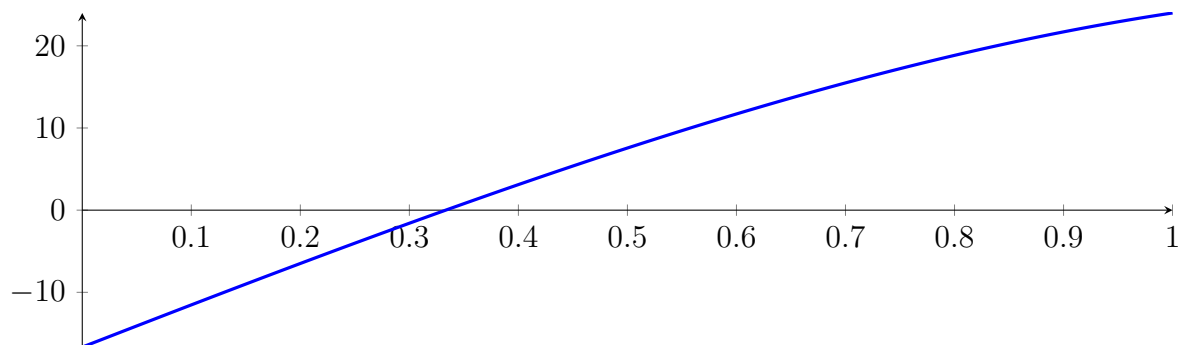
190.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Found root in interval $[0.333317, 0.333491]$ at recursion depth 3!

190.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333317, 0.333491]$$

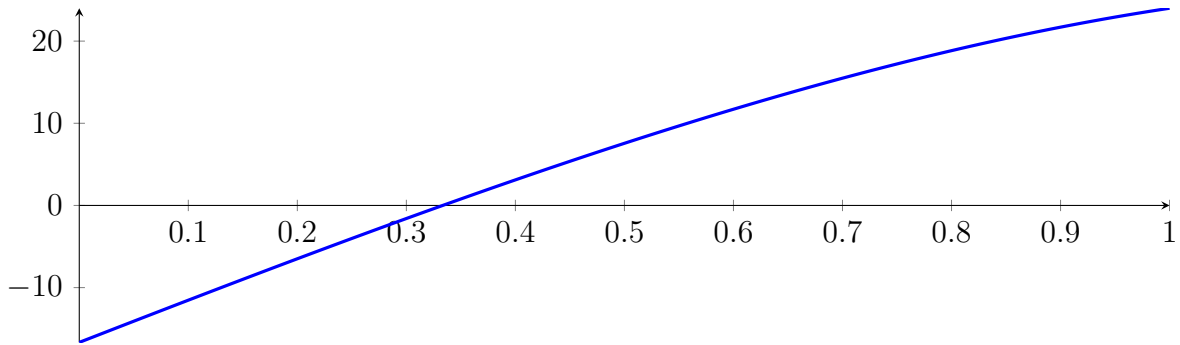
with precision $\varepsilon = 0.01$.

191 Running QuadClip on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

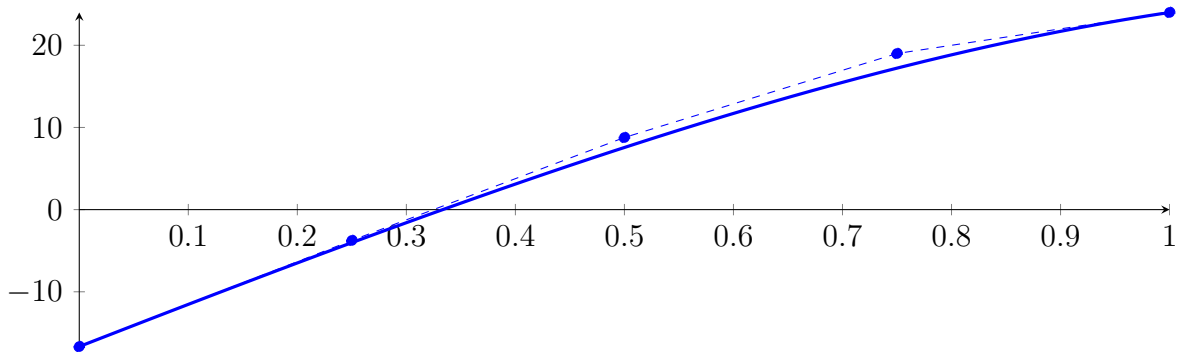
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



191.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

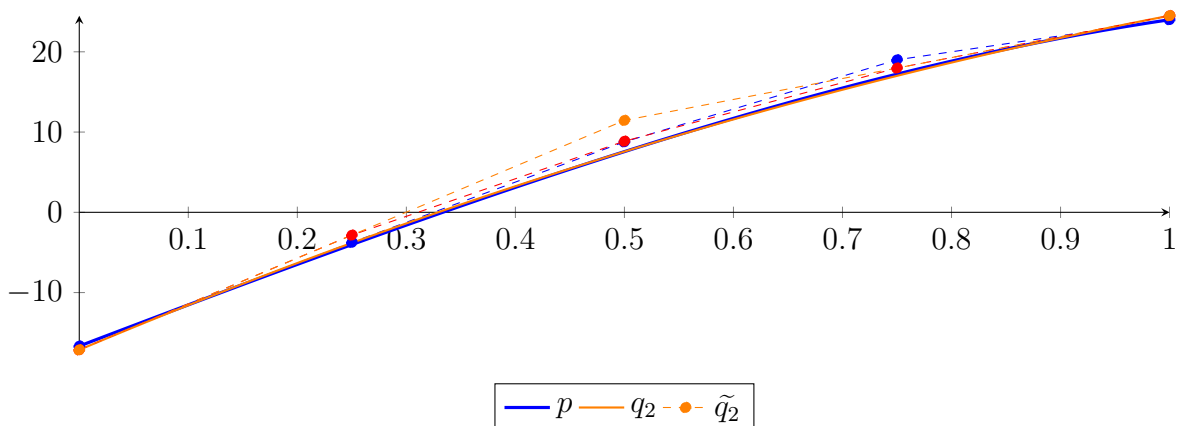
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -2.13607 \cdot 10^{-306}X^4 + 7.12024 \cdot 10^{-306}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

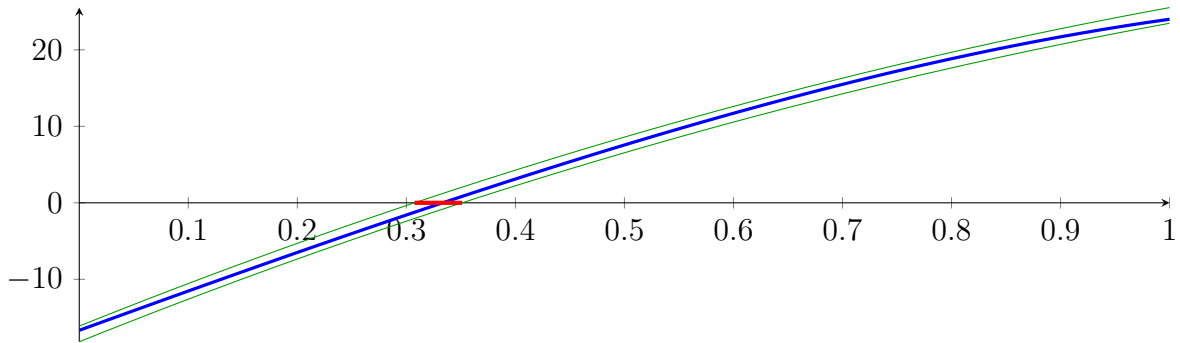
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

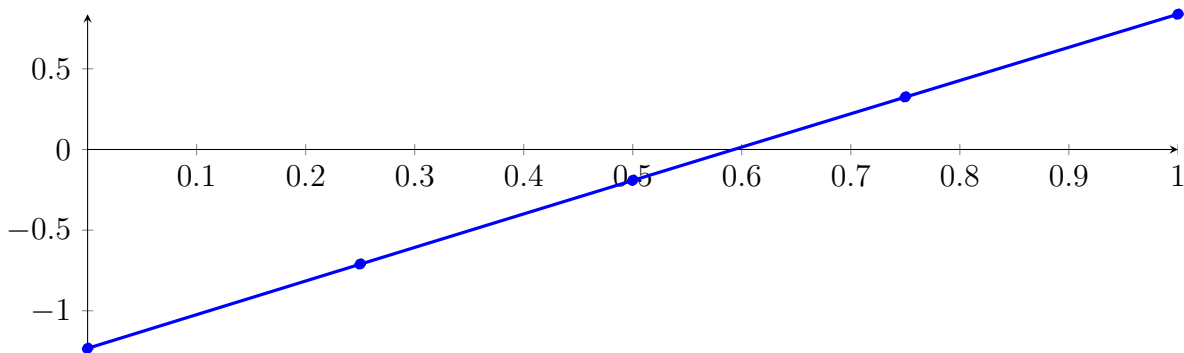
Longest intersection interval: 0.0436205

⇒ Selective recursion: interval 1: $[0.307477, 0.351097]$,

191.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

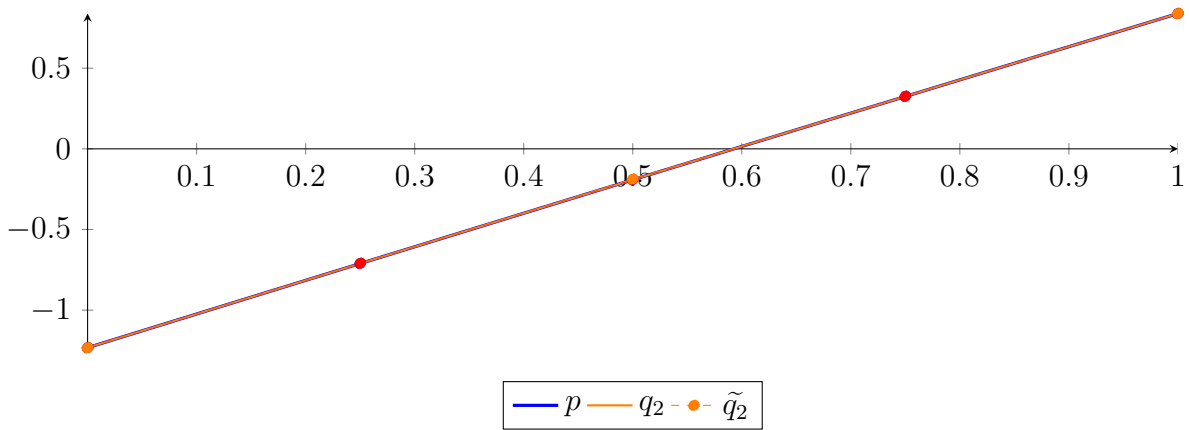
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -4.5614 \cdot 10^{-307} X^4 + 1.02353 \cdot 10^{-306} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

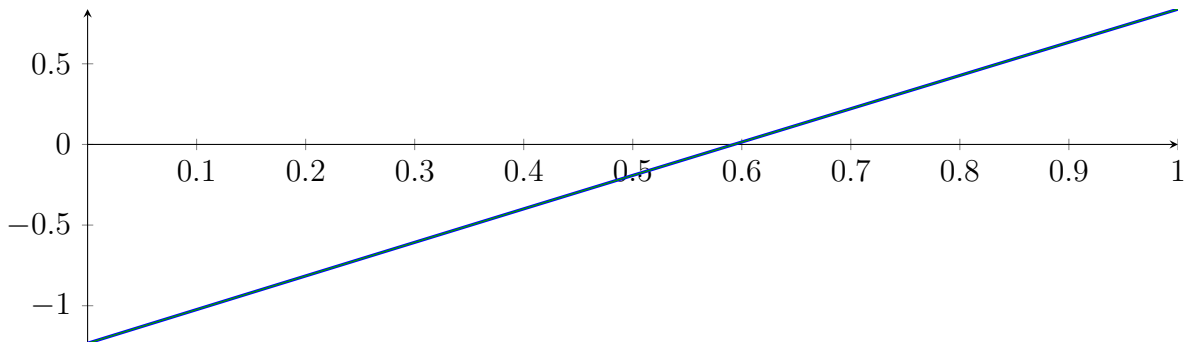
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

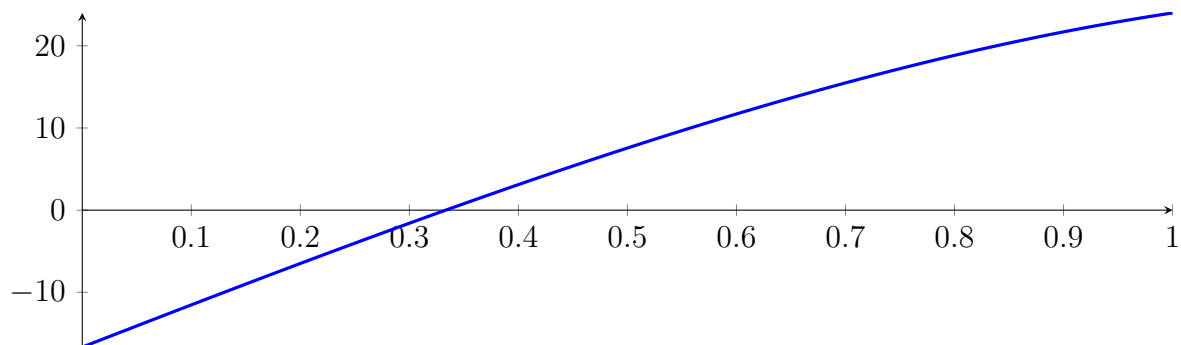
191.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Found root in interval $[0.333332, 0.333335]$ at recursion depth 3!

191.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333332, 0.333335]$$

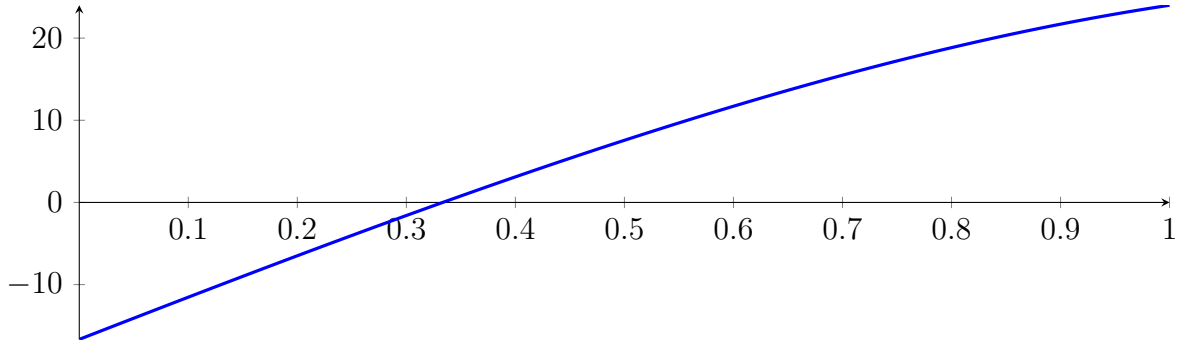
with precision $\varepsilon = 0.01$.

192 Running CubeClip on f_4 with epsilon 2

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

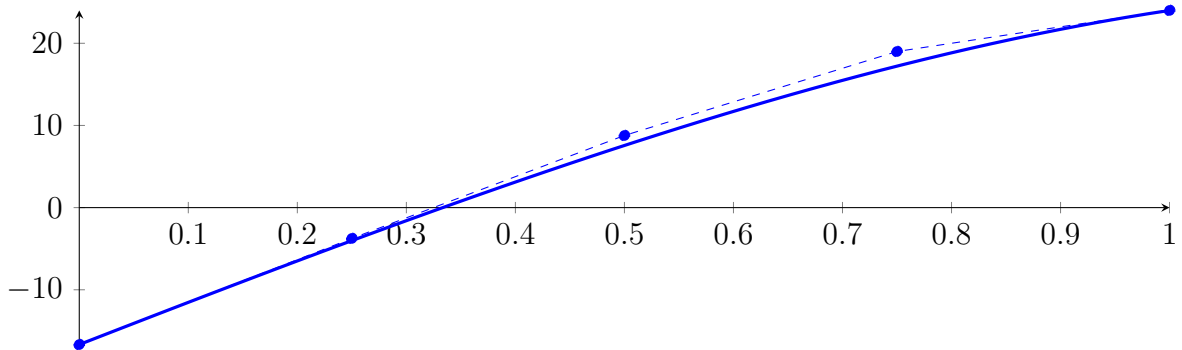
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



192.1 Recursion Branch 1 for Input Interval $[0, 1]$

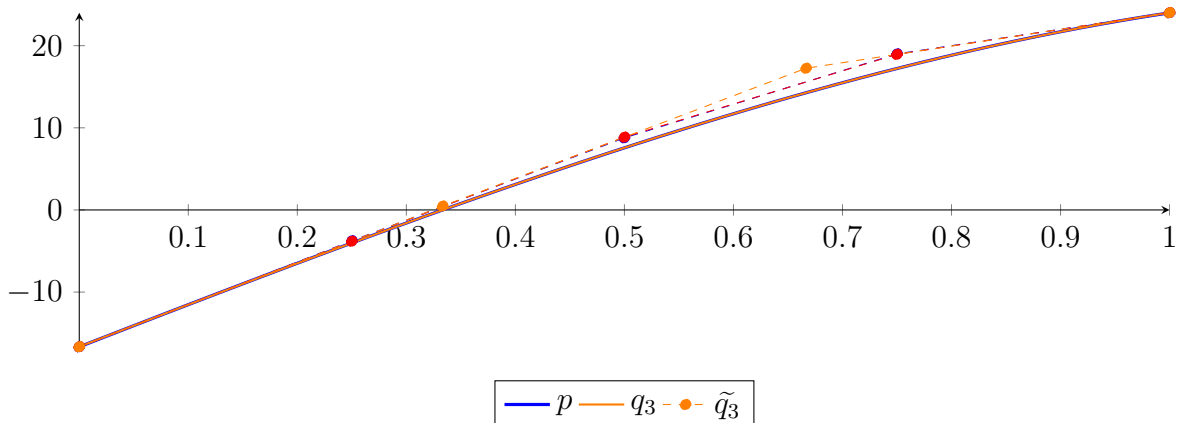
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= -3.56012 \cdot 10^{-307}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

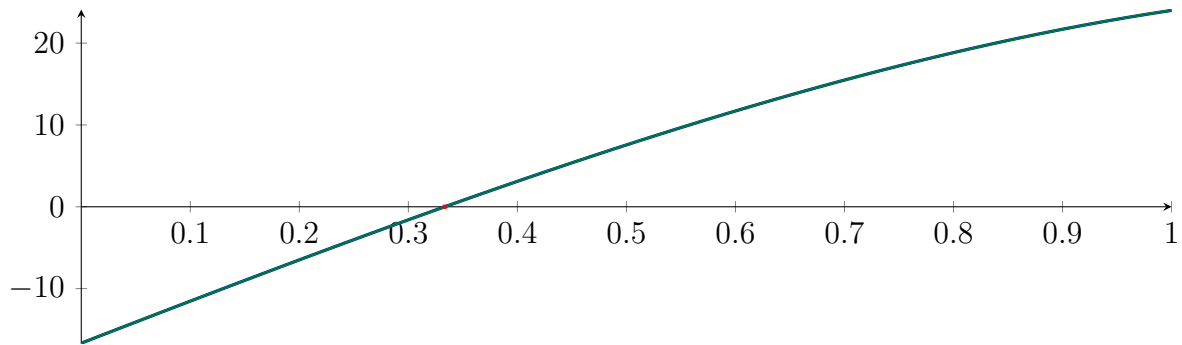
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

Longest intersection interval: 0.00361204

\implies Selective recursion: [interval 1: \[0.331524, 0.335136\]](#),

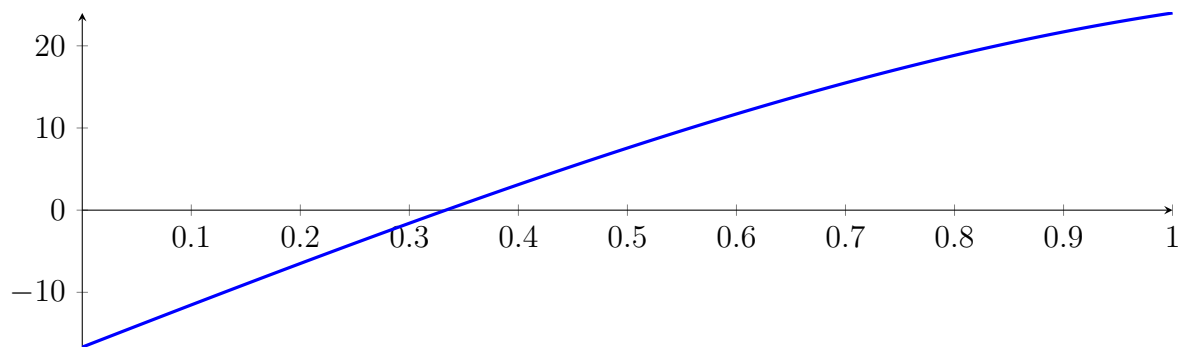
192.2 Recursion Branch 1 1 in Interval 1: [0.331524, 0.335136]

Found root in interval [0.331524, 0.335136] at recursion depth 2!

192.3 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.331524, 0.335136]$$

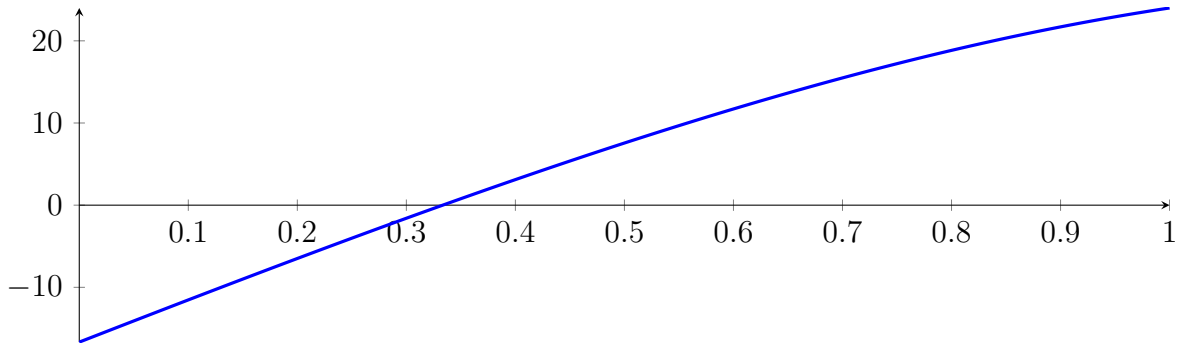
with precision $\varepsilon = 0.01$.

193 Running BezClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

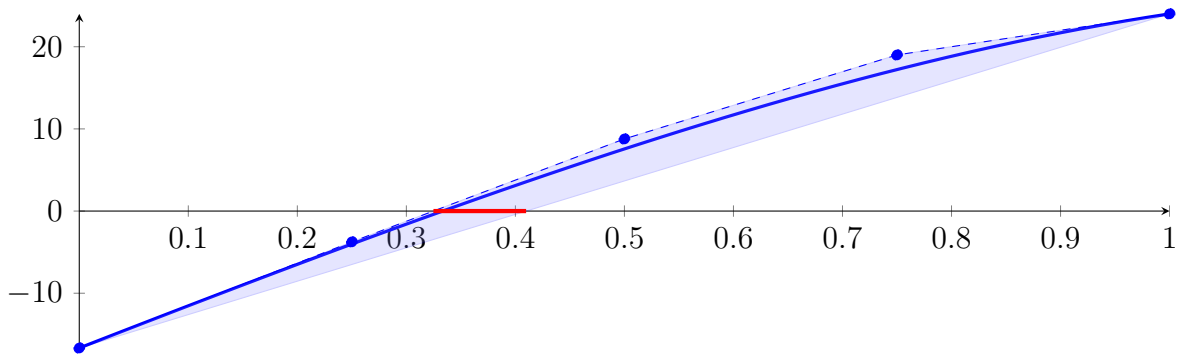
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



193.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

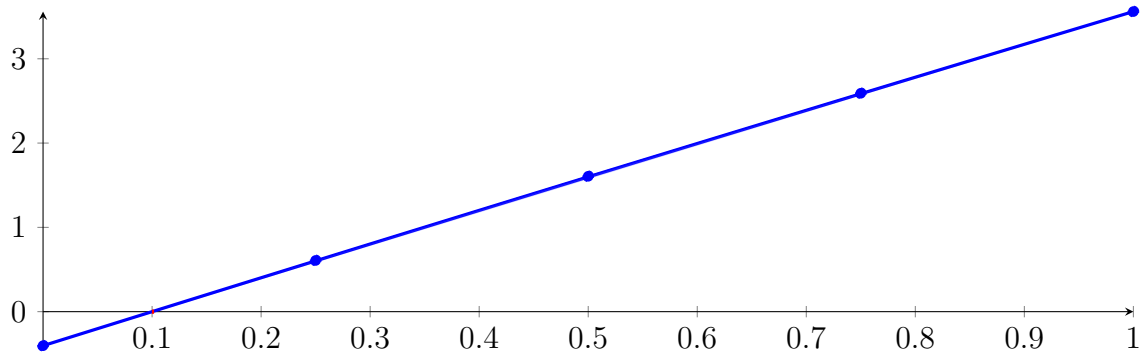
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

193.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

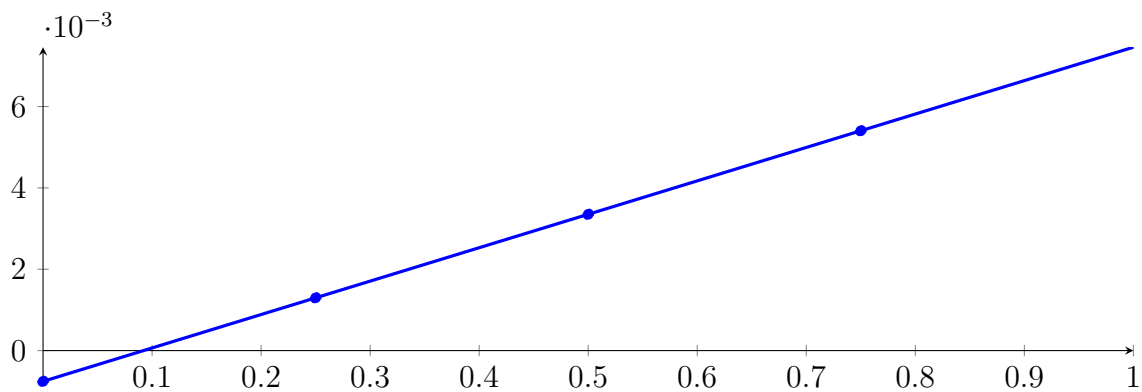
Longest intersection interval: 0.00203877

⇒ Selective recursion: interval 1: $[0.333317, 0.333491]$,

193.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01975 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

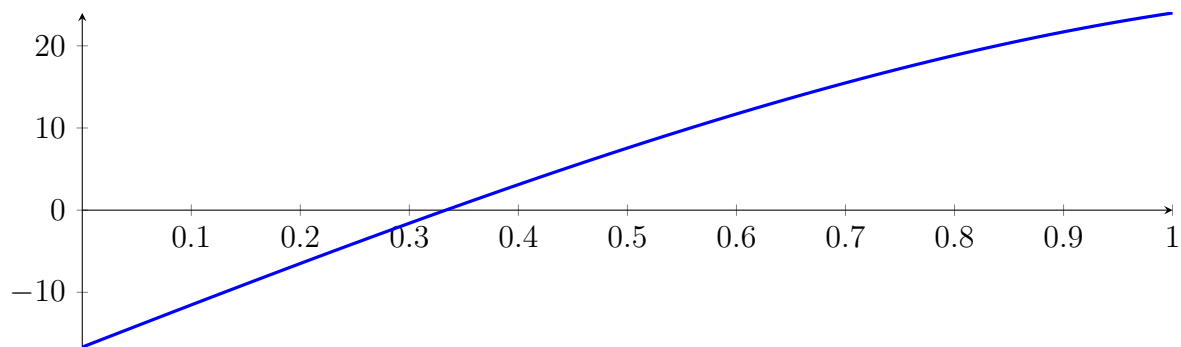
193.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

193.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

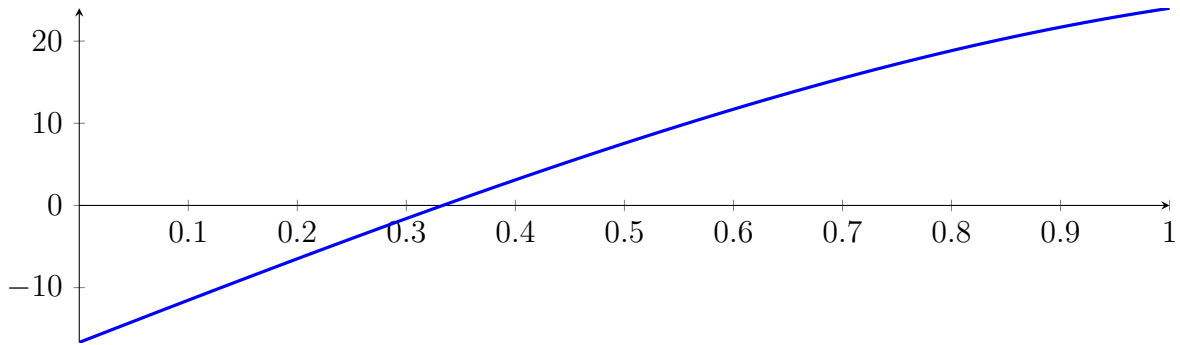
with precision $\varepsilon = 0.0001$.

194 Running QuadClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

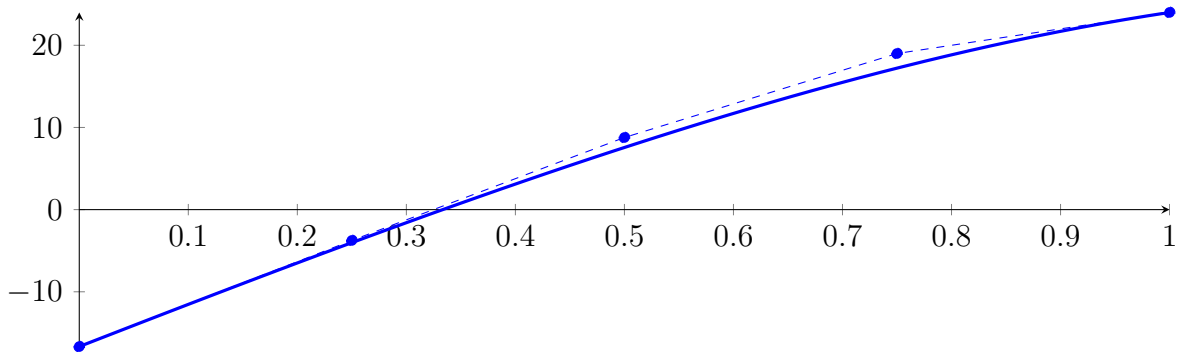
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



194.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

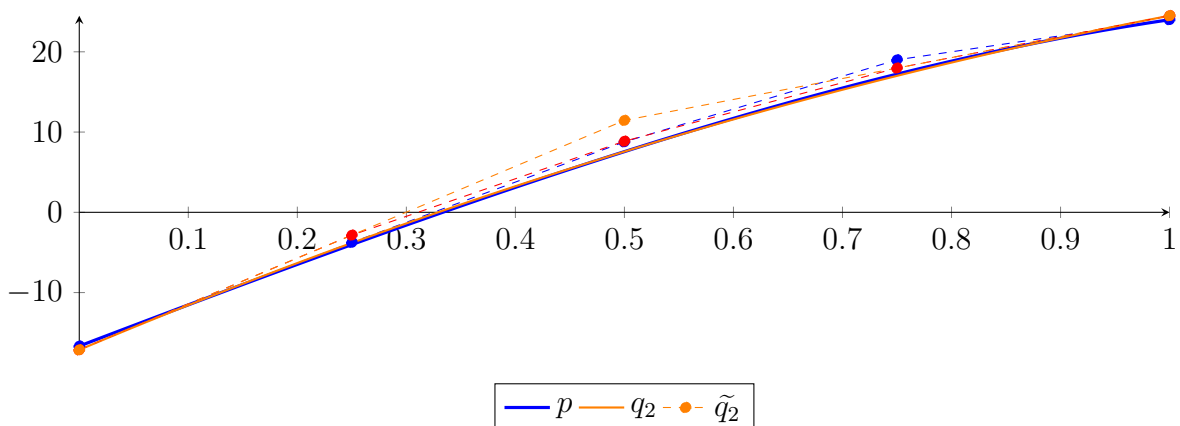
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -2.13607 \cdot 10^{-306}X^4 + 7.12024 \cdot 10^{-306}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

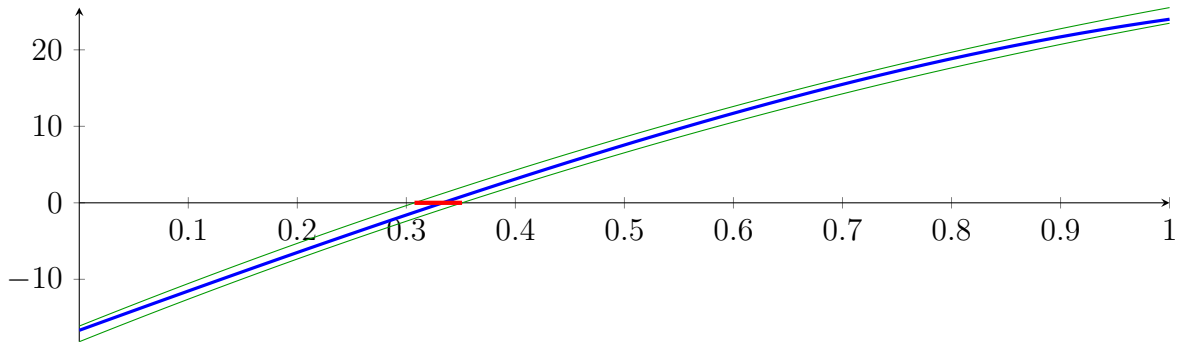
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

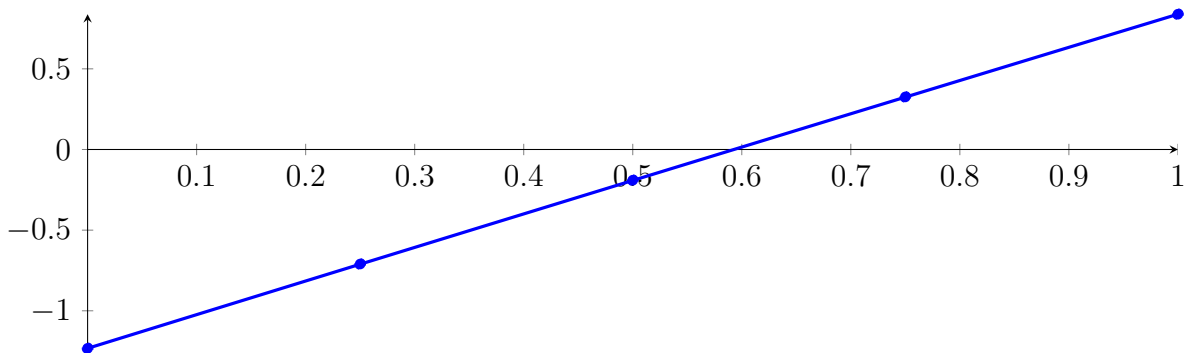
Longest intersection interval: 0.0436205

⇒ Selective recursion: interval 1: $[0.307477, 0.351097]$,

194.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

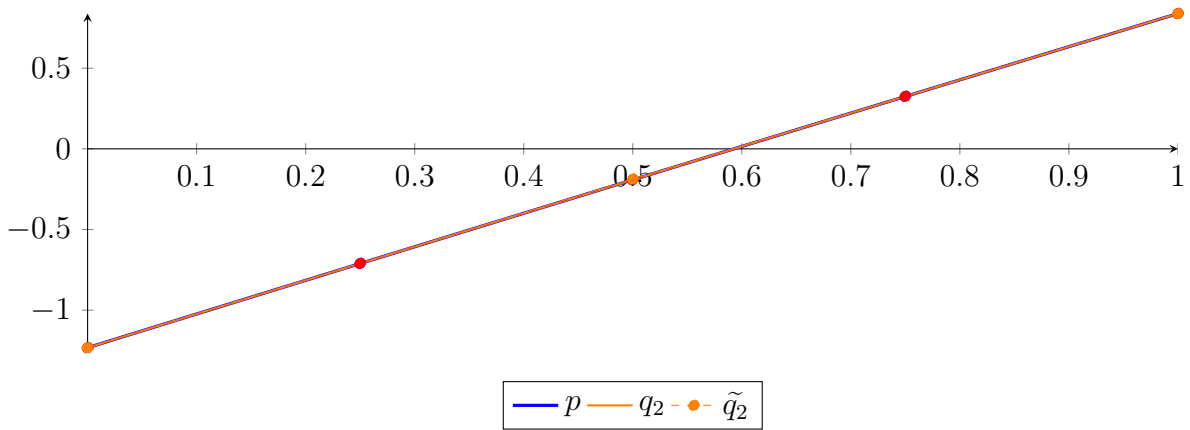
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -4.5614 \cdot 10^{-307} X^4 + 1.02353 \cdot 10^{-306} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

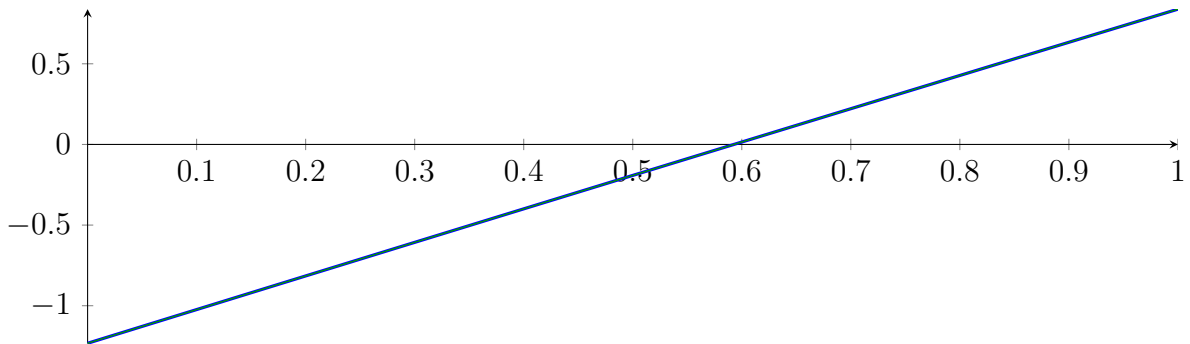
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

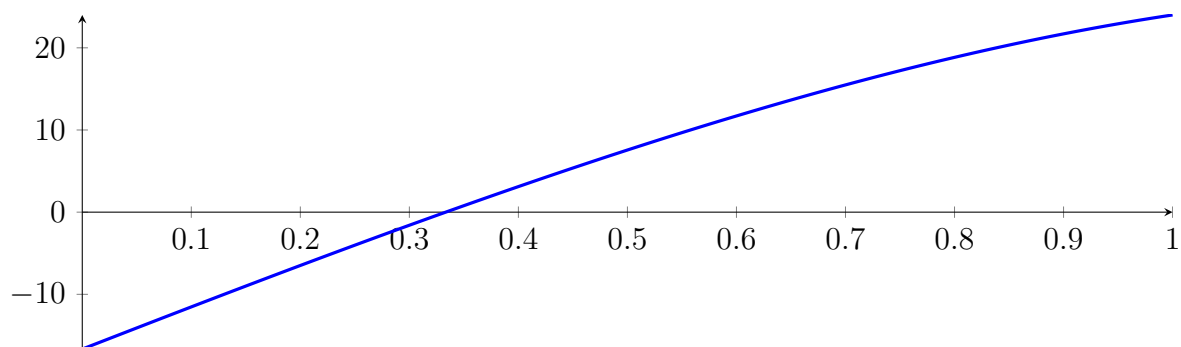
194.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Found root in interval $[0.333332, 0.333335]$ at recursion depth 3!

194.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333332, 0.333335]$$

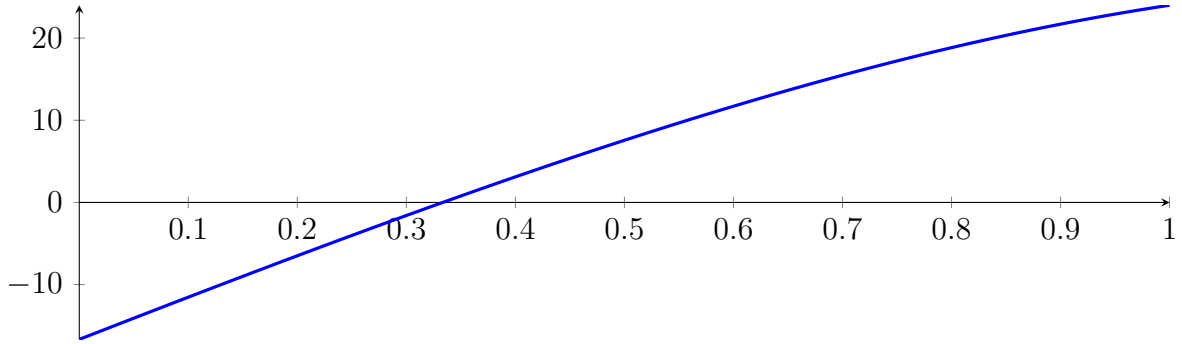
with precision $\varepsilon = 0.0001$.

195 Running CubeClip on f_4 with epsilon 4

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

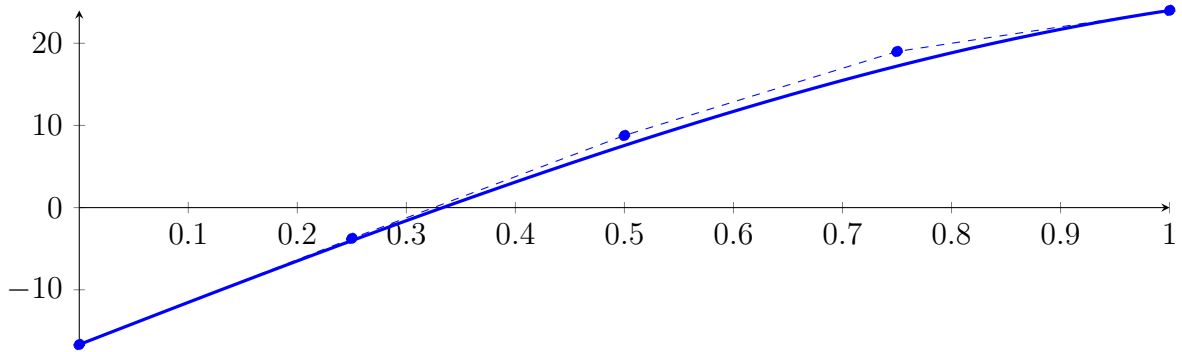
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



195.1 Recursion Branch 1 for Input Interval $[0, 1]$

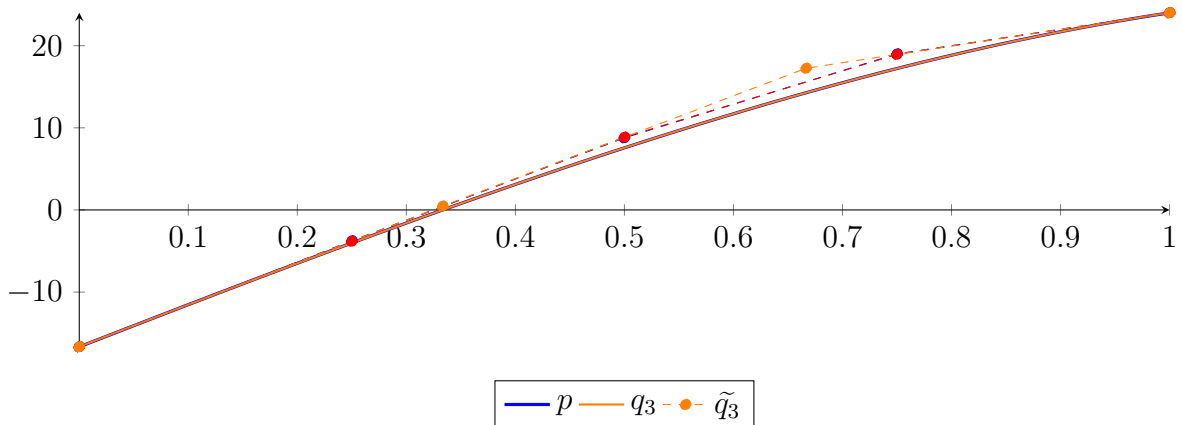
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= -3.56012 \cdot 10^{-307}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

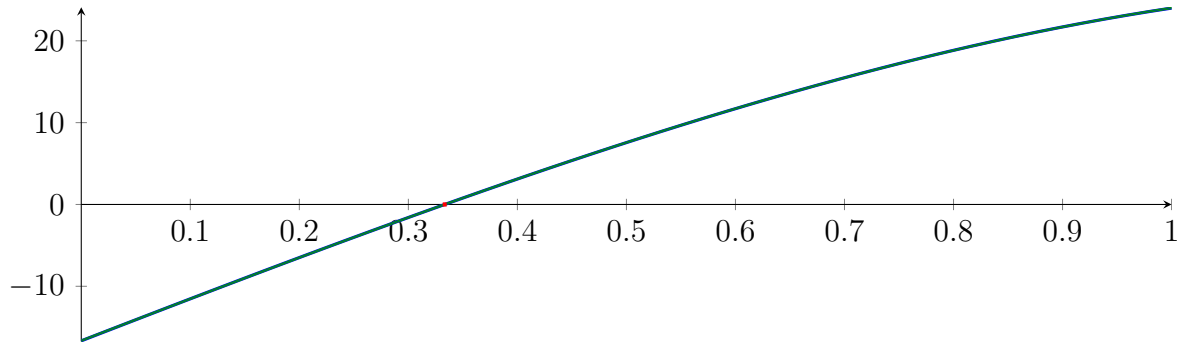
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

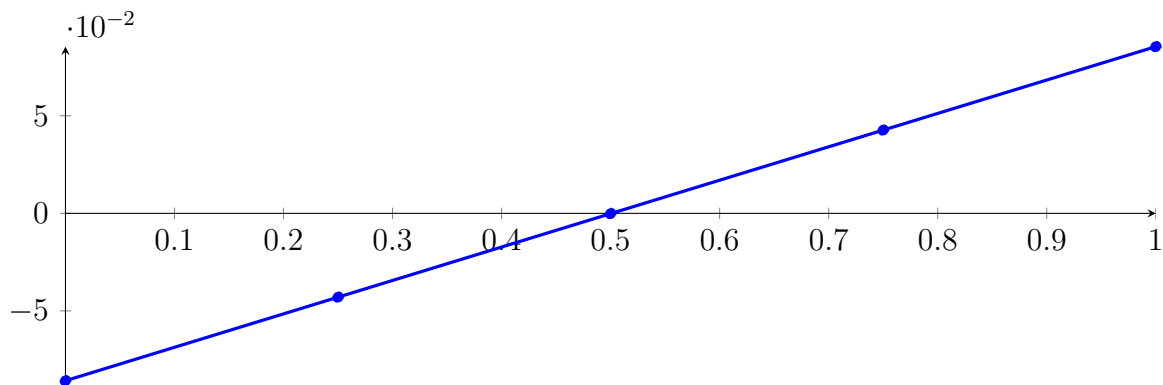
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

195.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

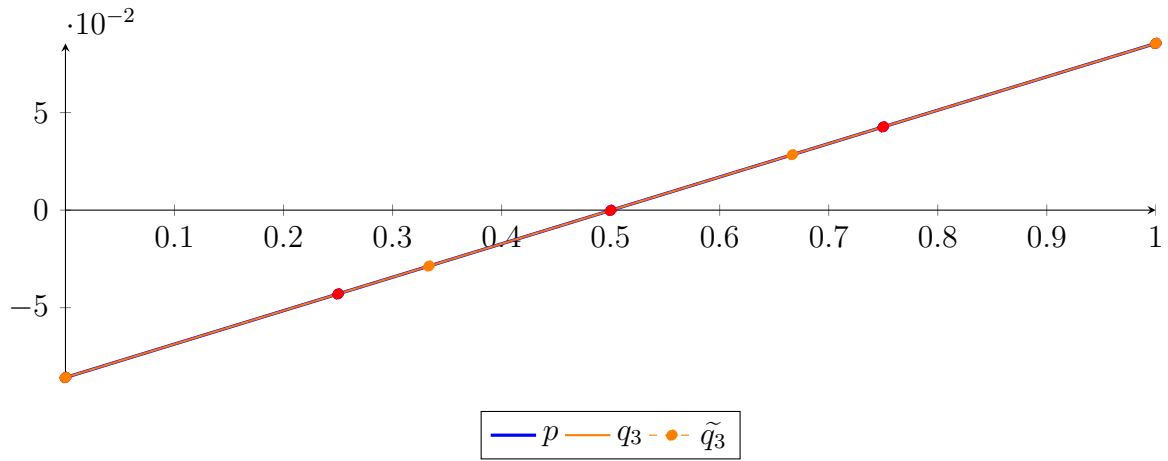
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -1.39067 \cdot 10^{-309} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m :

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

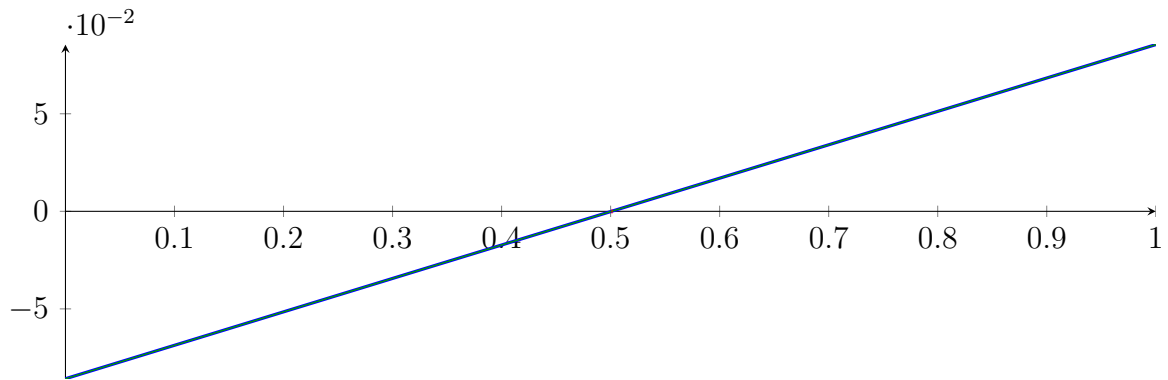
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\}$$

$$N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

Longest intersection interval: $1.7041 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

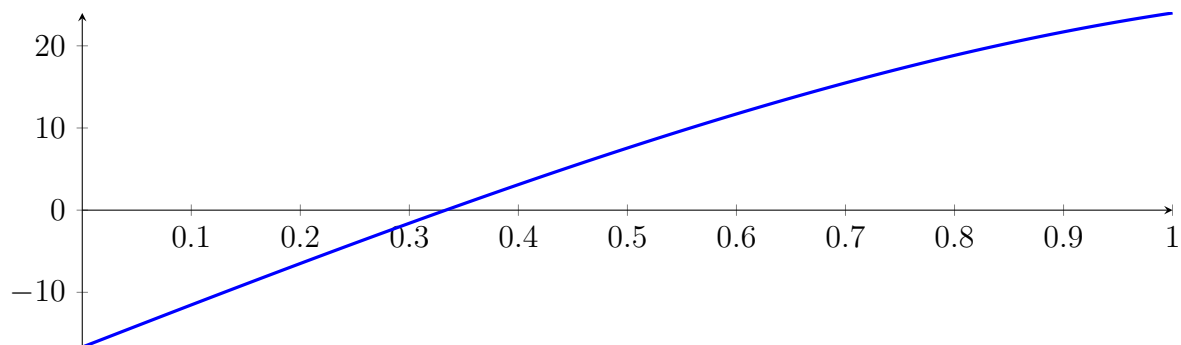
195.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

195.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

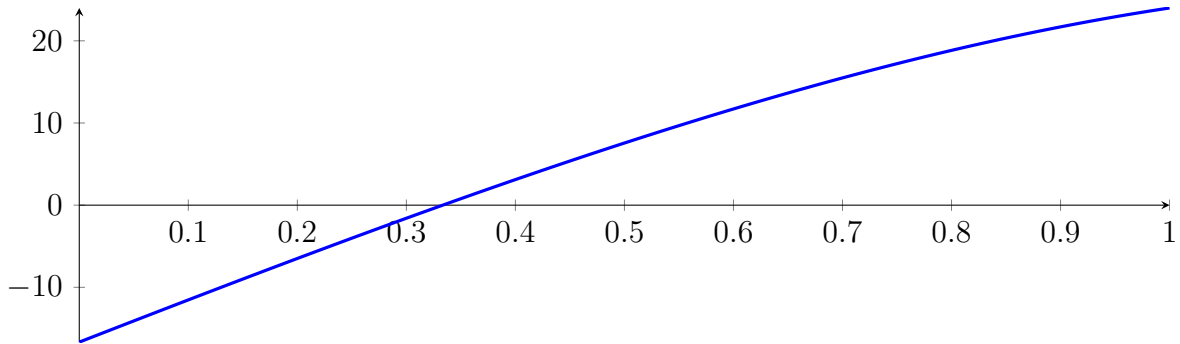
with precision $\varepsilon = 0.0001$.

196 Running BezClip on f_4 with epsilon 8

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

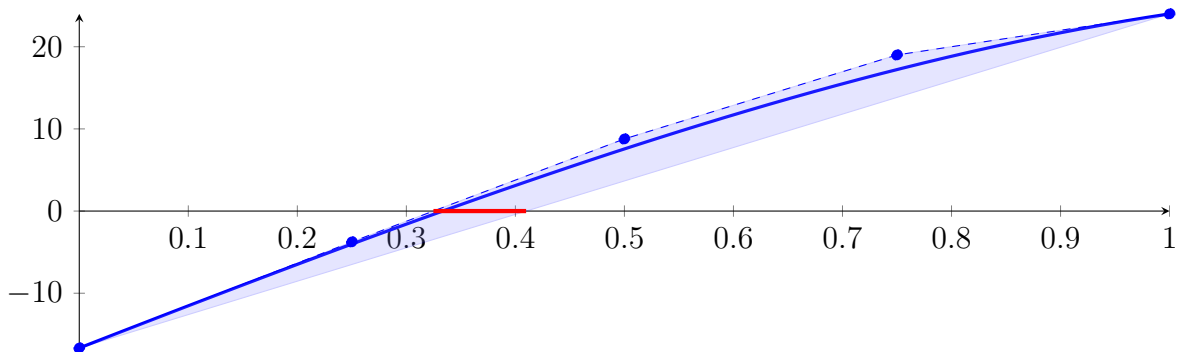
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



196.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

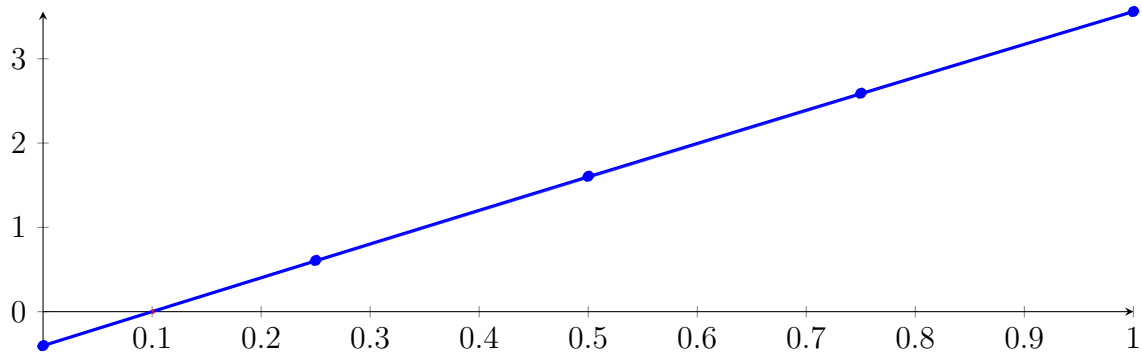
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

196.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

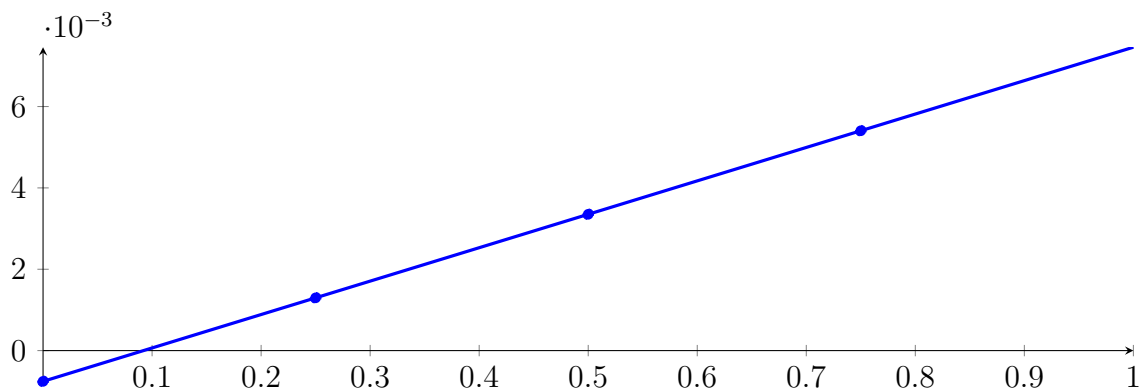
Longest intersection interval: 0.00203877

⇒ Selective recursion: interval 1: $[0.333317, 0.333491]$,

196.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01975 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

Longest intersection interval: $3.59185 \cdot 10^{-06}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

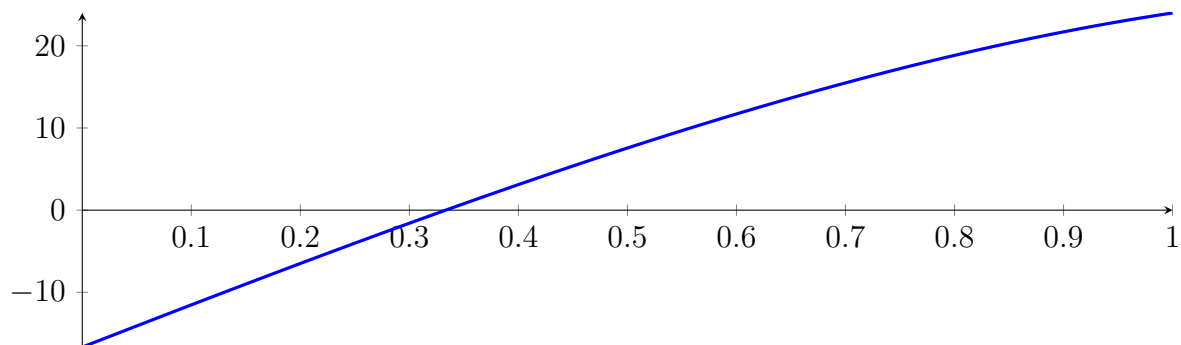
196.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

196.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

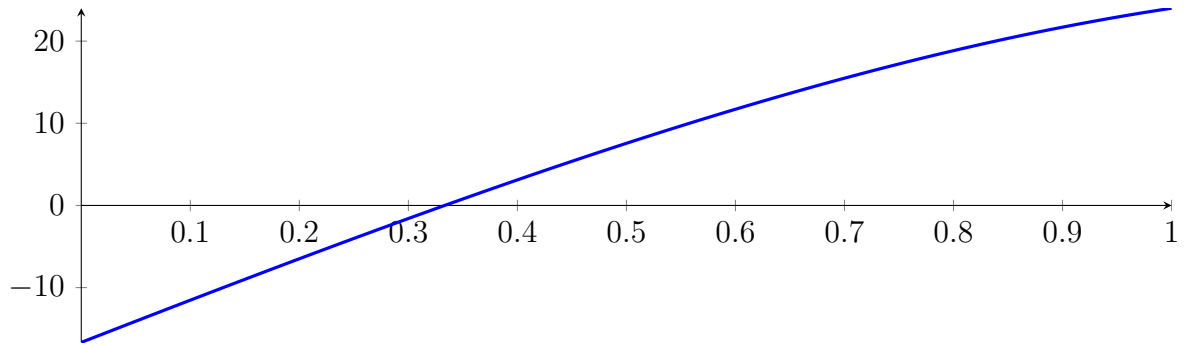
with precision $\varepsilon = 1 \cdot 10^{-08}$.

197 Running QuadClip on f_4 with epsilon 8

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

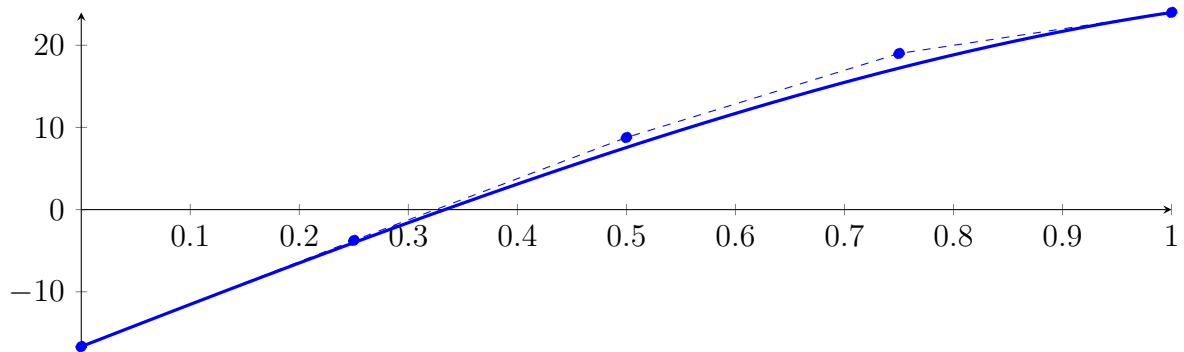
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



197.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

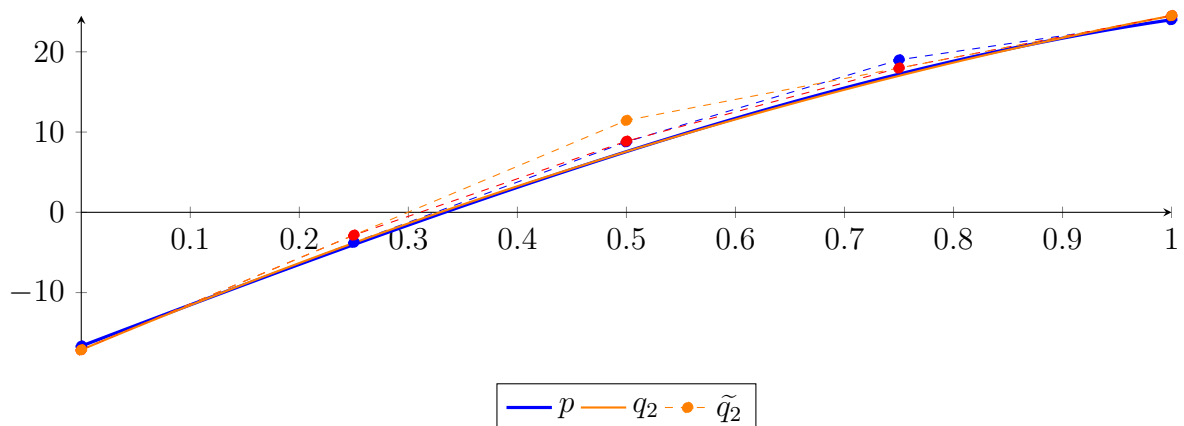
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -2.13607 \cdot 10^{-306}X^4 + 7.12024 \cdot 10^{-306}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

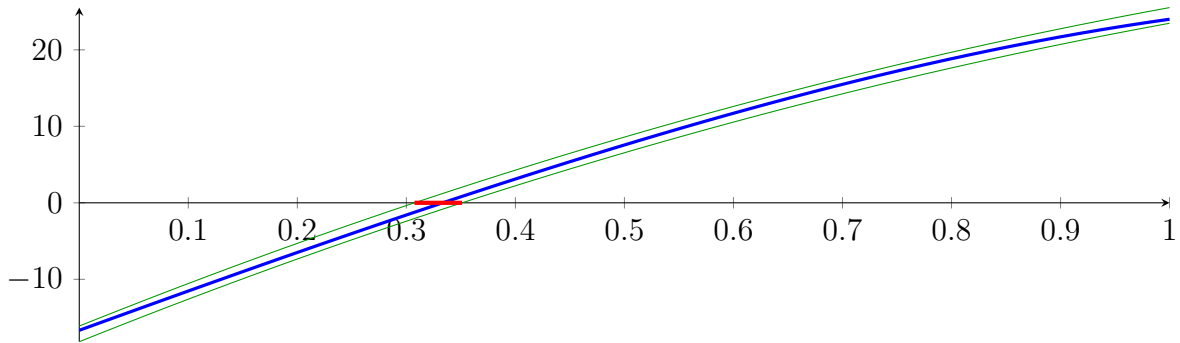
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

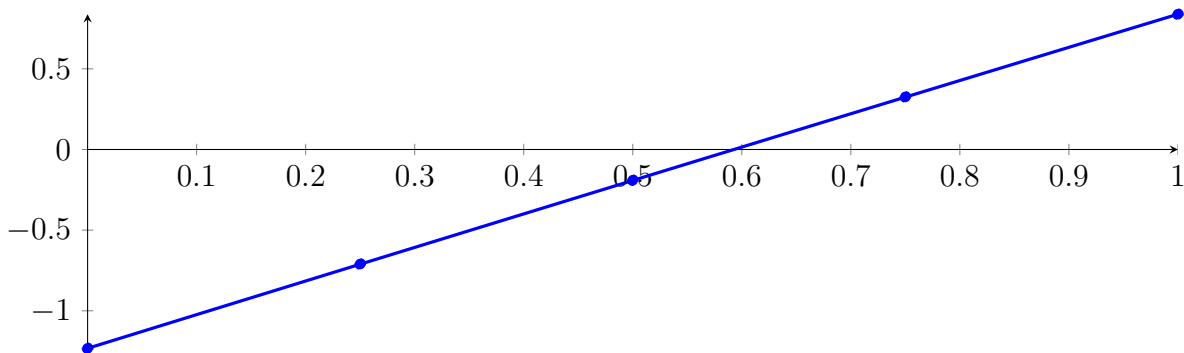
Longest intersection interval: 0.0436205

⇒ Selective recursion: interval 1: $[0.307477, 0.351097]$,

197.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

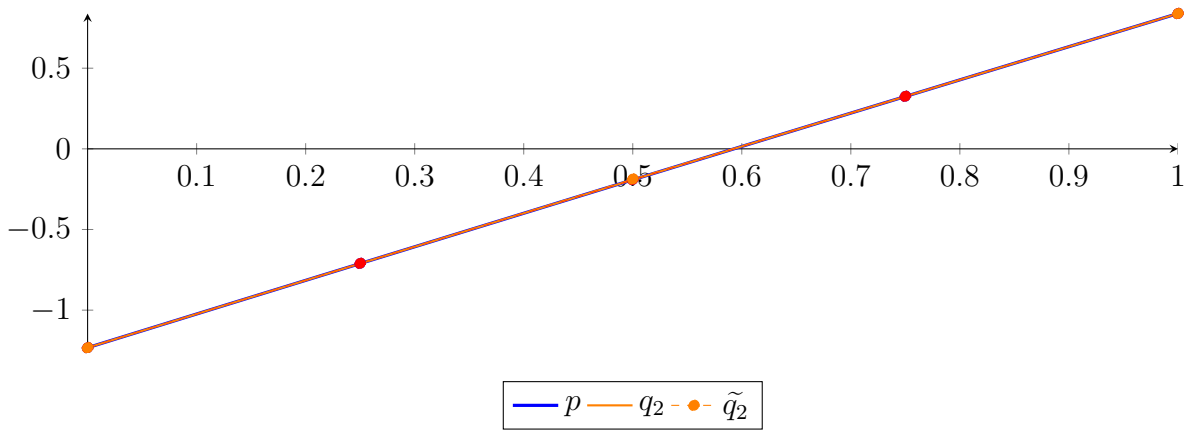
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -4.5614 \cdot 10^{-307} X^4 + 1.02353 \cdot 10^{-306} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

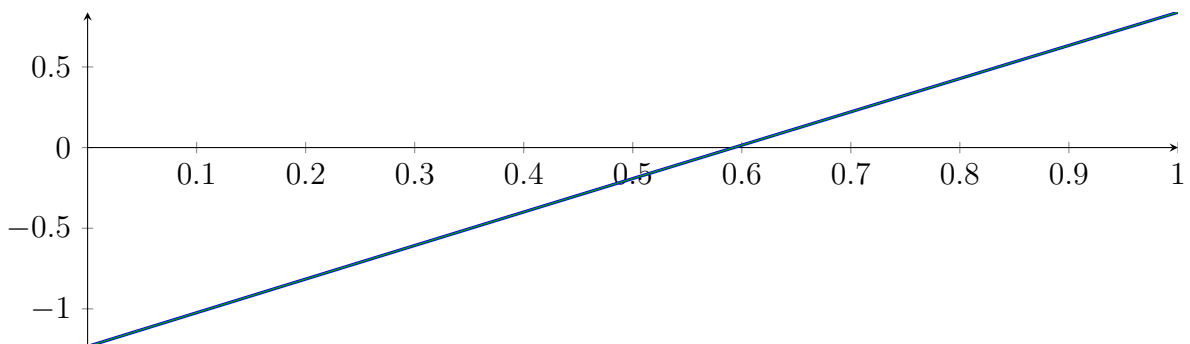
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

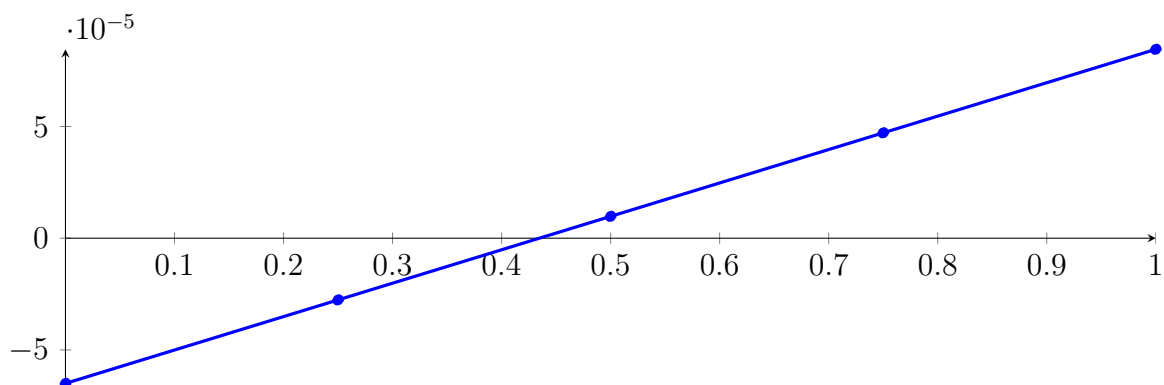
Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

197.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.9027 \cdot 10^{-23} X^4 - 2.82525 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05} \\ &= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \\ &\quad \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X) \end{aligned}$$



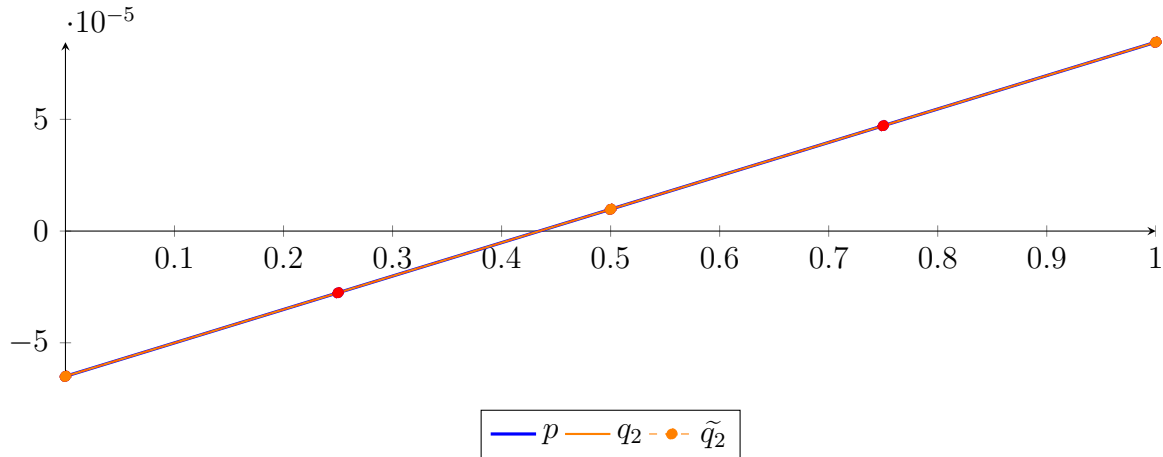
Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = 1.22227 \cdot 10^{-311} X^4 - 1.62969 \cdot 10^{-311} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.82526 \cdot 10^{-17}$.

Bounding polynomials M and m :

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

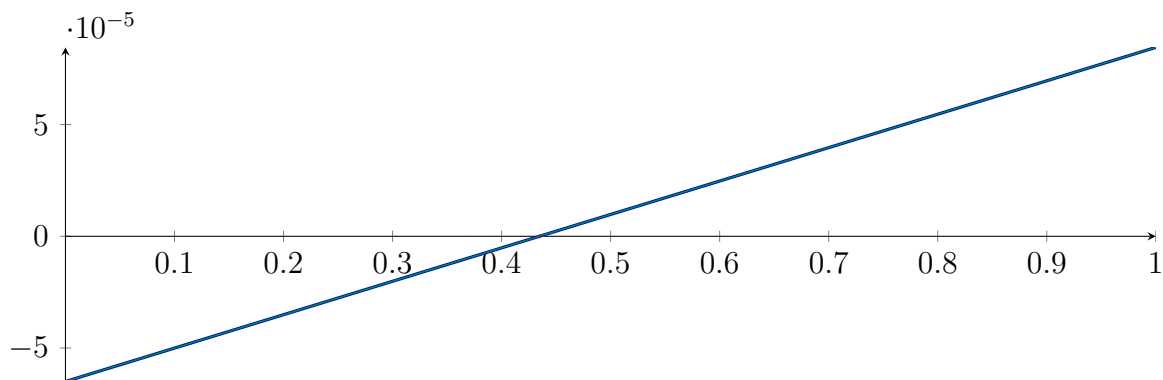
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.434685, 1.4089 \cdot 10^6\}$$

$$N(m) = \{0.434685, 1.4089 \cdot 10^6\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

Longest intersection interval: $3.77836 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

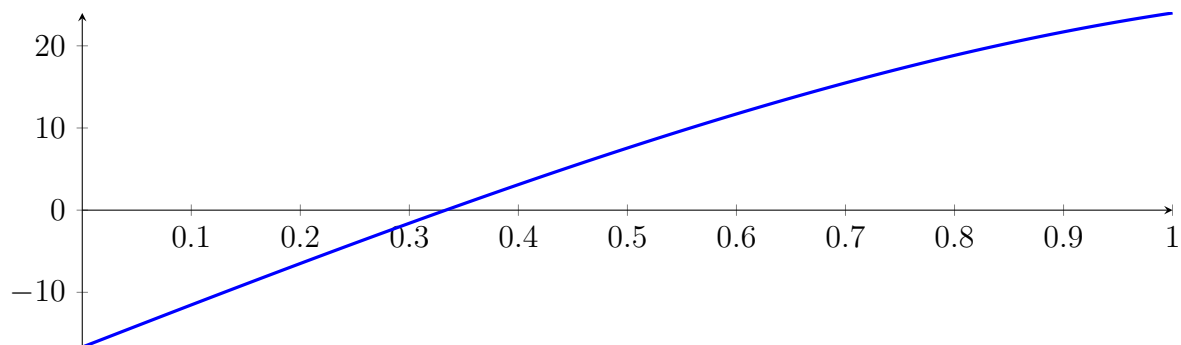
197.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

197.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

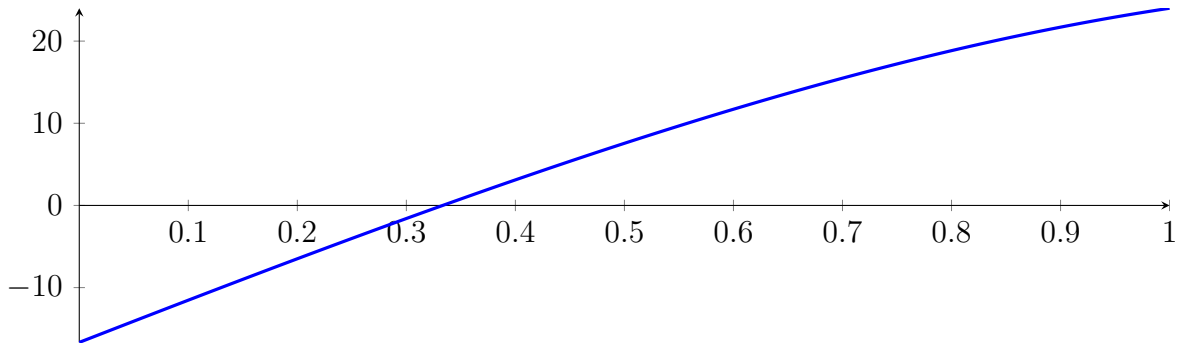
with precision $\varepsilon = 1 \cdot 10^{-08}$.

198 Running CubeClip on f_4 with epsilon 8

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

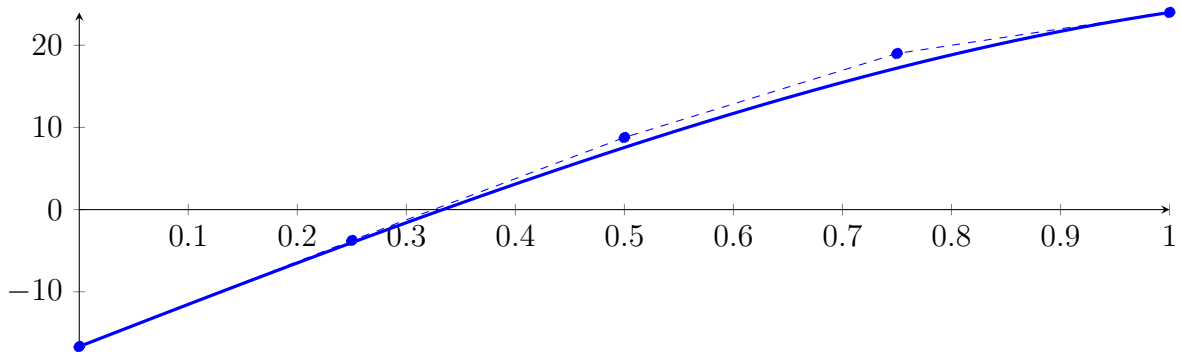
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



198.1 Recursion Branch 1 for Input Interval $[0, 1]$

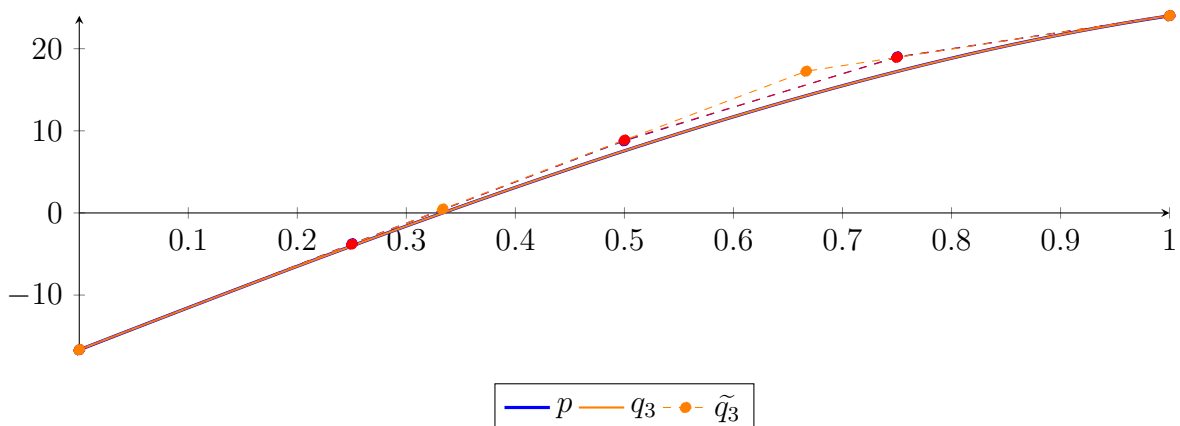
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= -3.56012 \cdot 10^{-307}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

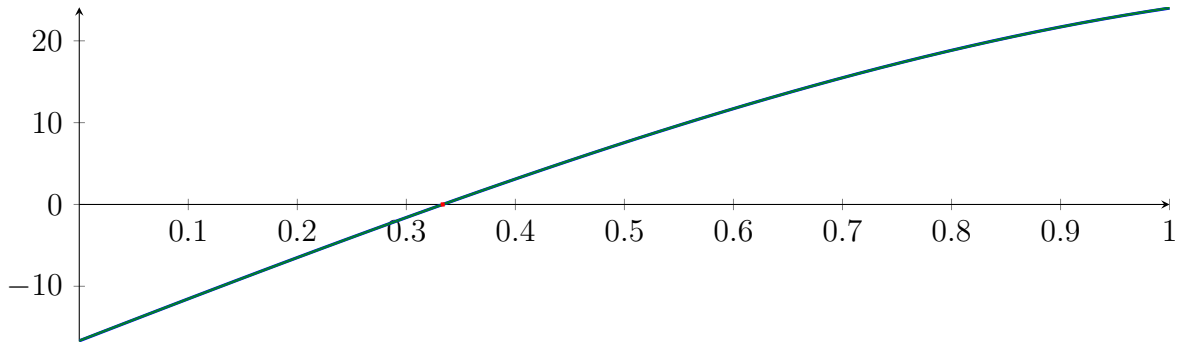
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

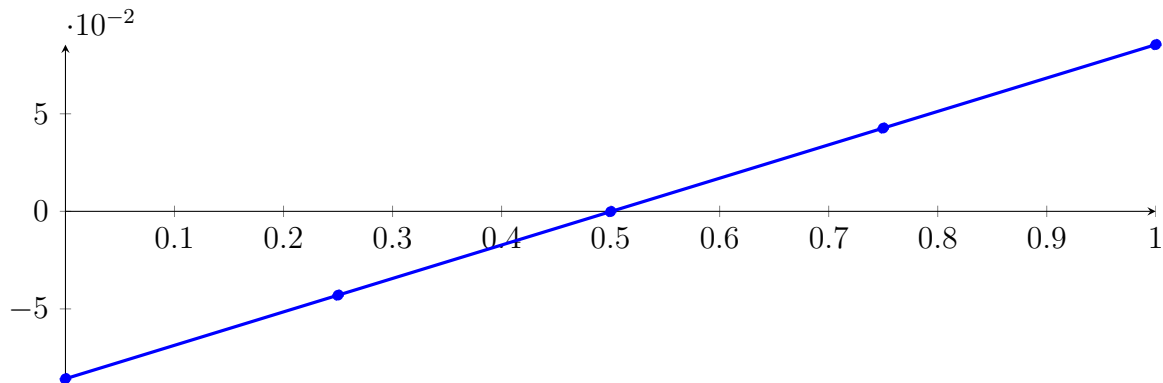
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

198.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

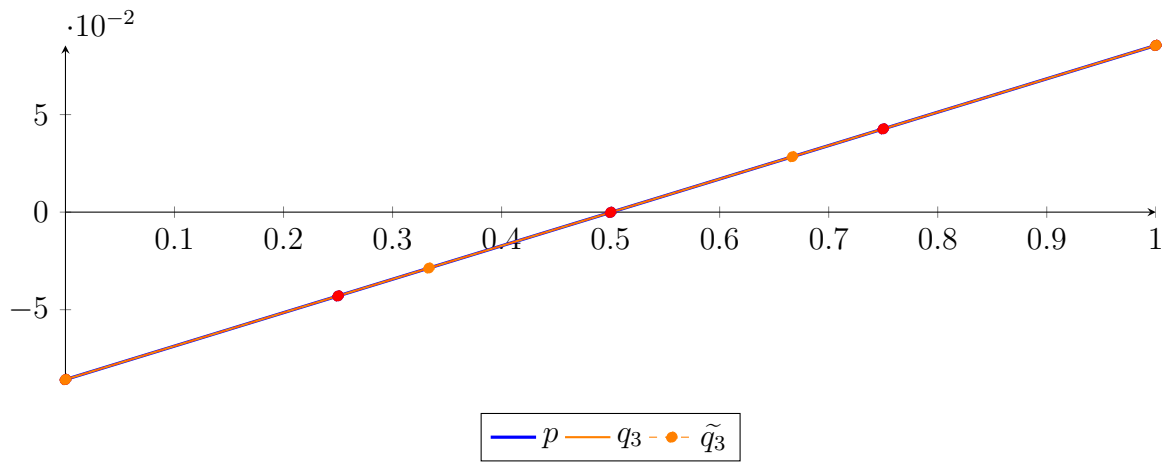
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -1.39067 \cdot 10^{-309} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m :

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

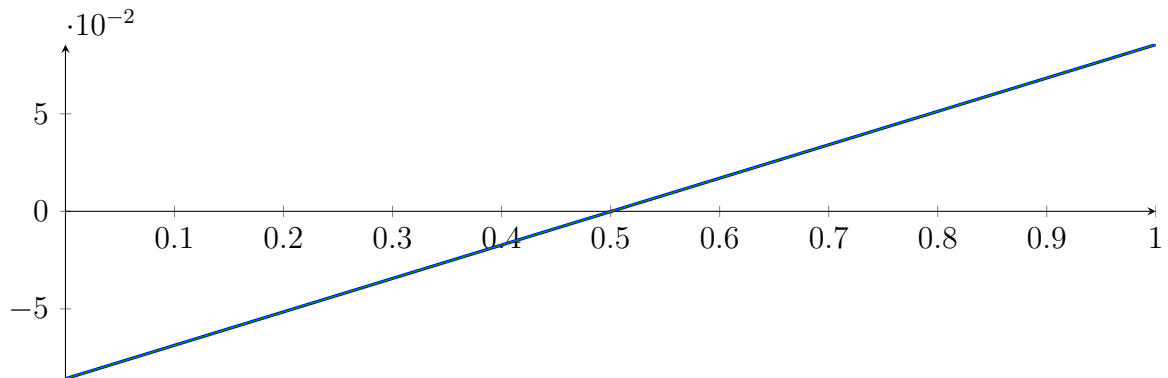
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\}$$

$$N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

Longest intersection interval: $1.7041 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

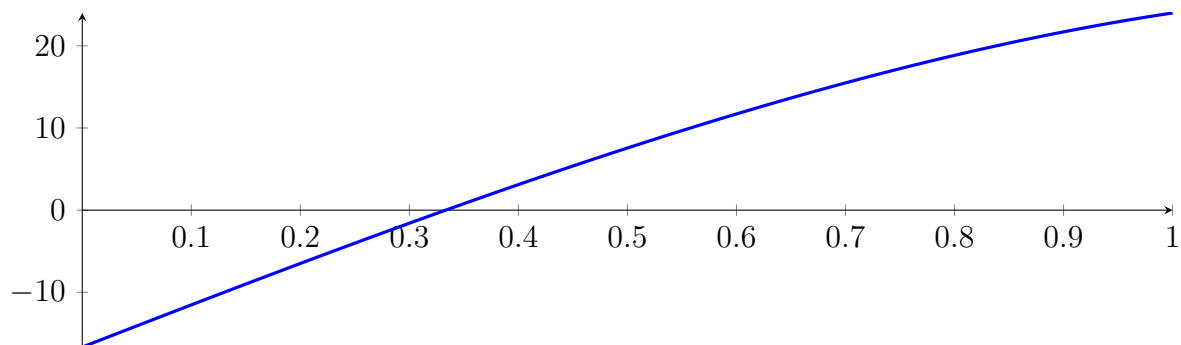
198.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

198.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

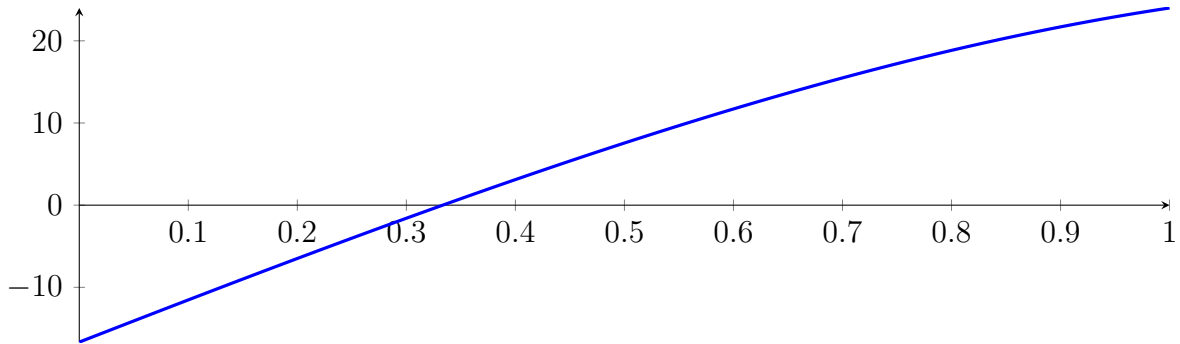
with precision $\varepsilon = 1 \cdot 10^{-08}$.

199 Running BezClip on f_4 with epsilon 16

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

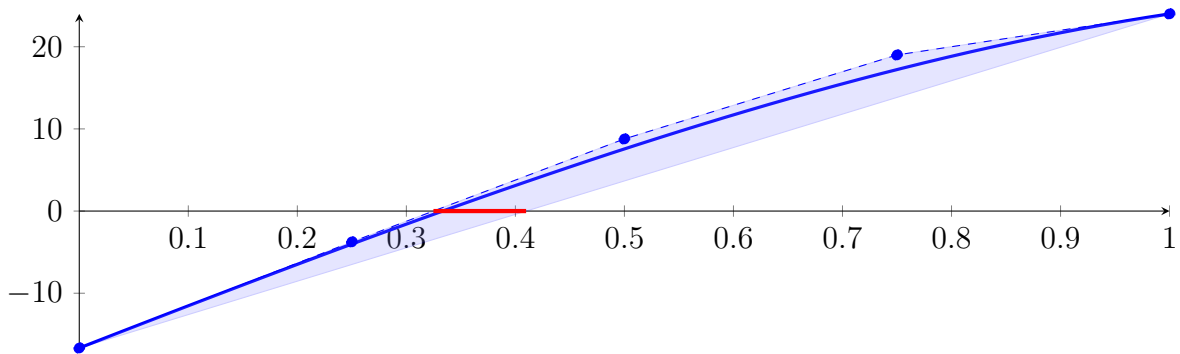
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



199.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

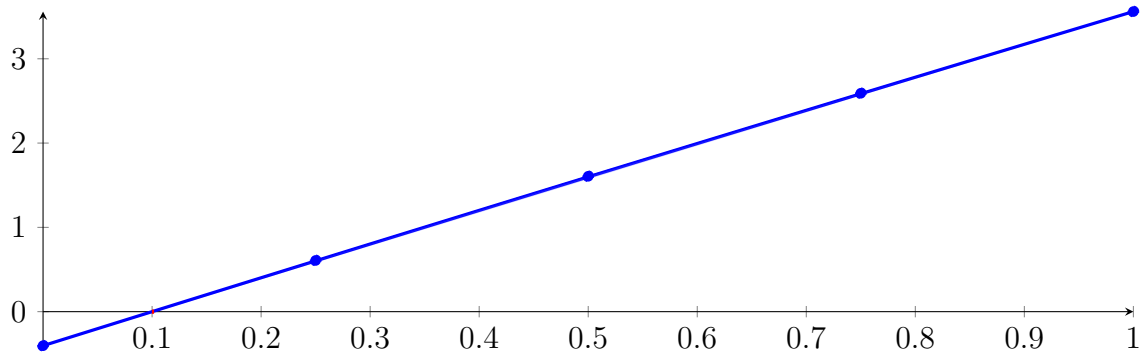
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

199.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

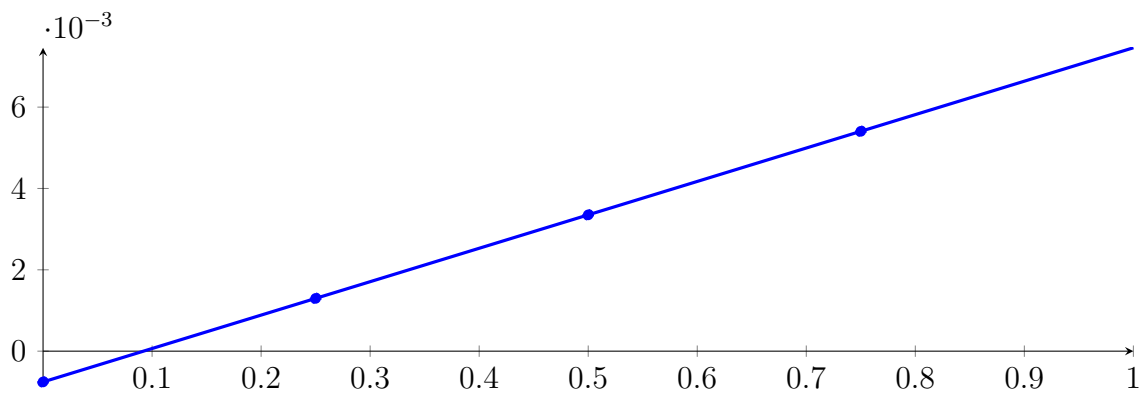
Longest intersection interval: 0.00203877

\implies Selective recursion: interval 1: $[0.333317, 0.333491]$,

199.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01975 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

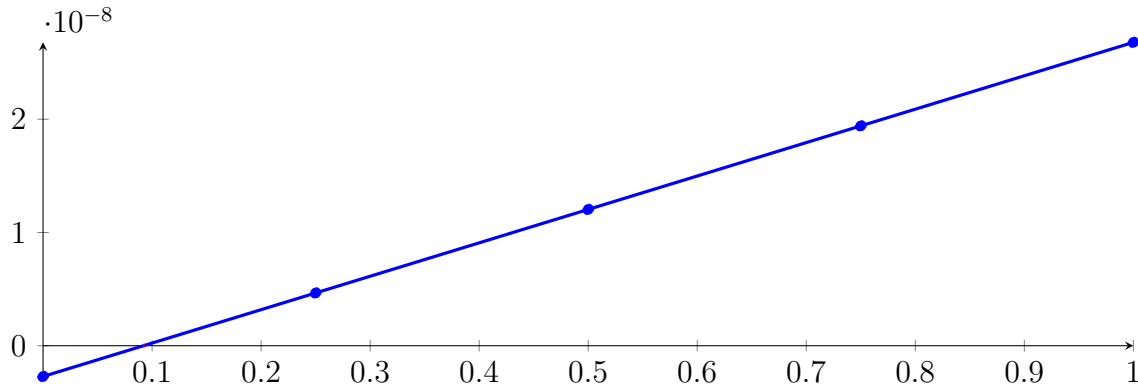
Longest intersection interval: $3.59185 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

199.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.50129 \cdot 10^{-37} X^4 - 2.17066 \cdot 10^{-27} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09} \\ &= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \\ &\quad \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.28975 \cdot 10^{-11}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

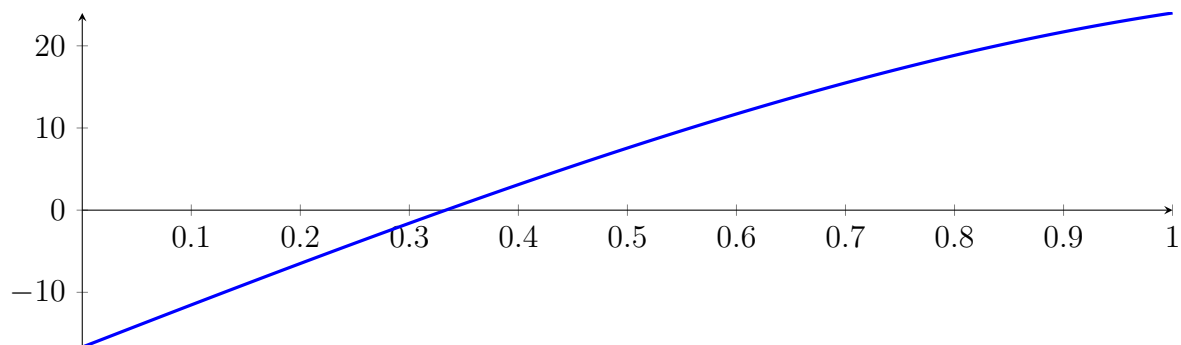
199.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

199.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

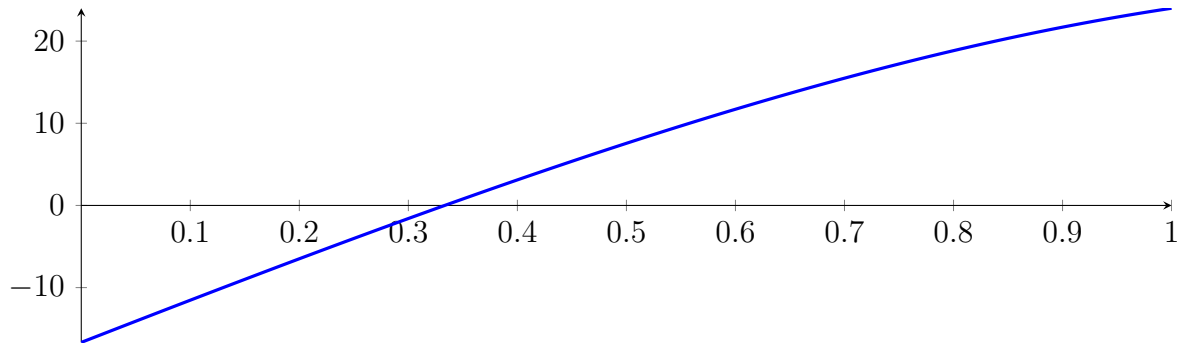
with precision $\varepsilon = 1 \cdot 10^{-16}$.

200 Running QuadClip on f_4 with epsilon 16

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

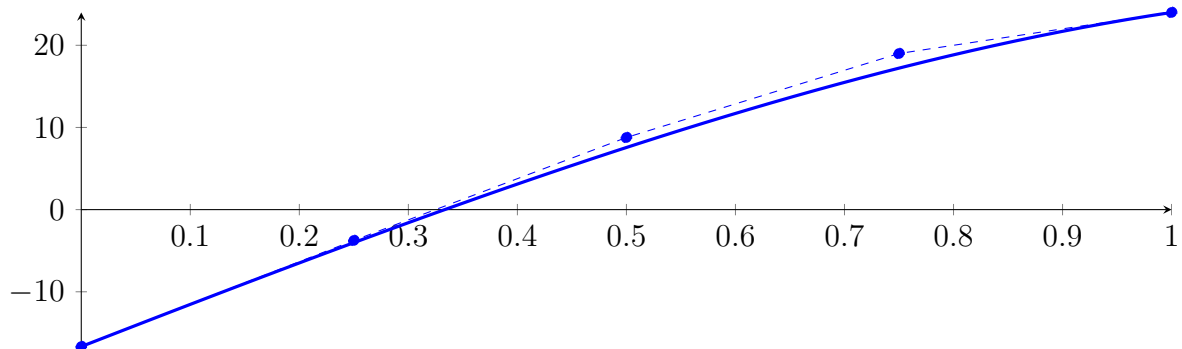
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



200.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

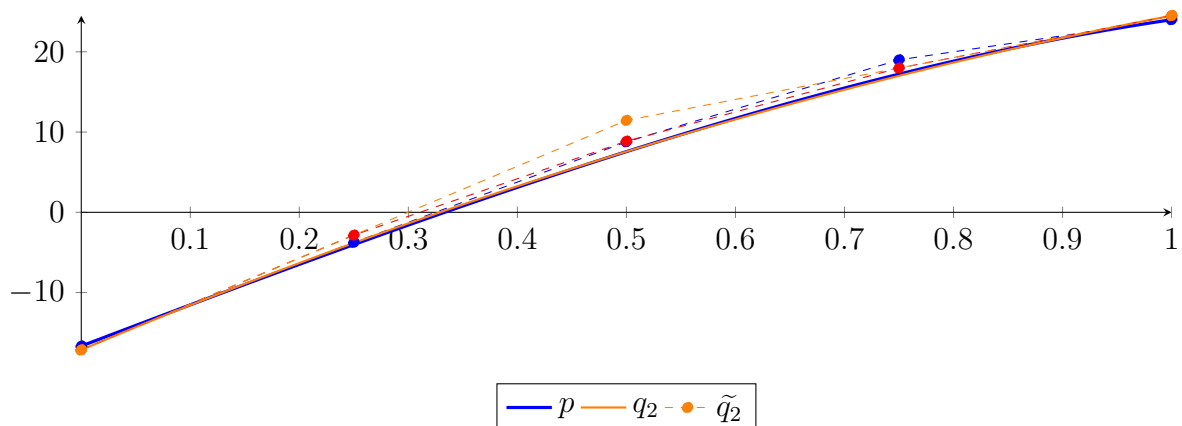
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -2.13607 \cdot 10^{-306}X^4 + 7.12024 \cdot 10^{-306}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

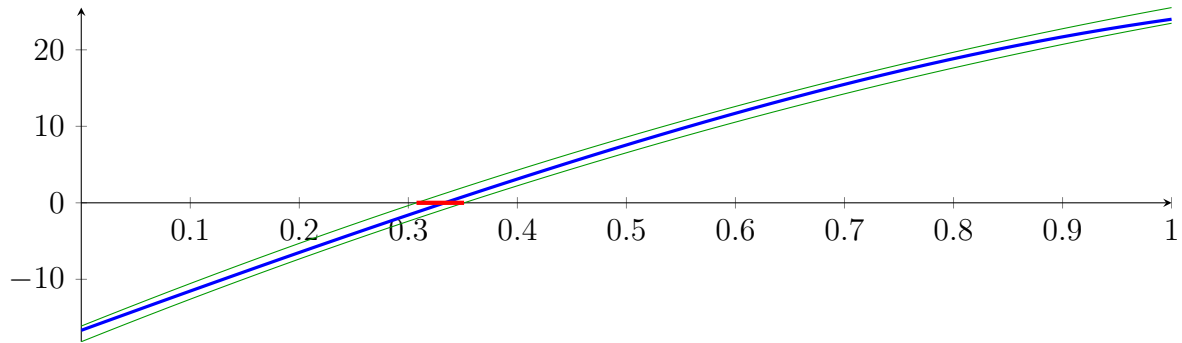
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

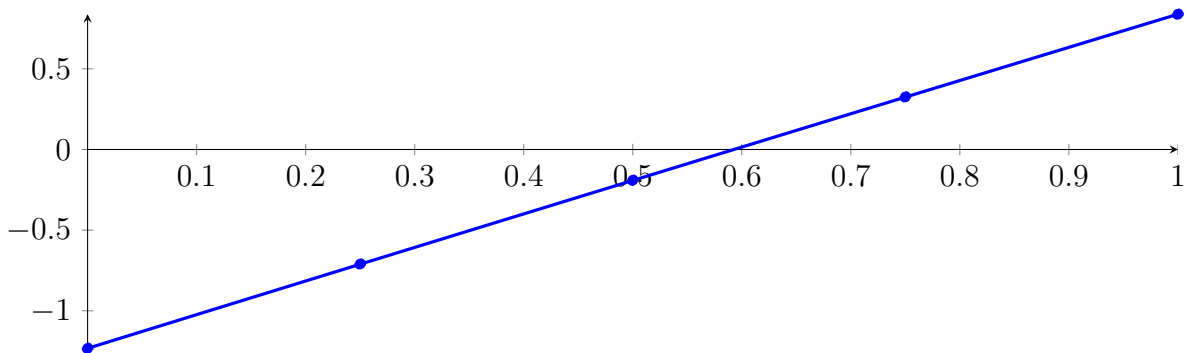
Longest intersection interval: 0.0436205

⇒ Selective recursion: **interval 1:** $[0.307477, 0.351097]$,

200.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

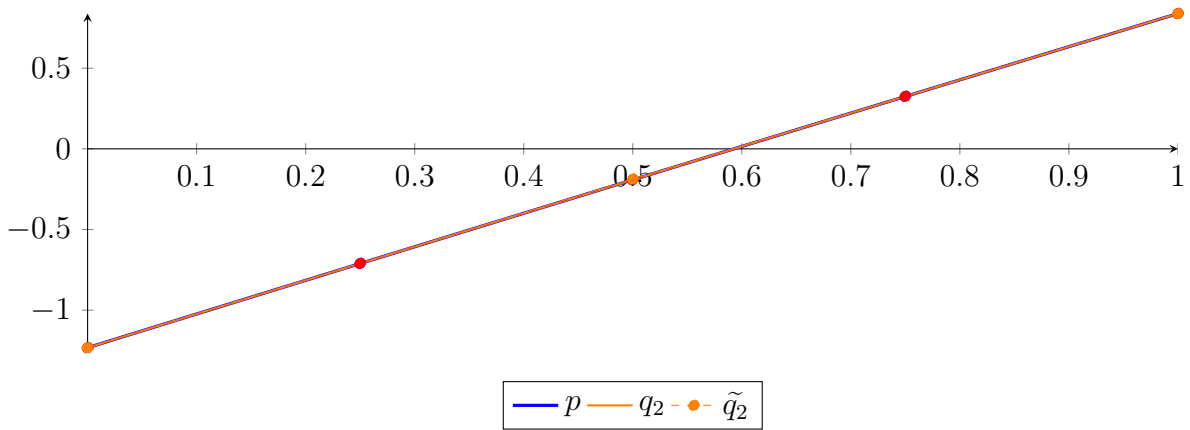
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -4.5614 \cdot 10^{-307} X^4 + 1.02353 \cdot 10^{-306} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

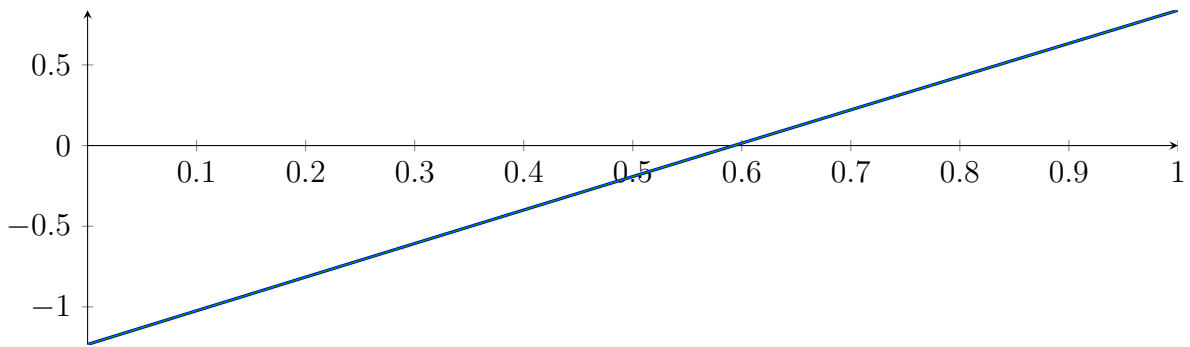
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

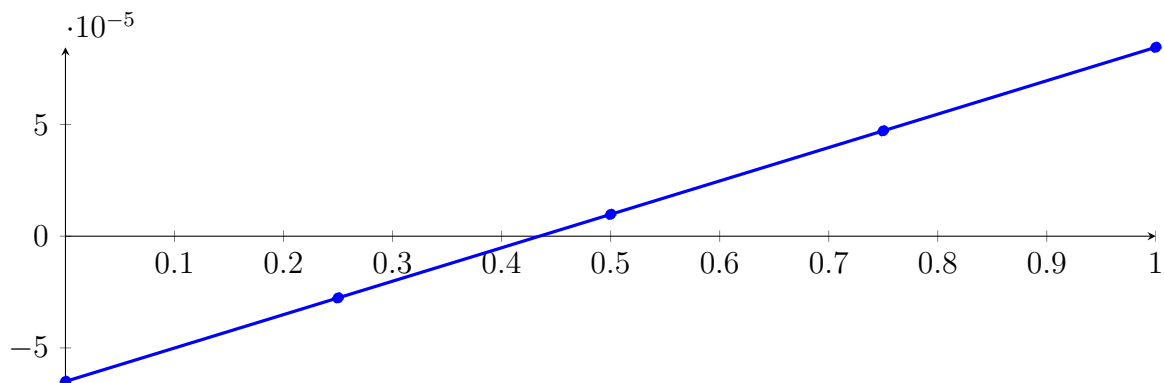
Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: **interval 1:** $[0.333332, 0.333335]$,

200.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.9027 \cdot 10^{-23} X^4 - 2.82525 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05} \\ &= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \\ &\quad \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X) \end{aligned}$$



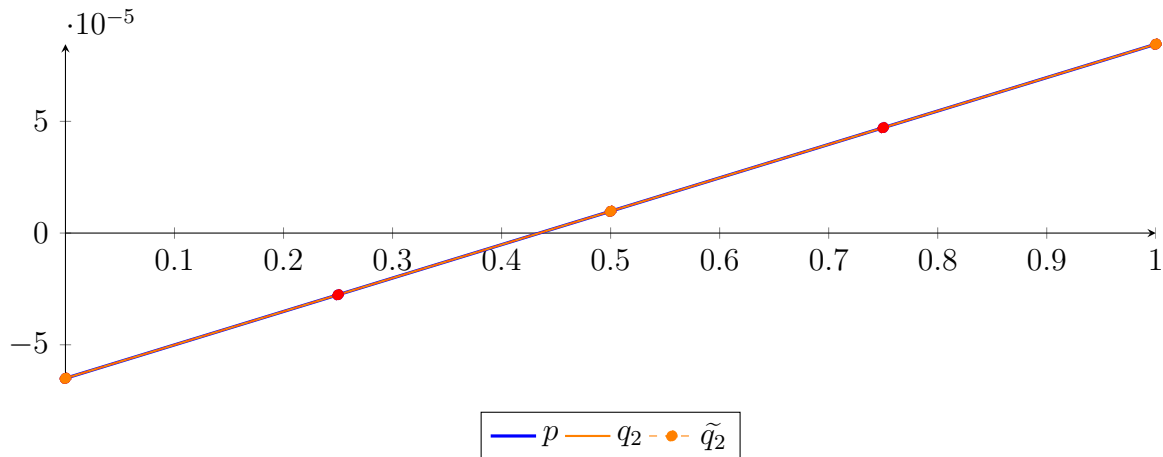
Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = 1.22227 \cdot 10^{-311} X^4 - 1.62969 \cdot 10^{-311} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.82526 \cdot 10^{-17}$.

Bounding polynomials M and m :

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

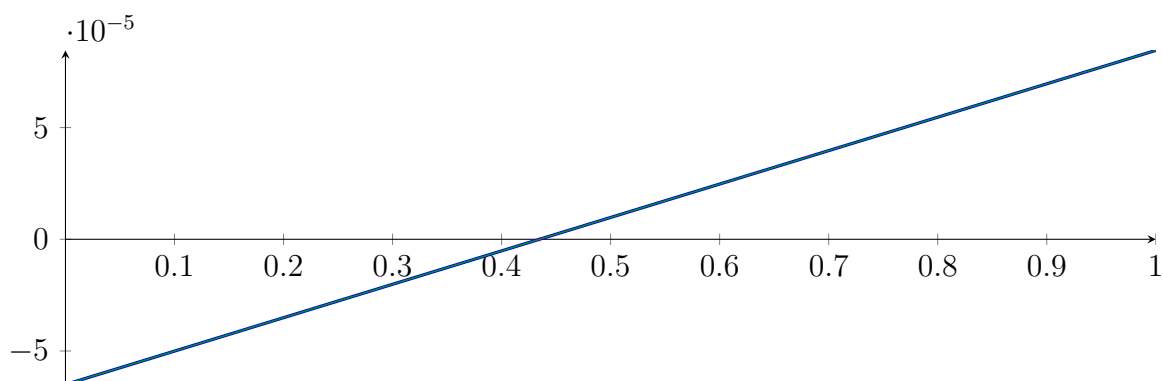
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.434685, 1.4089 \cdot 10^6\}$$

$$N(m) = \{0.434685, 1.4089 \cdot 10^6\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

Longest intersection interval: $3.77836 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

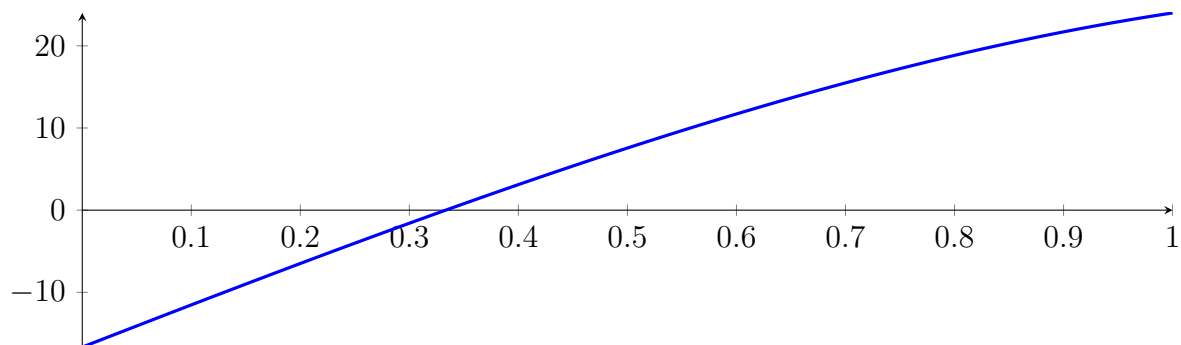
200.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

200.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

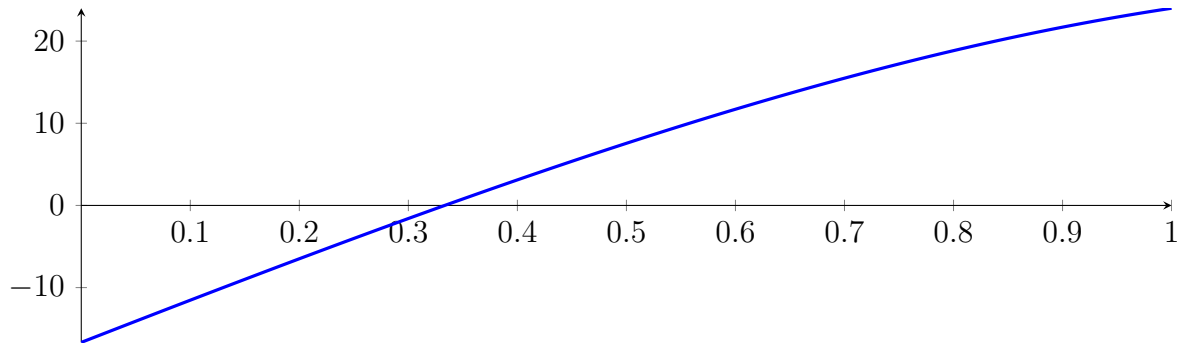
with precision $\varepsilon = 1 \cdot 10^{-16}$.

201 Running CubeClip on f_4 with epsilon 16

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

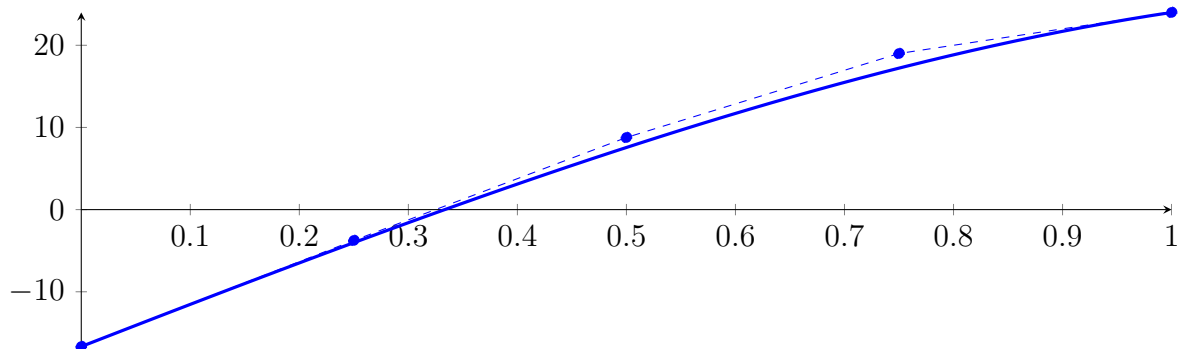
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



201.1 Recursion Branch 1 for Input Interval $[0, 1]$

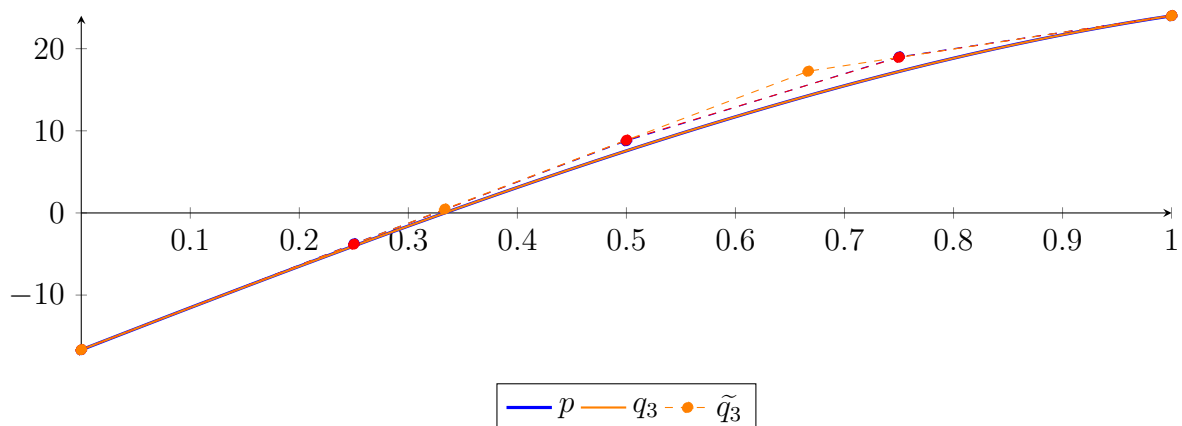
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= -3.56012 \cdot 10^{-307}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

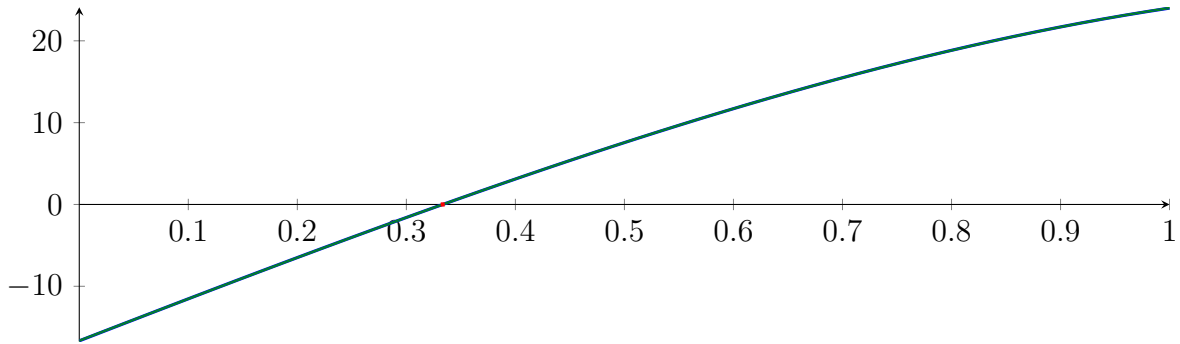
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

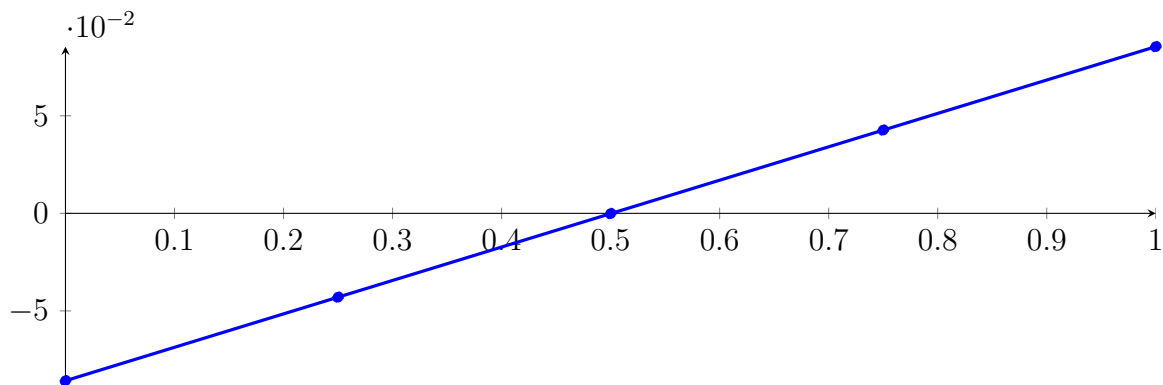
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

201.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

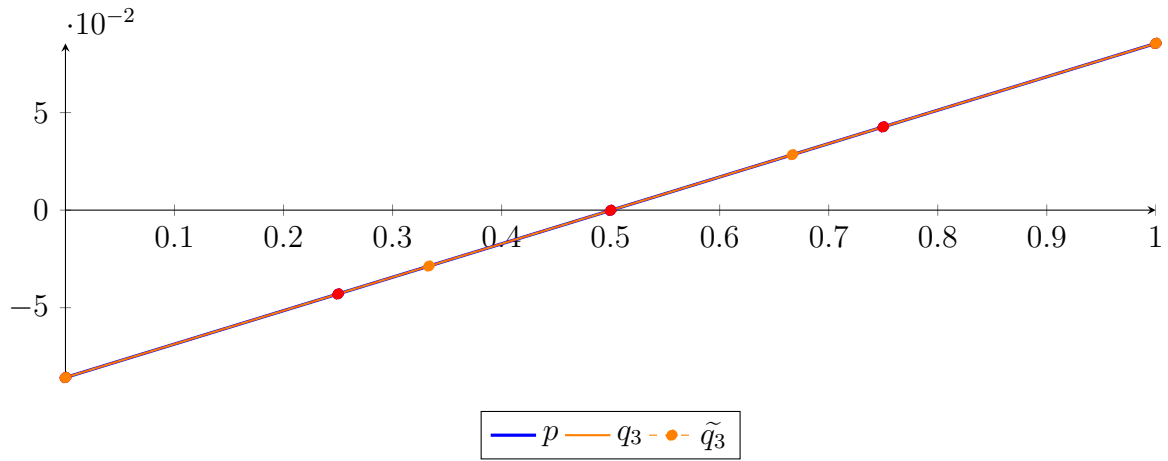
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -1.39067 \cdot 10^{-309} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m :

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

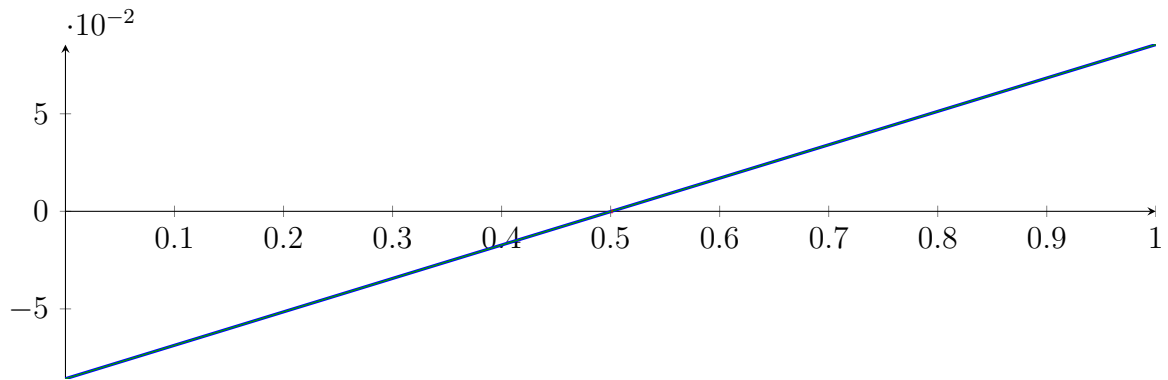
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\}$$

$$N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

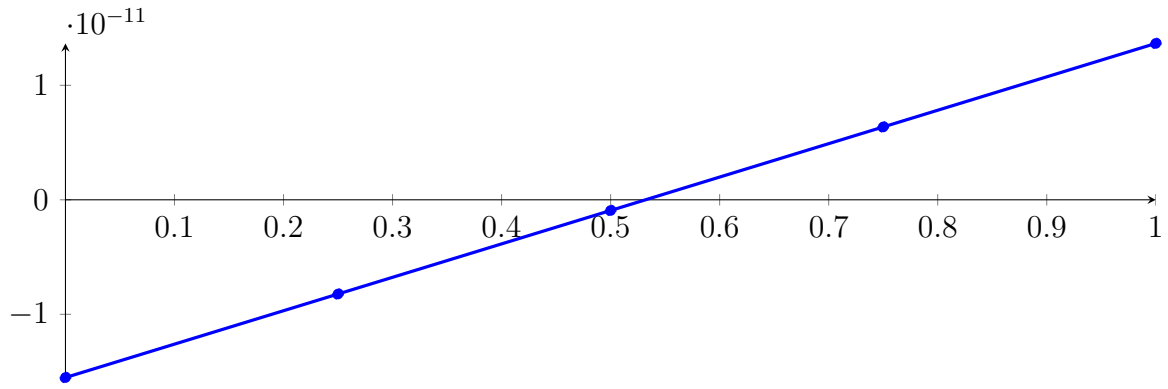
Longest intersection interval: $1.7041 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

201.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

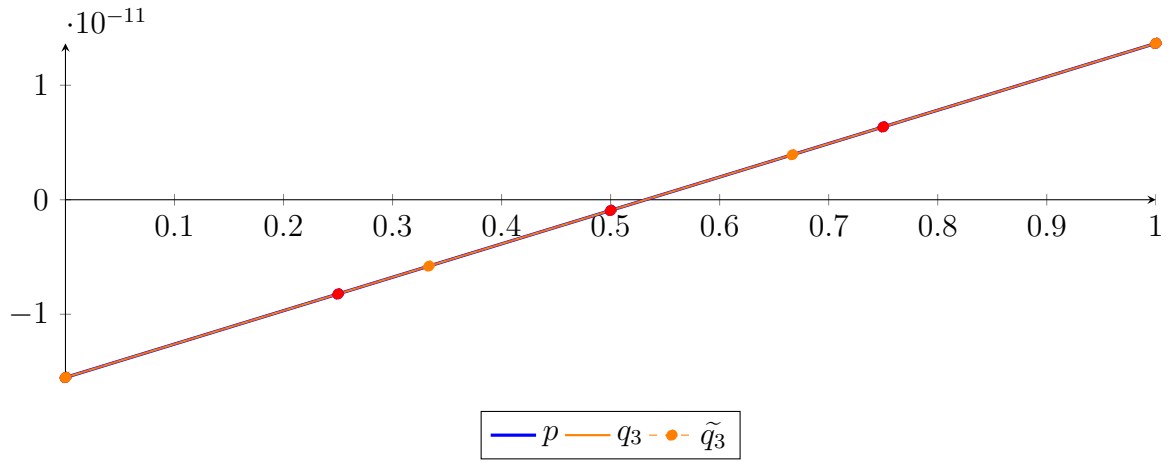
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.43544 \cdot 10^{-49} X^4 - 2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\ &= -1.55233 \cdot 10^{-11} B_{0,4}(X) - 8.22817 \cdot 10^{-12} B_{1,4}(X) - 9.33052 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36207 \cdot 10^{-12} B_{3,4}(X) + 1.36572 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 &= -1.55233 \cdot 10^{-11} B_{0,3} - 5.79646 \cdot 10^{-12} B_{1,3} + 3.93036 \cdot 10^{-12} B_{2,3} + 1.36572 \cdot 10^{-11} B_{3,3} \\
 \tilde{q}_3 &= -3.23791 \cdot 10^{-319} X^4 - 2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 &= -1.55233 \cdot 10^{-11} B_{0,4} - 8.22817 \cdot 10^{-12} B_{1,4} - 9.33052 \cdot 10^{-13} B_{2,4} + 6.36207 \cdot 10^{-12} B_{3,4} + 1.36572 \cdot 10^{-11} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.23038 \cdot 10^{-50}$.

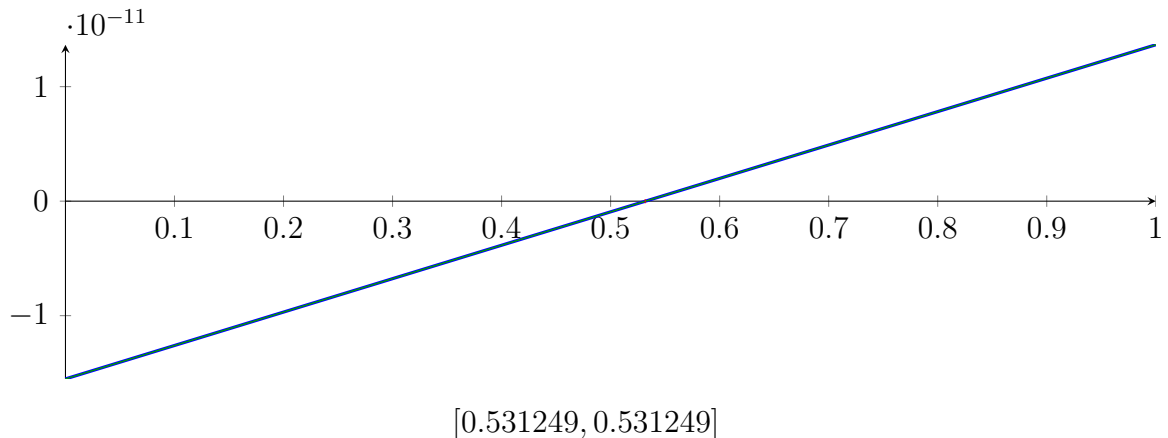
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 m &= -2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-4.81371 \cdot 10^{12}, 0.531249, 2.88823 \cdot 10^{12}\} \quad N(m) = \{-4.81371 \cdot 10^{12}, 0.531249, 2.88823 \cdot 10^{12}\}$$

Intersection intervals:



Longest intersection interval: $8.43287 \cdot 10^{-40}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

201.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

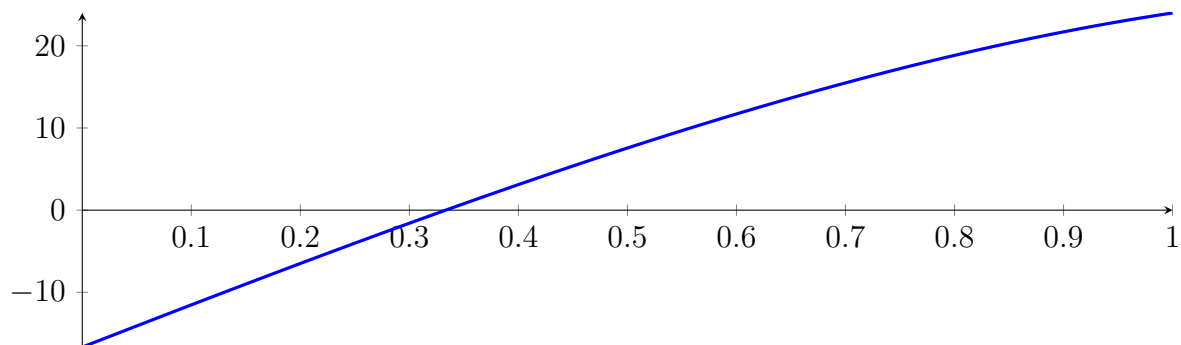
Reached interval [0.333333, 0.333333] **without sign change** at depth 4!

$p(0) = -2.11876e-14$ - $p(1) -2.11876e-14$

201.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

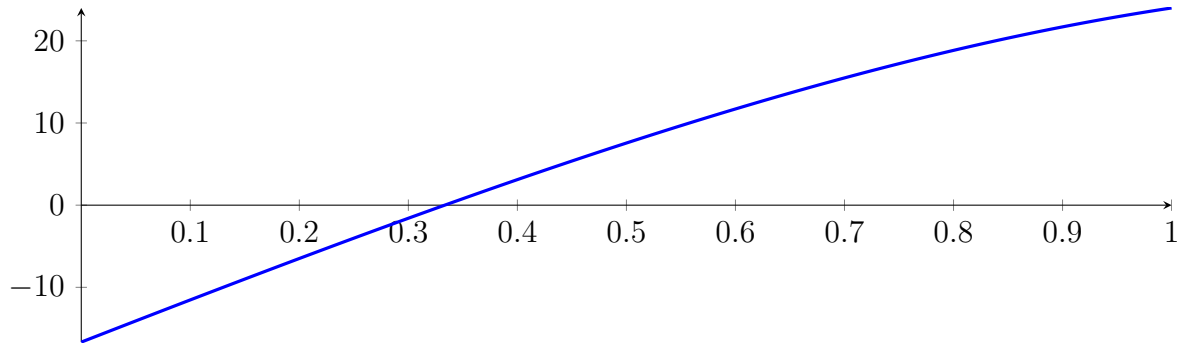
with precision $\varepsilon = 1 \cdot 10^{-16}$.

202 Running BezClip on f_4 with epsilon 32

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

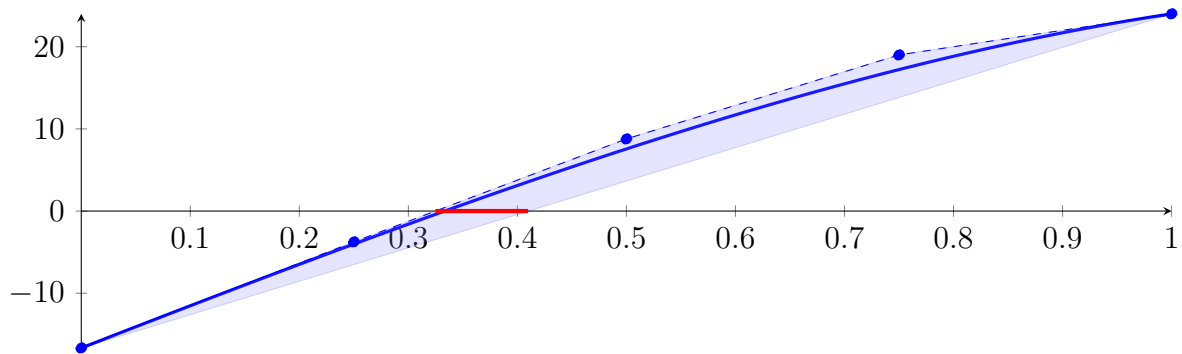
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



202.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

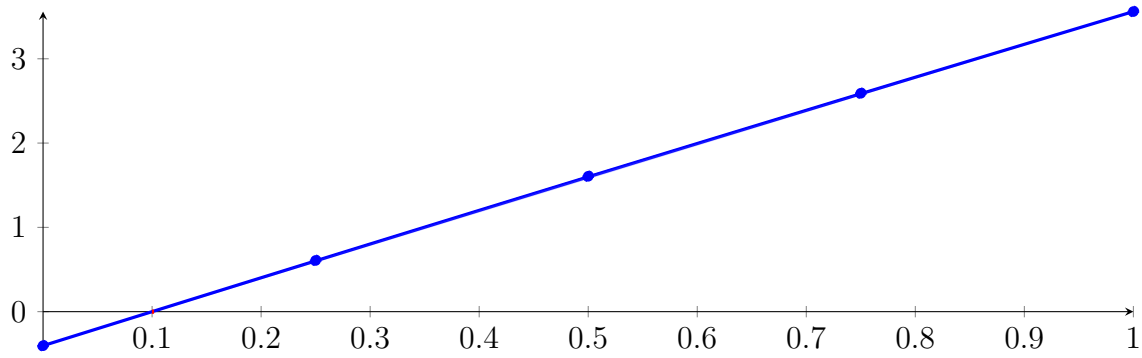
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

202.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

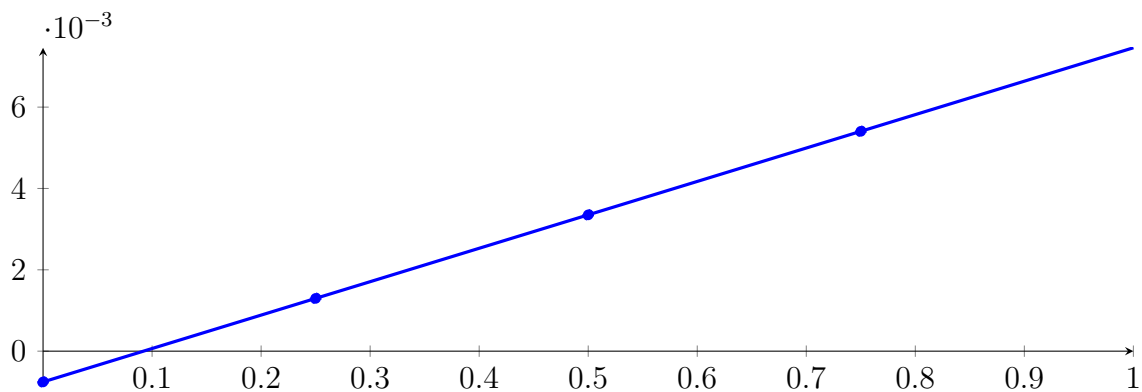
Longest intersection interval: 0.00203877

\implies Selective recursion: interval 1: $[0.333317, 0.333491]$,

202.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01975 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

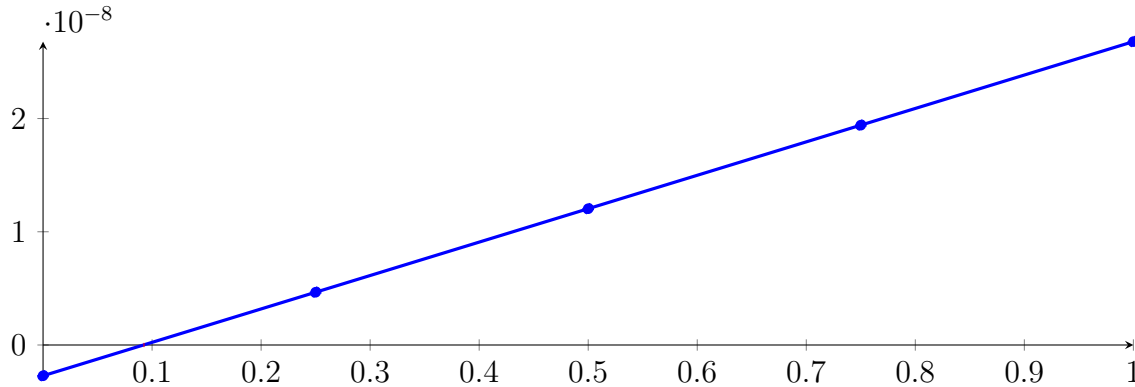
Longest intersection interval: $3.59185 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

202.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.50129 \cdot 10^{-37} X^4 - 2.17066 \cdot 10^{-27} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09} \\
 &= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \\
 &\quad \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

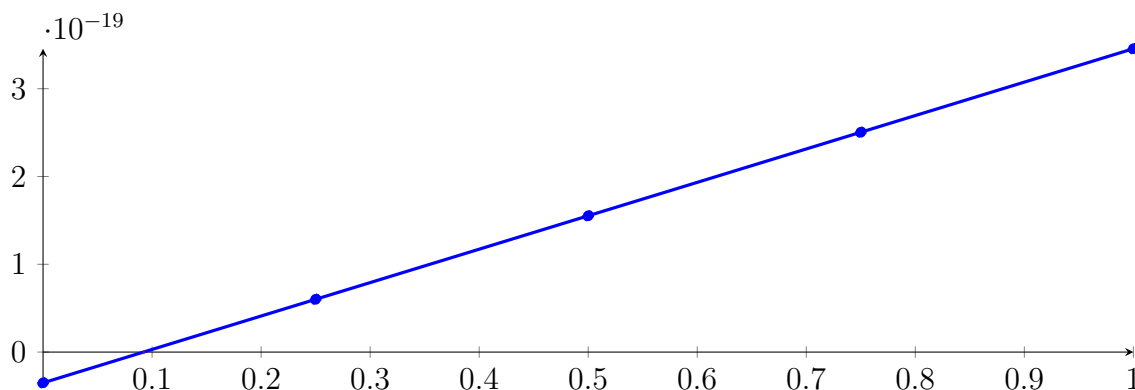
Longest intersection interval: $1.28975 \cdot 10^{-11}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

202.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -4.15417 \cdot 10^{-81} X^4 - 4.65699 \cdot 10^{-60} X^3 - 6.87497 \cdot 10^{-40} X^2 + 3.80599 \cdot 10^{-19} X - 3.50488 \cdot 10^{-20} \\
 &= -3.50488 \cdot 10^{-20} B_{0,4}(X) + 6.01009 \cdot 10^{-20} B_{1,4}(X) + 1.55251 \\
 &\quad \cdot 10^{-19} B_{2,4}(X) + 2.504 \cdot 10^{-19} B_{3,4}(X) + 3.4555 \cdot 10^{-19} B_{4,4}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $1.66345 \cdot 10^{-22}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

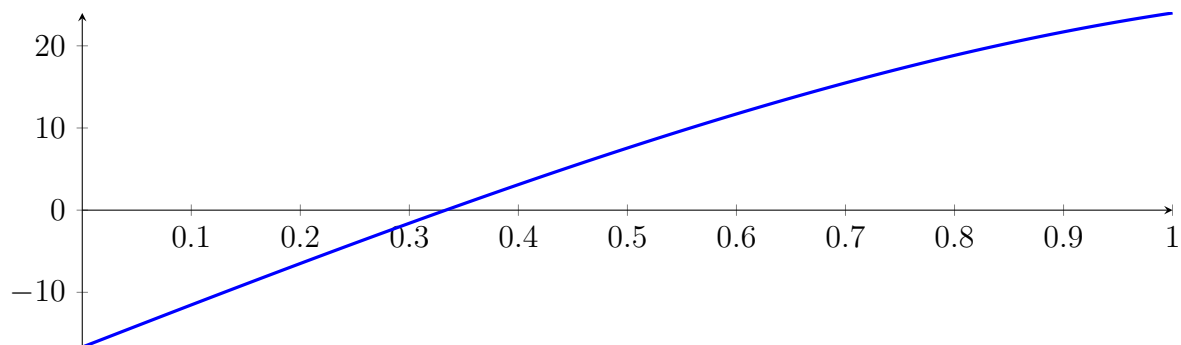
202.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

202.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

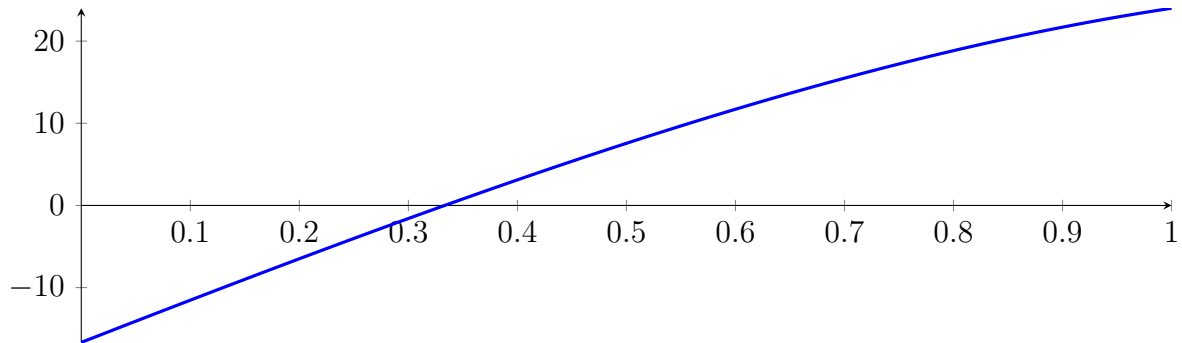
with precision $\varepsilon = 1 \cdot 10^{-32}$.

203 Running QuadClip on f_4 with epsilon 32

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

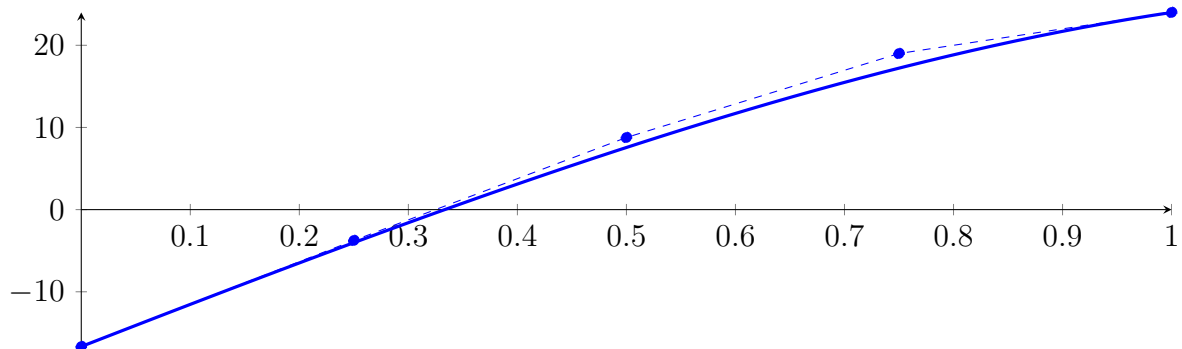
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



203.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

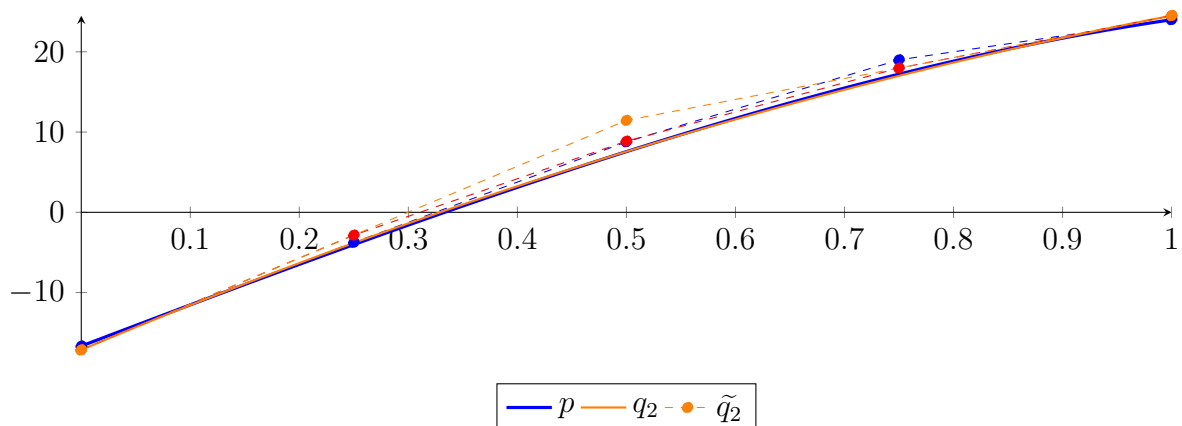
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -2.13607 \cdot 10^{-306}X^4 + 7.12024 \cdot 10^{-306}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

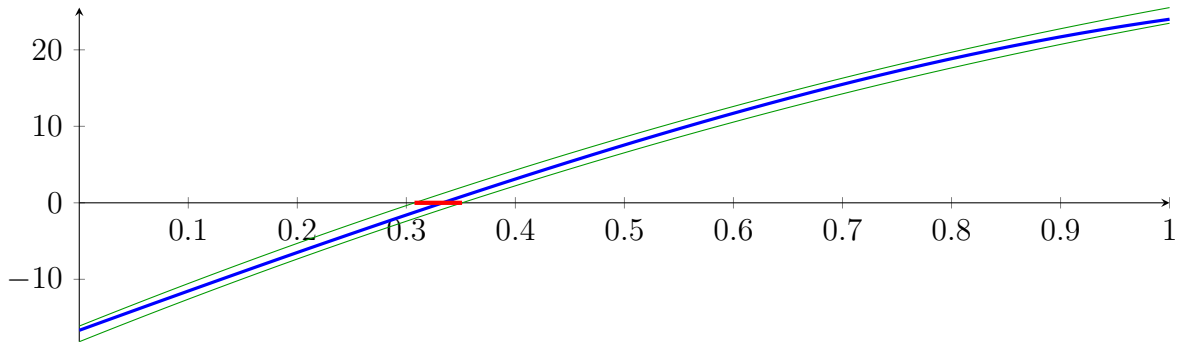
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

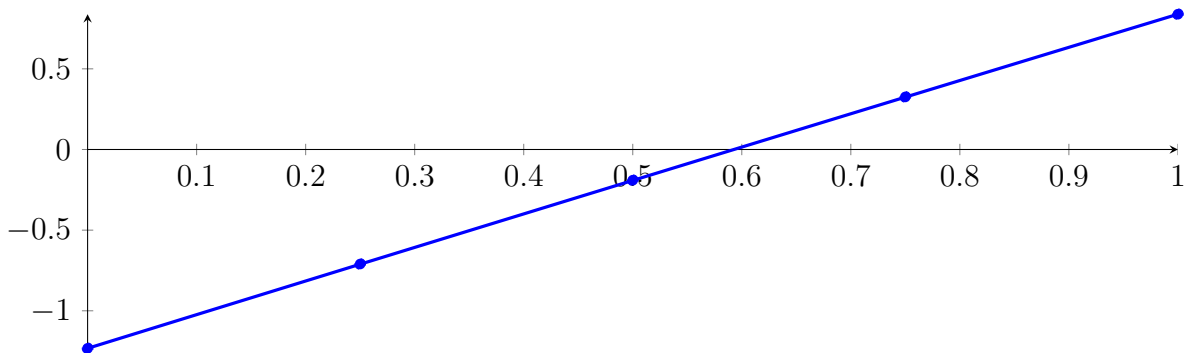
Longest intersection interval: 0.0436205

⇒ Selective recursion: interval 1: $[0.307477, 0.351097]$,

203.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

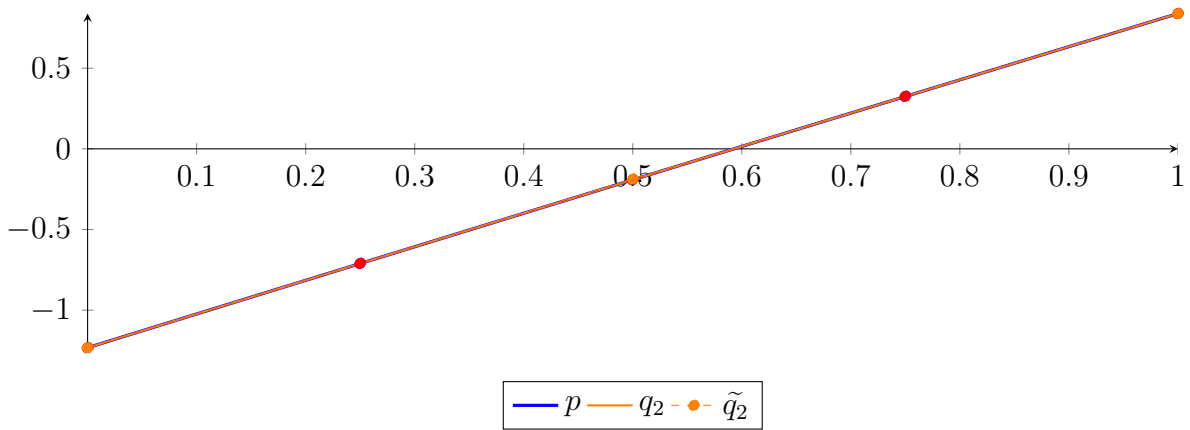
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -4.5614 \cdot 10^{-307} X^4 + 1.02353 \cdot 10^{-306} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

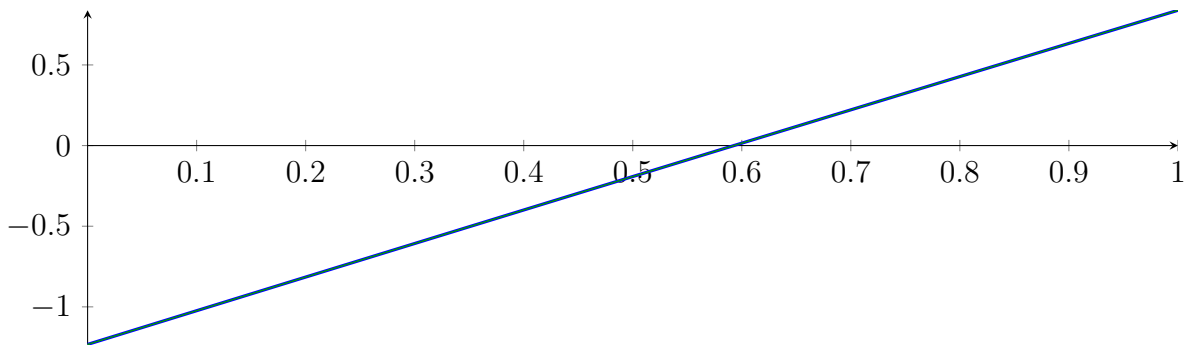
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

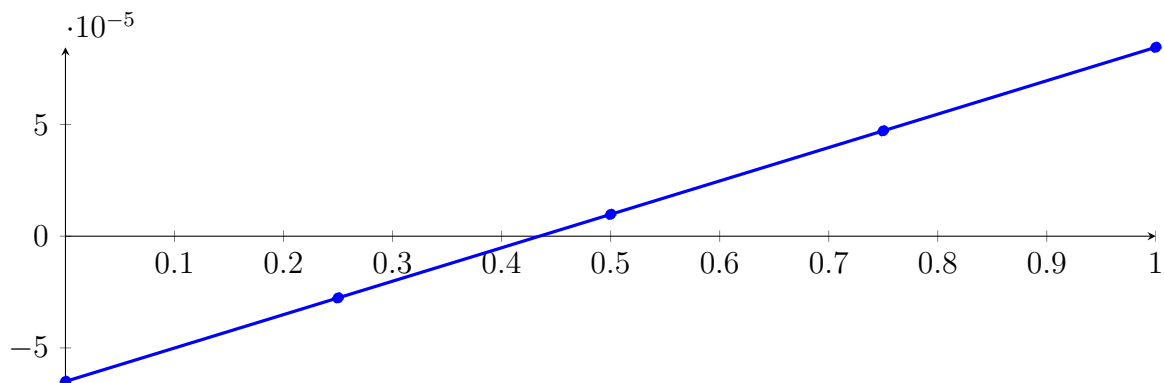
Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: **interval 1:** $[0.333332, 0.333335]$,

203.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.9027 \cdot 10^{-23} X^4 - 2.82525 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05} \\ &= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \\ &\quad \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X) \end{aligned}$$



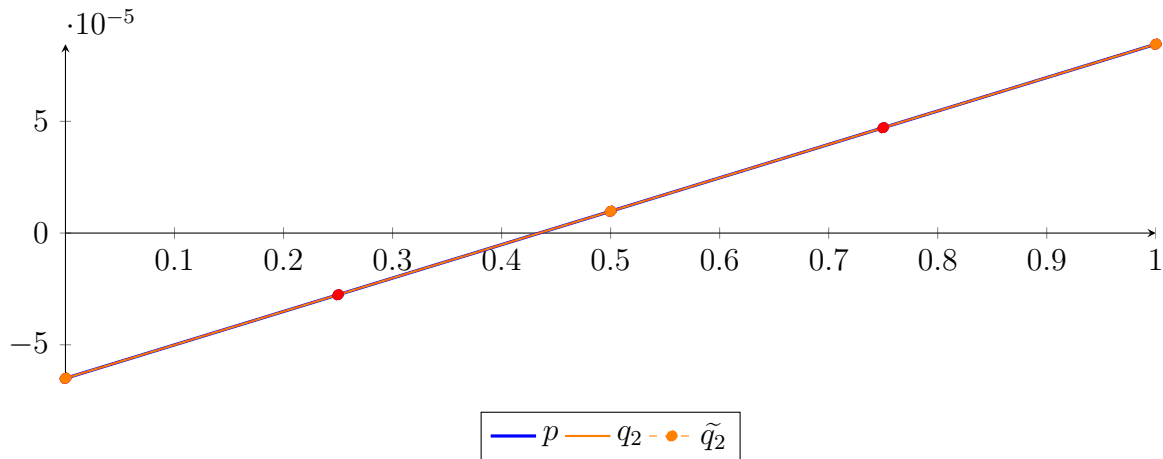
Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = 1.22227 \cdot 10^{-311} X^4 - 1.62969 \cdot 10^{-311} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.82526 \cdot 10^{-17}$.

Bounding polynomials M and m :

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

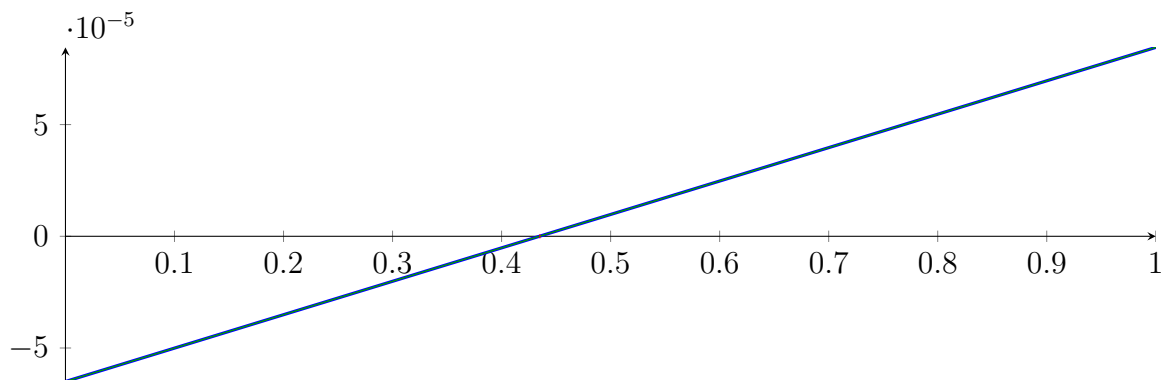
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.434685, 1.4089 \cdot 10^6\}$$

$$N(m) = \{0.434685, 1.4089 \cdot 10^6\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

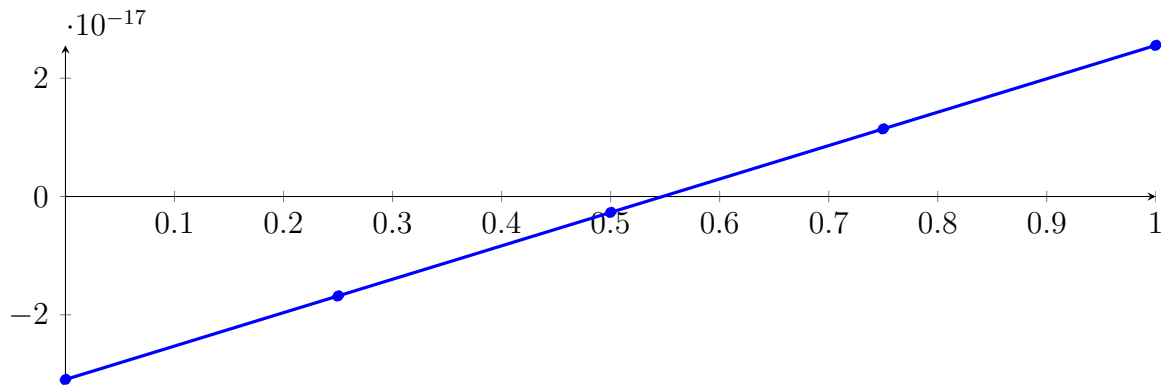
Longest intersection interval: $3.77836 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

203.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

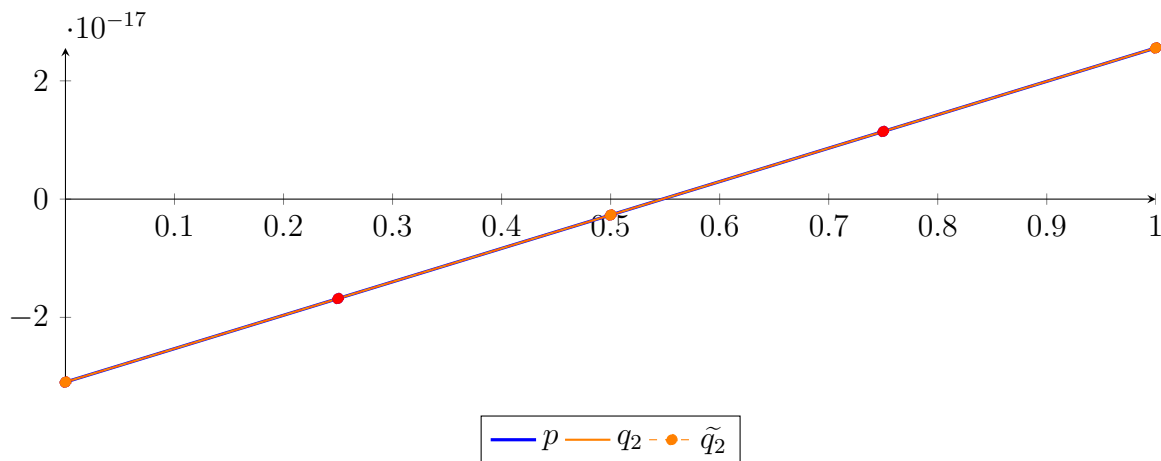
$$\begin{aligned} p &= -2.01821 \cdot 10^{-72} X^4 - 1.52394 \cdot 10^{-53} X^3 - 1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17} \\ &= -3.09418 \cdot 10^{-17} B_{0,4}(X) - 1.68155 \cdot 10^{-17} B_{1,4}(X) - 2.68924 \\ &\quad \cdot 10^{-18} B_{2,4}(X) + 1.1437 \cdot 10^{-17} B_{3,4}(X) + 2.55633 \cdot 10^{-17} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17} \\ &= -3.09418 \cdot 10^{-17} B_{0,2} - 2.68924 \cdot 10^{-18} B_{1,2} + 2.55633 \cdot 10^{-17} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -7.41098 \cdot 10^{-324} X^4 + 1.72923 \cdot 10^{-323} X^3 - 1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17} \\ &= -3.09418 \cdot 10^{-17} B_{0,4} - 1.68155 \cdot 10^{-17} B_{1,4} - 2.68924 \cdot 10^{-18} B_{2,4} + 1.1437 \cdot 10^{-17} B_{3,4} + 2.55633 \cdot 10^{-17} B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.52394 \cdot 10^{-54}$.

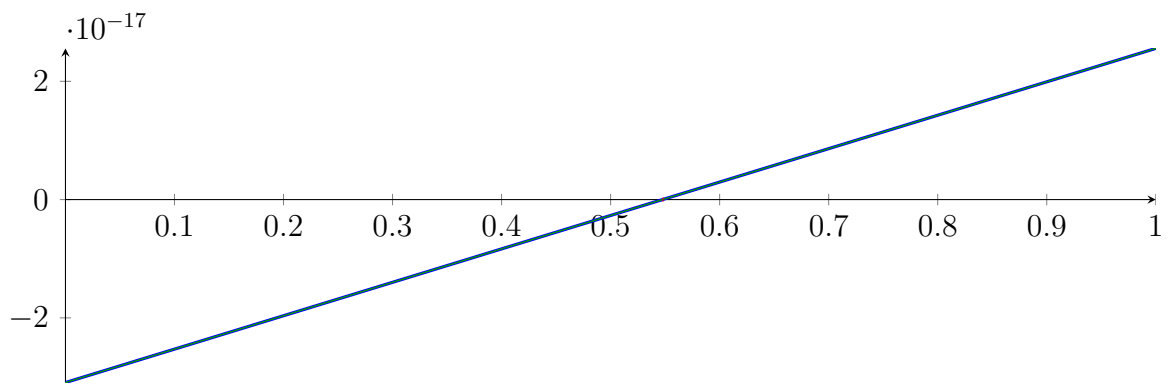
Bounding polynomials M and m :

$$\begin{aligned} M &= -1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17} \\ m &= -1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17} \end{aligned}$$

Root of M and m :

$$N(M) = \{0.547593, 3.72886 \cdot 10^{18}\} \quad N(m) = \{0.547593, 3.72886 \cdot 10^{18}\}$$

Intersection intervals:



[0.547593, 0.547593]

Longest intersection interval: $5.39398 \cdot 10^{-38}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

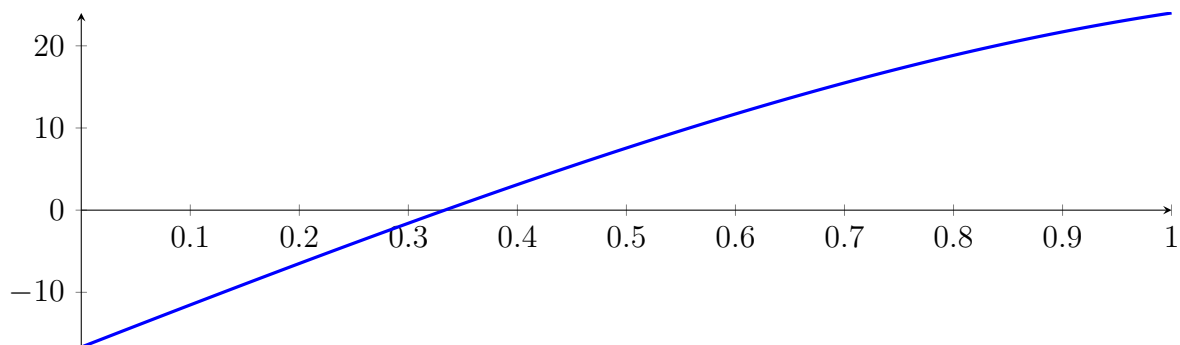
203.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

203.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

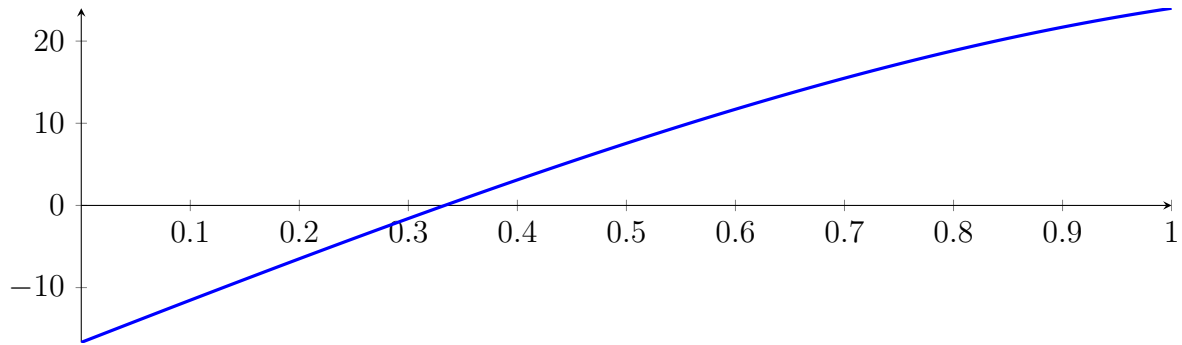
with precision $\varepsilon = 1 \cdot 10^{-32}$.

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$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

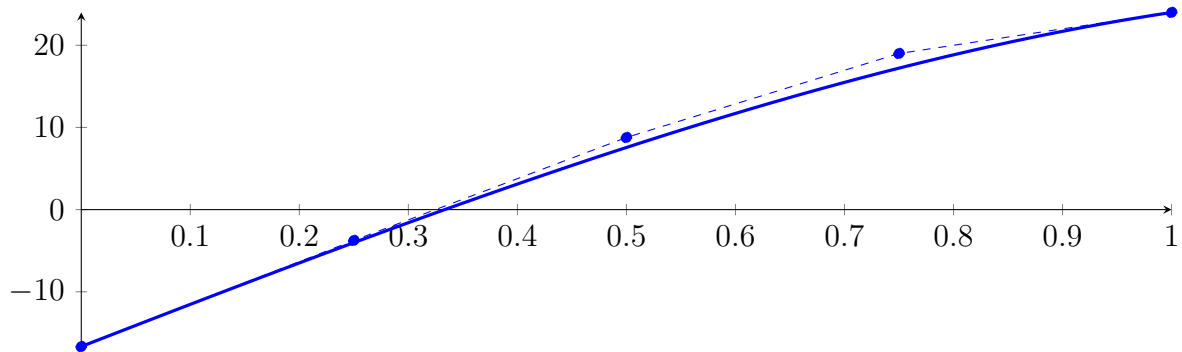
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



204.1 Recursion Branch 1 for Input Interval $[0, 1]$

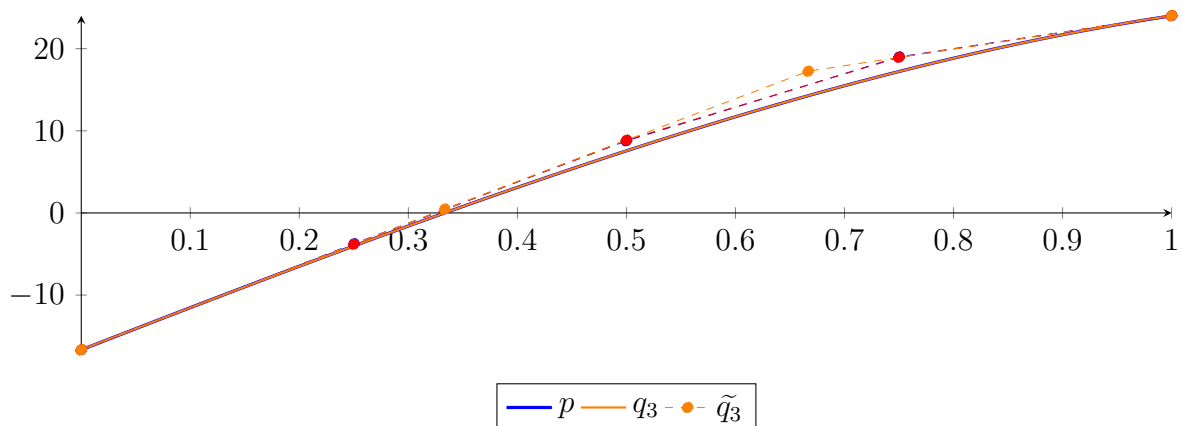
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= -3.56012 \cdot 10^{-307}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

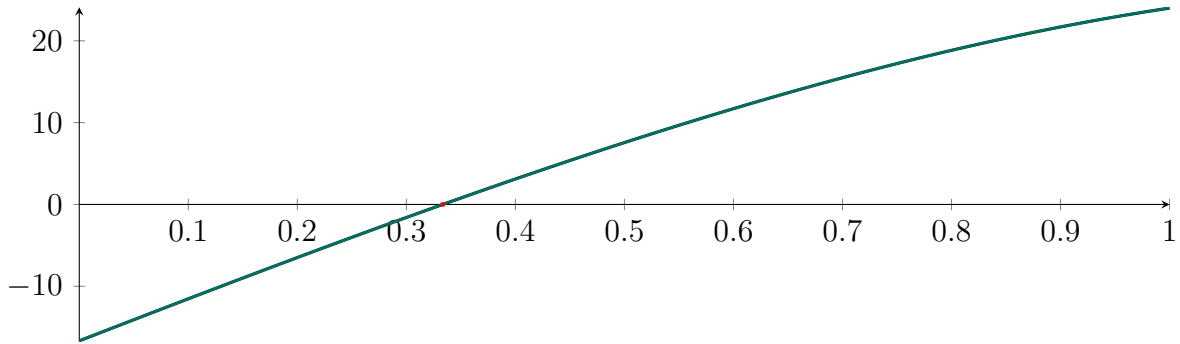
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

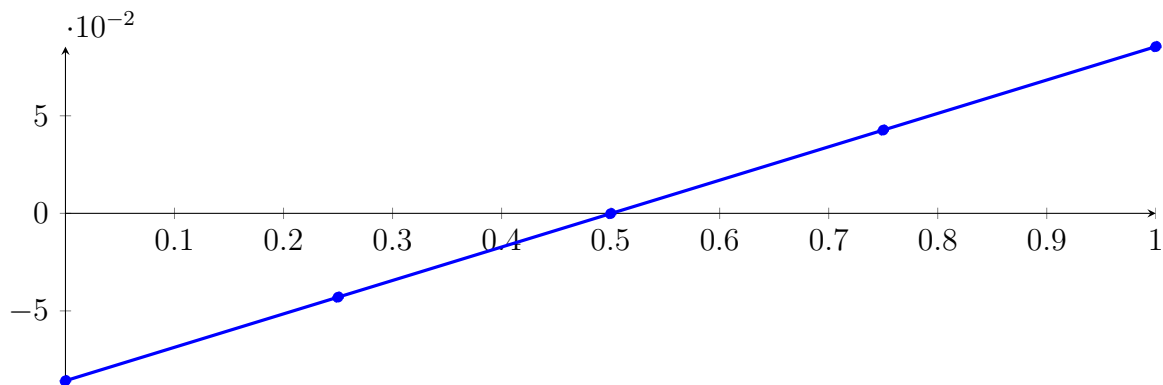
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

204.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

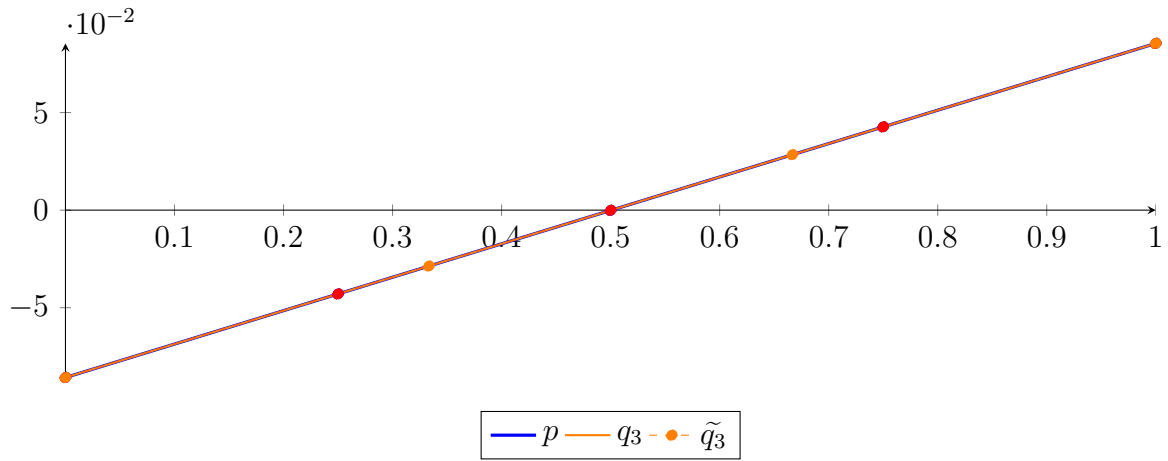
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -1.39067 \cdot 10^{-309} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m :

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

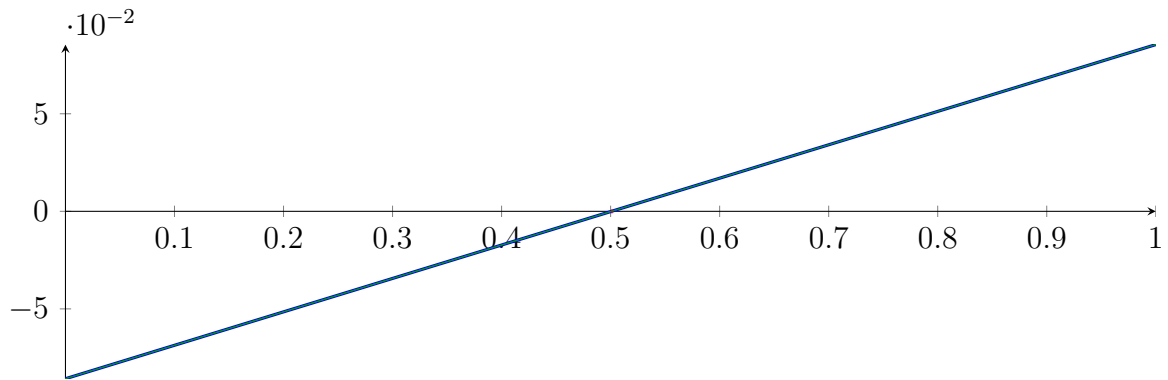
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\}$$

$$N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

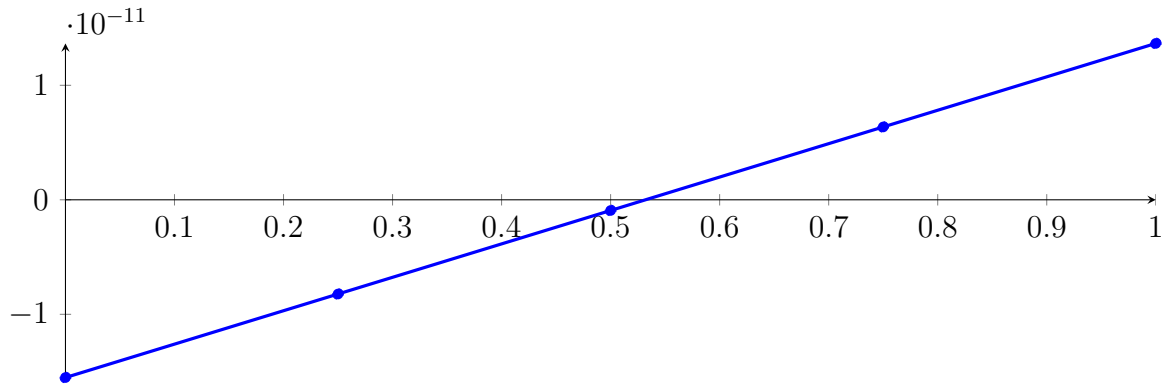
Longest intersection interval: $1.7041 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

204.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

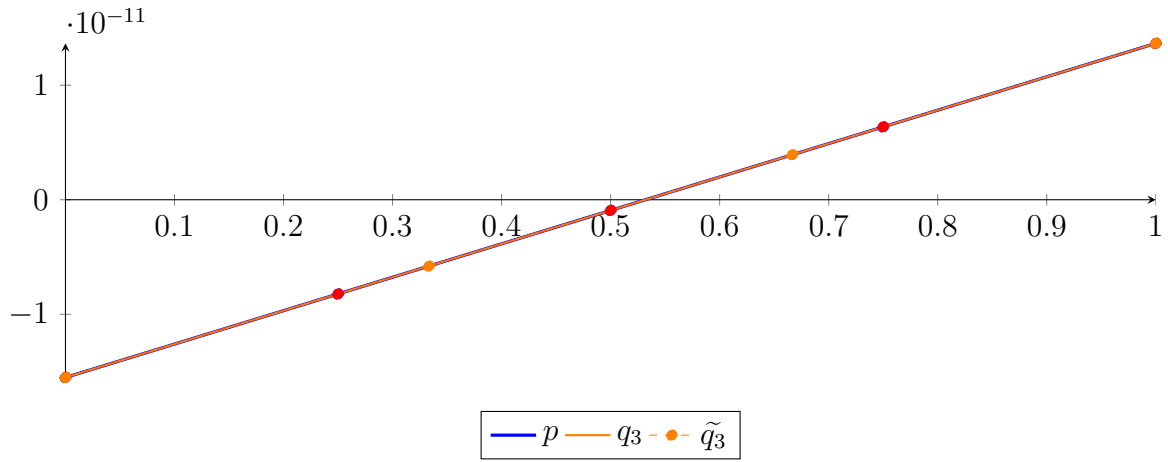
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.43544 \cdot 10^{-49} X^4 - 2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\ &= -1.55233 \cdot 10^{-11} B_{0,4}(X) - 8.22817 \cdot 10^{-12} B_{1,4}(X) - 9.33052 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36207 \cdot 10^{-12} B_{3,4}(X) + 1.36572 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 &= -1.55233 \cdot 10^{-11} B_{0,3} - 5.79646 \cdot 10^{-12} B_{1,3} + 3.93036 \cdot 10^{-12} B_{2,3} + 1.36572 \cdot 10^{-11} B_{3,3} \\
 \tilde{q}_3 &= -3.23791 \cdot 10^{-319} X^4 - 2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 &= -1.55233 \cdot 10^{-11} B_{0,4} - 8.22817 \cdot 10^{-12} B_{1,4} - 9.33052 \cdot 10^{-13} B_{2,4} + 6.36207 \cdot 10^{-12} B_{3,4} + 1.36572 \cdot 10^{-11} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.23038 \cdot 10^{-50}$.

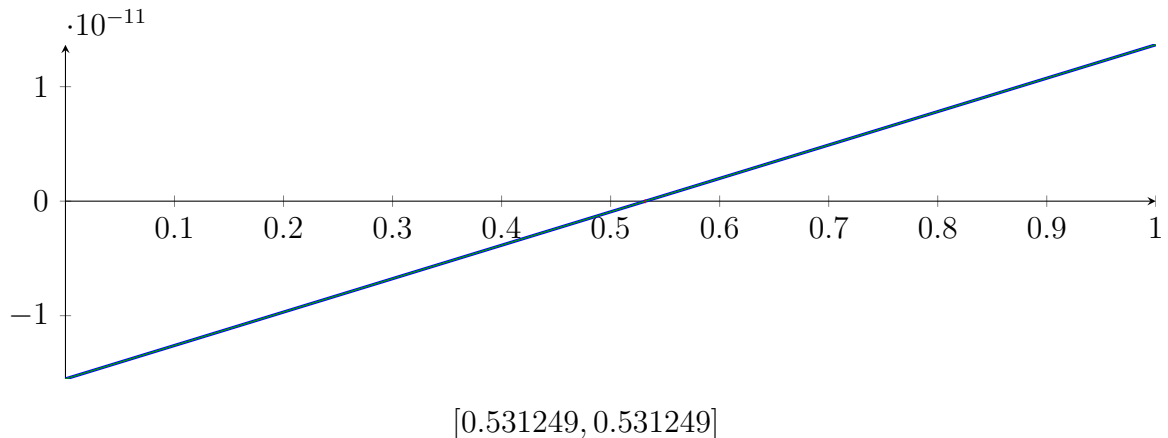
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 m &= -2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-4.81371 \cdot 10^{12}, 0.531249, 2.88823 \cdot 10^{12}\} \quad N(m) = \{-4.81371 \cdot 10^{12}, 0.531249, 2.88823 \cdot 10^{12}\}$$

Intersection intervals:



Longest intersection interval: $8.43287 \cdot 10^{-40}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

204.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

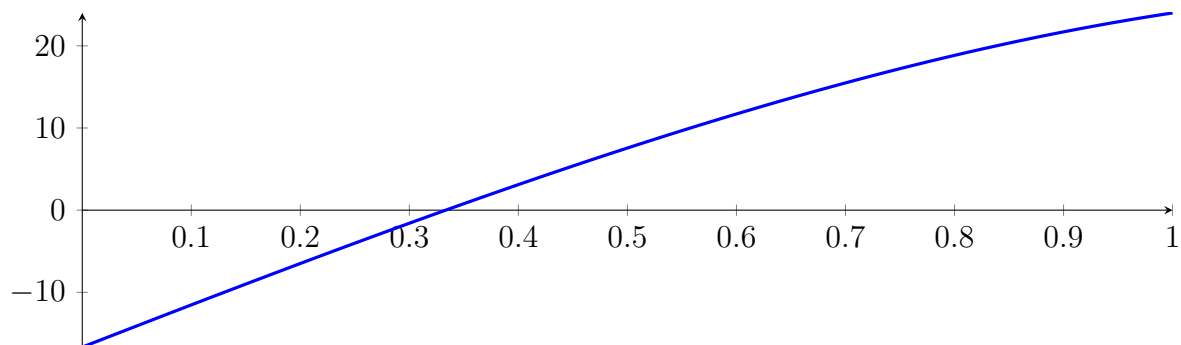
Reached interval [0.333333, 0.333333] **without sign change** at depth 4!

$p(0) = -2.11876e-14$ - $p(1) -2.11876e-14$

204.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

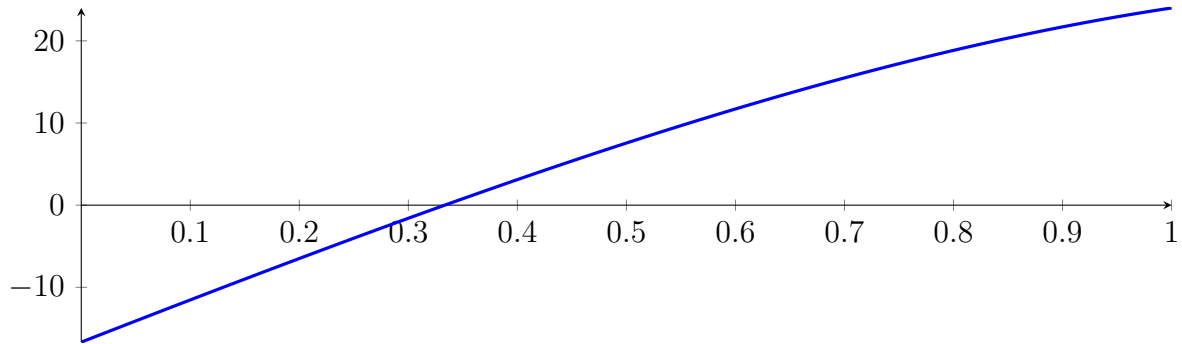
with precision $\varepsilon = 1 \cdot 10^{-32}$.

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$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

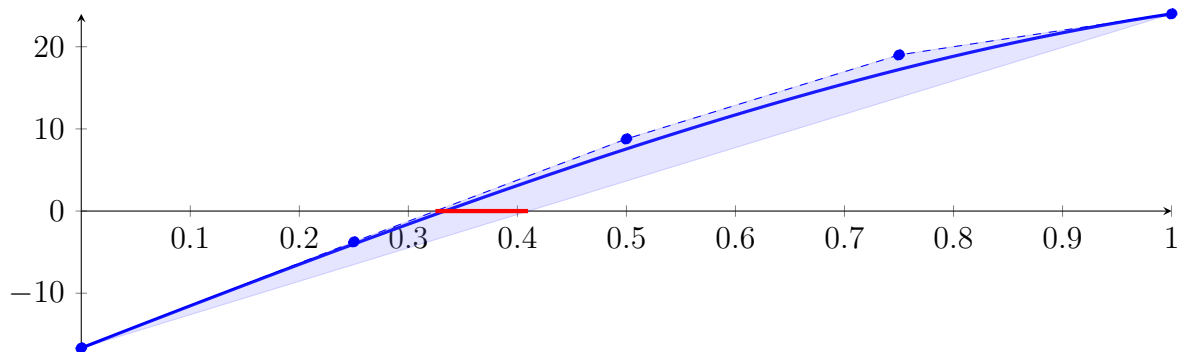
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



205.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

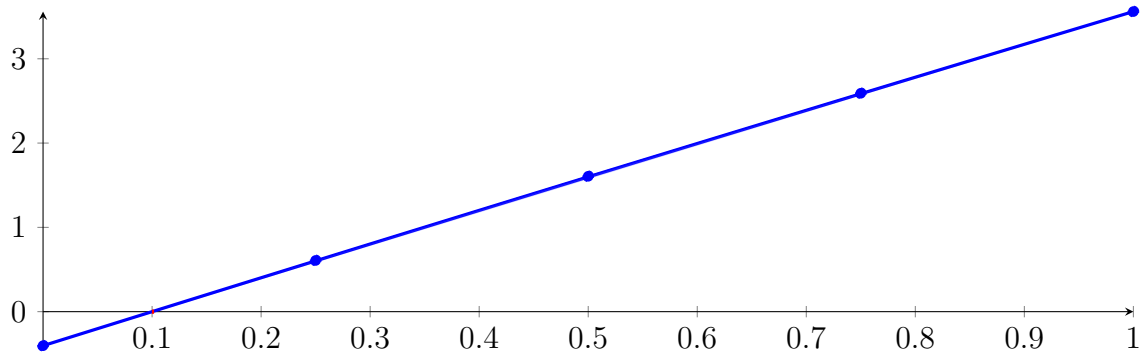
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

205.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

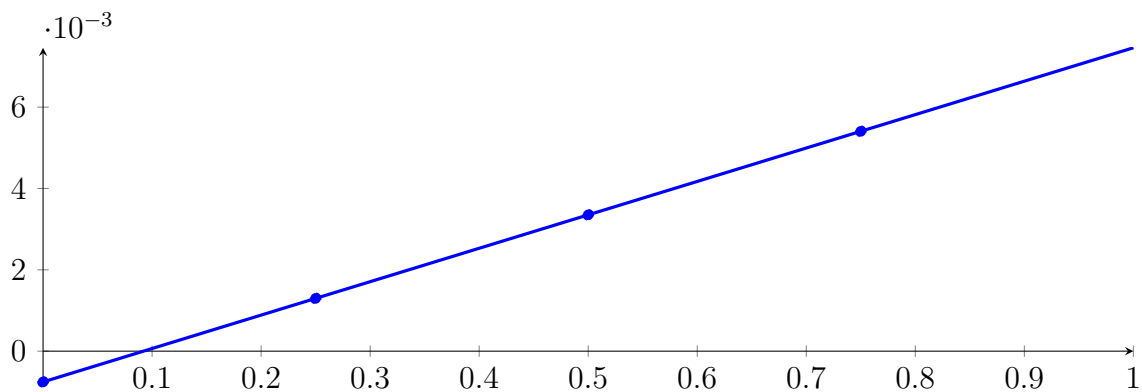
Longest intersection interval: 0.00203877

\implies Selective recursion: interval 1: $[0.333317, 0.333491]$,

205.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01975 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

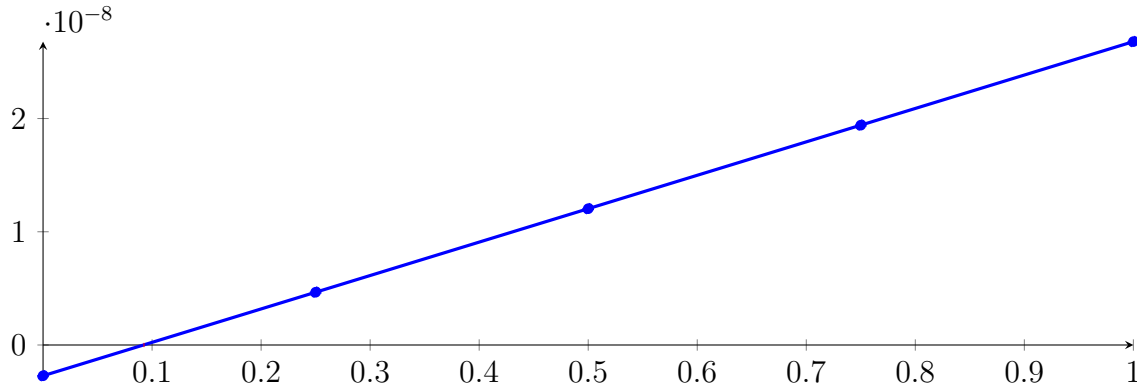
Longest intersection interval: $3.59185 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

205.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.50129 \cdot 10^{-37} X^4 - 2.17066 \cdot 10^{-27} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09} \\
 &= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \\
 &\quad \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

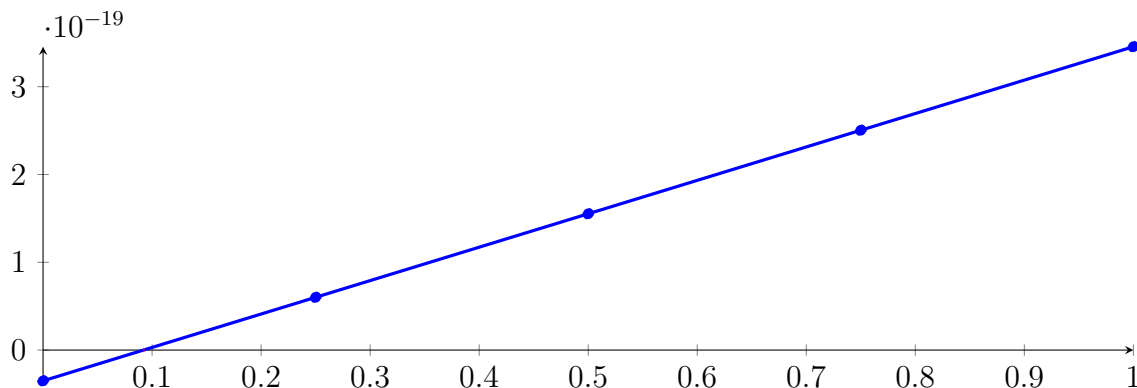
Longest intersection interval: $1.28975 \cdot 10^{-11}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

205.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -4.15417 \cdot 10^{-81} X^4 - 4.65699 \cdot 10^{-60} X^3 - 6.87497 \cdot 10^{-40} X^2 + 3.80599 \cdot 10^{-19} X - 3.50488 \cdot 10^{-20} \\
 &= -3.50488 \cdot 10^{-20} B_{0,4}(X) + 6.01009 \cdot 10^{-20} B_{1,4}(X) + 1.55251 \\
 &\quad \cdot 10^{-19} B_{2,4}(X) + 2.504 \cdot 10^{-19} B_{3,4}(X) + 3.4555 \cdot 10^{-19} B_{4,4}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

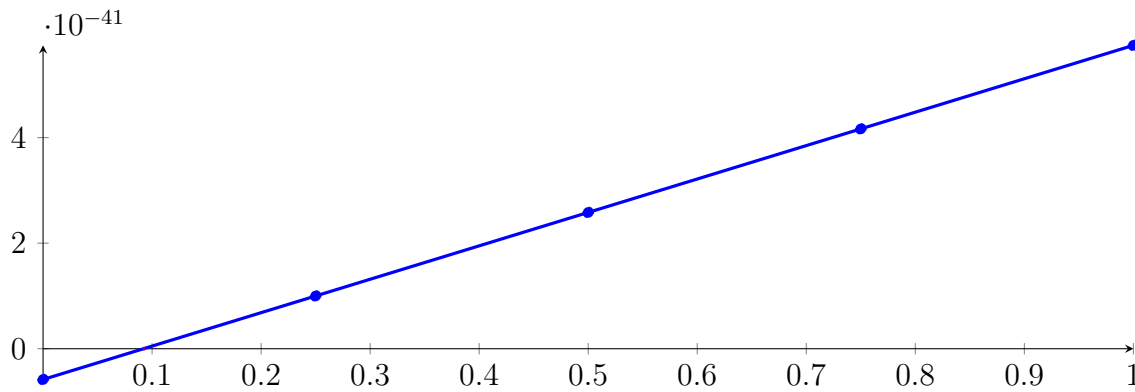
Longest intersection interval: $1.66345 \cdot 10^{-22}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

205.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.18068 \cdot 10^{-168} X^4 - 2.14355 \cdot 10^{-125} X^3 - 1.90234 \cdot 10^{-83} X^2 + 6.33106 \cdot 10^{-41} X - 5.83018 \cdot 10^{-42} \\
 &= -5.83018 \cdot 10^{-42} B_{0,4}(X) + 9.99747 \cdot 10^{-42} B_{1,4}(X) + 2.58251 \\
 &\quad \cdot 10^{-41} B_{2,4}(X) + 4.16528 \cdot 10^{-41} B_{3,4}(X) + 5.74804 \cdot 10^{-41} B_{4,4}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $2.76706 \cdot 10^{-44}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

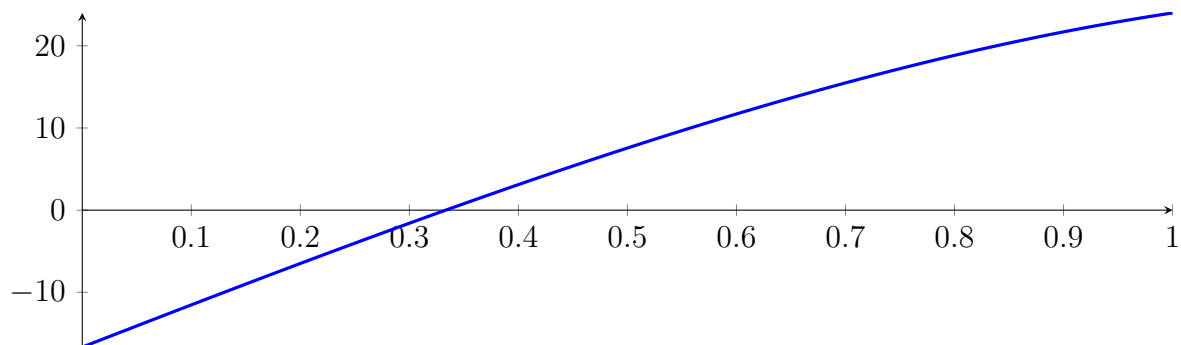
205.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 7!

205.8 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

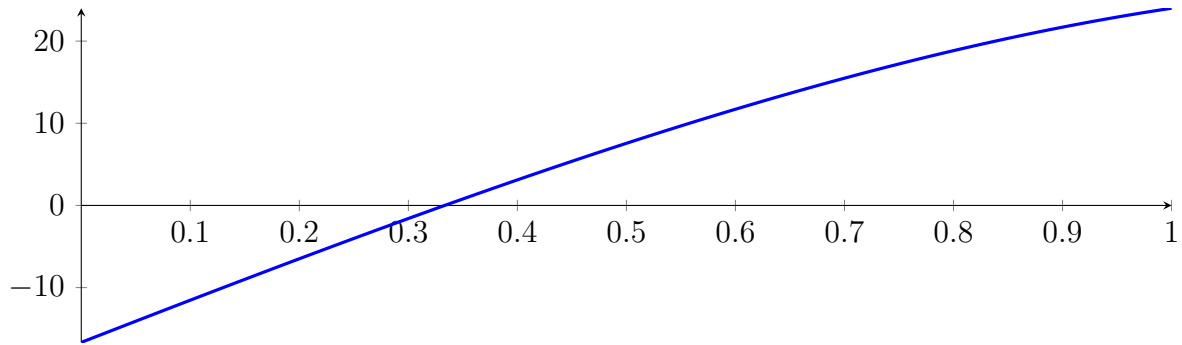
with precision $\varepsilon = 1 \cdot 10^{-64}$.

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$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

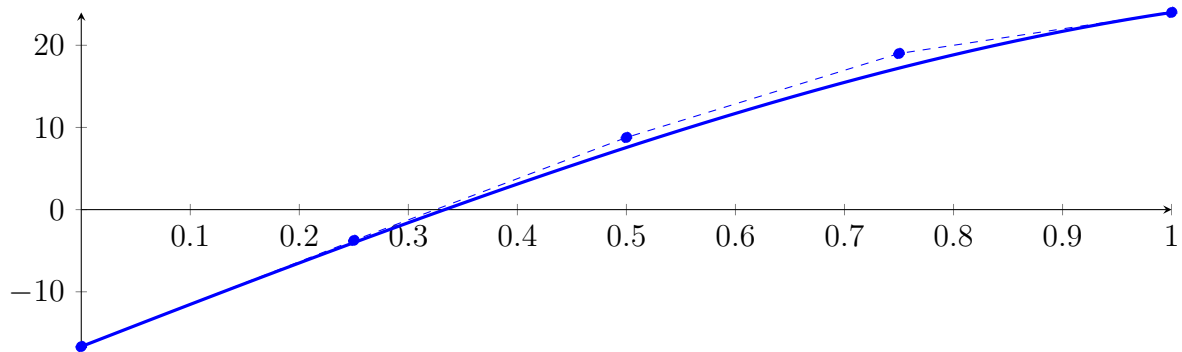
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



206.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

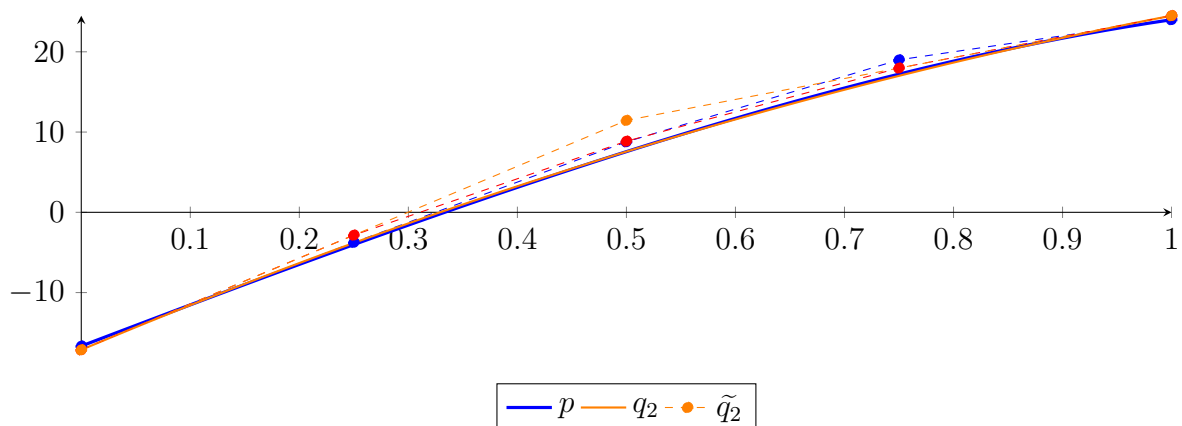
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -2.13607 \cdot 10^{-306}X^4 + 7.12024 \cdot 10^{-306}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

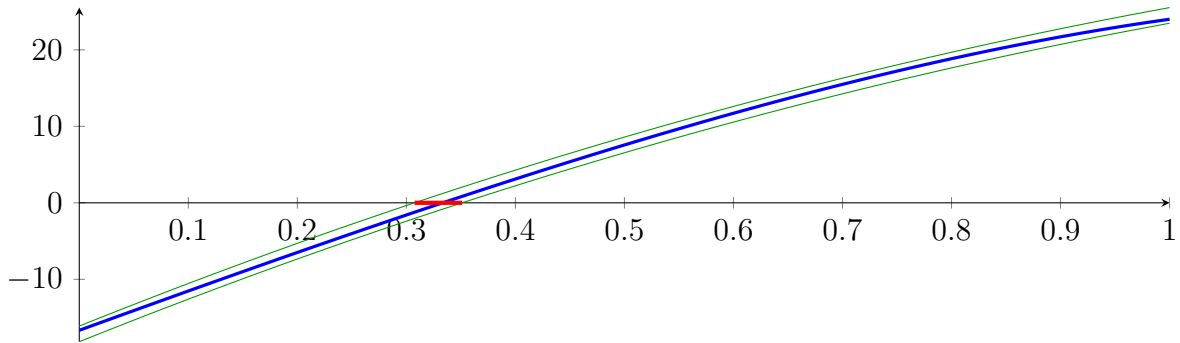
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

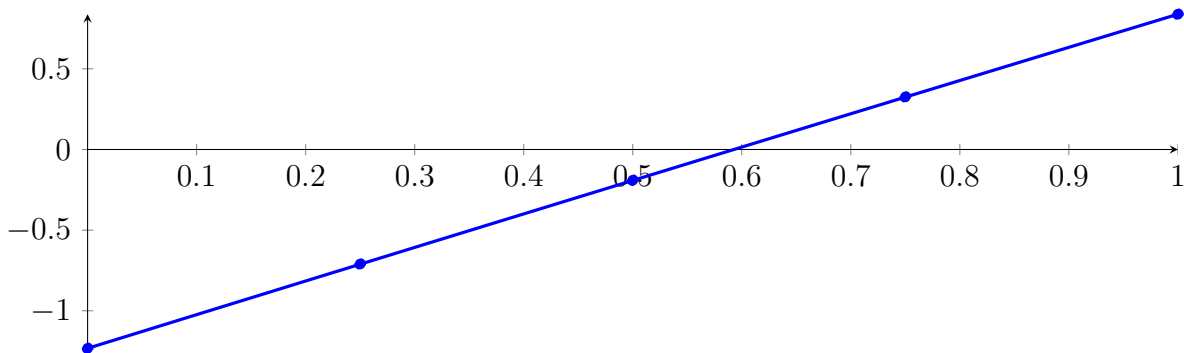
Longest intersection interval: 0.0436205

⇒ Selective recursion: interval 1: $[0.307477, 0.351097]$,

206.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

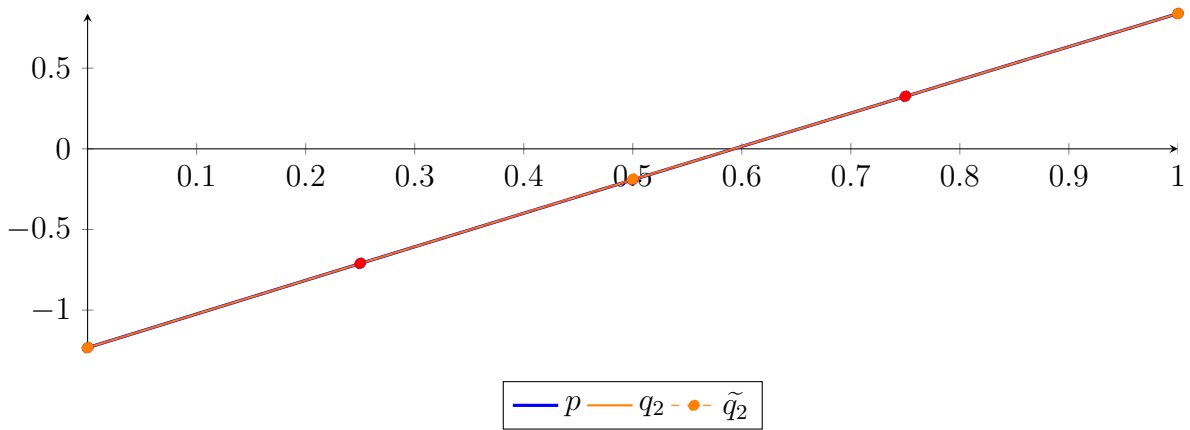
$$\begin{aligned} p &= -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278 \\ &= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -4.5614 \cdot 10^{-307} X^4 + 1.02353 \cdot 10^{-306} X^3 - 0.020089 X^2 + 2.09166 X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

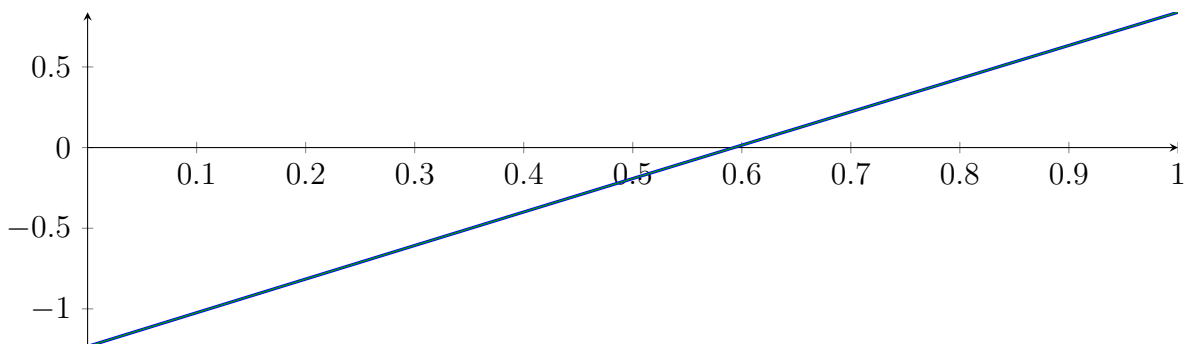
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

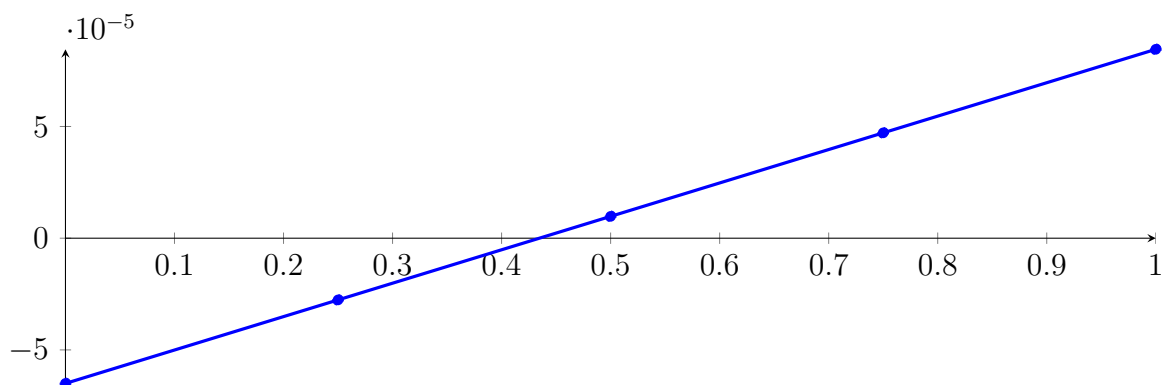
Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: **interval 1:** $[0.333332, 0.333335]$,

206.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.9027 \cdot 10^{-23} X^4 - 2.82525 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05} \\ &= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \\ &\quad \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X) \end{aligned}$$



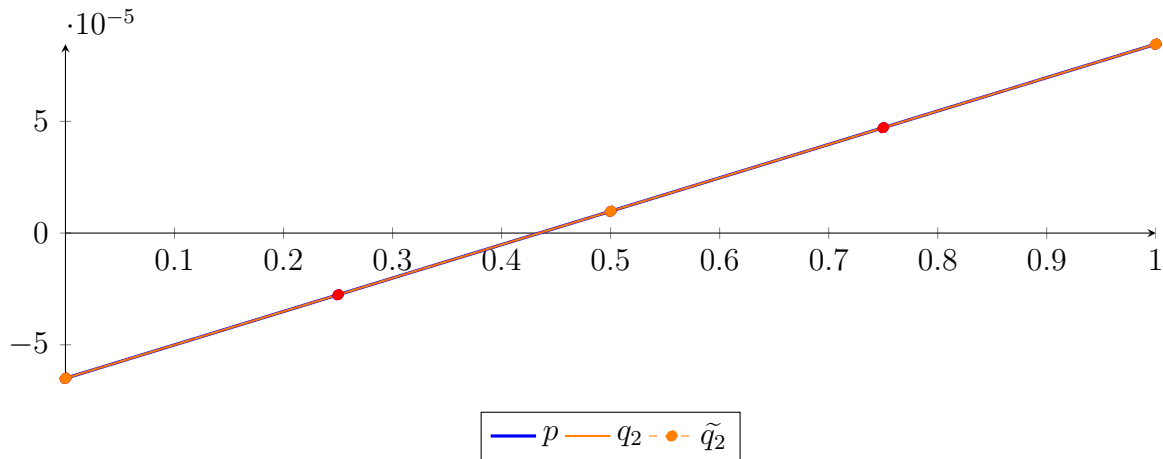
Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = 1.22227 \cdot 10^{-311} X^4 - 1.62969 \cdot 10^{-311} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.82526 \cdot 10^{-17}$.

Bounding polynomials M and m :

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

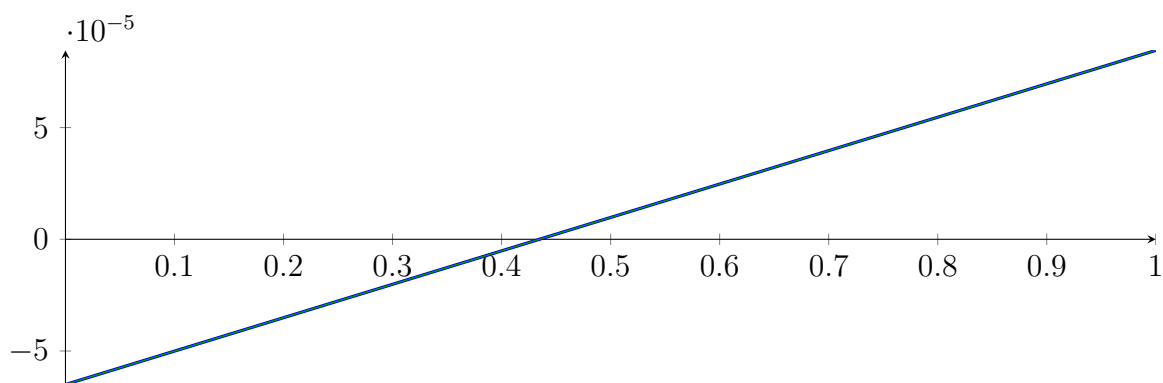
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.434685, 1.4089 \cdot 10^6\}$$

$$N(m) = \{0.434685, 1.4089 \cdot 10^6\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

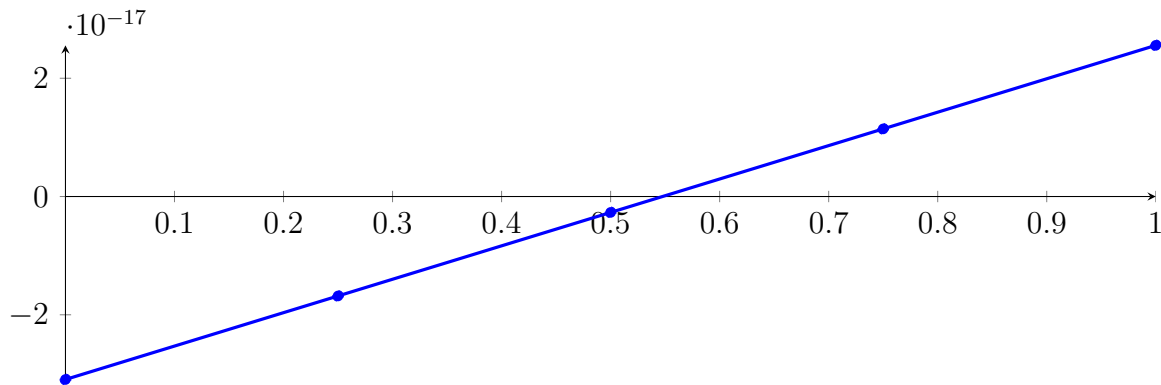
Longest intersection interval: $3.77836 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

206.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

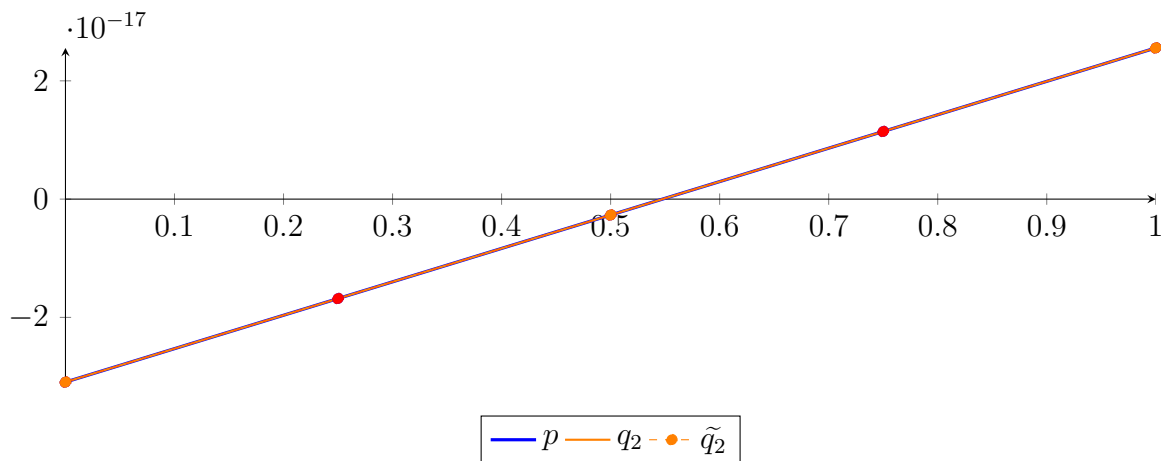
$$\begin{aligned}
 p &= -2.01821 \cdot 10^{-72} X^4 - 1.52394 \cdot 10^{-53} X^3 - 1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17} \\
 &= -3.09418 \cdot 10^{-17} B_{0,4}(X) - 1.68155 \cdot 10^{-17} B_{1,4}(X) - 2.68924 \\
 &\quad \cdot 10^{-18} B_{2,4}(X) + 1.1437 \cdot 10^{-17} B_{3,4}(X) + 2.55633 \cdot 10^{-17} B_{4,4}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17} \\
 &= -3.09418 \cdot 10^{-17} B_{0,2} - 2.68924 \cdot 10^{-18} B_{1,2} + 2.55633 \cdot 10^{-17} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -7.41098 \cdot 10^{-324} X^4 + 1.72923 \cdot 10^{-323} X^3 - 1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17} \\
 &= -3.09418 \cdot 10^{-17} B_{0,4} - 1.68155 \cdot 10^{-17} B_{1,4} - 2.68924 \cdot 10^{-18} B_{2,4} + 1.1437 \cdot 10^{-17} B_{3,4} + 2.55633 \cdot 10^{-17} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.52394 \cdot 10^{-54}$.

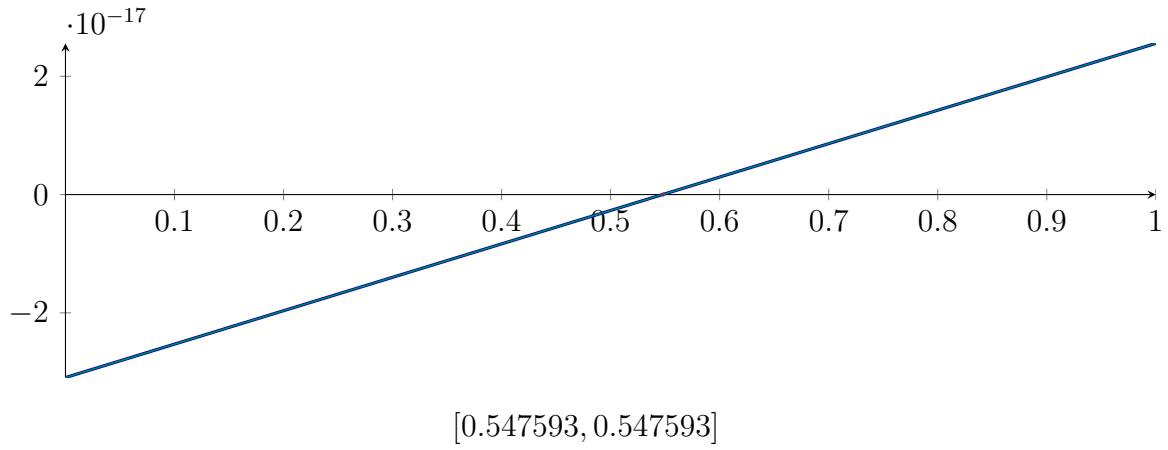
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17} \\
 m &= -1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.547593, 3.72886 \cdot 10^{18}\} \quad N(m) = \{0.547593, 3.72886 \cdot 10^{18}\}$$

Intersection intervals:

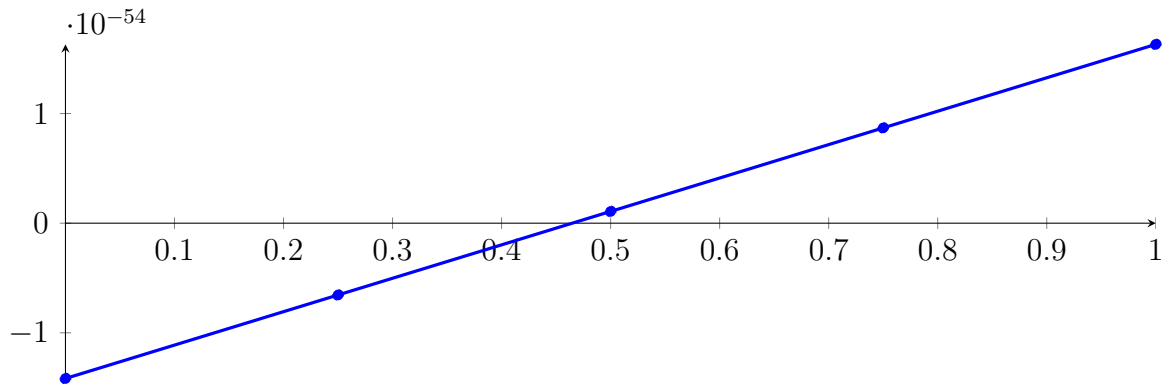


Longest intersection interval: $5.39398 \cdot 10^{-38}$
 \implies Selective recursion: interval 1: [0.333333, 0.333333],

206.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

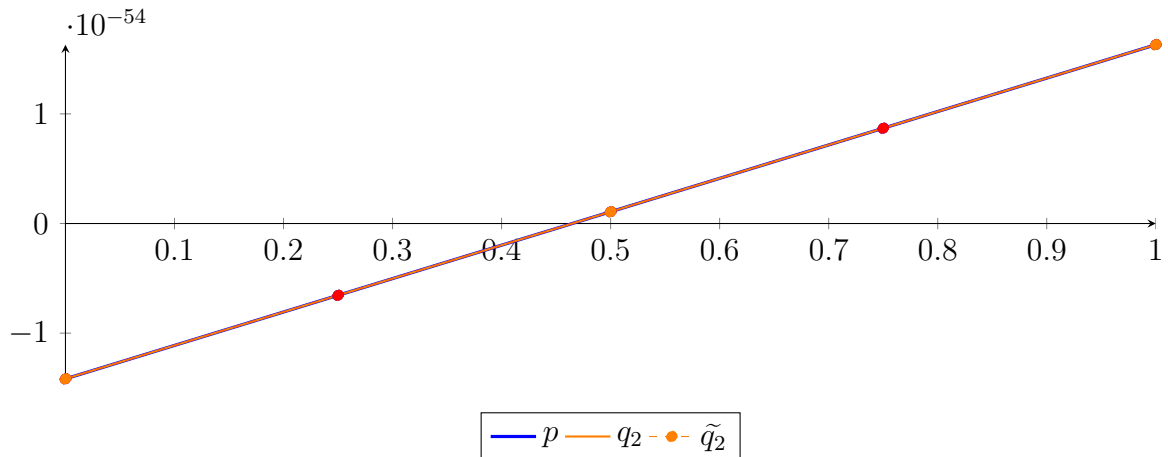
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.70846 \cdot 10^{-221} X^4 - 2.39164 \cdot 10^{-165} X^3 - 4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54} \\
 &= -1.41679 \cdot 10^{-54} B_{0,4}(X) - 6.54819 \cdot 10^{-55} B_{1,4}(X) + 1.0715 \\
 &\quad \cdot 10^{-55} B_{2,4}(X) + 8.6912 \cdot 10^{-55} B_{3,4}(X) + 1.63109 \cdot 10^{-54} B_{4,4}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54} \\
 &= -1.41679 \cdot 10^{-54} B_{0,2} + 1.0715 \cdot 10^{-55} B_{1,2} + 1.63109 \cdot 10^{-54} B_{2,2} \\
 \tilde{q}_2 &= 7.25964 \cdot 10^{-362} X^4 - 5.80771 \cdot 10^{-362} X^3 - 4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54} \\
 &= -1.41679 \cdot 10^{-54} B_{0,4} - 6.54819 \cdot 10^{-55} B_{1,4} + 1.0715 \cdot 10^{-55} B_{2,4} + 8.6912 \cdot 10^{-55} B_{3,4} + 1.63109 \cdot 10^{-54} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.39164 \cdot 10^{-166}$.

Bounding polynomials M and m :

$$M = -4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54}$$

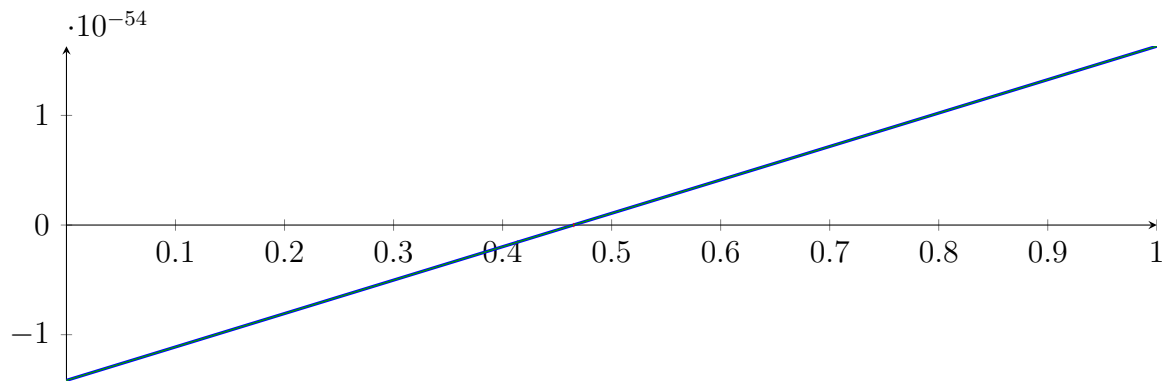
$$m = -4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54}$$

Root of M and m :

$$N(M) = \{0.464844, 6.91299 \cdot 10^{55}\}$$

$$N(m) = \{0.464844, 6.91299 \cdot 10^{55}\}$$

Intersection intervals:



$$[0.464844, 0.464844]$$

Longest intersection interval: $1.56938 \cdot 10^{-112}$

\implies Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

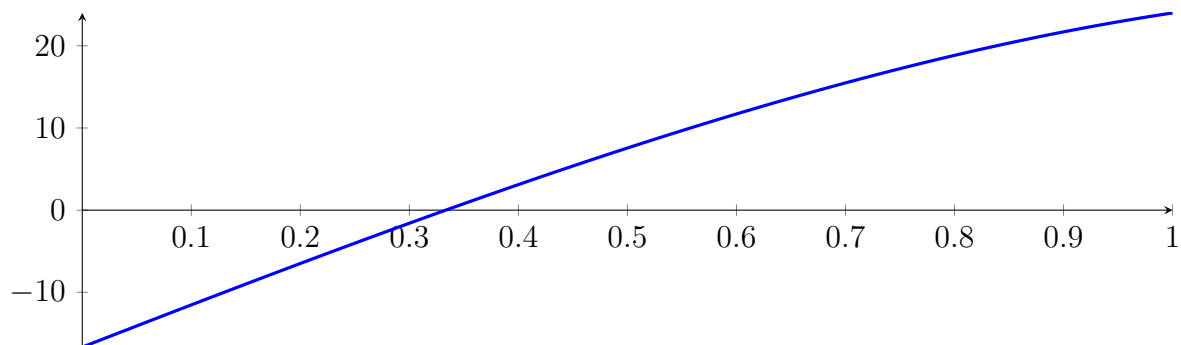
206.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

206.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

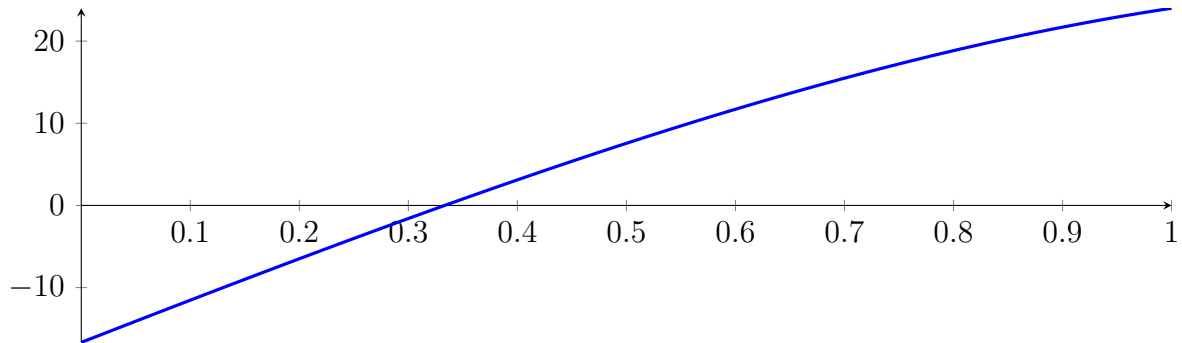
with precision $\varepsilon = 1 \cdot 10^{-64}$.

207 Running CubeClip on f_4 with epsilon 64

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

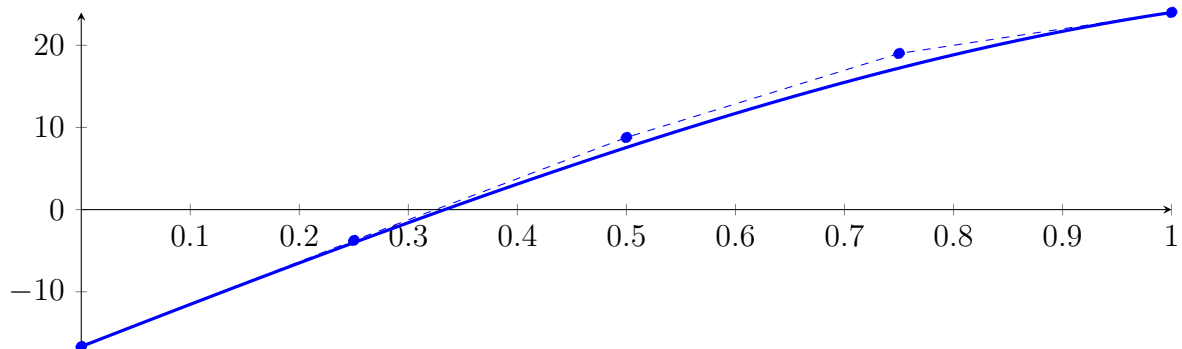
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



207.1 Recursion Branch 1 for Input Interval $[0, 1]$

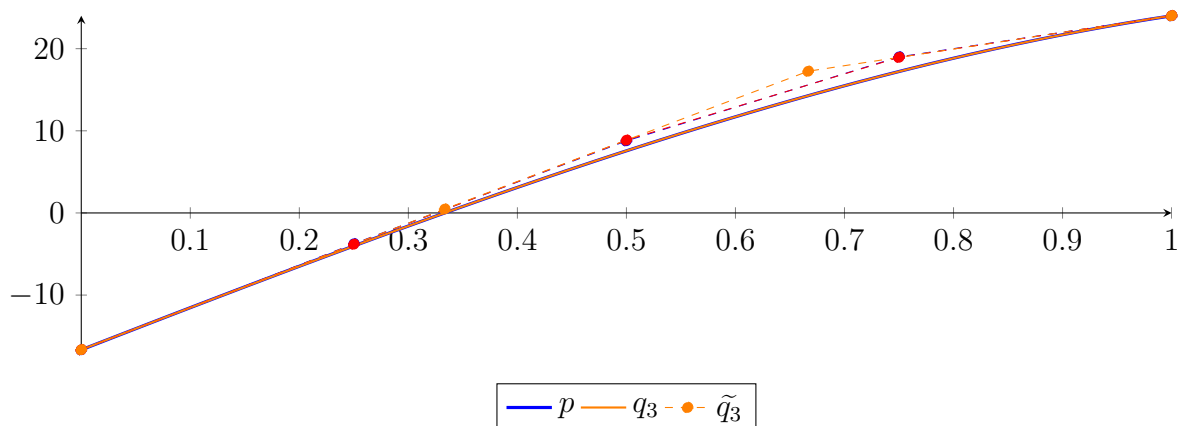
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= -3.56012 \cdot 10^{-307}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

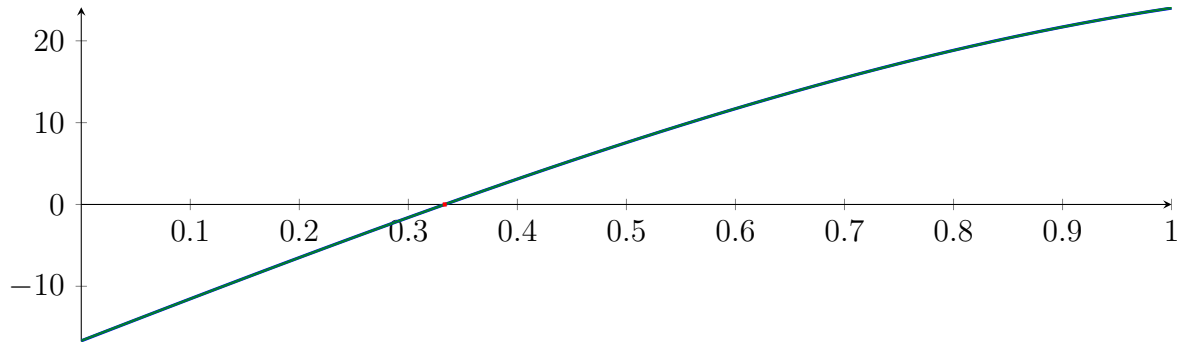
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

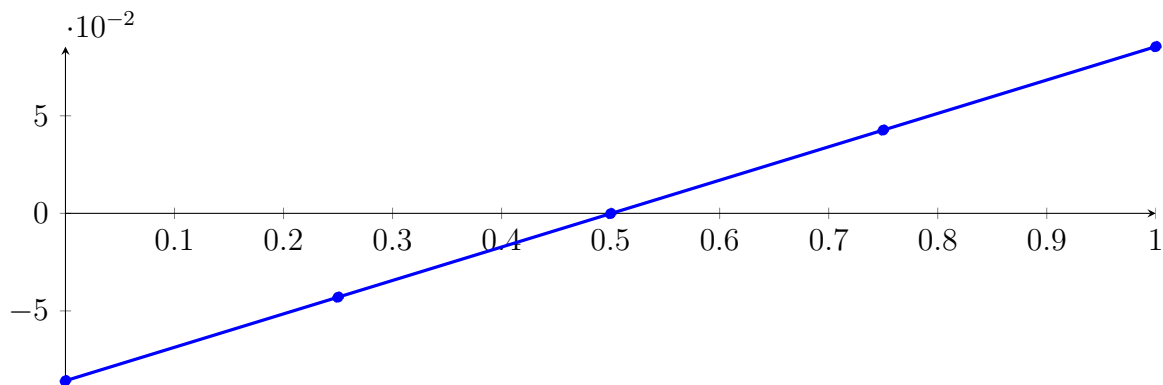
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

207.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

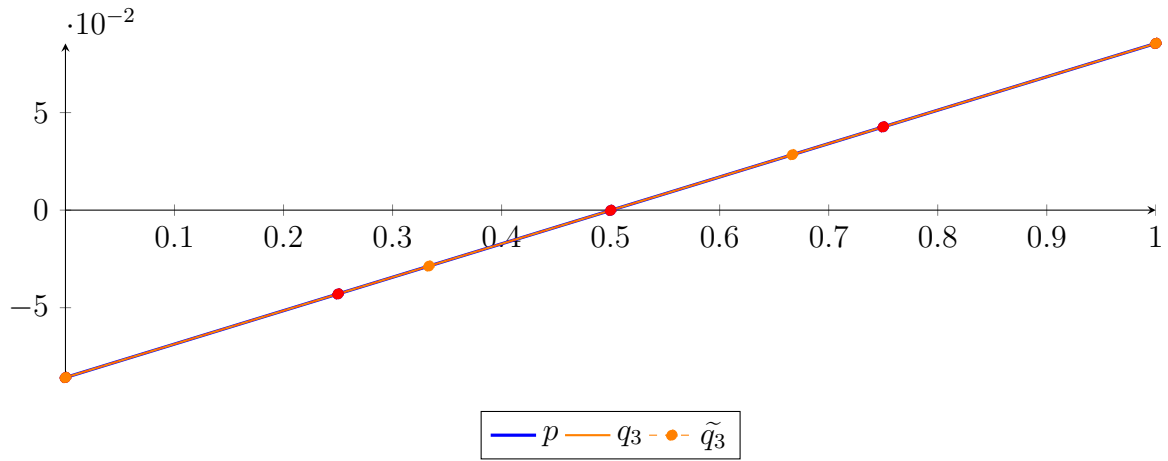
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -1.39067 \cdot 10^{-309} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m :

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

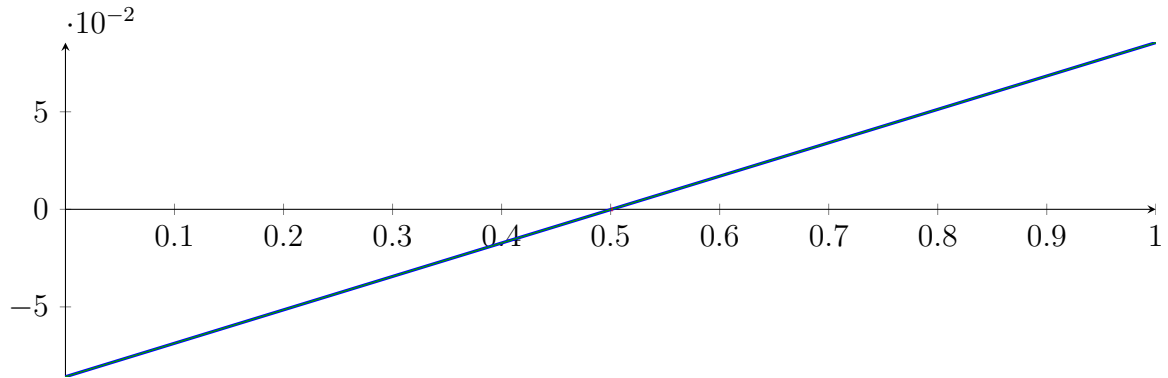
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\}$$

$$N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

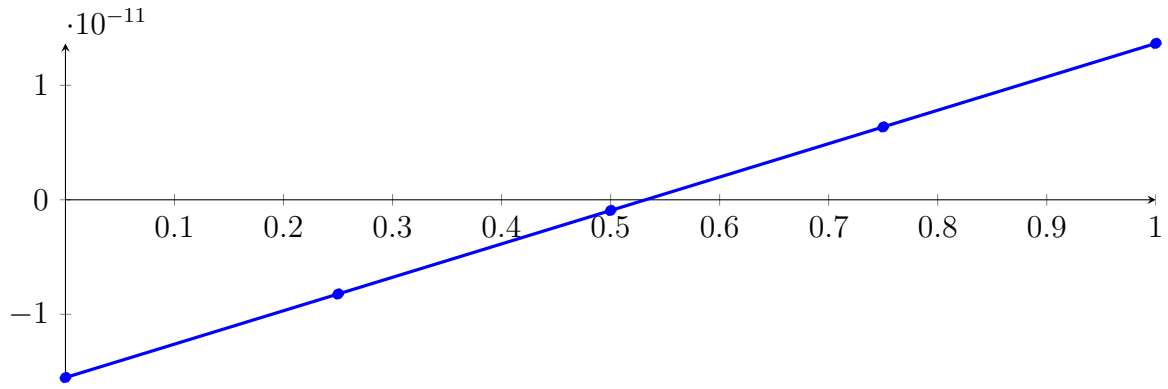
Longest intersection interval: $1.7041 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

207.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

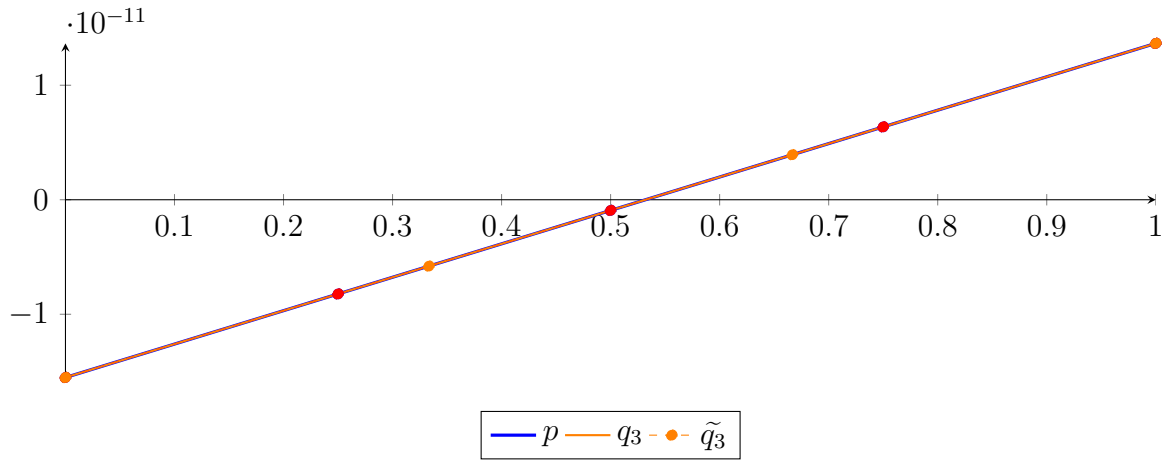
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.43544 \cdot 10^{-49} X^4 - 2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\ &= -1.55233 \cdot 10^{-11} B_{0,4}(X) - 8.22817 \cdot 10^{-12} B_{1,4}(X) - 9.33052 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36207 \cdot 10^{-12} B_{3,4}(X) + 1.36572 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 &= -1.55233 \cdot 10^{-11} B_{0,3} - 5.79646 \cdot 10^{-12} B_{1,3} + 3.93036 \cdot 10^{-12} B_{2,3} + 1.36572 \cdot 10^{-11} B_{3,3} \\
 \tilde{q}_3 &= -3.23791 \cdot 10^{-319} X^4 - 2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 &= -1.55233 \cdot 10^{-11} B_{0,4} - 8.22817 \cdot 10^{-12} B_{1,4} - 9.33052 \cdot 10^{-13} B_{2,4} + 6.36207 \cdot 10^{-12} B_{3,4} + 1.36572 \cdot 10^{-11} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.23038 \cdot 10^{-50}$.

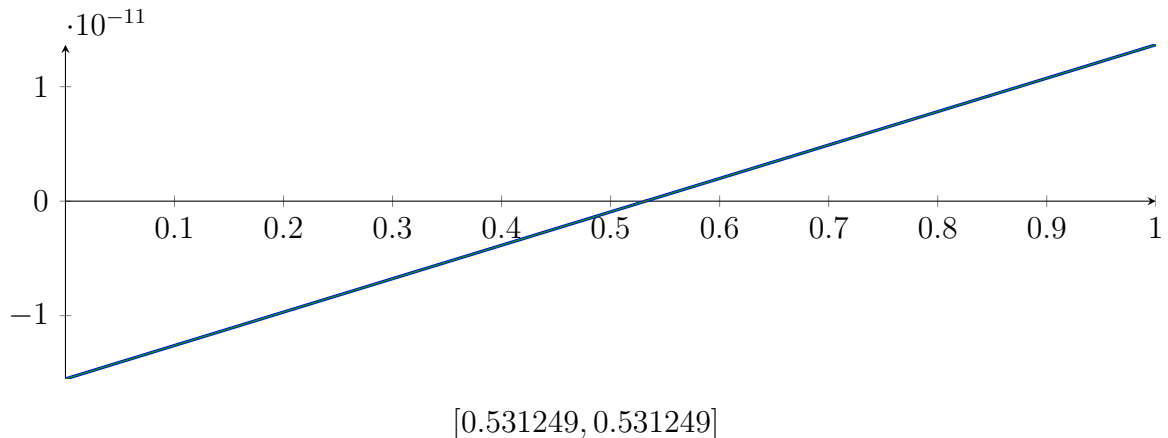
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 m &= -2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-4.81371 \cdot 10^{12}, 0.531249, 2.88823 \cdot 10^{12}\} \quad N(m) = \{-4.81371 \cdot 10^{12}, 0.531249, 2.88823 \cdot 10^{12}\}$$

Intersection intervals:



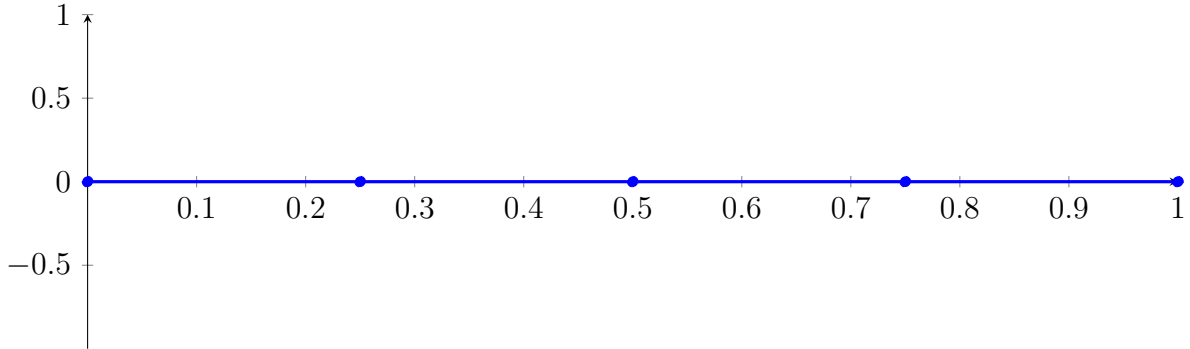
Longest intersection interval: $8.43287 \cdot 10^{-40}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

207.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

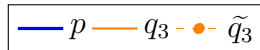
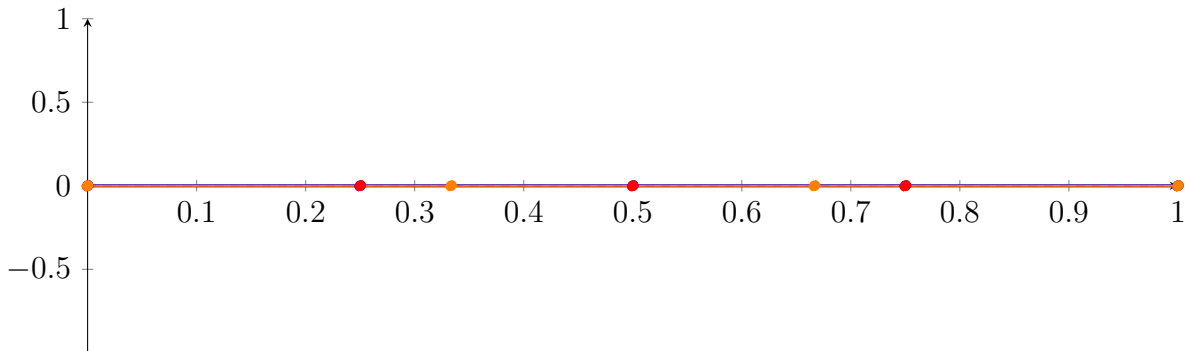
$$\begin{aligned} p &= -7.25914 \cdot 10^{-206} X^4 - 1.25865 \cdot 10^{-153} X^3 - 2.8739 \cdot 10^{-102} X^2 + 2.46075 \cdot 10^{-50} X - 2.11876 \cdot 10^{-14} \\ &= -2.11876 \cdot 10^{-14} B_{0,4}(X) - 2.11876 \cdot 10^{-14} B_{1,4}(X) - 2.11876 \\ &\quad \cdot 10^{-14} B_{2,4}(X) - 2.11876 \cdot 10^{-14} B_{3,4}(X) - 2.11876 \cdot 10^{-14} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.25865 \cdot 10^{-153} X^3 - 2.8739 \cdot 10^{-102} X^2 + 2.46075 \cdot 10^{-50} X - 2.11876 \cdot 10^{-14} \\ &= -2.11876 \cdot 10^{-14} B_{0,3} - 2.11876 \cdot 10^{-14} B_{1,3} - 2.11876 \cdot 10^{-14} B_{2,3} - 2.11876 \cdot 10^{-14} B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 6.32404 \cdot 10^{-322} X^4 - 1.25865 \cdot 10^{-153} X^3 - 2.8739 \cdot 10^{-102} X^2 + 2.46075 \cdot 10^{-50} X - 2.11876 \cdot 10^{-14} \\ &= -2.11876 \cdot 10^{-14} B_{0,4} - 2.11876 \cdot 10^{-14} B_{1,4} - 2.11876 \cdot 10^{-14} B_{2,4} - 2.11876 \cdot 10^{-14} B_{3,4} - 2.11876 \cdot 10^{-14} B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.22212 \cdot 10^{-207}$.

Bounding polynomials M and m :

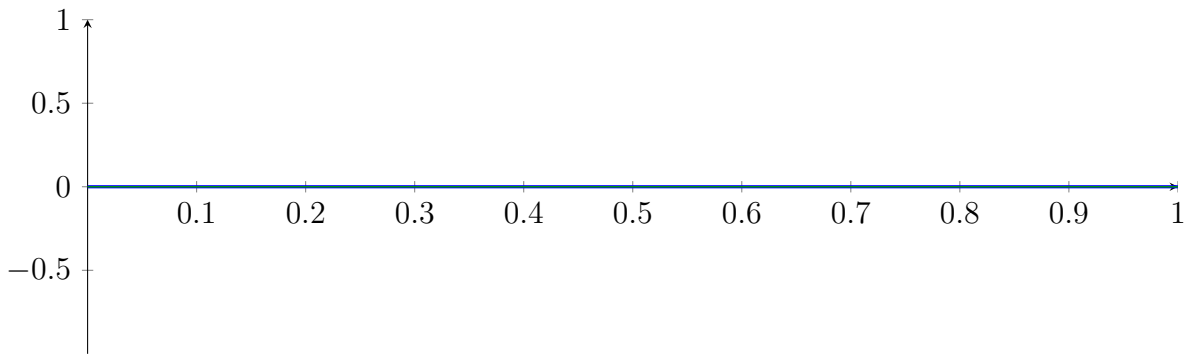
$$M = -1.25865 \cdot 10^{-153} X^3 - 2.8739 \cdot 10^{-102} X^2 + 2.46075 \cdot 10^{-50} X - 2.11876 \cdot 10^{-14}$$

$$m = -1.25865 \cdot 10^{-153} X^3 - 2.8739 \cdot 10^{-102} X^2 + 2.46075 \cdot 10^{-50} X - 2.11876 \cdot 10^{-14}$$

Root of M and m :

$$N(M) = \{-5.70827 \cdot 10^{51}, 3.91034 \cdot 10^{21}, 3.42496 \cdot 10^{51}\} \quad N(m) = \{-5.70827 \cdot 10^{51}, 3.91034 \cdot 10^{21}, 3.42496 \cdot 10^{51}\}$$

Intersection intervals:

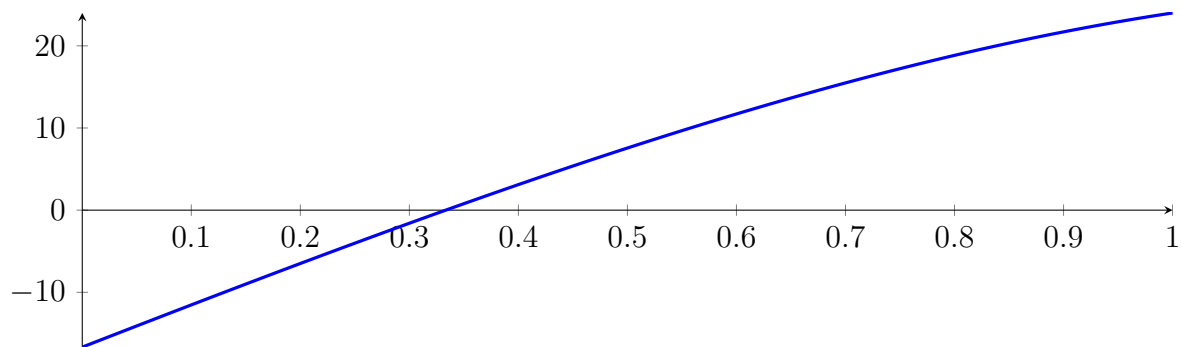


No intersection intervals with the x axis.

207.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

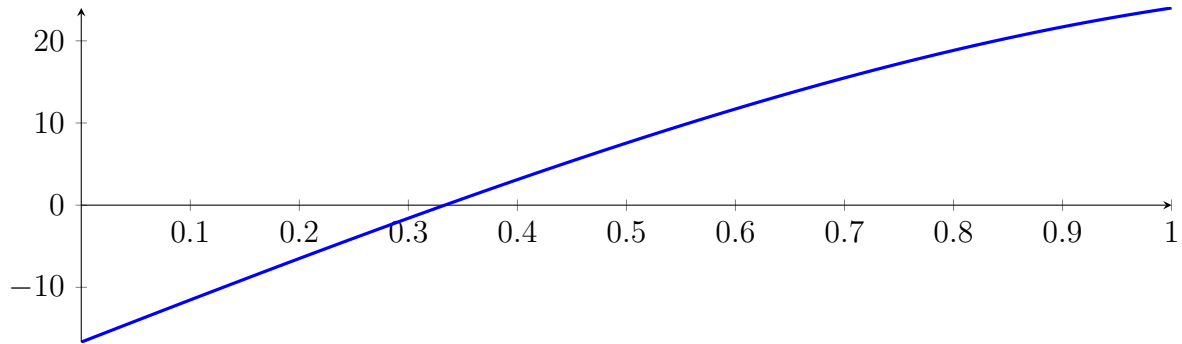
with precision $\varepsilon = 1 \cdot 10^{-64}$.

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$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called BezClip with input polynomial on interval $[0, 1]$:

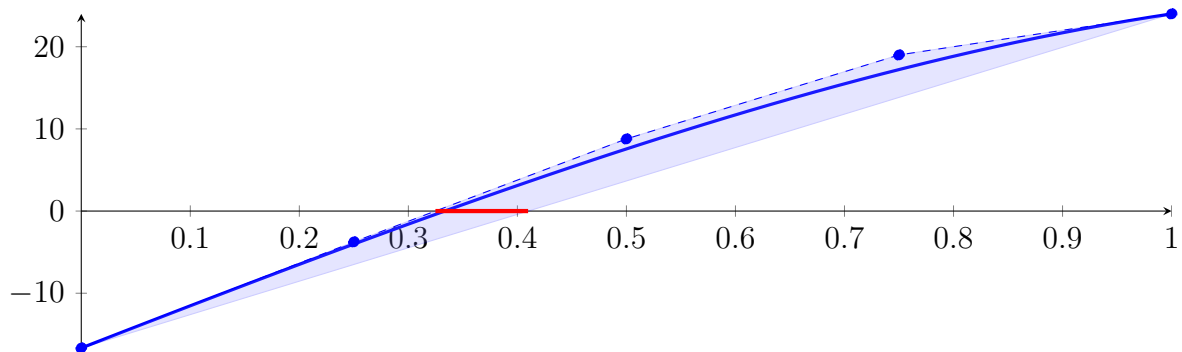
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



208.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.324834, 0.409836\}$$

Intersection intervals with the x axis:

$$[0.324834, 0.409836]$$

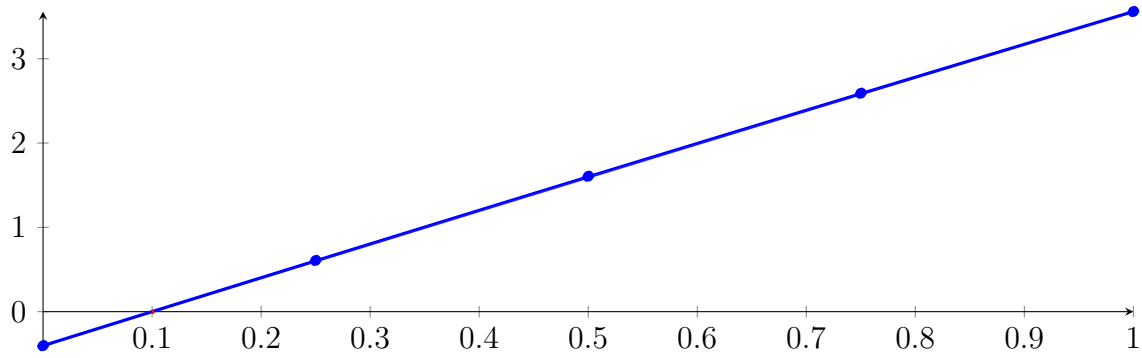
Longest intersection interval: 0.0850024

\implies Selective recursion: interval 1: $[0.324834, 0.409836]$,

208.2 Recursion Branch 1 1 in Interval 1: $[0.324834, 0.409836]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -5.22064 \cdot 10^{-05} X^4 - 0.0055067 X^3 - 0.0754159 X^2 + 4.04499 X - 0.403711 \\ &= -0.403711B_{0,4}(X) + 0.607537B_{1,4}(X) + 1.60621B_{2,4}(X) + 2.59095B_{3,4}(X) + 3.5603B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0998051, 0.101844\}$$

Intersection intervals with the x axis:

$$[0.0998051, 0.101844]$$

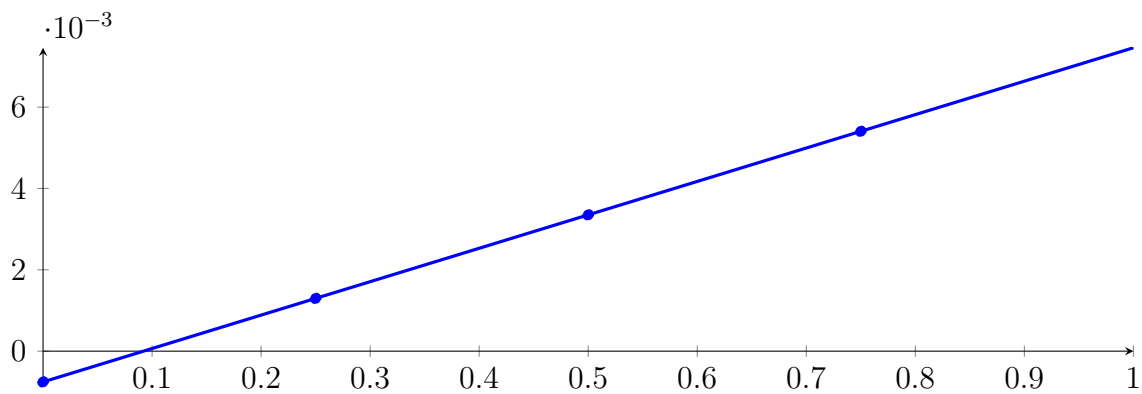
Longest intersection interval: 0.00203877

\implies Selective recursion: interval 1: $[0.333317, 0.333491]$,

208.3 Recursion Branch 1 1 1 in Interval 1: $[0.333317, 0.333491]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.01975 \cdot 10^{-16} X^4 - 4.6842 \cdot 10^{-11} X^3 - 3.20338 \cdot 10^{-07} X^2 + 0.00821576 X - 0.000756702 \\ &= -0.000756702 B_{0,4}(X) + 0.00129724 B_{1,4}(X) + 0.00335113 B_{2,4}(X) \\ &\quad + 0.00540496 B_{3,4}(X) + 0.00745874 B_{4,4}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0921037, 0.0921073\}$$

Intersection intervals with the x axis:

$$[0.0921037, 0.0921073]$$

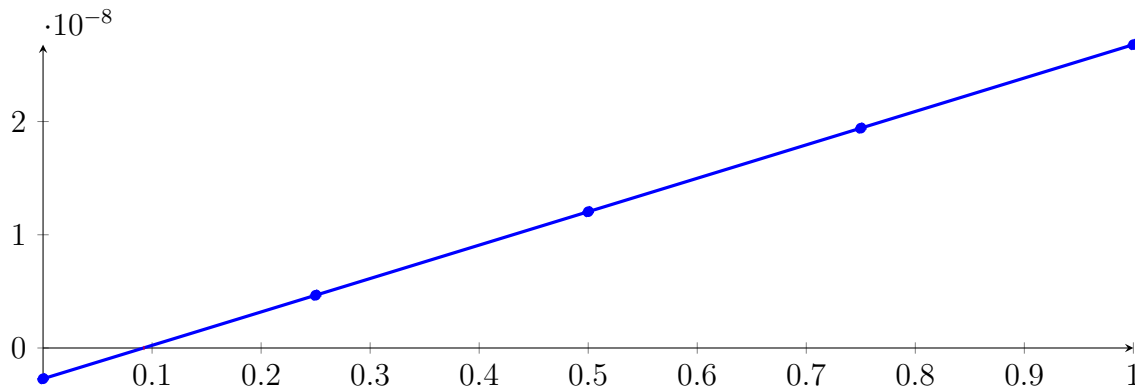
Longest intersection interval: $3.59185 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

208.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.50129 \cdot 10^{-37} X^4 - 2.17066 \cdot 10^{-27} X^3 - 4.13296 \cdot 10^{-18} X^2 + 2.95096 \cdot 10^{-08} X - 2.71749 \cdot 10^{-09} \\
 &= -2.71749 \cdot 10^{-09} B_{0,4}(X) + 4.6599 \cdot 10^{-09} B_{1,4}(X) + 1.20373 \\
 &\quad \cdot 10^{-08} B_{2,4}(X) + 1.94147 \cdot 10^{-08} B_{3,4}(X) + 2.67921 \cdot 10^{-08} B_{4,4}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

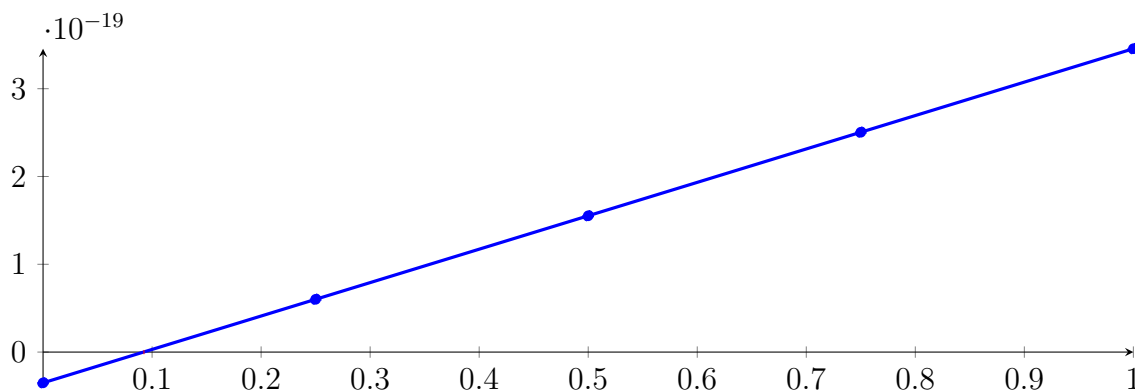
Longest intersection interval: $1.28975 \cdot 10^{-11}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

208.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -4.15417 \cdot 10^{-81} X^4 - 4.65699 \cdot 10^{-60} X^3 - 6.87497 \cdot 10^{-40} X^2 + 3.80599 \cdot 10^{-19} X - 3.50488 \cdot 10^{-20} \\
 &= -3.50488 \cdot 10^{-20} B_{0,4}(X) + 6.01009 \cdot 10^{-20} B_{1,4}(X) + 1.55251 \\
 &\quad \cdot 10^{-19} B_{2,4}(X) + 2.504 \cdot 10^{-19} B_{3,4}(X) + 3.4555 \cdot 10^{-19} B_{4,4}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

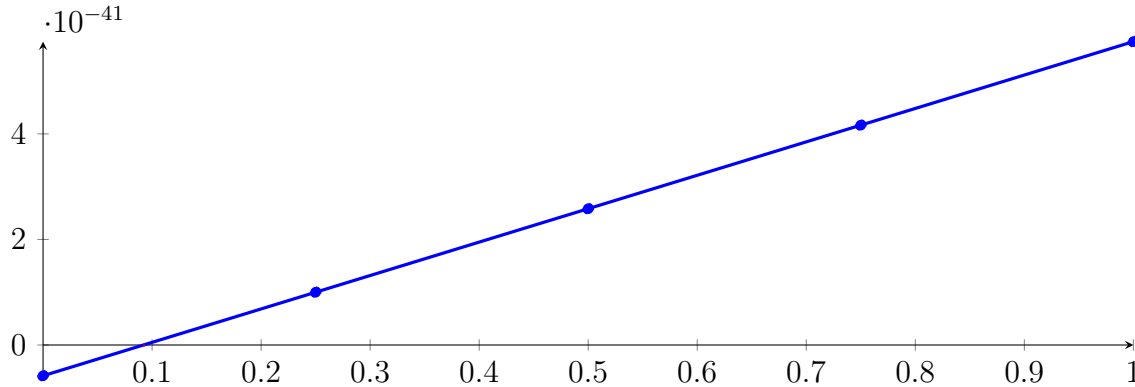
Longest intersection interval: $1.66345 \cdot 10^{-22}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

208.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.18068 \cdot 10^{-168} X^4 - 2.14355 \cdot 10^{-125} X^3 - 1.90234 \cdot 10^{-83} X^2 + 6.33106 \cdot 10^{-41} X - 5.83018 \cdot 10^{-42} \\
 &= -5.83018 \cdot 10^{-42} B_{0,4}(X) + 9.99747 \cdot 10^{-42} B_{1,4}(X) + 2.58251 \\
 &\quad \cdot 10^{-41} B_{2,4}(X) + 4.16528 \cdot 10^{-41} B_{3,4}(X) + 5.74804 \cdot 10^{-41} B_{4,4}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

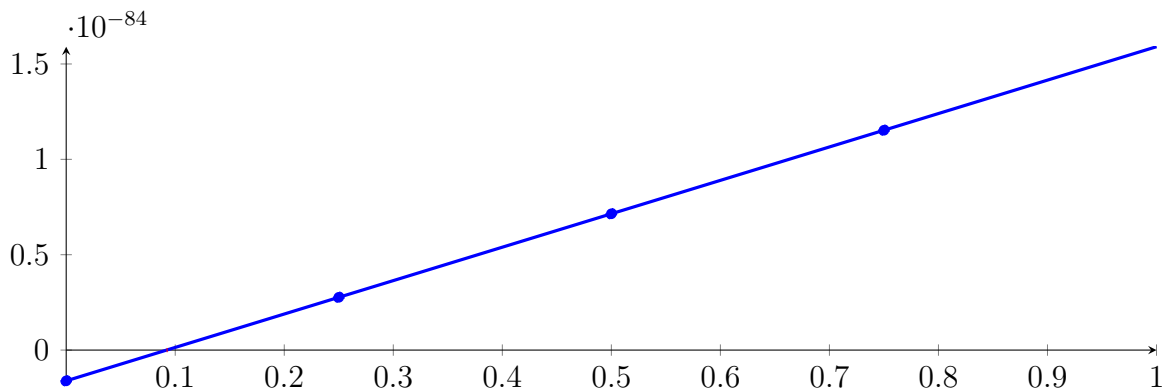
Longest intersection interval: $2.76706 \cdot 10^{-44}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

208.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.86463 \cdot 10^{-342} X^4 - 4.54137 \cdot 10^{-256} X^3 - 1.45655 \cdot 10^{-170} X^2 + 1.75184 \cdot 10^{-84} X - 1.61324 \cdot 10^{-85} \\
 &= -1.61324 \cdot 10^{-85} B_{0,4}(X) + 2.76636 \cdot 10^{-85} B_{1,4}(X) + 7.14596 \\
 &\quad \cdot 10^{-85} B_{2,4}(X) + 1.15256 \cdot 10^{-84} B_{3,4}(X) + 1.59052 \cdot 10^{-84} B_{4,4}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0920885, 0.0920885\}$$

Intersection intervals with the x axis:

$$[0.0920885, 0.0920885]$$

Longest intersection interval: $7.65661 \cdot 10^{-88}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

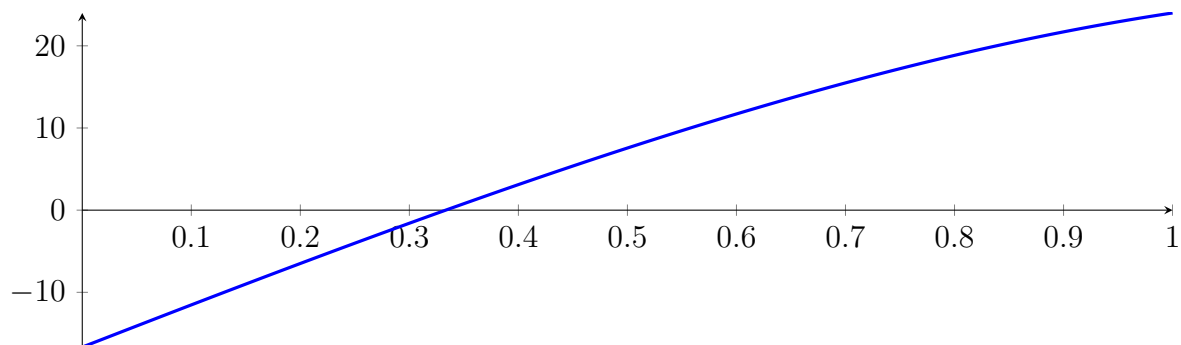
208.8 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 8!

208.9 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

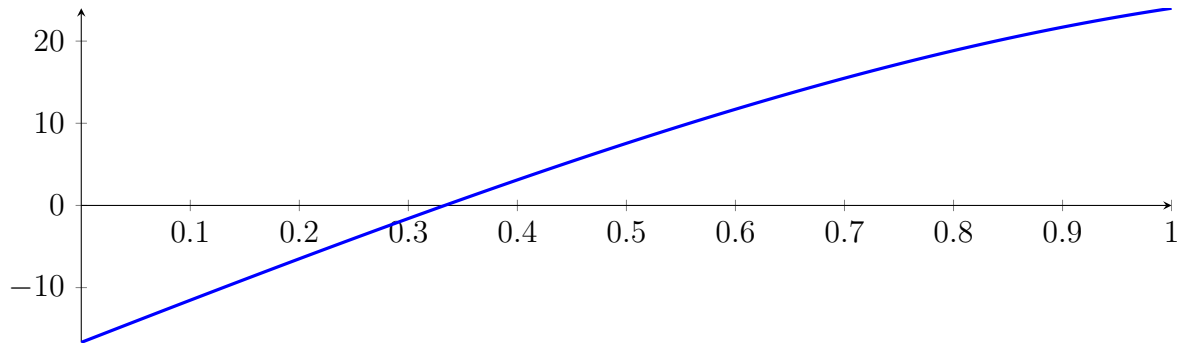
with precision $\varepsilon = 1 \cdot 10^{-128}$.

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$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called QuadClip with input polynomial on interval $[0, 1]$:

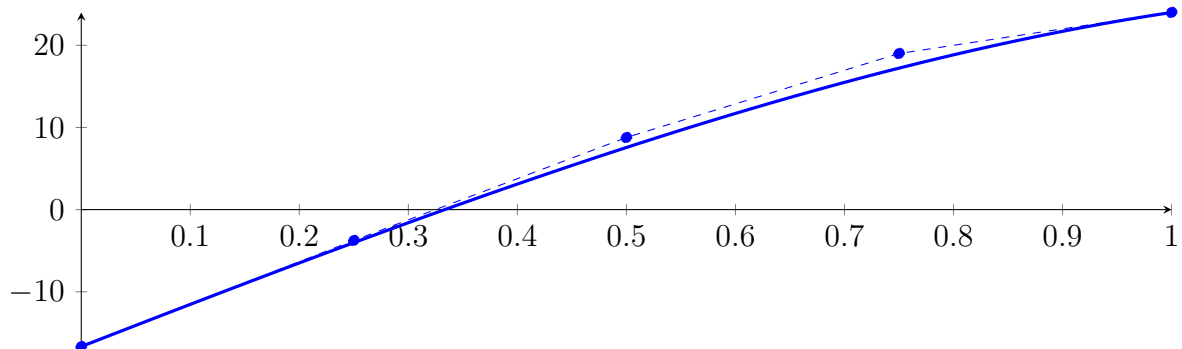
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



209.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

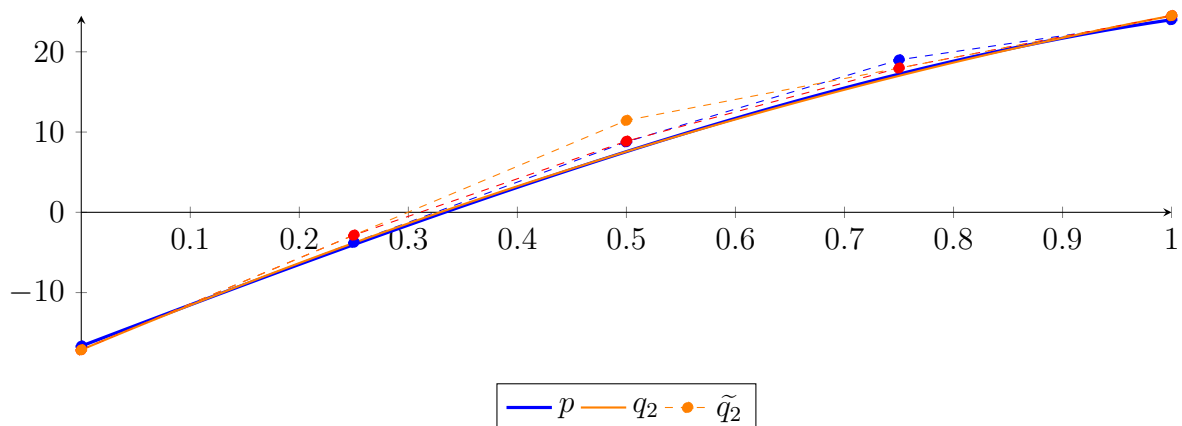
$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,2} + 11.4548B_{1,2} + 24.4976B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -2.13607 \cdot 10^{-306}X^4 + 7.12024 \cdot 10^{-306}X^3 - 15.5476X^2 + 57.181X - 17.1357 \\ &= -17.1357B_{0,4} - 2.84048B_{1,4} + 8.86349B_{2,4} + 17.9762B_{3,4} + 24.4976B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.02381$.

Bounding polynomials M and m :

$$M = -15.5476X^2 + 57.181X - 16.1119$$

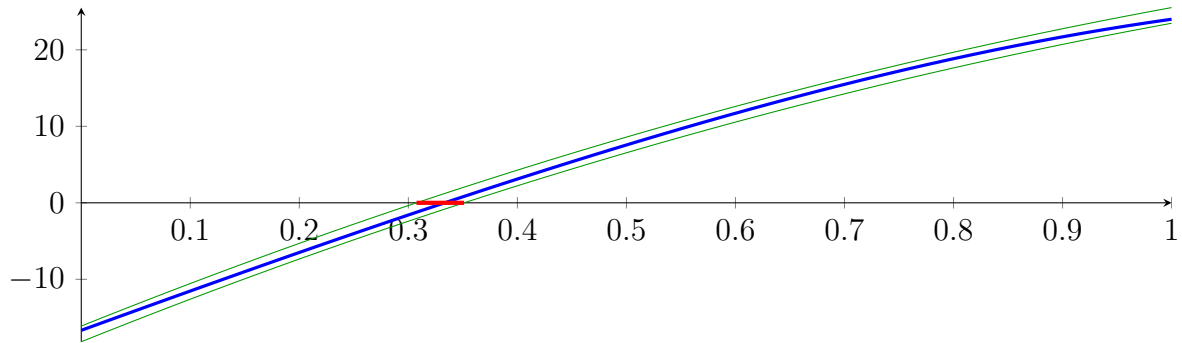
$$m = -15.5476X^2 + 57.181X - 18.1595$$

Root of M and m :

$$N(M) = \{0.307477, 3.37032\}$$

$$N(m) = \{0.351097, 3.3267\}$$

Intersection intervals:



$$[0.307477, 0.351097]$$

Longest intersection interval: 0.0436205

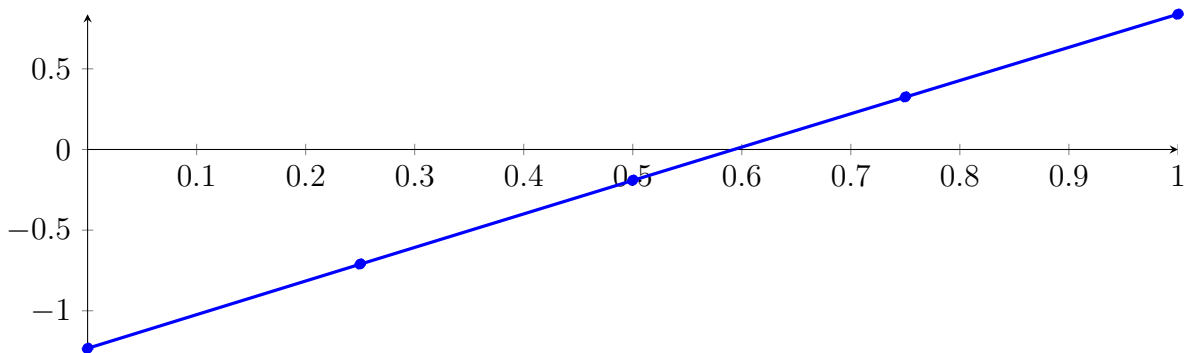
⇒ Selective recursion: **interval 1:** $[0.307477, 0.351097]$,

209.2 Recursion Branch 1 1 in Interval 1: $[0.307477, 0.351097]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.62044 \cdot 10^{-06} X^4 - 0.000738404 X^3 - 0.0189752 X^2 + 2.09121 X - 1.23278$$

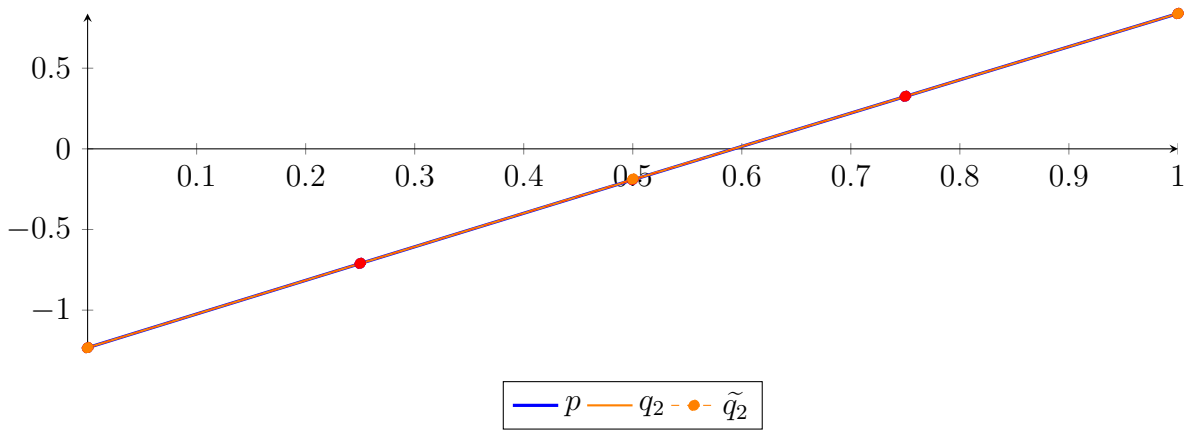
$$= -1.23278 B_{0,4}(X) - 0.709974 B_{1,4}(X) - 0.190333 B_{2,4}(X) + 0.325959 B_{3,4}(X) + 0.838717 B_{4,4}(X)$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281 B_{0,2} - 0.186985 B_{1,2} + 0.838754 B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -4.5614 \cdot 10^{-307} X^4 + 1.02353 \cdot 10^{-306} X^3 - 0.020089X^2 + 2.09166X - 1.23281 \\ &= -1.23281 B_{0,4} - 0.709899 B_{1,4} - 0.190333 B_{2,4} + 0.325885 B_{3,4} + 0.838754 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 7.47713 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -0.020089X^2 + 2.09166X - 1.23274$$

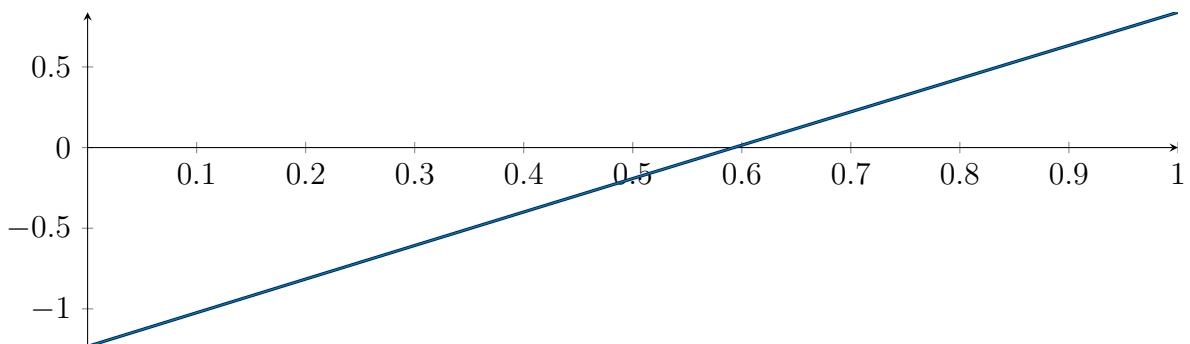
$$m = -0.020089X^2 + 2.09166X - 1.23289$$

Root of M and m :

$$N(M) = \{0.592734, 103.527\}$$

$$N(m) = \{0.592807, 103.527\}$$

Intersection intervals:



$$[0.592734, 0.592807]$$

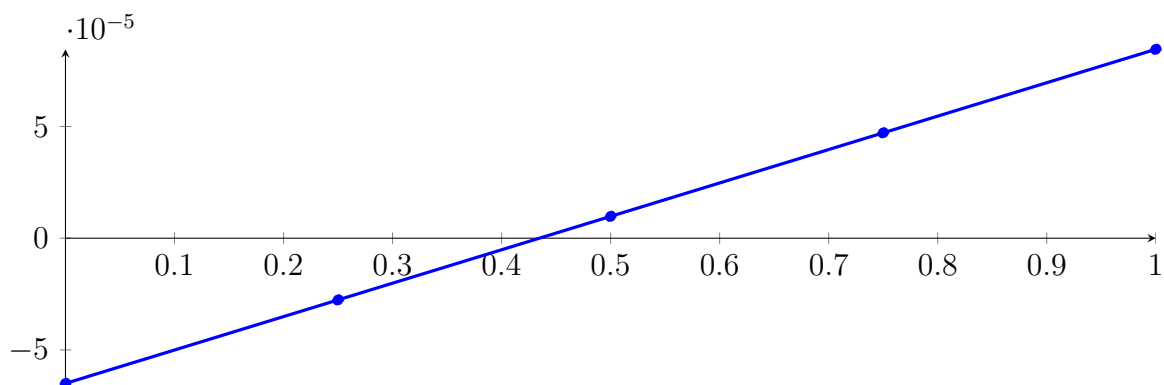
Longest intersection interval: $7.23183 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333332, 0.333335]$,

209.3 Recursion Branch 1 1 1 in Interval 1: $[0.333332, 0.333335]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -9.9027 \cdot 10^{-23} X^4 - 2.82525 \cdot 10^{-16} X^3 - 1.06146 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05} \\ &= -6.50069 \cdot 10^{-05} B_{0,4}(X) - 2.76196 \cdot 10^{-05} B_{1,4}(X) + 9.76777 \\ &\quad \cdot 10^{-06} B_{2,4}(X) + 4.71551 \cdot 10^{-05} B_{3,4}(X) + 8.45424 \cdot 10^{-05} B_{4,4}(X) \end{aligned}$$



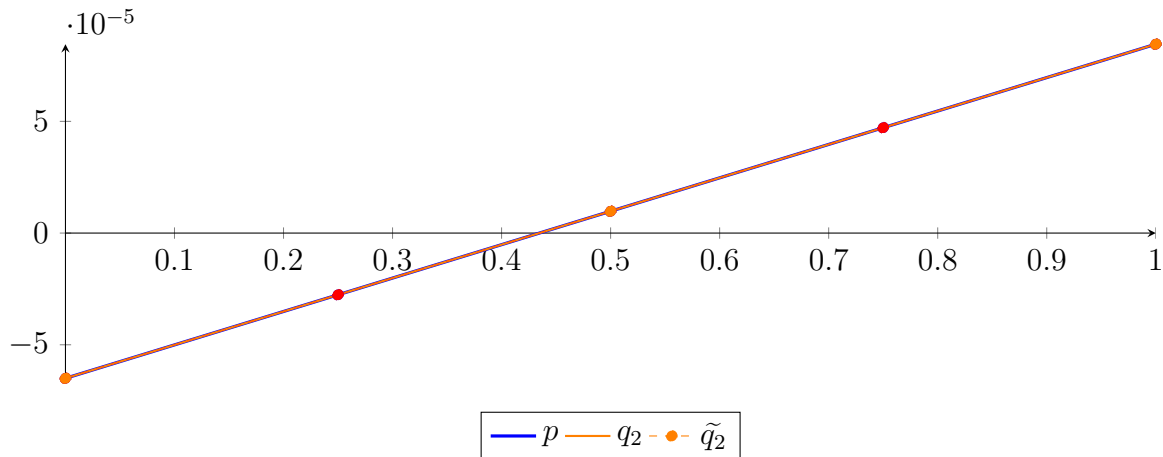
Degree reduction and raising:

$$q_2 = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,2} + 9.76779 \cdot 10^{-06} B_{1,2} + 8.45424 \cdot 10^{-05} B_{2,2}$$

$$\tilde{q}_2 = 1.22227 \cdot 10^{-311} X^4 - 1.62969 \cdot 10^{-311} X^3 - 1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

$$= -6.50069 \cdot 10^{-05} B_{0,4} - 2.76196 \cdot 10^{-05} B_{1,4} + 9.76777 \cdot 10^{-06} B_{2,4} + 4.71551 \cdot 10^{-05} B_{3,4} + 8.45424 \cdot 10^{-05} B_{4,4}$$



The maximum difference of the Bézier coefficients is $\delta = 2.82526 \cdot 10^{-17}$.

Bounding polynomials M and m :

$$M = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

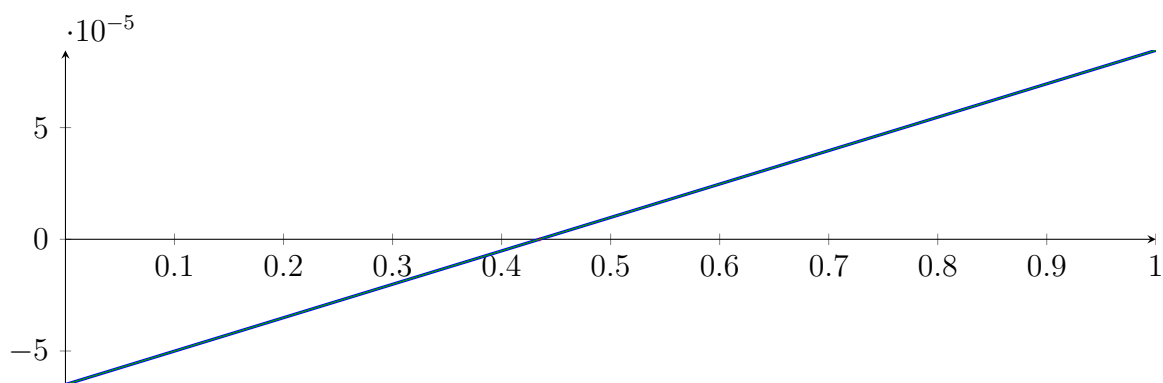
$$m = -1.06147 \cdot 10^{-10} X^2 + 0.000149549 X - 6.50069 \cdot 10^{-05}$$

Root of M and m :

$$N(M) = \{0.434685, 1.4089 \cdot 10^6\}$$

$$N(m) = \{0.434685, 1.4089 \cdot 10^6\}$$

Intersection intervals:



$$[0.434685, 0.434685]$$

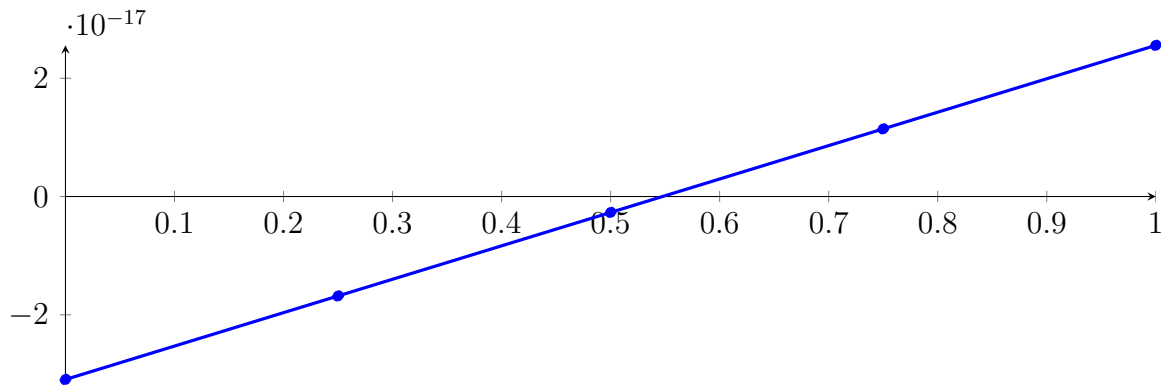
Longest intersection interval: $3.77836 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

209.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

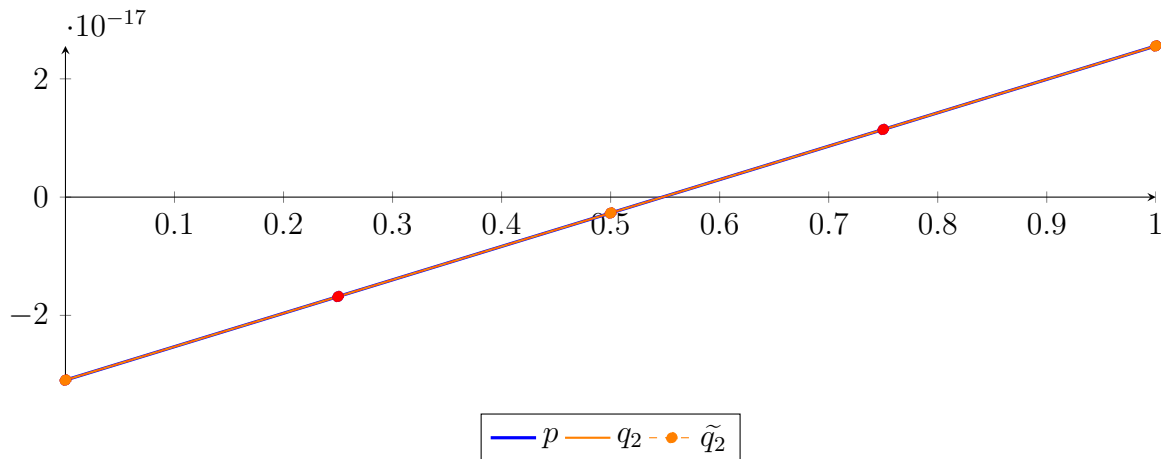
$$\begin{aligned} p &= -2.01821 \cdot 10^{-72} X^4 - 1.52394 \cdot 10^{-53} X^3 - 1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17} \\ &= -3.09418 \cdot 10^{-17} B_{0,4}(X) - 1.68155 \cdot 10^{-17} B_{1,4}(X) - 2.68924 \\ &\quad \cdot 10^{-18} B_{2,4}(X) + 1.1437 \cdot 10^{-17} B_{3,4}(X) + 2.55633 \cdot 10^{-17} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17} \\ &= -3.09418 \cdot 10^{-17} B_{0,2} - 2.68924 \cdot 10^{-18} B_{1,2} + 2.55633 \cdot 10^{-17} B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= -7.41098 \cdot 10^{-324} X^4 + 1.72923 \cdot 10^{-323} X^3 - 1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17} \\ &= -3.09418 \cdot 10^{-17} B_{0,4} - 1.68155 \cdot 10^{-17} B_{1,4} - 2.68924 \cdot 10^{-18} B_{2,4} + 1.1437 \cdot 10^{-17} B_{3,4} + 2.55633 \cdot 10^{-17} B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.52394 \cdot 10^{-54}$.

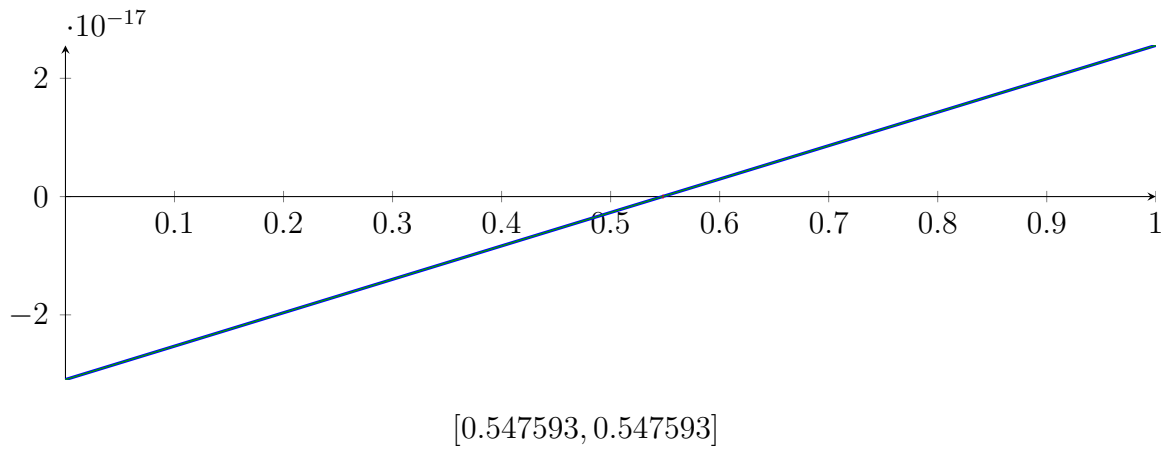
Bounding polynomials M and m :

$$\begin{aligned} M &= -1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17} \\ m &= -1.51535 \cdot 10^{-35} X^2 + 5.65051 \cdot 10^{-17} X - 3.09418 \cdot 10^{-17} \end{aligned}$$

Root of M and m :

$$N(M) = \{0.547593, 3.72886 \cdot 10^{18}\} \quad N(m) = \{0.547593, 3.72886 \cdot 10^{18}\}$$

Intersection intervals:

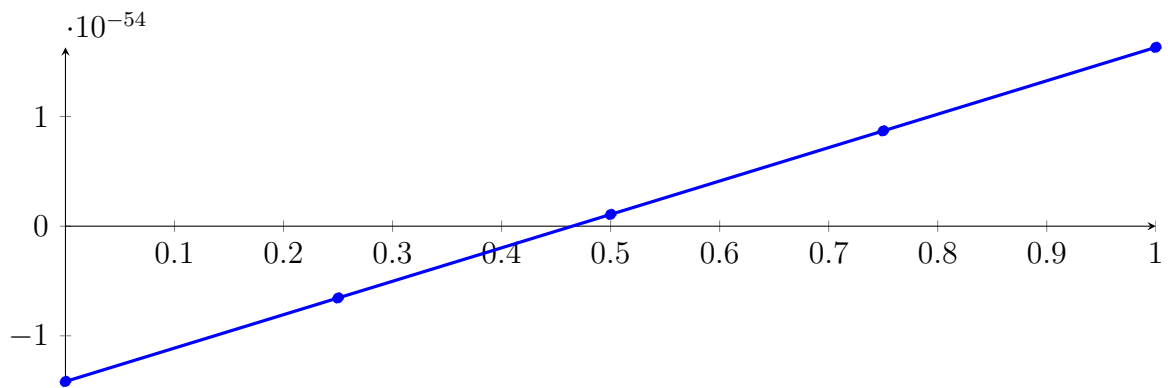


Longest intersection interval: $5.39398 \cdot 10^{-38}$
 \implies Selective recursion: interval 1: [0.333333, 0.333333],

209.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

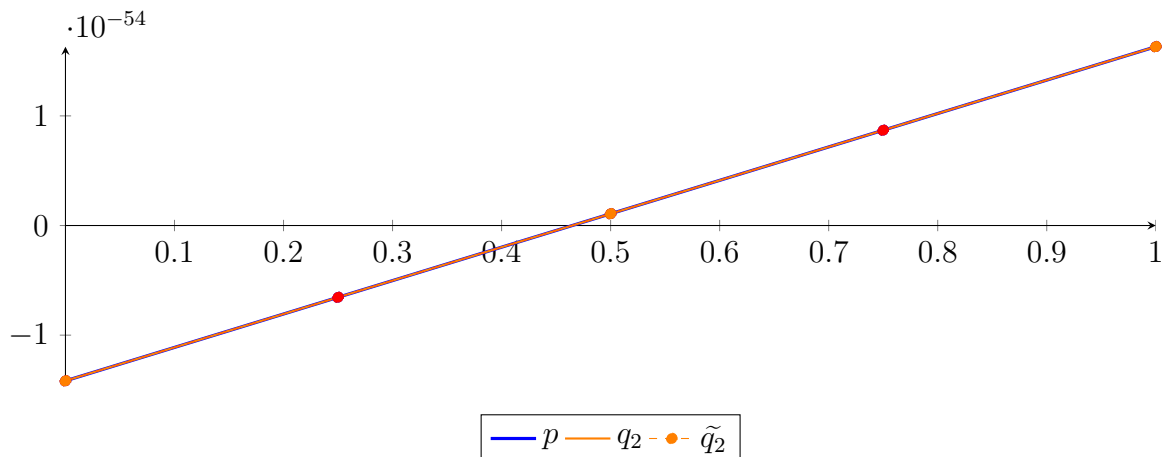
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.70846 \cdot 10^{-221} X^4 - 2.39164 \cdot 10^{-165} X^3 - 4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54} \\
 &= -1.41679 \cdot 10^{-54} B_{0,4}(X) - 6.54819 \cdot 10^{-55} B_{1,4}(X) + 1.0715 \\
 &\quad \cdot 10^{-55} B_{2,4}(X) + 8.6912 \cdot 10^{-55} B_{3,4}(X) + 1.63109 \cdot 10^{-54} B_{4,4}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54} \\
 &= -1.41679 \cdot 10^{-54} B_{0,2} + 1.0715 \cdot 10^{-55} B_{1,2} + 1.63109 \cdot 10^{-54} B_{2,2} \\
 \tilde{q}_2 &= 7.25964 \cdot 10^{-362} X^4 - 5.80771 \cdot 10^{-362} X^3 - 4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54} \\
 &= -1.41679 \cdot 10^{-54} B_{0,4} - 6.54819 \cdot 10^{-55} B_{1,4} + 1.0715 \cdot 10^{-55} B_{2,4} + 8.6912 \cdot 10^{-55} B_{3,4} + 1.63109 \cdot 10^{-54} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.39164 \cdot 10^{-166}$.

Bounding polynomials M and m :

$$M = -4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54}$$

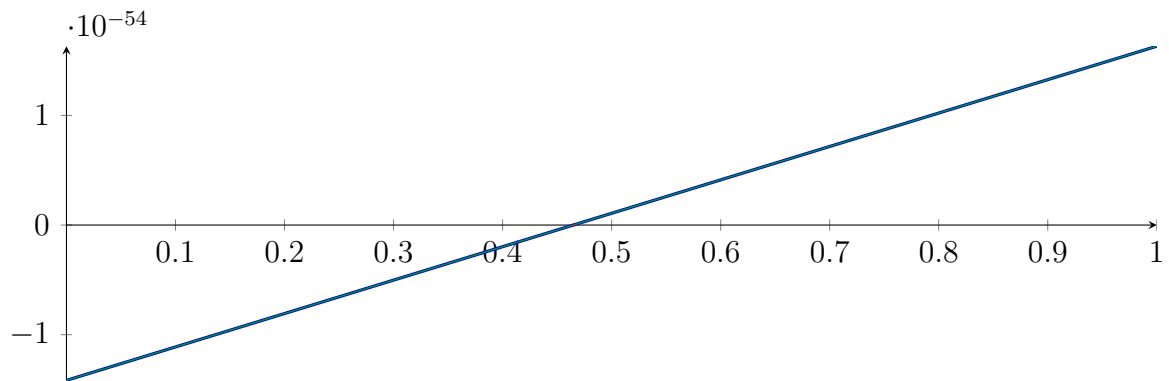
$$m = -4.40891 \cdot 10^{-110} X^2 + 3.04788 \cdot 10^{-54} X - 1.41679 \cdot 10^{-54}$$

Root of M and m :

$$N(M) = \{0.464844, 6.91299 \cdot 10^{55}\}$$

$$N(m) = \{0.464844, 6.91299 \cdot 10^{55}\}$$

Intersection intervals:



$$[0.464844, 0.464844]$$

Longest intersection interval: $1.56938 \cdot 10^{-112}$

\implies Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

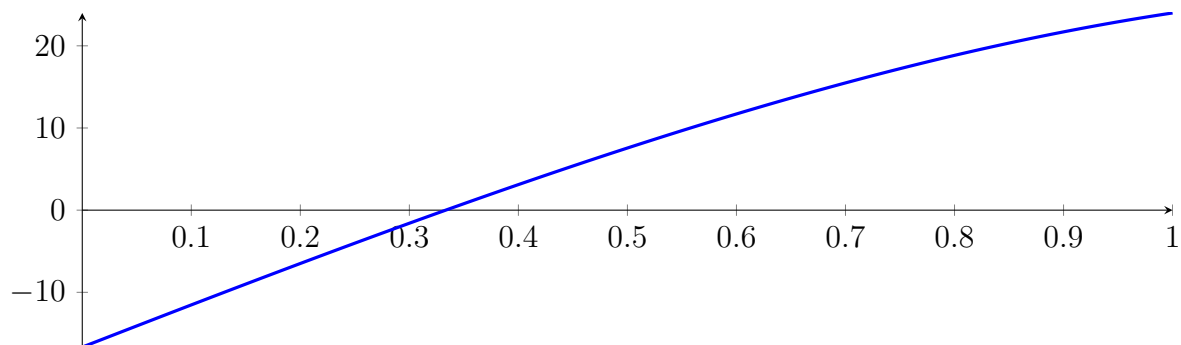
209.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

209.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

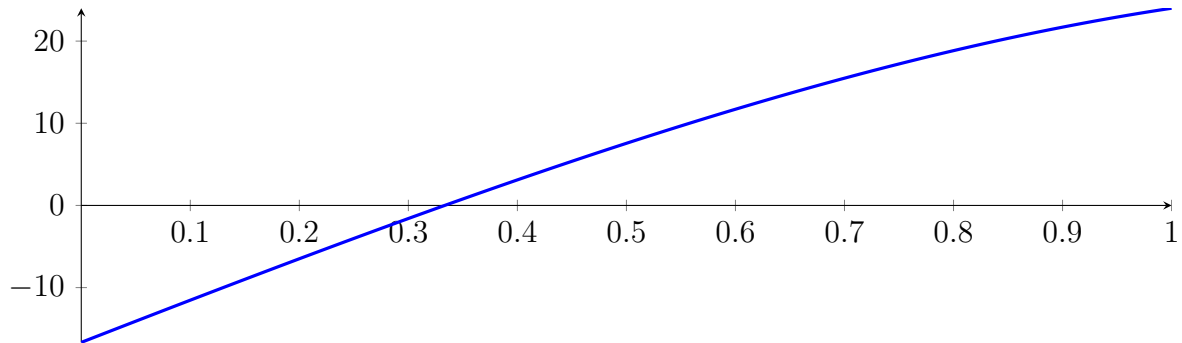
with precision $\varepsilon = 1 \cdot 10^{-128}$.

210 Running CubeClip on f_4 with epsilon 128

$$-1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$

Called CubeClip with input polynomial on interval $[0, 1]$:

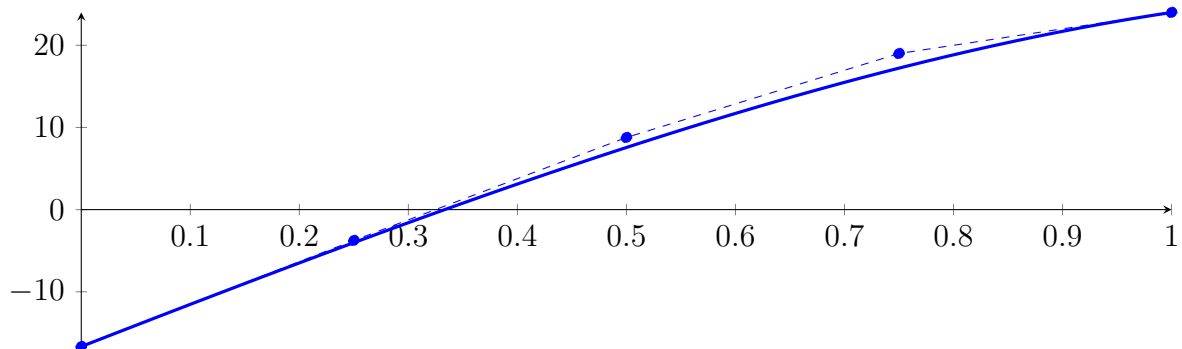
$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



210.1 Recursion Branch 1 for Input Interval $[0, 1]$

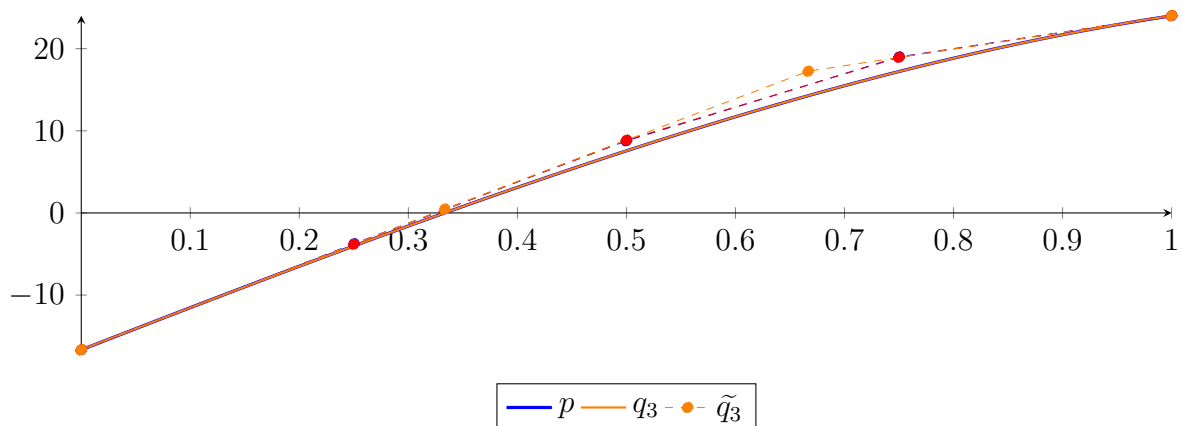
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667 \\ &= -16.6667B_{0,4}(X) - 3.75B_{1,4}(X) + 8.77778B_{2,4}(X) + 19B_{3,4}(X) + 24B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,3} + 0.474603B_{1,3} + 17.2524B_{2,3} + 24.0143B_{3,3} \\ \tilde{q}_3 &= -3.56012 \cdot 10^{-307}X^4 - 9.66667X^3 - 1.04762X^2 + 51.381X - 16.6524 \\ &= -16.6524B_{0,4} - 3.80714B_{1,4} + 8.86349B_{2,4} + 18.9429B_{3,4} + 24.0143B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.0857143$.

Bounding polynomials M and m :

$$M = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.5667$$

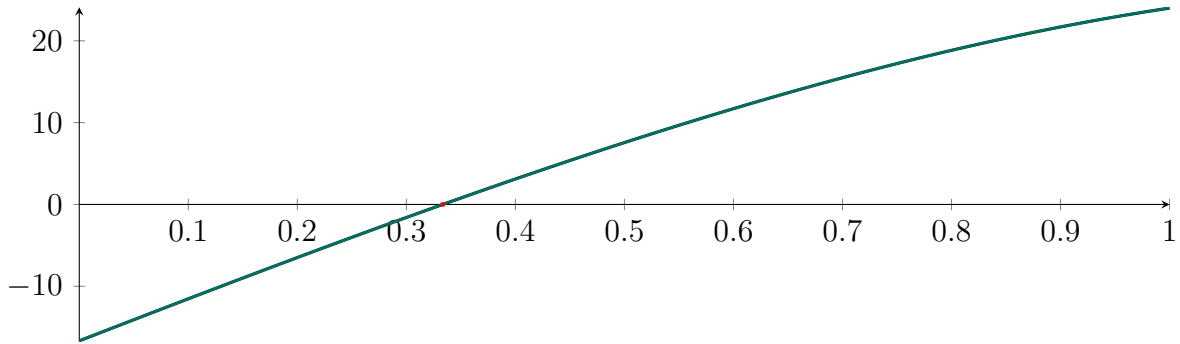
$$m = -9.66667X^3 - 1.04762X^2 + 51.381X - 16.7381$$

Root of M and m :

$$N(M) = \{-2.5042, 0.331524, 2.0643\}$$

$$N(m) = \{-2.50557, 0.335136, 2.06206\}$$

Intersection intervals:



$$[0.331524, 0.335136]$$

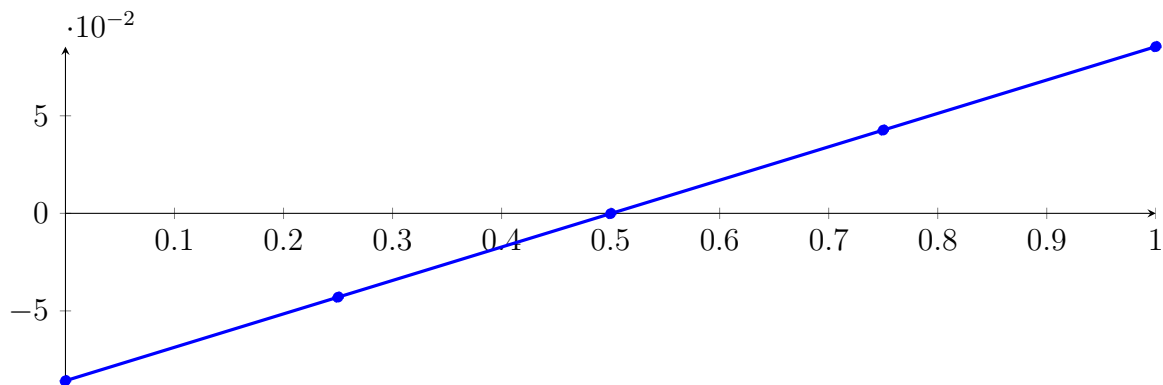
Longest intersection interval: 0.00361204

⇒ Selective recursion: **interval 1:** $[0.331524, 0.335136]$,

210.2 Recursion Branch 1 1 in Interval 1: $[0.331524, 0.335136]$

Normalized monomial und Bézier representations and the Bézier polygon:

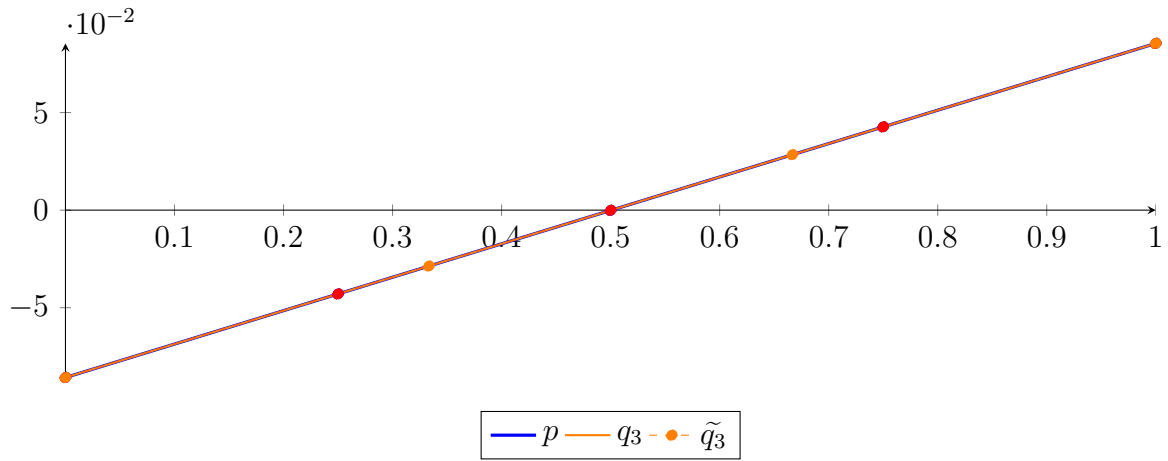
$$\begin{aligned} p &= -1.70219 \cdot 10^{-10} X^4 - 4.23789 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4}(X) - 0.0429507 B_{1,4}(X) - 0.000129666 B_{2,4}(X) \\ &\quad + 0.0426682 B_{3,4}(X) + 0.0854427 B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,3} - 0.0286693 B_{1,3} + 0.02841 B_{2,3} + 0.0854427 B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -1.39067 \cdot 10^{-309} X^4 - 4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948 \\ &= -0.0857948 B_{0,4} - 0.0429507 B_{1,4} - 0.000129666 B_{2,4} + 0.0426682 B_{3,4} + 0.0854427 B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.45902 \cdot 10^{-11}$.

Bounding polynomials M and m :

$$M = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

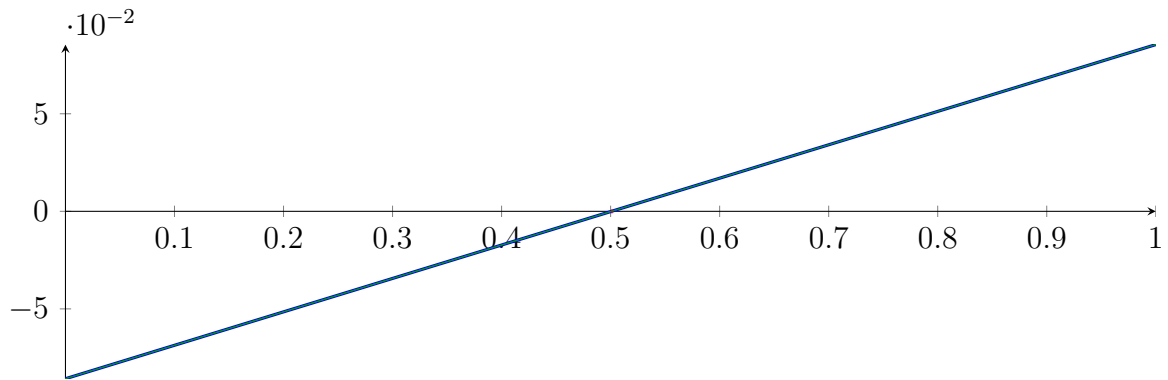
$$m = -4.2413 \cdot 10^{-07} X^3 - 0.000138529 X^2 + 0.171376 X - 0.0857948$$

Root of M and m :

$$N(M) = \{-819.802, 0.500825, 492.682\}$$

$$N(m) = \{-819.802, 0.500825, 492.682\}$$

Intersection intervals:



$$[0.500825, 0.500825]$$

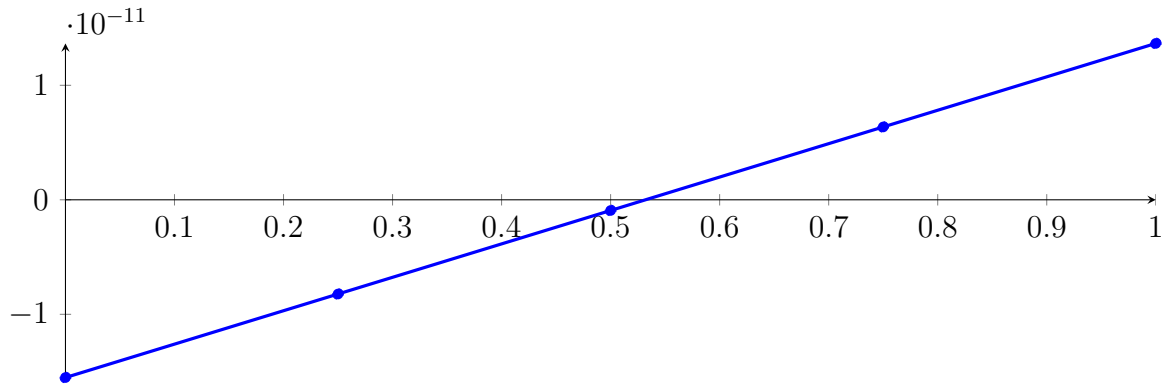
Longest intersection interval: $1.7041 \cdot 10^{-10}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

210.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

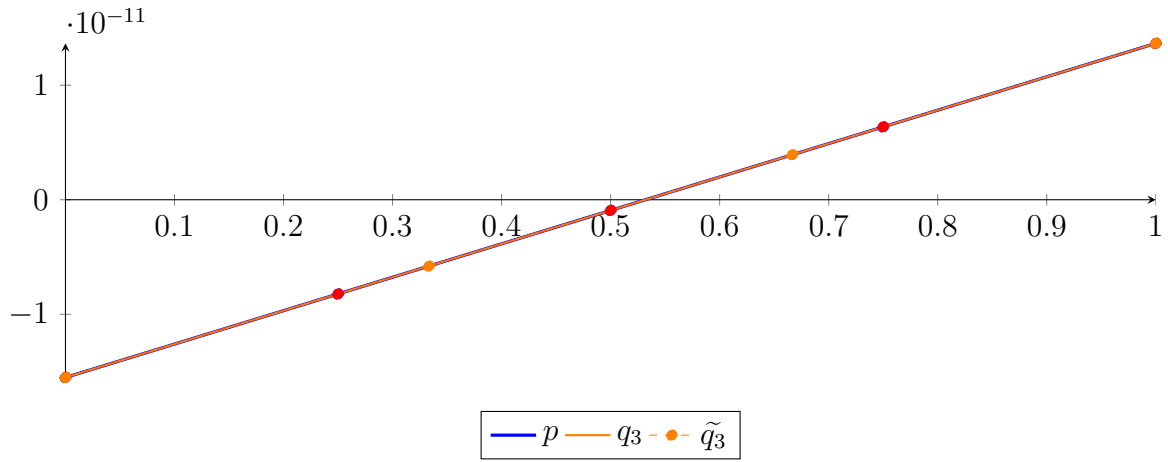
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.43544 \cdot 10^{-49} X^4 - 2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\ &= -1.55233 \cdot 10^{-11} B_{0,4}(X) - 8.22817 \cdot 10^{-12} B_{1,4}(X) - 9.33052 \\ &\quad \cdot 10^{-13} B_{2,4}(X) + 6.36207 \cdot 10^{-12} B_{3,4}(X) + 1.36572 \cdot 10^{-11} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 &= -1.55233 \cdot 10^{-11} B_{0,3} - 5.79646 \cdot 10^{-12} B_{1,3} + 3.93036 \cdot 10^{-12} B_{2,3} + 1.36572 \cdot 10^{-11} B_{3,3} \\
 \tilde{q}_3 &= -3.23791 \cdot 10^{-319} X^4 - 2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 &= -1.55233 \cdot 10^{-11} B_{0,4} - 8.22817 \cdot 10^{-12} B_{1,4} - 9.33052 \cdot 10^{-13} B_{2,4} + 6.36207 \cdot 10^{-12} B_{3,4} + 1.36572 \cdot 10^{-11} B_{4,4}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.23038 \cdot 10^{-50}$.

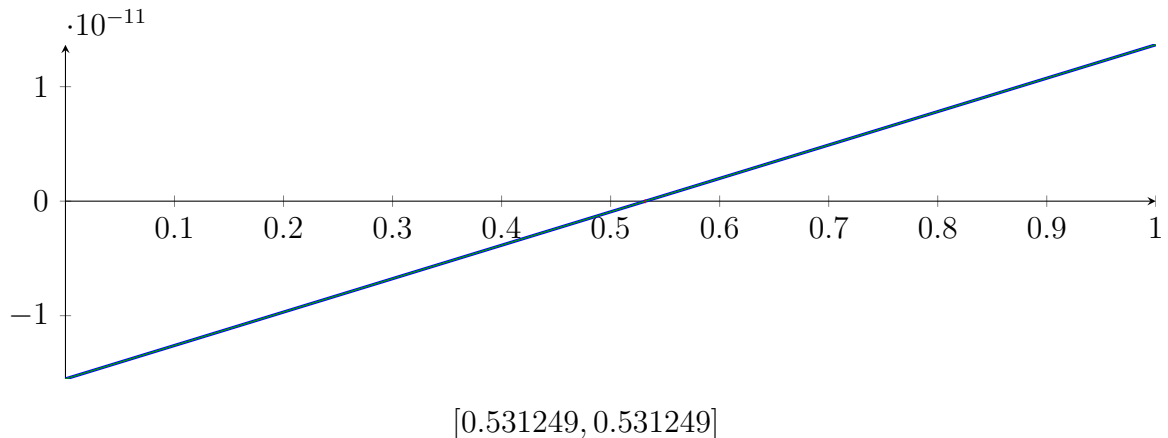
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11} \\
 m &= -2.09885 \cdot 10^{-36} X^3 - 4.0413 \cdot 10^{-24} X^2 + 2.91805 \cdot 10^{-11} X - 1.55233 \cdot 10^{-11}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-4.81371 \cdot 10^{12}, 0.531249, 2.88823 \cdot 10^{12}\} \quad N(m) = \{-4.81371 \cdot 10^{12}, 0.531249, 2.88823 \cdot 10^{12}\}$$

Intersection intervals:



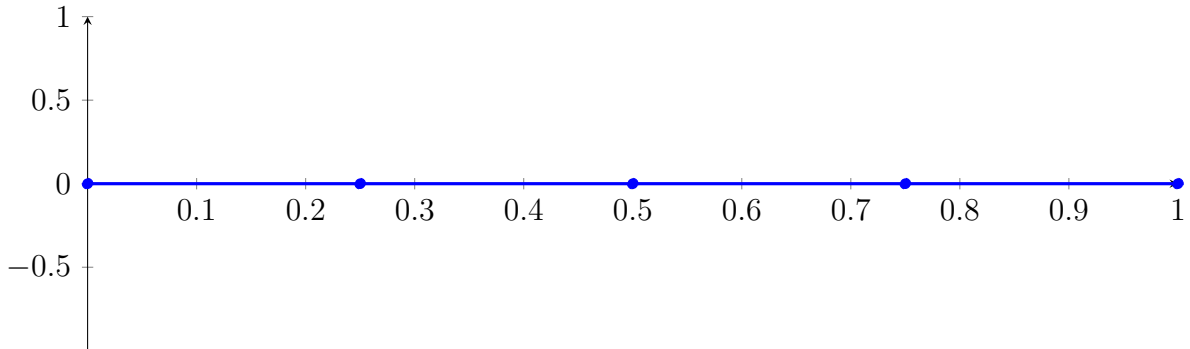
Longest intersection interval: $8.43287 \cdot 10^{-40}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

210.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

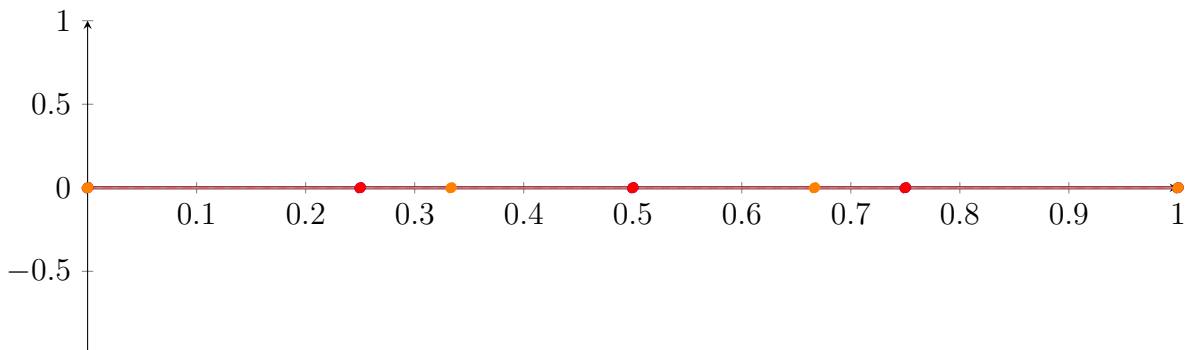
$$\begin{aligned} p &= -7.25914 \cdot 10^{-206} X^4 - 1.25865 \cdot 10^{-153} X^3 - 2.8739 \cdot 10^{-102} X^2 + 2.46075 \cdot 10^{-50} X - 2.11876 \cdot 10^{-14} \\ &= -2.11876 \cdot 10^{-14} B_{0,4}(X) - 2.11876 \cdot 10^{-14} B_{1,4}(X) - 2.11876 \\ &\quad \cdot 10^{-14} B_{2,4}(X) - 2.11876 \cdot 10^{-14} B_{3,4}(X) - 2.11876 \cdot 10^{-14} B_{4,4}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.25865 \cdot 10^{-153} X^3 - 2.8739 \cdot 10^{-102} X^2 + 2.46075 \cdot 10^{-50} X - 2.11876 \cdot 10^{-14} \\ &= -2.11876 \cdot 10^{-14} B_{0,3} - 2.11876 \cdot 10^{-14} B_{1,3} - 2.11876 \cdot 10^{-14} B_{2,3} - 2.11876 \cdot 10^{-14} B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= 6.32404 \cdot 10^{-322} X^4 - 1.25865 \cdot 10^{-153} X^3 - 2.8739 \cdot 10^{-102} X^2 + 2.46075 \cdot 10^{-50} X - 2.11876 \cdot 10^{-14} \\ &= -2.11876 \cdot 10^{-14} B_{0,4} - 2.11876 \cdot 10^{-14} B_{1,4} - 2.11876 \cdot 10^{-14} B_{2,4} - 2.11876 \cdot 10^{-14} B_{3,4} - 2.11876 \cdot 10^{-14} B_{4,4} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.22212 \cdot 10^{-207}$.

Bounding polynomials M and m :

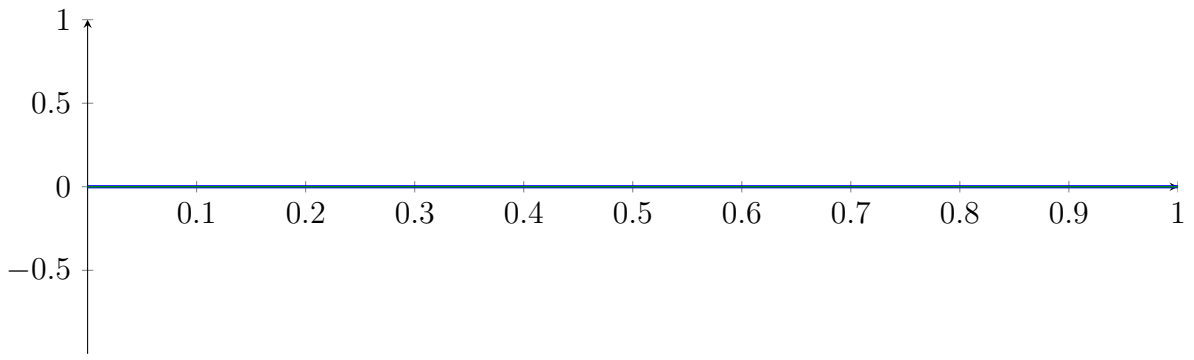
$$M = -1.25865 \cdot 10^{-153} X^3 - 2.8739 \cdot 10^{-102} X^2 + 2.46075 \cdot 10^{-50} X - 2.11876 \cdot 10^{-14}$$

$$m = -1.25865 \cdot 10^{-153} X^3 - 2.8739 \cdot 10^{-102} X^2 + 2.46075 \cdot 10^{-50} X - 2.11876 \cdot 10^{-14}$$

Root of M and m :

$$N(M) = \{-5.70827 \cdot 10^{51}, 3.91034 \cdot 10^{21}, 3.42496 \cdot 10^{51}\} \quad N(m) = \{-5.70827 \cdot 10^{51}, 3.91034 \cdot 10^{21}, 3.42496 \cdot 10^{51}\}$$

Intersection intervals:

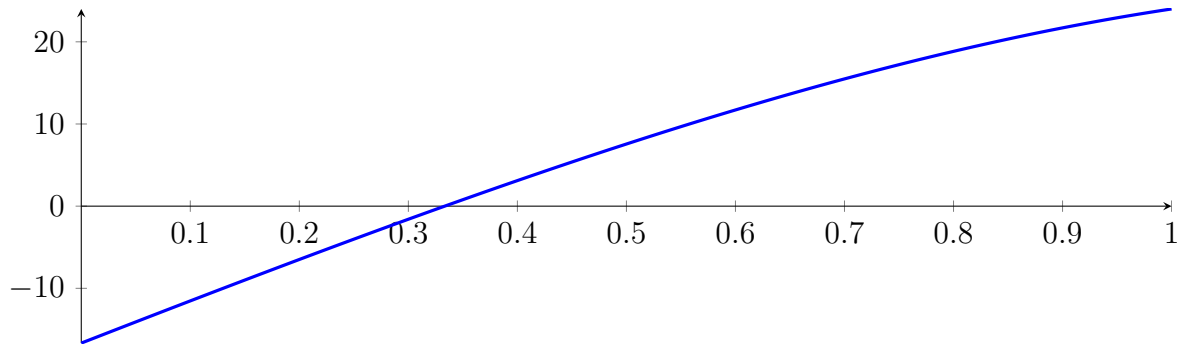


No intersection intervals with the x axis.

210.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^4 - 7.66667X^3 - 2.33333X^2 + 51.6667X - 16.6667$$



Result: Root Intervals

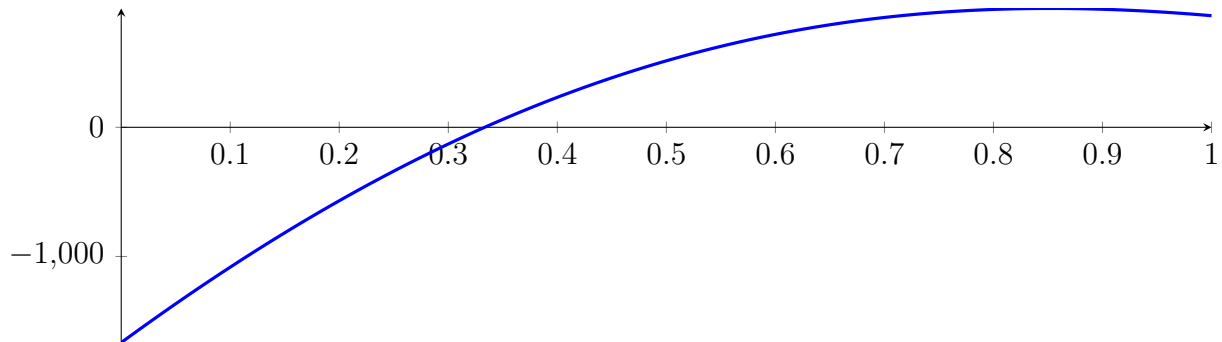
with precision $\varepsilon = 1 \cdot 10^{-128}$.

211 Running BezClip on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

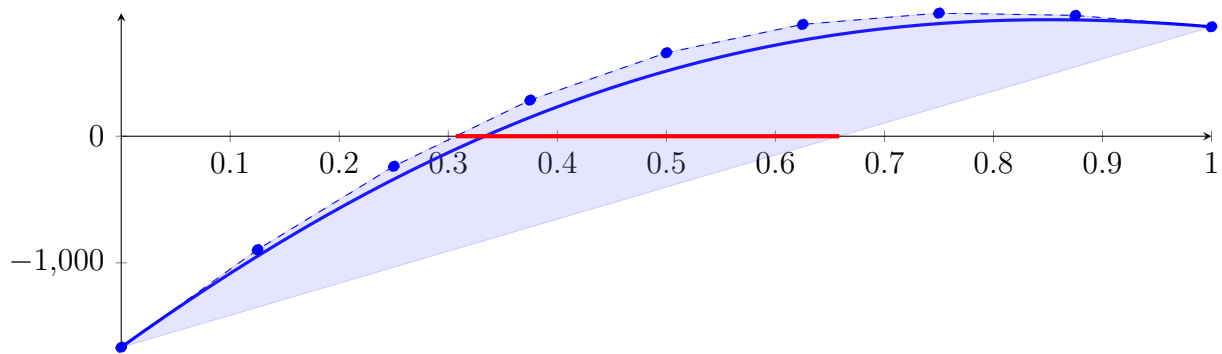
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



211.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

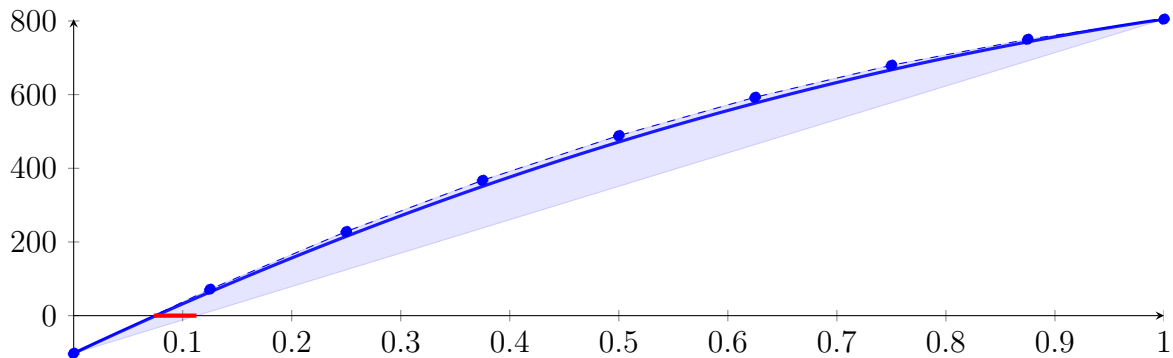
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

211.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

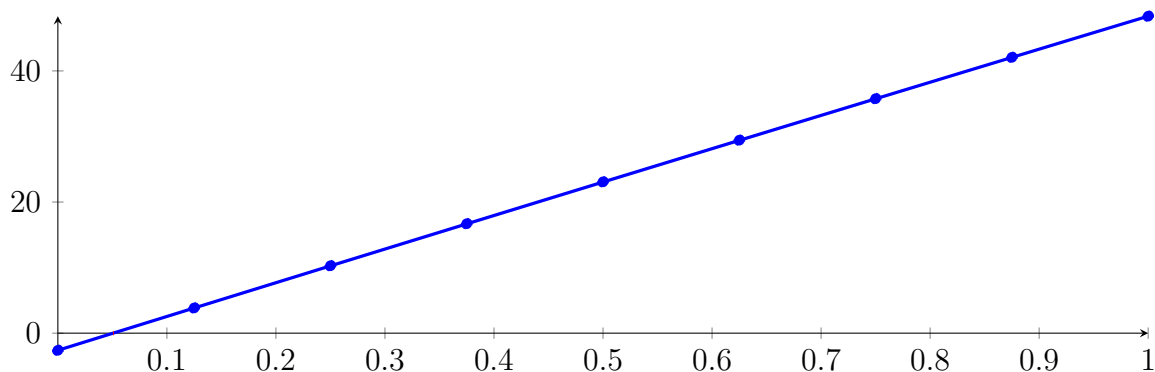
Longest intersection interval: 0.0391855

\implies Selective recursion: interval 1: [0.332635, 0.34642],

211.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.30406 \cdot 10^{-15} X^8 - 1.54459 \cdot 10^{-12} X^7 - 4.9583 \cdot 10^{-10} X^6 + 3.66751 \cdot 10^{-08} X^5 \\
 &\quad + 3.25466 \cdot 10^{-05} X^4 - 0.000586142 X^3 - 0.747315 X^2 + 51.7118 X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589

\implies Selective recursion: interval 1: [0.333333, 0.333343],

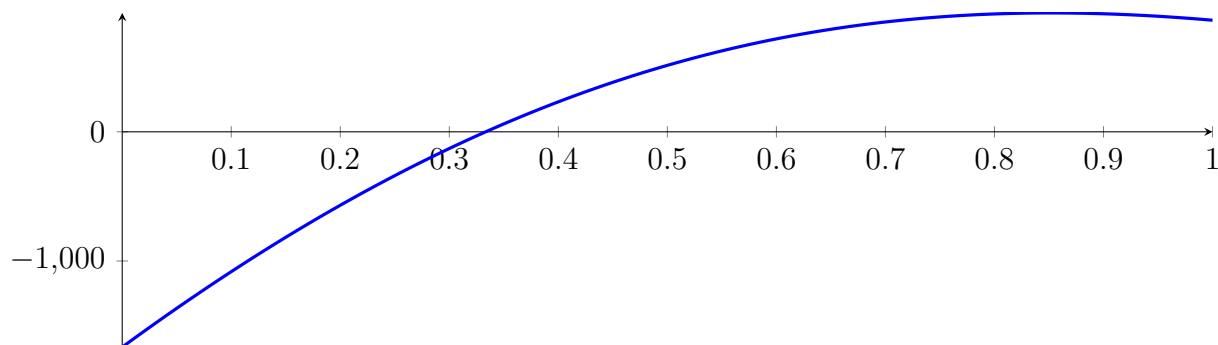
211.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Found root in interval [0.333333, 0.333343] at recursion depth 4!

211.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333343]$$

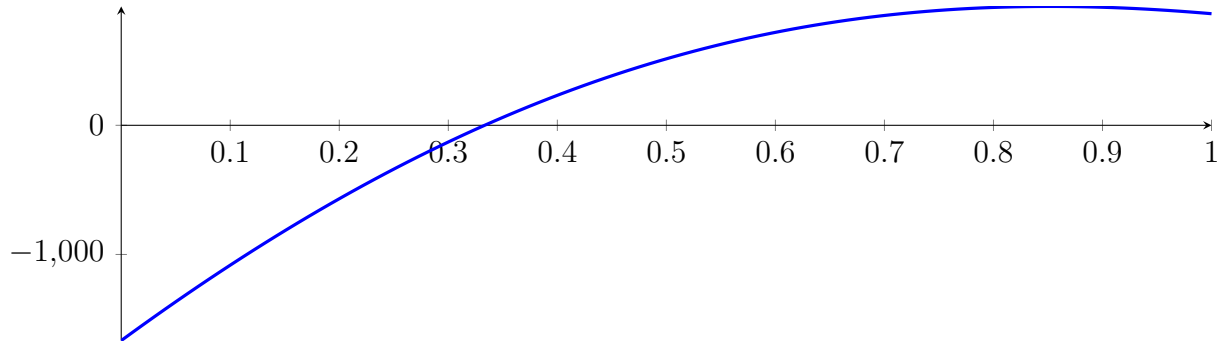
with precision $\varepsilon = 0.01$.

212 Running QuadClip on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

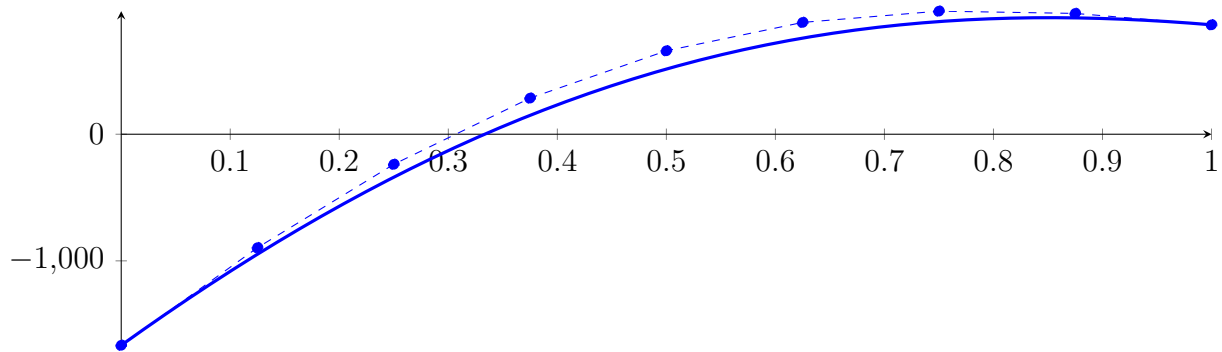
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



212.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

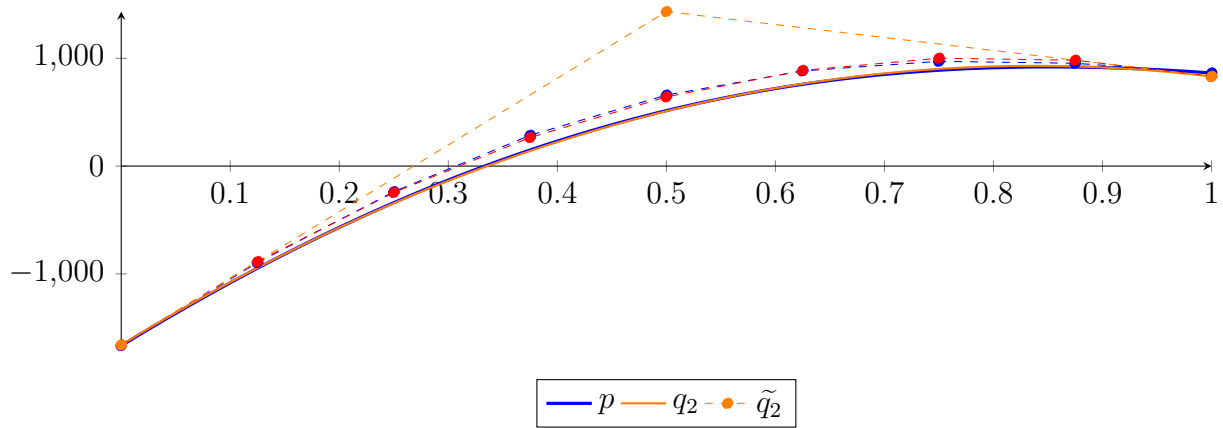
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 6.38292 \cdot 10^{-300}X^8 - 2.86993 \cdot 10^{-299}X^7 + 5.15342 \cdot 10^{-299}X^6 - 4.6928 \cdot 10^{-299}X^5 \\ &\quad + 2.29296 \cdot 10^{-299}X^4 - 6.02693 \cdot 10^{-300}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

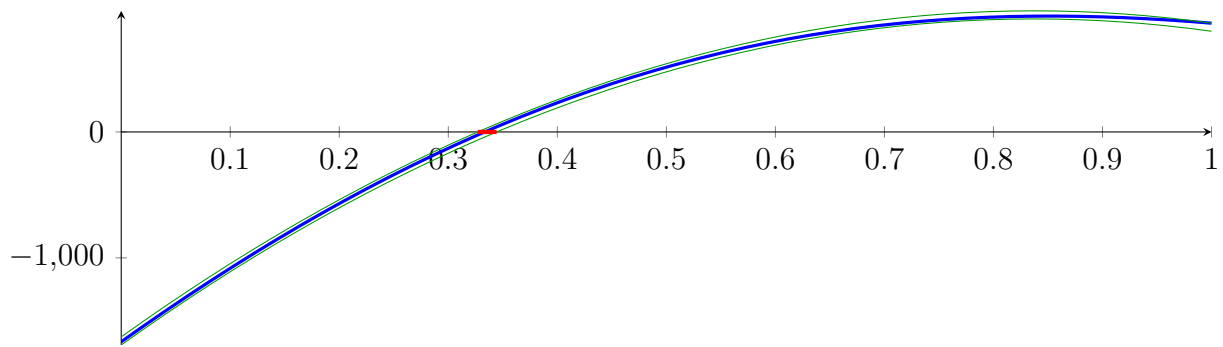
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

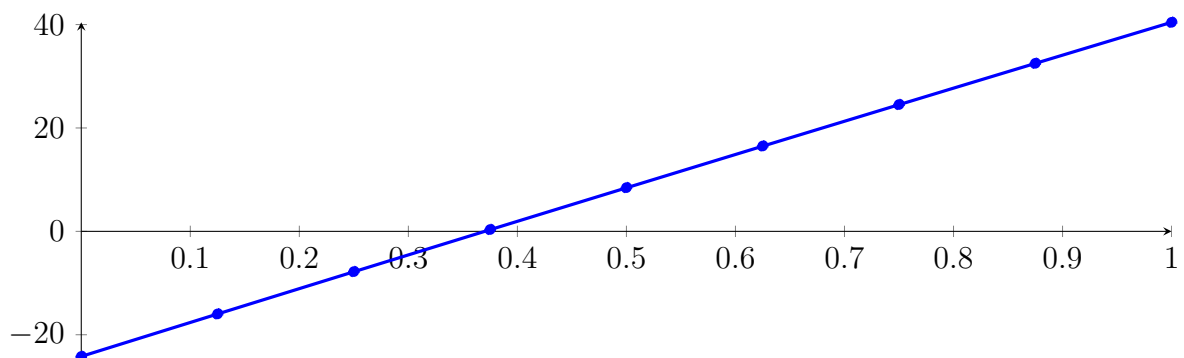
Longest intersection interval: 0.0173372

\implies Selective recursion: **interval 1:** $[0.326917, 0.344255]$,

212.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -8.16249 \cdot 10^{-15} X^8 - 7.66571 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

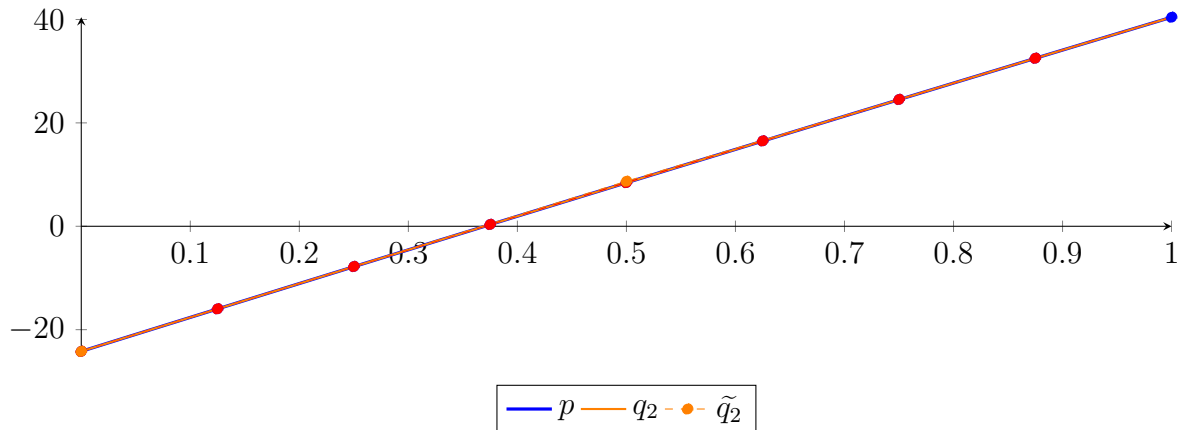
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = 4.27533 \cdot 10^{-302}X^8 - 2.42468 \cdot 10^{-301}X^7 + 4.90537 \cdot 10^{-301}X^6 - 4.6286 \cdot 10^{-301}X^5$$

$$+ 2.17546 \cdot 10^{-301}X^4 - 5.36695 \cdot 10^{-302}X^3 - 1.18261X^2 + 65.8162X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

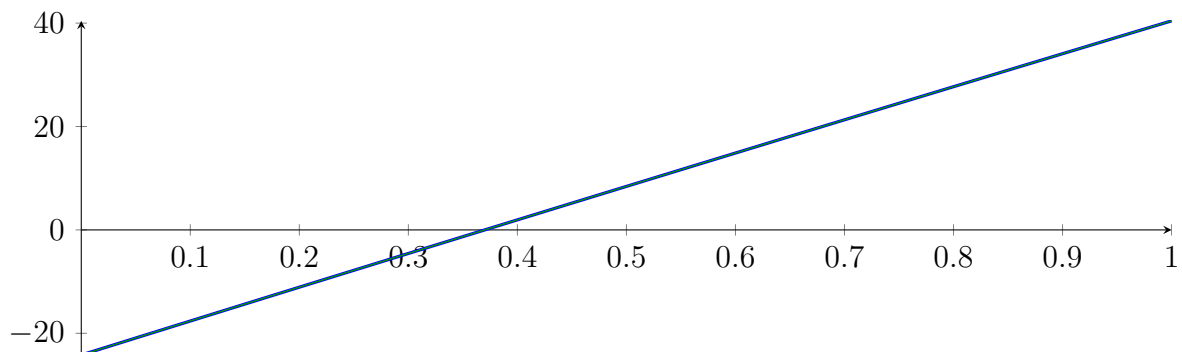
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\} \quad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

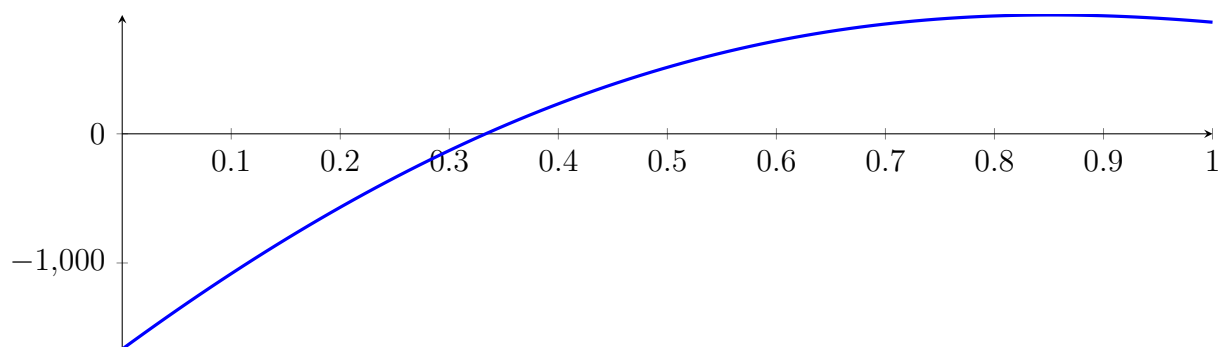
212.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

212.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

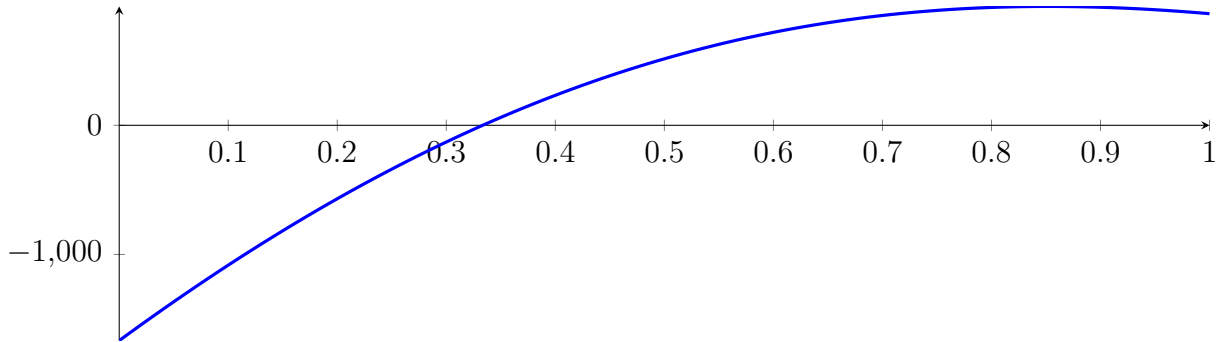
with precision $\varepsilon = 0.01$.

213 Running CubeClip on f_8 with epsilon 2

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

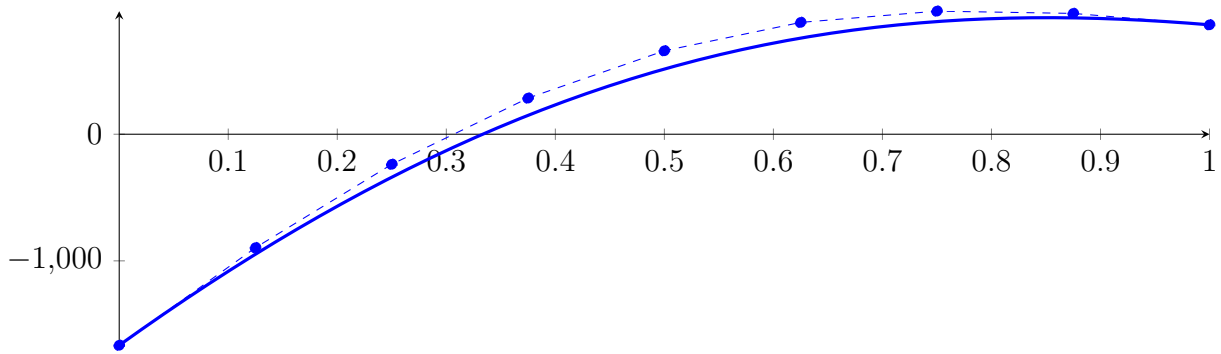
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



213.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

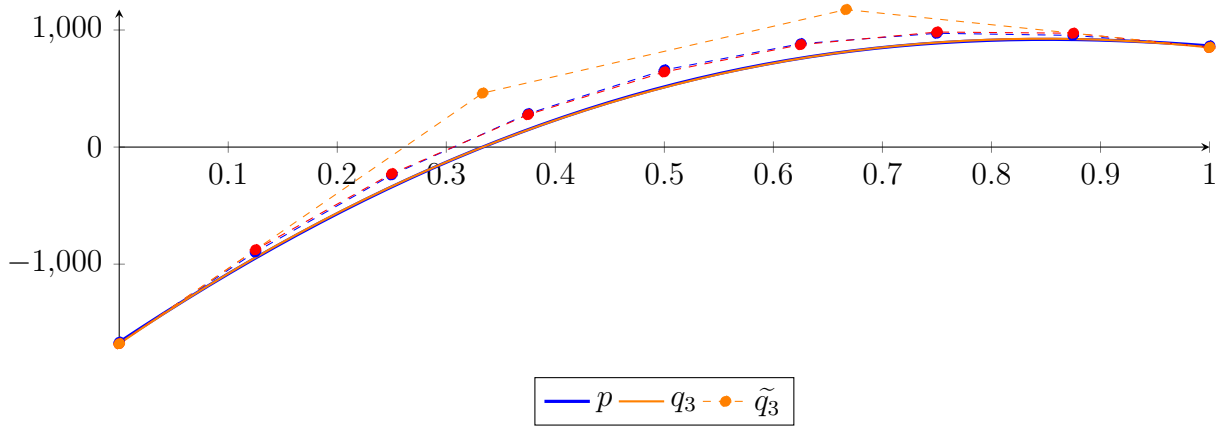
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.80826 \cdot 10^{-300}X^8 + 1.16606 \cdot 10^{-299}X^7 - 2.1197 \cdot 10^{-299}X^6 + 2.15003 \cdot 10^{-299}X^5 \\ &\quad - 1.20234 \cdot 10^{-299}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

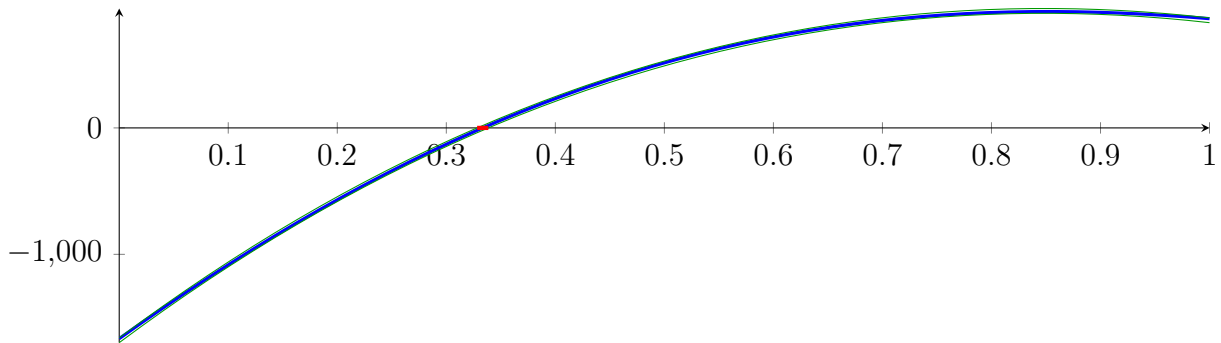
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

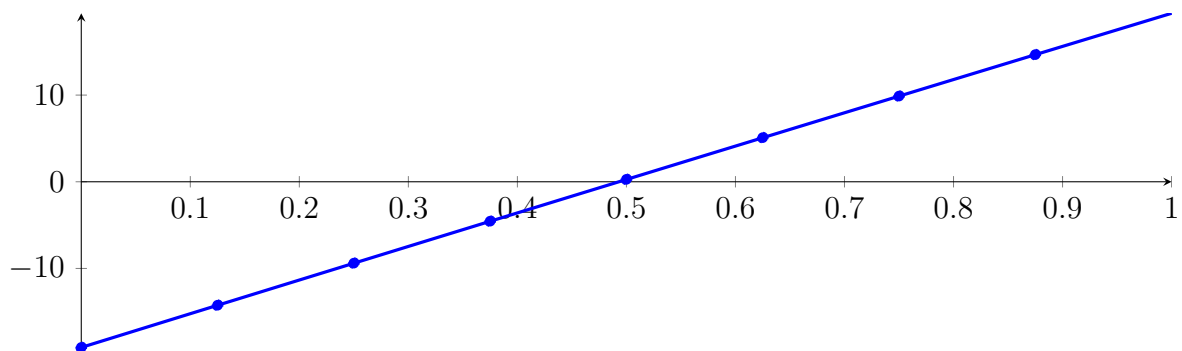
Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

213.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.25947 \cdot 10^{-16} X^8 - 1.9937 \cdot 10^{-13} X^7 - 8.53073 \cdot 10^{-11} X^6 + 8.72839 \cdot 10^{-09} X^5 \\
 &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\
 &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\
 &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

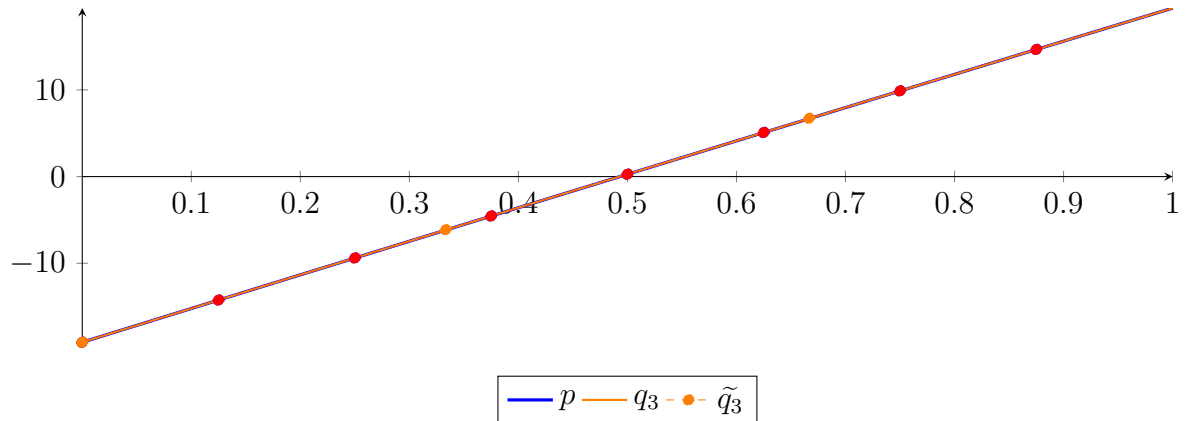
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = -1.96643 \cdot 10^{-303} X^8 + 1.82947 \cdot 10^{-302} X^7 - 4.89395 \cdot 10^{-302} X^6 + 5.49554 \cdot 10^{-302} X^5$$

$$- 2.47838 \cdot 10^{-302} X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m :

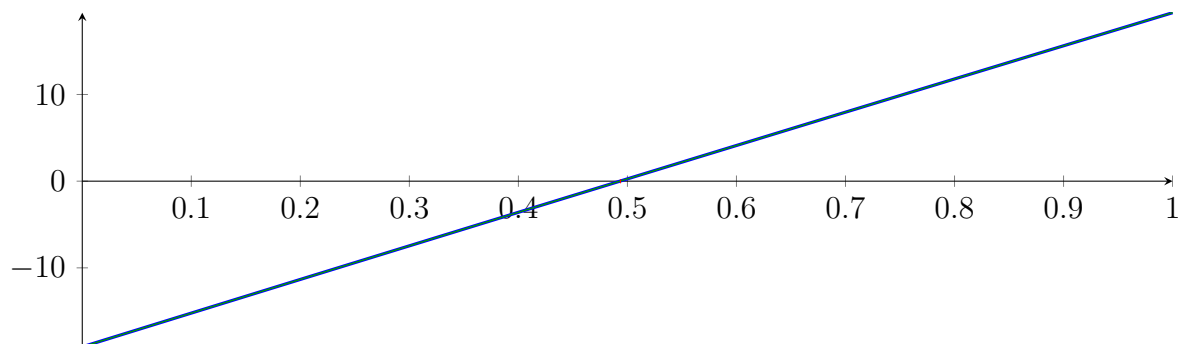
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \quad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.1252 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

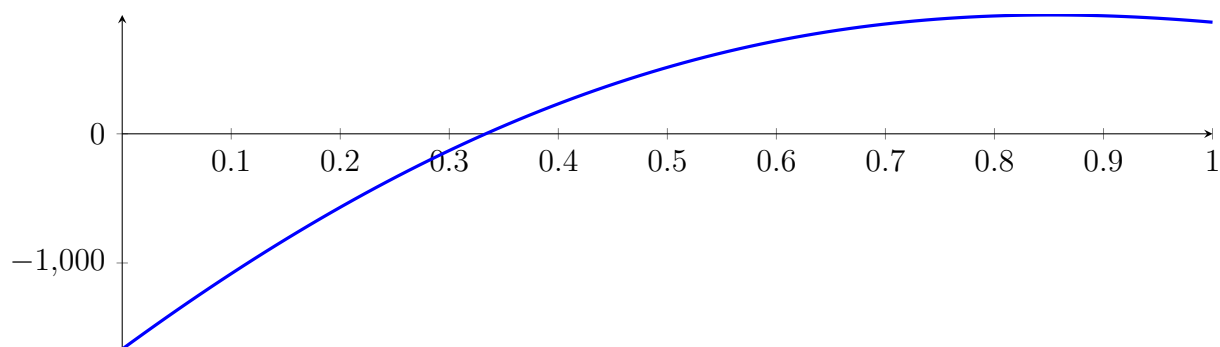
213.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

213.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

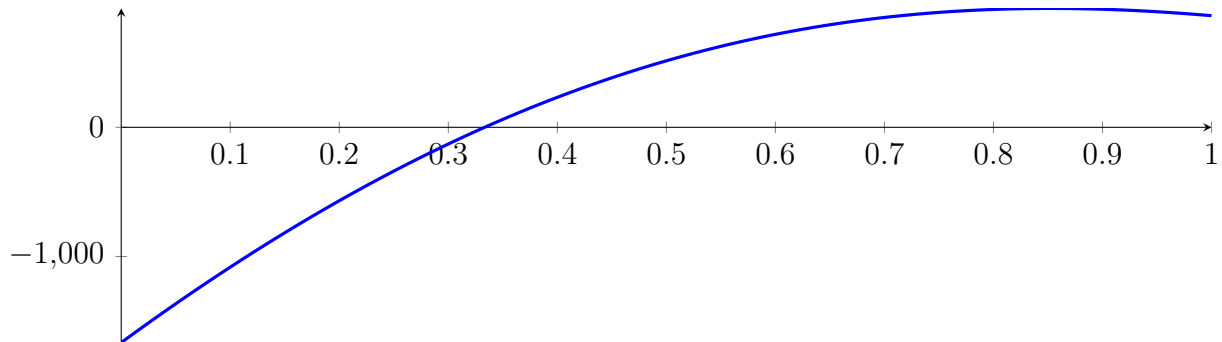
with precision $\varepsilon = 0.01$.

214 Running BezClip on f_8 with epsilon 4

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

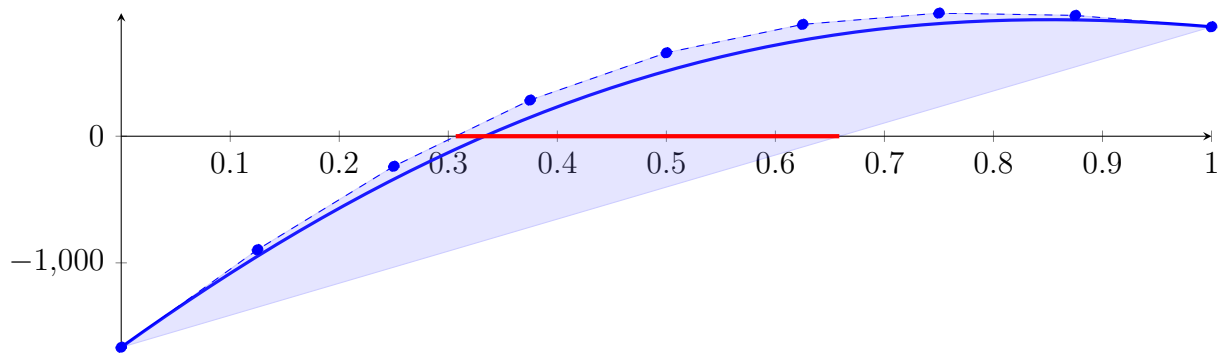
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



214.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

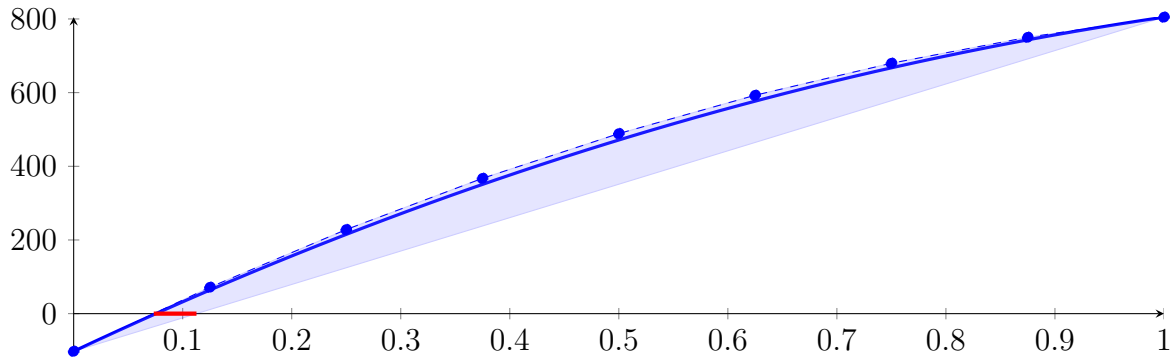
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

214.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

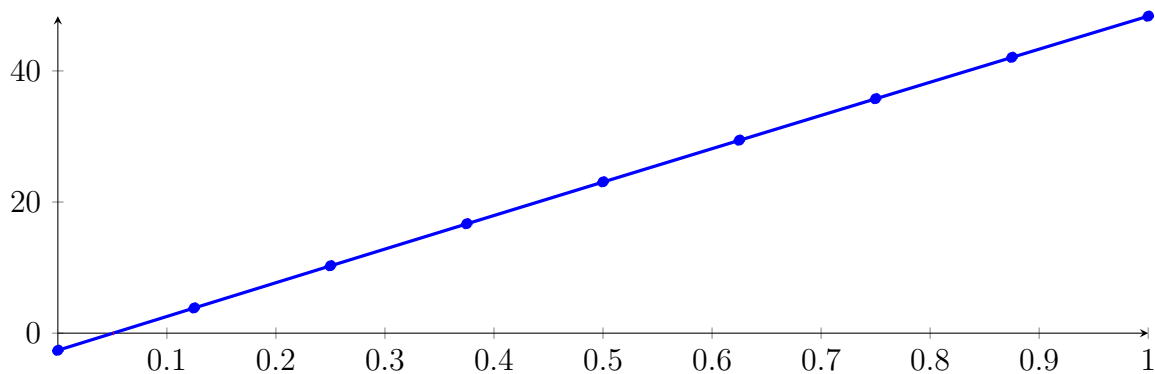
Longest intersection interval: 0.0391855

\implies Selective recursion: interval 1: [0.332635, 0.34642],

214.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.30406 \cdot 10^{-15} X^8 - 1.54459 \cdot 10^{-12} X^7 - 4.9583 \cdot 10^{-10} X^6 + 3.66751 \cdot 10^{-08} X^5 \\
 &\quad + 3.25466 \cdot 10^{-05} X^4 - 0.000586142 X^3 - 0.747315 X^2 + 51.7118 X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

Longest intersection interval: 0.000742589

\implies Selective recursion: interval 1: [0.333333, 0.333343],

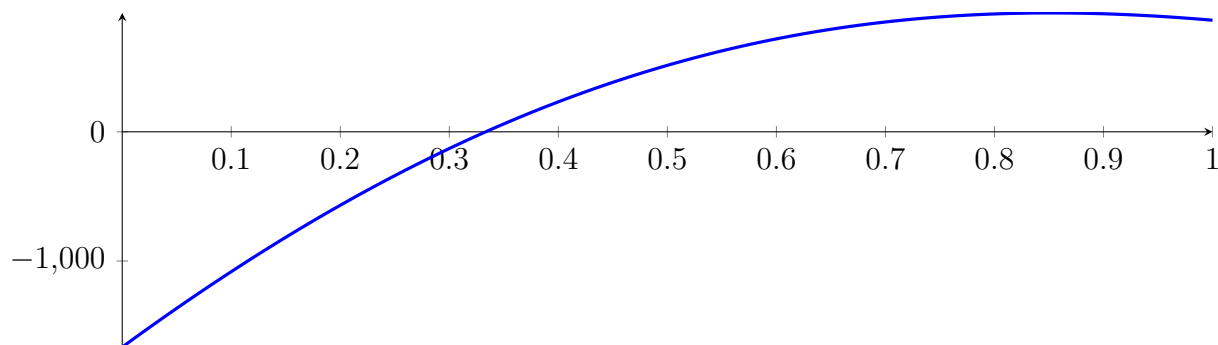
214.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Found root in interval [0.333333, 0.333343] at recursion depth 4!

214.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333343]$$

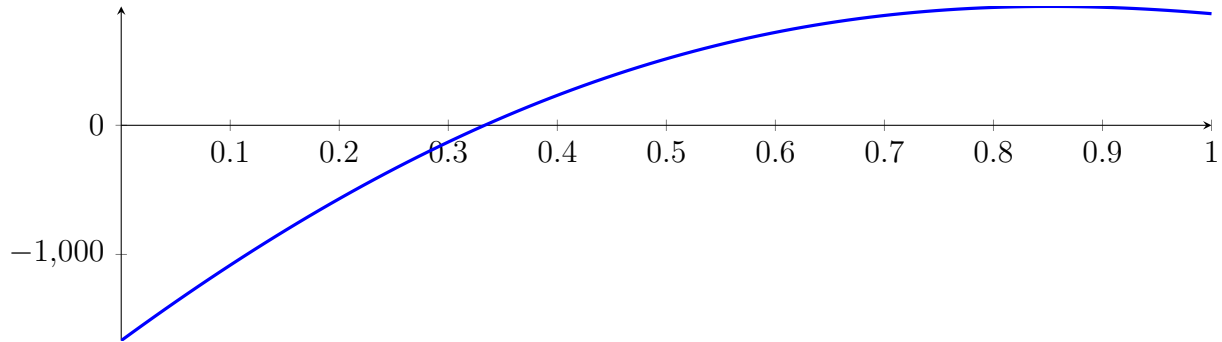
with precision $\varepsilon = 0.0001$.

215 Running QuadClip on f_8 with epsilon 4

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

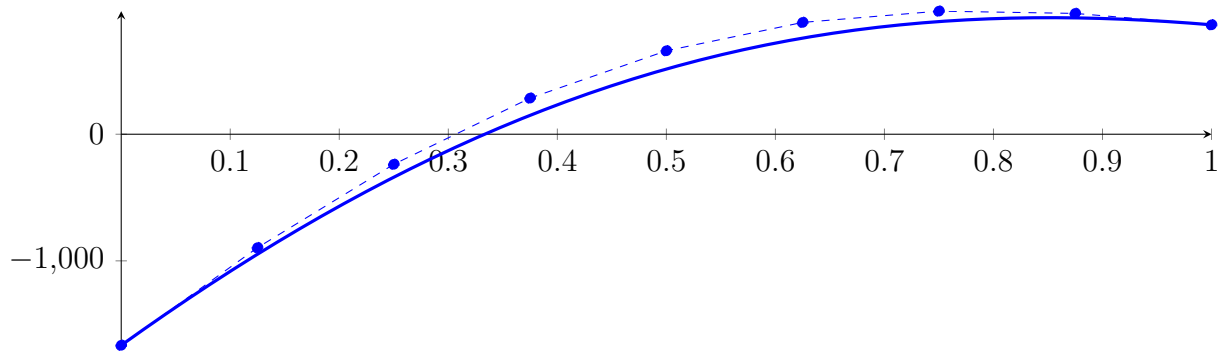
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



215.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

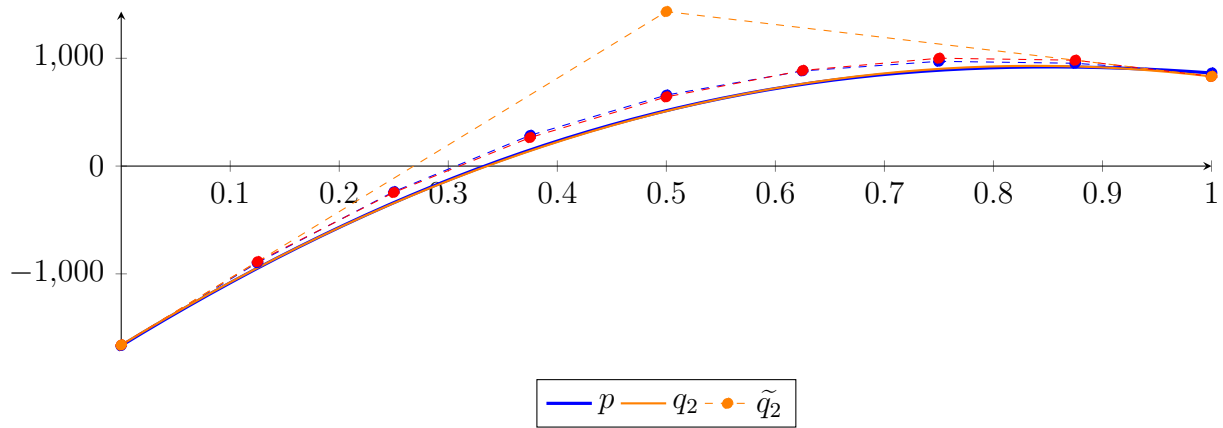
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 6.38292 \cdot 10^{-300}X^8 - 2.86993 \cdot 10^{-299}X^7 + 5.15342 \cdot 10^{-299}X^6 - 4.6928 \cdot 10^{-299}X^5 \\ &\quad + 2.29296 \cdot 10^{-299}X^4 - 6.02693 \cdot 10^{-300}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

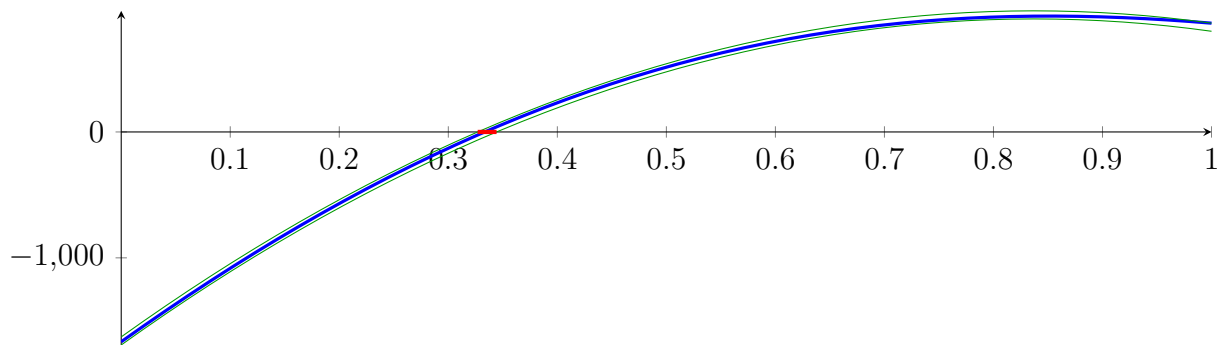
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

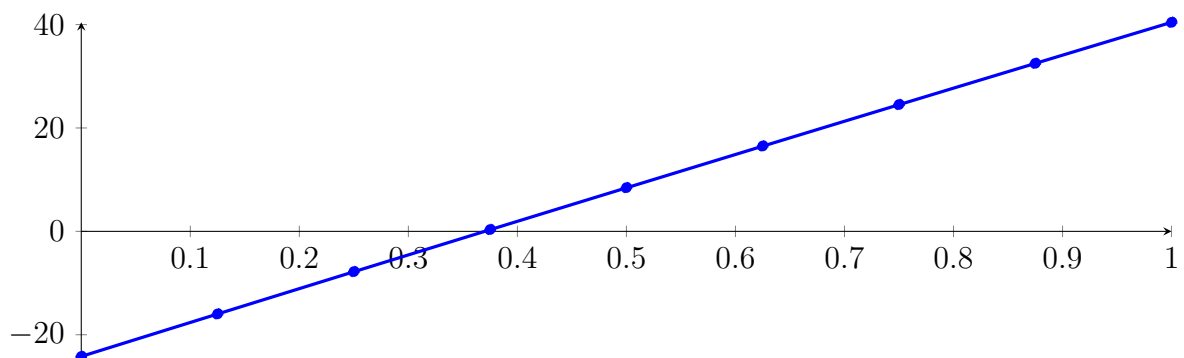
Longest intersection interval: 0.0173372

\implies Selective recursion: **interval 1:** $[0.326917, 0.344255]$,

215.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -8.16249 \cdot 10^{-15} X^8 - 7.66571 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

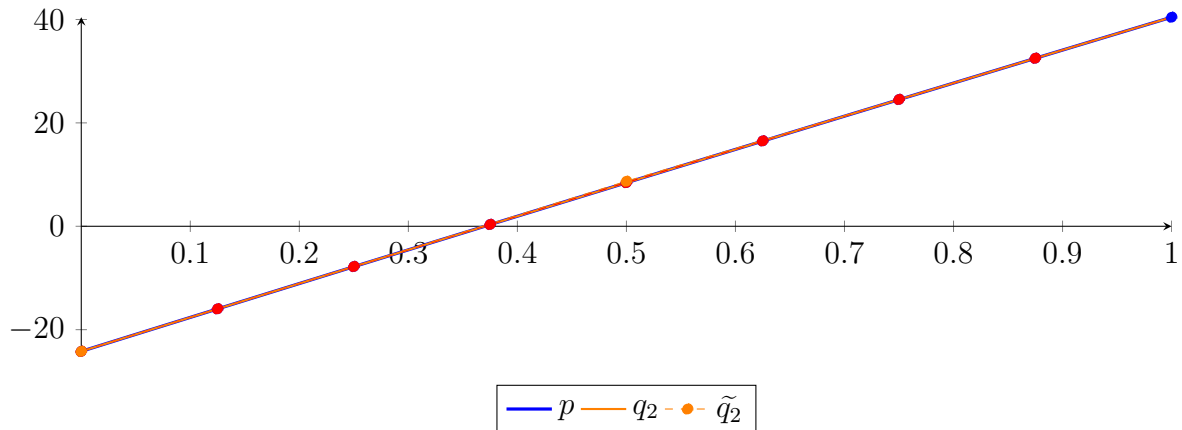
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = 4.27533 \cdot 10^{-302} X^8 - 2.42468 \cdot 10^{-301} X^7 + 4.90537 \cdot 10^{-301} X^6 - 4.6286 \cdot 10^{-301} X^5$$

$$+ 2.17546 \cdot 10^{-301} X^4 - 5.36695 \cdot 10^{-302} X^3 - 1.18261 X^2 + 65.8162 X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

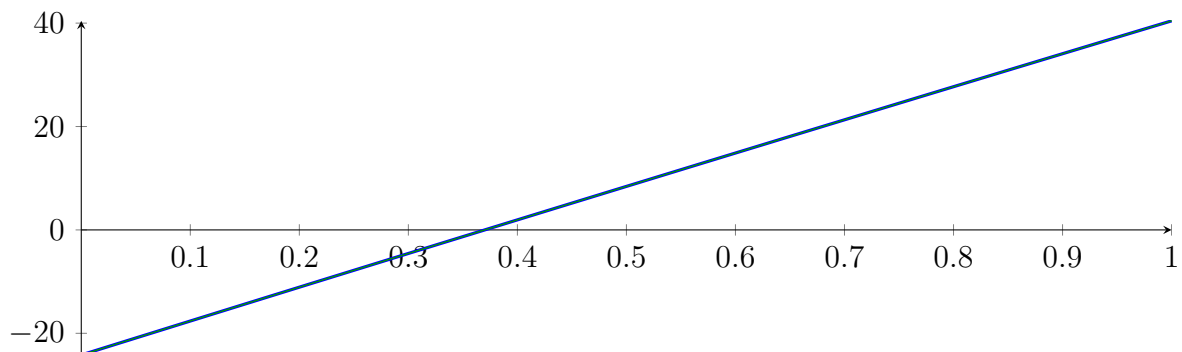
$$M = -1.18261X^2 + 65.8162X - 24.1945$$

$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\} \quad N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

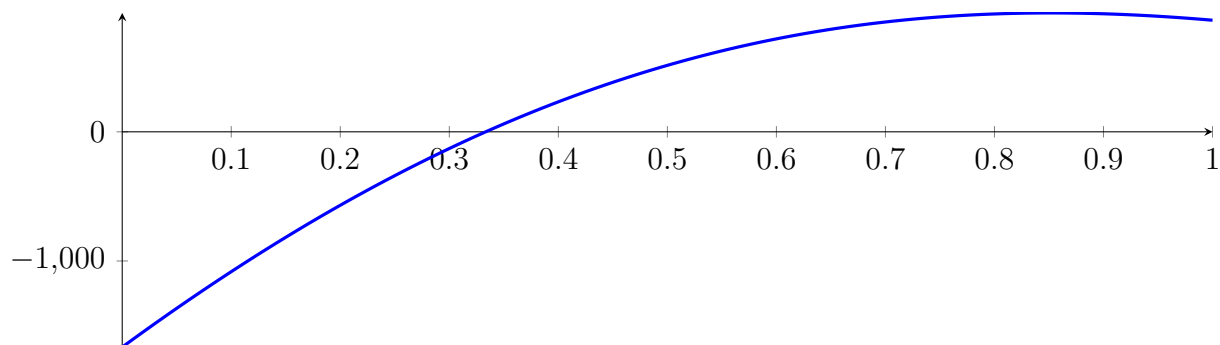
215.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

215.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

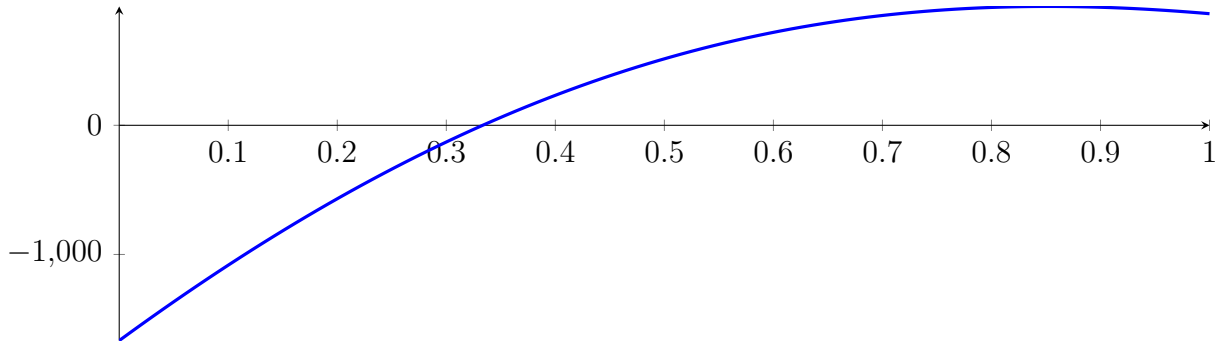
with precision $\varepsilon = 0.0001$.

216 Running CubeClip on f_8 with epsilon 4

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

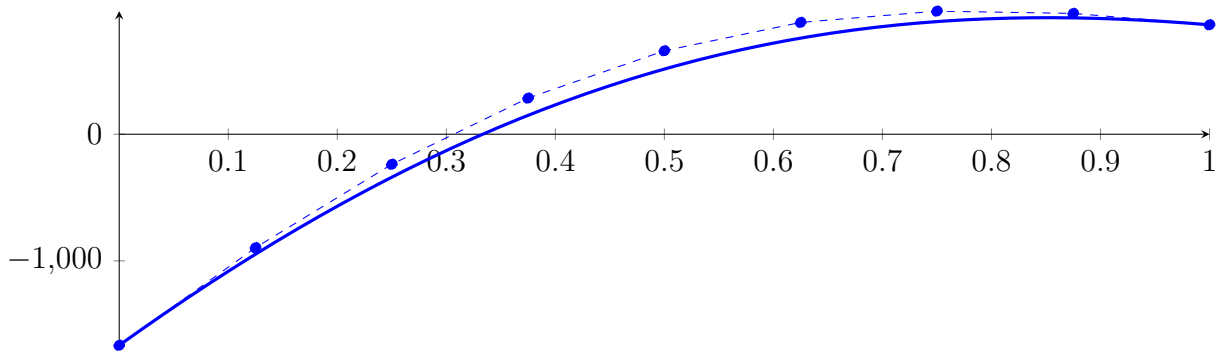
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



216.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

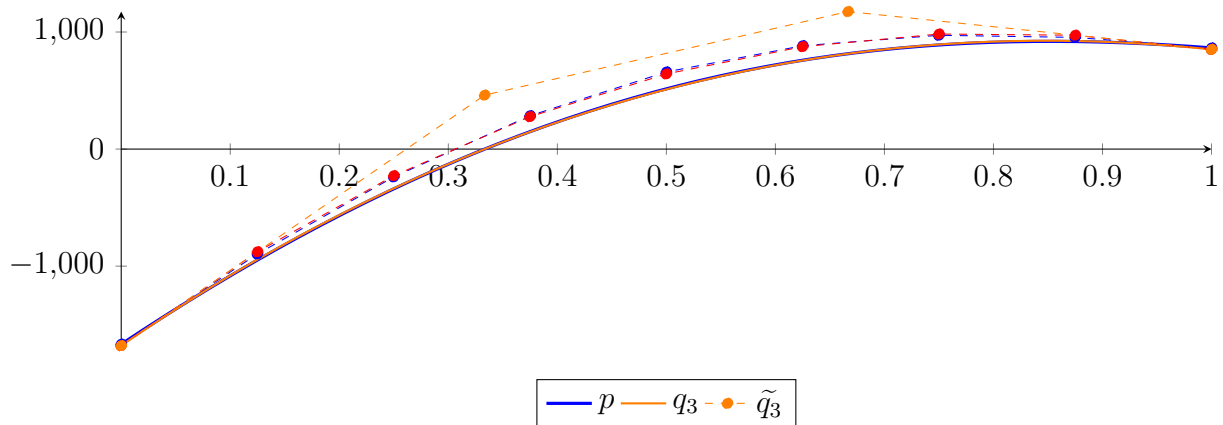
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.80826 \cdot 10^{-300}X^8 + 1.16606 \cdot 10^{-299}X^7 - 2.1197 \cdot 10^{-299}X^6 + 2.15003 \cdot 10^{-299}X^5 \\ &\quad - 1.20234 \cdot 10^{-299}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

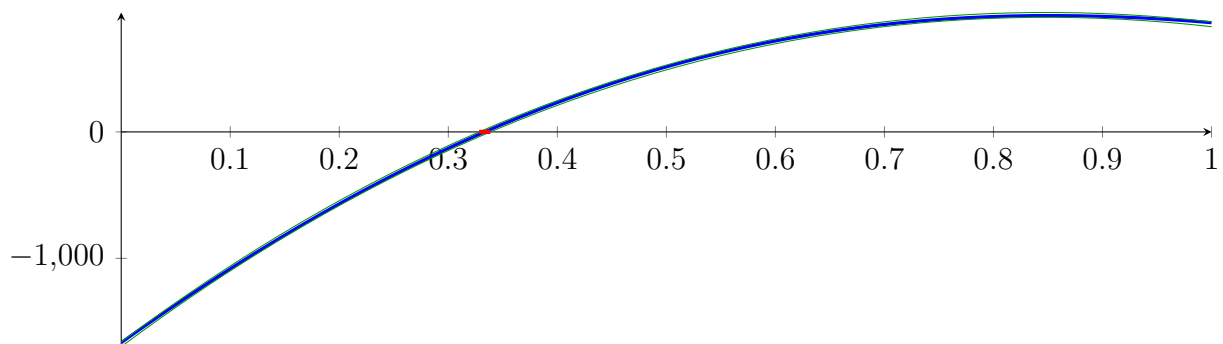
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

Longest intersection interval: 0.0102926

⇒ Selective recursion: interval 1: $[0.328258, 0.338551]$,

216.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

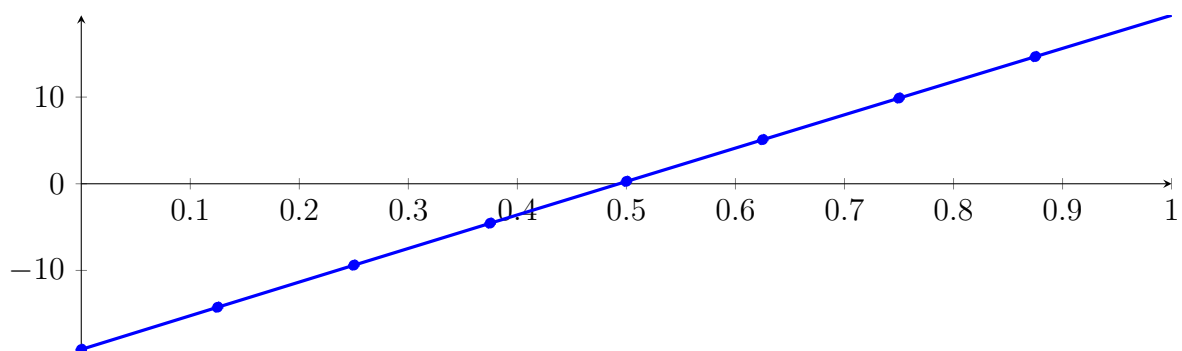
Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.25947 \cdot 10^{-16} X^8 - 1.9937 \cdot 10^{-13} X^7 - 8.53073 \cdot 10^{-11} X^6 + 8.72839 \cdot 10^{-09} X^5$$

$$+ 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124$$

$$= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X)$$

$$+ 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

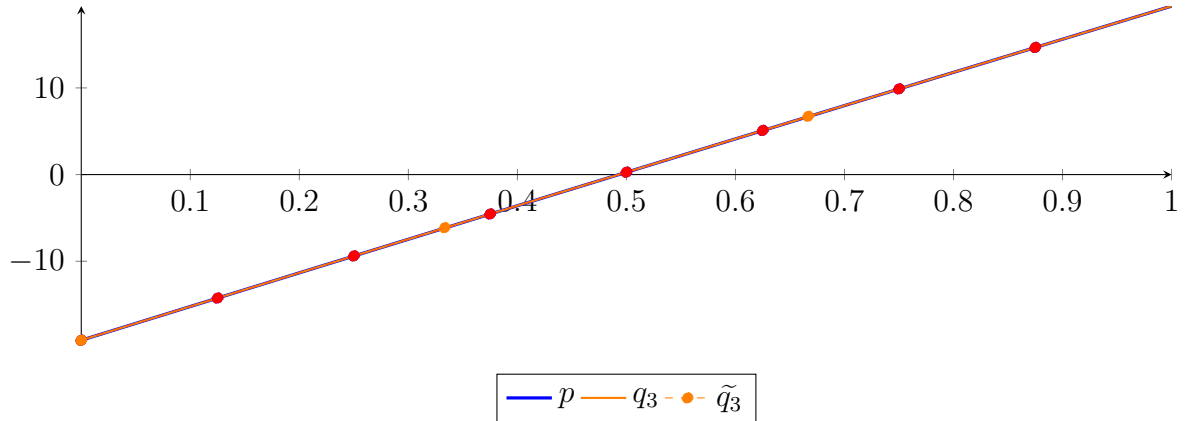
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = -1.96643 \cdot 10^{-303} X^8 + 1.82947 \cdot 10^{-302} X^7 - 4.89395 \cdot 10^{-302} X^6 + 5.49554 \cdot 10^{-302} X^5$$

$$- 2.47838 \cdot 10^{-302} X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m :

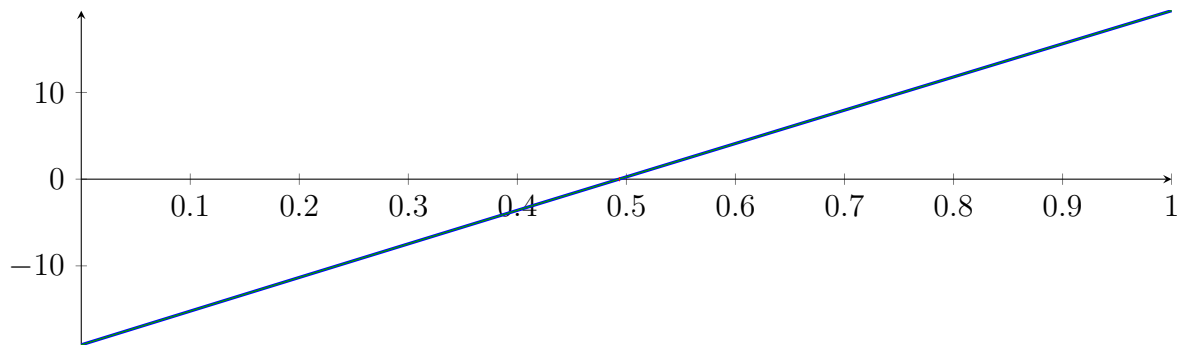
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \quad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.1252 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

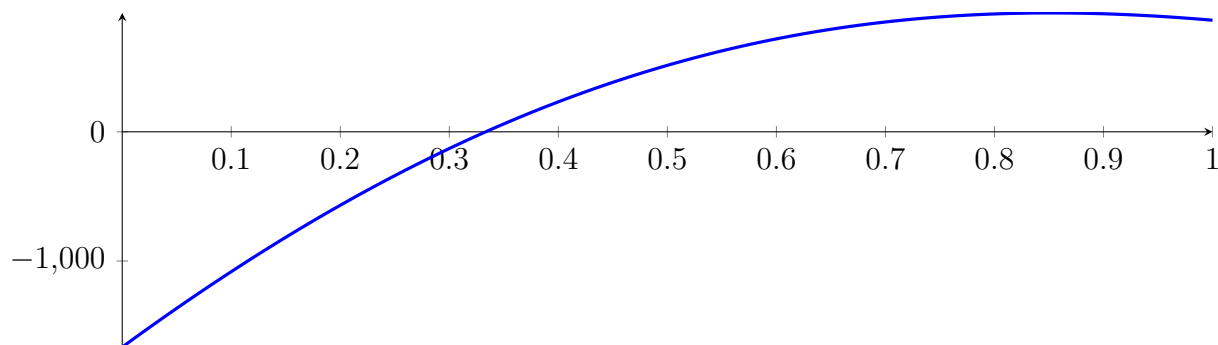
216.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

216.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

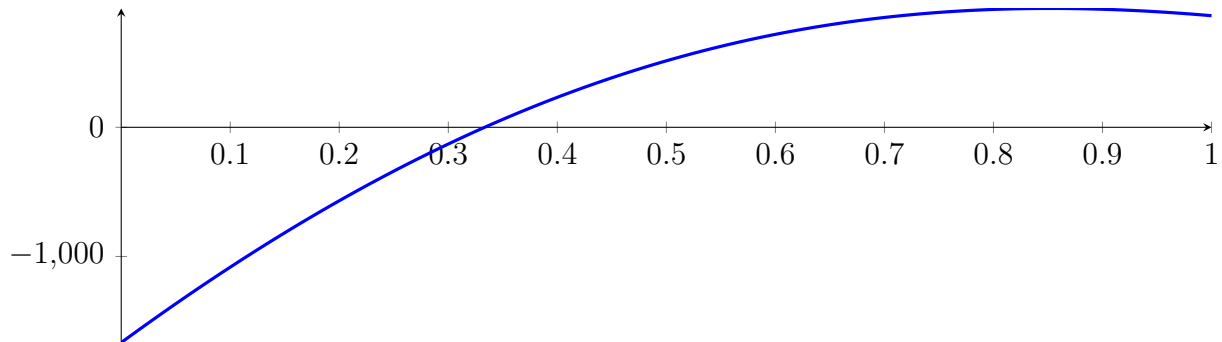
with precision $\varepsilon = 0.0001$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

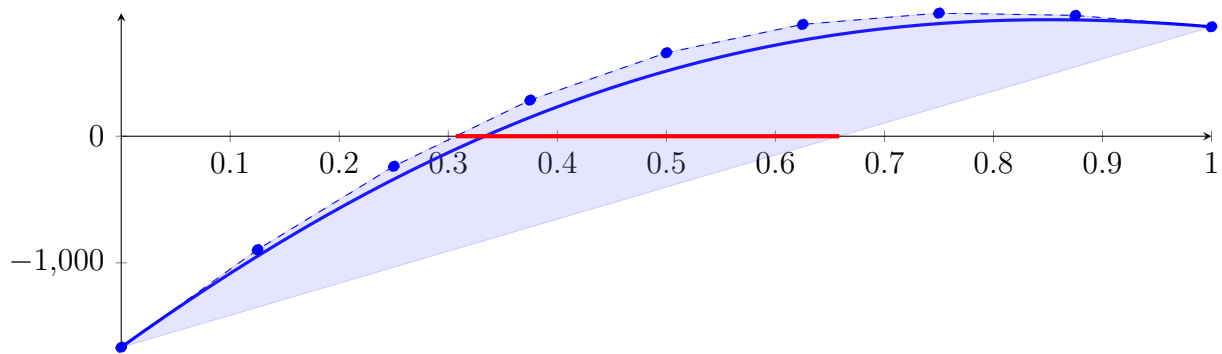
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



217.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

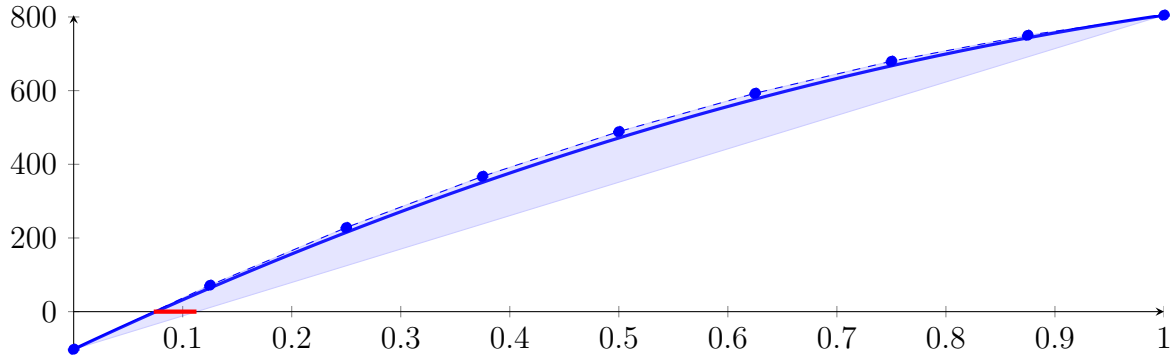
Longest intersection interval: 0.351792

\Rightarrow Selective recursion: interval 1: $[0.306796, 0.658588]$,

217.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

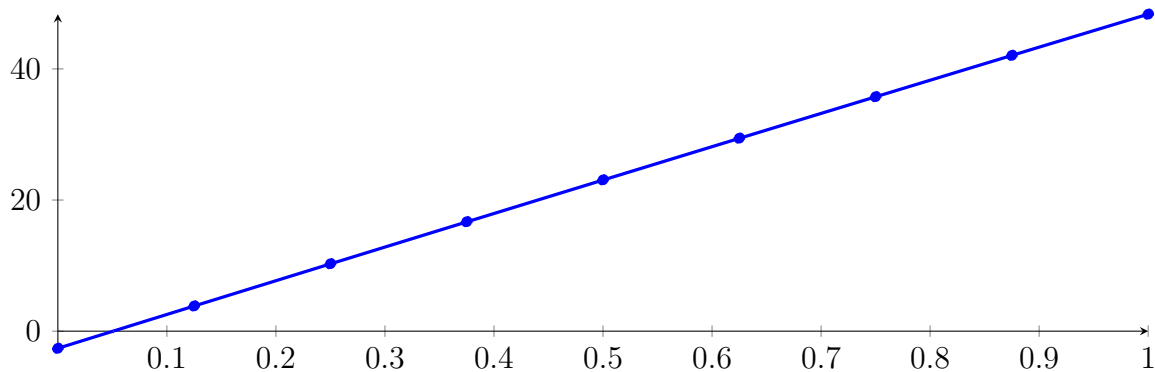
Longest intersection interval: 0.0391855

\implies Selective recursion: interval 1: [0.332635, 0.34642],

217.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.30406 \cdot 10^{-15} X^8 - 1.54459 \cdot 10^{-12} X^7 - 4.9583 \cdot 10^{-10} X^6 + 3.66751 \cdot 10^{-08} X^5 \\
 &\quad + 3.25466 \cdot 10^{-05} X^4 - 0.000586142 X^3 - 0.747315 X^2 + 51.7118 X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

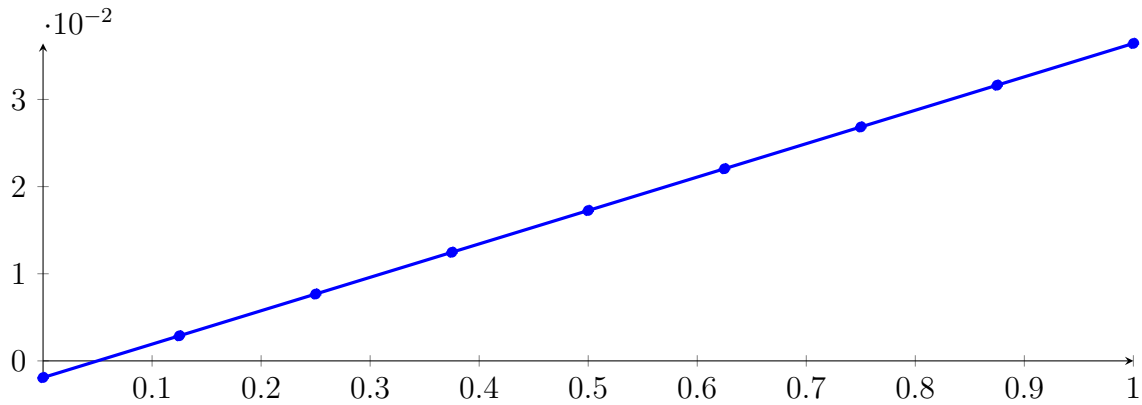
Longest intersection interval: 0.000742589

\implies Selective recursion: interval 1: [0.333333, 0.333343],

217.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.20583 \cdot 10^{-40} X^8 - 1.92397 \cdot 10^{-34} X^7 - 8.32342 \cdot 10^{-29} X^6 + 8.24755 \cdot 10^{-24} X^5 \\ &\quad + 9.89972 \cdot 10^{-18} X^4 - 2.37322 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\ &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\ &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\ &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

Longest intersection interval: $5.36469 \cdot 10^{-07}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

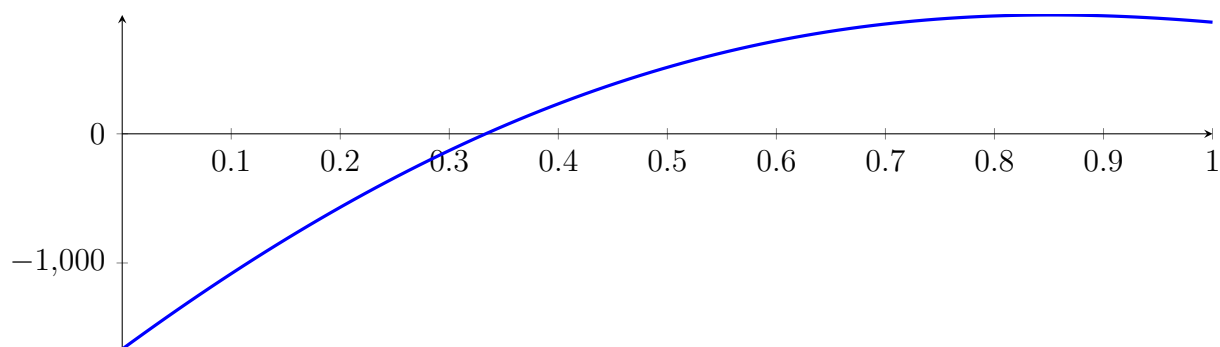
217.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

217.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

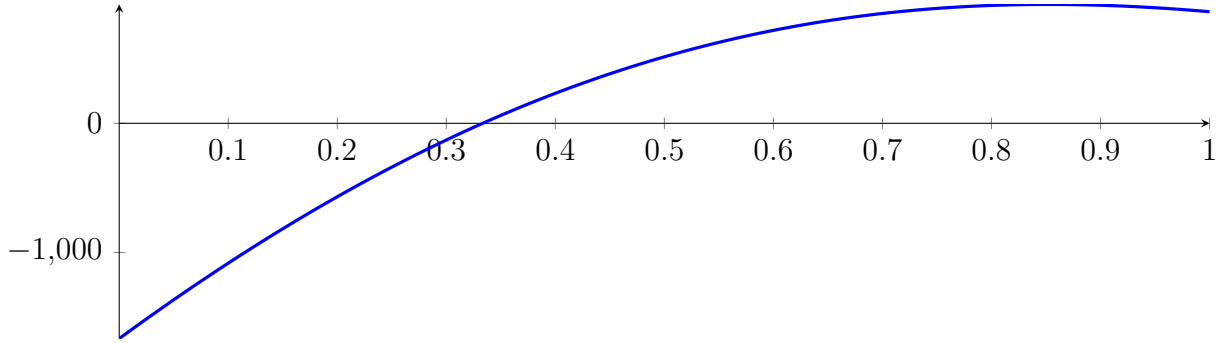
with precision $\varepsilon = 1 \cdot 10^{-08}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

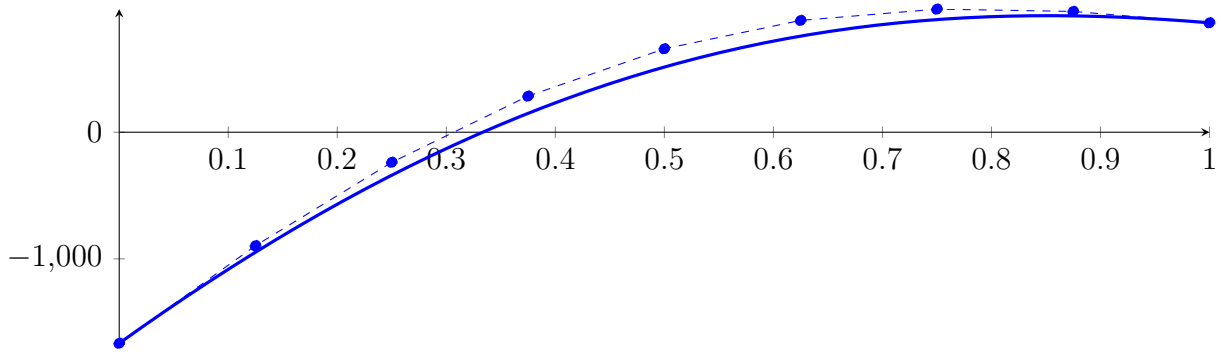
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



218.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

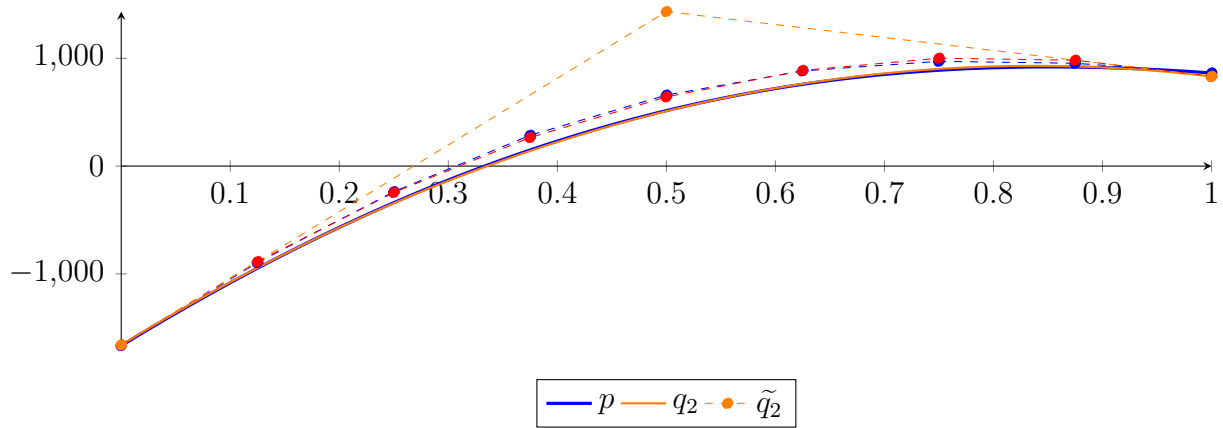
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 6.38292 \cdot 10^{-300}X^8 - 2.86993 \cdot 10^{-299}X^7 + 5.15342 \cdot 10^{-299}X^6 - 4.6928 \cdot 10^{-299}X^5 \\ &\quad + 2.29296 \cdot 10^{-299}X^4 - 6.02693 \cdot 10^{-300}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

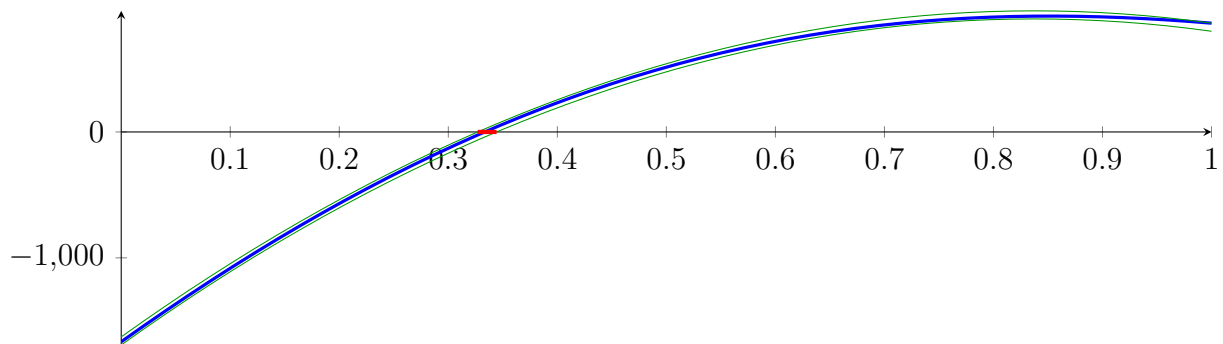
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

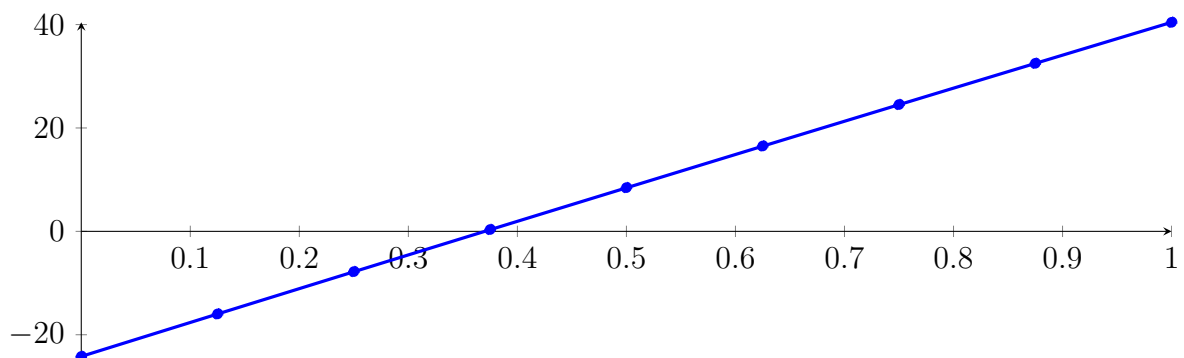
Longest intersection interval: 0.0173372

\implies Selective recursion: interval 1: $[0.326917, 0.344255]$,

218.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -8.16249 \cdot 10^{-15} X^8 - 7.66571 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

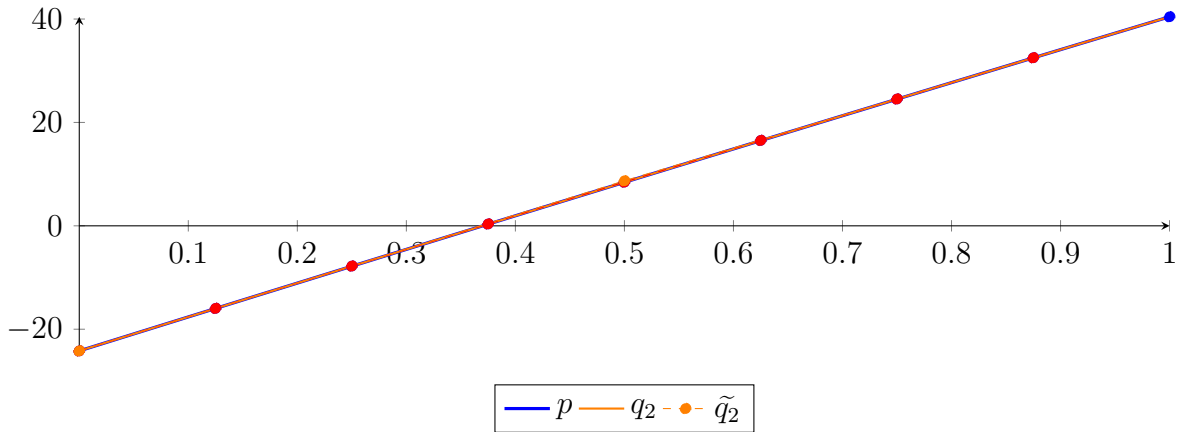
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = 4.27533 \cdot 10^{-302}X^8 - 2.42468 \cdot 10^{-301}X^7 + 4.90537 \cdot 10^{-301}X^6 - 4.6286 \cdot 10^{-301}X^5$$

$$+ 2.17546 \cdot 10^{-301}X^4 - 5.36695 \cdot 10^{-302}X^3 - 1.18261X^2 + 65.8162X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

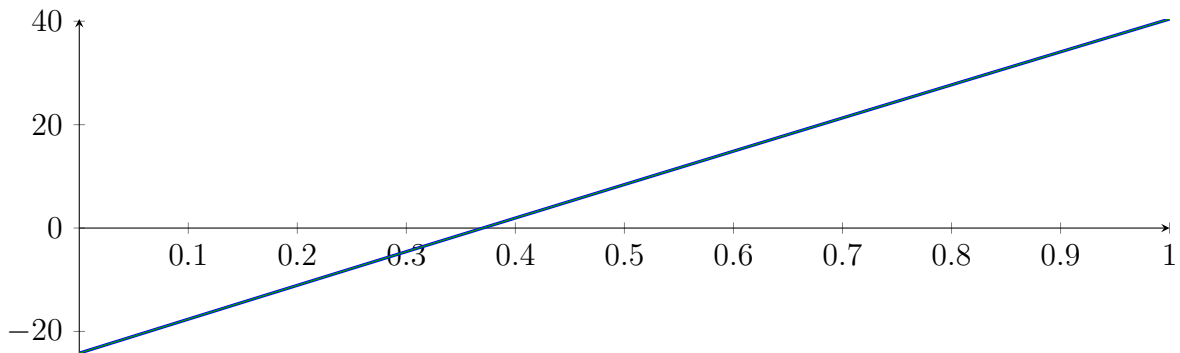
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

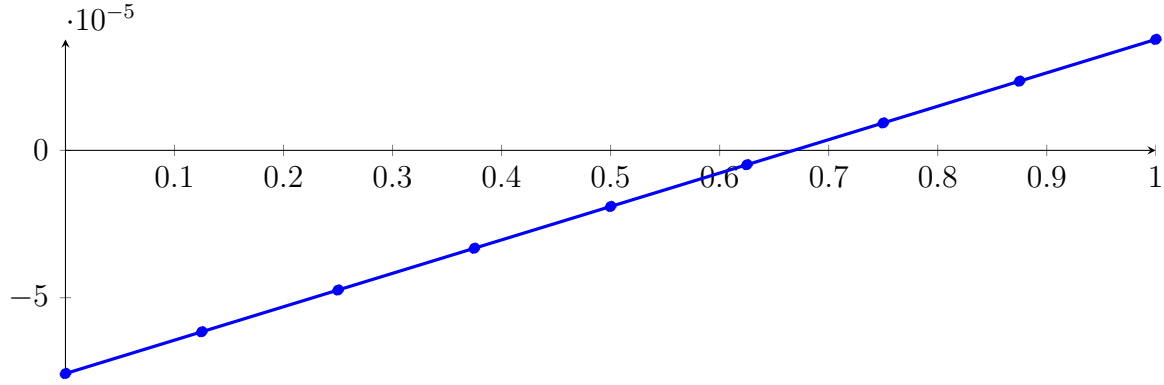
Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

218.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

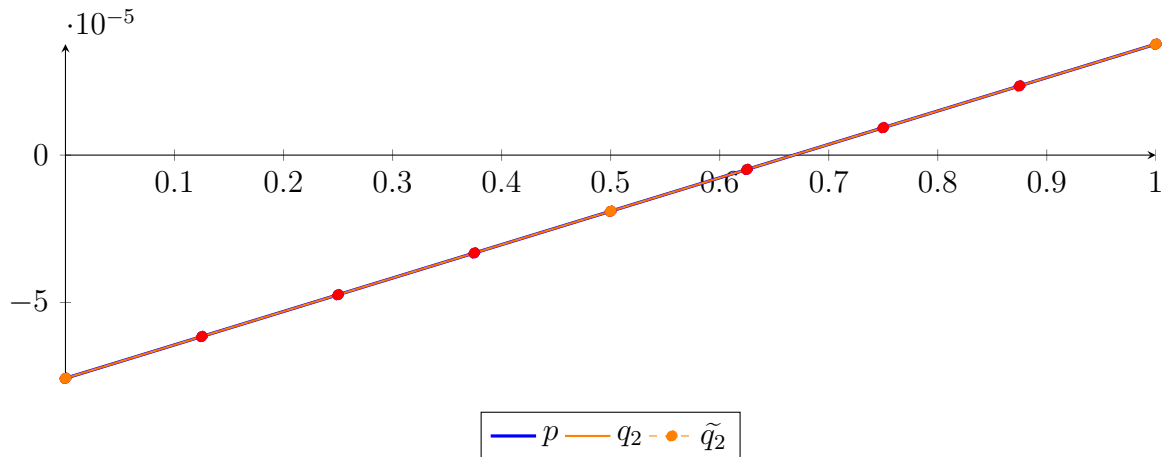
$$\begin{aligned}
 p &= -7.04578 \cdot 10^{-61} X^8 - 3.80201 \cdot 10^{-52} X^7 - 5.5627 \cdot 10^{-44} X^6 + 1.86413 \cdot 10^{-36} X^5 + 7.56737 \\
 &\quad \cdot 10^{-28} X^4 - 6.13517 \cdot 10^{-21} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) \\
 &\quad - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) \\
 &\quad + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -3.62396 \cdot 10^{-308} X^8 + 2.32992 \cdot 10^{-308} X^7 + 2.61753 \cdot 10^{-307} X^6 - 5.97049 \cdot 10^{-307} X^5 + 5.13401 \\
 &\quad \cdot 10^{-307} X^4 - 1.82868 \cdot 10^{-307} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \\
 &\quad \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.06758 \cdot 10^{-22}$.

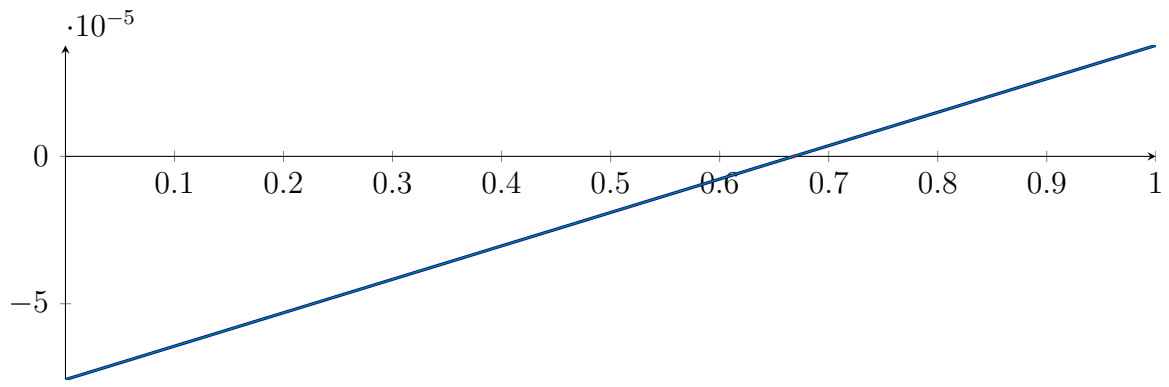
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 m &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.667955, 3.14644 \cdot 10^7\} \quad N(m) = \{0.667955, 3.14644 \cdot 10^7\}$$

Intersection intervals:



[0.667955, 0.667955]

Longest intersection interval: $5.41121 \cdot 10^{-18}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

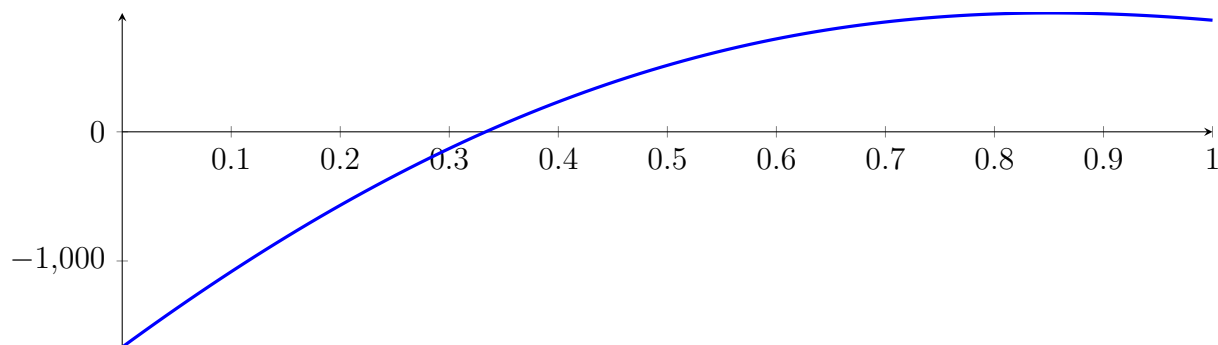
218.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

218.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

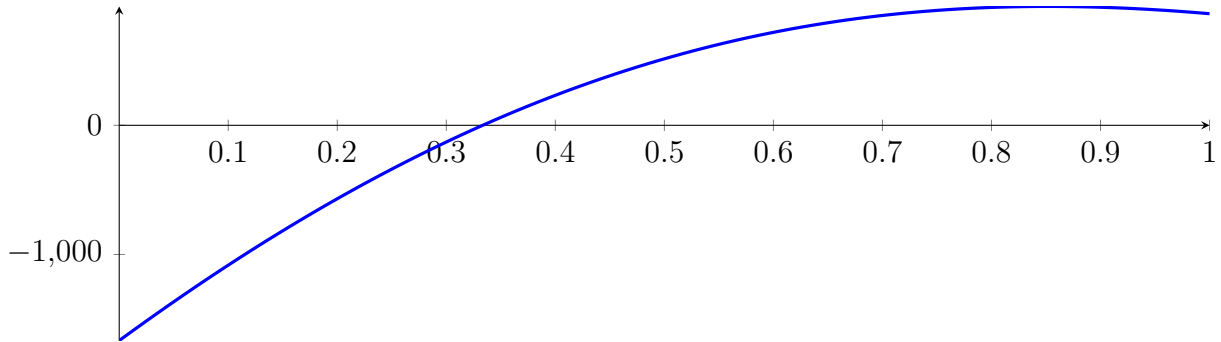
with precision $\varepsilon = 1 \cdot 10^{-08}$.

219 Running CubeClip on f_8 with epsilon 8

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

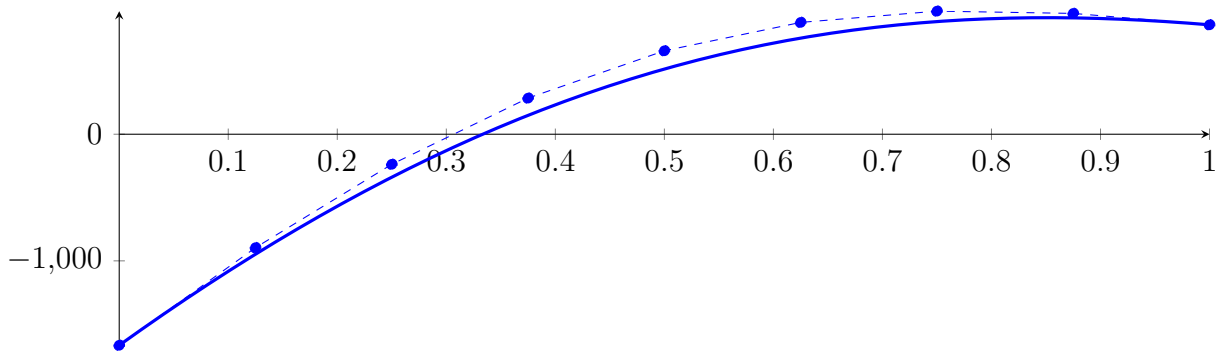
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



219.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

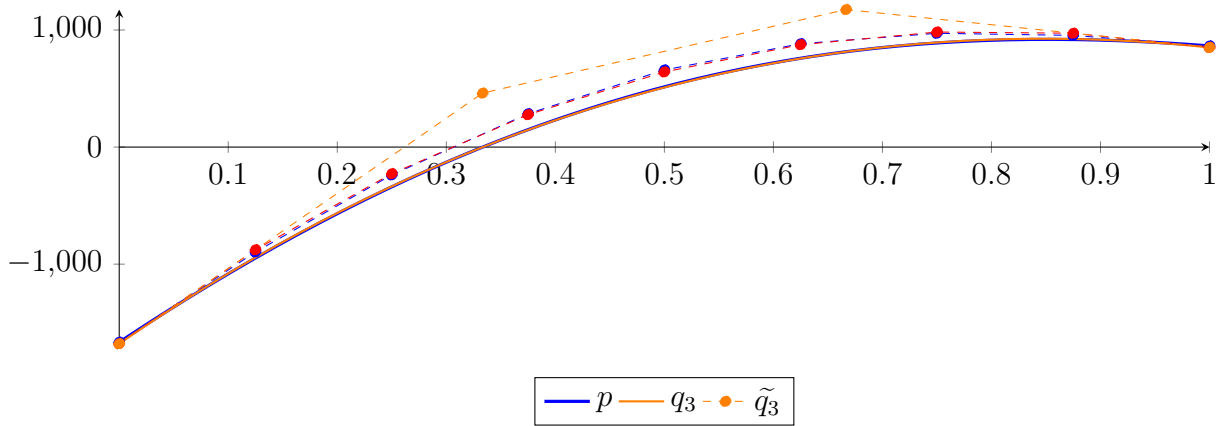
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.80826 \cdot 10^{-300}X^8 + 1.16606 \cdot 10^{-299}X^7 - 2.1197 \cdot 10^{-299}X^6 + 2.15003 \cdot 10^{-299}X^5 \\ &\quad - 1.20234 \cdot 10^{-299}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

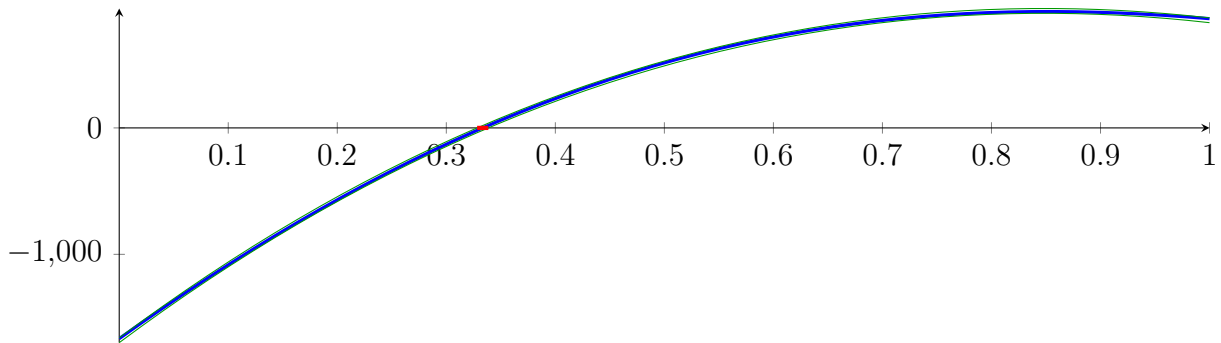
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

219.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

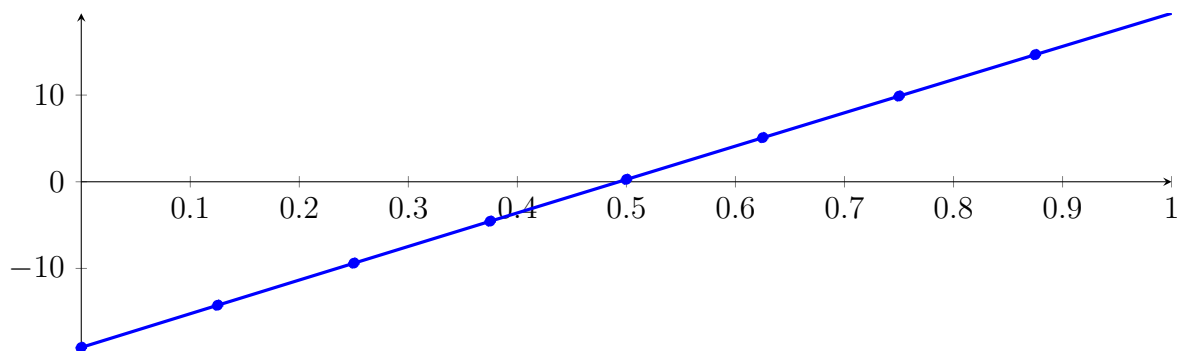
Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.25947 \cdot 10^{-16} X^8 - 1.9937 \cdot 10^{-13} X^7 - 8.53073 \cdot 10^{-11} X^6 + 8.72839 \cdot 10^{-09} X^5$$

$$+ 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124$$

$$= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X)$$

$$+ 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

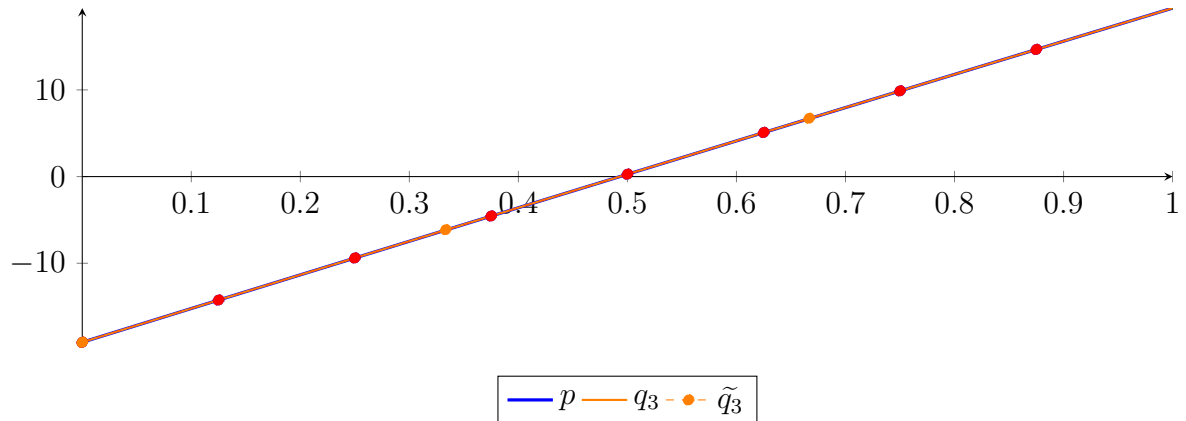
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = -1.96643 \cdot 10^{-303} X^8 + 1.82947 \cdot 10^{-302} X^7 - 4.89395 \cdot 10^{-302} X^6 + 5.49554 \cdot 10^{-302} X^5$$

$$- 2.47838 \cdot 10^{-302} X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m :

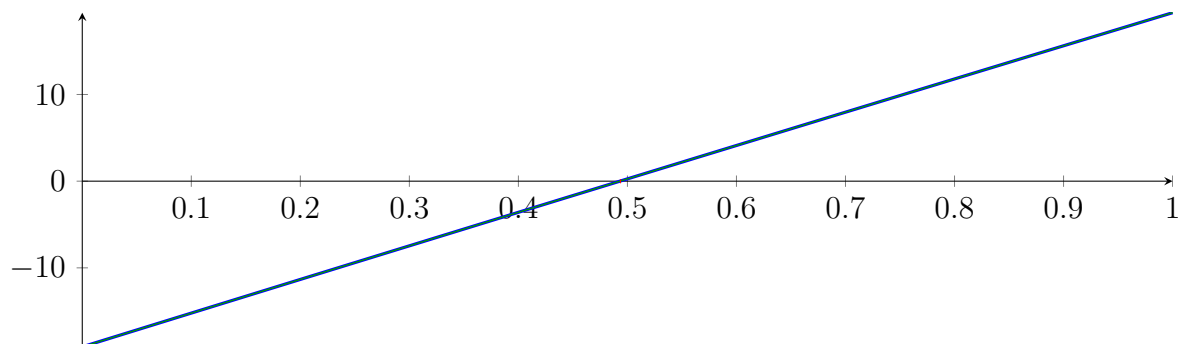
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \quad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

Longest intersection interval: $1.1252 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

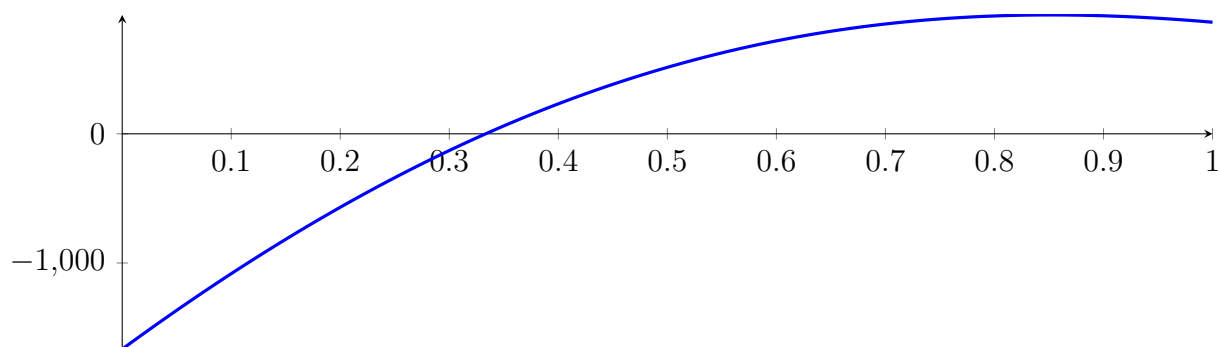
219.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

219.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

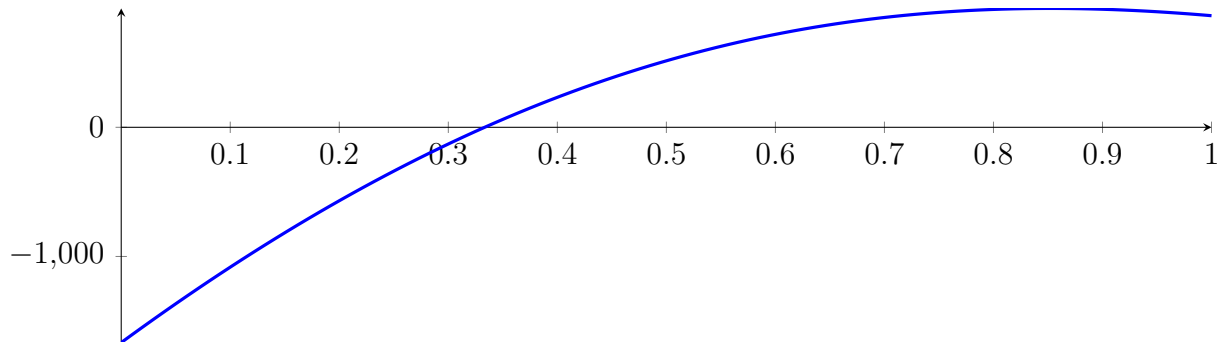
with precision $\varepsilon = 1 \cdot 10^{-08}$.

220 Running BezClip on f_8 with epsilon 16

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

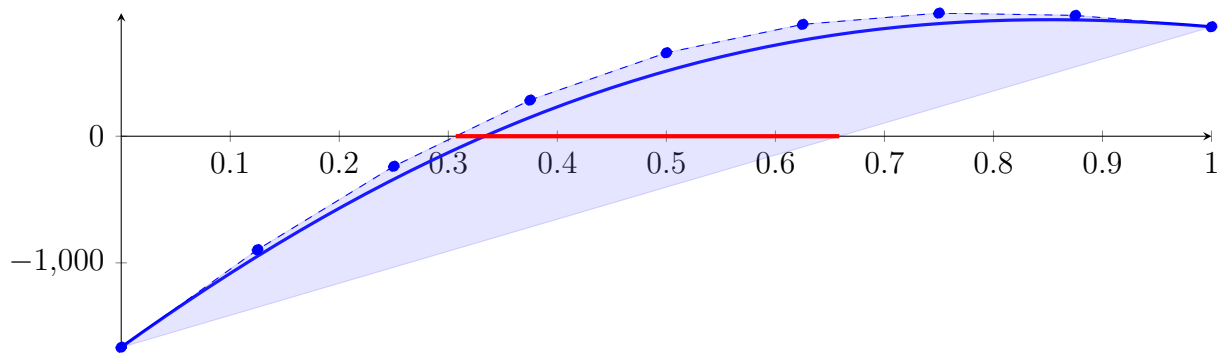
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



220.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

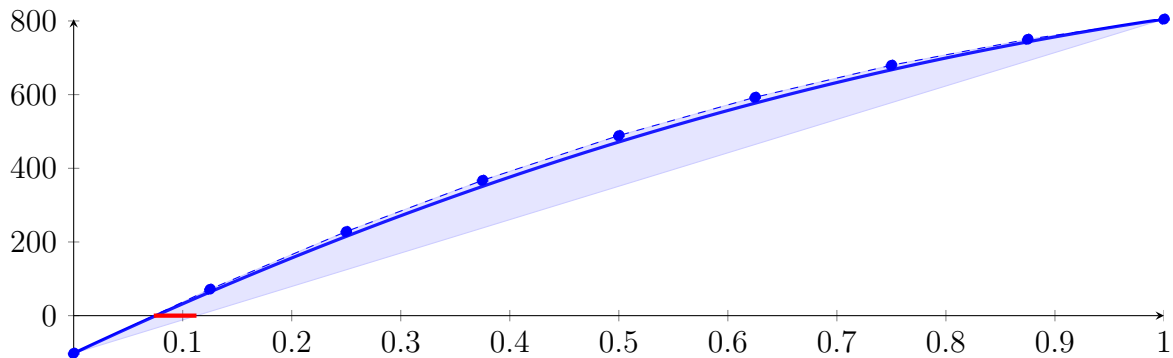
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

220.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

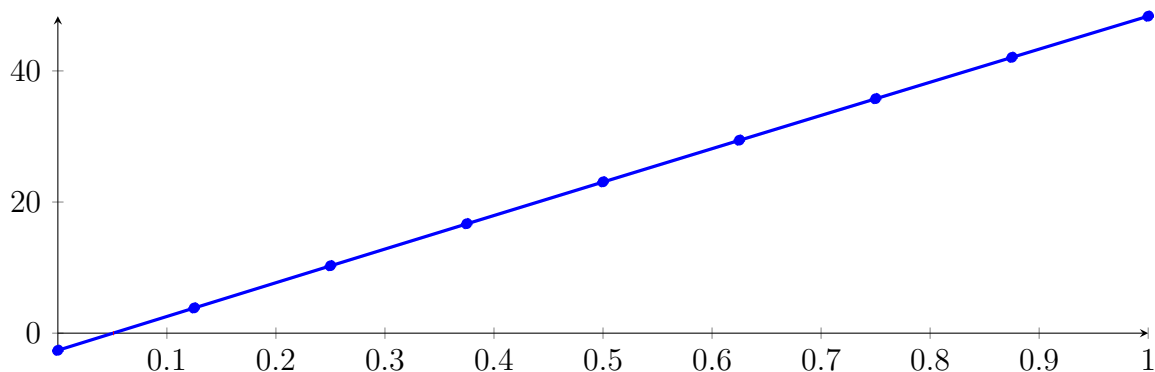
Longest intersection interval: 0.0391855

\implies Selective recursion: interval 1: [0.332635, 0.34642],

220.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.30406 \cdot 10^{-15} X^8 - 1.54459 \cdot 10^{-12} X^7 - 4.9583 \cdot 10^{-10} X^6 + 3.66751 \cdot 10^{-08} X^5 \\
 &\quad + 3.25466 \cdot 10^{-05} X^4 - 0.000586142 X^3 - 0.747315 X^2 + 51.7118 X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

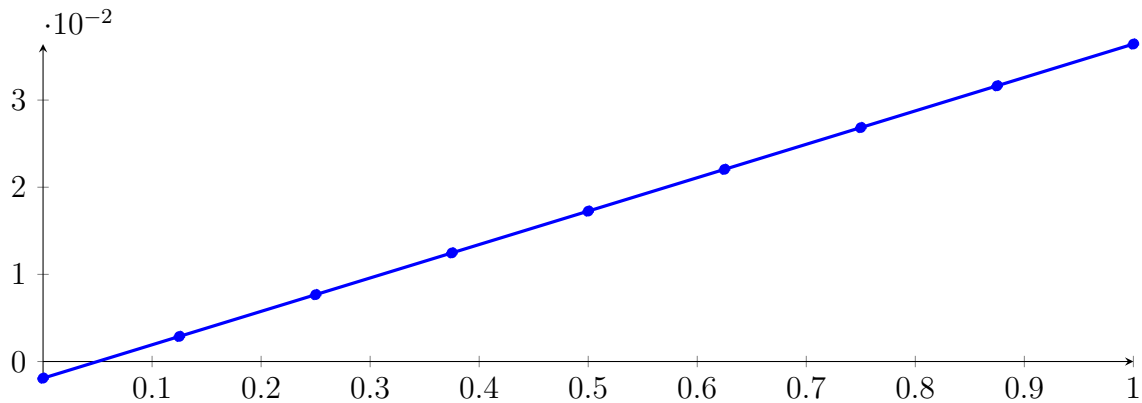
Longest intersection interval: 0.000742589

\implies Selective recursion: interval 1: [0.333333, 0.333343],

220.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.20583 \cdot 10^{-40} X^8 - 1.92397 \cdot 10^{-34} X^7 - 8.32342 \cdot 10^{-29} X^6 + 8.24755 \cdot 10^{-24} X^5 \\
 &\quad + 9.89972 \cdot 10^{-18} X^4 - 2.37322 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\
 &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\
 &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\
 &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

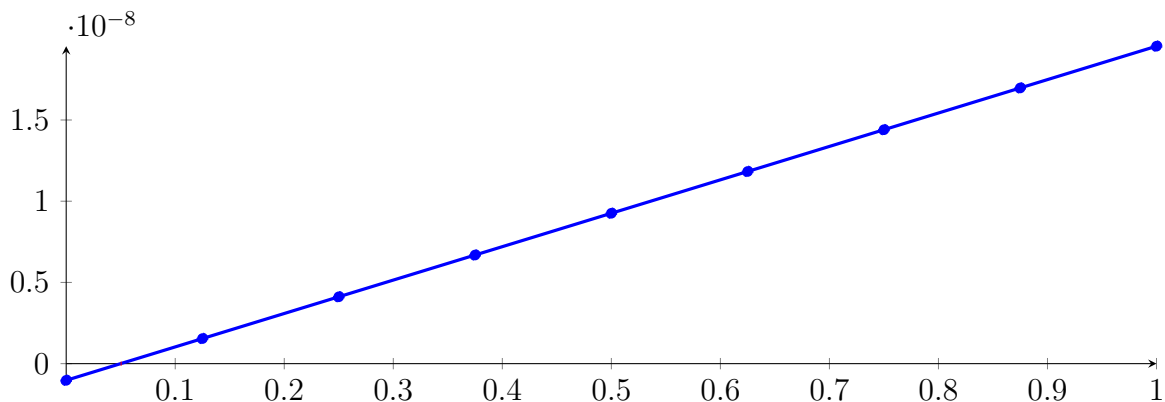
Longest intersection interval: $5.36469 \cdot 10^{-07}$

⇒ Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

220.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -8.27263 \cdot 10^{-91} X^8 - 2.46044 \cdot 10^{-78} X^7 - 1.98413 \cdot 10^{-66} X^6 + 3.66478 \cdot 10^{-55} X^5 + 8.19978 \\
 &\quad \cdot 10^{-43} X^4 - 3.66412 \cdot 10^{-32} X^3 - 1.18615 \cdot 10^{-19} X^2 + 2.05706 \cdot 10^{-08} X - 1.02667 \cdot 10^{-09} \\
 &= -1.02667 \cdot 10^{-09} B_{0,8}(X) + 1.54465 \cdot 10^{-09} B_{1,8}(X) + 4.11597 \cdot 10^{-09} B_{2,8}(X) \\
 &\quad + 6.68729 \cdot 10^{-09} B_{3,8}(X) + 9.25862 \cdot 10^{-09} B_{4,8}(X) + 1.18299 \cdot 10^{-08} B_{5,8}(X) \\
 &\quad + 1.44013 \cdot 10^{-08} B_{6,8}(X) + 1.69726 \cdot 10^{-08} B_{7,8}(X) + 1.95439 \cdot 10^{-08} B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $2.87793 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

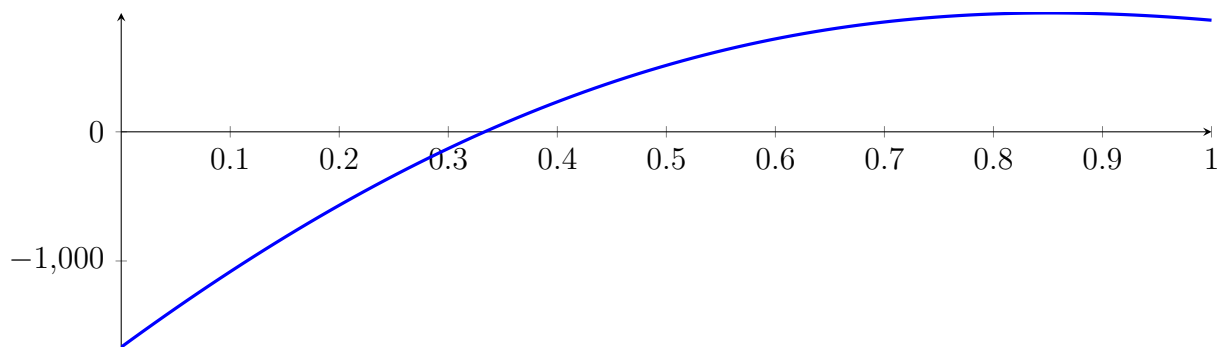
220.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

220.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

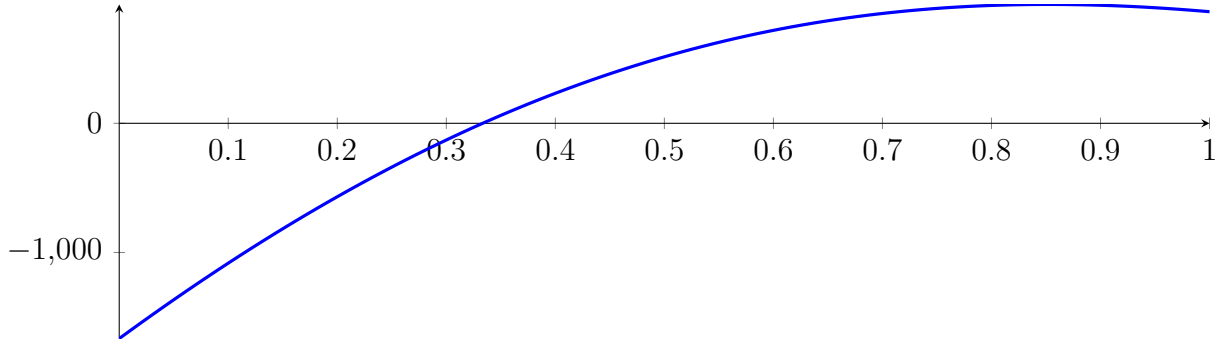
with precision $\varepsilon = 1 \cdot 10^{-16}$.

221 Running QuadClip on f_8 with epsilon 16

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

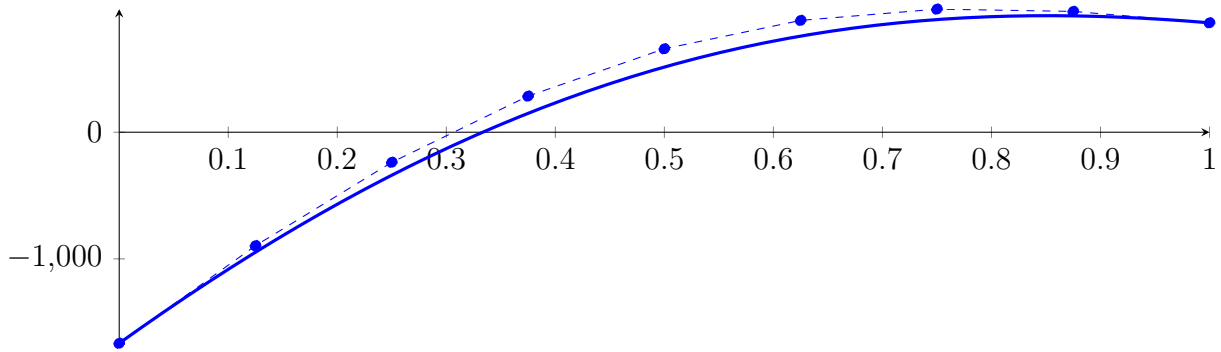
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



221.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

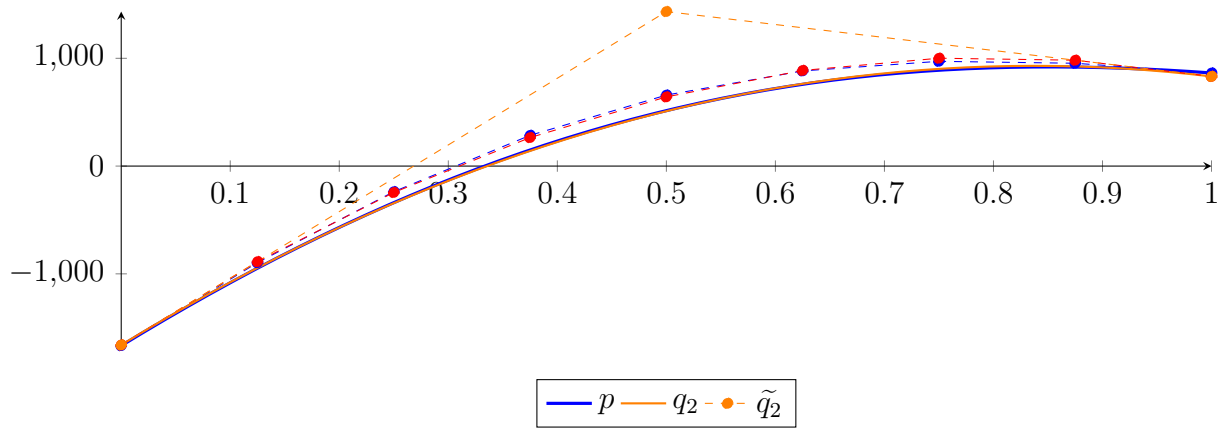
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 6.38292 \cdot 10^{-300}X^8 - 2.86993 \cdot 10^{-299}X^7 + 5.15342 \cdot 10^{-299}X^6 - 4.6928 \cdot 10^{-299}X^5 \\ &\quad + 2.29296 \cdot 10^{-299}X^4 - 6.02693 \cdot 10^{-300}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

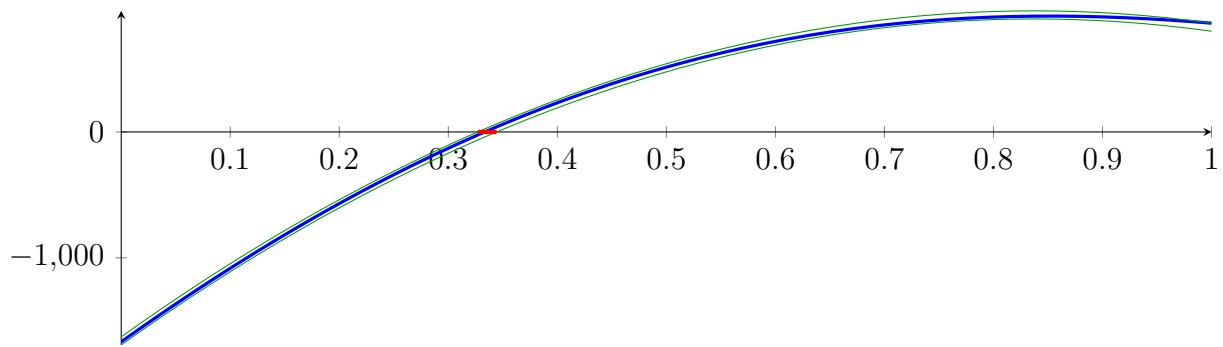
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

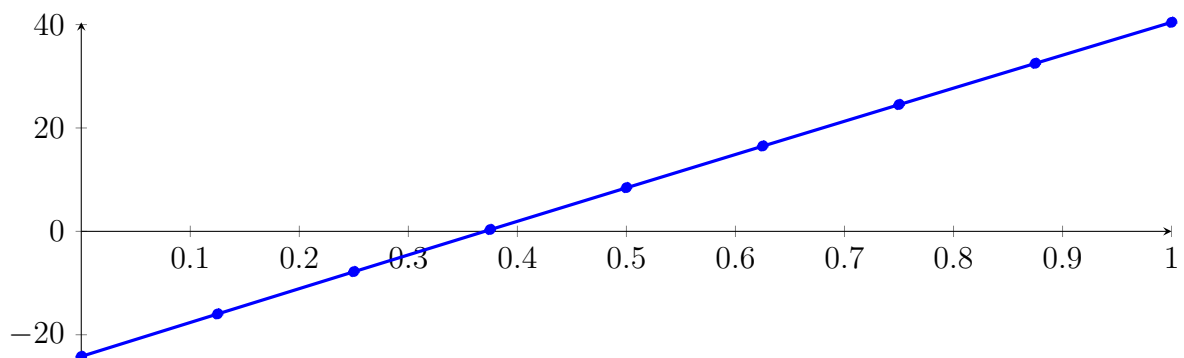
Longest intersection interval: 0.0173372

\implies Selective recursion: interval 1: $[0.326917, 0.344255]$,

221.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -8.16249 \cdot 10^{-15} X^8 - 7.66571 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

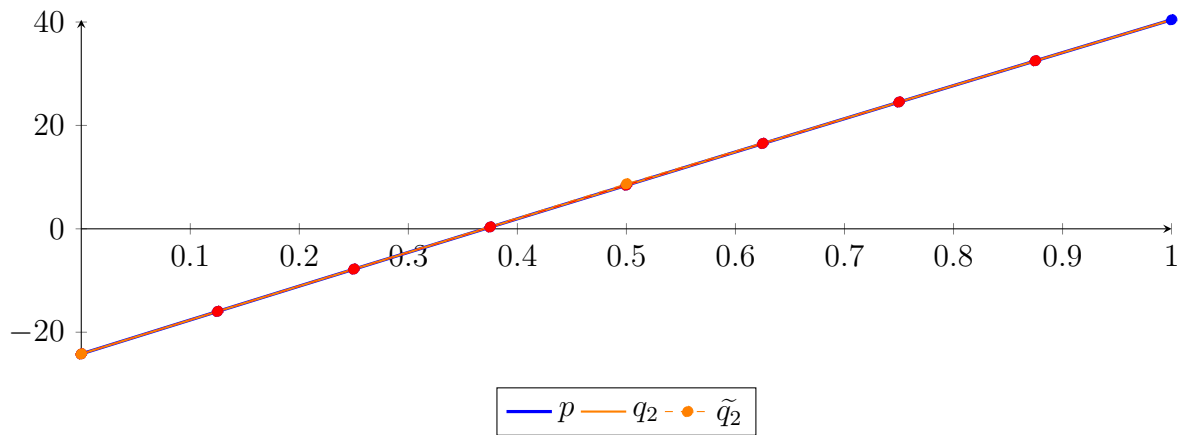
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = 4.27533 \cdot 10^{-302}X^8 - 2.42468 \cdot 10^{-301}X^7 + 4.90537 \cdot 10^{-301}X^6 - 4.6286 \cdot 10^{-301}X^5$$

$$+ 2.17546 \cdot 10^{-301}X^4 - 5.36695 \cdot 10^{-302}X^3 - 1.18261X^2 + 65.8162X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

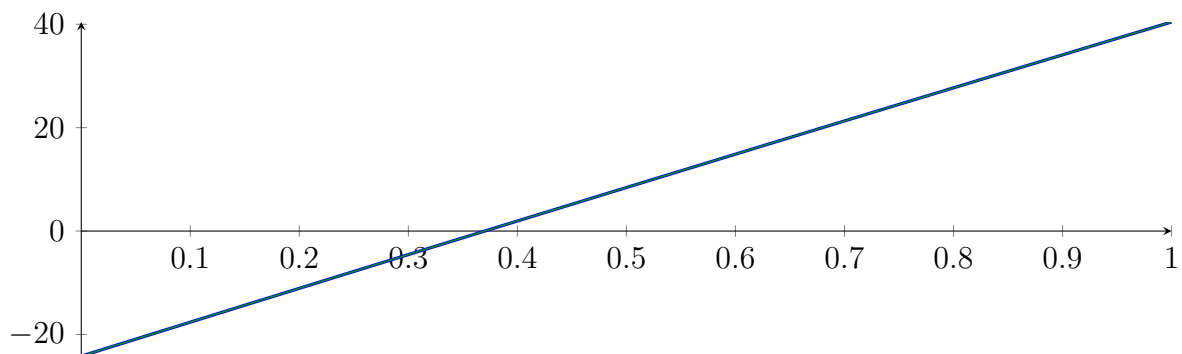
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

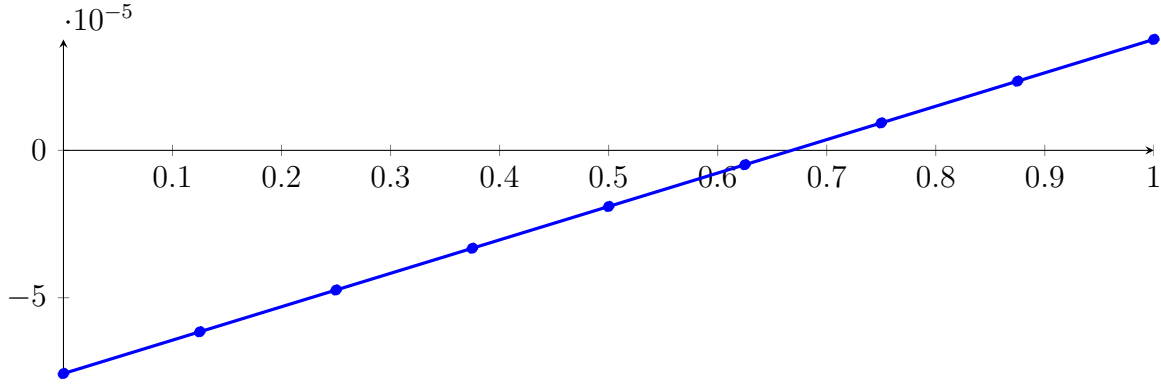
Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

221.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

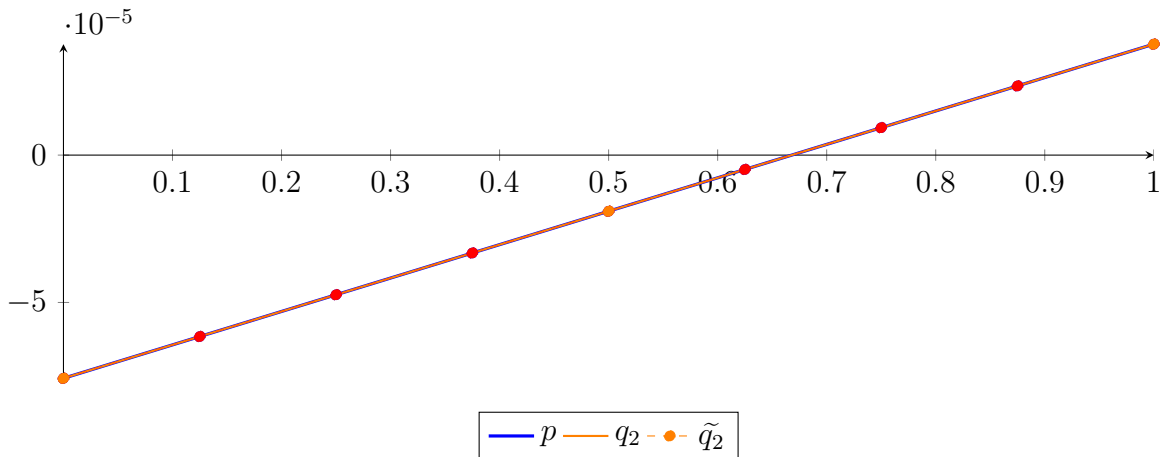
$$\begin{aligned}
 p &= -7.04578 \cdot 10^{-61} X^8 - 3.80201 \cdot 10^{-52} X^7 - 5.5627 \cdot 10^{-44} X^6 + 1.86413 \cdot 10^{-36} X^5 + 7.56737 \\
 &\quad \cdot 10^{-28} X^4 - 6.13517 \cdot 10^{-21} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) \\
 &\quad - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) \\
 &\quad + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -3.62396 \cdot 10^{-308} X^8 + 2.32992 \cdot 10^{-308} X^7 + 2.61753 \cdot 10^{-307} X^6 - 5.97049 \cdot 10^{-307} X^5 + 5.13401 \\
 &\quad \cdot 10^{-307} X^4 - 1.82868 \cdot 10^{-307} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \\
 &\quad \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.06758 \cdot 10^{-22}$.

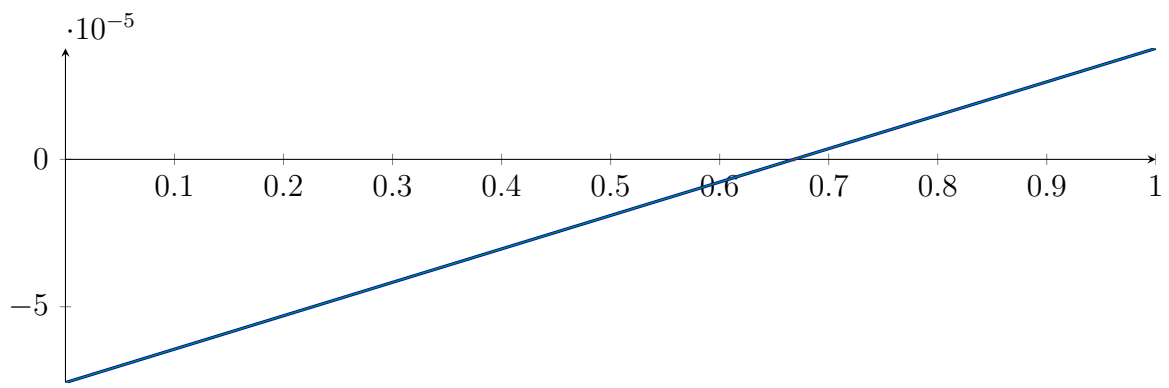
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 m &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.667955, 3.14644 \cdot 10^7\} \qquad N(m) = \{0.667955, 3.14644 \cdot 10^7\}$$

Intersection intervals:



[0.667955, 0.667955]

Longest intersection interval: $5.41121 \cdot 10^{-18}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

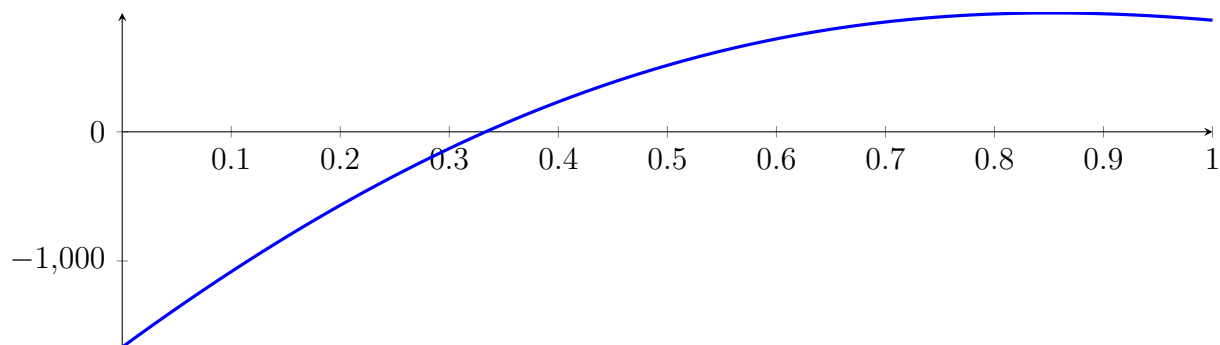
221.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 4!

221.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

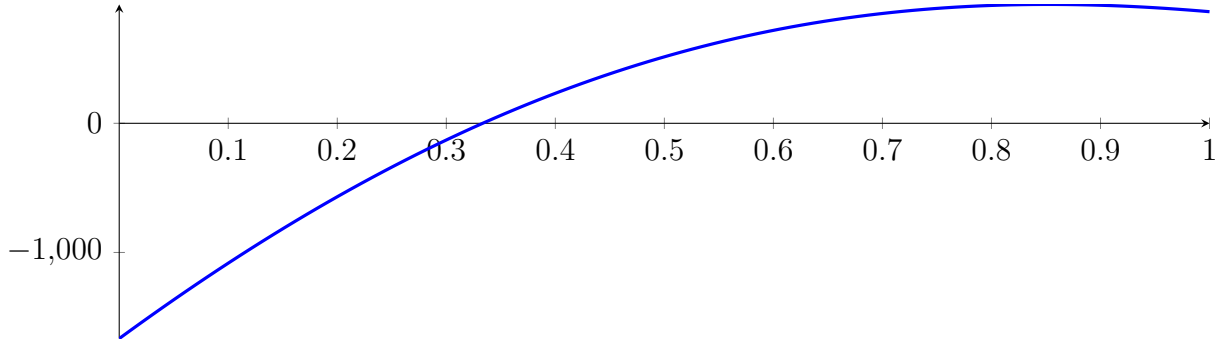
with precision $\varepsilon = 1 \cdot 10^{-16}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

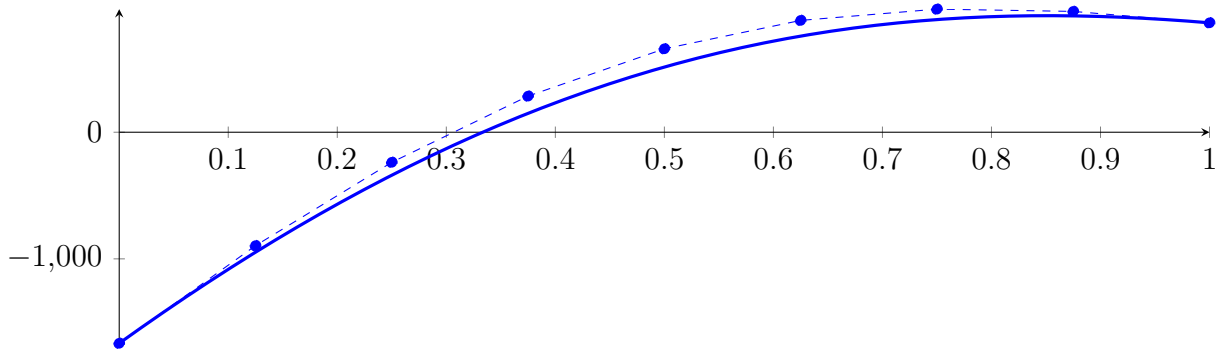
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



222.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

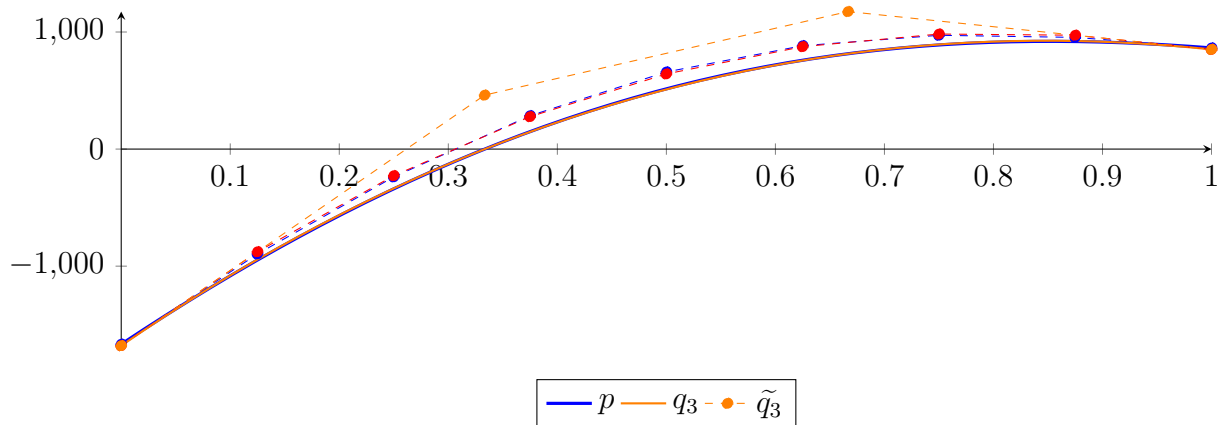
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.80826 \cdot 10^{-300}X^8 + 1.16606 \cdot 10^{-299}X^7 - 2.1197 \cdot 10^{-299}X^6 + 2.15003 \cdot 10^{-299}X^5 \\ &\quad - 1.20234 \cdot 10^{-299}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

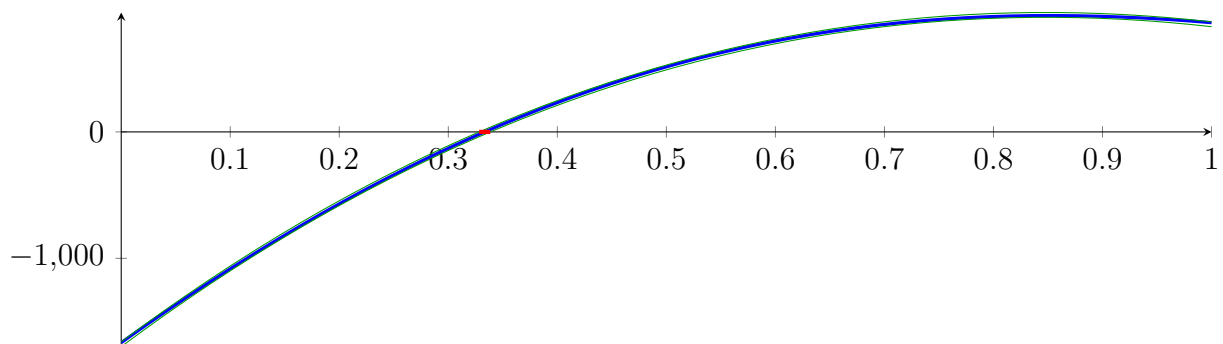
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

222.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

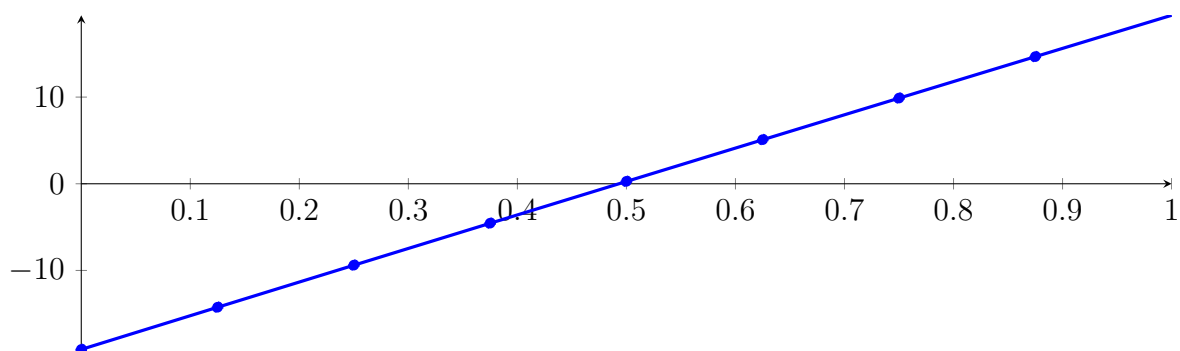
Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.25947 \cdot 10^{-16} X^8 - 1.9937 \cdot 10^{-13} X^7 - 8.53073 \cdot 10^{-11} X^6 + 8.72839 \cdot 10^{-09} X^5$$

$$+ 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124$$

$$= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X)$$

$$+ 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

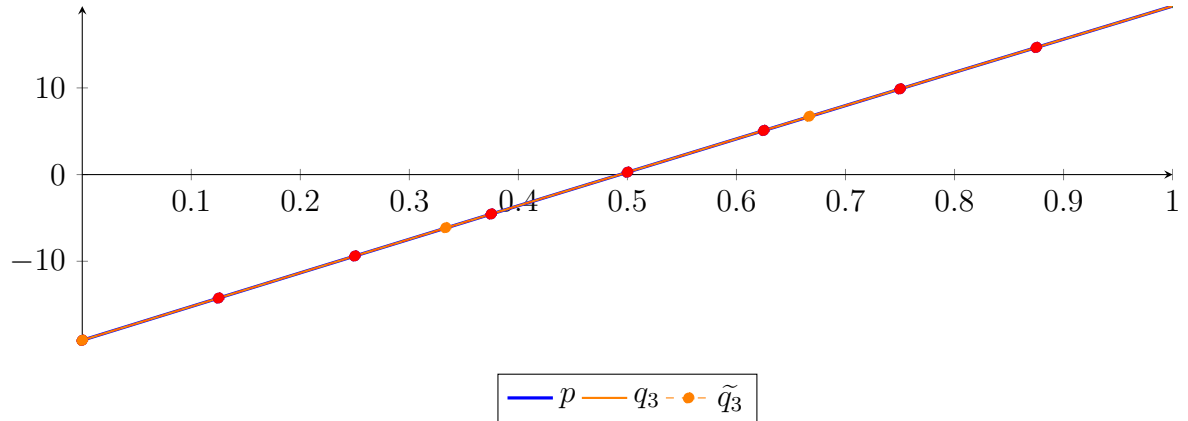
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = -1.96643 \cdot 10^{-303} X^8 + 1.82947 \cdot 10^{-302} X^7 - 4.89395 \cdot 10^{-302} X^6 + 5.49554 \cdot 10^{-302} X^5$$

$$- 2.47838 \cdot 10^{-302} X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m :

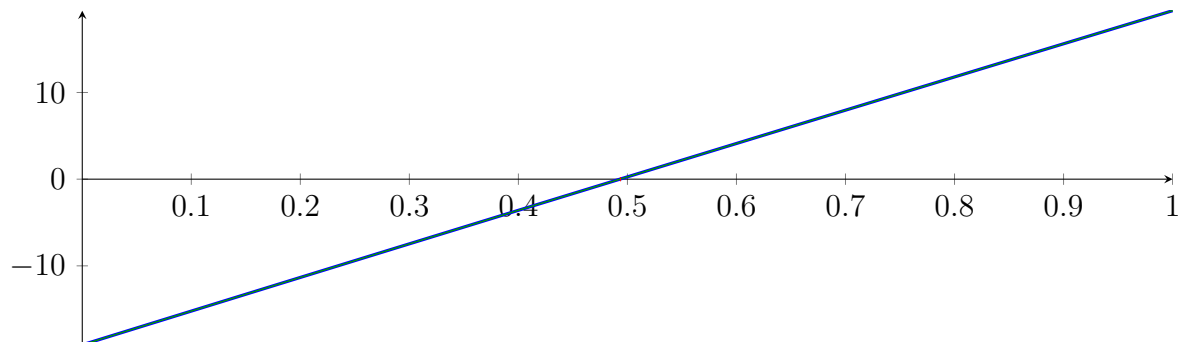
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \quad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

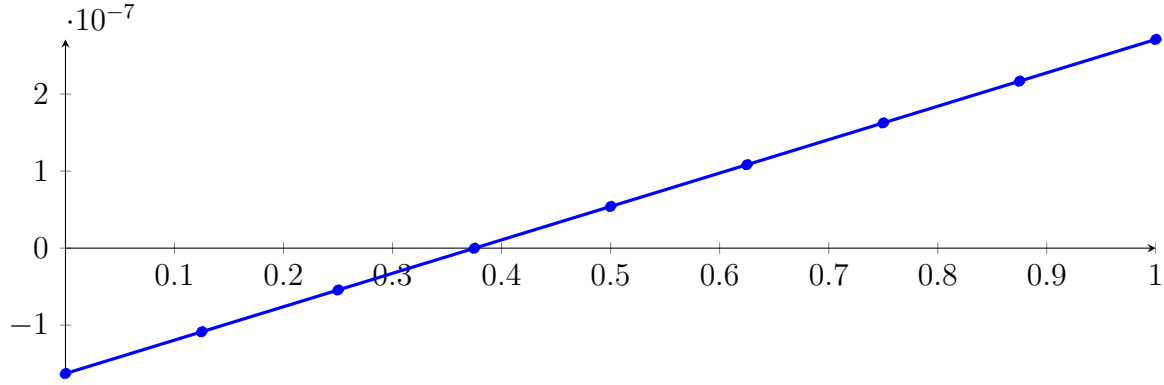
Longest intersection interval: $1.1252 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

222.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

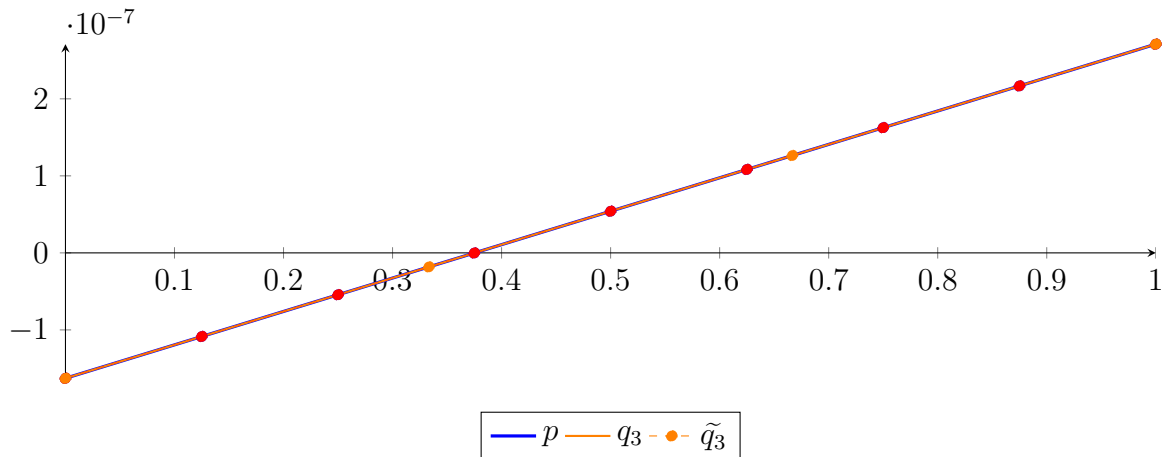
$$\begin{aligned}
 p &= -3.2361 \cdot 10^{-80} X^8 - 4.56398 \cdot 10^{-69} X^7 - 1.74524 \cdot 10^{-58} X^6 + 1.52857 \cdot 10^{-48} X^5 + 1.62178 \\
 &\quad \cdot 10^{-37} X^4 - 3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,8}(X) - 1.08585 \cdot 10^{-07} B_{1,8}(X) - 5.43592 \cdot 10^{-08} B_{2,8}(X) \\
 &\quad - 1.33711 \cdot 10^{-10} B_{3,8}(X) + 5.40918 \cdot 10^{-08} B_{4,8}(X) + 1.08317 \cdot 10^{-07} B_{5,8}(X) \\
 &\quad + 1.62543 \cdot 10^{-07} B_{6,8}(X) + 2.16768 \cdot 10^{-07} B_{7,8}(X) + 2.70994 \cdot 10^{-07} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,3} - 1.82089 \cdot 10^{-08} B_{1,3} + 1.26393 \cdot 10^{-07} B_{2,3} + 2.70994 \cdot 10^{-07} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 8.86066 \cdot 10^{-312} X^8 + 7.41744 \cdot 10^{-311} X^7 - 5.24902 \cdot 10^{-310} X^6 + 1.01267 \cdot 10^{-309} X^5 - 7.8921 \\
 &\quad \cdot 10^{-310} X^4 - 3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,8} - 1.08585 \cdot 10^{-07} B_{1,8} - 5.43592 \cdot 10^{-08} B_{2,8} - 1.33711 \cdot 10^{-10} B_{3,8} + 5.40918 \\
 &\quad \cdot 10^{-08} B_{4,8} + 1.08317 \cdot 10^{-07} B_{5,8} + 1.62543 \cdot 10^{-07} B_{6,8} + 2.16768 \cdot 10^{-07} B_{7,8} + 2.70994 \cdot 10^{-07} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.47524 \cdot 10^{-39}$.

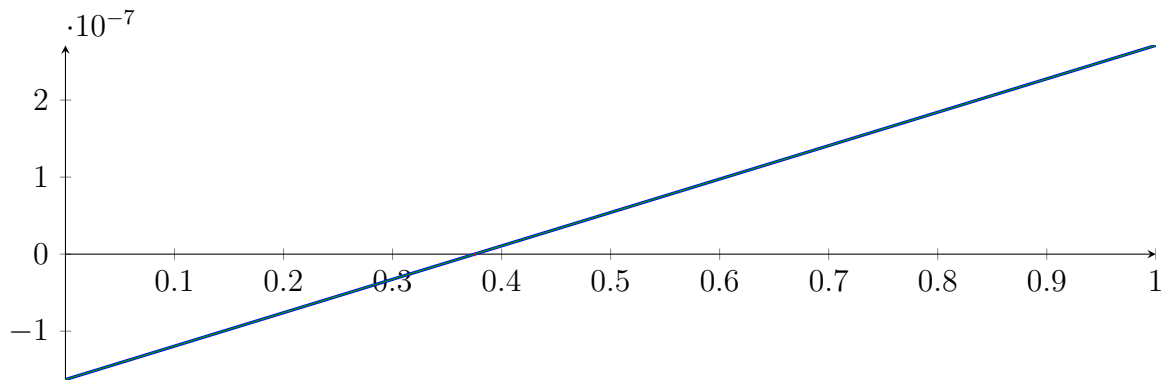
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 m &= -3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-1.6133 \cdot 10^{11}, 0.375292, 7.82468 \cdot 10^9\} \quad N(m) = \{-1.6133 \cdot 10^{11}, 0.375292, 7.82468 \cdot 10^9\}$$

Intersection intervals:



[0.375292, 0.375292]

Longest intersection interval: $1.60221 \cdot 10^{-32}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

222.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

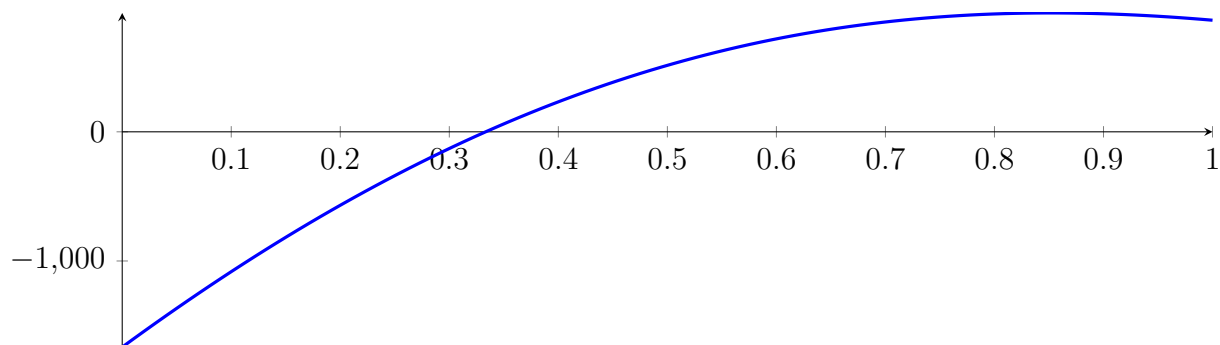
Reached interval [0.333333, 0.333333] **without sign change** at depth 4!

$p(0) = -6.9178e-12$ - $p(1) -6.9178e-12$

222.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

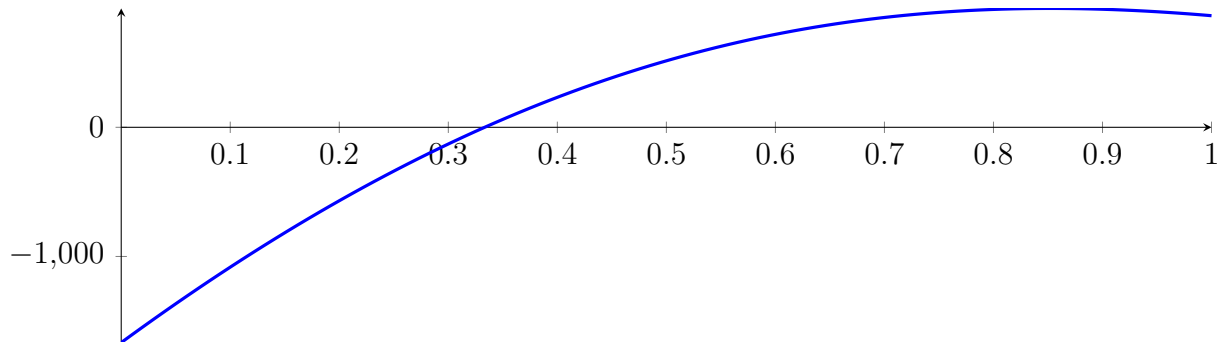
with precision $\varepsilon = 1 \cdot 10^{-16}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

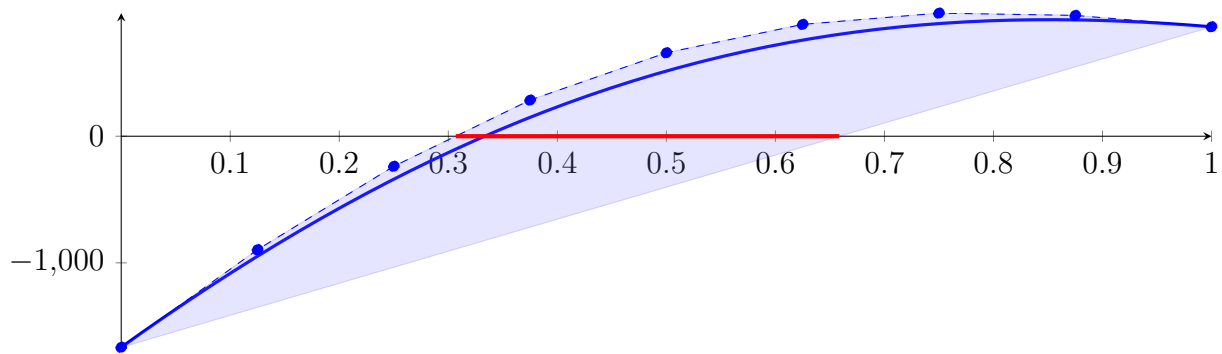
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



223.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

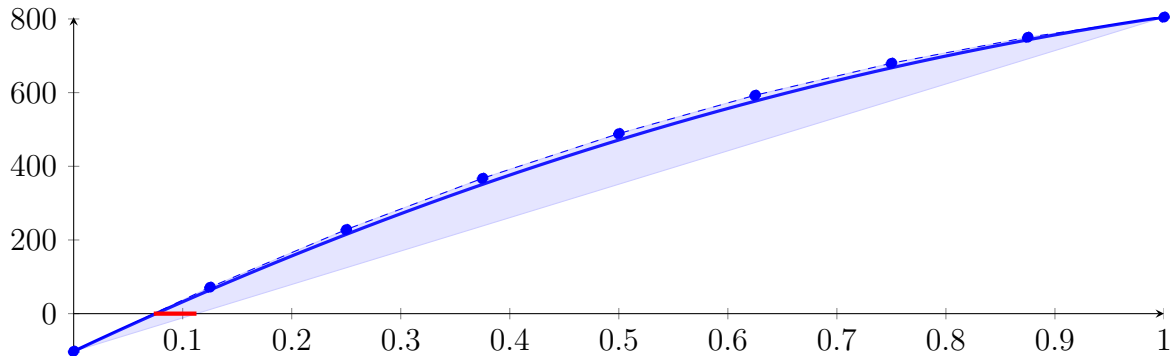
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

223.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

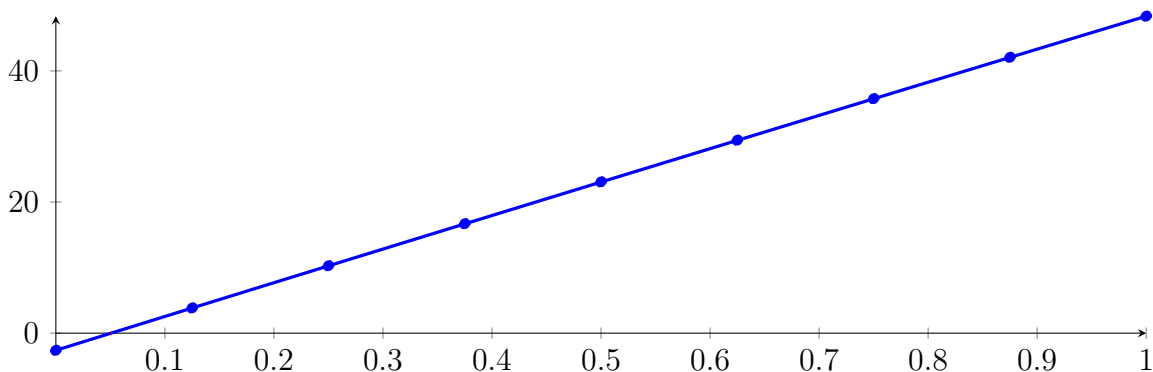
Longest intersection interval: 0.0391855

\implies Selective recursion: interval 1: [0.332635, 0.34642],

223.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.30406 \cdot 10^{-15} X^8 - 1.54459 \cdot 10^{-12} X^7 - 4.9583 \cdot 10^{-10} X^6 + 3.66751 \cdot 10^{-08} X^5 \\
 &\quad + 3.25466 \cdot 10^{-05} X^4 - 0.000586142 X^3 - 0.747315 X^2 + 51.7118 X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

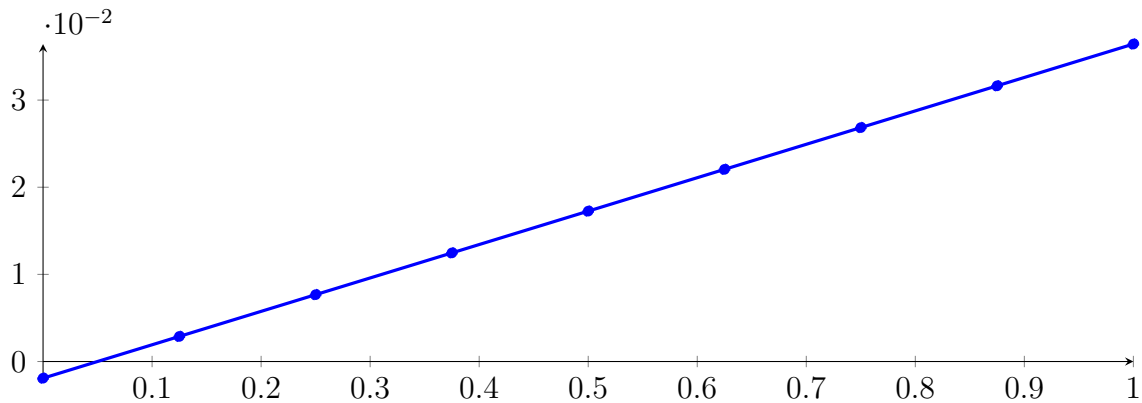
Longest intersection interval: 0.000742589

\implies Selective recursion: interval 1: [0.333333, 0.333343],

223.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.20583 \cdot 10^{-40} X^8 - 1.92397 \cdot 10^{-34} X^7 - 8.32342 \cdot 10^{-29} X^6 + 8.24755 \cdot 10^{-24} X^5 \\
 &\quad + 9.89972 \cdot 10^{-18} X^4 - 2.37322 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\
 &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\
 &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\
 &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

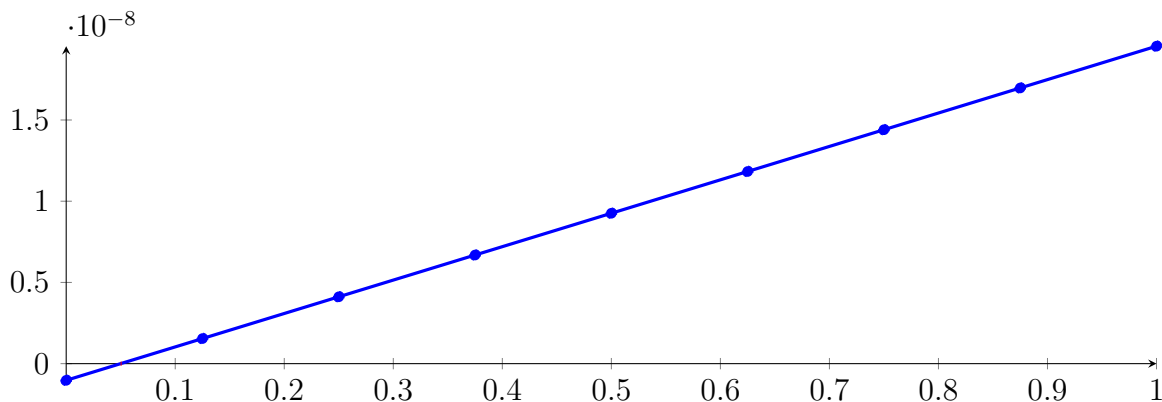
Longest intersection interval: $5.36469 \cdot 10^{-07}$

⇒ Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

223.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -8.27263 \cdot 10^{-91} X^8 - 2.46044 \cdot 10^{-78} X^7 - 1.98413 \cdot 10^{-66} X^6 + 3.66478 \cdot 10^{-55} X^5 + 8.19978 \\
 &\quad \cdot 10^{-43} X^4 - 3.66412 \cdot 10^{-32} X^3 - 1.18615 \cdot 10^{-19} X^2 + 2.05706 \cdot 10^{-08} X - 1.02667 \cdot 10^{-09} \\
 &= -1.02667 \cdot 10^{-09} B_{0,8}(X) + 1.54465 \cdot 10^{-09} B_{1,8}(X) + 4.11597 \cdot 10^{-09} B_{2,8}(X) \\
 &\quad + 6.68729 \cdot 10^{-09} B_{3,8}(X) + 9.25862 \cdot 10^{-09} B_{4,8}(X) + 1.18299 \cdot 10^{-08} B_{5,8}(X) \\
 &\quad + 1.44013 \cdot 10^{-08} B_{6,8}(X) + 1.69726 \cdot 10^{-08} B_{7,8}(X) + 1.95439 \cdot 10^{-08} B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

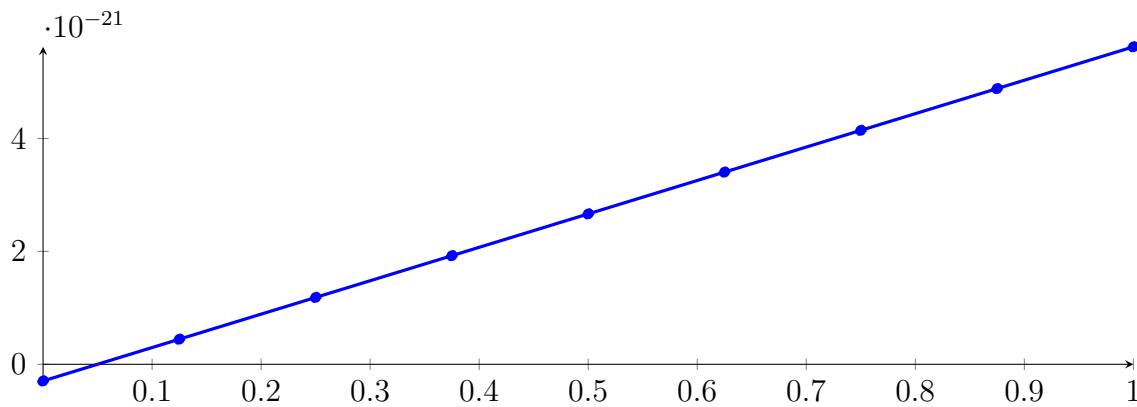
Longest intersection interval: $2.87793 \cdot 10^{-13}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

223.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -3.89305 \cdot 10^{-191} X^8 - 4.02327 \cdot 10^{-166} X^7 - 1.12734 \cdot 10^{-141} X^6 + 7.23523 \cdot 10^{-118} X^5 + 5.62504 \\ &\quad \cdot 10^{-93} X^4 - 8.73397 \cdot 10^{-70} X^3 - 9.82433 \cdot 10^{-45} X^2 + 5.92008 \cdot 10^{-21} X - 2.9547 \cdot 10^{-22} \\ &= -2.9547 \cdot 10^{-22} B_{0,8}(X) + 4.4454 \cdot 10^{-22} B_{1,8}(X) + 1.18455 \cdot 10^{-21} B_{2,8}(X) \\ &\quad + 1.92456 \cdot 10^{-21} B_{3,8}(X) + 2.66457 \cdot 10^{-21} B_{4,8}(X) + 3.40458 \cdot 10^{-21} B_{5,8}(X) \\ &\quad + 4.14459 \cdot 10^{-21} B_{6,8}(X) + 4.8846 \cdot 10^{-21} B_{7,8}(X) + 5.62461 \cdot 10^{-21} B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $8.28251 \cdot 10^{-26}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

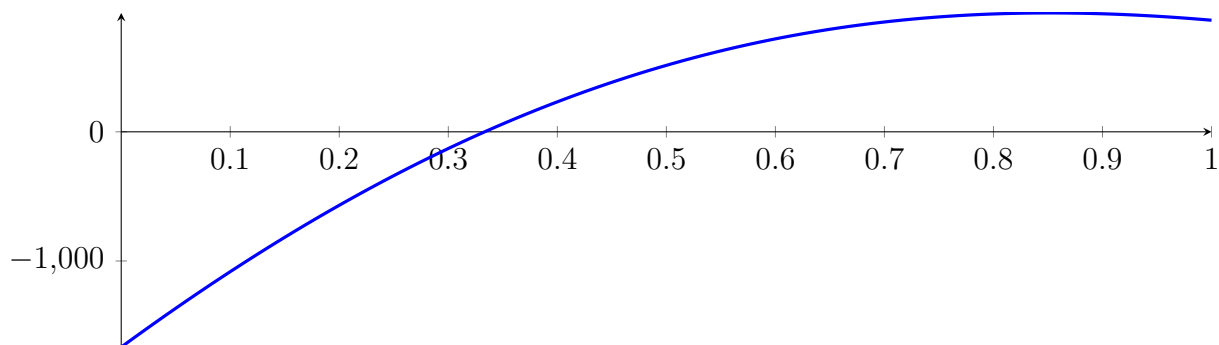
223.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 7!

223.8 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

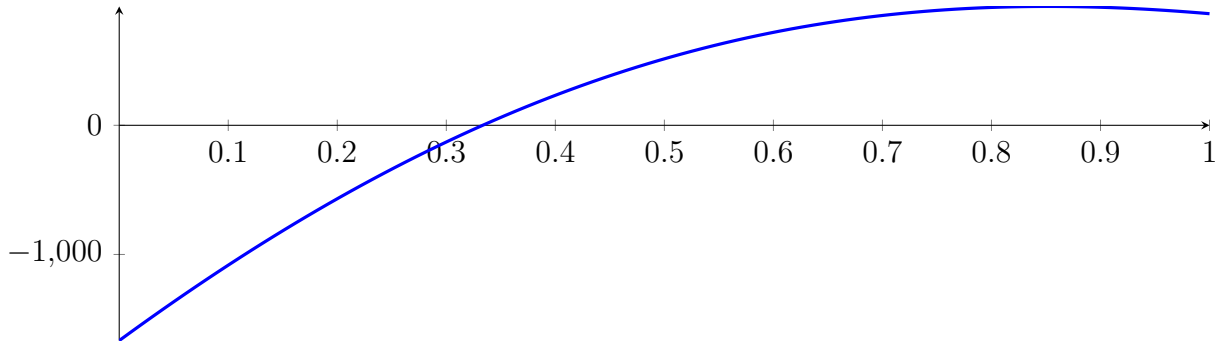
with precision $\varepsilon = 1 \cdot 10^{-32}$.

224 Running QuadClip on f_8 with epsilon 32

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

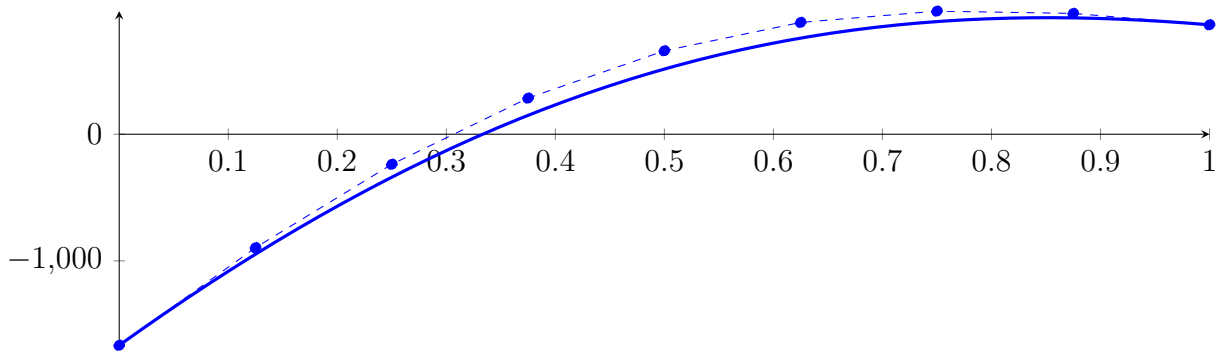
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



224.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

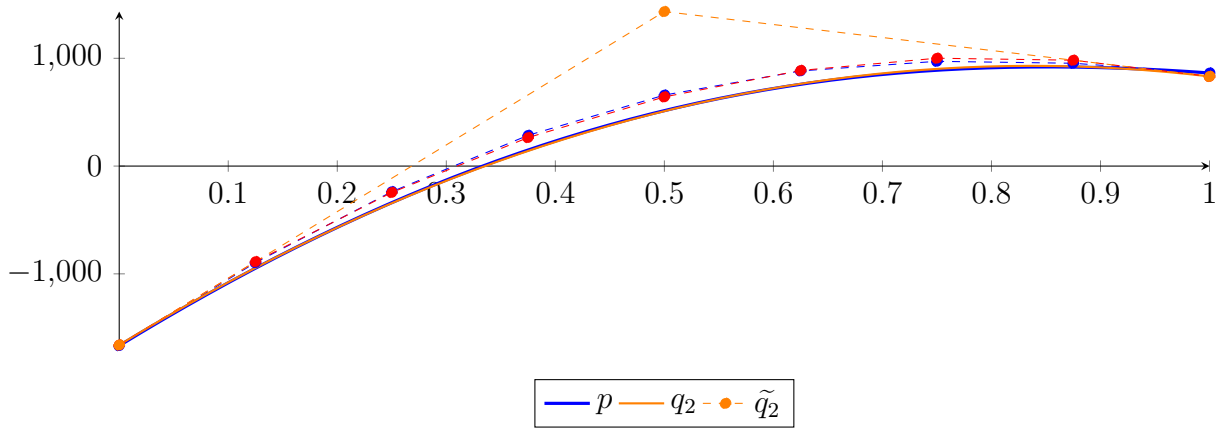
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 6.38292 \cdot 10^{-300}X^8 - 2.86993 \cdot 10^{-299}X^7 + 5.15342 \cdot 10^{-299}X^6 - 4.6928 \cdot 10^{-299}X^5 \\ &\quad + 2.29296 \cdot 10^{-299}X^4 - 6.02693 \cdot 10^{-300}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

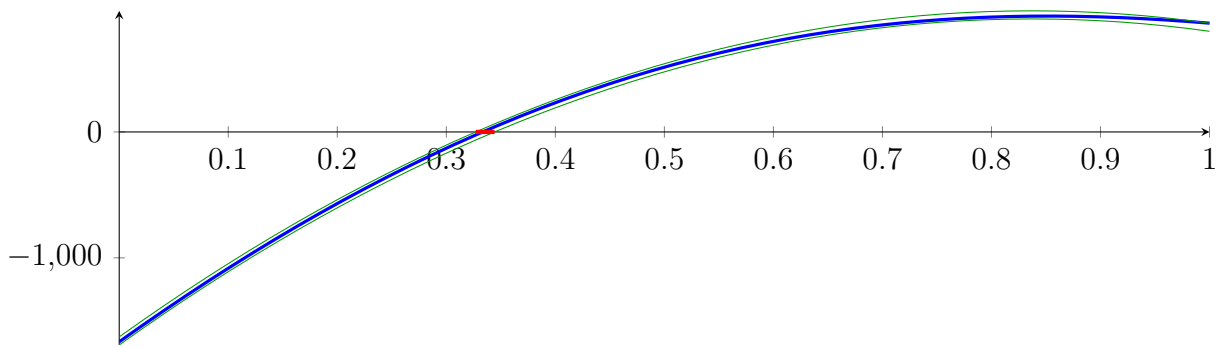
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

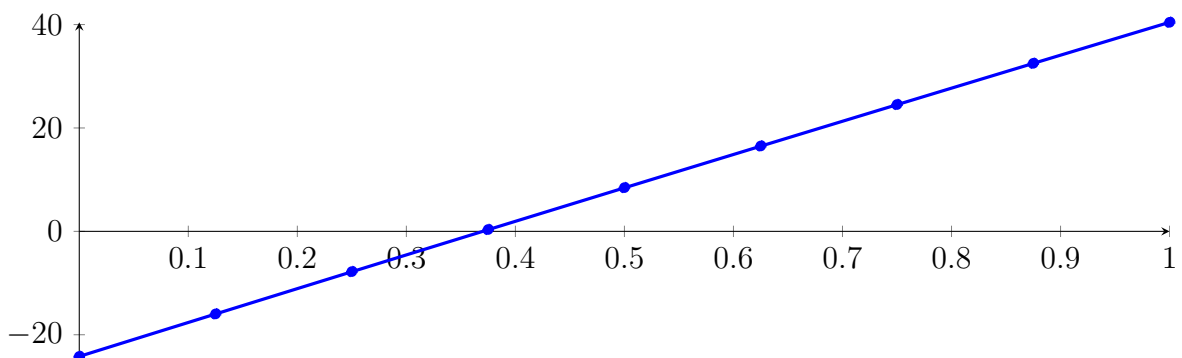
Longest intersection interval: 0.0173372

\implies Selective recursion: interval 1: $[0.326917, 0.344255]$,

224.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -8.16249 \cdot 10^{-15} X^8 - 7.66571 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

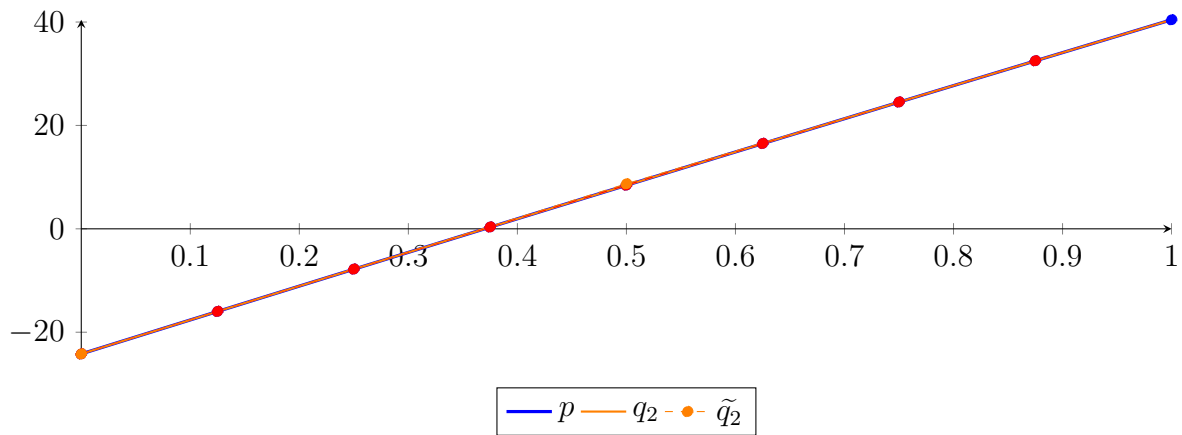
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = 4.27533 \cdot 10^{-302}X^8 - 2.42468 \cdot 10^{-301}X^7 + 4.90537 \cdot 10^{-301}X^6 - 4.6286 \cdot 10^{-301}X^5$$

$$+ 2.17546 \cdot 10^{-301}X^4 - 5.36695 \cdot 10^{-302}X^3 - 1.18261X^2 + 65.8162X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

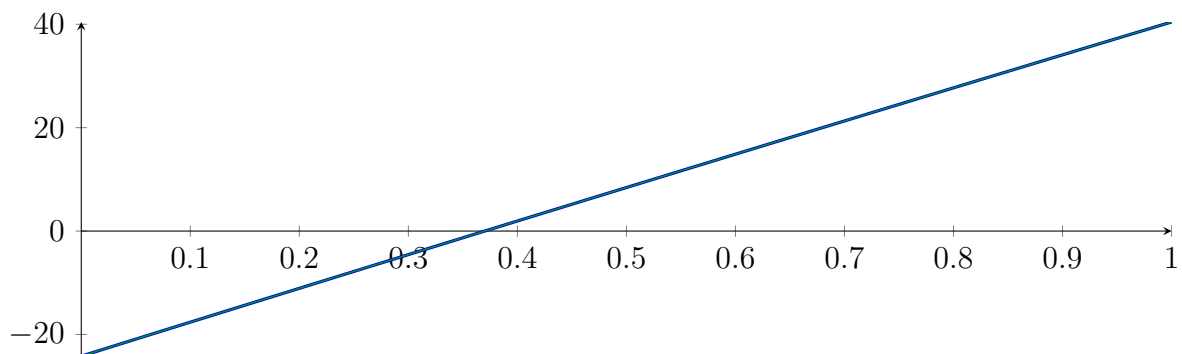
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

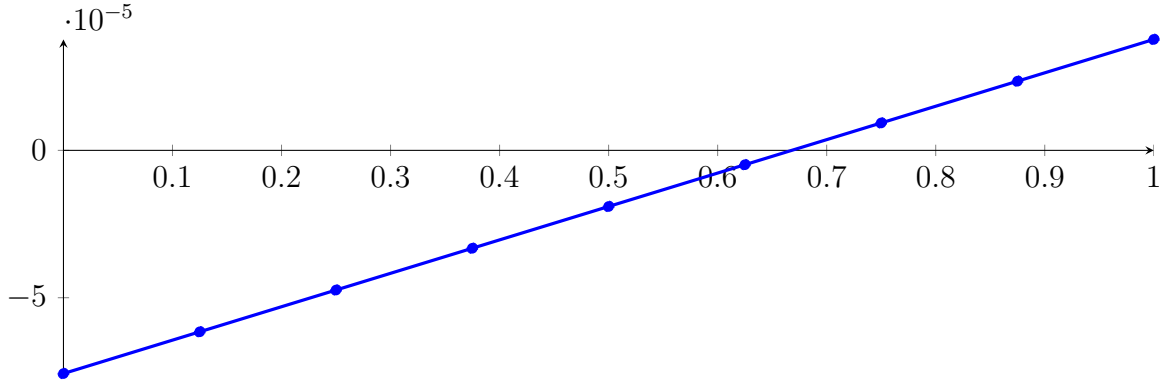
Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

224.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

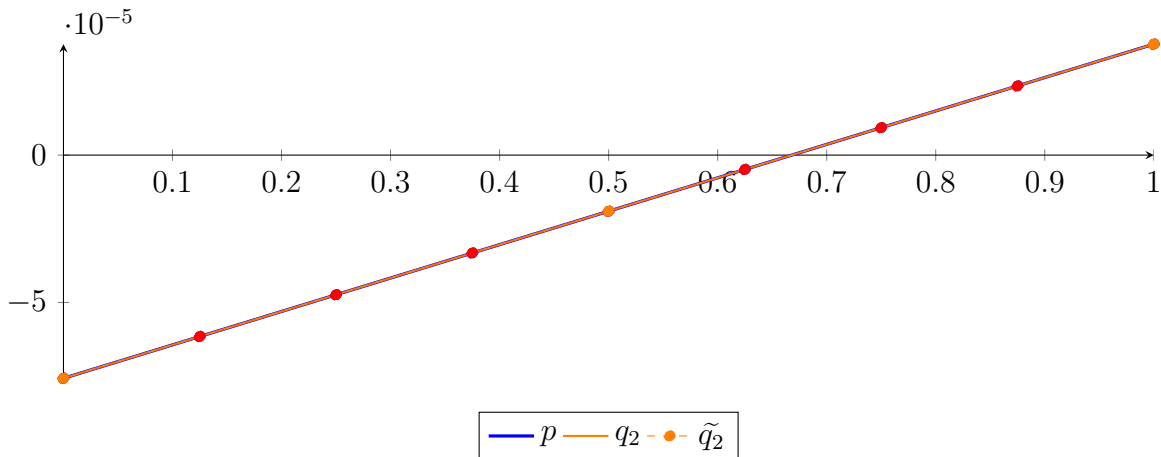
$$\begin{aligned}
 p &= -7.04578 \cdot 10^{-61} X^8 - 3.80201 \cdot 10^{-52} X^7 - 5.5627 \cdot 10^{-44} X^6 + 1.86413 \cdot 10^{-36} X^5 + 7.56737 \\
 &\quad \cdot 10^{-28} X^4 - 6.13517 \cdot 10^{-21} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) \\
 &\quad - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) \\
 &\quad + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -3.62396 \cdot 10^{-308} X^8 + 2.32992 \cdot 10^{-308} X^7 + 2.61753 \cdot 10^{-307} X^6 - 5.97049 \cdot 10^{-307} X^5 + 5.13401 \\
 &\quad \cdot 10^{-307} X^4 - 1.82868 \cdot 10^{-307} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \\
 &\quad \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.06758 \cdot 10^{-22}$.

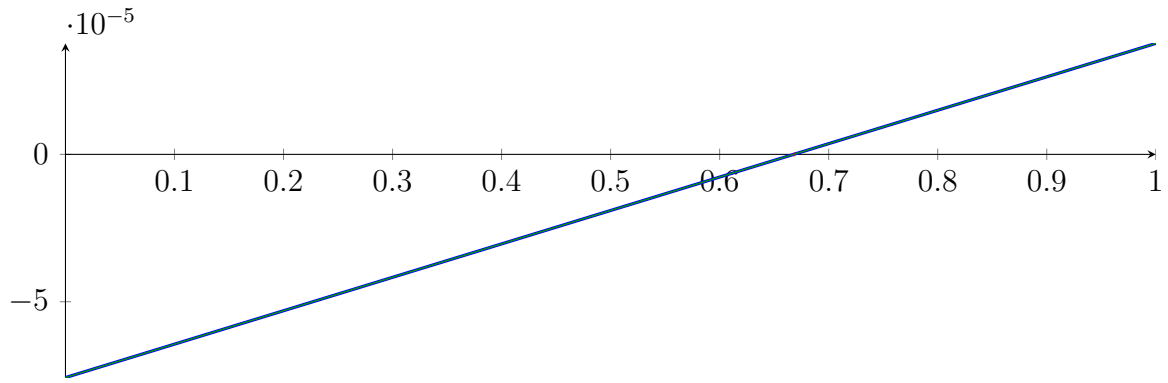
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 m &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.667955, 3.14644 \cdot 10^7\} \quad N(m) = \{0.667955, 3.14644 \cdot 10^7\}$$

Intersection intervals:



[0.667955, 0.667955]

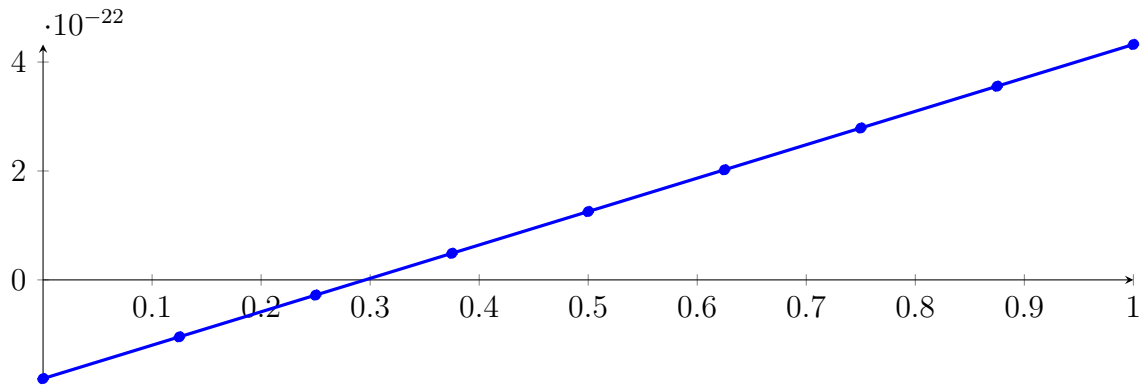
Longest intersection interval: $5.41121 \cdot 10^{-18}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

224.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

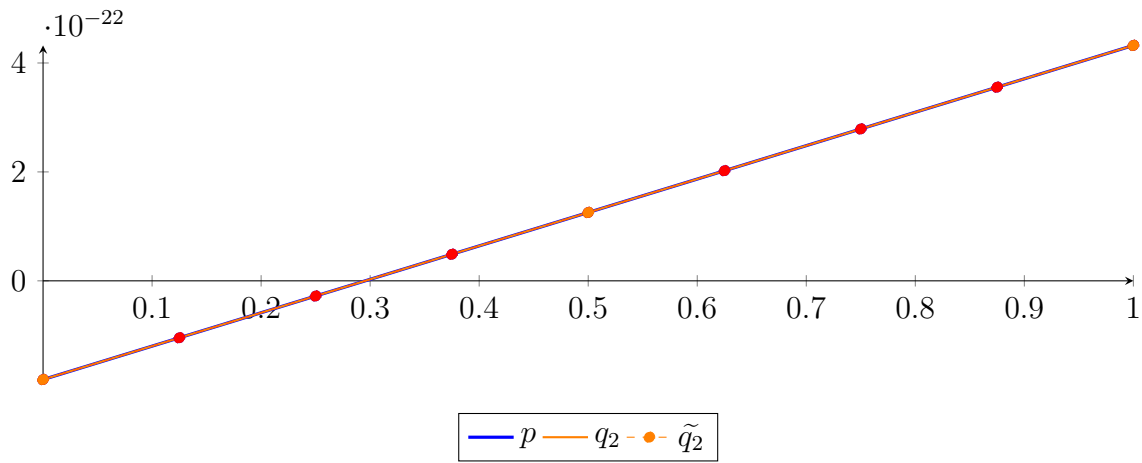
$$\begin{aligned}
 p &= -5.17944 \cdot 10^{-199} X^8 - 5.16502 \cdot 10^{-173} X^7 - 1.39653 \cdot 10^{-147} X^6 + 8.64863 \cdot 10^{-123} X^5 + 6.48817 \\
 &\quad \cdot 10^{-97} X^4 - 9.72096 \cdot 10^{-73} X^3 - 1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22} \\
 &= -1.81261 \cdot 10^{-22} B_{0,8}(X) - 1.04571 \cdot 10^{-22} B_{1,8}(X) - 2.78818 \cdot 10^{-23} B_{2,8}(X) \\
 &\quad + 4.88078 \cdot 10^{-23} B_{3,8}(X) + 1.25497 \cdot 10^{-22} B_{4,8}(X) + 2.02187 \cdot 10^{-22} B_{5,8}(X) \\
 &\quad + 2.78877 \cdot 10^{-22} B_{6,8}(X) + 3.55566 \cdot 10^{-22} B_{7,8}(X) + 4.32256 \cdot 10^{-22} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22} \\
 &= -1.81261 \cdot 10^{-22} B_{0,2} + 1.25497 \cdot 10^{-22} B_{1,2} + 4.32256 \cdot 10^{-22} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 5.39888 \cdot 10^{-325} X^8 - 2.84119 \cdot 10^{-324} X^7 + 5.35011 \cdot 10^{-324} X^6 - 4.57499 \cdot 10^{-324} X^5 + 1.82797 \\
 &\quad \cdot 10^{-324} X^4 - 3.72306 \cdot 10^{-325} X^3 - 1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22} \\
 &= -1.81261 \cdot 10^{-22} B_{0,8} - 1.04571 \cdot 10^{-22} B_{1,8} - 2.78818 \cdot 10^{-23} B_{2,8} + 4.88078 \cdot 10^{-23} B_{3,8} + 1.25497 \\
 &\quad \cdot 10^{-22} B_{4,8} + 2.02187 \cdot 10^{-22} B_{5,8} + 2.78877 \cdot 10^{-22} B_{6,8} + 3.55566 \cdot 10^{-22} B_{7,8} + 4.32256 \cdot 10^{-22} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.86048 \cdot 10^{-74}$.

Bounding polynomials M and m :

$$M = -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22}$$

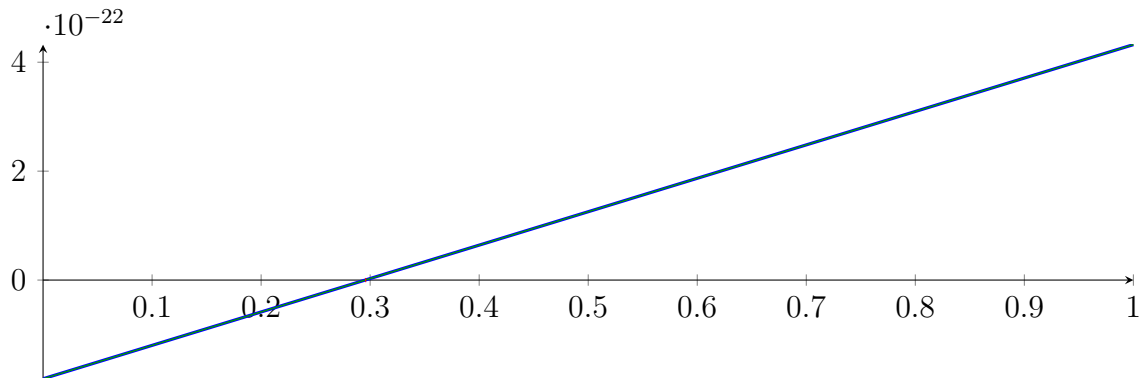
$$m = -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22}$$

Root of M and m :

$$N(M) = \{0.295446, 5.81467 \cdot 10^{24}\}$$

$$N(m) = \{0.295446, 5.81467 \cdot 10^{24}\}$$

Intersection intervals:



$$[0.295446, 0.295446]$$

Longest intersection interval: $1.58446 \cdot 10^{-52}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

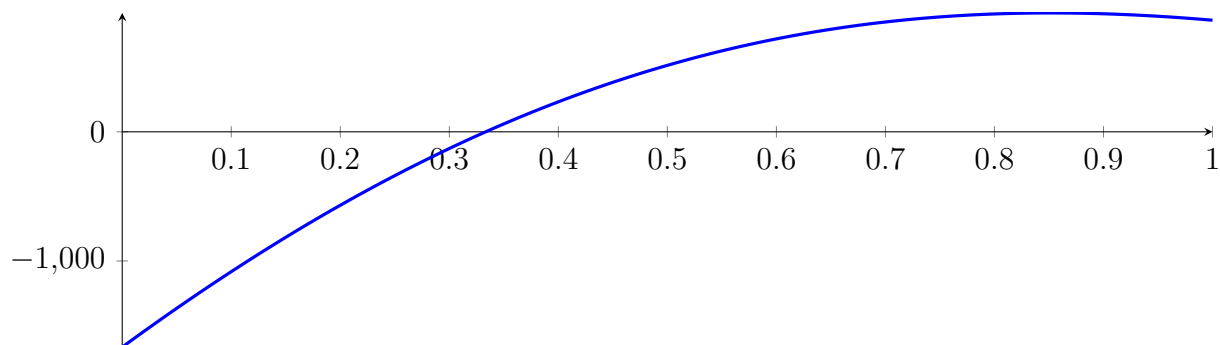
224.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

224.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

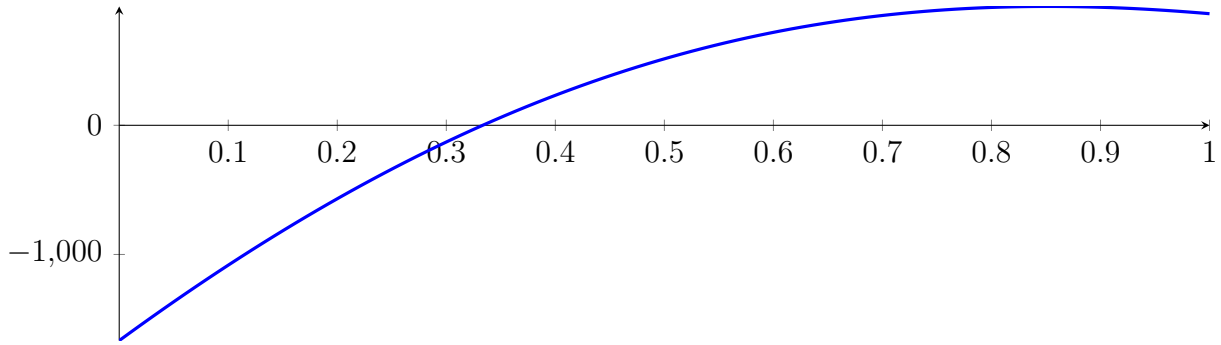
with precision $\varepsilon = 1 \cdot 10^{-32}$.

225 Running CubeClip on f_8 with epsilon 32

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

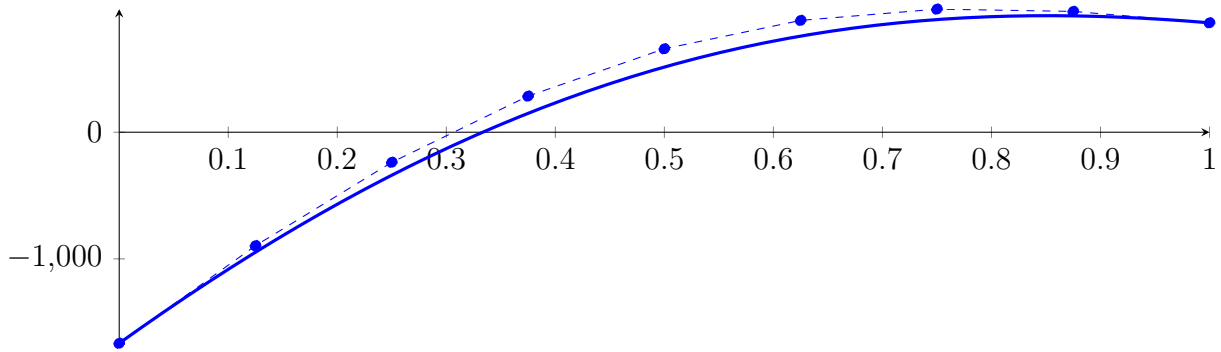
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



225.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

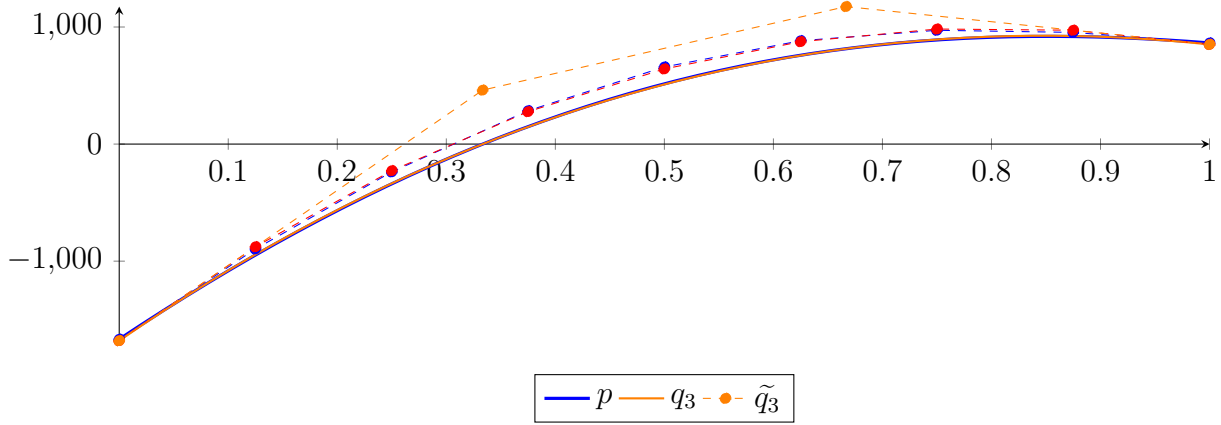
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.80826 \cdot 10^{-300}X^8 + 1.16606 \cdot 10^{-299}X^7 - 2.1197 \cdot 10^{-299}X^6 + 2.15003 \cdot 10^{-299}X^5 \\ &\quad - 1.20234 \cdot 10^{-299}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

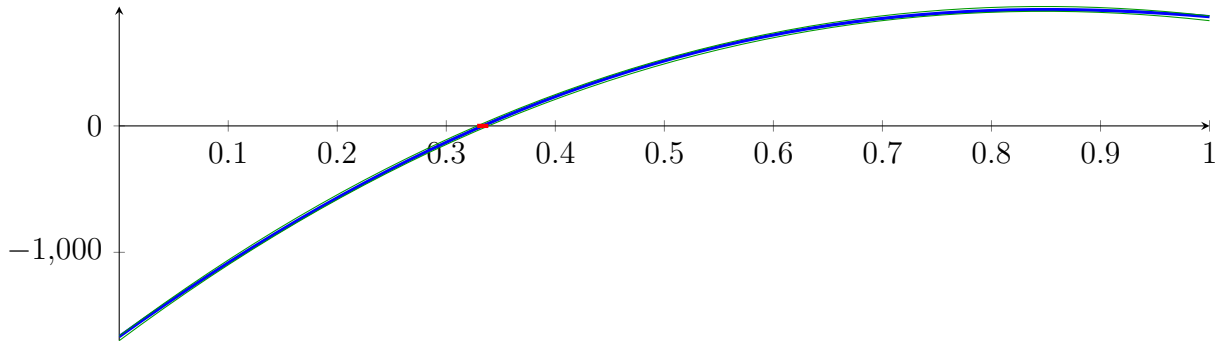
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

225.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

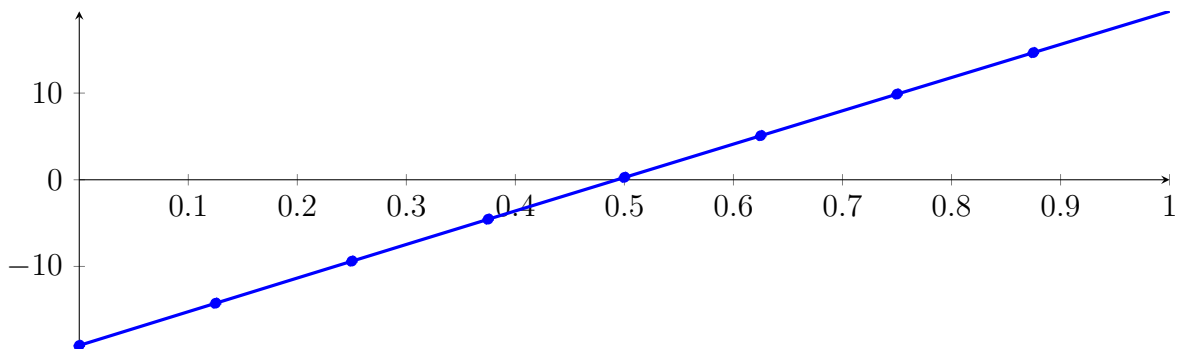
Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.25947 \cdot 10^{-16} X^8 - 1.9937 \cdot 10^{-13} X^7 - 8.53073 \cdot 10^{-11} X^6 + 8.72839 \cdot 10^{-09} X^5$$

$$+ 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124$$

$$= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X)$$

$$+ 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

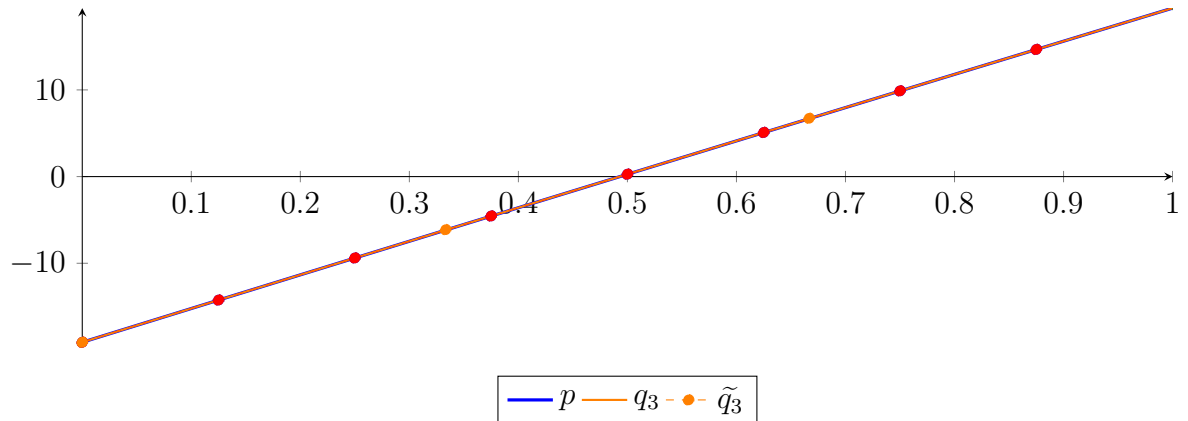
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = -1.96643 \cdot 10^{-303} X^8 + 1.82947 \cdot 10^{-302} X^7 - 4.89395 \cdot 10^{-302} X^6 + 5.49554 \cdot 10^{-302} X^5$$

$$- 2.47838 \cdot 10^{-302} X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m :

$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

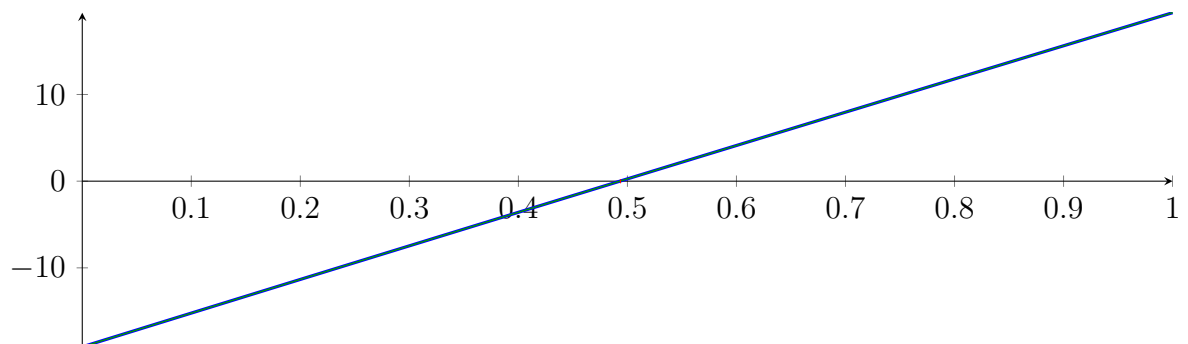
$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\}$$

$$N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

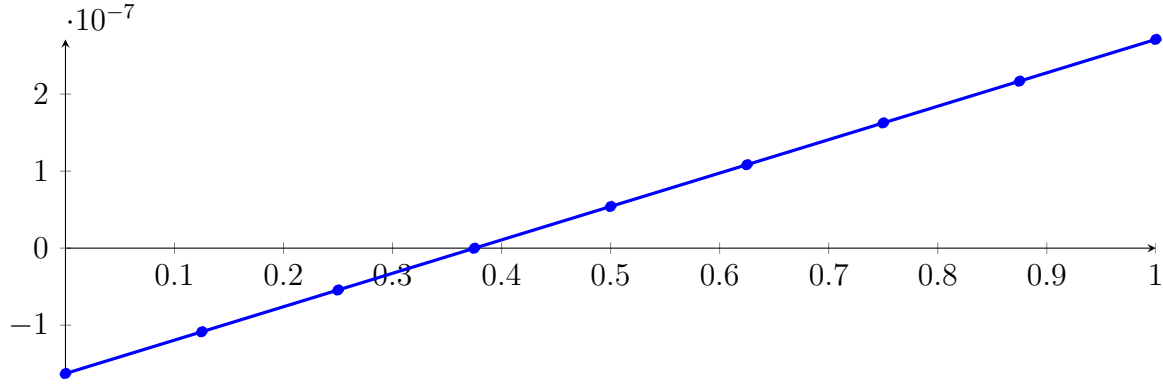
Longest intersection interval: $1.1252 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

225.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

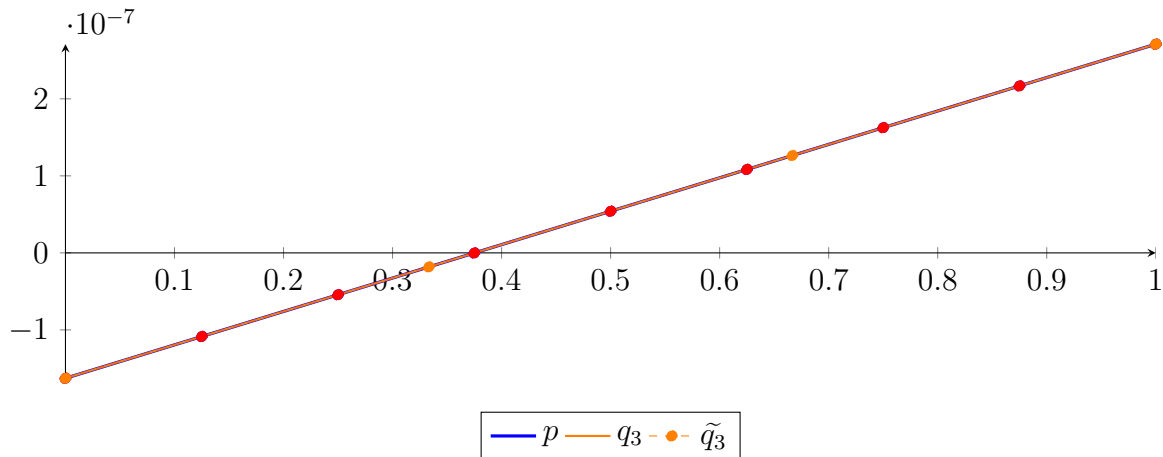
$$\begin{aligned}
 p &= -3.2361 \cdot 10^{-80} X^8 - 4.56398 \cdot 10^{-69} X^7 - 1.74524 \cdot 10^{-58} X^6 + 1.52857 \cdot 10^{-48} X^5 + 1.62178 \\
 &\quad \cdot 10^{-37} X^4 - 3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,8}(X) - 1.08585 \cdot 10^{-07} B_{1,8}(X) - 5.43592 \cdot 10^{-08} B_{2,8}(X) \\
 &\quad - 1.33711 \cdot 10^{-10} B_{3,8}(X) + 5.40918 \cdot 10^{-08} B_{4,8}(X) + 1.08317 \cdot 10^{-07} B_{5,8}(X) \\
 &\quad + 1.62543 \cdot 10^{-07} B_{6,8}(X) + 2.16768 \cdot 10^{-07} B_{7,8}(X) + 2.70994 \cdot 10^{-07} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,3} - 1.82089 \cdot 10^{-08} B_{1,3} + 1.26393 \cdot 10^{-07} B_{2,3} + 2.70994 \cdot 10^{-07} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 8.86066 \cdot 10^{-312} X^8 + 7.41744 \cdot 10^{-311} X^7 - 5.24902 \cdot 10^{-310} X^6 + 1.01267 \cdot 10^{-309} X^5 - 7.8921 \\
 &\quad \cdot 10^{-310} X^4 - 3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,8} - 1.08585 \cdot 10^{-07} B_{1,8} - 5.43592 \cdot 10^{-08} B_{2,8} - 1.33711 \cdot 10^{-10} B_{3,8} + 5.40918 \\
 &\quad \cdot 10^{-08} B_{4,8} + 1.08317 \cdot 10^{-07} B_{5,8} + 1.62543 \cdot 10^{-07} B_{6,8} + 2.16768 \cdot 10^{-07} B_{7,8} + 2.70994 \cdot 10^{-07} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.47524 \cdot 10^{-39}$.

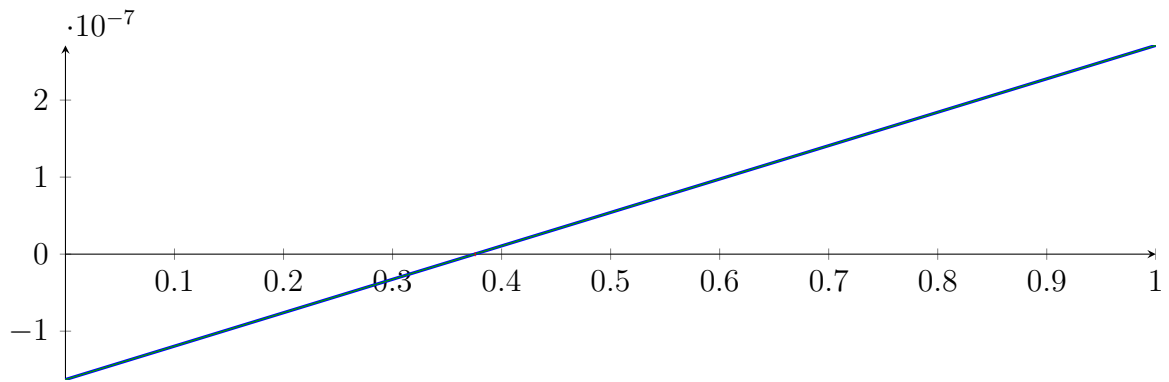
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 m &= -3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-1.6133 \cdot 10^{11}, 0.375292, 7.82468 \cdot 10^9\} \quad N(m) = \{-1.6133 \cdot 10^{11}, 0.375292, 7.82468 \cdot 10^9\}$$

Intersection intervals:



[0.375292, 0.375292]

Longest intersection interval: $1.60221 \cdot 10^{-32}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

225.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

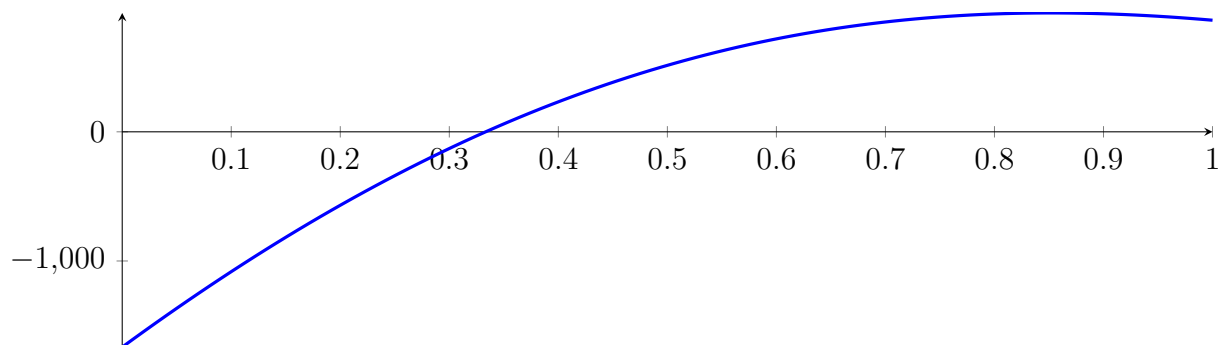
Reached interval [0.333333, 0.333333] **without sign change** at depth 4!

$p(0) = -6.9178e-12$ - $p(1) -6.9178e-12$

225.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

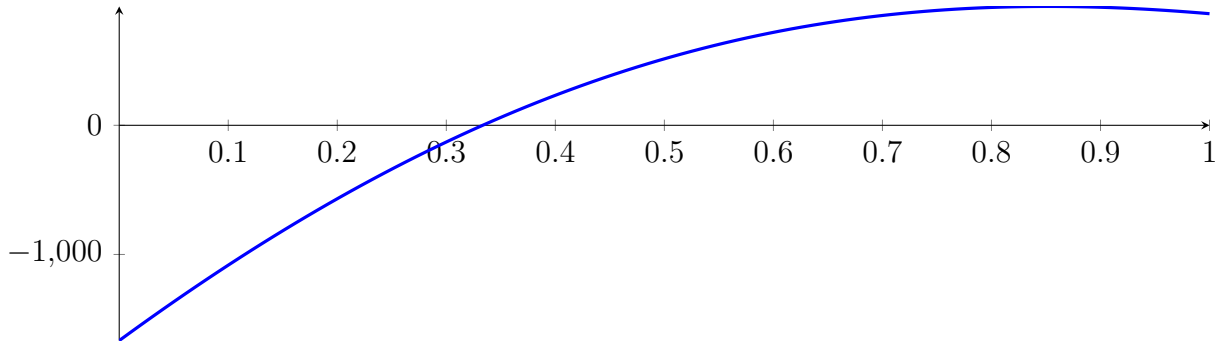
with precision $\varepsilon = 1 \cdot 10^{-32}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

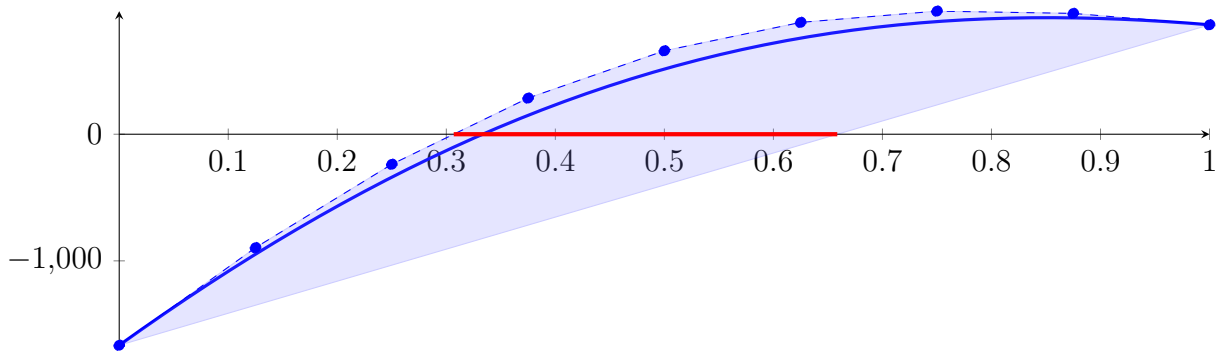
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



226.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

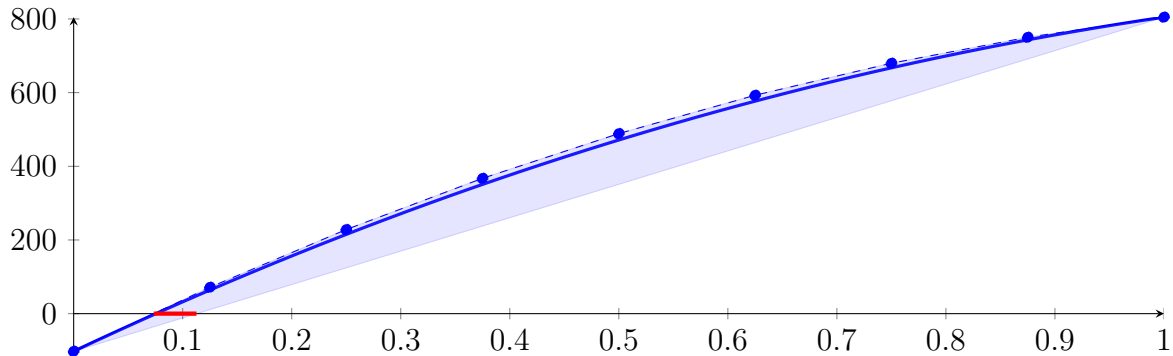
Longest intersection interval: 0.351792

\implies Selective recursion: interval 1: $[0.306796, 0.658588]$,

226.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

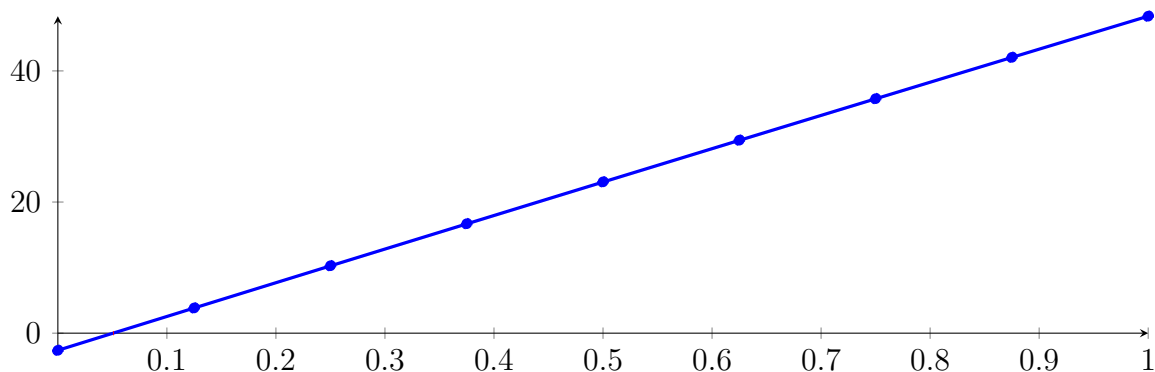
Longest intersection interval: 0.0391855

\implies Selective recursion: interval 1: [0.332635, 0.34642],

226.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.30406 \cdot 10^{-15} X^8 - 1.54459 \cdot 10^{-12} X^7 - 4.9583 \cdot 10^{-10} X^6 + 3.66751 \cdot 10^{-08} X^5 \\
 &\quad + 3.25466 \cdot 10^{-05} X^4 - 0.000586142 X^3 - 0.747315 X^2 + 51.7118 X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

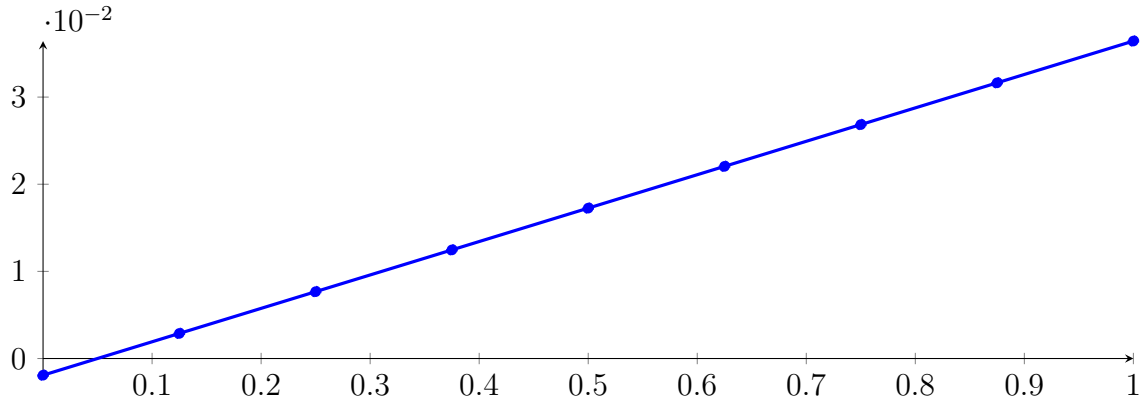
Longest intersection interval: 0.000742589

\implies Selective recursion: interval 1: [0.333333, 0.333343],

226.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.20583 \cdot 10^{-40} X^8 - 1.92397 \cdot 10^{-34} X^7 - 8.32342 \cdot 10^{-29} X^6 + 8.24755 \cdot 10^{-24} X^5 \\
 &\quad + 9.89972 \cdot 10^{-18} X^4 - 2.37322 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\
 &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\
 &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\
 &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

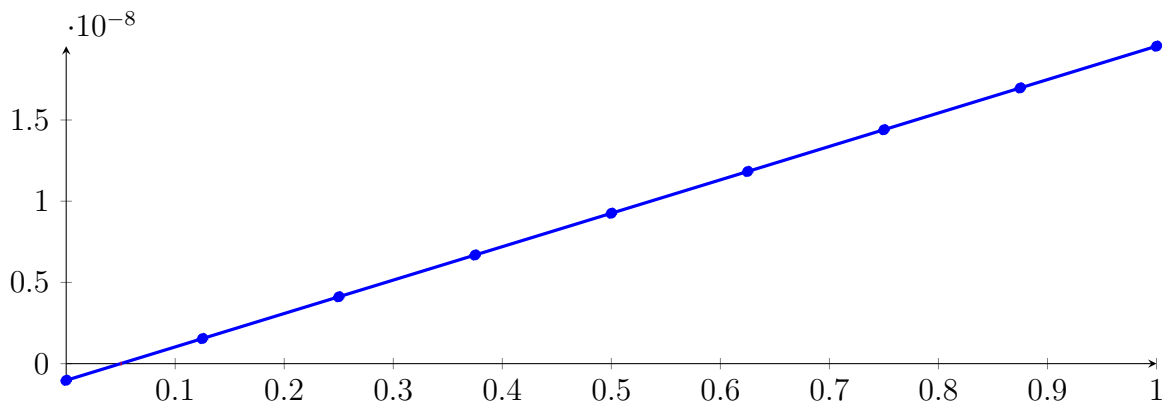
Longest intersection interval: $5.36469 \cdot 10^{-07}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

226.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -8.27263 \cdot 10^{-91} X^8 - 2.46044 \cdot 10^{-78} X^7 - 1.98413 \cdot 10^{-66} X^6 + 3.66478 \cdot 10^{-55} X^5 + 8.19978 \\
 &\quad \cdot 10^{-43} X^4 - 3.66412 \cdot 10^{-32} X^3 - 1.18615 \cdot 10^{-19} X^2 + 2.05706 \cdot 10^{-08} X - 1.02667 \cdot 10^{-09} \\
 &= -1.02667 \cdot 10^{-09} B_{0,8}(X) + 1.54465 \cdot 10^{-09} B_{1,8}(X) + 4.11597 \cdot 10^{-09} B_{2,8}(X) \\
 &\quad + 6.68729 \cdot 10^{-09} B_{3,8}(X) + 9.25862 \cdot 10^{-09} B_{4,8}(X) + 1.18299 \cdot 10^{-08} B_{5,8}(X) \\
 &\quad + 1.44013 \cdot 10^{-08} B_{6,8}(X) + 1.69726 \cdot 10^{-08} B_{7,8}(X) + 1.95439 \cdot 10^{-08} B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

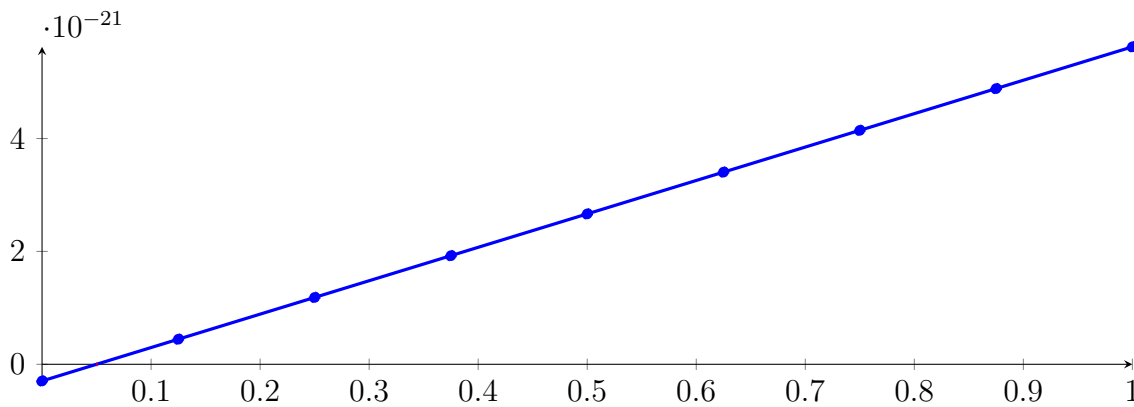
Longest intersection interval: $2.87793 \cdot 10^{-13}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

226.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -3.89305 \cdot 10^{-191} X^8 - 4.02327 \cdot 10^{-166} X^7 - 1.12734 \cdot 10^{-141} X^6 + 7.23523 \cdot 10^{-118} X^5 + 5.62504 \\ &\quad \cdot 10^{-93} X^4 - 8.73397 \cdot 10^{-70} X^3 - 9.82433 \cdot 10^{-45} X^2 + 5.92008 \cdot 10^{-21} X - 2.9547 \cdot 10^{-22} \\ &= -2.9547 \cdot 10^{-22} B_{0,8}(X) + 4.4454 \cdot 10^{-22} B_{1,8}(X) + 1.18455 \cdot 10^{-21} B_{2,8}(X) \\ &\quad + 1.92456 \cdot 10^{-21} B_{3,8}(X) + 2.66457 \cdot 10^{-21} B_{4,8}(X) + 3.40458 \cdot 10^{-21} B_{5,8}(X) \\ &\quad + 4.14459 \cdot 10^{-21} B_{6,8}(X) + 4.8846 \cdot 10^{-21} B_{7,8}(X) + 5.62461 \cdot 10^{-21} B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

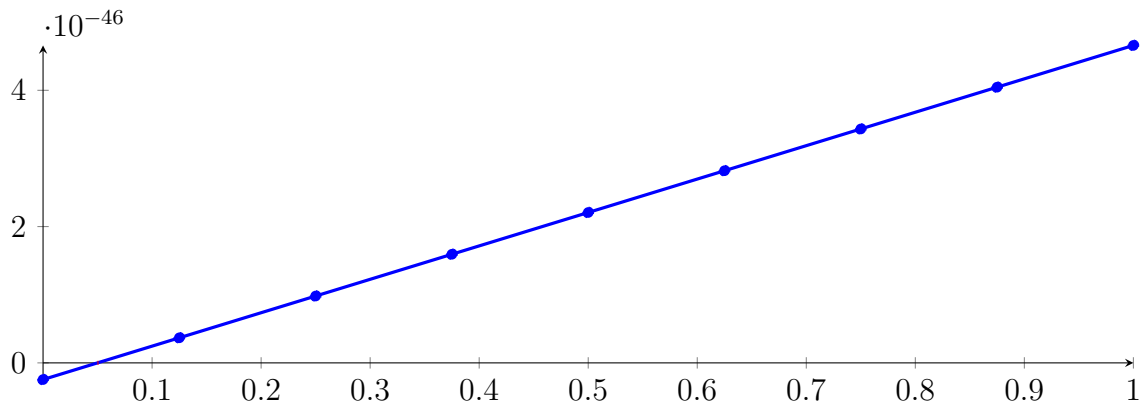
Longest intersection interval: $8.28251 \cdot 10^{-26}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

226.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.13027 \cdot 10^{-352} X^8 - 1.07575 \cdot 10^{-341} X^7 - 3.63937 \cdot 10^{-292} X^6 + 2.82008 \cdot 10^{-243} X^5 + 2.64711 \\ &\quad \cdot 10^{-193} X^4 - 4.96246 \cdot 10^{-145} X^3 - 6.73948 \cdot 10^{-95} X^2 + 4.90331 \cdot 10^{-46} X - 2.44723 \cdot 10^{-47} \\ &= -2.44723 \cdot 10^{-47} B_{0,8}(X) + 3.6819 \cdot 10^{-47} B_{1,8}(X) + 9.81104 \cdot 10^{-47} B_{2,8}(X) \\ &\quad + 1.59402 \cdot 10^{-46} B_{3,8}(X) + 2.20693 \cdot 10^{-46} B_{4,8}(X) + 2.81984 \cdot 10^{-46} B_{5,8}(X) \\ &\quad + 3.43276 \cdot 10^{-46} B_{6,8}(X) + 4.04567 \cdot 10^{-46} B_{7,8}(X) + 4.65858 \cdot 10^{-46} B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $6.85999 \cdot 10^{-51}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

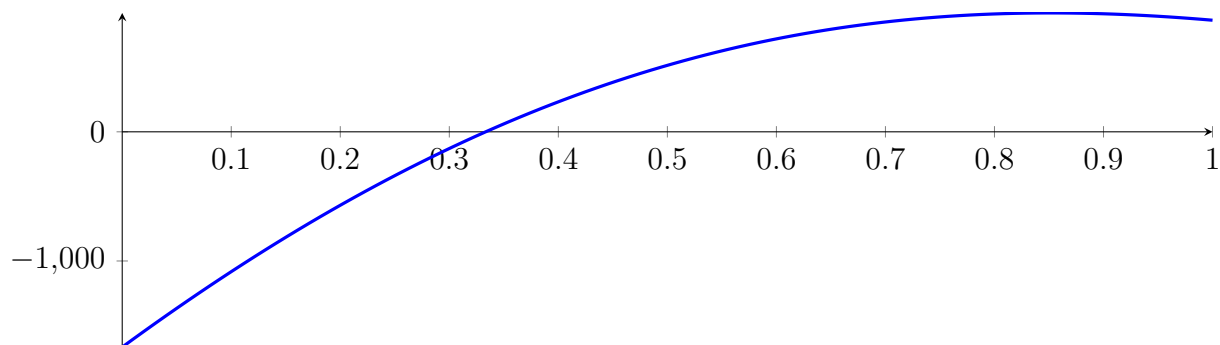
226.8 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 8!

226.9 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

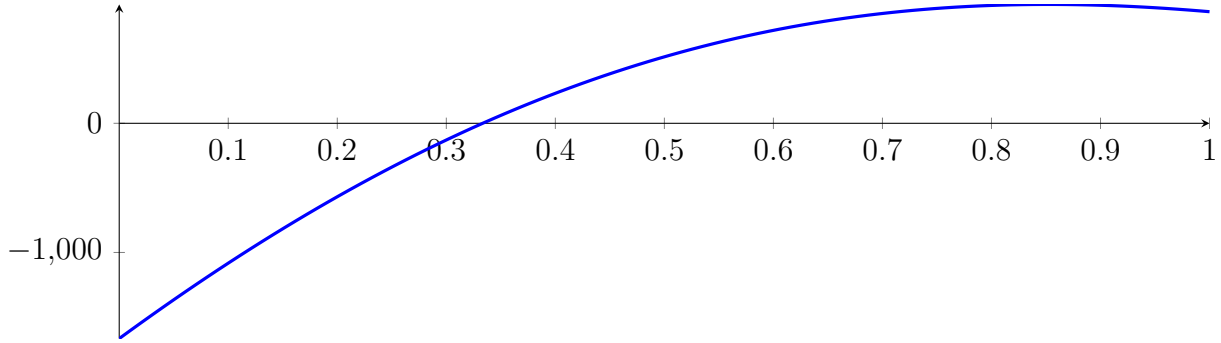
with precision $\varepsilon = 1 \cdot 10^{-64}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

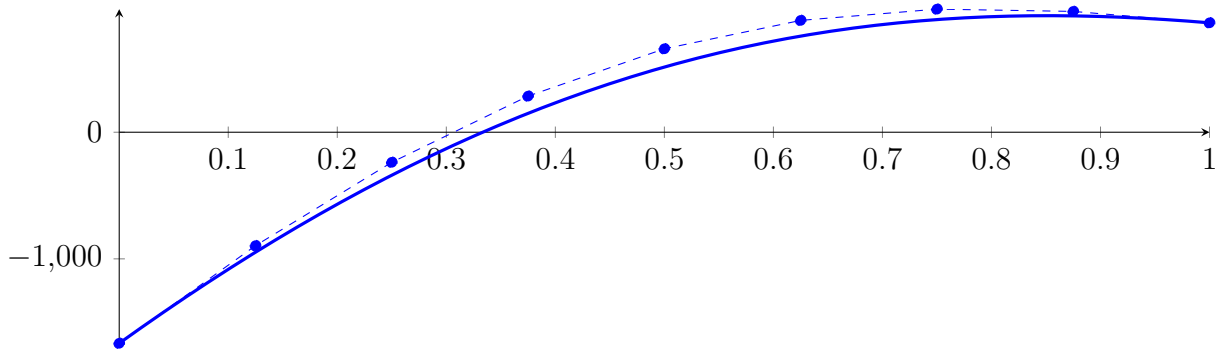
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



227.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

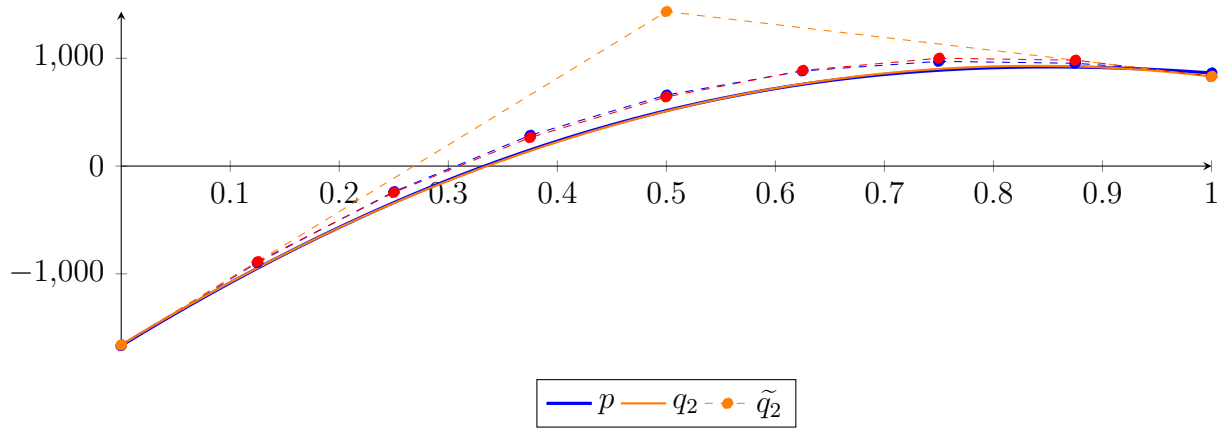
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 6.38292 \cdot 10^{-300}X^8 - 2.86993 \cdot 10^{-299}X^7 + 5.15342 \cdot 10^{-299}X^6 - 4.6928 \cdot 10^{-299}X^5 \\ &\quad + 2.29296 \cdot 10^{-299}X^4 - 6.02693 \cdot 10^{-300}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

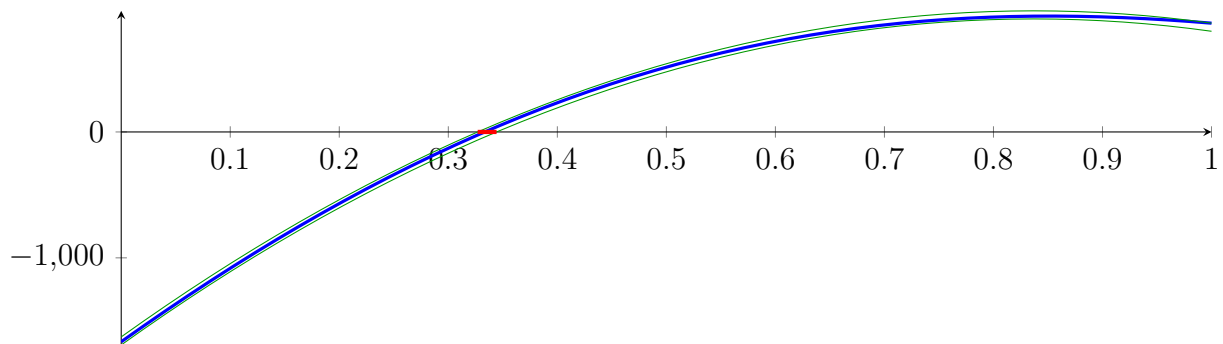
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

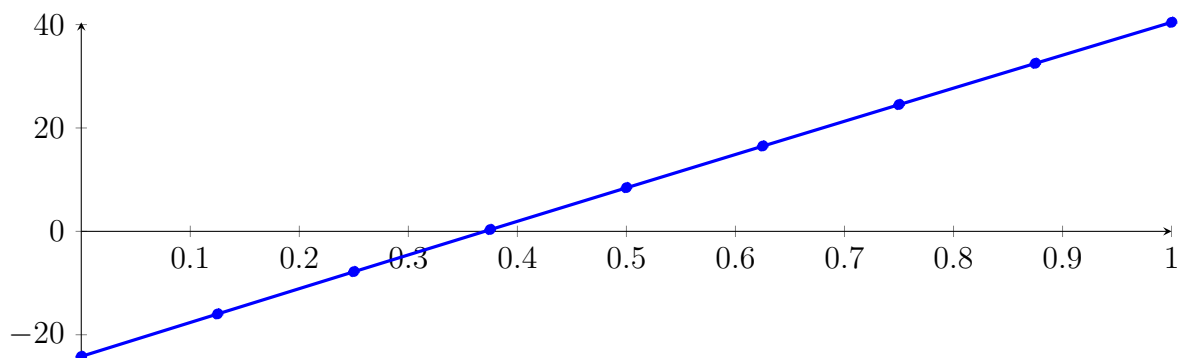
Longest intersection interval: 0.0173372

\implies Selective recursion: **interval 1:** $[0.326917, 0.344255]$,

227.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -8.16249 \cdot 10^{-15} X^8 - 7.66571 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

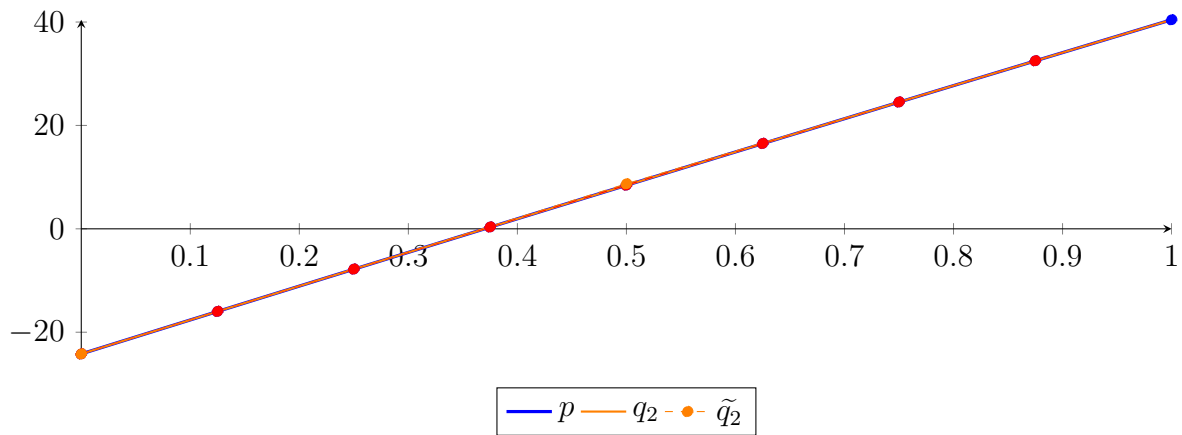
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = 4.27533 \cdot 10^{-302}X^8 - 2.42468 \cdot 10^{-301}X^7 + 4.90537 \cdot 10^{-301}X^6 - 4.6286 \cdot 10^{-301}X^5$$

$$+ 2.17546 \cdot 10^{-301}X^4 - 5.36695 \cdot 10^{-302}X^3 - 1.18261X^2 + 65.8162X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

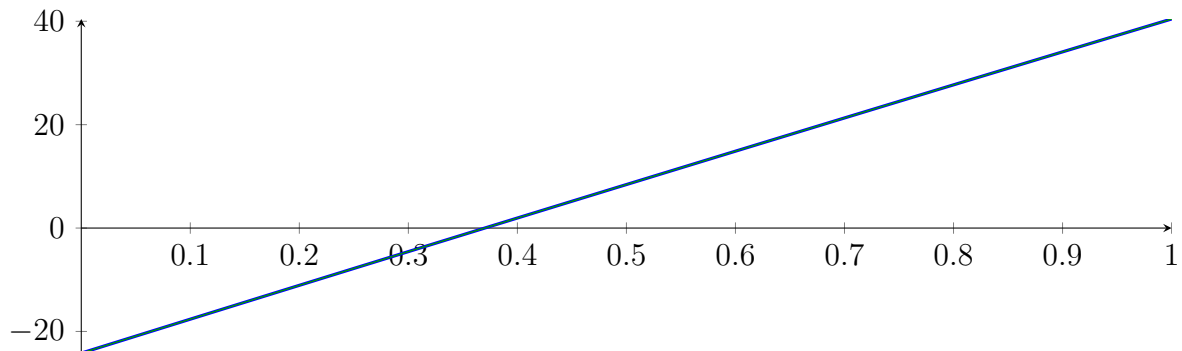
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

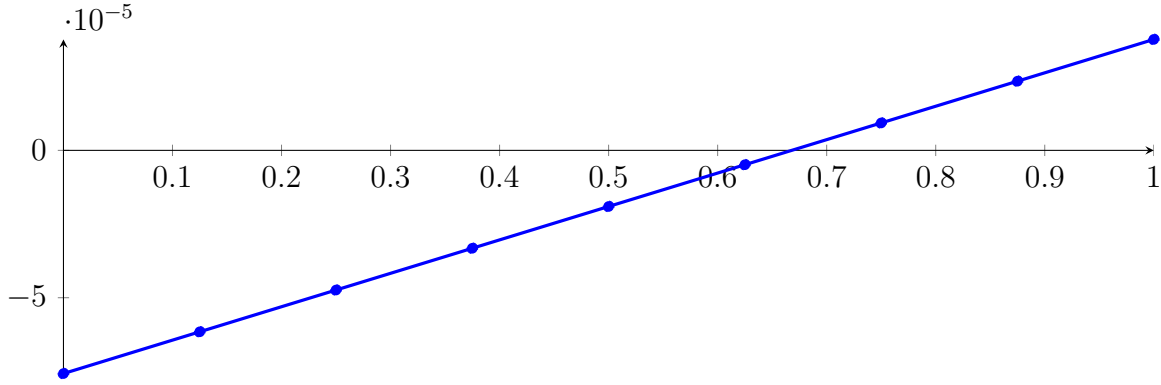
Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

227.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

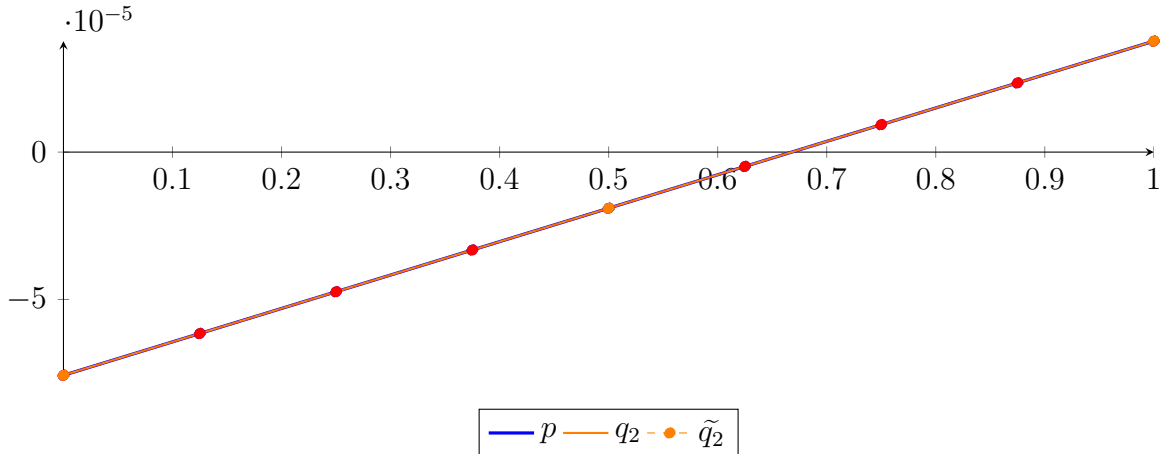
$$\begin{aligned}
 p &= -7.04578 \cdot 10^{-61} X^8 - 3.80201 \cdot 10^{-52} X^7 - 5.5627 \cdot 10^{-44} X^6 + 1.86413 \cdot 10^{-36} X^5 + 7.56737 \\
 &\quad \cdot 10^{-28} X^4 - 6.13517 \cdot 10^{-21} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) \\
 &\quad - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) \\
 &\quad + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -3.62396 \cdot 10^{-308} X^8 + 2.32992 \cdot 10^{-308} X^7 + 2.61753 \cdot 10^{-307} X^6 - 5.97049 \cdot 10^{-307} X^5 + 5.13401 \\
 &\quad \cdot 10^{-307} X^4 - 1.82868 \cdot 10^{-307} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \\
 &\quad \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.06758 \cdot 10^{-22}$.

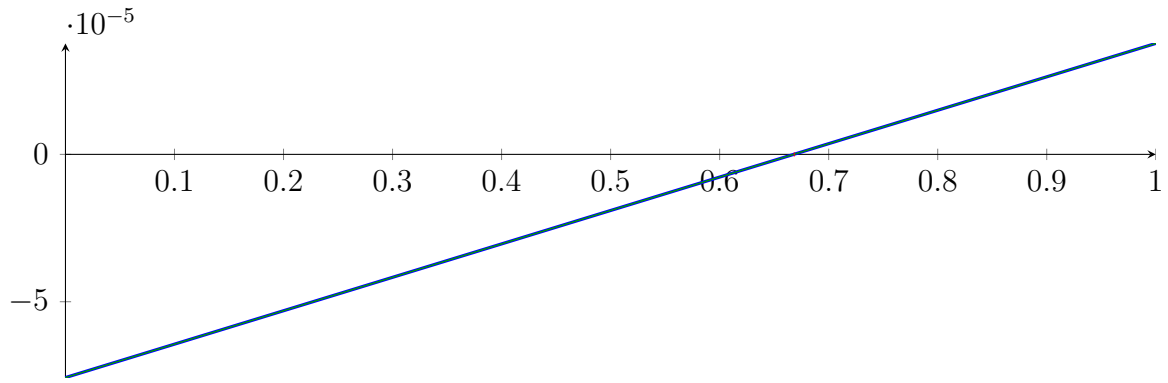
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 m &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.667955, 3.14644 \cdot 10^7\} \quad N(m) = \{0.667955, 3.14644 \cdot 10^7\}$$

Intersection intervals:



[0.667955, 0.667955]

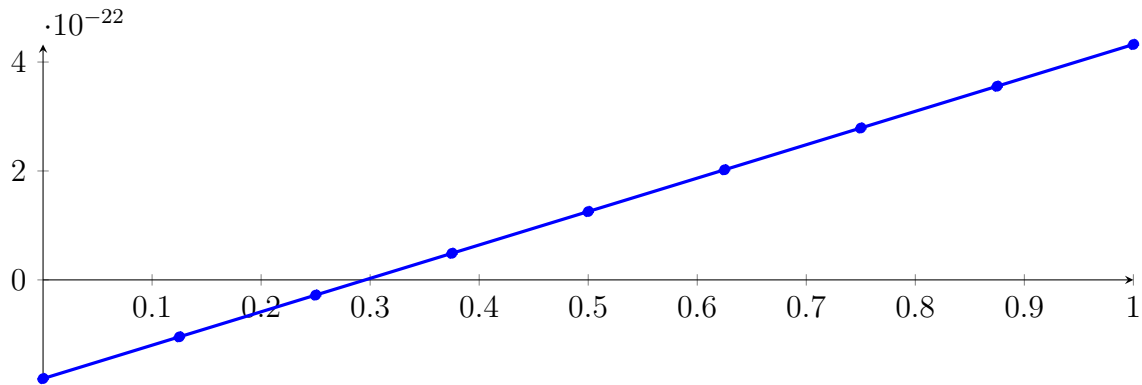
Longest intersection interval: $5.41121 \cdot 10^{-18}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

227.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

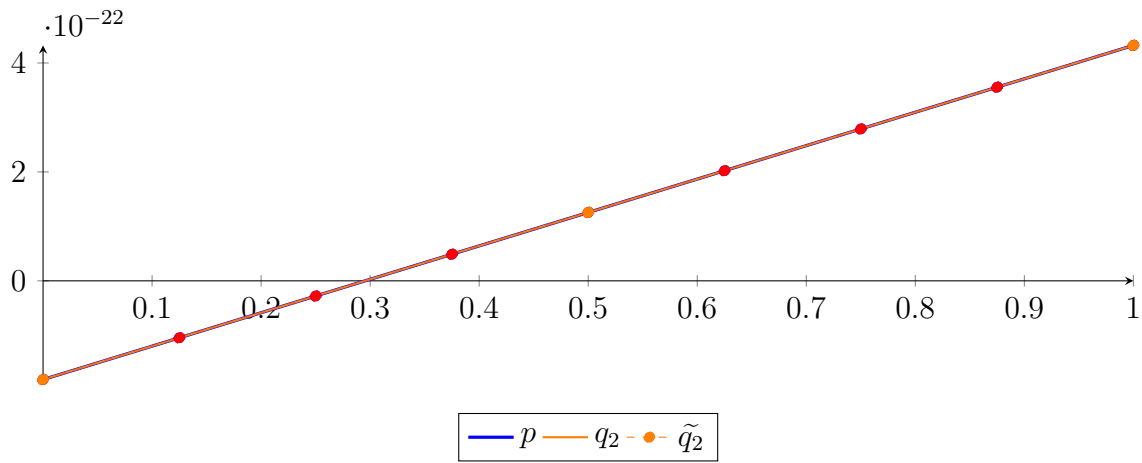
$$\begin{aligned}
 p &= -5.17944 \cdot 10^{-199} X^8 - 5.16502 \cdot 10^{-173} X^7 - 1.39653 \cdot 10^{-147} X^6 + 8.64863 \cdot 10^{-123} X^5 + 6.48817 \\
 &\quad \cdot 10^{-97} X^4 - 9.72096 \cdot 10^{-73} X^3 - 1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22} \\
 &= -1.81261 \cdot 10^{-22} B_{0,8}(X) - 1.04571 \cdot 10^{-22} B_{1,8}(X) - 2.78818 \cdot 10^{-23} B_{2,8}(X) \\
 &\quad + 4.88078 \cdot 10^{-23} B_{3,8}(X) + 1.25497 \cdot 10^{-22} B_{4,8}(X) + 2.02187 \cdot 10^{-22} B_{5,8}(X) \\
 &\quad + 2.78877 \cdot 10^{-22} B_{6,8}(X) + 3.55566 \cdot 10^{-22} B_{7,8}(X) + 4.32256 \cdot 10^{-22} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22} \\
 &= -1.81261 \cdot 10^{-22} B_{0,2} + 1.25497 \cdot 10^{-22} B_{1,2} + 4.32256 \cdot 10^{-22} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 5.39888 \cdot 10^{-325} X^8 - 2.84119 \cdot 10^{-324} X^7 + 5.35011 \cdot 10^{-324} X^6 - 4.57499 \cdot 10^{-324} X^5 + 1.82797 \\
 &\quad \cdot 10^{-324} X^4 - 3.72306 \cdot 10^{-325} X^3 - 1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22} \\
 &= -1.81261 \cdot 10^{-22} B_{0,8} - 1.04571 \cdot 10^{-22} B_{1,8} - 2.78818 \cdot 10^{-23} B_{2,8} + 4.88078 \cdot 10^{-23} B_{3,8} + 1.25497 \\
 &\quad \cdot 10^{-22} B_{4,8} + 2.02187 \cdot 10^{-22} B_{5,8} + 2.78877 \cdot 10^{-22} B_{6,8} + 3.55566 \cdot 10^{-22} B_{7,8} + 4.32256 \cdot 10^{-22} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.86048 \cdot 10^{-74}$.

Bounding polynomials M and m :

$$M = -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22}$$

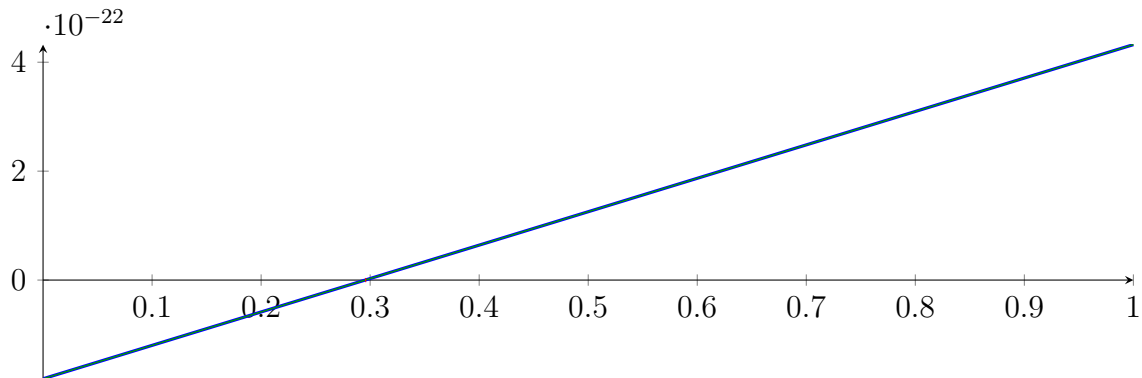
$$m = -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22}$$

Root of M and m :

$$N(M) = \{0.295446, 5.81467 \cdot 10^{24}\}$$

$$N(m) = \{0.295446, 5.81467 \cdot 10^{24}\}$$

Intersection intervals:



$$[0.295446, 0.295446]$$

Longest intersection interval: $1.58446 \cdot 10^{-52}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

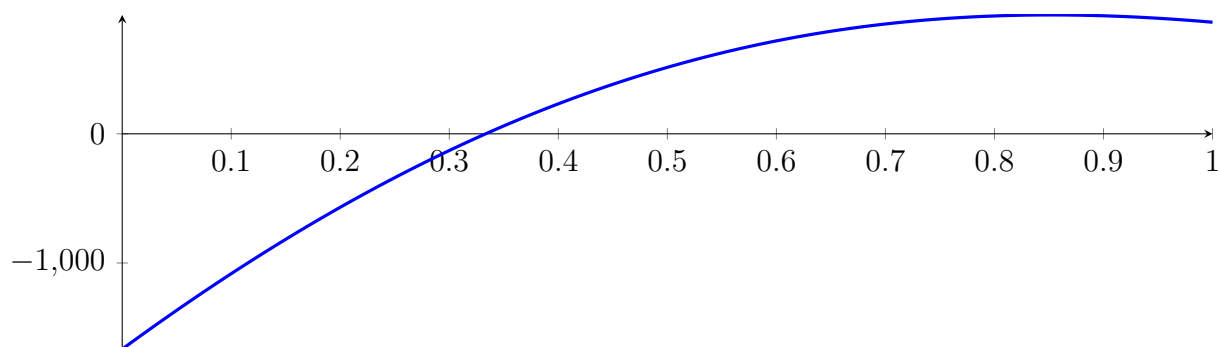
227.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

227.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

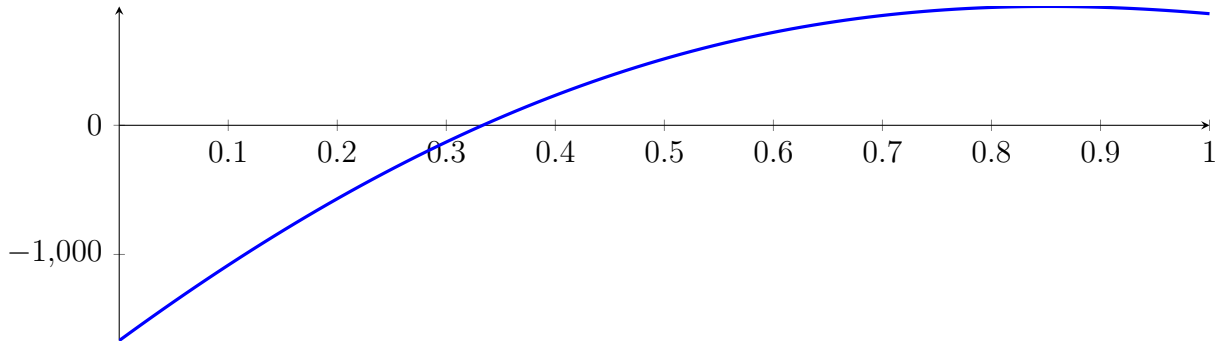
with precision $\varepsilon = 1 \cdot 10^{-64}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

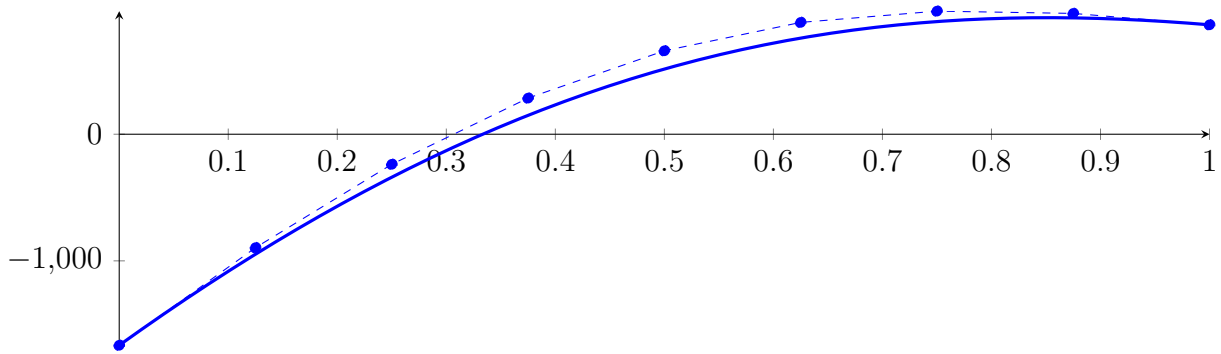
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



228.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

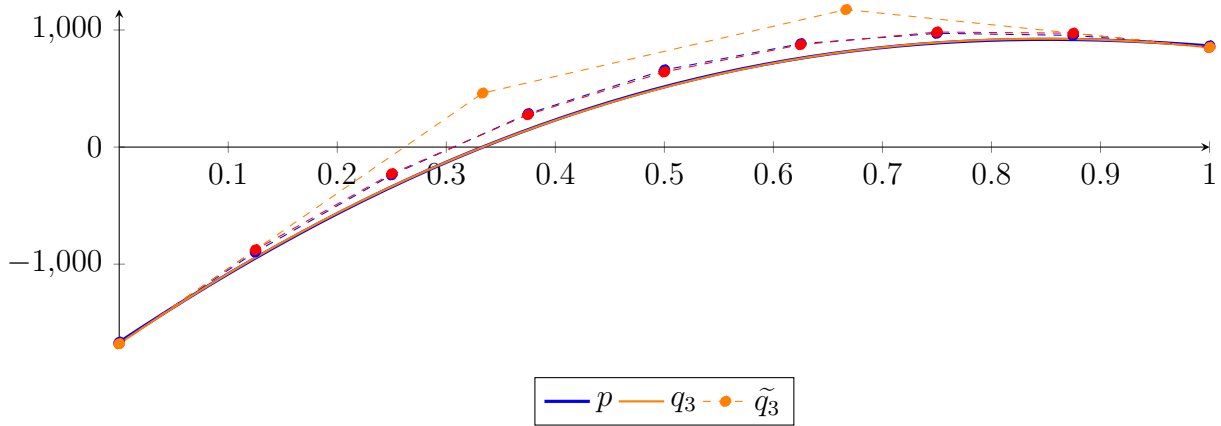
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.80826 \cdot 10^{-300}X^8 + 1.16606 \cdot 10^{-299}X^7 - 2.1197 \cdot 10^{-299}X^6 + 2.15003 \cdot 10^{-299}X^5 \\ &\quad - 1.20234 \cdot 10^{-299}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

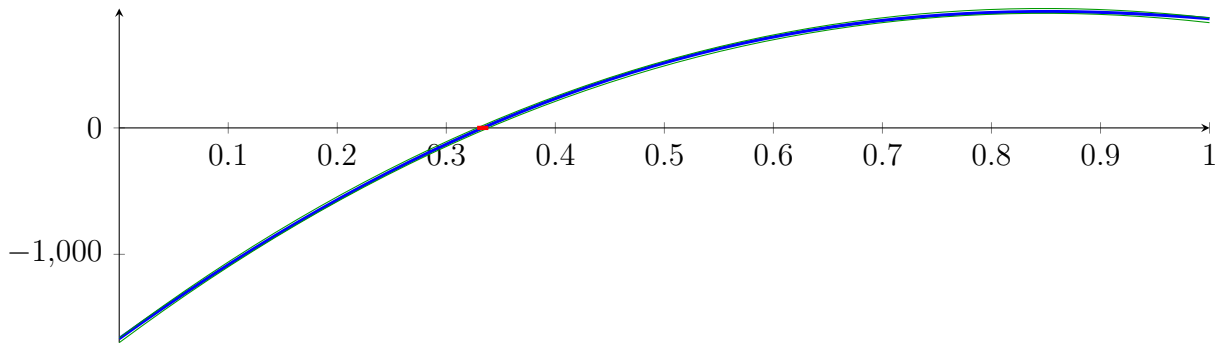
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

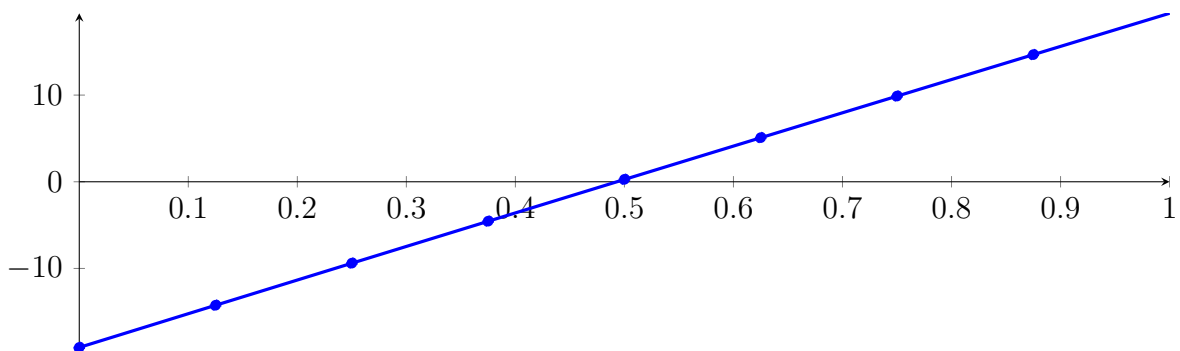
Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

228.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.25947 \cdot 10^{-16} X^8 - 1.9937 \cdot 10^{-13} X^7 - 8.53073 \cdot 10^{-11} X^6 + 8.72839 \cdot 10^{-09} X^5 \\ &\quad + 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124 \\ &= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X) \\ &\quad + 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

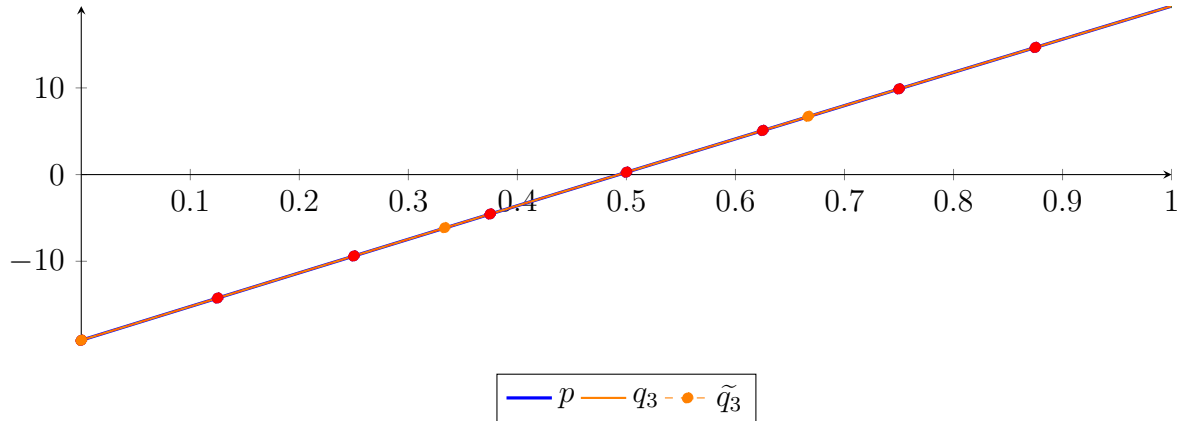
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = -1.96643 \cdot 10^{-303} X^8 + 1.82947 \cdot 10^{-302} X^7 - 4.89395 \cdot 10^{-302} X^6 + 5.49554 \cdot 10^{-302} X^5$$

$$- 2.47838 \cdot 10^{-302} X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m :

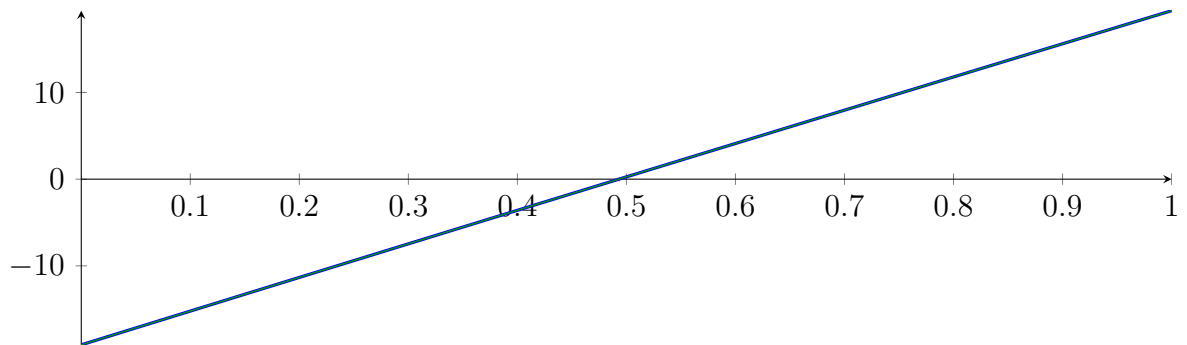
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \quad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

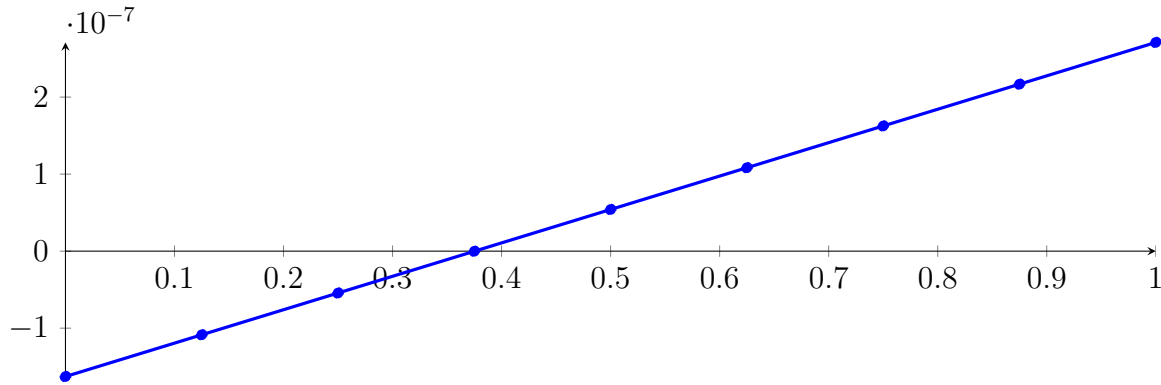
Longest intersection interval: $1.1252 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

228.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

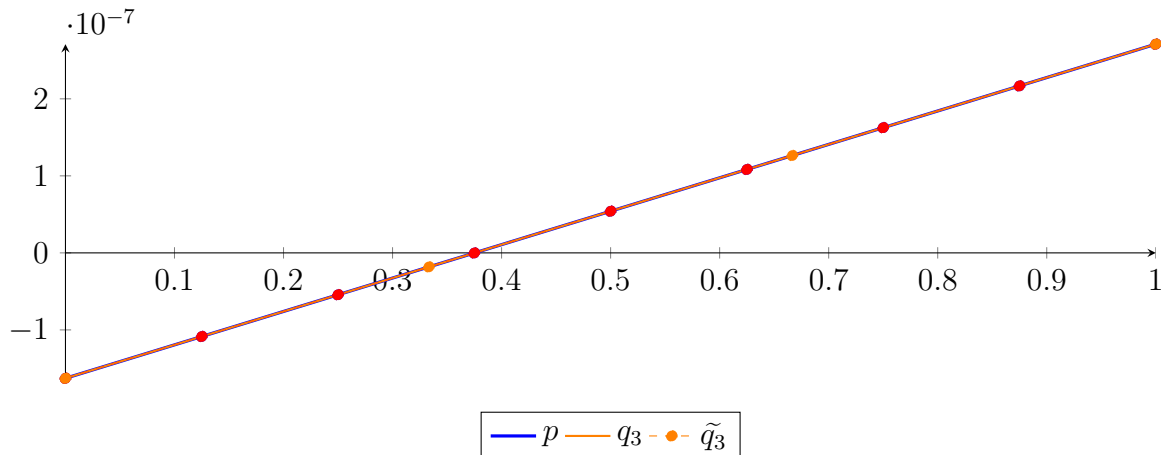
$$\begin{aligned}
 p &= -3.2361 \cdot 10^{-80} X^8 - 4.56398 \cdot 10^{-69} X^7 - 1.74524 \cdot 10^{-58} X^6 + 1.52857 \cdot 10^{-48} X^5 + 1.62178 \\
 &\quad \cdot 10^{-37} X^4 - 3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,8}(X) - 1.08585 \cdot 10^{-07} B_{1,8}(X) - 5.43592 \cdot 10^{-08} B_{2,8}(X) \\
 &\quad - 1.33711 \cdot 10^{-10} B_{3,8}(X) + 5.40918 \cdot 10^{-08} B_{4,8}(X) + 1.08317 \cdot 10^{-07} B_{5,8}(X) \\
 &\quad + 1.62543 \cdot 10^{-07} B_{6,8}(X) + 2.16768 \cdot 10^{-07} B_{7,8}(X) + 2.70994 \cdot 10^{-07} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,3} - 1.82089 \cdot 10^{-08} B_{1,3} + 1.26393 \cdot 10^{-07} B_{2,3} + 2.70994 \cdot 10^{-07} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 8.86066 \cdot 10^{-312} X^8 + 7.41744 \cdot 10^{-311} X^7 - 5.24902 \cdot 10^{-310} X^6 + 1.01267 \cdot 10^{-309} X^5 - 7.8921 \\
 &\quad \cdot 10^{-310} X^4 - 3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,8} - 1.08585 \cdot 10^{-07} B_{1,8} - 5.43592 \cdot 10^{-08} B_{2,8} - 1.33711 \cdot 10^{-10} B_{3,8} + 5.40918 \\
 &\quad \cdot 10^{-08} B_{4,8} + 1.08317 \cdot 10^{-07} B_{5,8} + 1.62543 \cdot 10^{-07} B_{6,8} + 2.16768 \cdot 10^{-07} B_{7,8} + 2.70994 \cdot 10^{-07} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.47524 \cdot 10^{-39}$.

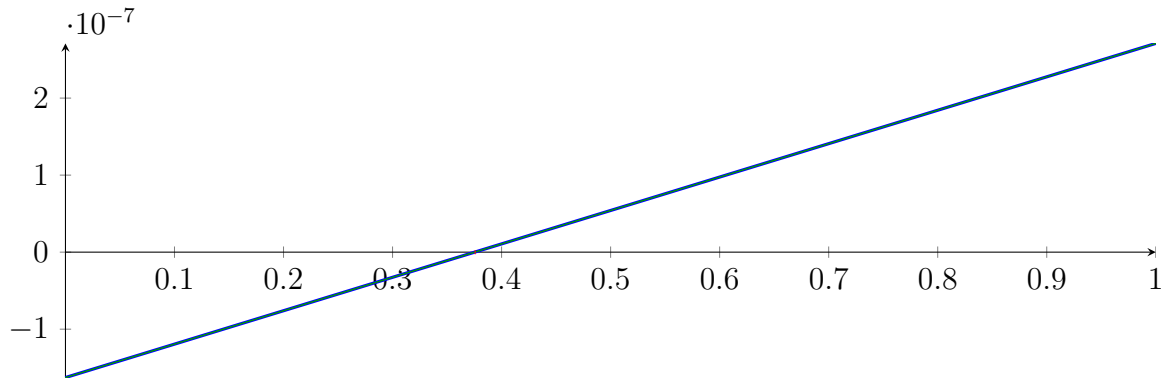
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 m &= -3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-1.6133 \cdot 10^{11}, 0.375292, 7.82468 \cdot 10^9\} \quad N(m) = \{-1.6133 \cdot 10^{11}, 0.375292, 7.82468 \cdot 10^9\}$$

Intersection intervals:



[0.375292, 0.375292]

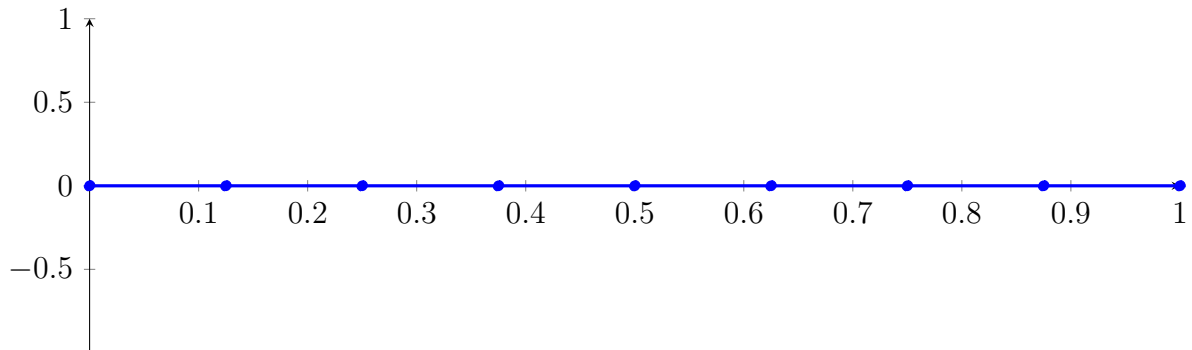
Longest intersection interval: $1.60221 \cdot 10^{-32}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

228.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

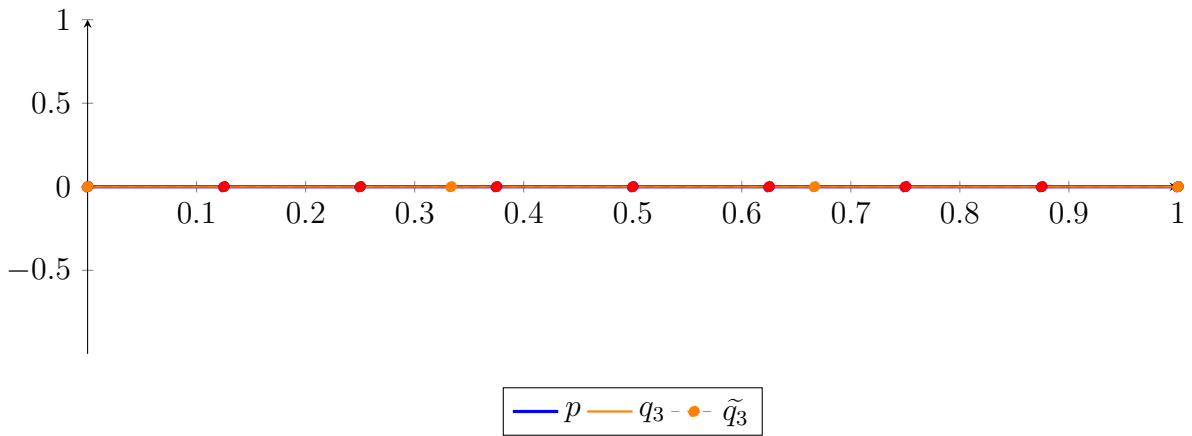
$$\begin{aligned}
 p &= 3.5617 \cdot 10^{-318} X^8 - 1.23705 \cdot 10^{-291} X^7 - 2.95243 \cdot 10^{-249} X^6 + 1.61394 \cdot 10^{-207} X^5 + 1.06875 \\
 &\quad \cdot 10^{-164} X^4 - 1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12} \\
 &= -6.9178 \cdot 10^{-12} B_{0,8}(X) - 6.9178 \cdot 10^{-12} B_{1,8}(X) - 6.9178 \cdot 10^{-12} B_{2,8}(X) \\
 &\quad - 6.9178 \cdot 10^{-12} B_{3,8}(X) - 6.9178 \cdot 10^{-12} B_{4,8}(X) - 6.9178 \cdot 10^{-12} B_{5,8}(X) \\
 &\quad - 6.9178 \cdot 10^{-12} B_{6,8}(X) - 6.9178 \cdot 10^{-12} B_{7,8}(X) - 6.9178 \cdot 10^{-12} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12} \\
 &= -6.9178 \cdot 10^{-12} B_{0,3} - 6.9178 \cdot 10^{-12} B_{1,3} - 6.9178 \cdot 10^{-12} B_{2,3} - 6.9178 \cdot 10^{-12} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= -3.69134 \cdot 10^{-315} X^8 + 1.58227 \cdot 10^{-314} X^7 - 4.11376 \cdot 10^{-316} X^6 - 5.57319 \cdot 10^{-314} X^5 + 6.99255 \\
 &\quad \cdot 10^{-314} X^4 - 1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12} \\
 &= -6.9178 \cdot 10^{-12} B_{0,8} - 6.9178 \cdot 10^{-12} B_{1,8} - 6.9178 \cdot 10^{-12} B_{2,8} - 6.9178 \cdot 10^{-12} B_{3,8} - 6.9178 \\
 &\quad \cdot 10^{-12} B_{4,8} - 6.9178 \cdot 10^{-12} B_{5,8} - 6.9178 \cdot 10^{-12} B_{6,8} - 6.9178 \cdot 10^{-12} B_{7,8} - 6.9178 \cdot 10^{-12} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.29017 \cdot 10^{-166}$.

Bounding polynomials M and m :

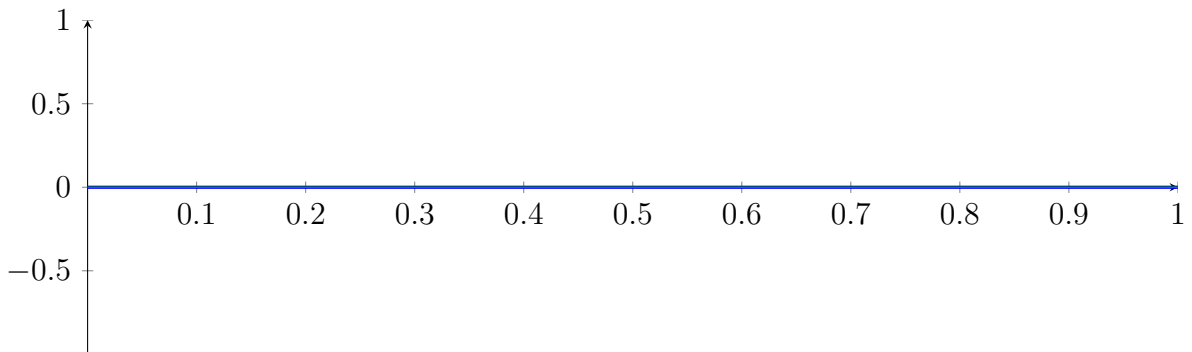
$$M = -1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12}$$

$$m = -1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12}$$

Root of M and m :

$$N(M) = \{-1.00692 \cdot 10^{43}, 2.13634 \cdot 10^{17}, 4.88366 \cdot 10^{41}\} \quad N(m) = \{-1.00692 \cdot 10^{43}, 2.13634 \cdot 10^{17}, 4.88366 \cdot 10^{41}\}$$

Intersection intervals:

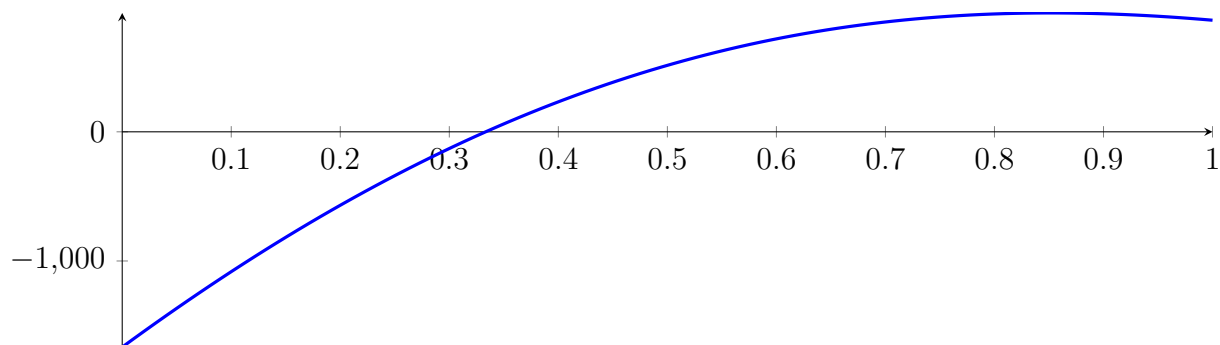


No intersection intervals with the x axis.

228.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

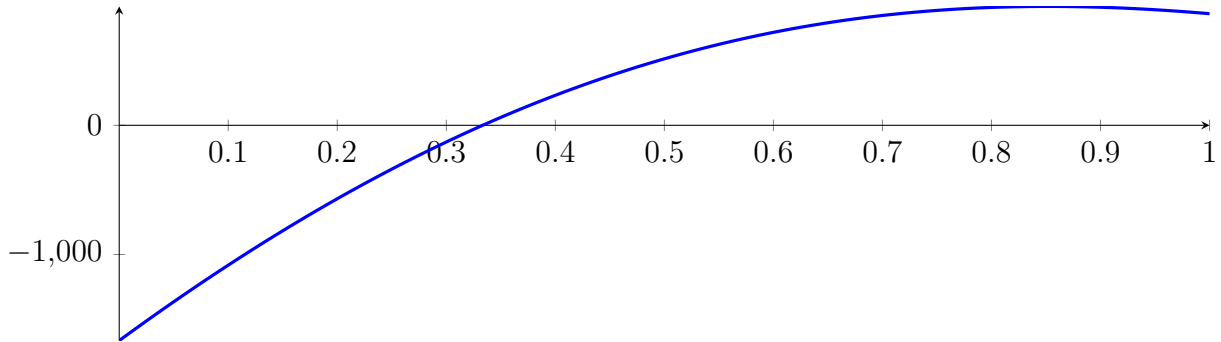
with precision $\varepsilon = 1 \cdot 10^{-64}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called BezClip with input polynomial on interval $[0, 1]$:

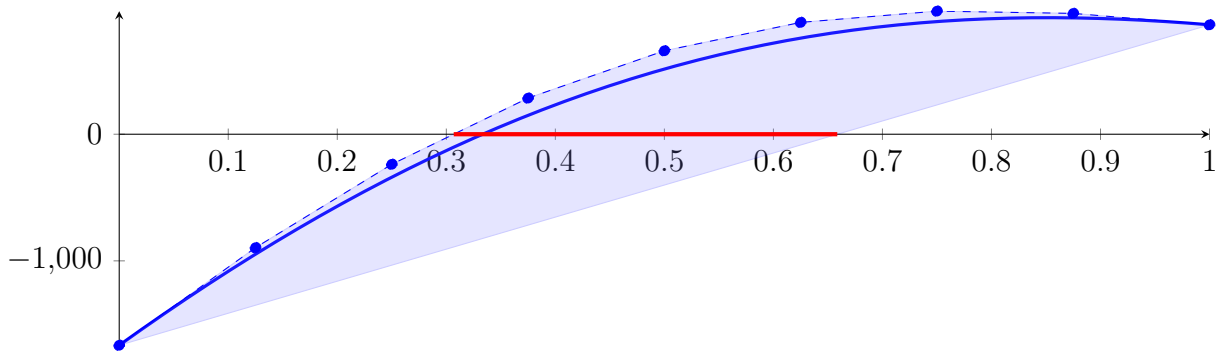
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



229.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.306796, 0.658588\}$$

Intersection intervals with the x axis:

$$[0.306796, 0.658588]$$

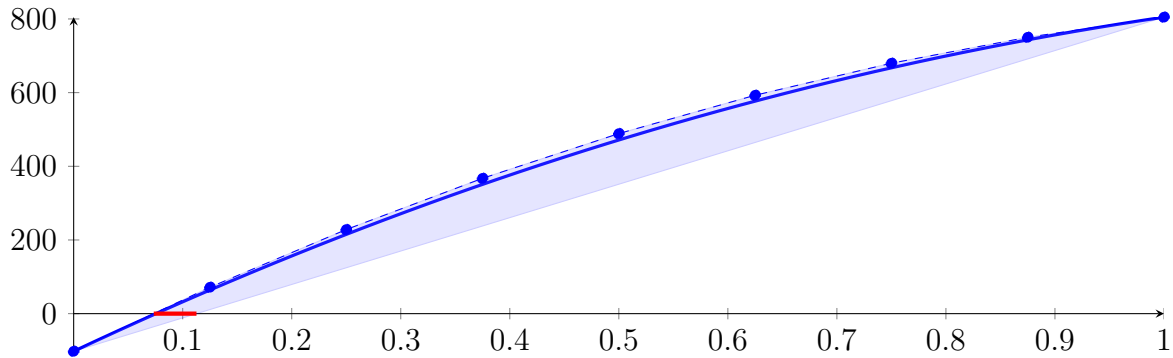
Longest intersection interval: 0.351792

⇒ Selective recursion: interval 1: $[0.306796, 0.658588]$,

229.2 Recursion Branch 1 1 in Interval 1: [0.306796, 0.658588]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -0.000234581X^8 - 0.0107498X^7 - 0.131393X^6 + 0.456085X^5 \\
 &\quad + 13.6472X^4 - 13.7747X^3 - 484.098X^2 + 1390.98X - 102.17 \\
 &= -102.17B_{0,8}(X) + 71.703B_{1,8}(X) + 228.287B_{2,8}(X) + 367.335B_{3,8}(X) \\
 &\quad + 488.797B_{4,8}(X) + 592.825B_{5,8}(X) + 679.778B_{6,8}(X) + 750.218B_{7,8}(X) + 804.901B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0734515, 0.112637\}$$

Intersection intervals with the x axis:

$$[0.0734515, 0.112637]$$

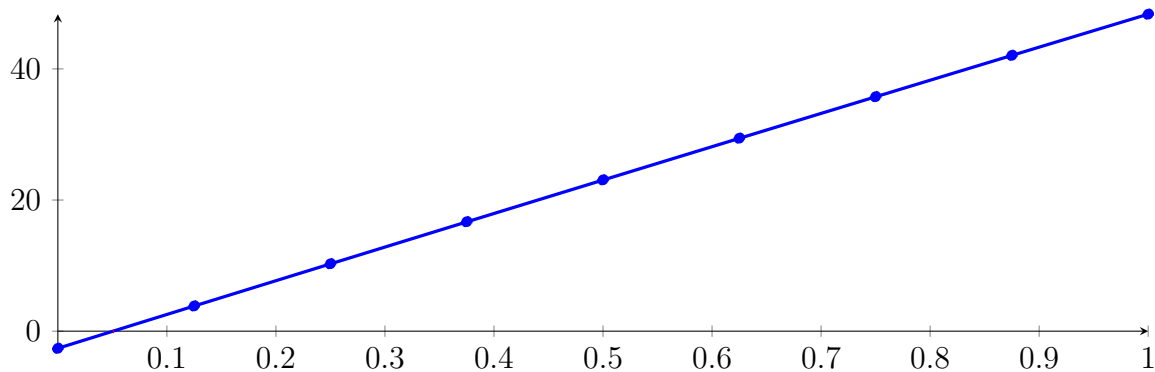
Longest intersection interval: 0.0391855

\implies Selective recursion: interval 1: [0.332635, 0.34642],

229.3 Recursion Branch 1 1 1 in Interval 1: [0.332635, 0.34642]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.30406 \cdot 10^{-15} X^8 - 1.54459 \cdot 10^{-12} X^7 - 4.9583 \cdot 10^{-10} X^6 + 3.66751 \cdot 10^{-08} X^5 \\
 &\quad + 3.25466 \cdot 10^{-05} X^4 - 0.000586142 X^3 - 0.747315 X^2 + 51.7118 X - 2.61683 \\
 &= -2.61683B_{0,8}(X) + 3.84714B_{1,8}(X) + 10.2844B_{2,8}(X) + 16.695B_{3,8}(X) + 23.0789B_{4,8}(X) \\
 &\quad + 29.436B_{5,8}(X) + 35.7665B_{6,8}(X) + 42.0702B_{7,8}(X) + 48.3471B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0506041, 0.0513467\}$$

Intersection intervals with the x axis:

$$[0.0506041, 0.0513467]$$

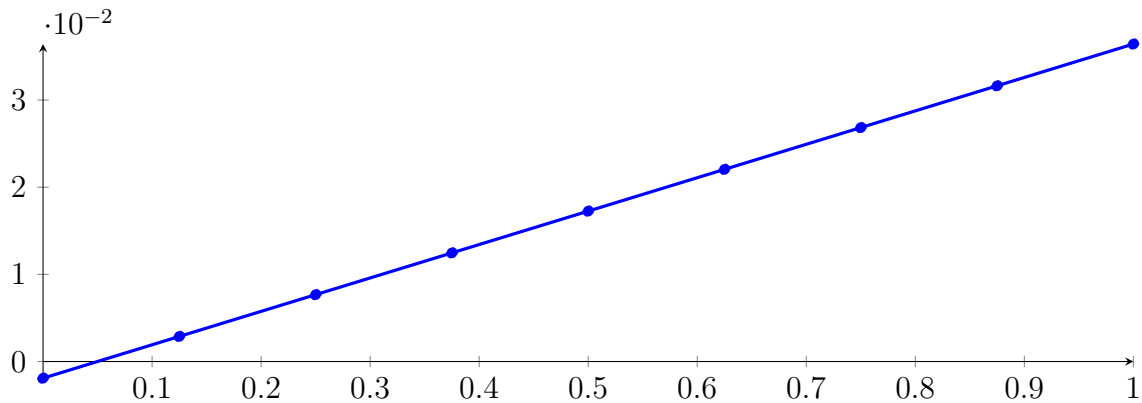
Longest intersection interval: 0.000742589

\implies Selective recursion: interval 1: [0.333333, 0.333343],

229.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333343]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.20583 \cdot 10^{-40} X^8 - 1.92397 \cdot 10^{-34} X^7 - 8.32342 \cdot 10^{-29} X^6 + 8.24755 \cdot 10^{-24} X^5 \\
 &\quad + 9.89972 \cdot 10^{-18} X^4 - 2.37322 \cdot 10^{-13} X^3 - 4.12146 \cdot 10^{-07} X^2 + 0.0383444 X - 0.00191378 \\
 &= -0.00191378 B_{0,8}(X) + 0.00287927 B_{1,8}(X) + 0.00767231 B_{2,8}(X) \\
 &\quad + 0.0124653 B_{3,8}(X) + 0.0172583 B_{4,8}(X) + 0.0220513 B_{5,8}(X) \\
 &\quad + 0.0268443 B_{6,8}(X) + 0.0316373 B_{7,8}(X) + 0.0364302 B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499103, 0.0499109\}$$

Intersection intervals with the x axis:

$$[0.0499103, 0.0499109]$$

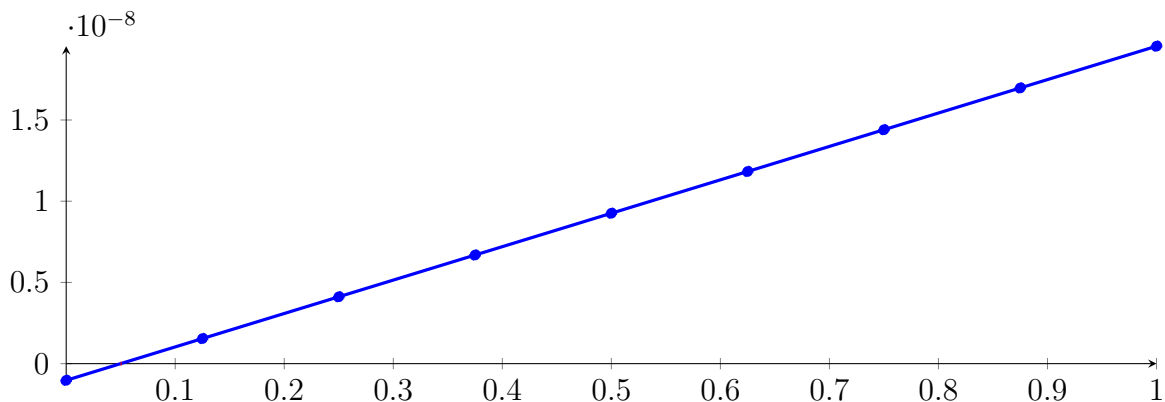
Longest intersection interval: $5.36469 \cdot 10^{-07}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

229.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -8.27263 \cdot 10^{-91} X^8 - 2.46044 \cdot 10^{-78} X^7 - 1.98413 \cdot 10^{-66} X^6 + 3.66478 \cdot 10^{-55} X^5 + 8.19978 \\
 &\quad \cdot 10^{-43} X^4 - 3.66412 \cdot 10^{-32} X^3 - 1.18615 \cdot 10^{-19} X^2 + 2.05706 \cdot 10^{-08} X - 1.02667 \cdot 10^{-09} \\
 &= -1.02667 \cdot 10^{-09} B_{0,8}(X) + 1.54465 \cdot 10^{-09} B_{1,8}(X) + 4.11597 \cdot 10^{-09} B_{2,8}(X) \\
 &\quad + 6.68729 \cdot 10^{-09} B_{3,8}(X) + 9.25862 \cdot 10^{-09} B_{4,8}(X) + 1.18299 \cdot 10^{-08} B_{5,8}(X) \\
 &\quad + 1.44013 \cdot 10^{-08} B_{6,8}(X) + 1.69726 \cdot 10^{-08} B_{7,8}(X) + 1.95439 \cdot 10^{-08} B_{8,8}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

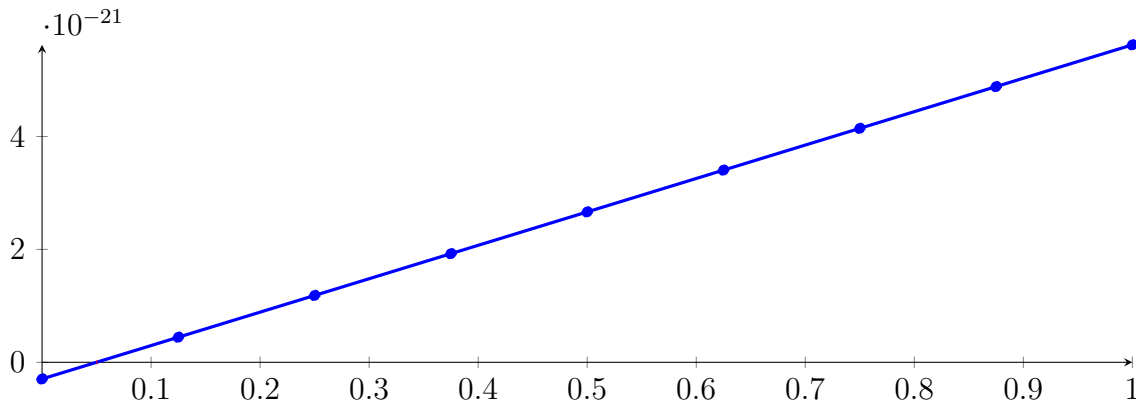
Longest intersection interval: $2.87793 \cdot 10^{-13}$

⇒ Selective recursion: interval 1: [\[0.333333, 0.333333\]](#),

229.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -3.89305 \cdot 10^{-191} X^8 - 4.02327 \cdot 10^{-166} X^7 - 1.12734 \cdot 10^{-141} X^6 + 7.23523 \cdot 10^{-118} X^5 + 5.62504 \\ &\quad \cdot 10^{-93} X^4 - 8.73397 \cdot 10^{-70} X^3 - 9.82433 \cdot 10^{-45} X^2 + 5.92008 \cdot 10^{-21} X - 2.9547 \cdot 10^{-22} \\ &= -2.9547 \cdot 10^{-22} B_{0,8}(X) + 4.4454 \cdot 10^{-22} B_{1,8}(X) + 1.18455 \cdot 10^{-21} B_{2,8}(X) \\ &\quad + 1.92456 \cdot 10^{-21} B_{3,8}(X) + 2.66457 \cdot 10^{-21} B_{4,8}(X) + 3.40458 \cdot 10^{-21} B_{5,8}(X) \\ &\quad + 4.14459 \cdot 10^{-21} B_{6,8}(X) + 4.8846 \cdot 10^{-21} B_{7,8}(X) + 5.62461 \cdot 10^{-21} B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

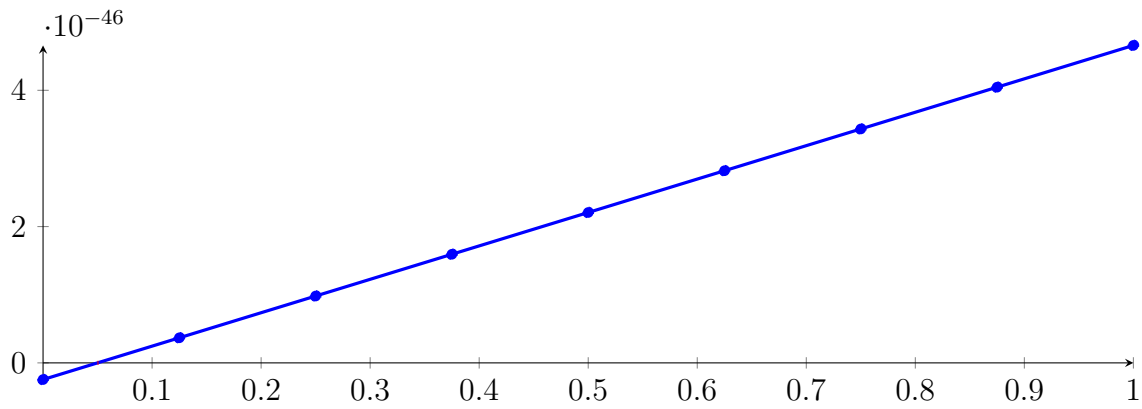
Longest intersection interval: $8.28251 \cdot 10^{-26}$

⇒ Selective recursion: interval 1: [\[0.333333, 0.333333\]](#),

229.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.13027 \cdot 10^{-352} X^8 - 1.07575 \cdot 10^{-341} X^7 - 3.63937 \cdot 10^{-292} X^6 + 2.82008 \cdot 10^{-243} X^5 + 2.64711 \\ &\quad \cdot 10^{-193} X^4 - 4.96246 \cdot 10^{-145} X^3 - 6.73948 \cdot 10^{-95} X^2 + 4.90331 \cdot 10^{-46} X - 2.44723 \cdot 10^{-47} \\ &= -2.44723 \cdot 10^{-47} B_{0,8}(X) + 3.6819 \cdot 10^{-47} B_{1,8}(X) + 9.81104 \cdot 10^{-47} B_{2,8}(X) \\ &\quad + 1.59402 \cdot 10^{-46} B_{3,8}(X) + 2.20693 \cdot 10^{-46} B_{4,8}(X) + 2.81984 \cdot 10^{-46} B_{5,8}(X) \\ &\quad + 3.43276 \cdot 10^{-46} B_{6,8}(X) + 4.04567 \cdot 10^{-46} B_{7,8}(X) + 4.65858 \cdot 10^{-46} B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

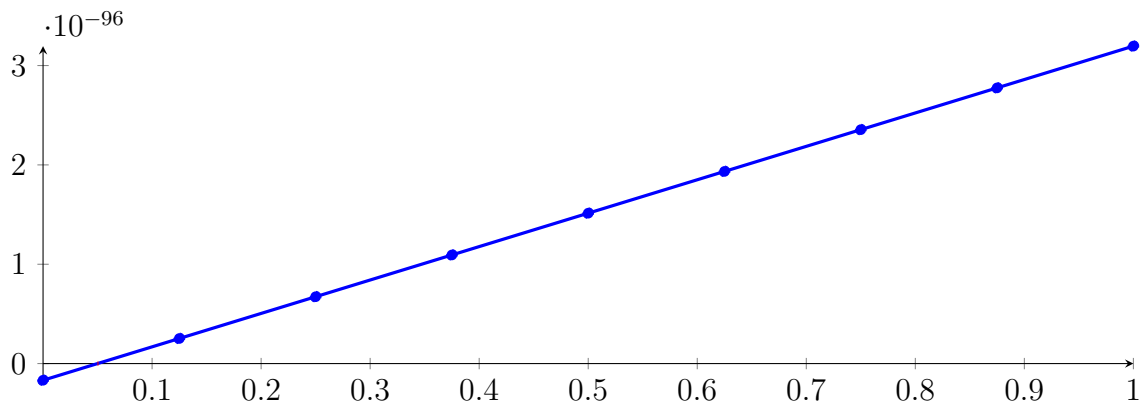
Longest intersection interval: $6.85999 \cdot 10^{-51}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

229.8 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 1.12504 \cdot 10^{-402} X^8 - 2.5001 \cdot 10^{-402} X^7 + 2.6251 \cdot 10^{-402} X^6 - 7.00027 \cdot 10^{-402} X^5 + 5.86228 \\ &\quad \cdot 10^{-394} X^4 - 1.60202 \cdot 10^{-295} X^3 - 3.17156 \cdot 10^{-195} X^2 + 3.36366 \cdot 10^{-96} X - 1.6788 \cdot 10^{-97} \\ &= -1.6788 \cdot 10^{-97} B_{0,8}(X) + 2.52578 \cdot 10^{-97} B_{1,8}(X) + 6.73036 \cdot 10^{-97} B_{2,8}(X) \\ &\quad + 1.09349 \cdot 10^{-96} B_{3,8}(X) + 1.51395 \cdot 10^{-96} B_{4,8}(X) + 1.93441 \cdot 10^{-96} B_{5,8}(X) \\ &\quad + 2.35487 \cdot 10^{-96} B_{6,8}(X) + 2.77533 \cdot 10^{-96} B_{7,8}(X) + 3.19578 \cdot 10^{-96} B_{8,8}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0499098, 0.0499098\}$$

Intersection intervals with the x axis:

$$[0.0499098, 0.0499098]$$

Longest intersection interval: $4.70595 \cdot 10^{-101}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

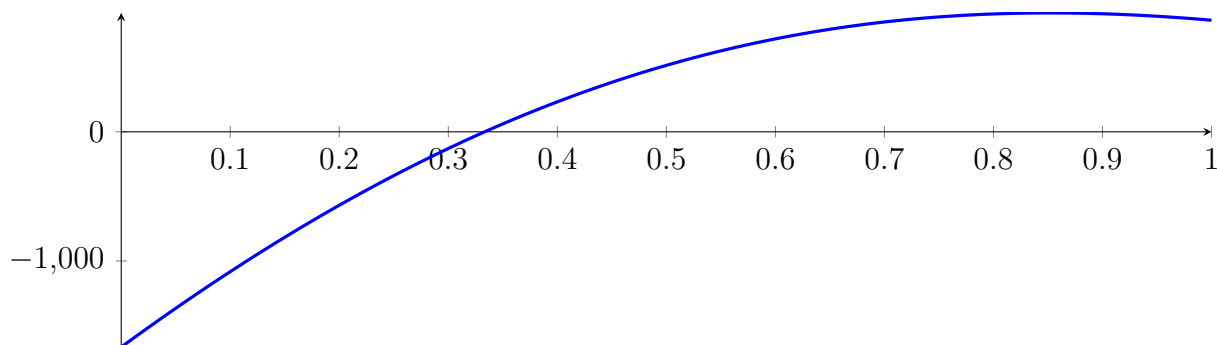
229.9 Recursion Branch 1 1 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 9!

229.10 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

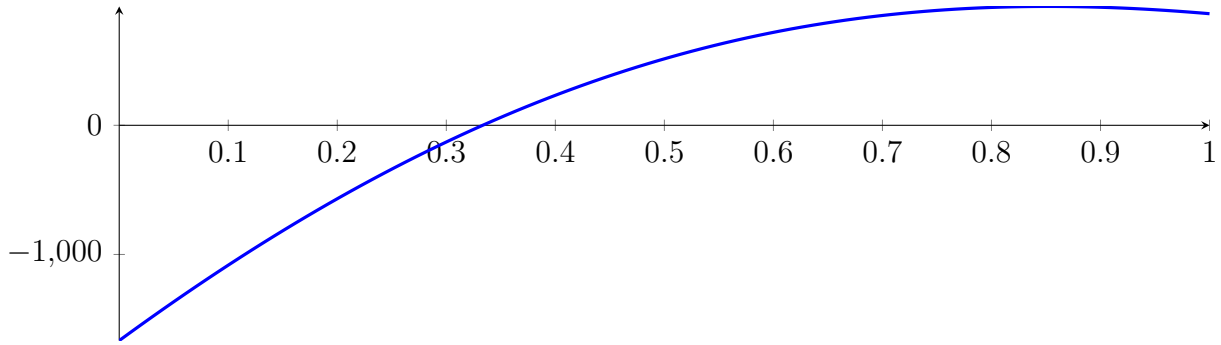
with precision $\varepsilon = 1 \cdot 10^{-128}$.

230 Running QuadClip on f_8 with epsilon 128

$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called QuadClip with input polynomial on interval $[0, 1]$:

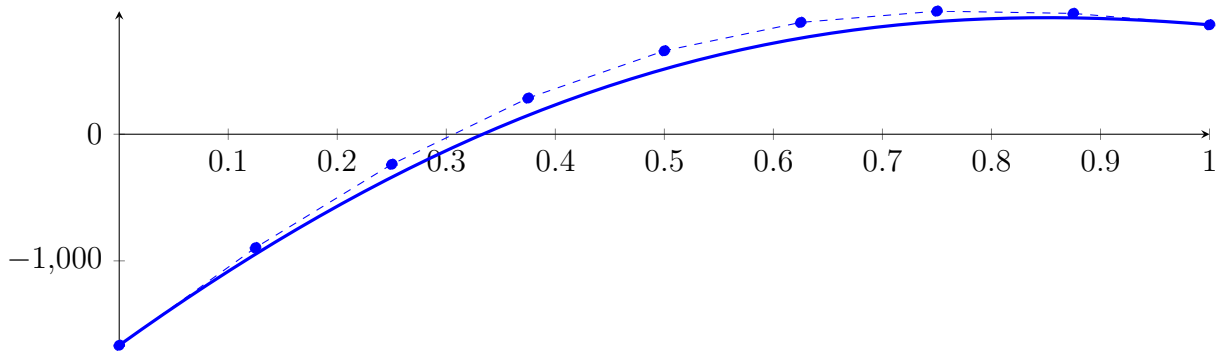
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



230.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

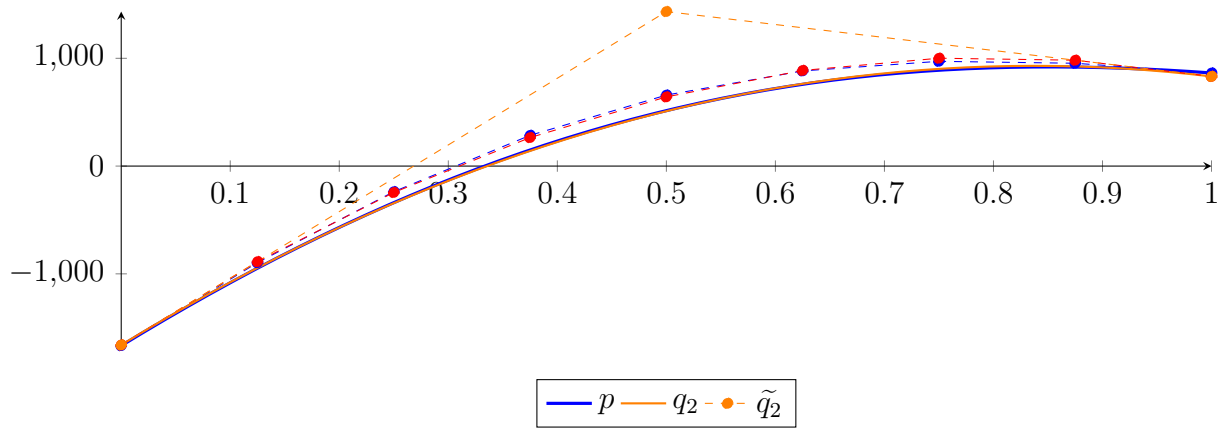
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,2} + 1433.82B_{1,2} + 831.864B_{2,2} \end{aligned}$$

$$\begin{aligned} \tilde{q}_2 &= 6.38292 \cdot 10^{-300}X^8 - 2.86993 \cdot 10^{-299}X^7 + 5.15342 \cdot 10^{-299}X^6 - 4.6928 \cdot 10^{-299}X^5 \\ &\quad + 2.29296 \cdot 10^{-299}X^4 - 6.02693 \cdot 10^{-300}X^3 - 3695.78X^2 + 6187.64X - 1660 \\ &= -1660B_{0,8} - 886.542B_{1,8} - 245.079B_{2,8} + 264.392B_{3,8} + 641.871B_{4,8} \\ &\quad + 887.358B_{5,8} + 1000.85B_{6,8} + 982.354B_{7,8} + 831.864B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 32.1356$.

Bounding polynomials M and m :

$$M = -3695.78X^2 + 6187.64X - 1627.86$$

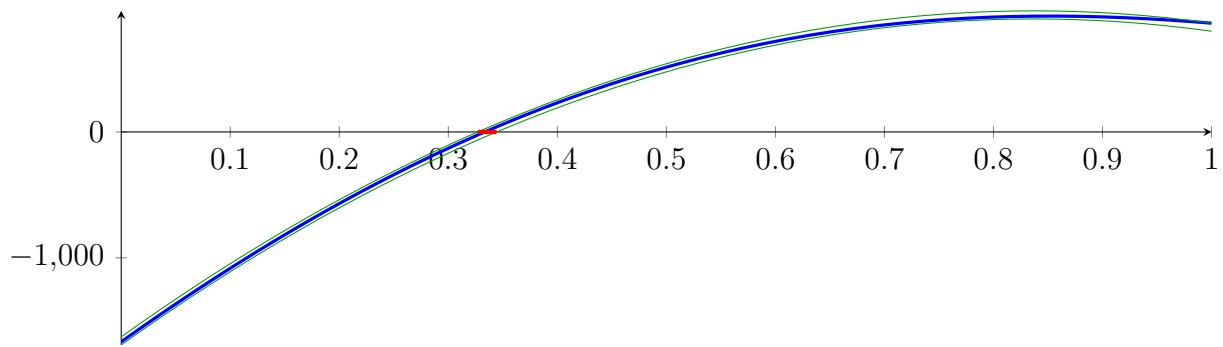
$$m = -3695.78X^2 + 6187.64X - 1692.13$$

Root of M and m :

$$N(M) = \{0.326917, 1.34733\}$$

$$N(m) = \{0.344255, 1.32999\}$$

Intersection intervals:



$$[0.326917, 0.344255]$$

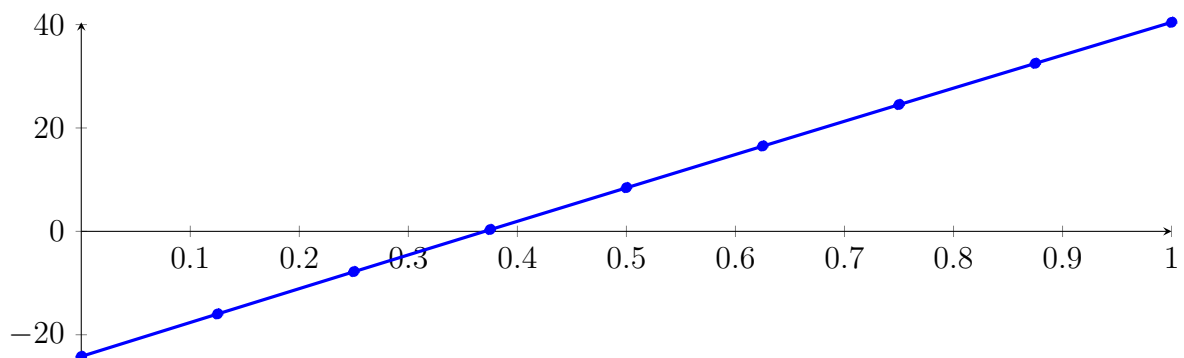
Longest intersection interval: 0.0173372

\implies Selective recursion: **interval 1:** $[0.326917, 0.344255]$,

230.2 Recursion Branch 1 1 in Interval 1: $[0.326917, 0.344255]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -8.16249 \cdot 10^{-15} X^8 - 7.66571 \cdot 10^{-12} X^7 - 1.9444 \cdot 10^{-09} X^6 + 1.19263 \cdot 10^{-07} X^5 \\ &\quad + 8.12335 \cdot 10^{-05} X^4 - 0.0012733 X^3 - 1.18084 X^2 + 65.8155 X - 24.1945 \\ &= -24.1945 B_{0,8}(X) - 15.9676 B_{1,8}(X) - 7.78282 B_{2,8}(X) + 0.35975 B_{3,8}(X) \\ &\quad + 8.4601 B_{4,8}(X) + 16.5182 B_{5,8}(X) + 24.5341 B_{6,8}(X) + 32.5077 B_{7,8}(X) + 40.4389 B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -1.18261X^2 + 65.8162X - 24.1946$$

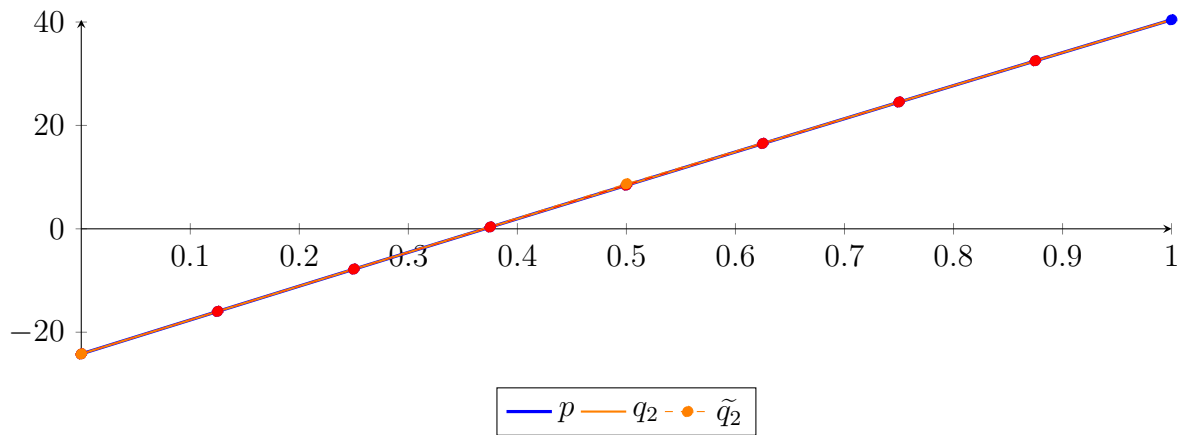
$$= -24.1946B_{0,2} + 8.71352B_{1,2} + 40.439B_{2,2}$$

$$\tilde{q}_2 = 4.27533 \cdot 10^{-302}X^8 - 2.42468 \cdot 10^{-301}X^7 + 4.90537 \cdot 10^{-301}X^6 - 4.6286 \cdot 10^{-301}X^5$$

$$+ 2.17546 \cdot 10^{-301}X^4 - 5.36695 \cdot 10^{-302}X^3 - 1.18261X^2 + 65.8162X - 24.1946$$

$$= -24.1946B_{0,8} - 15.9676B_{1,8} - 7.78277B_{2,8} + 0.359785B_{3,8}$$

$$+ 8.4601B_{4,8} + 16.5182B_{5,8} + 24.534B_{6,8} + 32.5076B_{7,8} + 40.439B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 5.66894 \cdot 10^{-05}$.

Bounding polynomials M and m :

$$M = -1.18261X^2 + 65.8162X - 24.1945$$

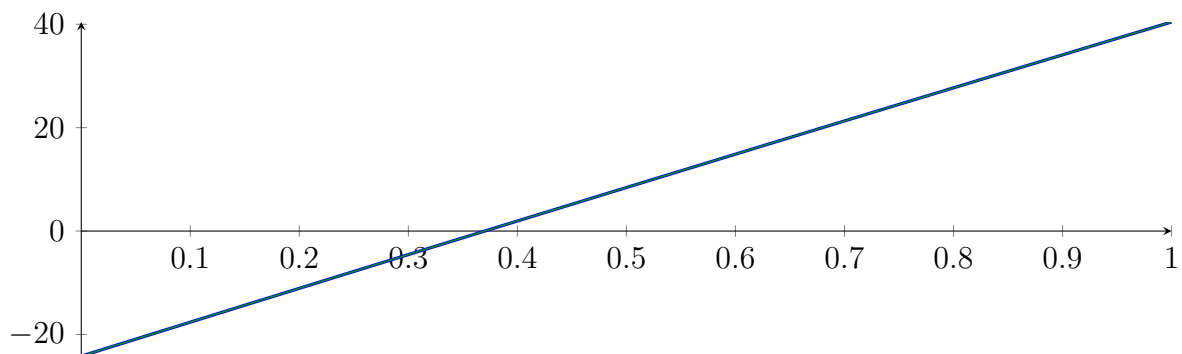
$$m = -1.18261X^2 + 65.8162X - 24.1946$$

Root of M and m :

$$N(M) = \{0.370068, 55.2832\}$$

$$N(m) = \{0.37007, 55.2832\}$$

Intersection intervals:



$$[0.370068, 0.37007]$$

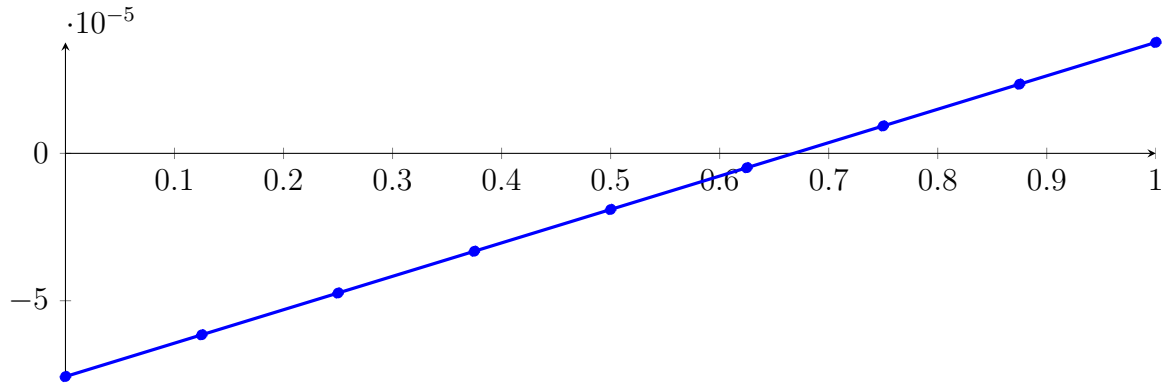
Longest intersection interval: $1.74588 \cdot 10^{-06}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

230.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

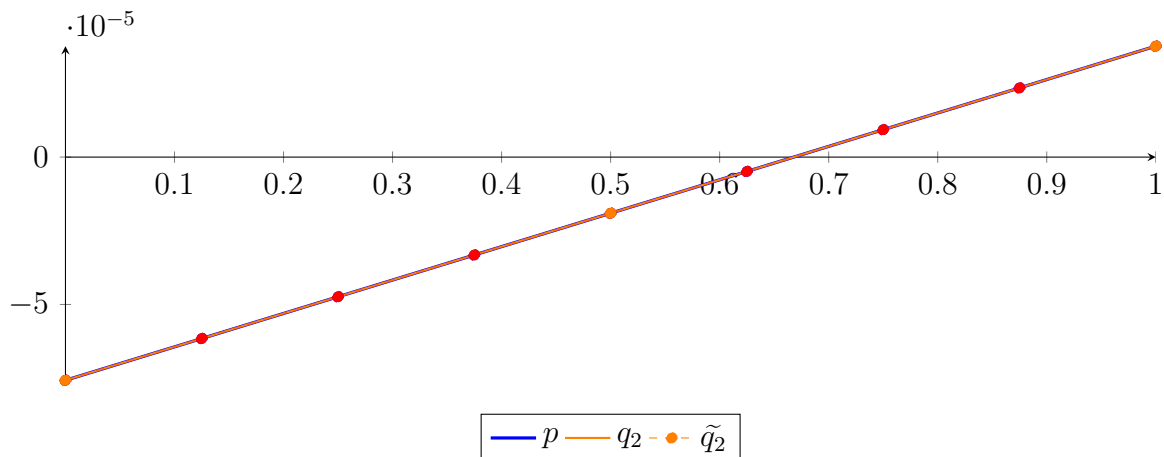
$$\begin{aligned}
 p &= -7.04578 \cdot 10^{-61} X^8 - 3.80201 \cdot 10^{-52} X^7 - 5.5627 \cdot 10^{-44} X^6 + 1.86413 \cdot 10^{-36} X^5 + 7.56737 \\
 &\quad \cdot 10^{-28} X^4 - 6.13517 \cdot 10^{-21} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8}(X) - 6.15596 \cdot 10^{-05} B_{1,8}(X) - 4.73873 \cdot 10^{-05} B_{2,8}(X) \\
 &\quad - 3.32149 \cdot 10^{-05} B_{3,8}(X) - 1.90425 \cdot 10^{-05} B_{4,8}(X) - 4.87016 \cdot 10^{-06} B_{5,8}(X) \\
 &\quad + 9.3022 \cdot 10^{-06} B_{6,8}(X) + 2.34746 \cdot 10^{-05} B_{7,8}(X) + 3.76469 \cdot 10^{-05} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,2} - 1.90425 \cdot 10^{-05} B_{1,2} + 3.76469 \cdot 10^{-05} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -3.62396 \cdot 10^{-308} X^8 + 2.32992 \cdot 10^{-308} X^7 + 2.61753 \cdot 10^{-307} X^6 - 5.97049 \cdot 10^{-307} X^5 + 5.13401 \\
 &\quad \cdot 10^{-307} X^4 - 1.82868 \cdot 10^{-307} X^3 - 3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 &= -7.5732 \cdot 10^{-05} B_{0,8} - 6.15596 \cdot 10^{-05} B_{1,8} - 4.73873 \cdot 10^{-05} B_{2,8} - 3.32149 \cdot 10^{-05} B_{3,8} - 1.90425 \\
 &\quad \cdot 10^{-05} B_{4,8} - 4.87016 \cdot 10^{-06} B_{5,8} + 9.3022 \cdot 10^{-06} B_{6,8} + 2.34746 \cdot 10^{-05} B_{7,8} + 3.76469 \cdot 10^{-05} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.06758 \cdot 10^{-22}$.

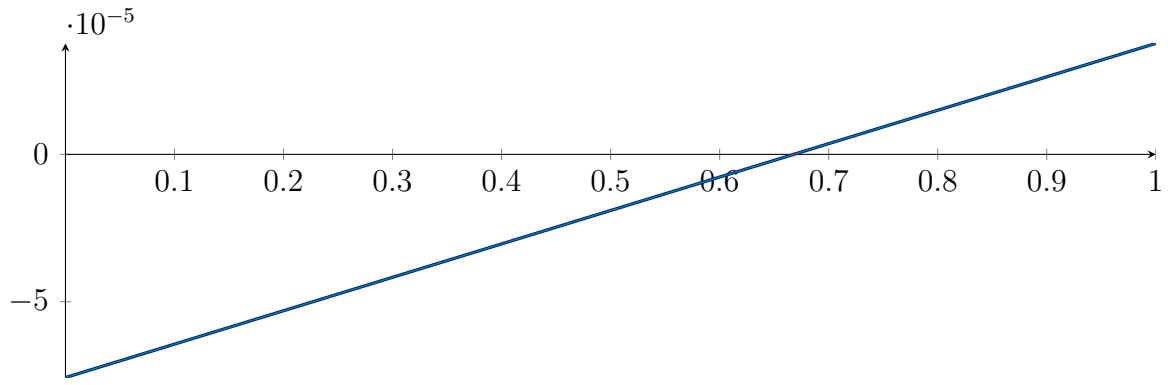
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05} \\
 m &= -3.6034 \cdot 10^{-12} X^2 + 0.000113379 X - 7.5732 \cdot 10^{-05}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.667955, 3.14644 \cdot 10^7\} \quad N(m) = \{0.667955, 3.14644 \cdot 10^7\}$$

Intersection intervals:



[0.667955, 0.667955]

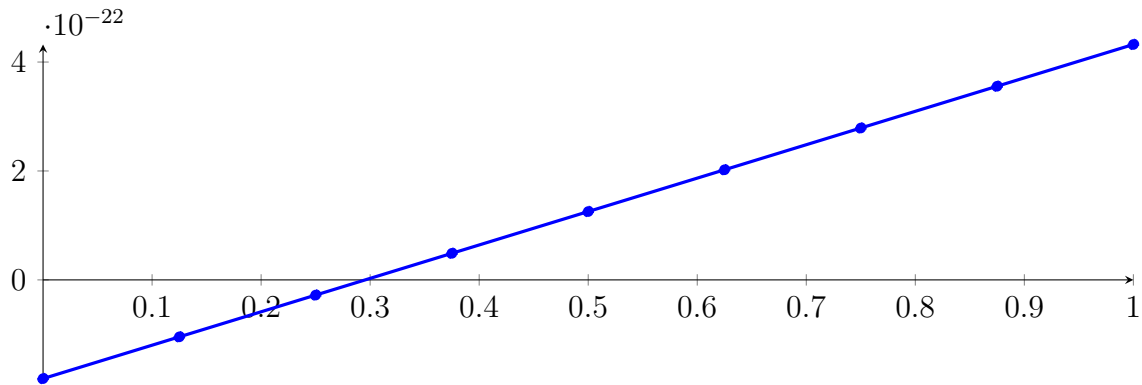
Longest intersection interval: $5.41121 \cdot 10^{-18}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

230.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

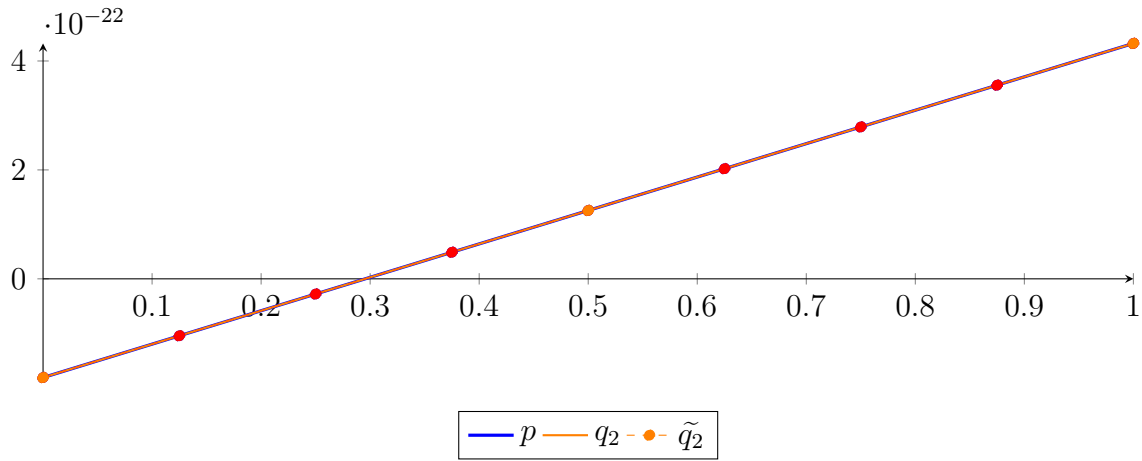
$$\begin{aligned}
 p &= -5.17944 \cdot 10^{-199} X^8 - 5.16502 \cdot 10^{-173} X^7 - 1.39653 \cdot 10^{-147} X^6 + 8.64863 \cdot 10^{-123} X^5 + 6.48817 \\
 &\quad \cdot 10^{-97} X^4 - 9.72096 \cdot 10^{-73} X^3 - 1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22} \\
 &= -1.81261 \cdot 10^{-22} B_{0,8}(X) - 1.04571 \cdot 10^{-22} B_{1,8}(X) - 2.78818 \cdot 10^{-23} B_{2,8}(X) \\
 &\quad + 4.88078 \cdot 10^{-23} B_{3,8}(X) + 1.25497 \cdot 10^{-22} B_{4,8}(X) + 2.02187 \cdot 10^{-22} B_{5,8}(X) \\
 &\quad + 2.78877 \cdot 10^{-22} B_{6,8}(X) + 3.55566 \cdot 10^{-22} B_{7,8}(X) + 4.32256 \cdot 10^{-22} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22} \\
 &= -1.81261 \cdot 10^{-22} B_{0,2} + 1.25497 \cdot 10^{-22} B_{1,2} + 4.32256 \cdot 10^{-22} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= 5.39888 \cdot 10^{-325} X^8 - 2.84119 \cdot 10^{-324} X^7 + 5.35011 \cdot 10^{-324} X^6 - 4.57499 \cdot 10^{-324} X^5 + 1.82797 \\
 &\quad \cdot 10^{-324} X^4 - 3.72306 \cdot 10^{-325} X^3 - 1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22} \\
 &= -1.81261 \cdot 10^{-22} B_{0,8} - 1.04571 \cdot 10^{-22} B_{1,8} - 2.78818 \cdot 10^{-23} B_{2,8} + 4.88078 \cdot 10^{-23} B_{3,8} + 1.25497 \\
 &\quad \cdot 10^{-22} B_{4,8} + 2.02187 \cdot 10^{-22} B_{5,8} + 2.78877 \cdot 10^{-22} B_{6,8} + 3.55566 \cdot 10^{-22} B_{7,8} + 4.32256 \cdot 10^{-22} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 4.86048 \cdot 10^{-74}$.

Bounding polynomials M and m :

$$M = -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22}$$

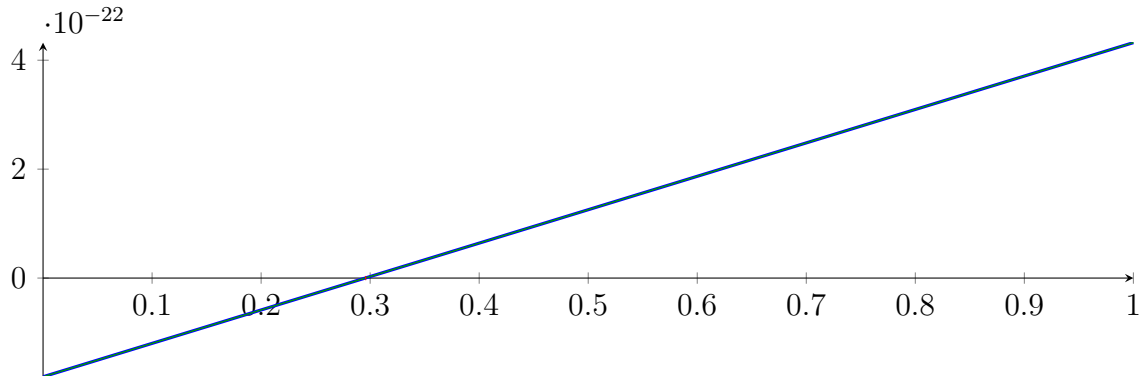
$$m = -1.05512 \cdot 10^{-46} X^2 + 6.13517 \cdot 10^{-22} X - 1.81261 \cdot 10^{-22}$$

Root of M and m :

$$N(M) = \{0.295446, 5.81467 \cdot 10^{24}\}$$

$$N(m) = \{0.295446, 5.81467 \cdot 10^{24}\}$$

Intersection intervals:



$$[0.295446, 0.295446]$$

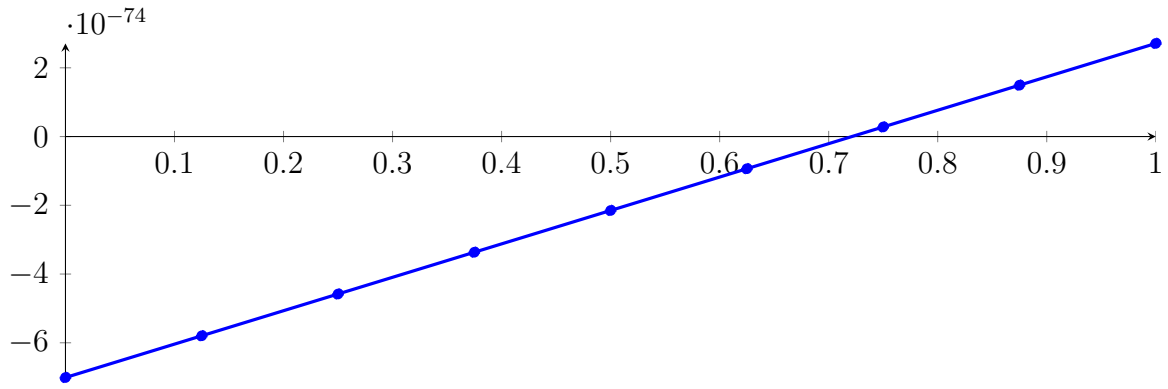
Longest intersection interval: $1.58446 \cdot 10^{-52}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

230.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= 6.29674 \cdot 10^{-381} X^8 - 2.51869 \cdot 10^{-380} X^7 - 8.81543 \cdot 10^{-380} X^6 + 4.08933 \cdot 10^{-304} X^4 \\ &\quad - 3.86684 \cdot 10^{-228} X^3 - 2.6489 \cdot 10^{-150} X^2 + 9.72096 \cdot 10^{-74} X - 7.01115 \cdot 10^{-74} \\ &= -7.01115 \cdot 10^{-74} B_{0,8}(X) - 5.79603 \cdot 10^{-74} B_{1,8}(X) - 4.58091 \cdot 10^{-74} B_{2,8}(X) \\ &\quad - 3.36579 \cdot 10^{-74} B_{3,8}(X) - 2.15067 \cdot 10^{-74} B_{4,8}(X) - 9.35553 \cdot 10^{-75} B_{5,8}(X) \\ &\quad + 2.79566 \cdot 10^{-75} B_{6,8}(X) + 1.49469 \cdot 10^{-74} B_{7,8}(X) + 2.70981 \cdot 10^{-74} B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.6489 \cdot 10^{-150} X^2 + 9.72096 \cdot 10^{-74} X - 7.01115 \cdot 10^{-74}$$

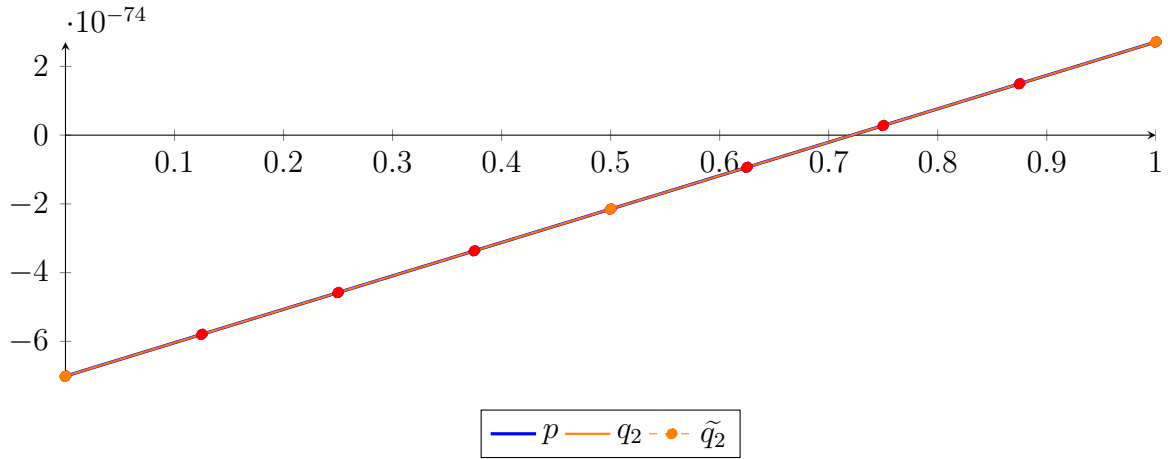
$$= -7.01115 \cdot 10^{-74} B_{0,2} - 2.15067 \cdot 10^{-74} B_{1,2} + 2.70981 \cdot 10^{-74} B_{2,2}$$

$$\tilde{q}_2 = -4.77607 \cdot 10^{-377} X^8 + 8.70965 \cdot 10^{-377} X^7 + 1.35581 \cdot 10^{-376} X^6 - 4.8194 \cdot 10^{-376} X^5 + 4.61929$$

$$\cdot 10^{-376} X^4 - 1.70887 \cdot 10^{-376} X^3 - 2.6489 \cdot 10^{-150} X^2 + 9.72096 \cdot 10^{-74} X - 7.01115 \cdot 10^{-74}$$

$$= -7.01115 \cdot 10^{-74} B_{0,8} - 5.79603 \cdot 10^{-74} B_{1,8} - 4.58091 \cdot 10^{-74} B_{2,8} - 3.36579 \cdot 10^{-74} B_{3,8} - 2.15067$$

$$\cdot 10^{-74} B_{4,8} - 9.35553 \cdot 10^{-75} B_{5,8} + 2.79566 \cdot 10^{-75} B_{6,8} + 1.49469 \cdot 10^{-74} B_{7,8} + 2.70981 \cdot 10^{-74} B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 1.93342 \cdot 10^{-229}$.

Bounding polynomials M and m :

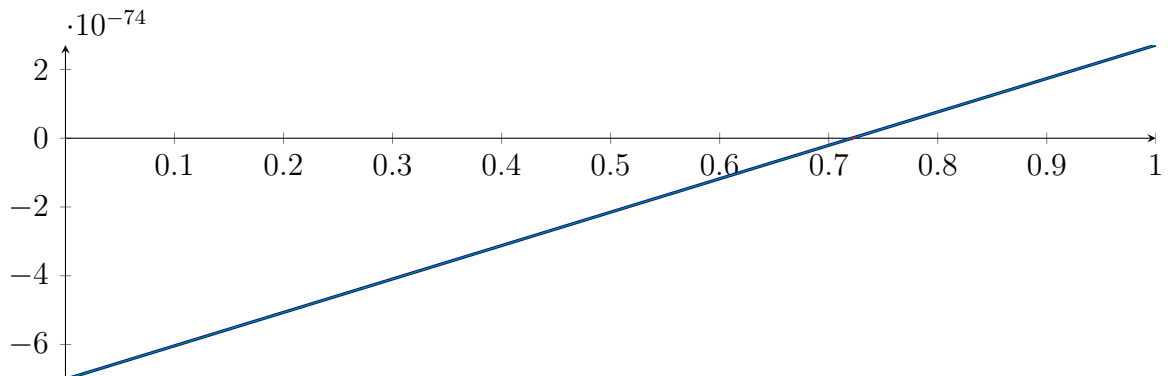
$$M = -2.6489 \cdot 10^{-150} X^2 + 9.72096 \cdot 10^{-74} X - 7.01115 \cdot 10^{-74}$$

$$m = -2.6489 \cdot 10^{-150} X^2 + 9.72096 \cdot 10^{-74} X - 7.01115 \cdot 10^{-74}$$

Root of M and m :

$$N(M) = \{0.721241, 3.6698 \cdot 10^{76}\} \quad N(m) = \{0.721241, 3.6698 \cdot 10^{76}\}$$

Intersection intervals:



[0.721241, 0.721241]

Longest intersection interval: $3.97784 \cdot 10^{-156}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

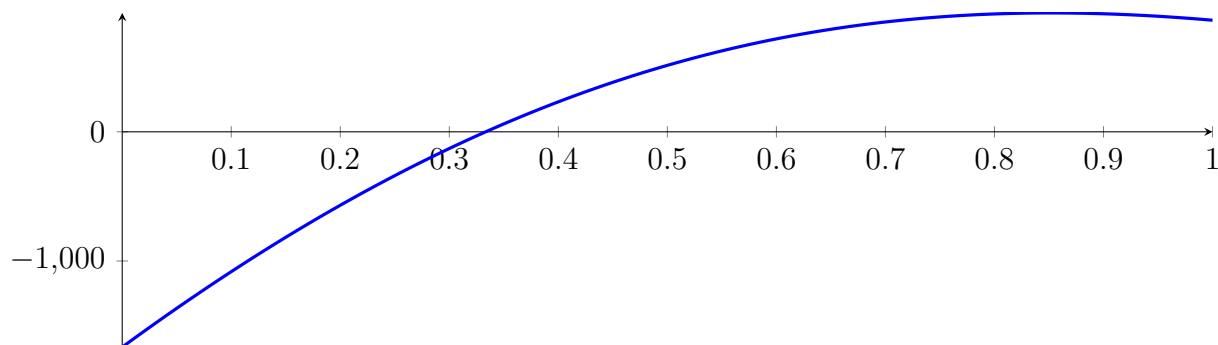
230.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

230.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

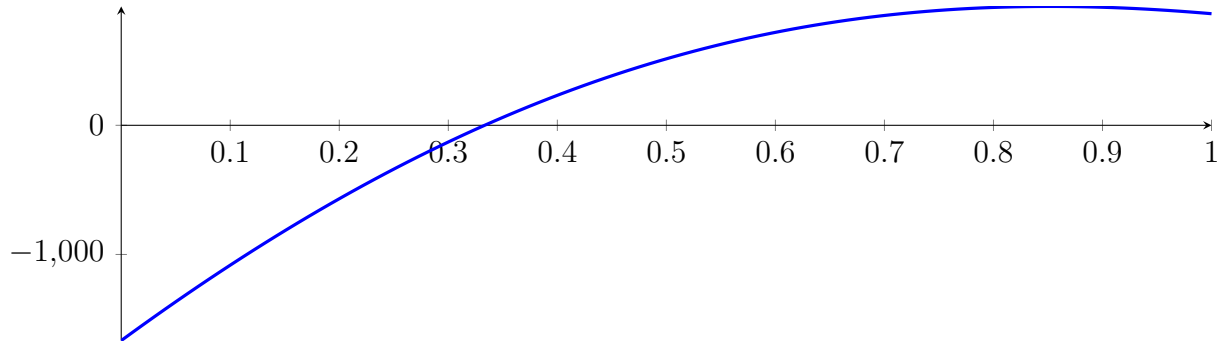
with precision $\varepsilon = 1 \cdot 10^{-128}$.

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$$-1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$

Called CubeClip with input polynomial on interval $[0, 1]$:

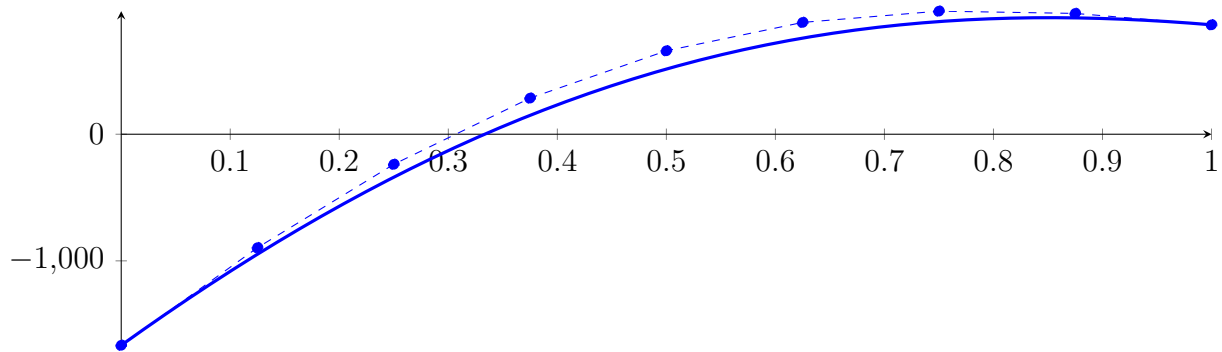
$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



231.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

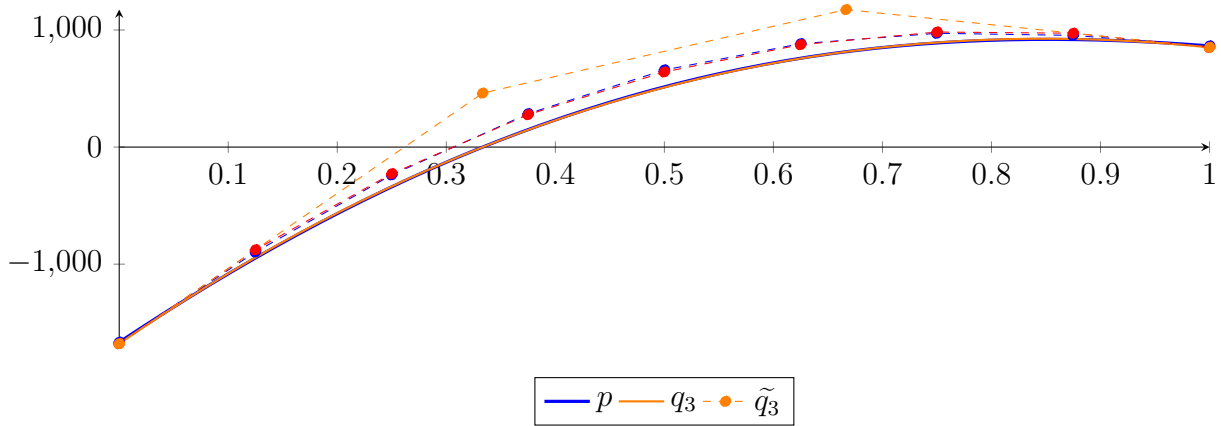
$$\begin{aligned} p &= -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67 \\ &= -1666.67B_{0,8}(X) - 895.833B_{1,8}(X) - 237.5B_{2,8}(X) + 285.208B_{3,8}(X) \\ &\quad + 658.867B_{4,8}(X) + 883B_{5,8}(X) + 972B_{6,8}(X) + 954B_{7,8}(X) + 864B_{8,8}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,3} + 460.963B_{1,3} + 1174.76B_{2,3} + 851.335B_{3,3} \end{aligned}$$

$$\begin{aligned} \tilde{q}_3 &= -2.80826 \cdot 10^{-300}X^8 + 1.16606 \cdot 10^{-299}X^7 - 2.1197 \cdot 10^{-299}X^6 + 2.15003 \cdot 10^{-299}X^5 \\ &\quad - 1.20234 \cdot 10^{-299}X^4 + 389.419X^3 - 4279.91X^2 + 6421.29X - 1679.47 \\ &= -1679.47B_{0,8} - 876.806B_{1,8} - 226.998B_{2,8} + 276.909B_{3,8} + 641.871B_{4,8} \\ &\quad + 874.841B_{5,8} + 982.772B_{6,8} + 972.619B_{7,8} + 851.335B_{8,8} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 19.0273$.

Bounding polynomials M and m :

$$M = 389.419X^3 - 4279.91X^2 + 6421.29X - 1660.44$$

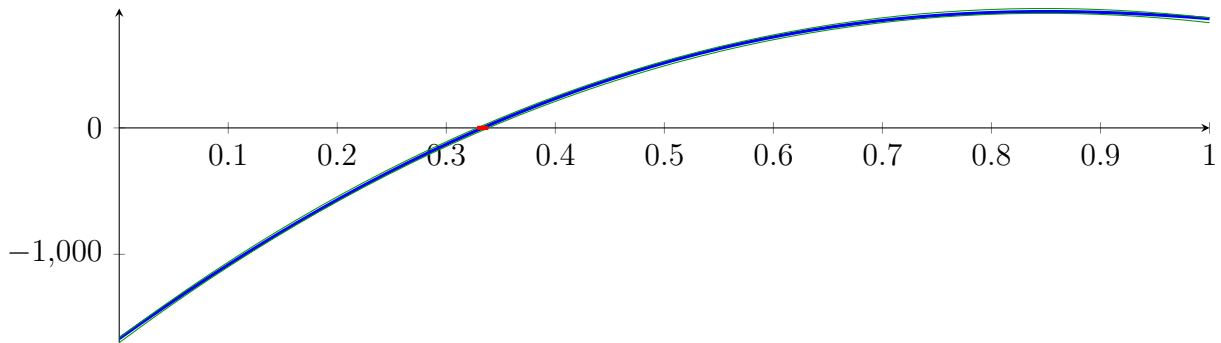
$$m = 389.419X^3 - 4279.91X^2 + 6421.29X - 1698.49$$

Root of M and m :

$$N(M) = \{0.328258, 1.40284, 9.2594\}$$

$$N(m) = \{0.338551, 1.39115, 9.26079\}$$

Intersection intervals:



$$[0.328258, 0.338551]$$

Longest intersection interval: 0.0102926

\implies Selective recursion: interval 1: $[0.328258, 0.338551]$,

231.2 Recursion Branch 1 1 in Interval 1: $[0.328258, 0.338551]$

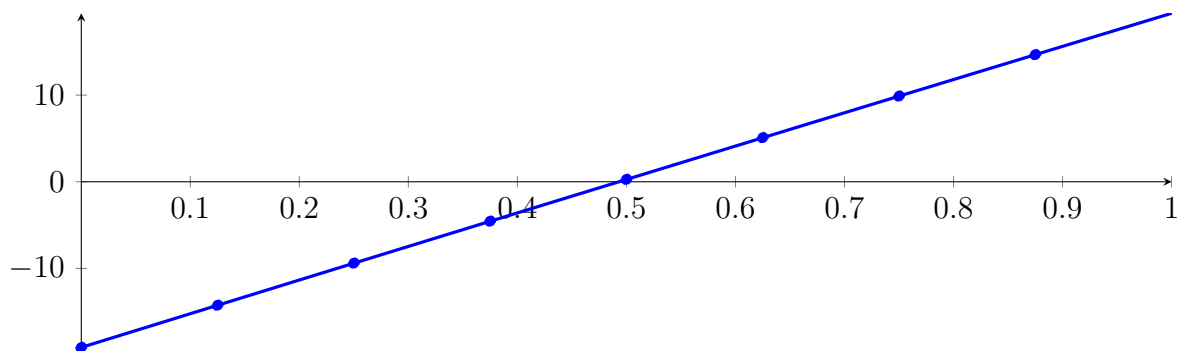
Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -1.25947 \cdot 10^{-16} X^8 - 1.9937 \cdot 10^{-13} X^7 - 8.53073 \cdot 10^{-11} X^6 + 8.72839 \cdot 10^{-09} X^5$$

$$+ 1.00963 \cdot 10^{-05} X^4 - 0.000261161 X^3 - 0.416284 X^2 + 38.9643 X - 19.1124$$

$$= -19.1124 B_{0,8}(X) - 14.2419 B_{1,8}(X) - 9.38619 B_{2,8}(X) - 4.54539 B_{3,8}(X)$$

$$+ 0.280537 B_{4,8}(X) + 5.09158 B_{5,8}(X) + 9.88774 B_{6,8}(X) + 14.669 B_{7,8}(X) + 19.4354 B_{8,8}(X)$$



Degree reduction and raising:

$$q_3 = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

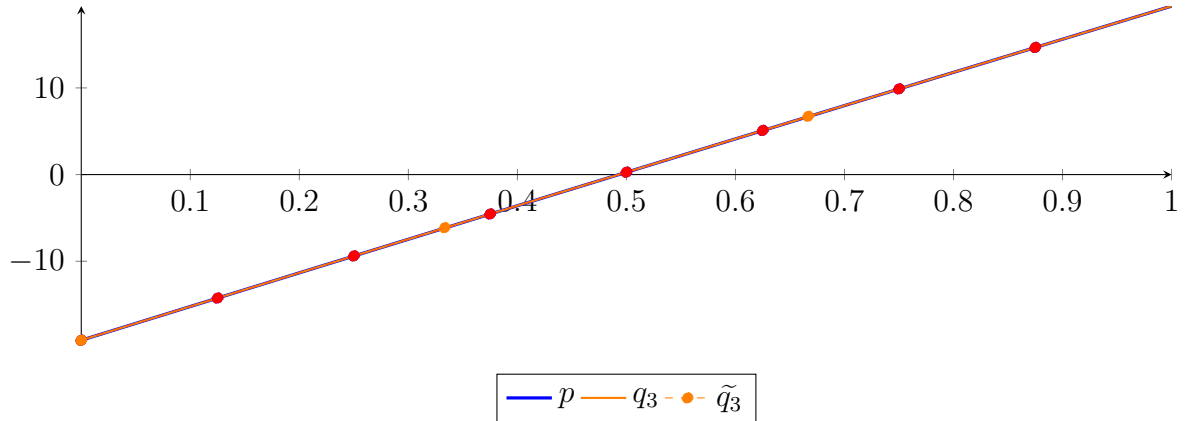
$$= -19.1124B_{0,3} - 6.12429B_{1,3} + 6.72505B_{2,3} + 19.4354B_{3,3}$$

$$\tilde{q}_3 = -1.96643 \cdot 10^{-303} X^8 + 1.82947 \cdot 10^{-302} X^7 - 4.89395 \cdot 10^{-302} X^6 + 5.49554 \cdot 10^{-302} X^5$$

$$- 2.47838 \cdot 10^{-302} X^4 - 0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$= -19.1124B_{0,8} - 14.2419B_{1,8} - 9.38619B_{2,8} - 4.54539B_{3,8}$$

$$+ 0.280537B_{4,8} + 5.09158B_{5,8} + 9.88774B_{6,8} + 14.669B_{7,8} + 19.4354B_{8,8}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16902 \cdot 10^{-07}$.

Bounding polynomials M and m :

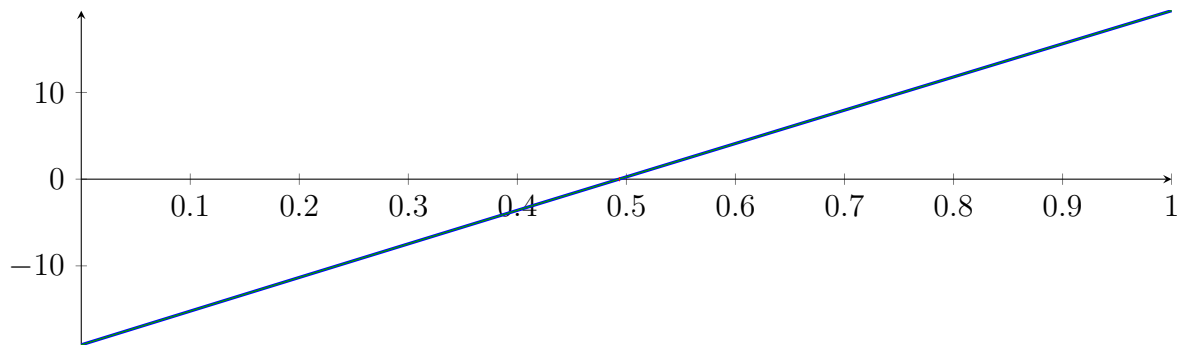
$$M = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

$$m = -0.000240945X^3 - 0.416297X^2 + 38.9643X - 19.1124$$

Root of M and m :

$$N(M) = \{-1816.81, 0.493109, 88.5414\} \quad N(m) = \{-1816.81, 0.493109, 88.5414\}$$

Intersection intervals:



$$[0.493109, 0.493109]$$

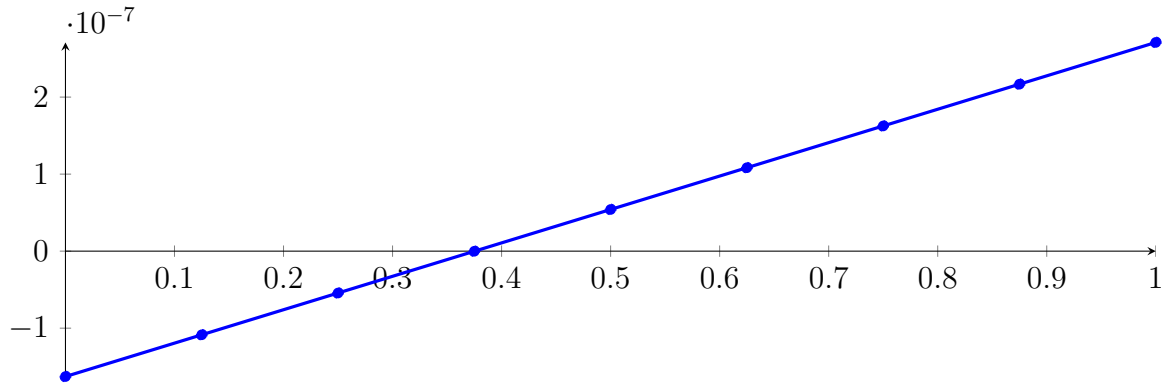
Longest intersection interval: $1.1252 \cdot 10^{-08}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

231.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

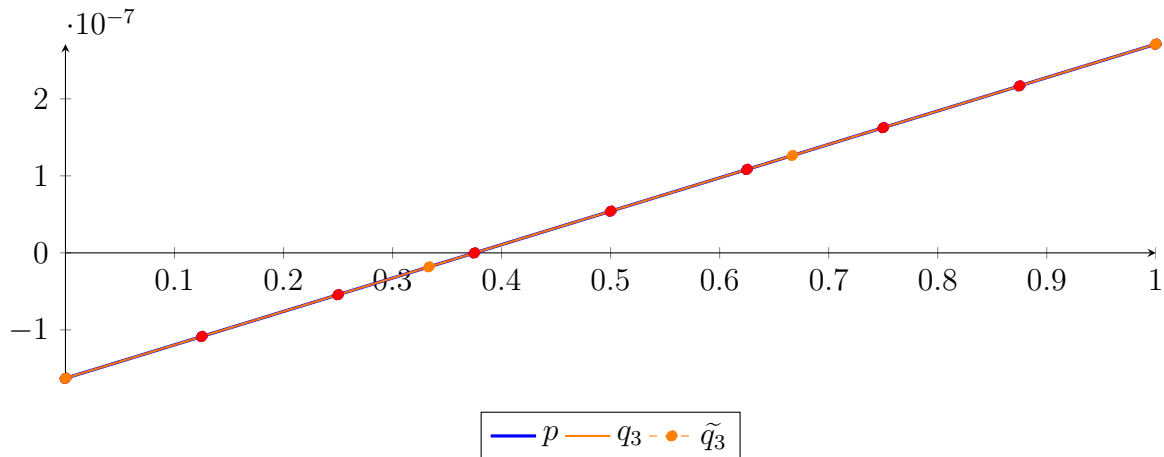
$$\begin{aligned}
 p &= -3.2361 \cdot 10^{-80} X^8 - 4.56398 \cdot 10^{-69} X^7 - 1.74524 \cdot 10^{-58} X^6 + 1.52857 \cdot 10^{-48} X^5 + 1.62178 \\
 &\quad \cdot 10^{-37} X^4 - 3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,8}(X) - 1.08585 \cdot 10^{-07} B_{1,8}(X) - 5.43592 \cdot 10^{-08} B_{2,8}(X) \\
 &\quad - 1.33711 \cdot 10^{-10} B_{3,8}(X) + 5.40918 \cdot 10^{-08} B_{4,8}(X) + 1.08317 \cdot 10^{-07} B_{5,8}(X) \\
 &\quad + 1.62543 \cdot 10^{-07} B_{6,8}(X) + 2.16768 \cdot 10^{-07} B_{7,8}(X) + 2.70994 \cdot 10^{-07} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,3} - 1.82089 \cdot 10^{-08} B_{1,3} + 1.26393 \cdot 10^{-07} B_{2,3} + 2.70994 \cdot 10^{-07} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 8.86066 \cdot 10^{-312} X^8 + 7.41744 \cdot 10^{-311} X^7 - 5.24902 \cdot 10^{-310} X^6 + 1.01267 \cdot 10^{-309} X^5 - 7.8921 \\
 &\quad \cdot 10^{-310} X^4 - 3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 &= -1.6281 \cdot 10^{-07} B_{0,8} - 1.08585 \cdot 10^{-07} B_{1,8} - 5.43592 \cdot 10^{-08} B_{2,8} - 1.33711 \cdot 10^{-10} B_{3,8} + 5.40918 \\
 &\quad \cdot 10^{-08} B_{4,8} + 1.08317 \cdot 10^{-07} B_{5,8} + 1.62543 \cdot 10^{-07} B_{6,8} + 2.16768 \cdot 10^{-07} B_{7,8} + 2.70994 \cdot 10^{-07} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 3.47524 \cdot 10^{-39}$.

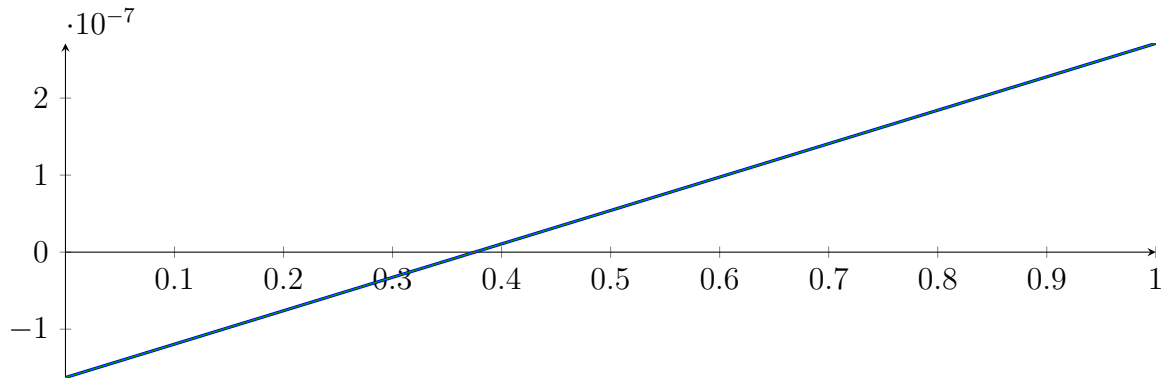
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07} \\
 m &= -3.43646 \cdot 10^{-28} X^3 - 5.27516 \cdot 10^{-17} X^2 + 4.33804 \cdot 10^{-07} X - 1.6281 \cdot 10^{-07}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{-1.6133 \cdot 10^{11}, 0.375292, 7.82468 \cdot 10^9\} \quad N(m) = \{-1.6133 \cdot 10^{11}, 0.375292, 7.82468 \cdot 10^9\}$$

Intersection intervals:



[0.375292, 0.375292]

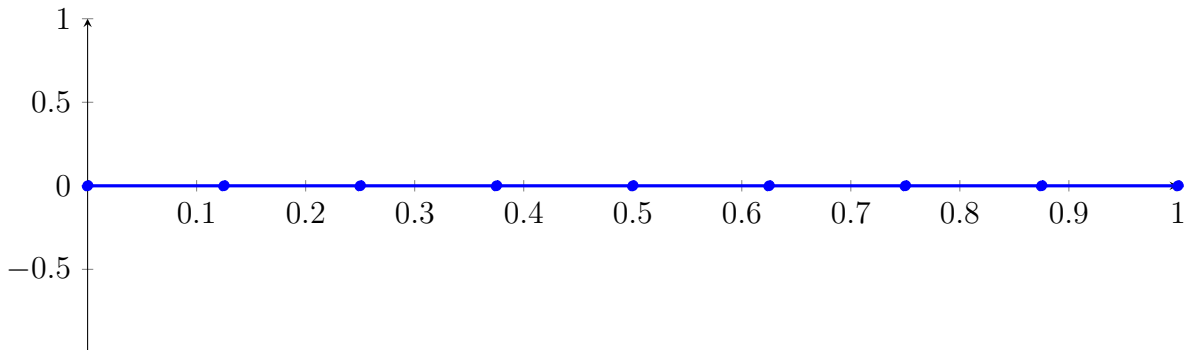
Longest intersection interval: $1.60221 \cdot 10^{-32}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

231.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

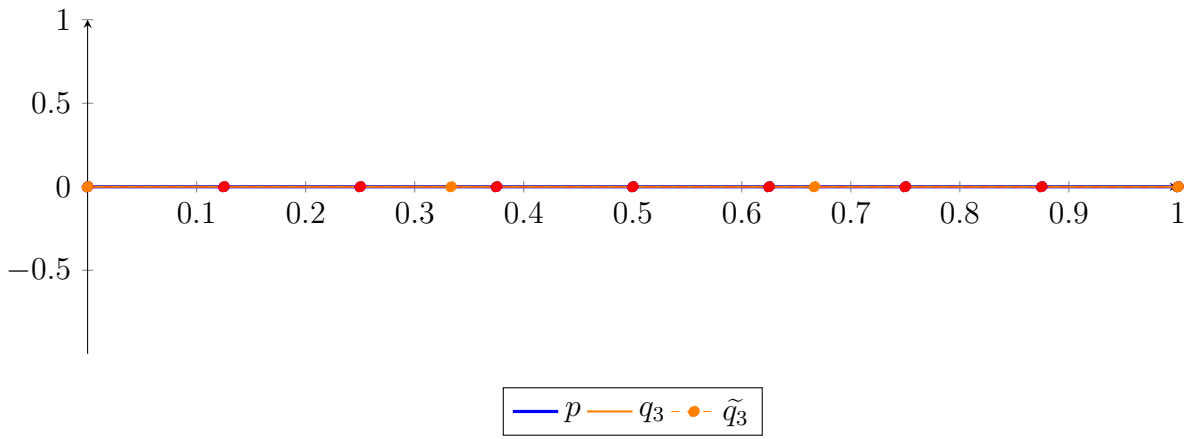
$$\begin{aligned}
 p &= 3.5617 \cdot 10^{-318} X^8 - 1.23705 \cdot 10^{-291} X^7 - 2.95243 \cdot 10^{-249} X^6 + 1.61394 \cdot 10^{-207} X^5 + 1.06875 \\
 &\quad \cdot 10^{-164} X^4 - 1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12} \\
 &= -6.9178 \cdot 10^{-12} B_{0,8}(X) - 6.9178 \cdot 10^{-12} B_{1,8}(X) - 6.9178 \cdot 10^{-12} B_{2,8}(X) \\
 &\quad - 6.9178 \cdot 10^{-12} B_{3,8}(X) - 6.9178 \cdot 10^{-12} B_{4,8}(X) - 6.9178 \cdot 10^{-12} B_{5,8}(X) \\
 &\quad - 6.9178 \cdot 10^{-12} B_{6,8}(X) - 6.9178 \cdot 10^{-12} B_{7,8}(X) - 6.9178 \cdot 10^{-12} B_{8,8}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12} \\
 &= -6.9178 \cdot 10^{-12} B_{0,3} - 6.9178 \cdot 10^{-12} B_{1,3} - 6.9178 \cdot 10^{-12} B_{2,3} - 6.9178 \cdot 10^{-12} B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= -3.69134 \cdot 10^{-315} X^8 + 1.58227 \cdot 10^{-314} X^7 - 4.11376 \cdot 10^{-316} X^6 - 5.57319 \cdot 10^{-314} X^5 + 6.99255 \\
 &\quad \cdot 10^{-314} X^4 - 1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12} \\
 &= -6.9178 \cdot 10^{-12} B_{0,8} - 6.9178 \cdot 10^{-12} B_{1,8} - 6.9178 \cdot 10^{-12} B_{2,8} - 6.9178 \cdot 10^{-12} B_{3,8} - 6.9178 \\
 &\quad \cdot 10^{-12} B_{4,8} - 6.9178 \cdot 10^{-12} B_{5,8} - 6.9178 \cdot 10^{-12} B_{6,8} - 6.9178 \cdot 10^{-12} B_{7,8} - 6.9178 \cdot 10^{-12} B_{8,8}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.29017 \cdot 10^{-166}$.

Bounding polynomials M and m :

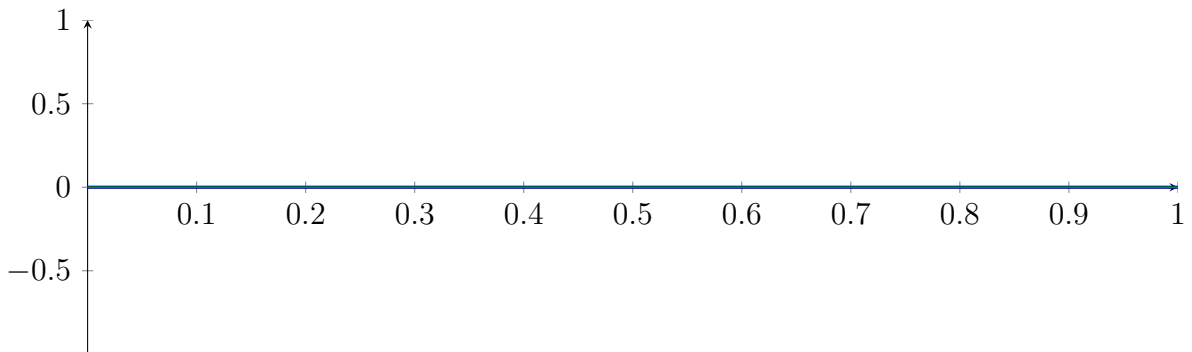
$$M = -1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12}$$

$$m = -1.41343 \cdot 10^{-123} X^3 - 1.35418 \cdot 10^{-80} X^2 + 6.95048 \cdot 10^{-39} X - 6.9178 \cdot 10^{-12}$$

Root of M and m :

$$N(M) = \{-1.00692 \cdot 10^{43}, 2.13634 \cdot 10^{17}, 4.88366 \cdot 10^{41}\} \quad N(m) = \{-1.00692 \cdot 10^{43}, 2.13634 \cdot 10^{17}, 4.88366 \cdot 10^{41}\}$$

Intersection intervals:

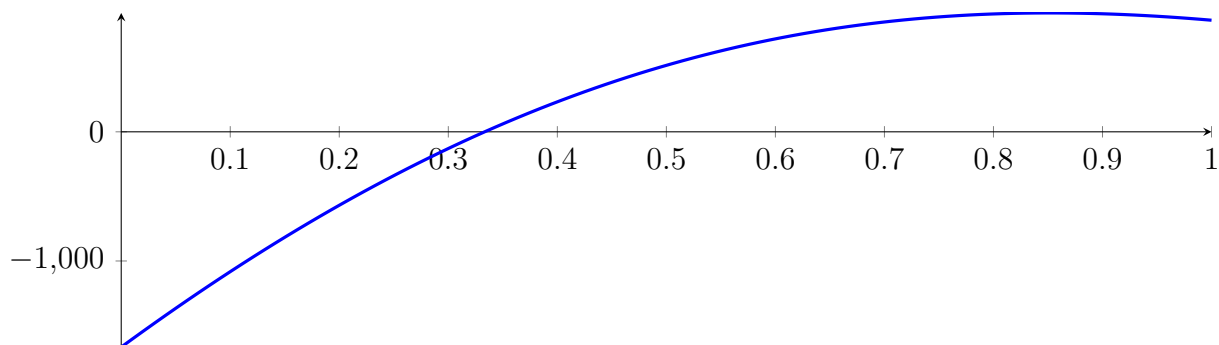


No intersection intervals with the x axis.

231.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^8 - 13.6667X^7 - 37.3333X^6 + 182X^5 + 679X^4 - 1295X^3 - 3150X^2 + 6166.67X - 1666.67$$



Result: Root Intervals

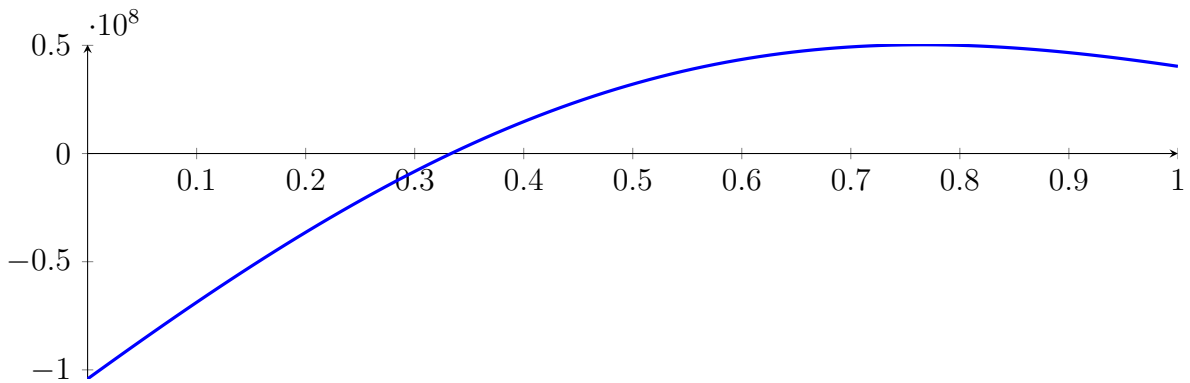
with precision $\varepsilon = 1 \cdot 10^{-128}$.

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$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

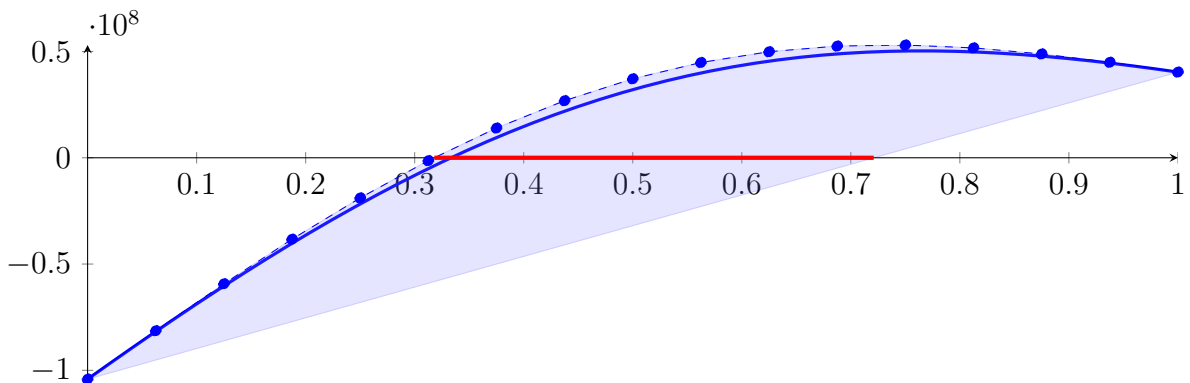
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



232.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

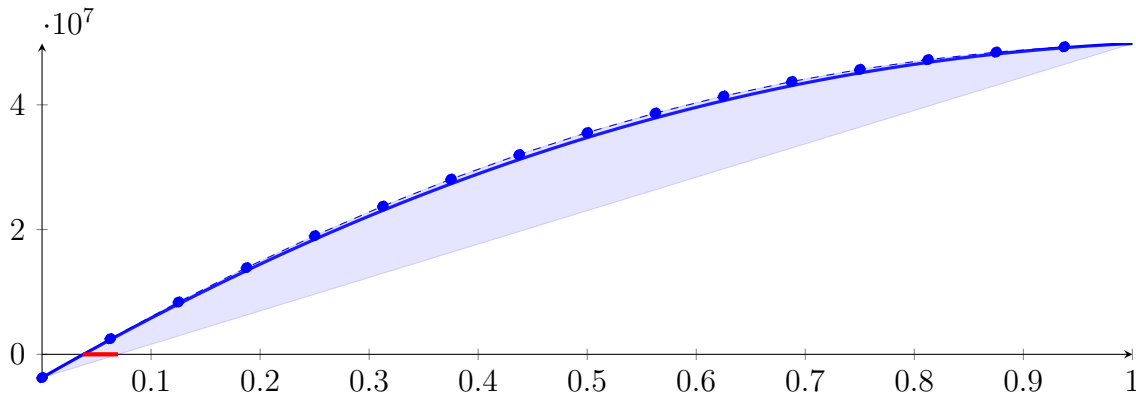
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

232.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.83858 \cdot 10^{-07} X^{16} - 5.37355 \cdot 10^{-05} X^{15} - 0.00254146 X^{14} - 0.064977 X^{13} - 0.909205 X^{12} \\ &\quad - 5.03924 X^{11} + 36.4549 X^{10} + 692.921 X^9 + 1886.97 X^8 - 25792 X^7 - 149671 X^6 + 492605 X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

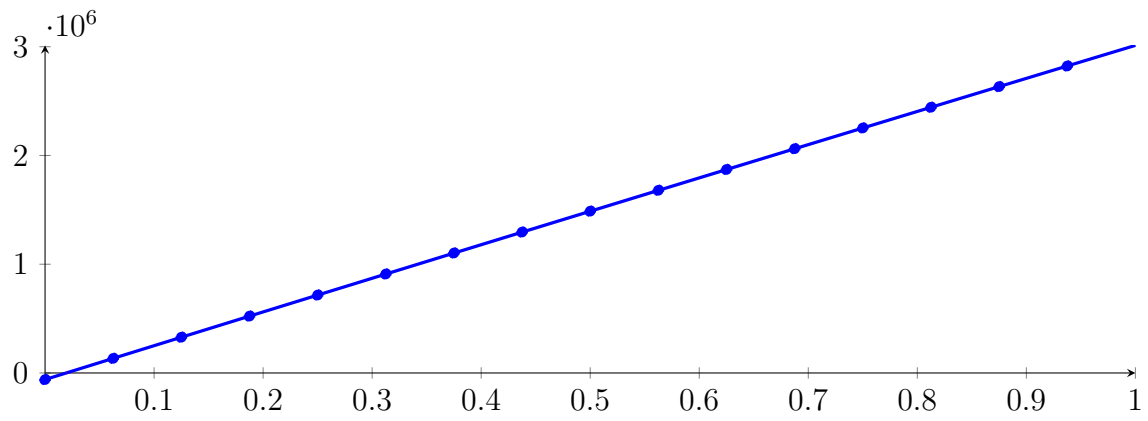
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

232.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -6.78234 \cdot 10^{-31} X^{16} - 2.34473 \cdot 10^{-27} X^{15} - 3.45581 \cdot 10^{-24} X^{14} - 2.75921 \cdot 10^{-21} X^{13} - 1.21242 \cdot 10^{-18} X^{12} \\ &\quad - 2.17572 \cdot 10^{-16} X^{11} + 4.23544 \cdot 10^{-14} X^{10} + 2.70025 \cdot 10^{-11} X^9 + 2.51306 \cdot 10^{-09} X^8 - 9.23474 \cdot 10^{-07} X^7 \\ &\quad - 0.000177459 X^6 + 0.0161038 X^5 + 4.36155 X^4 - 234.216 X^3 - 45622.2 X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554

\implies Selective recursion: interval 1: $[0.333333, 0.333337]$,

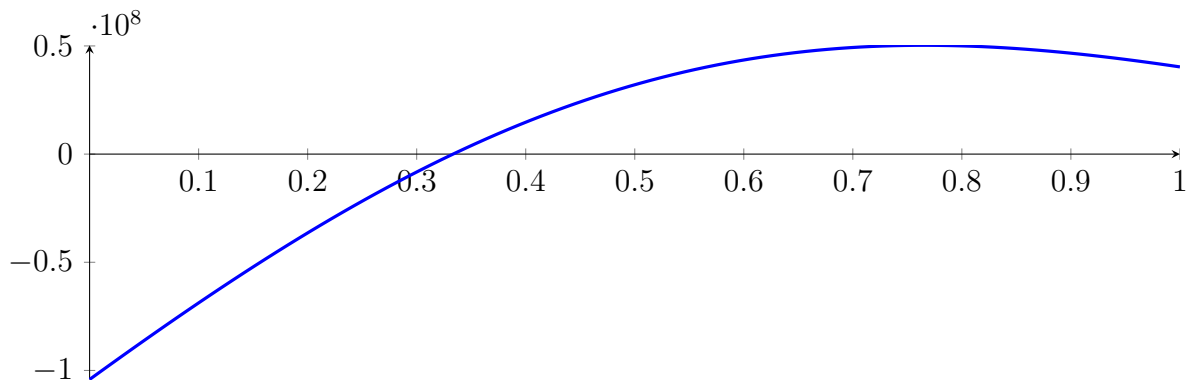
232.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Found root in interval $[0.333333, 0.333337]$ at recursion depth 4!

232.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333337]$$

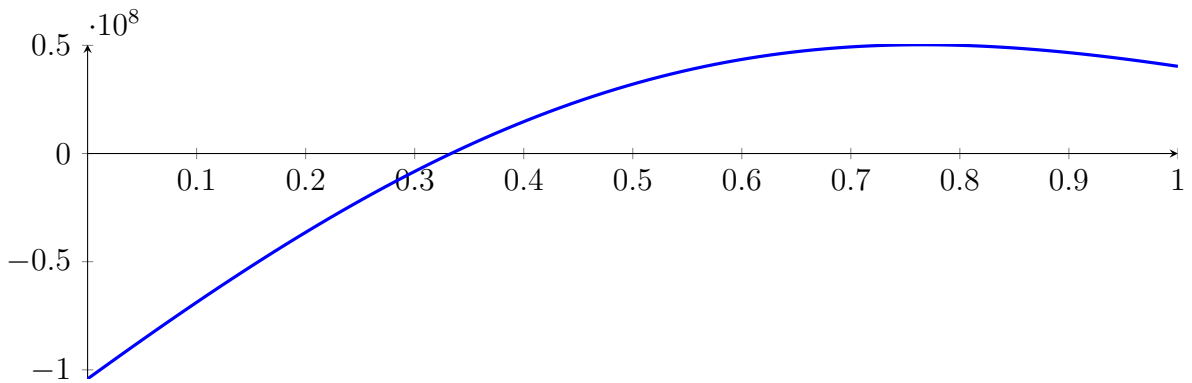
with precision $\varepsilon = 0.01$.

233 Running QuadClip on f_{16} with epsilon 2

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

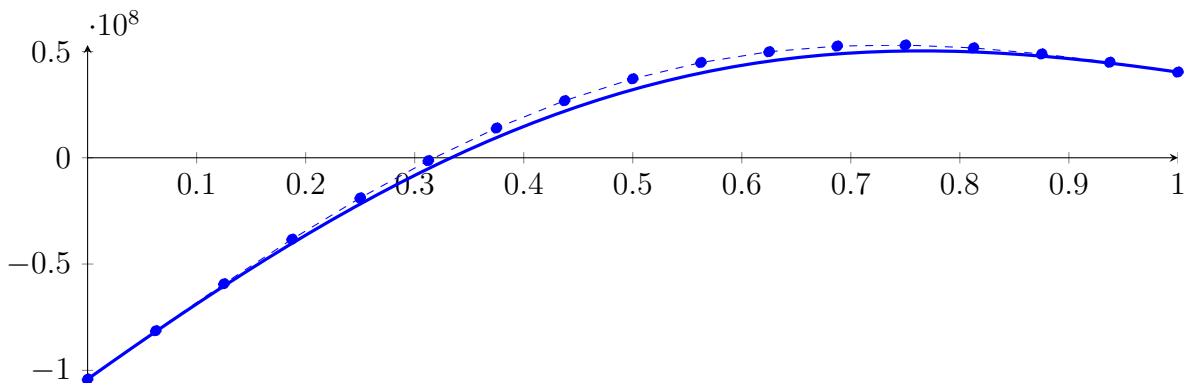
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



233.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = 3.74473 \cdot 10^{-288} X^{16} - 3.08889 \cdot 10^{-287} X^{15} + 1.17443 \cdot 10^{-286} X^{14} - 2.71603 \cdot 10^{-286} X^{13}$$

$$+ 4.23554 \cdot 10^{-286} X^{12} - 4.66107 \cdot 10^{-286} X^{11} + 3.6845 \cdot 10^{-286} X^{10} - 2.09811 \cdot 10^{-286} X^9$$

$$+ 8.58708 \cdot 10^{-287} X^8 - 2.54063 \cdot 10^{-287} X^7 + 5.57852 \cdot 10^{-288} X^6 - 9.21534 \cdot 10^{-289} X^5$$

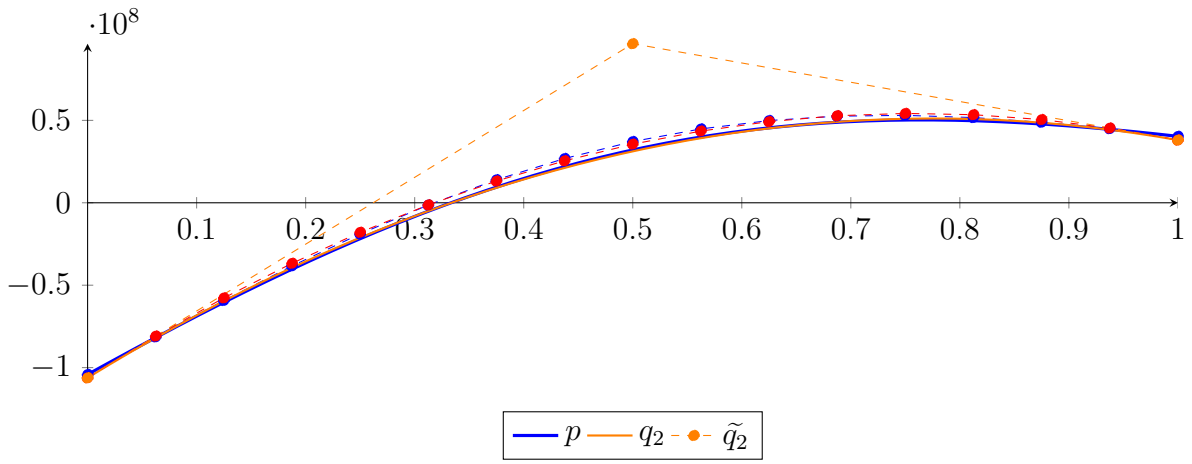
$$+ 1.02122 \cdot 10^{-289} X^4 - 5.95624 \cdot 10^{-291} X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

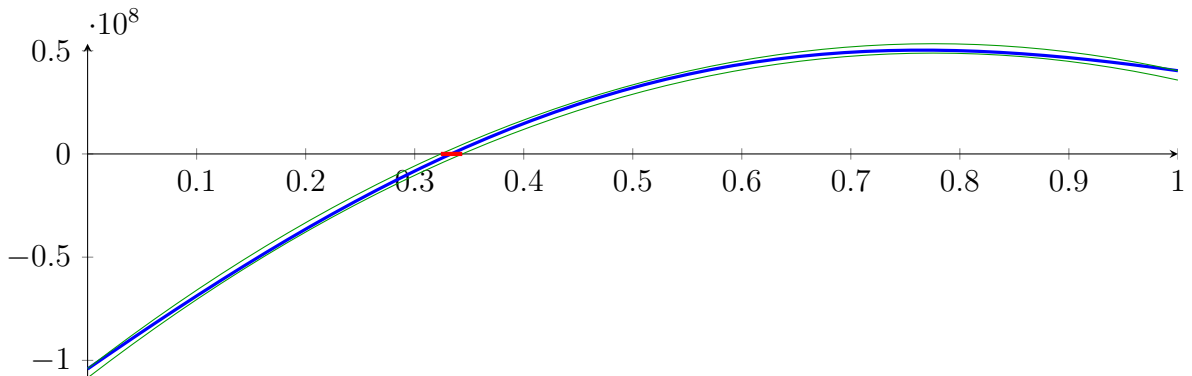
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

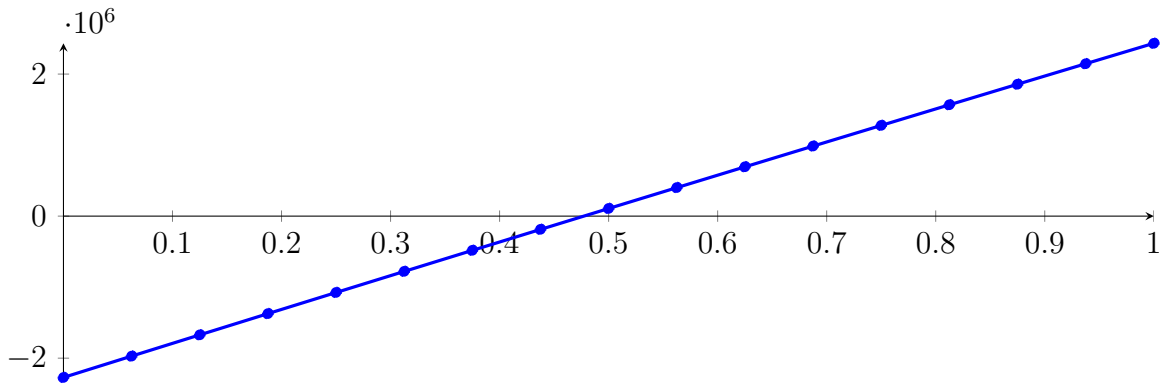
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

233.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

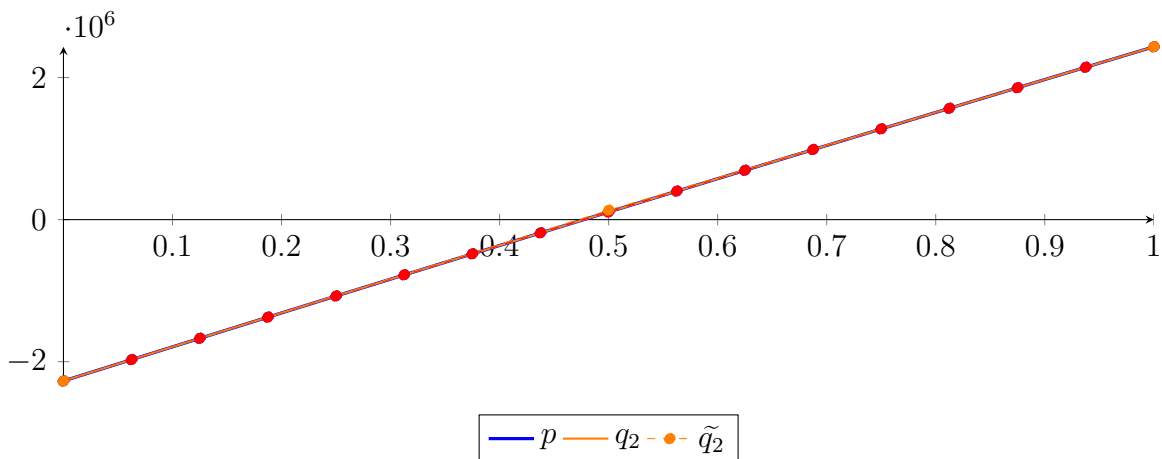
$$\begin{aligned}
 p &= -5.0162 \cdot 10^{-28} X^{16} - 1.14383 \cdot 10^{-24} X^{15} - 1.11125 \cdot 10^{-21} X^{14} - 5.84096 \cdot 10^{-19} X^{13} - 1.684 \cdot 10^{-16} X^{12} \\
 &\quad - 1.94707 \cdot 10^{-14} X^{11} + 2.73227 \cdot 10^{-12} X^{10} + 1.09727 \cdot 10^{-9} X^9 + 6.37314 \cdot 10^{-8} X^8 - 1.68645 \cdot 10^{-5} X^7 \\
 &\quad - 0.00205892 X^6 + 0.132721 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -8.18461 \cdot 10^{-291} X^{16} + 4.13745 \cdot 10^{-290} X^{15} - 5.8254 \cdot 10^{-290} X^{14} - 6.80919 \cdot 10^{-290} X^{13} \\
 &\quad + 3.48059 \cdot 10^{-289} X^{12} - 5.47245 \cdot 10^{-289} X^{11} + 4.8261 \cdot 10^{-289} X^{10} - 2.67867 \cdot 10^{-289} X^9 \\
 &\quad + 1.00314 \cdot 10^{-289} X^8 - 2.84766 \cdot 10^{-290} X^7 + 6.85072 \cdot 10^{-291} X^6 - 1.12501 \cdot 10^{-291} X^5 \\
 &\quad + 1.24731 \cdot 10^{-293} X^4 + 2.25944 \cdot 10^{-293} X^3 - 104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} \\
 &\quad + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

$$M = -104265X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

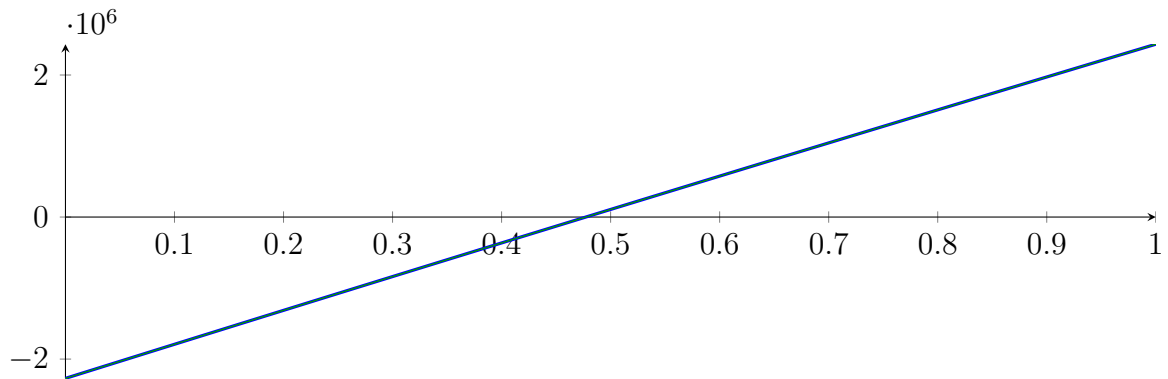
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\}$$

$$N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301 \cdot 10^{-05}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

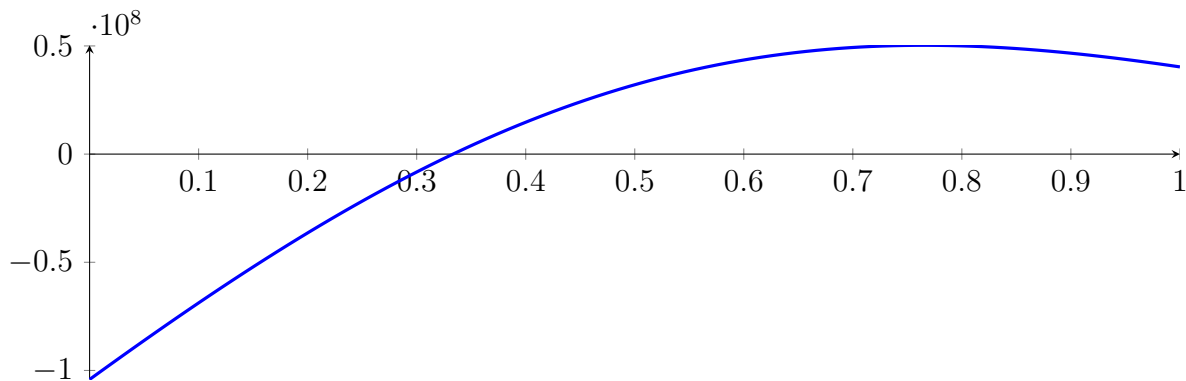
233.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

233.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

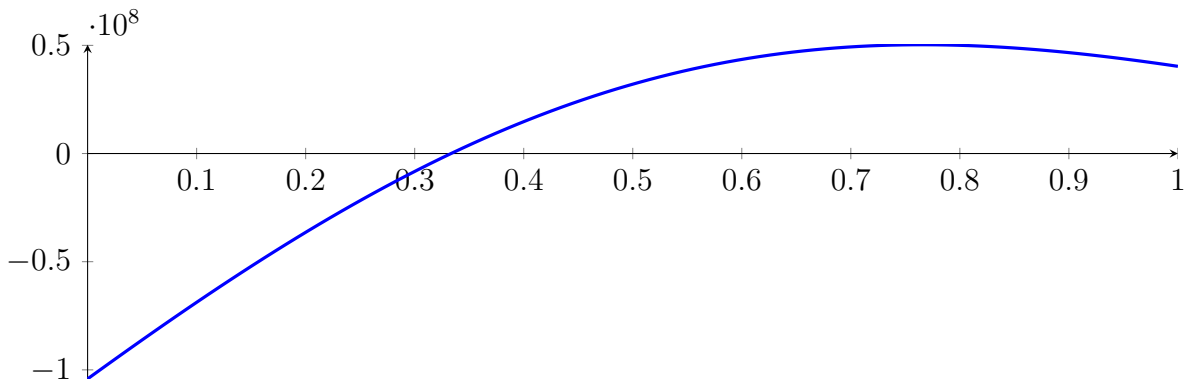
with precision $\varepsilon = 0.01$.

234 Running CubeClip on f_{16} with epsilon 2

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

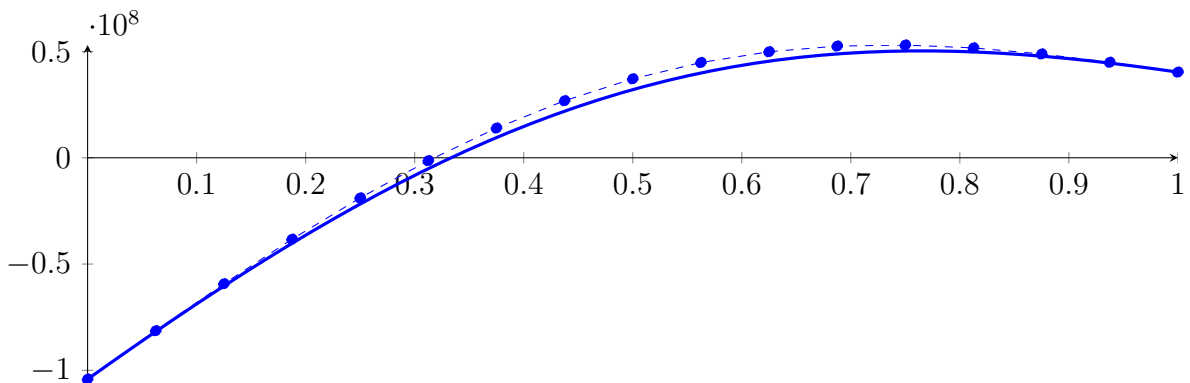
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



234.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 1.89955 \cdot 10^{-288} X^{16} - 1.4851 \cdot 10^{-287} X^{15} + 5.12896 \cdot 10^{-287} X^{14} - 1.02224 \cdot 10^{-286} X^{13}$$

$$+ 1.29027 \cdot 10^{-286} X^{12} - 1.07058 \cdot 10^{-286} X^{11} + 6.00739 \cdot 10^{-287} X^{10} - 2.54352 \cdot 10^{-287} X^9$$

$$+ 1.12089 \cdot 10^{-287} X^8 - 5.8673 \cdot 10^{-288} X^7 + 2.52637 \cdot 10^{-288} X^6 - 6.79624 \cdot 10^{-289} X^5$$

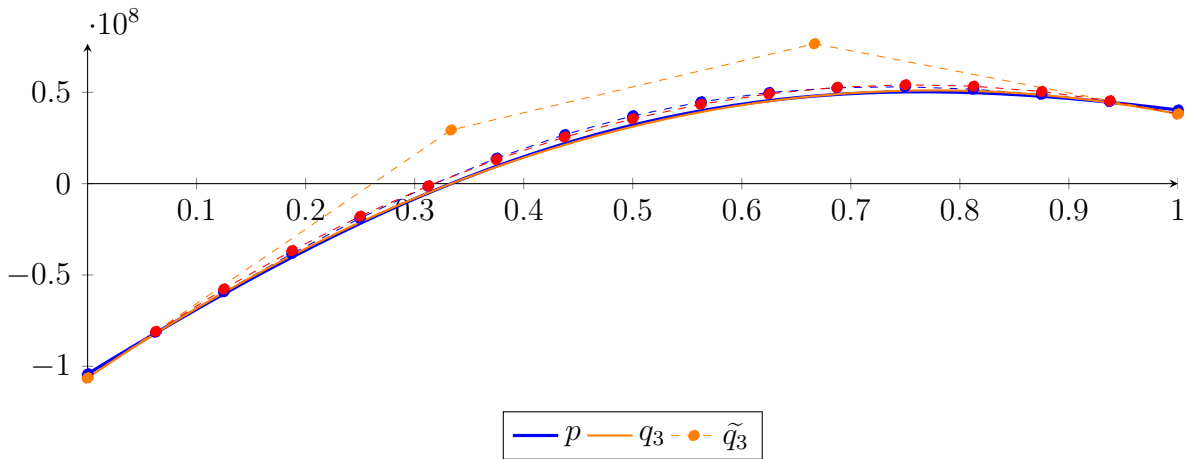
$$+ 9.36341 \cdot 10^{-290} X^4 + 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18551 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91815 \cdot 10^7 B_{10,16} + 5.27353 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

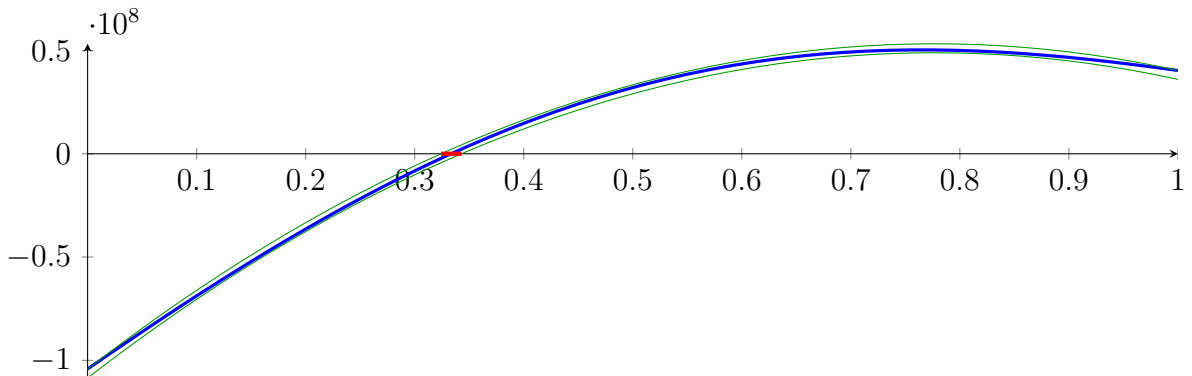
$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\}$$

$$N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

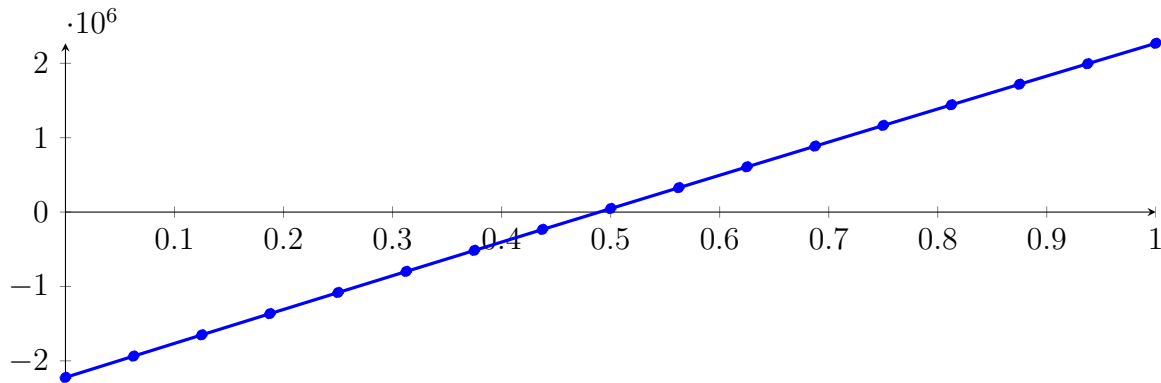
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

234.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

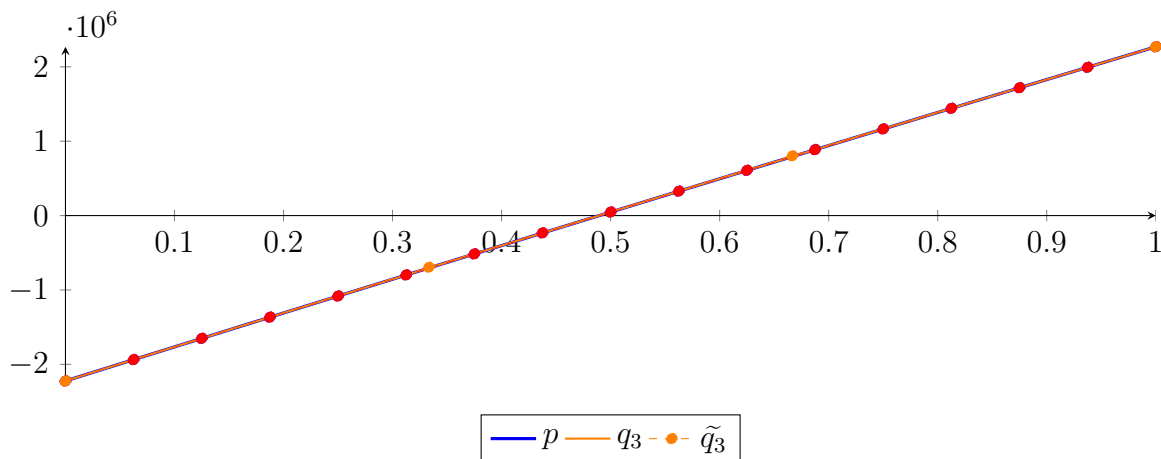
$$\begin{aligned}
 p &= -2.37433 \cdot 10^{-28} X^{16} - 5.67363 \cdot 10^{-25} X^{15} - 5.77631 \cdot 10^{-22} X^{14} - 3.1818 \cdot 10^{-19} X^{13} - 9.6142 \cdot 10^{-17} X^{12} \\
 &\quad - 1.16549 \cdot 10^{-14} X^{11} + 1.71065 \cdot 10^{-12} X^{10} + 7.20611 \cdot 10^{-10} X^9 + 4.39147 \cdot 10^{-08} X^8 - 1.21542 \cdot 10^{-05} X^7 \\
 &\quad - 0.00155624 X^6 + 0.10496 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 4.41595 \cdot 10^{-291} X^{16} - 1.48598 \cdot 10^{-290} X^{15} - 2.81102 \cdot 10^{-290} X^{14} + 2.65129 \cdot 10^{-289} X^{13} \\
 &\quad - 7.34593 \cdot 10^{-289} X^{12} + 1.17654 \cdot 10^{-288} X^{11} - 1.23119 \cdot 10^{-288} X^{10} + 8.63493 \cdot 10^{-289} X^9 \\
 &\quad - 3.92909 \cdot 10^{-289} X^8 + 1.01265 \cdot 10^{-289} X^7 - 6.13644 \cdot 10^{-291} X^6 - 3.92664 \cdot 10^{-291} X^5 \\
 &\quad + 9.04488 \cdot 10^{-292} X^4 - 700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} \\
 &\quad + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m :

$$M = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

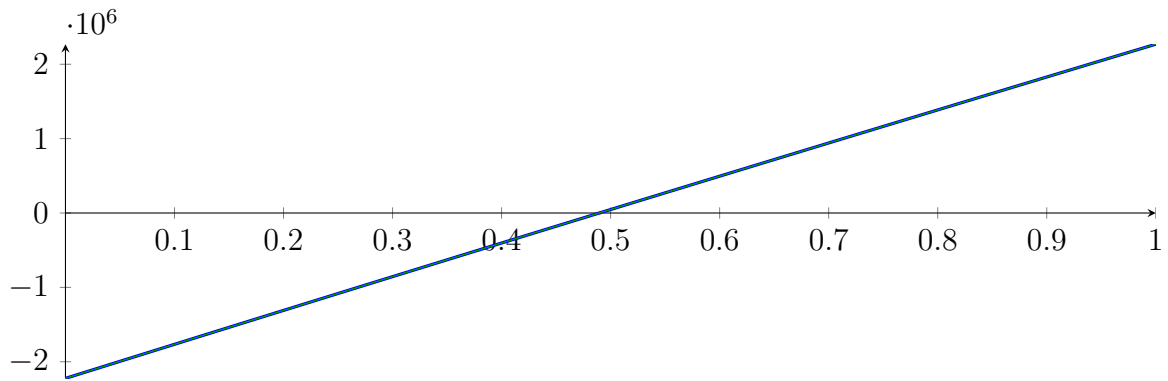
$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\}$$

$$N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

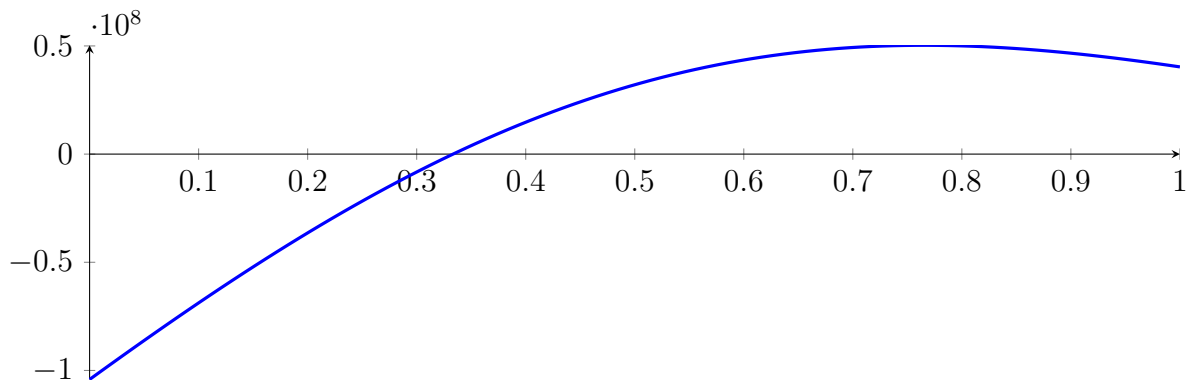
234.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

234.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

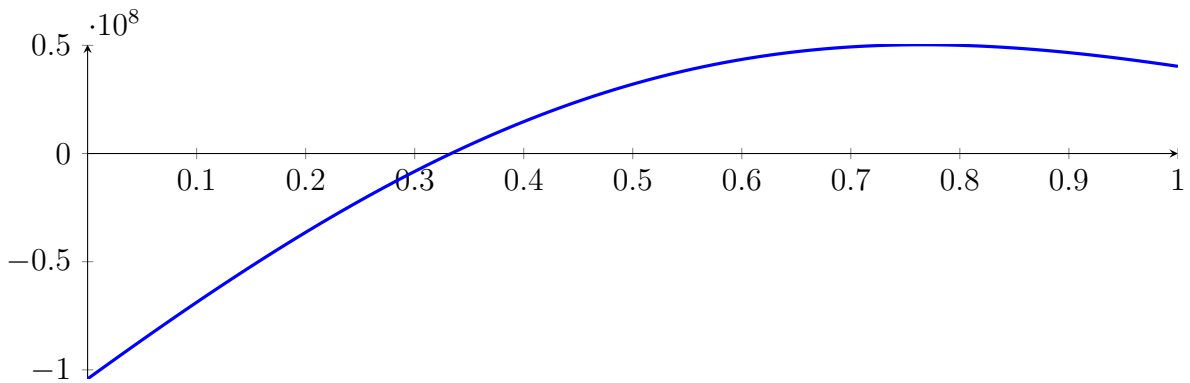
with precision $\varepsilon = 0.01$.

235 Running BezClip on f_{16} with epsilon 4

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

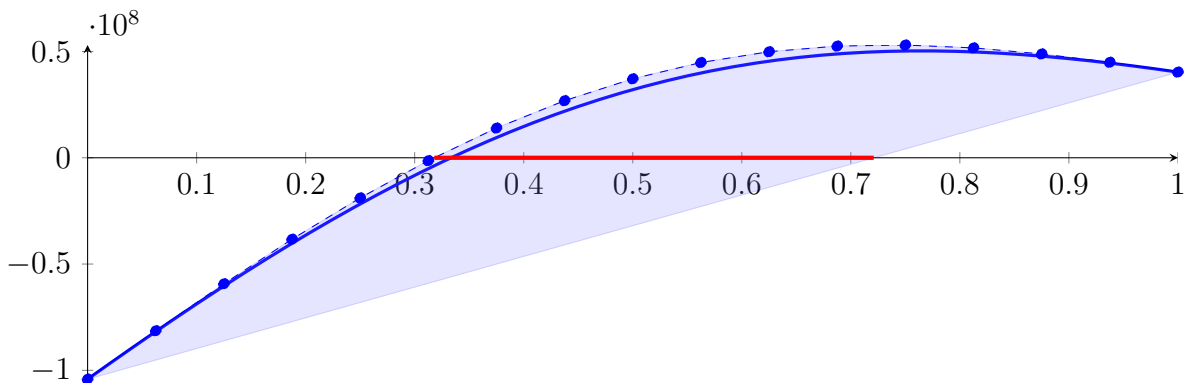
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



235.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

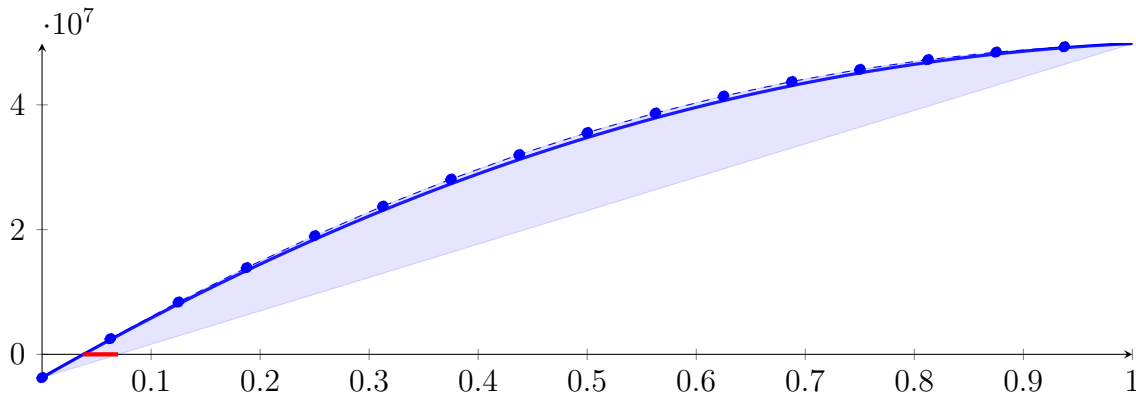
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

235.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.83858 \cdot 10^{-07} X^{16} - 5.37355 \cdot 10^{-05} X^{15} - 0.00254146 X^{14} - 0.064977 X^{13} - 0.909205 X^{12} \\ &\quad - 5.03924 X^{11} + 36.4549 X^{10} + 692.921 X^9 + 1886.97 X^8 - 25792 X^7 - 149671 X^6 + 492605 X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

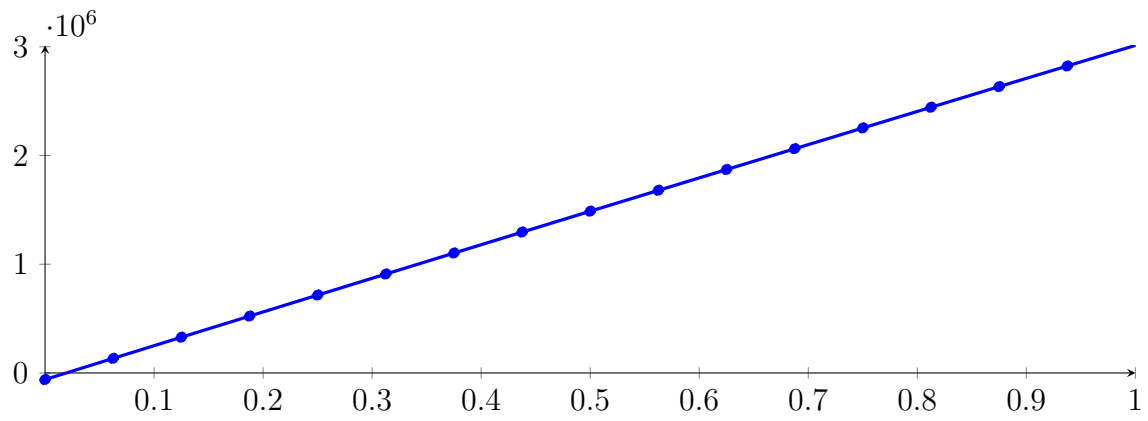
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

235.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -6.78234 \cdot 10^{-31} X^{16} - 2.34473 \cdot 10^{-27} X^{15} - 3.45581 \cdot 10^{-24} X^{14} - 2.75921 \cdot 10^{-21} X^{13} - 1.21242 \cdot 10^{-18} X^{12} \\ &\quad - 2.17572 \cdot 10^{-16} X^{11} + 4.23544 \cdot 10^{-14} X^{10} + 2.70025 \cdot 10^{-11} X^9 + 2.51306 \cdot 10^{-09} X^8 - 9.23474 \cdot 10^{-07} X^7 \\ &\quad - 0.000177459 X^6 + 0.0161038 X^5 + 4.36155 X^4 - 234.216 X^3 - 45622.2 X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554

⇒ Selective recursion: interval 1: $[0.333333, 0.333337]$,

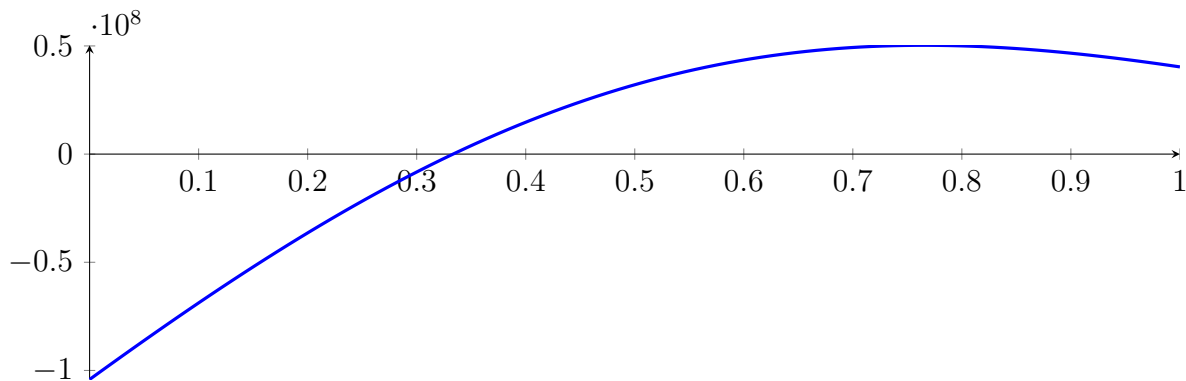
235.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Found root in interval $[0.333333, 0.333337]$ at recursion depth 4!

235.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333337]$$

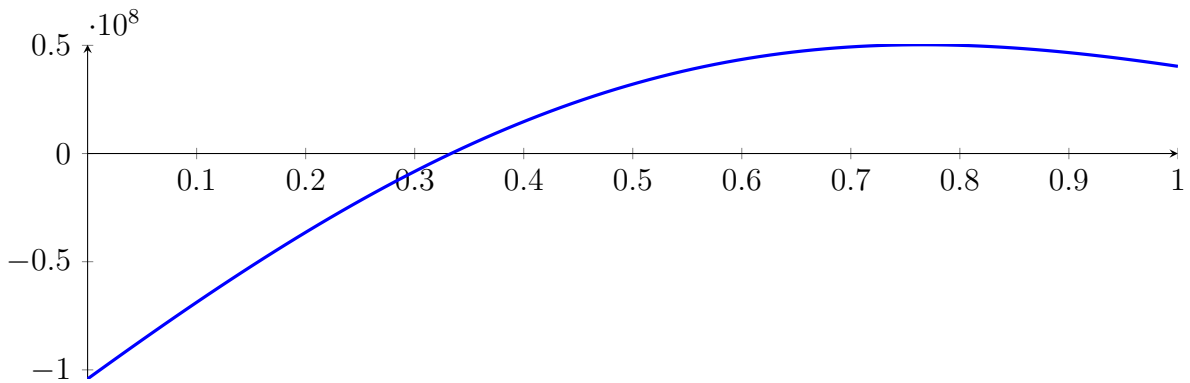
with precision $\varepsilon = 0.0001$.

236 Running QuadClip on f_{16} with epsilon 4

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

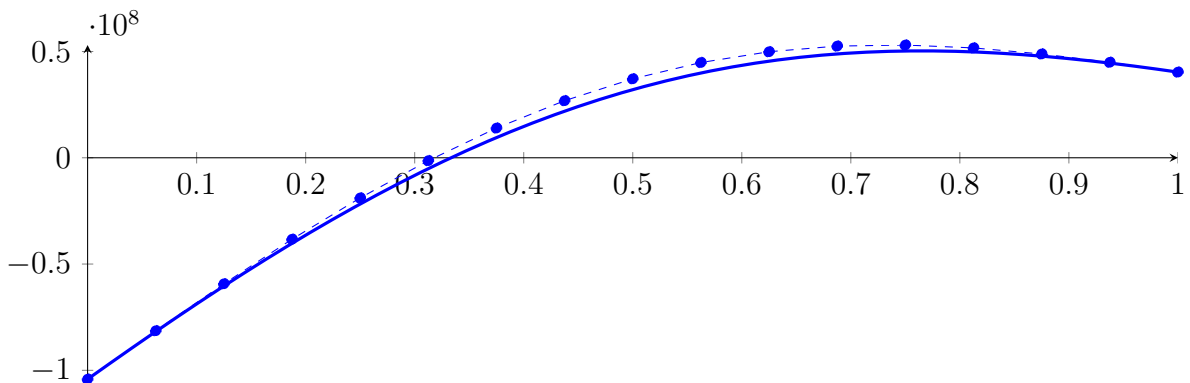
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



236.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = 3.74473 \cdot 10^{-288} X^{16} - 3.08889 \cdot 10^{-287} X^{15} + 1.17443 \cdot 10^{-286} X^{14} - 2.71603 \cdot 10^{-286} X^{13}$$

$$+ 4.23554 \cdot 10^{-286} X^{12} - 4.66107 \cdot 10^{-286} X^{11} + 3.6845 \cdot 10^{-286} X^{10} - 2.09811 \cdot 10^{-286} X^9$$

$$+ 8.58708 \cdot 10^{-287} X^8 - 2.54063 \cdot 10^{-287} X^7 + 5.57852 \cdot 10^{-288} X^6 - 9.21534 \cdot 10^{-289} X^5$$

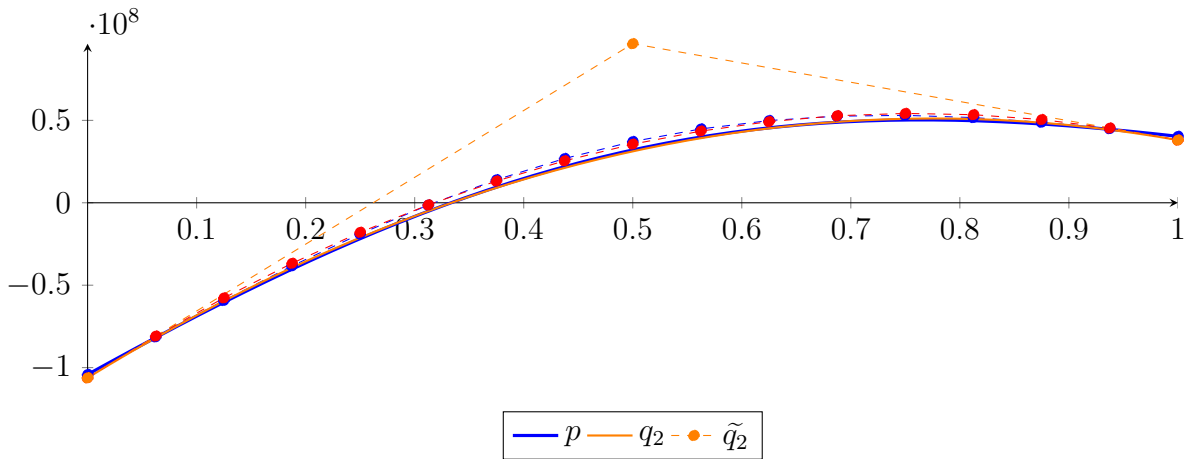
$$+ 1.02122 \cdot 10^{-289} X^4 - 5.95624 \cdot 10^{-291} X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

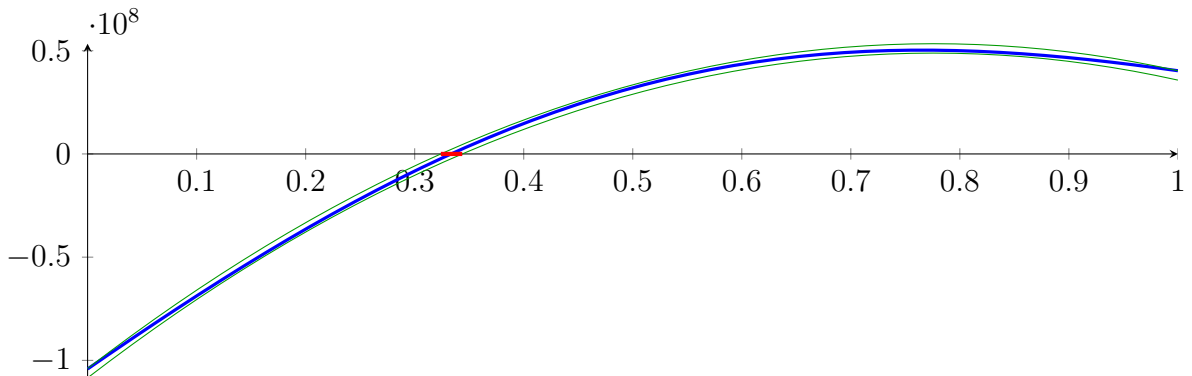
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

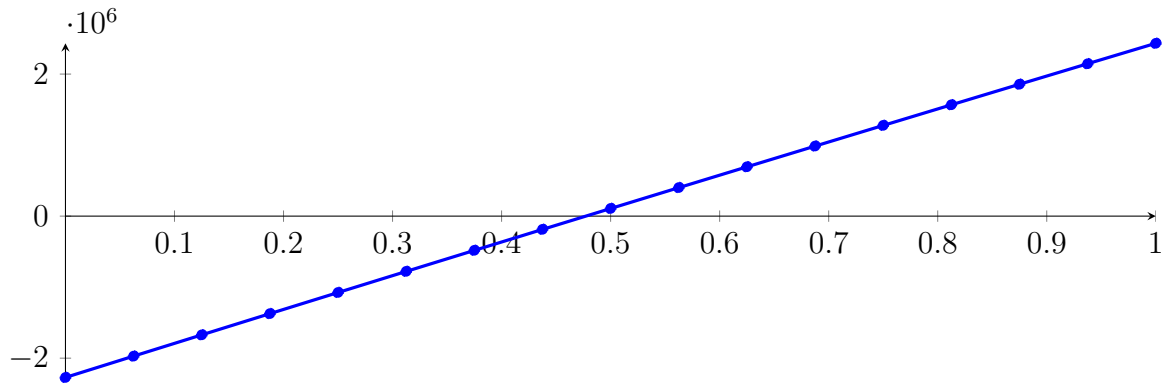
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

236.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

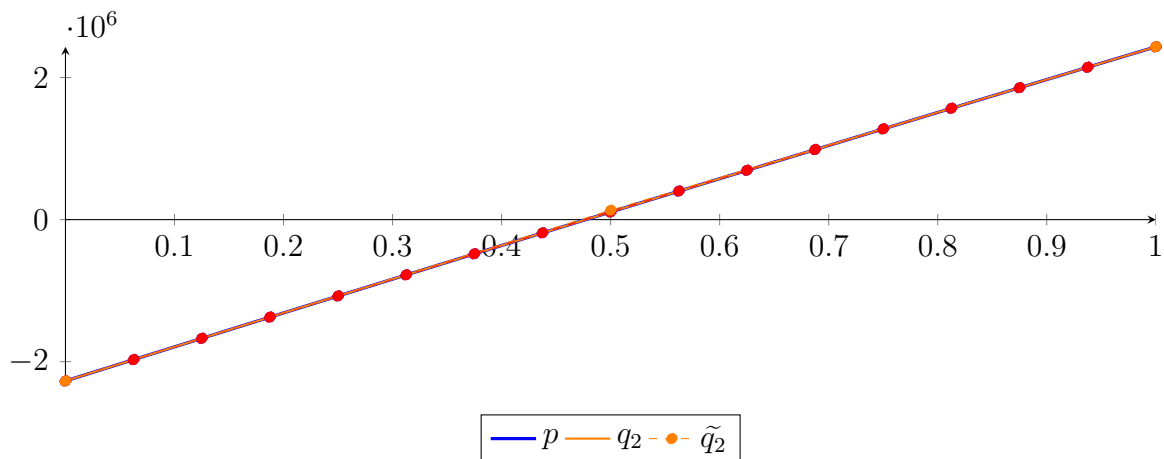
$$\begin{aligned}
 p &= -5.0162 \cdot 10^{-28} X^{16} - 1.14383 \cdot 10^{-24} X^{15} - 1.11125 \cdot 10^{-21} X^{14} - 5.84096 \cdot 10^{-19} X^{13} - 1.684 \cdot 10^{-16} X^{12} \\
 &\quad - 1.94707 \cdot 10^{-14} X^{11} + 2.73227 \cdot 10^{-12} X^{10} + 1.09727 \cdot 10^{-9} X^9 + 6.37314 \cdot 10^{-8} X^8 - 1.68645 \cdot 10^{-5} X^7 \\
 &\quad - 0.00205892 X^6 + 0.132721 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -8.18461 \cdot 10^{-291} X^{16} + 4.13745 \cdot 10^{-290} X^{15} - 5.8254 \cdot 10^{-290} X^{14} - 6.80919 \cdot 10^{-290} X^{13} \\
 &\quad + 3.48059 \cdot 10^{-289} X^{12} - 5.47245 \cdot 10^{-289} X^{11} + 4.8261 \cdot 10^{-289} X^{10} - 2.67867 \cdot 10^{-289} X^9 \\
 &\quad + 1.00314 \cdot 10^{-289} X^8 - 2.84766 \cdot 10^{-290} X^7 + 6.85072 \cdot 10^{-291} X^6 - 1.12501 \cdot 10^{-291} X^5 \\
 &\quad + 1.24731 \cdot 10^{-293} X^4 + 2.25944 \cdot 10^{-293} X^3 - 104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} \\
 &\quad + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

$$M = -104265X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

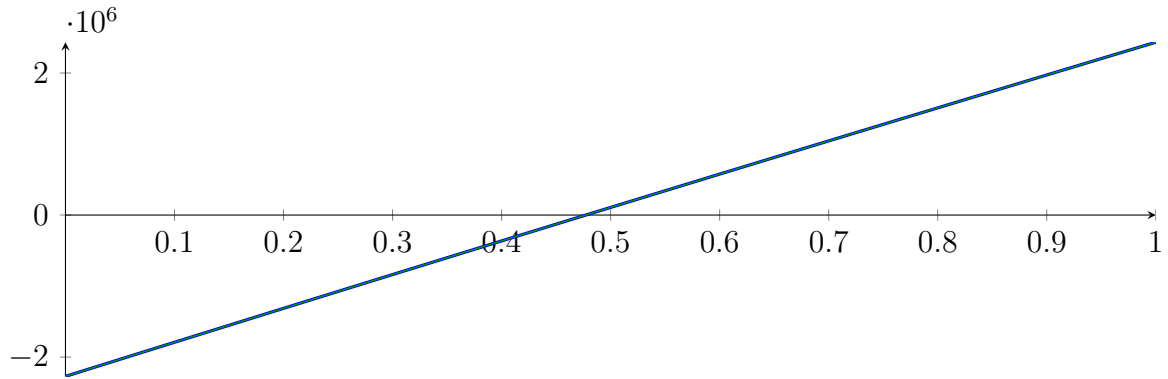
$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\}$$

$$N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301 \cdot 10^{-05}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

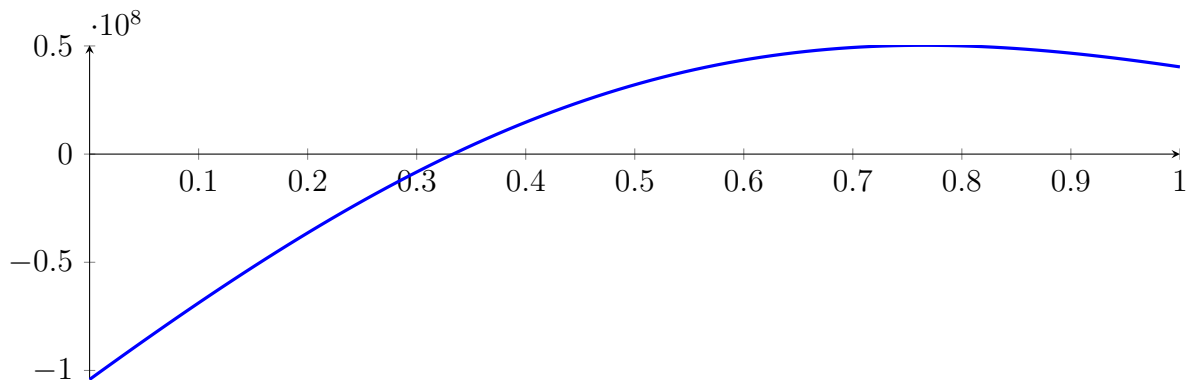
236.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

236.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

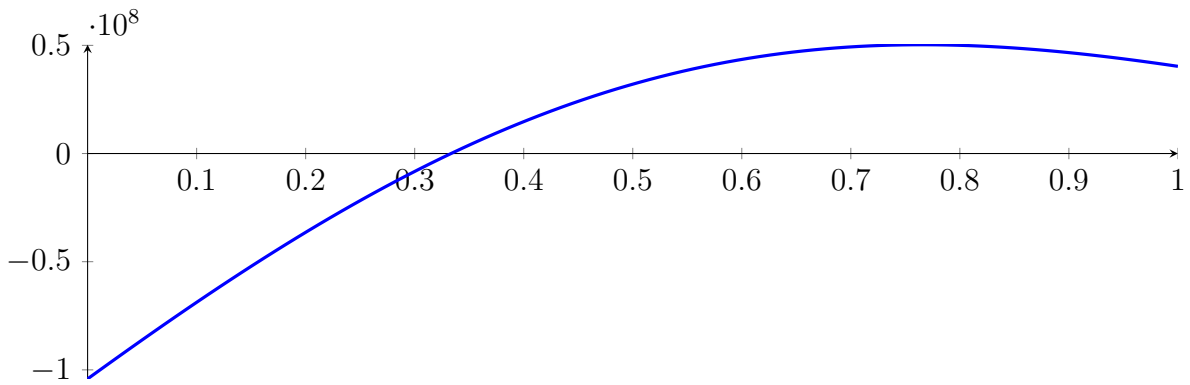
with precision $\varepsilon = 0.0001$.

237 Running CubeClip on f_{16} with epsilon 4

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

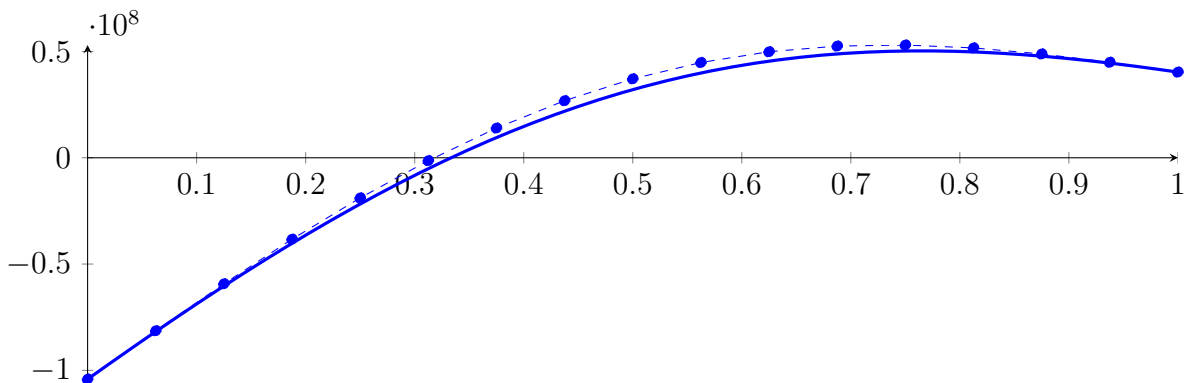
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



237.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 1.89955 \cdot 10^{-288} X^{16} - 1.4851 \cdot 10^{-287} X^{15} + 5.12896 \cdot 10^{-287} X^{14} - 1.02224 \cdot 10^{-286} X^{13}$$

$$+ 1.29027 \cdot 10^{-286} X^{12} - 1.07058 \cdot 10^{-286} X^{11} + 6.00739 \cdot 10^{-287} X^{10} - 2.54352 \cdot 10^{-287} X^9$$

$$+ 1.12089 \cdot 10^{-287} X^8 - 5.8673 \cdot 10^{-288} X^7 + 2.52637 \cdot 10^{-288} X^6 - 6.79624 \cdot 10^{-289} X^5$$

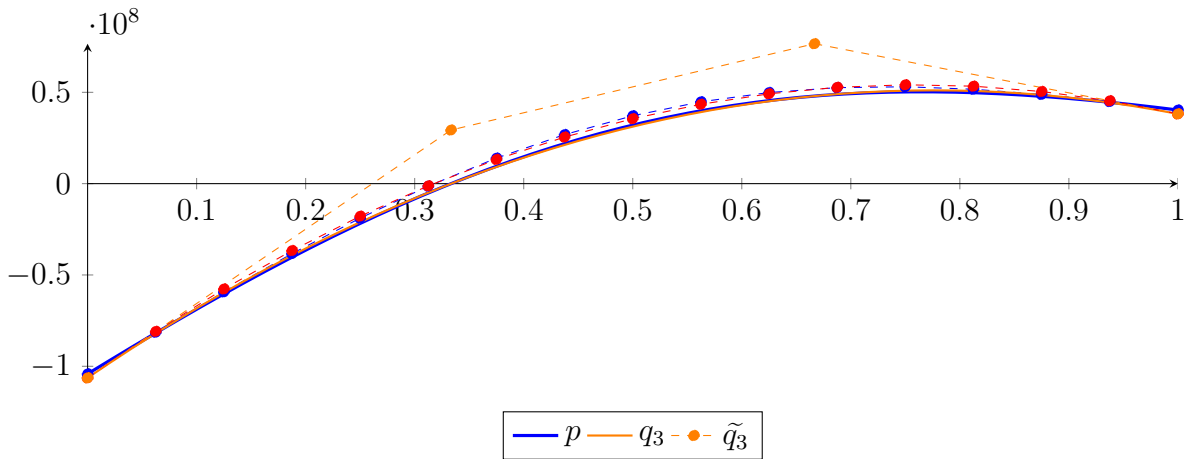
$$+ 9.36341 \cdot 10^{-290} X^4 + 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18551 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91815 \cdot 10^7 B_{10,16} + 5.27353 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

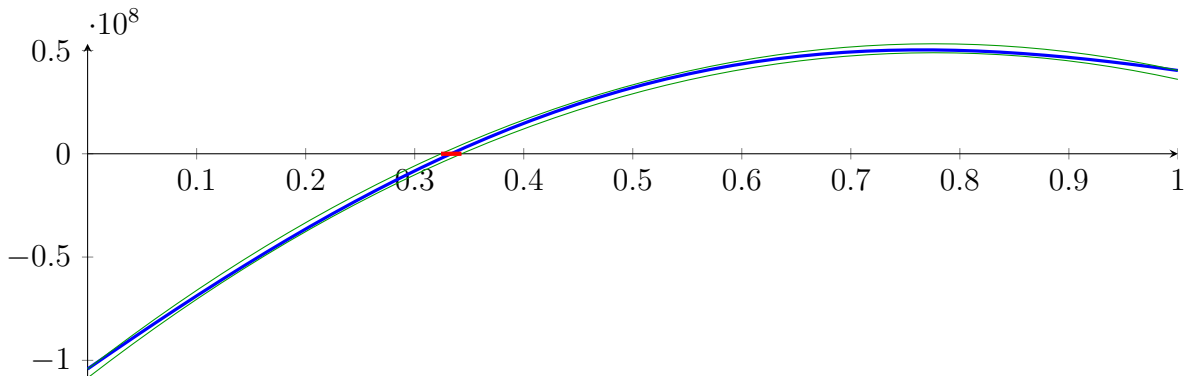
$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\}$$

$$N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

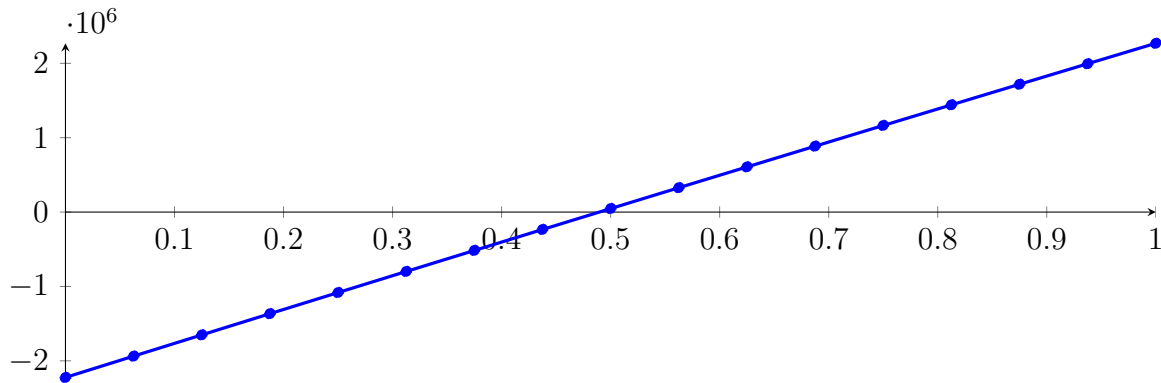
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

237.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

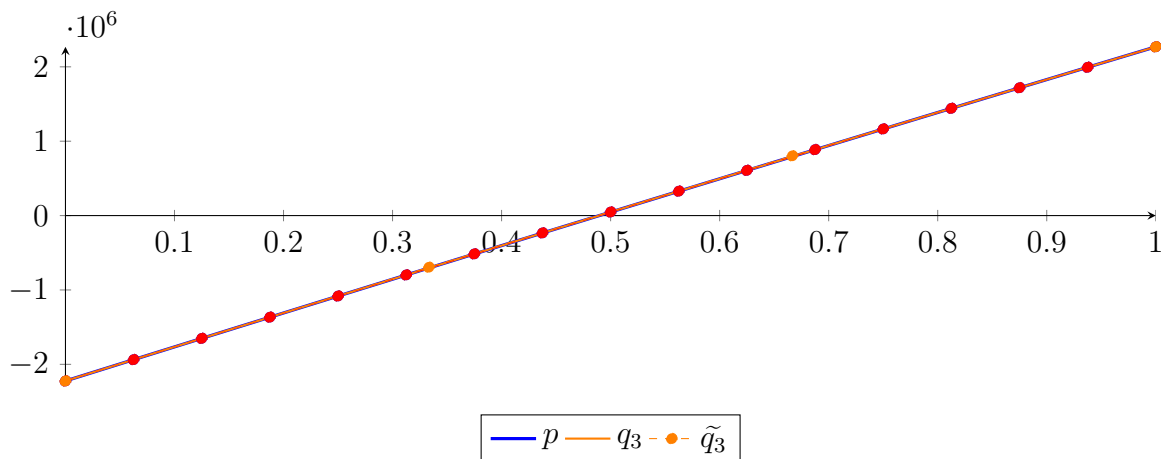
$$\begin{aligned}
 p &= -2.37433 \cdot 10^{-28} X^{16} - 5.67363 \cdot 10^{-25} X^{15} - 5.77631 \cdot 10^{-22} X^{14} - 3.1818 \cdot 10^{-19} X^{13} - 9.6142 \cdot 10^{-17} X^{12} \\
 &\quad - 1.16549 \cdot 10^{-14} X^{11} + 1.71065 \cdot 10^{-12} X^{10} + 7.20611 \cdot 10^{-10} X^9 + 4.39147 \cdot 10^{-08} X^8 - 1.21542 \cdot 10^{-05} X^7 \\
 &\quad - 0.00155624 X^6 + 0.10496 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 4.41595 \cdot 10^{-291} X^{16} - 1.48598 \cdot 10^{-290} X^{15} - 2.81102 \cdot 10^{-290} X^{14} + 2.65129 \cdot 10^{-289} X^{13} \\
 &\quad - 7.34593 \cdot 10^{-289} X^{12} + 1.17654 \cdot 10^{-288} X^{11} - 1.23119 \cdot 10^{-288} X^{10} + 8.63493 \cdot 10^{-289} X^9 \\
 &\quad - 3.92909 \cdot 10^{-289} X^8 + 1.01265 \cdot 10^{-289} X^7 - 6.13644 \cdot 10^{-291} X^6 - 3.92664 \cdot 10^{-291} X^5 \\
 &\quad + 9.04488 \cdot 10^{-292} X^4 - 700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} \\
 &\quad + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m :

$$M = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

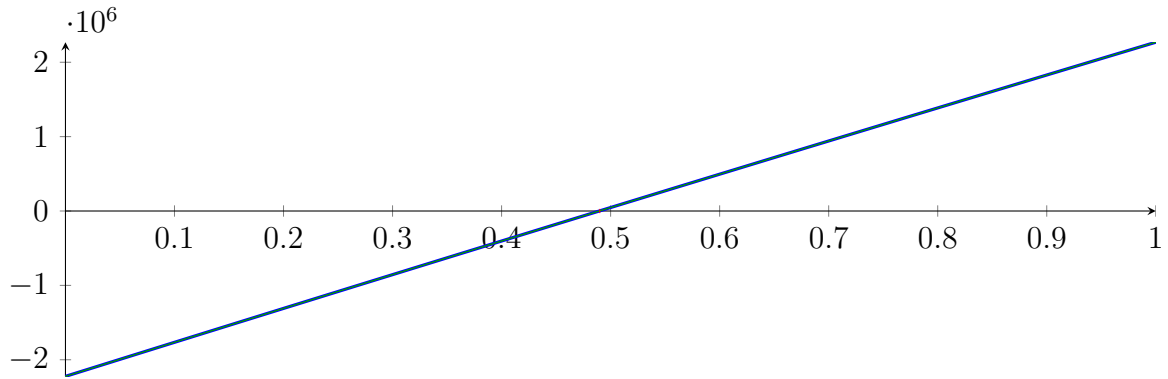
$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\}$$

$$N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174 \cdot 10^{-07}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

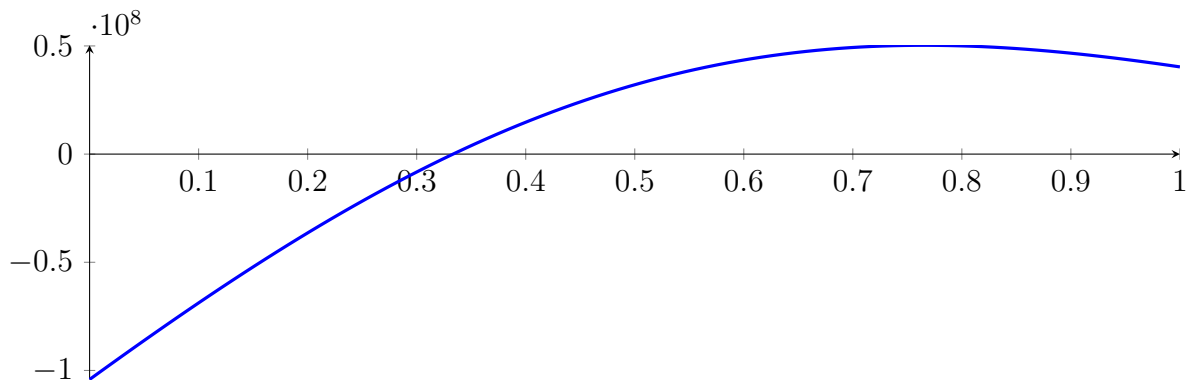
237.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

237.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

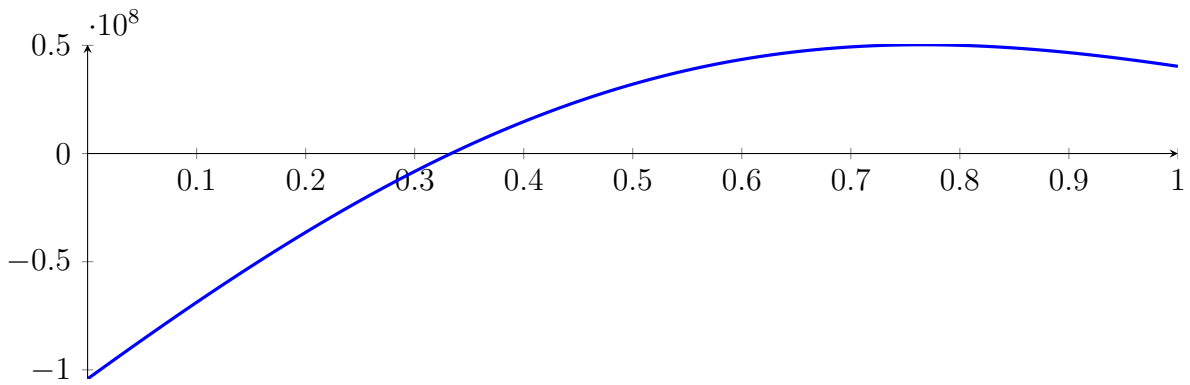
with precision $\varepsilon = 0.0001$.

238 Running BezClip on f_{16} with epsilon 8

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

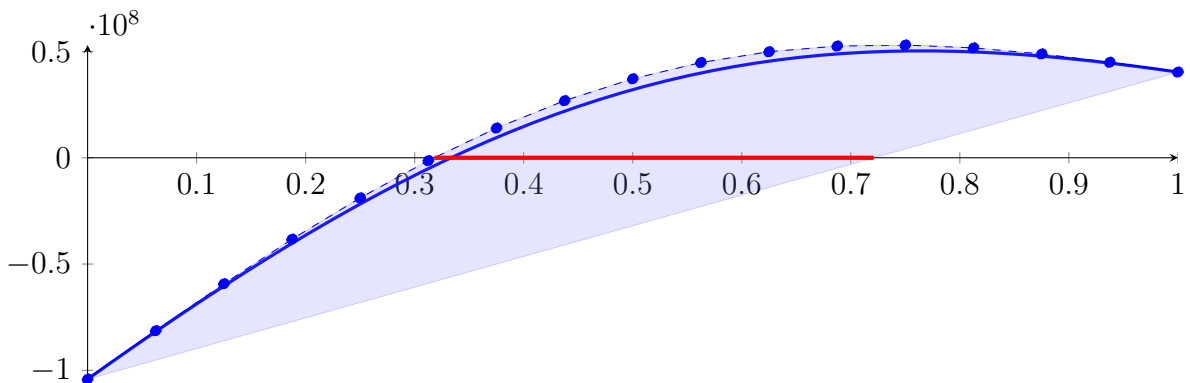
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



238.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

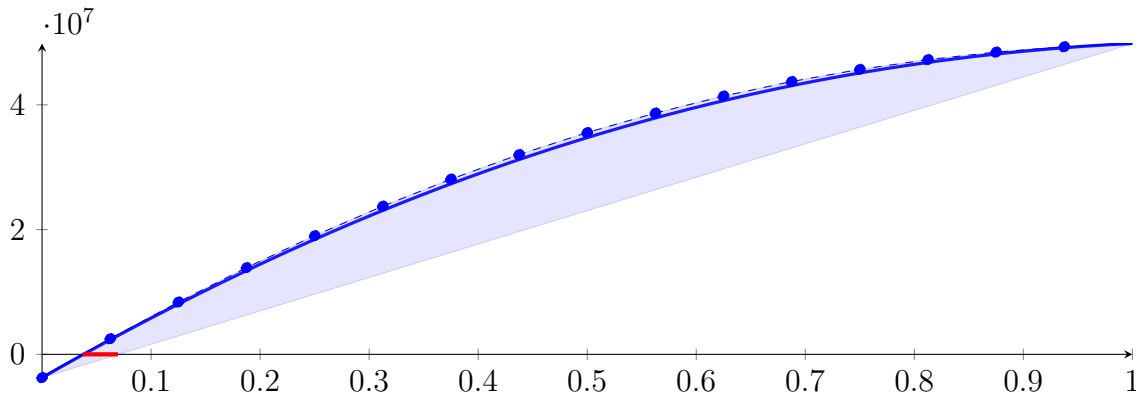
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

238.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.83858 \cdot 10^{-07} X^{16} - 5.37355 \cdot 10^{-05} X^{15} - 0.00254146 X^{14} - 0.064977 X^{13} - 0.909205 X^{12} \\ &\quad - 5.03924 X^{11} + 36.4549 X^{10} + 692.921 X^9 + 1886.97 X^8 - 25792 X^7 - 149671 X^6 + 492605 X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

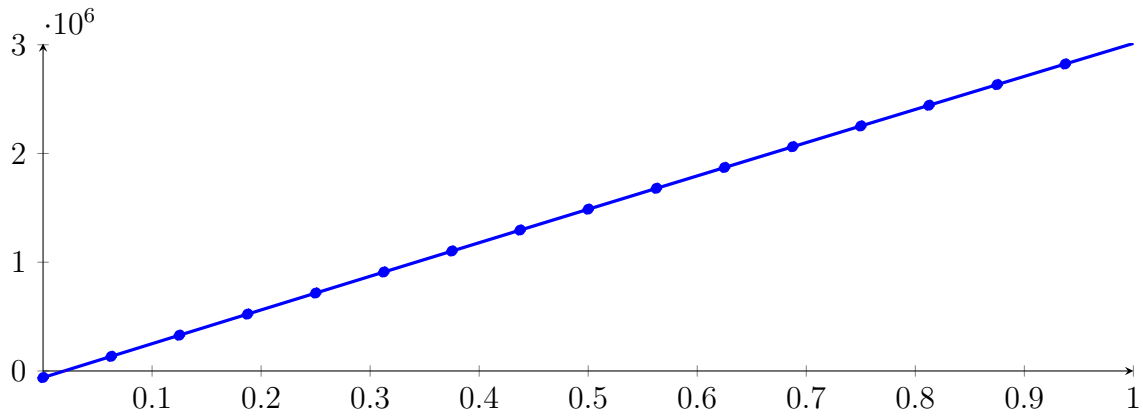
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

238.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -6.78234 \cdot 10^{-31} X^{16} - 2.34473 \cdot 10^{-27} X^{15} - 3.45581 \cdot 10^{-24} X^{14} - 2.75921 \cdot 10^{-21} X^{13} - 1.21242 \cdot 10^{-18} X^{12} \\ &\quad - 2.17572 \cdot 10^{-16} X^{11} + 4.23544 \cdot 10^{-14} X^{10} + 2.70025 \cdot 10^{-11} X^9 + 2.51306 \cdot 10^{-09} X^8 - 9.23474 \cdot 10^{-07} X^7 \\ &\quad - 0.000177459 X^6 + 0.0161038 X^5 + 4.36155 X^4 - 234.216 X^3 - 45622.2 X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

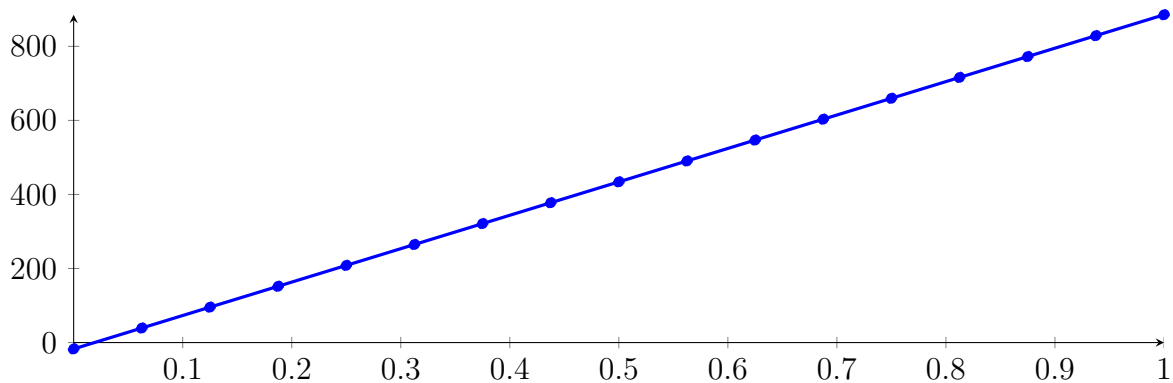
Longest intersection interval: 0.000289554

⇒ Selective recursion: interval 1: $[0.333333, 0.333337]$,

238.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.65599 \cdot 10^{-87} X^{16} - 1.97733 \cdot 10^{-80} X^{15} - 1.00659 \cdot 10^{-73} X^{14} - 2.77601 \cdot 10^{-67} X^{13} \\
 &\quad - 4.21367 \cdot 10^{-61} X^{12} - 2.61333 \cdot 10^{-55} X^{11} + 1.75275 \cdot 10^{-49} X^{10} + 3.8646 \cdot 10^{-43} X^9 \\
 &\quad + 1.2441 \cdot 10^{-37} X^8 - 1.57525 \cdot 10^{-31} X^7 - 1.04661 \cdot 10^{-25} X^6 + 3.27355 \cdot 10^{-20} X^5 \\
 &\quad + 3.06701 \cdot 10^{-14} X^4 - 5.67777 \cdot 10^{-09} X^3 - 0.00382618 X^2 + 902.448 X - 17.178 \\
 &= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
 &\quad + 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
 &\quad + 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
 &\quad + 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045 \cdot 10^{-08}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

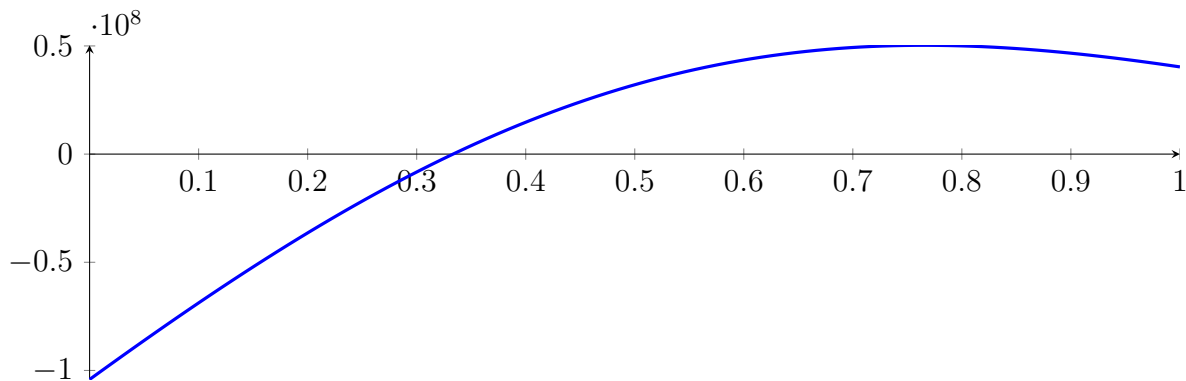
238.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

238.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

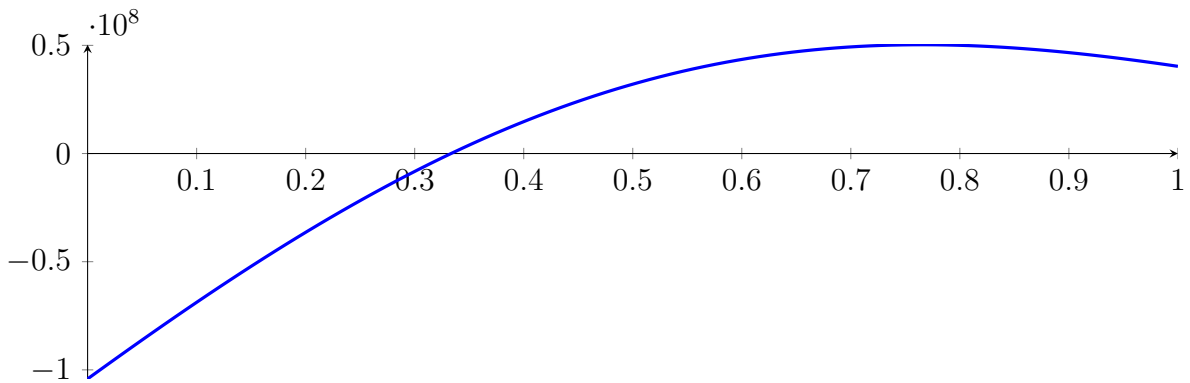
with precision $\varepsilon = 1 \cdot 10^{-08}$.

239 Running QuadClip on f_{16} with epsilon 8

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

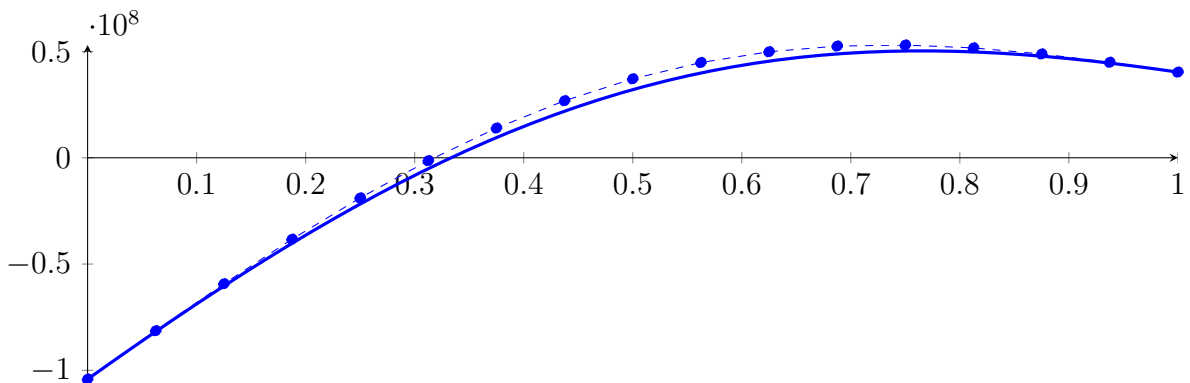
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



239.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = 3.74473 \cdot 10^{-288} X^{16} - 3.08889 \cdot 10^{-287} X^{15} + 1.17443 \cdot 10^{-286} X^{14} - 2.71603 \cdot 10^{-286} X^{13}$$

$$+ 4.23554 \cdot 10^{-286} X^{12} - 4.66107 \cdot 10^{-286} X^{11} + 3.6845 \cdot 10^{-286} X^{10} - 2.09811 \cdot 10^{-286} X^9$$

$$+ 8.58708 \cdot 10^{-287} X^8 - 2.54063 \cdot 10^{-287} X^7 + 5.57852 \cdot 10^{-288} X^6 - 9.21534 \cdot 10^{-289} X^5$$

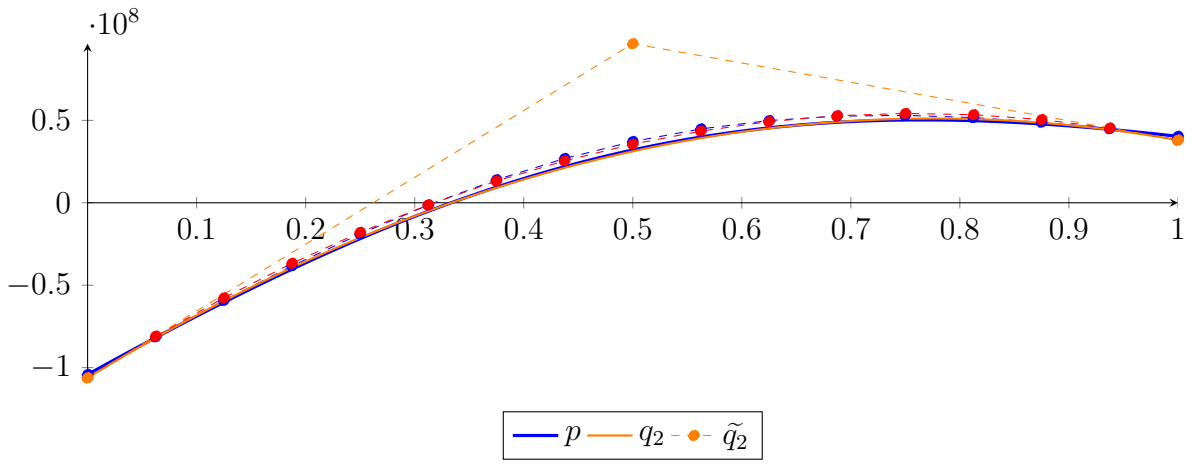
$$+ 1.02122 \cdot 10^{-289} X^4 - 5.95624 \cdot 10^{-291} X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

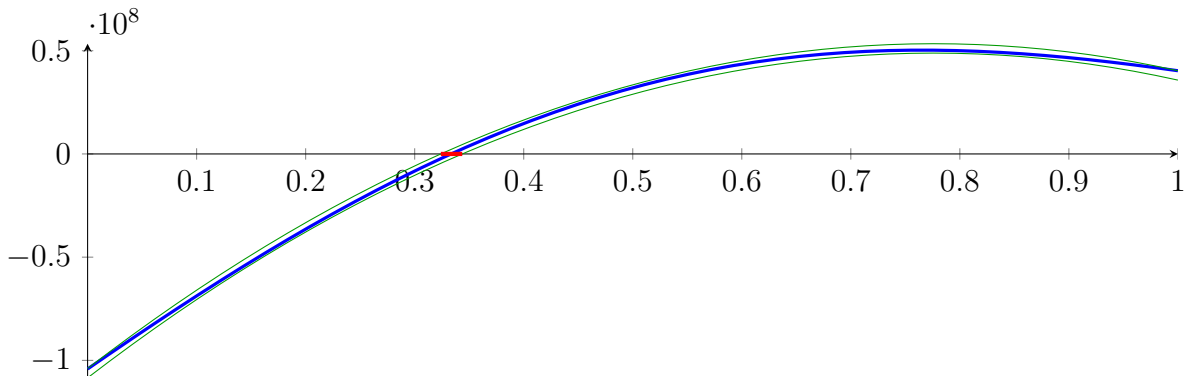
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

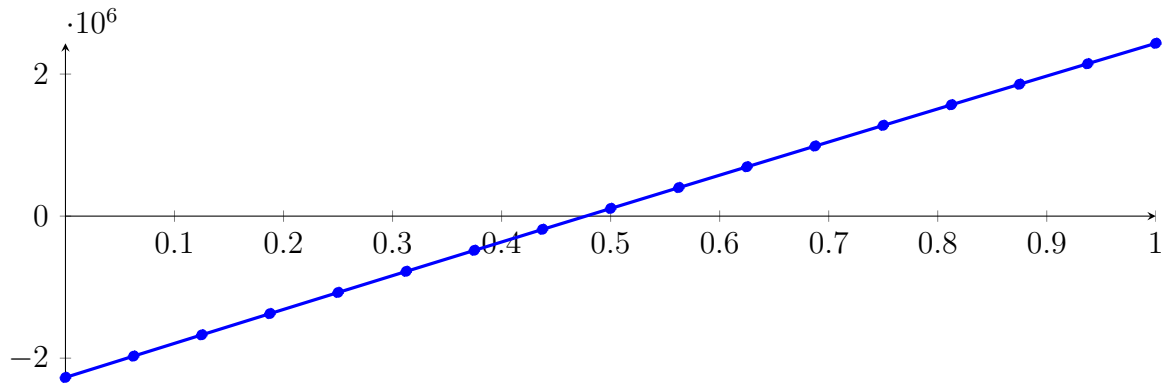
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

239.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

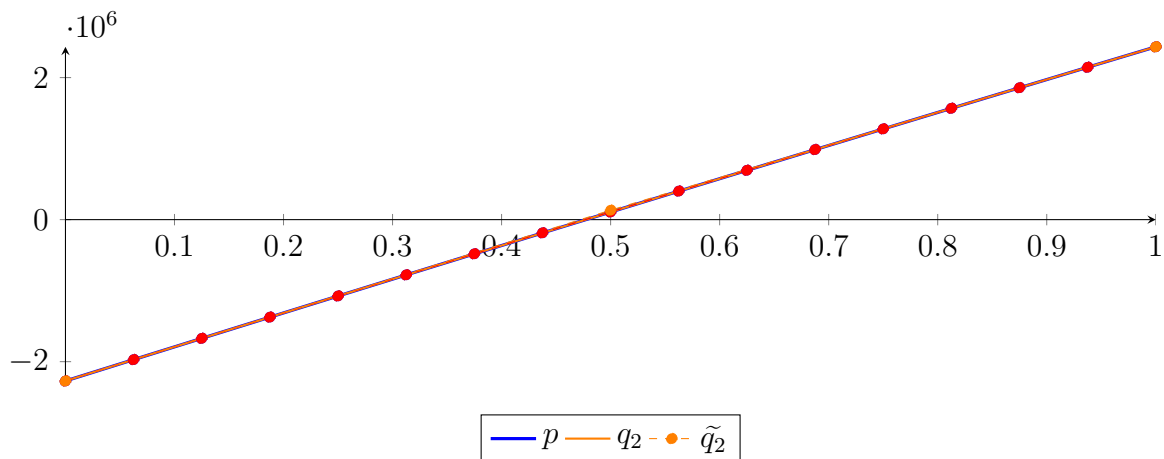
$$\begin{aligned}
 p &= -5.0162 \cdot 10^{-28} X^{16} - 1.14383 \cdot 10^{-24} X^{15} - 1.11125 \cdot 10^{-21} X^{14} - 5.84096 \cdot 10^{-19} X^{13} - 1.684 \cdot 10^{-16} X^{12} \\
 &\quad - 1.94707 \cdot 10^{-14} X^{11} + 2.73227 \cdot 10^{-12} X^{10} + 1.09727 \cdot 10^{-9} X^9 + 6.37314 \cdot 10^{-8} X^8 - 1.68645 \cdot 10^{-5} X^7 \\
 &\quad - 0.00205892 X^6 + 0.132721 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -8.18461 \cdot 10^{-291} X^{16} + 4.13745 \cdot 10^{-290} X^{15} - 5.8254 \cdot 10^{-290} X^{14} - 6.80919 \cdot 10^{-290} X^{13} \\
 &\quad + 3.48059 \cdot 10^{-289} X^{12} - 5.47245 \cdot 10^{-289} X^{11} + 4.8261 \cdot 10^{-289} X^{10} - 2.67867 \cdot 10^{-289} X^9 \\
 &\quad + 1.00314 \cdot 10^{-289} X^8 - 2.84766 \cdot 10^{-290} X^7 + 6.85072 \cdot 10^{-291} X^6 - 1.12501 \cdot 10^{-291} X^5 \\
 &\quad + 1.24731 \cdot 10^{-293} X^4 + 2.25944 \cdot 10^{-293} X^3 - 104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} \\
 &\quad + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

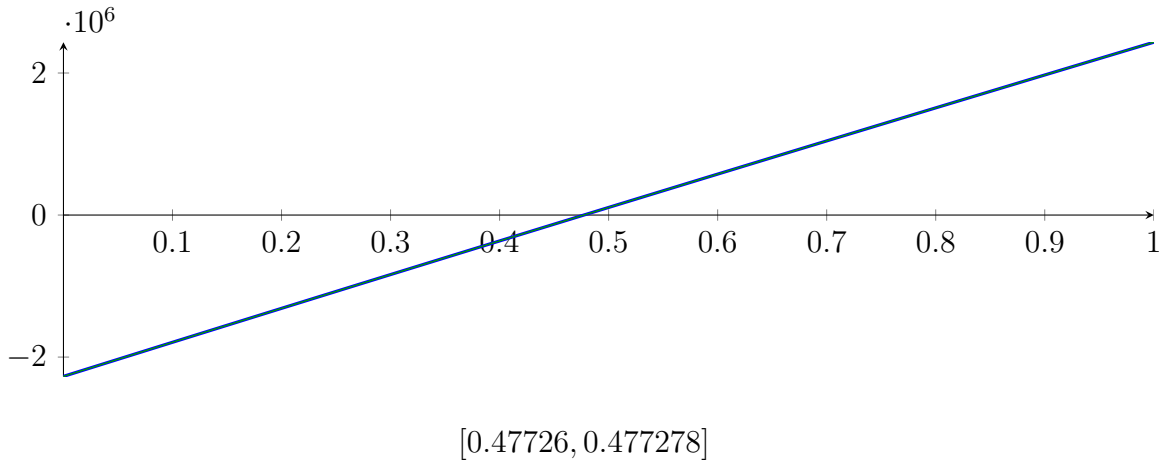
$$M = -104265X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:

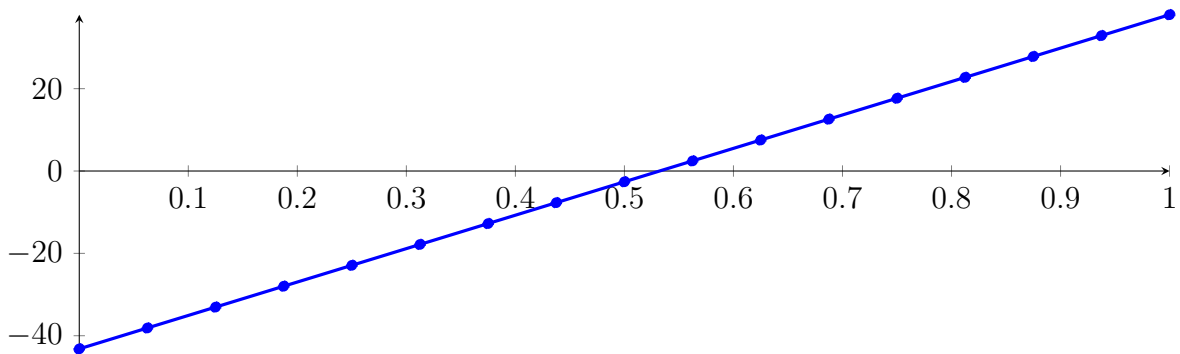


Longest intersection interval: $1.72301 \cdot 10^{-05}$
 \implies Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

239.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.02667 \cdot 10^{-104} X^{16} - 4.019 \cdot 10^{-96} X^{15} - 2.27522 \cdot 10^{-88} X^{14} - 6.97783 \cdot 10^{-81} X^{13} \\
 &\quad - 1.17785 \cdot 10^{-73} X^{12} - 8.12373 \cdot 10^{-67} X^{11} + 6.05916 \cdot 10^{-60} X^{10} + 1.48569 \cdot 10^{-52} X^9 \\
 &\quad + 5.31875 \cdot 10^{-46} X^8 - 7.48919 \cdot 10^{-39} X^7 - 5.53349 \cdot 10^{-32} X^6 + 1.92471 \cdot 10^{-25} X^5 \\
 &\quad + 2.00536 \cdot 10^{-18} X^4 - 4.12844 \cdot 10^{-12} X^3 - 3.09388 \cdot 10^{-05} X^2 + 81.1506X - 43.1911 \\
 &= -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\
 &\quad - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X) \\
 &\quad + 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X) \\
 &\quad + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

$$= -43.1911 B_{0,2} - 2.61586 B_{1,2} + 37.9594 B_{2,2}$$

$$\tilde{q}_2 = -1.88281 \cdot 10^{-295} X^{16} + 1.09893 \cdot 10^{-294} X^{15} - 2.26419 \cdot 10^{-294} X^{14} + 8.08223 \cdot 10^{-295} X^{13}$$

$$+ 4.92626 \cdot 10^{-294} X^{12} - 1.07605 \cdot 10^{-293} X^{11} + 1.11858 \cdot 10^{-293} X^{10} - 6.87288 \cdot 10^{-294} X^9$$

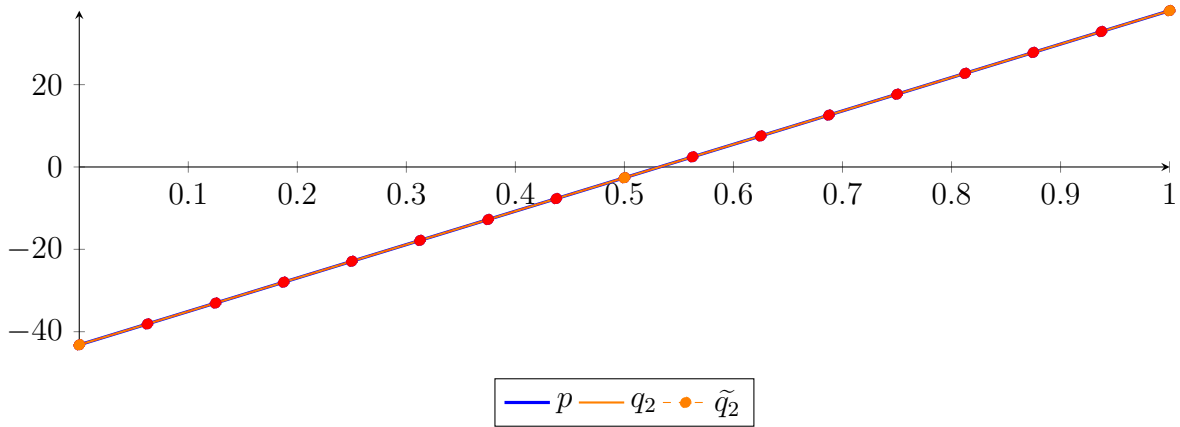
$$+ 2.54873 \cdot 10^{-294} X^8 - 5.2305 \cdot 10^{-295} X^7 + 3.18923 \cdot 10^{-296} X^6 + 1.34092 \cdot 10^{-296} X^5$$

$$- 4.89549 \cdot 10^{-297} X^4 + 5.89947 \cdot 10^{-298} X^3 - 3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

$$= -43.1911 B_{0,16} - 38.1192 B_{1,16} - 33.0473 B_{2,16} - 27.9754 B_{3,16} - 22.9035 B_{4,16} - 17.8316 B_{5,16}$$

$$- 12.7597 B_{6,16} - 7.68778 B_{7,16} - 2.61587 B_{8,16} + 2.45604 B_{9,16} + 7.52795 B_{10,16} + 12.5999 B_{11,16}$$

$$+ 17.6718 B_{12,16} + 22.7437 B_{13,16} + 27.8156 B_{14,16} + 32.8875 B_{15,16} + 37.9594 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.06422 \cdot 10^{-13}$.

Bounding polynomials M and m :

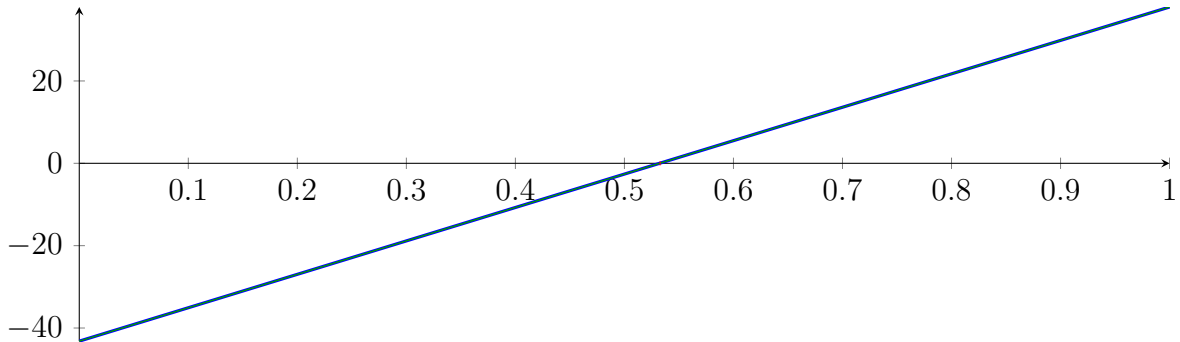
$$M = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

$$m = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

Root of M and m :

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\} \quad N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

Longest intersection interval: $5.08738 \cdot 10^{-15}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

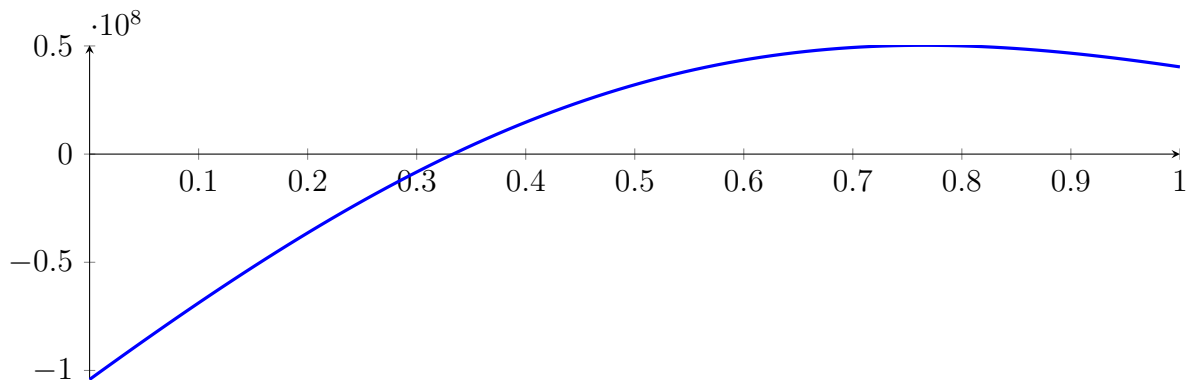
239.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

239.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

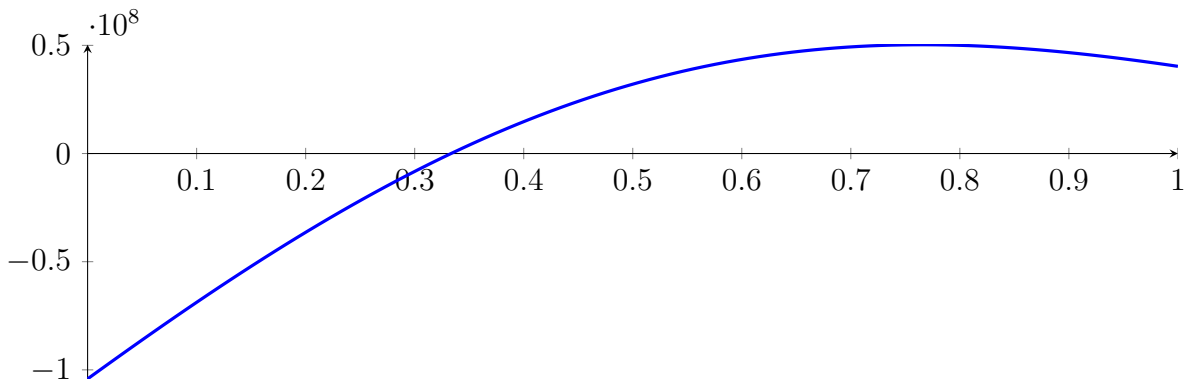
with precision $\varepsilon = 1 \cdot 10^{-08}$.

240 Running CubeClip on f_{16} with epsilon 8

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

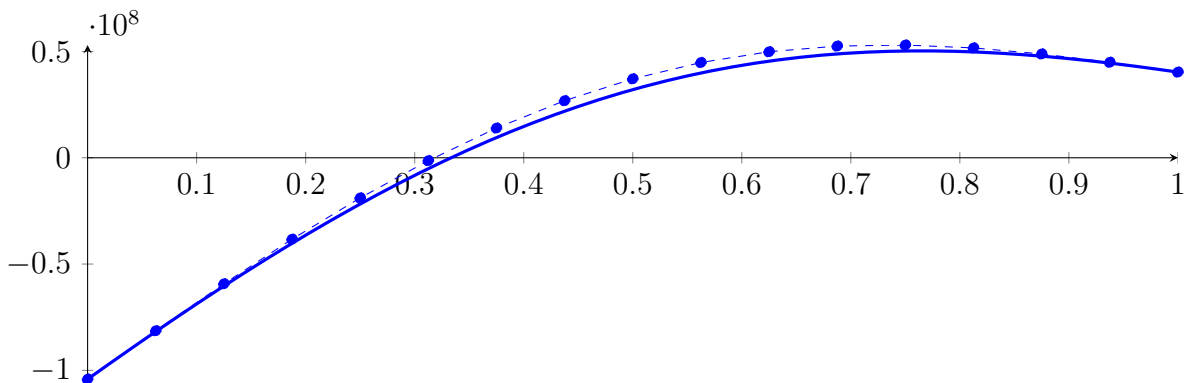
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



240.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 1.89955 \cdot 10^{-288} X^{16} - 1.4851 \cdot 10^{-287} X^{15} + 5.12896 \cdot 10^{-287} X^{14} - 1.02224 \cdot 10^{-286} X^{13}$$

$$+ 1.29027 \cdot 10^{-286} X^{12} - 1.07058 \cdot 10^{-286} X^{11} + 6.00739 \cdot 10^{-287} X^{10} - 2.54352 \cdot 10^{-287} X^9$$

$$+ 1.12089 \cdot 10^{-287} X^8 - 5.8673 \cdot 10^{-288} X^7 + 2.52637 \cdot 10^{-288} X^6 - 6.79624 \cdot 10^{-289} X^5$$

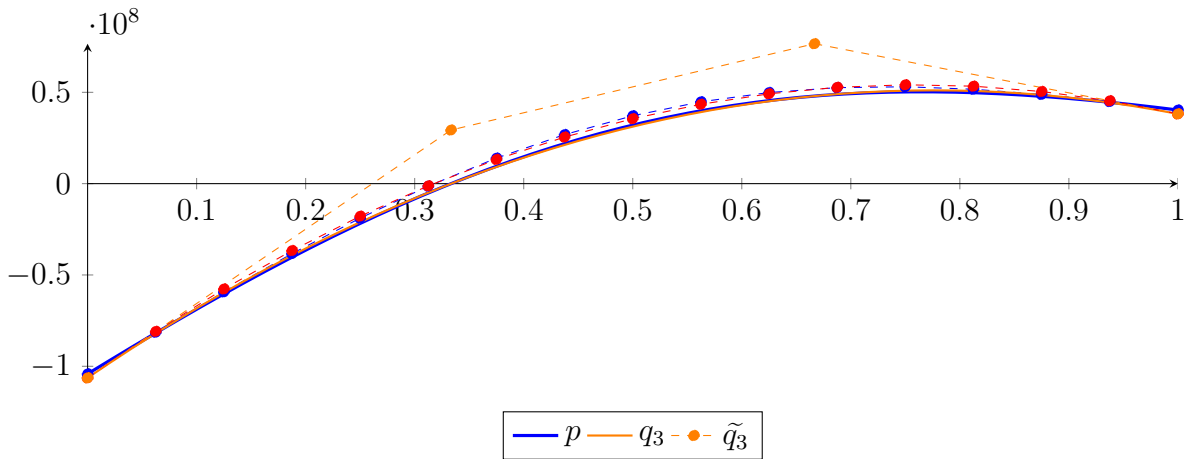
$$+ 9.36341 \cdot 10^{-290} X^4 + 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18551 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91815 \cdot 10^7 B_{10,16} + 5.27353 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

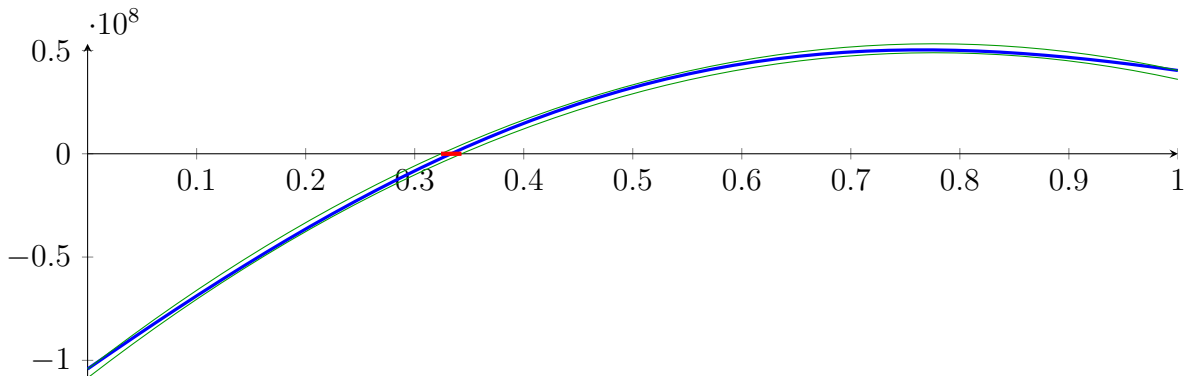
$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\}$$

$$N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

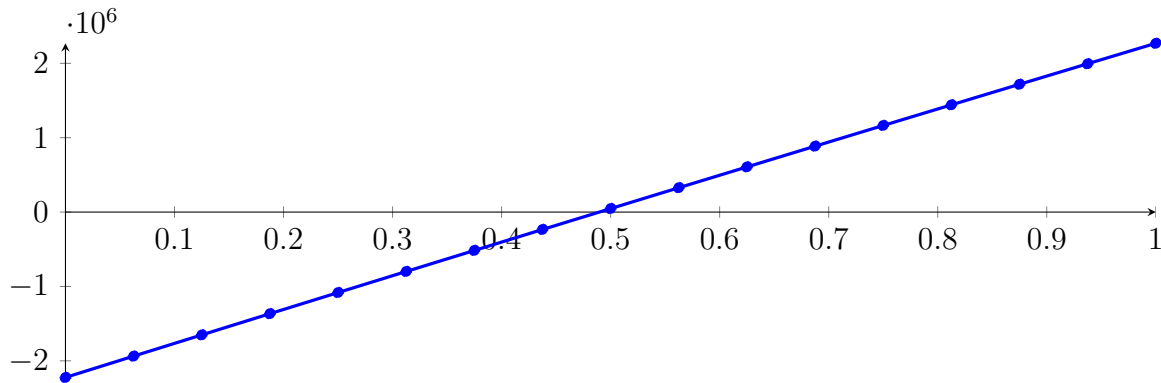
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

240.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

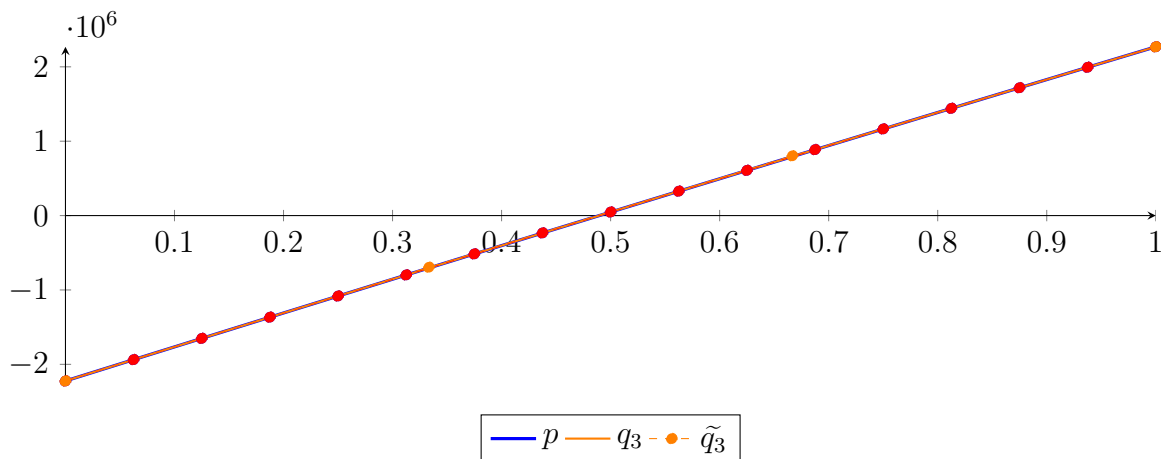
$$\begin{aligned}
 p &= -2.37433 \cdot 10^{-28} X^{16} - 5.67363 \cdot 10^{-25} X^{15} - 5.77631 \cdot 10^{-22} X^{14} - 3.1818 \cdot 10^{-19} X^{13} - 9.6142 \cdot 10^{-17} X^{12} \\
 &\quad - 1.16549 \cdot 10^{-14} X^{11} + 1.71065 \cdot 10^{-12} X^{10} + 7.20611 \cdot 10^{-10} X^9 + 4.39147 \cdot 10^{-08} X^8 - 1.21542 \cdot 10^{-05} X^7 \\
 &\quad - 0.00155624 X^6 + 0.10496 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 4.41595 \cdot 10^{-291} X^{16} - 1.48598 \cdot 10^{-290} X^{15} - 2.81102 \cdot 10^{-290} X^{14} + 2.65129 \cdot 10^{-289} X^{13} \\
 &\quad - 7.34593 \cdot 10^{-289} X^{12} + 1.17654 \cdot 10^{-288} X^{11} - 1.23119 \cdot 10^{-288} X^{10} + 8.63493 \cdot 10^{-289} X^9 \\
 &\quad - 3.92909 \cdot 10^{-289} X^8 + 1.01265 \cdot 10^{-289} X^7 - 6.13644 \cdot 10^{-291} X^6 - 3.92664 \cdot 10^{-291} X^5 \\
 &\quad + 9.04488 \cdot 10^{-292} X^4 - 700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} \\
 &\quad + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m :

$$M = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

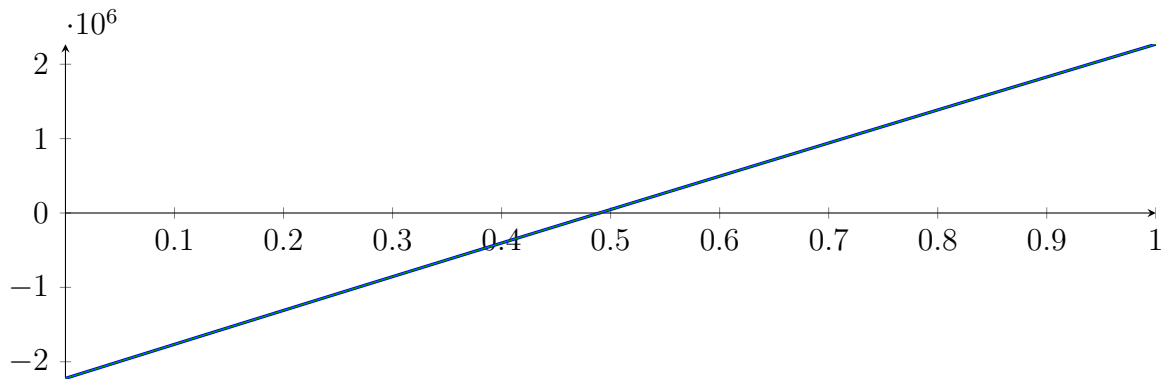
$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\}$$

$$N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174 \cdot 10^{-07}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

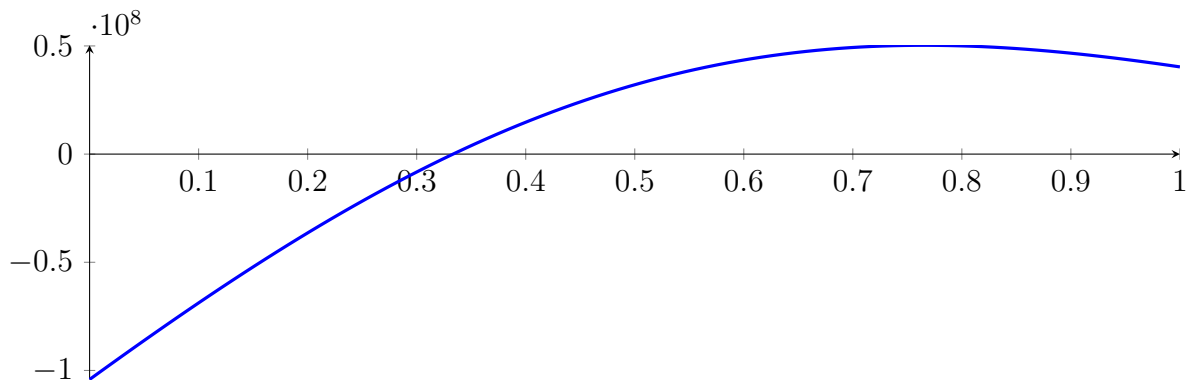
240.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 3!

240.4 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

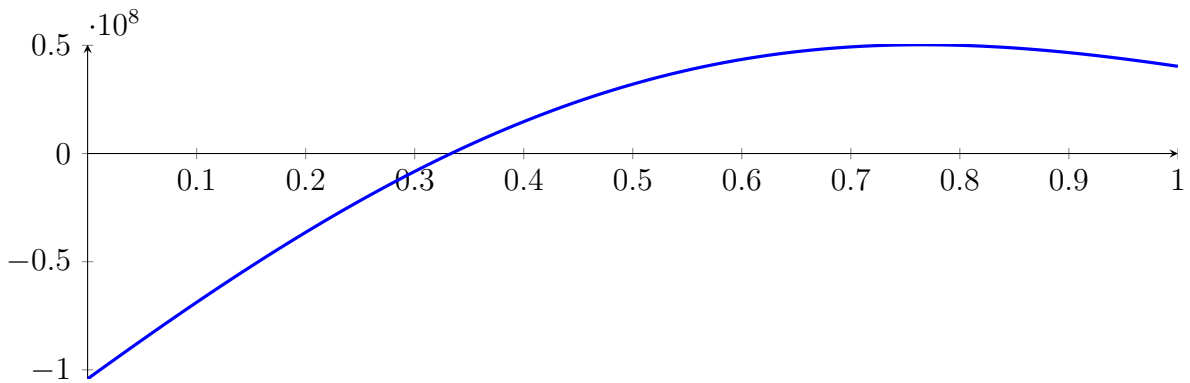
with precision $\varepsilon = 1 \cdot 10^{-08}$.

241 Running BezClip on f_{16} with epsilon 16

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

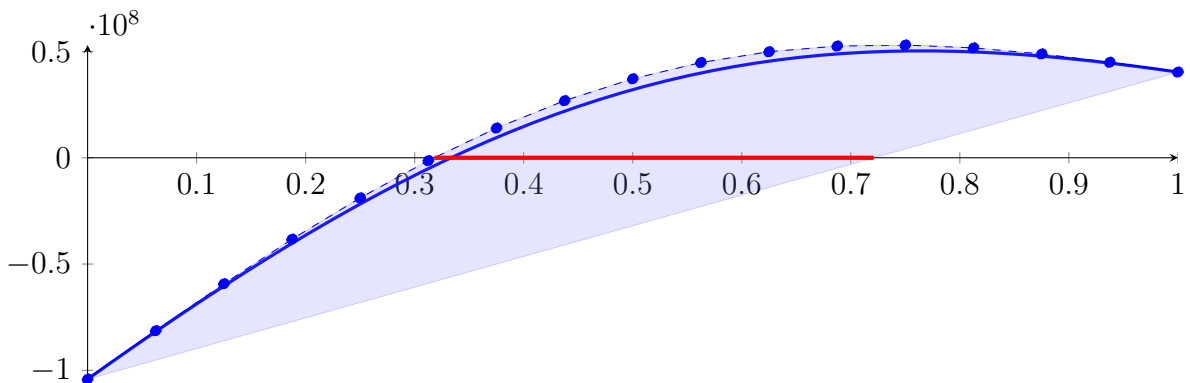
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



241.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

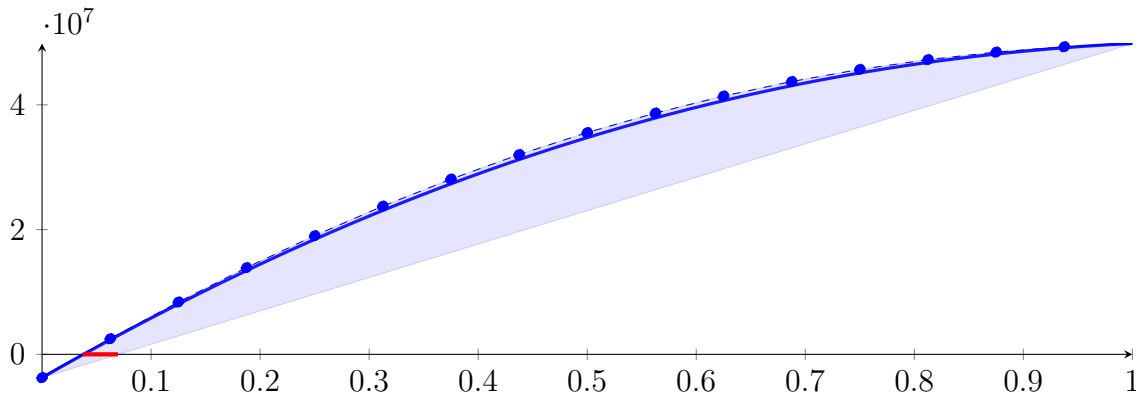
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

241.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.83858 \cdot 10^{-07} X^{16} - 5.37355 \cdot 10^{-05} X^{15} - 0.00254146 X^{14} - 0.064977 X^{13} - 0.909205 X^{12} \\ &\quad - 5.03924 X^{11} + 36.4549 X^{10} + 692.921 X^9 + 1886.97 X^8 - 25792 X^7 - 149671 X^6 + 492605 X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

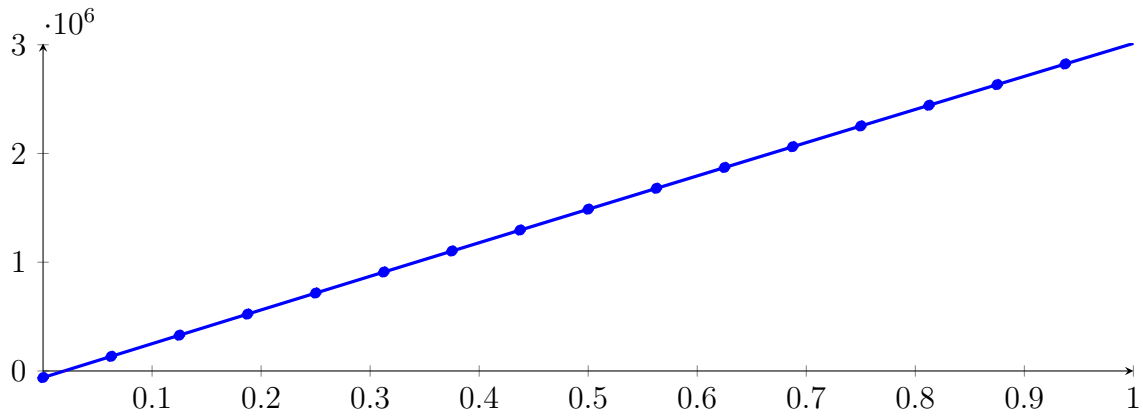
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

241.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -6.78234 \cdot 10^{-31} X^{16} - 2.34473 \cdot 10^{-27} X^{15} - 3.45581 \cdot 10^{-24} X^{14} - 2.75921 \cdot 10^{-21} X^{13} - 1.21242 \cdot 10^{-18} X^{12} \\ &\quad - 2.17572 \cdot 10^{-16} X^{11} + 4.23544 \cdot 10^{-14} X^{10} + 2.70025 \cdot 10^{-11} X^9 + 2.51306 \cdot 10^{-09} X^8 - 9.23474 \cdot 10^{-07} X^7 \\ &\quad - 0.000177459 X^6 + 0.0161038 X^5 + 4.36155 X^4 - 234.216 X^3 - 45622.2 X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

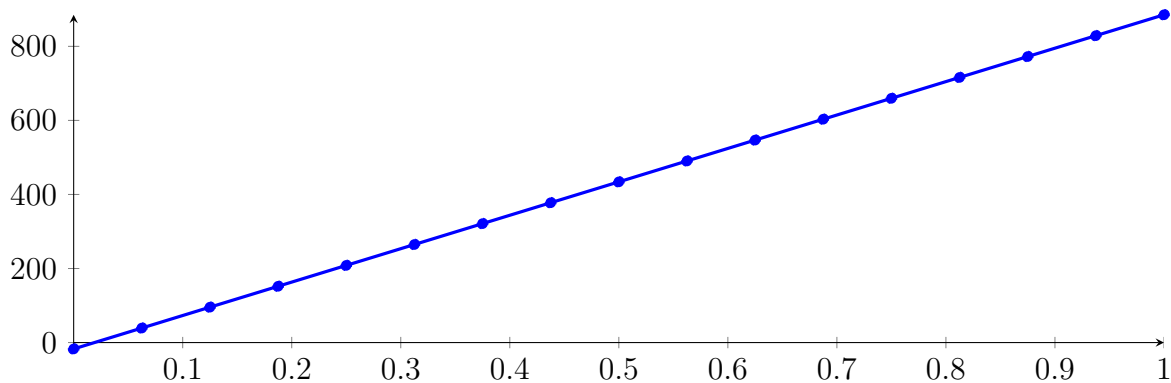
Longest intersection interval: 0.000289554

⇒ Selective recursion: interval 1: $[0.333333, 0.333337]$,

241.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.65599 \cdot 10^{-87} X^{16} - 1.97733 \cdot 10^{-80} X^{15} - 1.00659 \cdot 10^{-73} X^{14} - 2.77601 \cdot 10^{-67} X^{13} \\
 &\quad - 4.21367 \cdot 10^{-61} X^{12} - 2.61333 \cdot 10^{-55} X^{11} + 1.75275 \cdot 10^{-49} X^{10} + 3.8646 \cdot 10^{-43} X^9 \\
 &\quad + 1.2441 \cdot 10^{-37} X^8 - 1.57525 \cdot 10^{-31} X^7 - 1.04661 \cdot 10^{-25} X^6 + 3.27355 \cdot 10^{-20} X^5 \\
 &\quad + 3.06701 \cdot 10^{-14} X^4 - 5.67777 \cdot 10^{-09} X^3 - 0.00382618 X^2 + 902.448 X - 17.178 \\
 &= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
 &\quad + 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
 &\quad + 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
 &\quad + 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

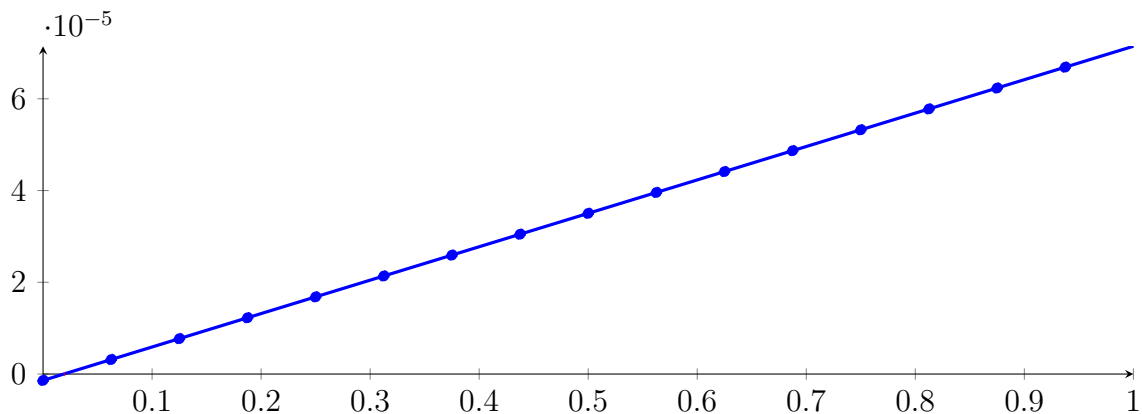
Longest intersection interval: $8.07045 \cdot 10^{-08}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

241.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -5.36315 \cdot 10^{-201} X^{16} - 7.93495 \cdot 10^{-187} X^{15} - 5.0052 \cdot 10^{-173} X^{14} - 1.71037 \cdot 10^{-159} X^{13} \\
 &\quad - 3.21686 \cdot 10^{-146} X^{12} - 2.47211 \cdot 10^{-133} X^{11} + 2.05446 \cdot 10^{-120} X^{10} + 5.61285 \cdot 10^{-107} X^9 \\
 &\quad + 2.23891 \cdot 10^{-94} X^8 - 3.51264 \cdot 10^{-81} X^7 - 2.89181 \cdot 10^{-68} X^6 + 1.12075 \cdot 10^{-55} X^5 + 1.30109 \\
 &\quad \cdot 10^{-42} X^4 - 2.98449 \cdot 10^{-30} X^3 - 2.49207 \cdot 10^{-17} X^2 + 7.28316 \cdot 10^{-05} X - 1.38634 \cdot 10^{-06} \\
 &= -1.38634 \cdot 10^{-06} B_{0,16}(X) + 3.16564 \cdot 10^{-06} B_{1,16}(X) + 7.71761 \cdot 10^{-06} B_{2,16}(X) + 1.22696 \\
 &\quad \cdot 10^{-05} B_{3,16}(X) + 1.68216 \cdot 10^{-05} B_{4,16}(X) + 2.13735 \cdot 10^{-05} B_{5,16}(X) + 2.59255 \cdot 10^{-05} B_{6,16}(X) \\
 &\quad + 3.04775 \cdot 10^{-05} B_{7,16}(X) + 3.50295 \cdot 10^{-05} B_{8,16}(X) + 3.95814 \cdot 10^{-05} B_{9,16}(X) + 4.41334 \\
 &\quad \cdot 10^{-05} B_{10,16}(X) + 4.86854 \cdot 10^{-05} B_{11,16}(X) + 5.32374 \cdot 10^{-05} B_{12,16}(X) + 5.77893 \cdot 10^{-05} B_{13,16}(X) \\
 &\quad + 6.23413 \cdot 10^{-05} B_{14,16}(X) + 6.68933 \cdot 10^{-05} B_{15,16}(X) + 7.14453 \cdot 10^{-05} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.51314 \cdot 10^{-15}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

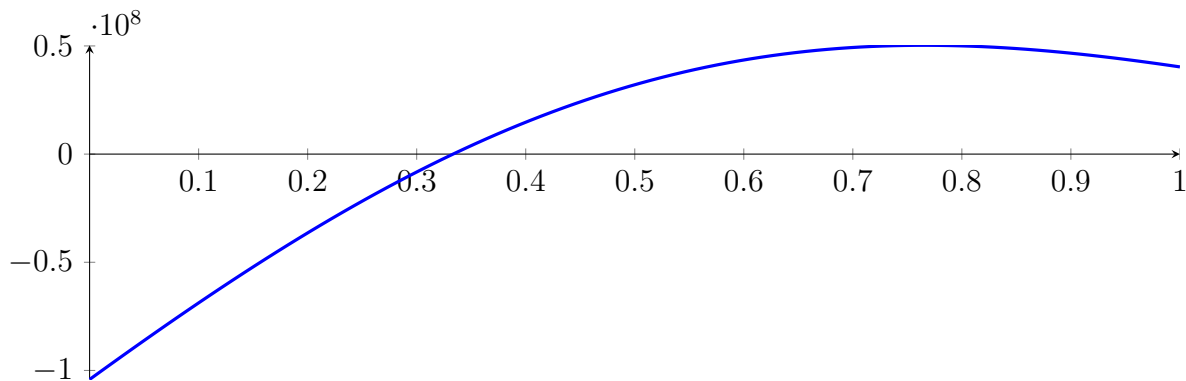
241.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 6!

241.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

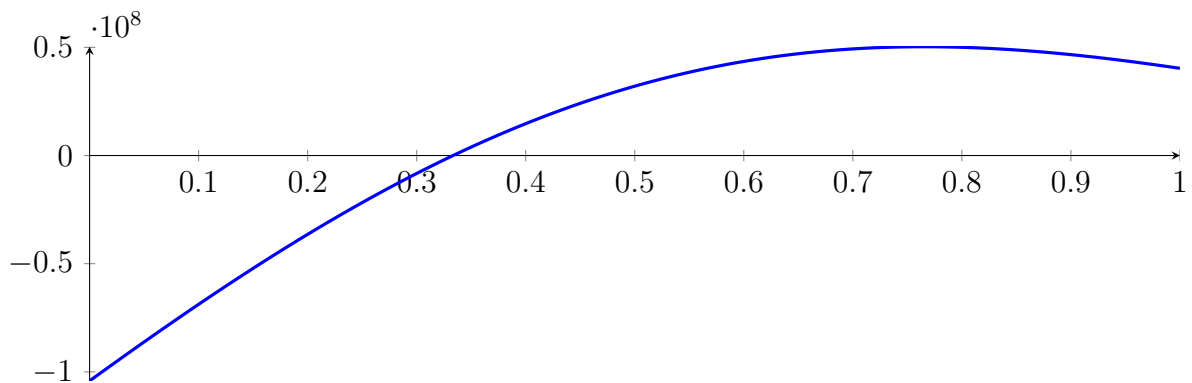
with precision $\varepsilon = 1 \cdot 10^{-16}$.

242 Running QuadClip on f_{16} with epsilon 16

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

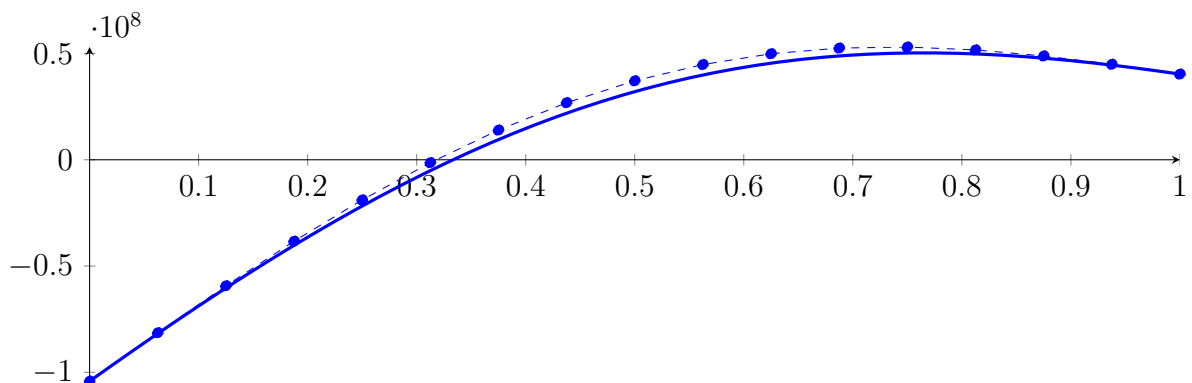
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



242.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = 3.74473 \cdot 10^{-288} X^{16} - 3.08889 \cdot 10^{-287} X^{15} + 1.17443 \cdot 10^{-286} X^{14} - 2.71603 \cdot 10^{-286} X^{13}$$

$$+ 4.23554 \cdot 10^{-286} X^{12} - 4.66107 \cdot 10^{-286} X^{11} + 3.6845 \cdot 10^{-286} X^{10} - 2.09811 \cdot 10^{-286} X^9$$

$$+ 8.58708 \cdot 10^{-287} X^8 - 2.54063 \cdot 10^{-287} X^7 + 5.57852 \cdot 10^{-288} X^6 - 9.21534 \cdot 10^{-289} X^5$$

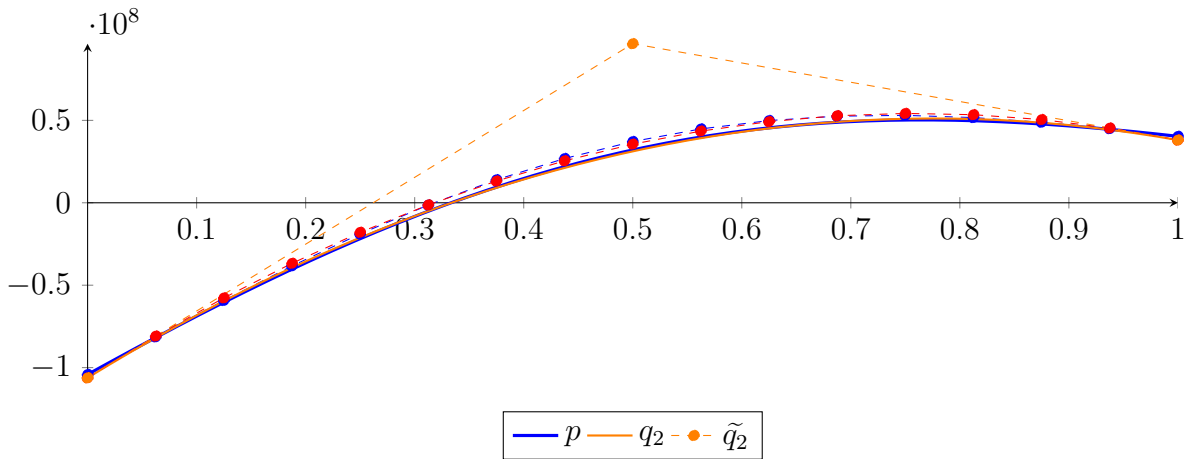
$$+ 1.02122 \cdot 10^{-289} X^4 - 5.95624 \cdot 10^{-291} X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

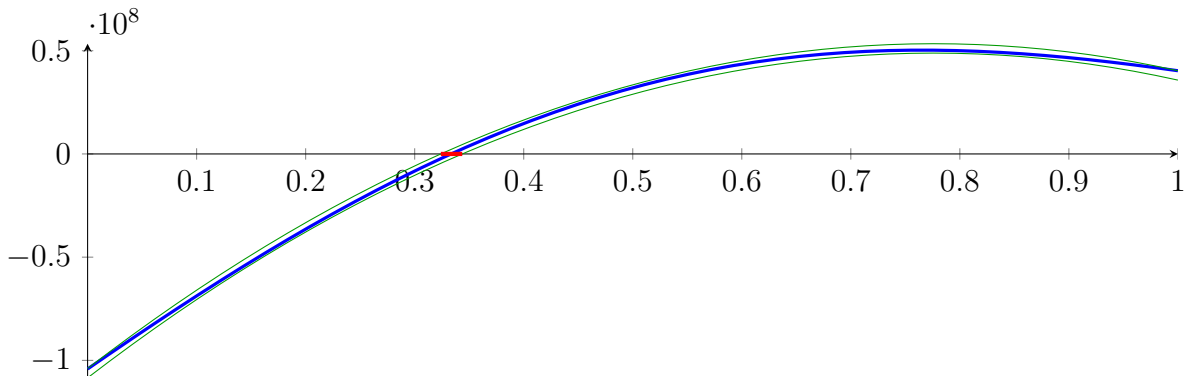
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

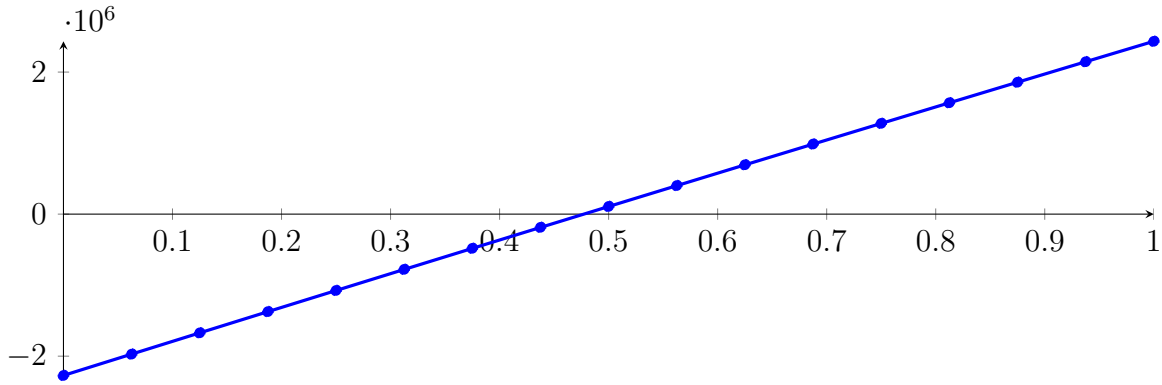
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

242.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

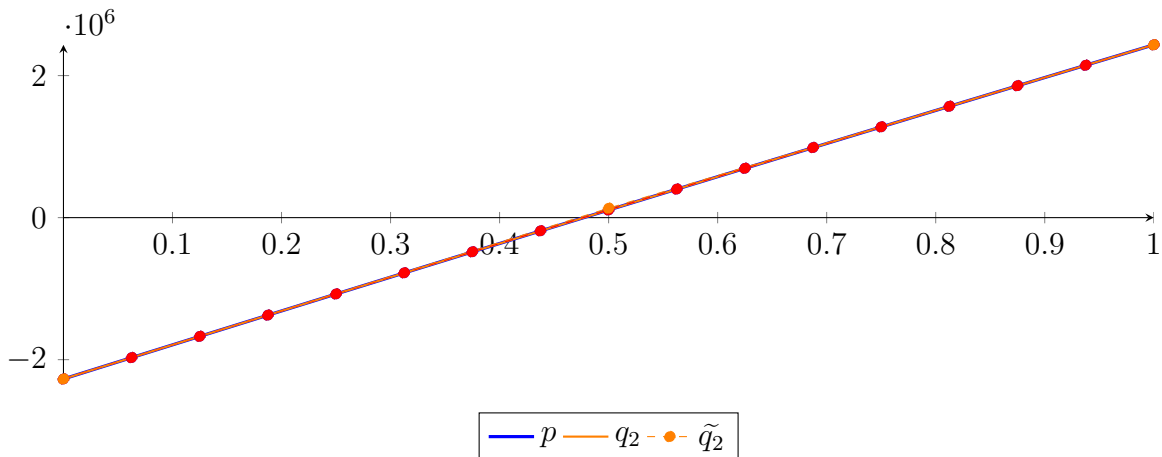
$$\begin{aligned}
 p &= -5.0162 \cdot 10^{-28} X^{16} - 1.14383 \cdot 10^{-24} X^{15} - 1.11125 \cdot 10^{-21} X^{14} - 5.84096 \cdot 10^{-19} X^{13} - 1.684 \cdot 10^{-16} X^{12} \\
 &\quad - 1.94707 \cdot 10^{-14} X^{11} + 2.73227 \cdot 10^{-12} X^{10} + 1.09727 \cdot 10^{-9} X^9 + 6.37314 \cdot 10^{-8} X^8 - 1.68645 \cdot 10^{-5} X^7 \\
 &\quad - 0.00205892 X^6 + 0.132721 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -8.18461 \cdot 10^{-291} X^{16} + 4.13745 \cdot 10^{-290} X^{15} - 5.8254 \cdot 10^{-290} X^{14} - 6.80919 \cdot 10^{-290} X^{13} \\
 &\quad + 3.48059 \cdot 10^{-289} X^{12} - 5.47245 \cdot 10^{-289} X^{11} + 4.8261 \cdot 10^{-289} X^{10} - 2.67867 \cdot 10^{-289} X^9 \\
 &\quad + 1.00314 \cdot 10^{-289} X^8 - 2.84766 \cdot 10^{-290} X^7 + 6.85072 \cdot 10^{-291} X^6 - 1.12501 \cdot 10^{-291} X^5 \\
 &\quad + 1.24731 \cdot 10^{-293} X^4 + 2.25944 \cdot 10^{-293} X^3 - 104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} \\
 &\quad + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

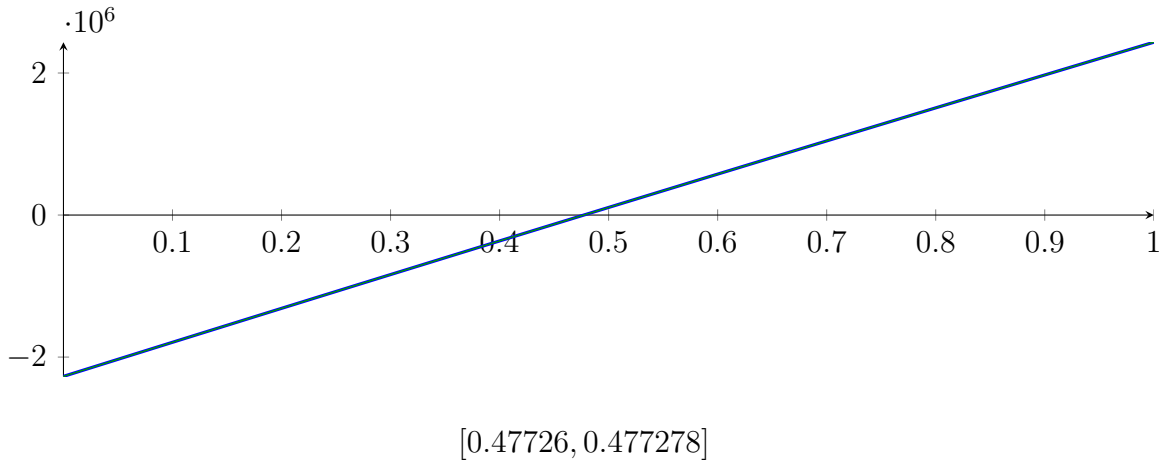
$$M = -104265X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



Longest intersection interval: $1.72301 \cdot 10^{-05}$
 \implies Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

242.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.02667 \cdot 10^{-104} X^{16} - 4.019 \cdot 10^{-96} X^{15} - 2.27522 \cdot 10^{-88} X^{14} - 6.97783 \cdot 10^{-81} X^{13}$$

$$- 1.17785 \cdot 10^{-73} X^{12} - 8.12373 \cdot 10^{-67} X^{11} + 6.05916 \cdot 10^{-60} X^{10} + 1.48569 \cdot 10^{-52} X^9$$

$$+ 5.31875 \cdot 10^{-46} X^8 - 7.48919 \cdot 10^{-39} X^7 - 5.53349 \cdot 10^{-32} X^6 + 1.92471 \cdot 10^{-25} X^5$$

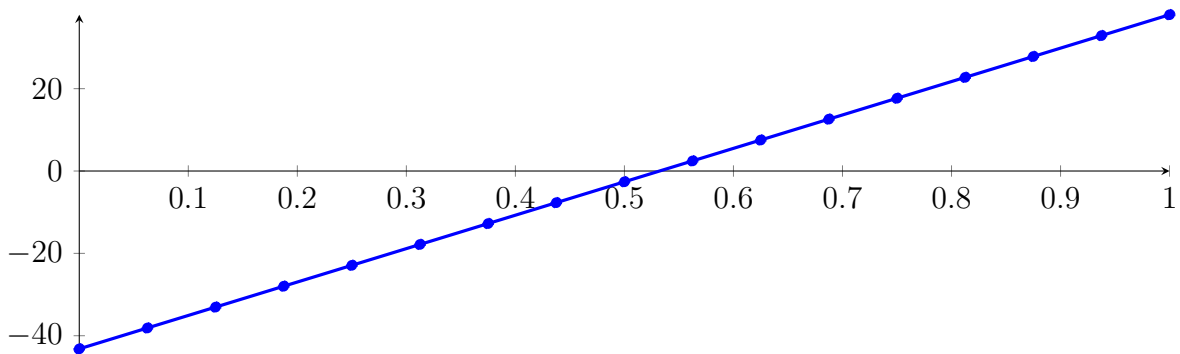
$$+ 2.00536 \cdot 10^{-18} X^4 - 4.12844 \cdot 10^{-12} X^3 - 3.09388 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$

$$= -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X)$$

$$- 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X)$$

$$+ 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X)$$

$$+ 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

$$= -43.1911 B_{0,2} - 2.61586 B_{1,2} + 37.9594 B_{2,2}$$

$$\tilde{q}_2 = -1.88281 \cdot 10^{-295} X^{16} + 1.09893 \cdot 10^{-294} X^{15} - 2.26419 \cdot 10^{-294} X^{14} + 8.08223 \cdot 10^{-295} X^{13}$$

$$+ 4.92626 \cdot 10^{-294} X^{12} - 1.07605 \cdot 10^{-293} X^{11} + 1.11858 \cdot 10^{-293} X^{10} - 6.87288 \cdot 10^{-294} X^9$$

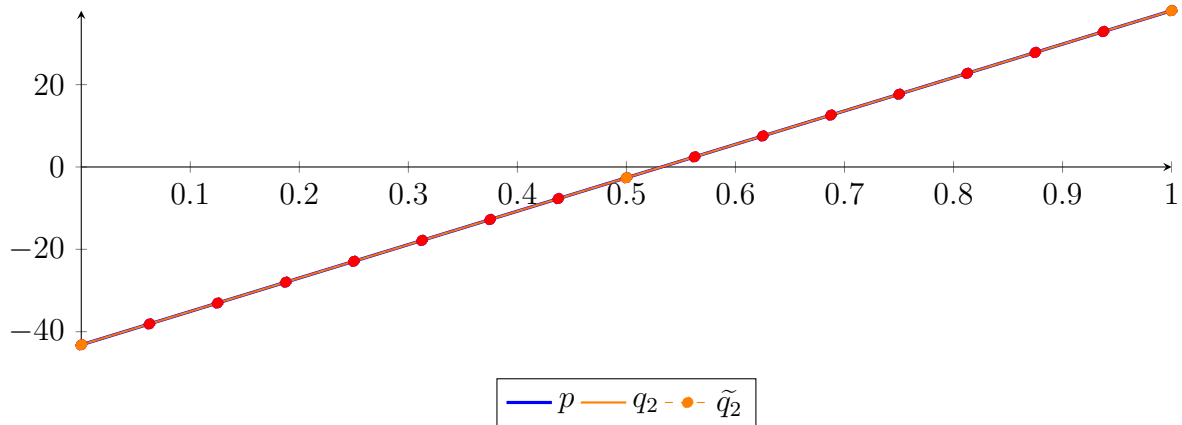
$$+ 2.54873 \cdot 10^{-294} X^8 - 5.2305 \cdot 10^{-295} X^7 + 3.18923 \cdot 10^{-296} X^6 + 1.34092 \cdot 10^{-296} X^5$$

$$- 4.89549 \cdot 10^{-297} X^4 + 5.89947 \cdot 10^{-298} X^3 - 3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

$$= -43.1911 B_{0,16} - 38.1192 B_{1,16} - 33.0473 B_{2,16} - 27.9754 B_{3,16} - 22.9035 B_{4,16} - 17.8316 B_{5,16}$$

$$- 12.7597 B_{6,16} - 7.68778 B_{7,16} - 2.61587 B_{8,16} + 2.45604 B_{9,16} + 7.52795 B_{10,16} + 12.5999 B_{11,16}$$

$$+ 17.6718 B_{12,16} + 22.7437 B_{13,16} + 27.8156 B_{14,16} + 32.8875 B_{15,16} + 37.9594 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.06422 \cdot 10^{-13}$.

Bounding polynomials M and m :

$$M = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

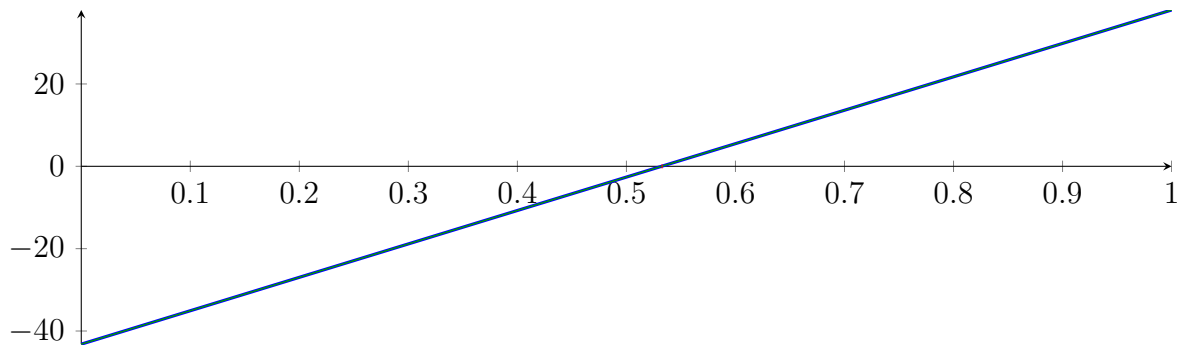
$$m = -3.09389 \cdot 10^{-05} X^2 + 81.1506 X - 43.1911$$

Root of M and m :

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\}$$

$$N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

Longest intersection interval: $5.08738 \cdot 10^{-15}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

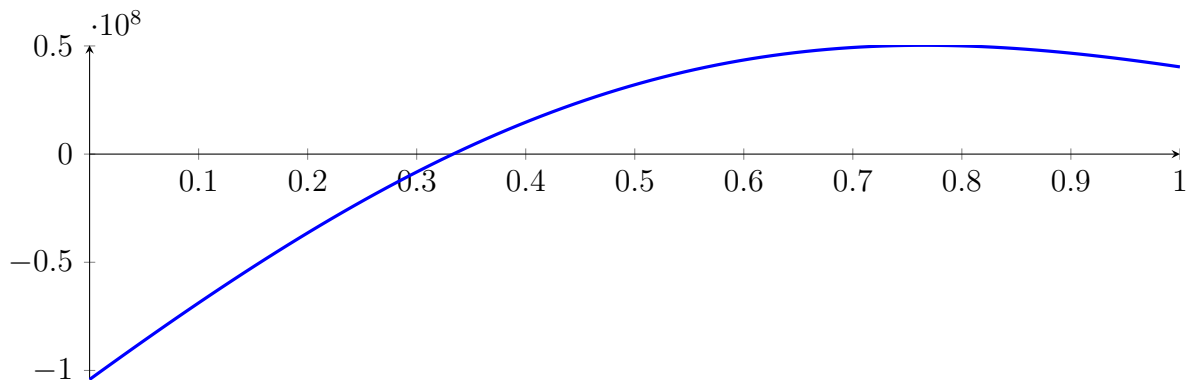
242.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 4!

242.5 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

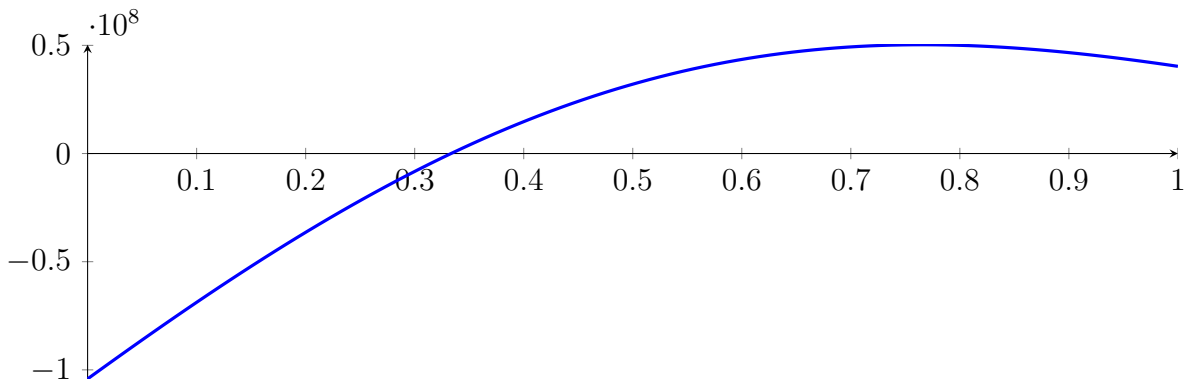
with precision $\varepsilon = 1 \cdot 10^{-16}$.

243 Running CubeClip on f_{16} with epsilon 16

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

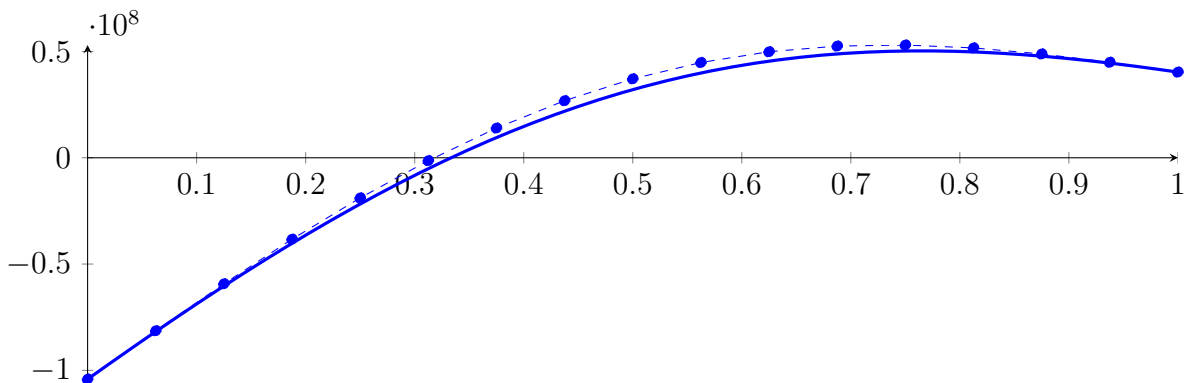
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



243.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 1.89955 \cdot 10^{-288} X^{16} - 1.4851 \cdot 10^{-287} X^{15} + 5.12896 \cdot 10^{-287} X^{14} - 1.02224 \cdot 10^{-286} X^{13}$$

$$+ 1.29027 \cdot 10^{-286} X^{12} - 1.07058 \cdot 10^{-286} X^{11} + 6.00739 \cdot 10^{-287} X^{10} - 2.54352 \cdot 10^{-287} X^9$$

$$+ 1.12089 \cdot 10^{-287} X^8 - 5.8673 \cdot 10^{-288} X^7 + 2.52637 \cdot 10^{-288} X^6 - 6.79624 \cdot 10^{-289} X^5$$

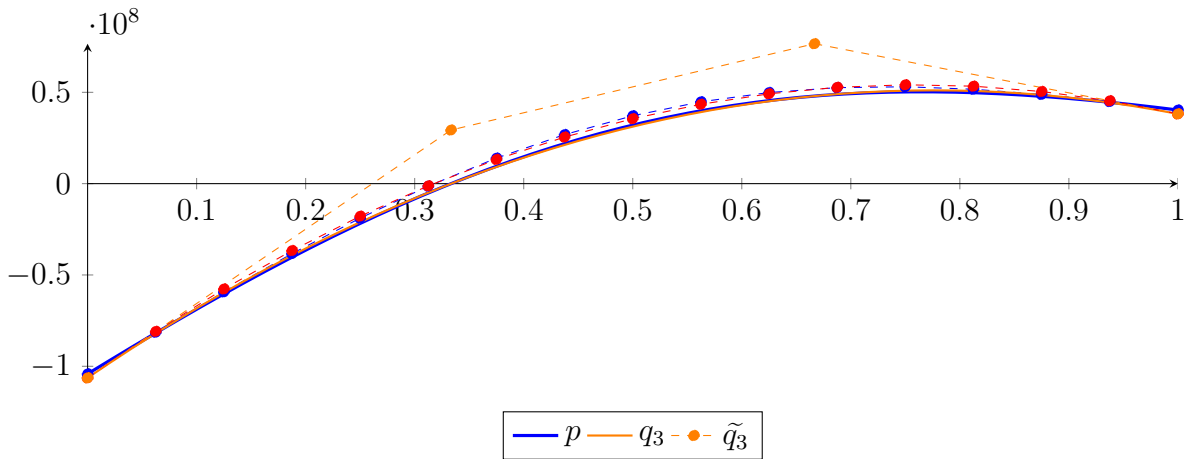
$$+ 9.36341 \cdot 10^{-290} X^4 + 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18551 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91815 \cdot 10^7 B_{10,16} + 5.27353 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

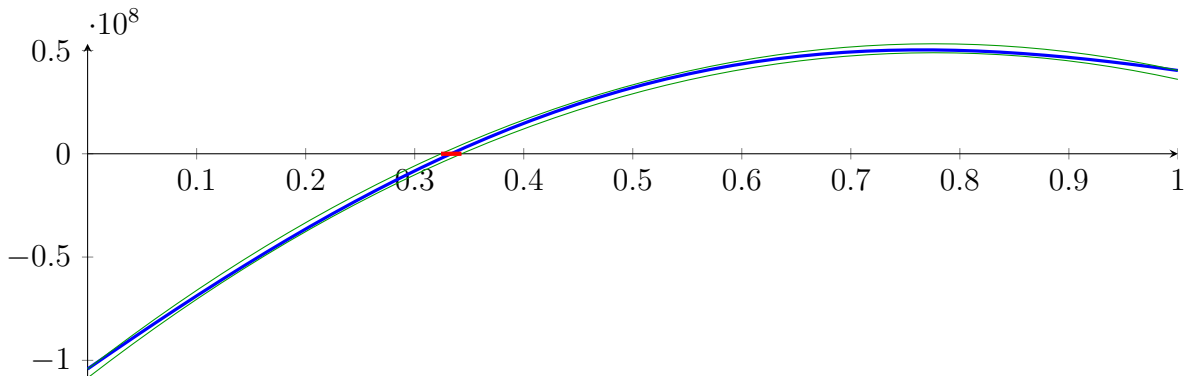
$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\}$$

$$N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

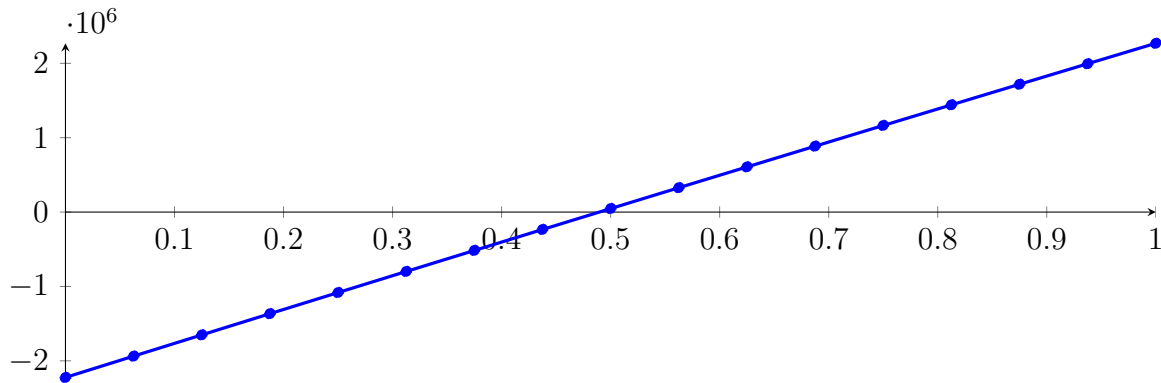
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

243.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

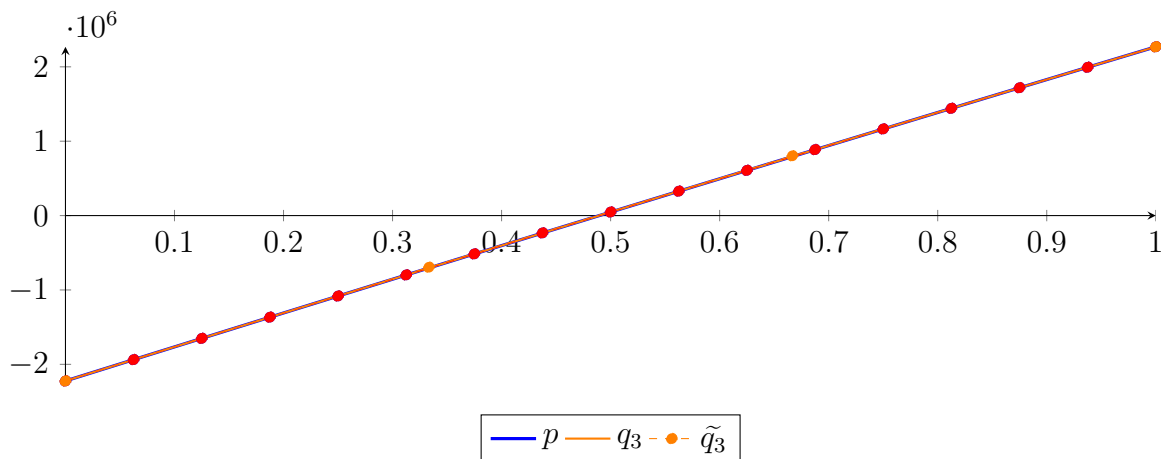
$$\begin{aligned}
 p &= -2.37433 \cdot 10^{-28} X^{16} - 5.67363 \cdot 10^{-25} X^{15} - 5.77631 \cdot 10^{-22} X^{14} - 3.1818 \cdot 10^{-19} X^{13} - 9.6142 \cdot 10^{-17} X^{12} \\
 &\quad - 1.16549 \cdot 10^{-14} X^{11} + 1.71065 \cdot 10^{-12} X^{10} + 7.20611 \cdot 10^{-10} X^9 + 4.39147 \cdot 10^{-08} X^8 - 1.21542 \cdot 10^{-05} X^7 \\
 &\quad - 0.00155624 X^6 + 0.10496 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 4.41595 \cdot 10^{-291} X^{16} - 1.48598 \cdot 10^{-290} X^{15} - 2.81102 \cdot 10^{-290} X^{14} + 2.65129 \cdot 10^{-289} X^{13} \\
 &\quad - 7.34593 \cdot 10^{-289} X^{12} + 1.17654 \cdot 10^{-288} X^{11} - 1.23119 \cdot 10^{-288} X^{10} + 8.63493 \cdot 10^{-289} X^9 \\
 &\quad - 3.92909 \cdot 10^{-289} X^8 + 1.01265 \cdot 10^{-289} X^7 - 6.13644 \cdot 10^{-291} X^6 - 3.92664 \cdot 10^{-291} X^5 \\
 &\quad + 9.04488 \cdot 10^{-292} X^4 - 700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} \\
 &\quad + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m :

$$M = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

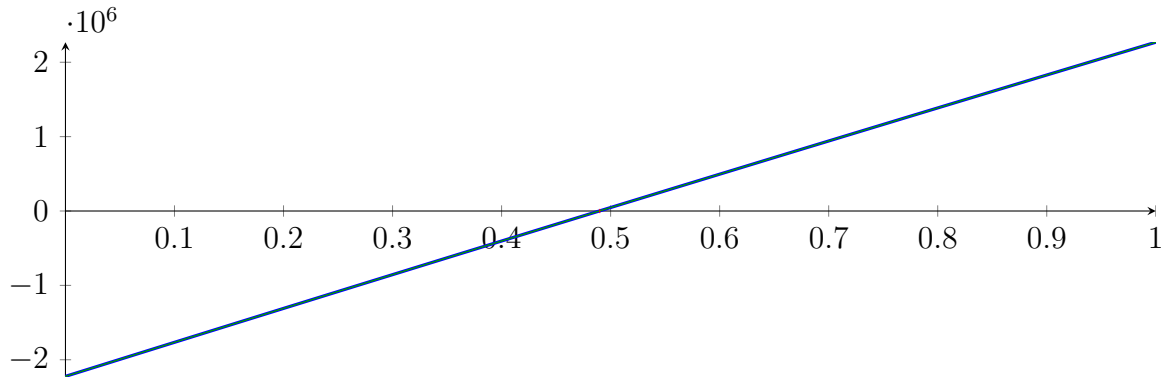
$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\}$$

$$N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

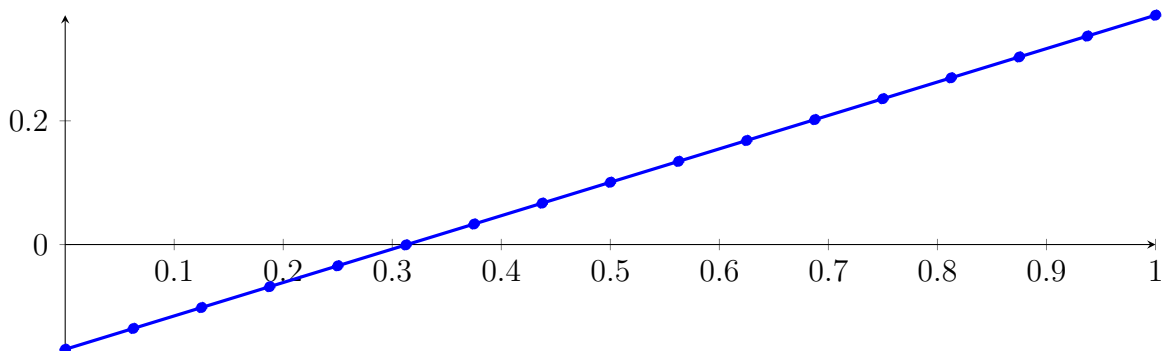
Longest intersection interval: $1.20174 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

243.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.49274 \cdot 10^{-139} X^{16} - 8.96277 \cdot 10^{-129} X^{15} - 7.623 \cdot 10^{-119} X^{14} - 3.51238 \cdot 10^{-109} X^{13} \\ &\quad - 8.90739 \cdot 10^{-100} X^{12} - 9.22984 \cdot 10^{-91} X^{11} + 1.03426 \cdot 10^{-81} X^{10} + 3.80998 \cdot 10^{-72} X^9 \\ &\quad + 2.04919 \cdot 10^{-63} X^8 - 4.33497 \cdot 10^{-54} X^7 - 4.81204 \cdot 10^{-45} X^6 + 2.51462 \cdot 10^{-36} X^5 \\ &\quad + 3.93622 \cdot 10^{-27} X^4 - 1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148X - 0.169396 \\ &= -0.169396B_{0,16}(X) - 0.135637B_{1,16}(X) - 0.101877B_{2,16}(X) - 0.068118B_{3,16}(X) \\ &\quad - 0.0343588B_{4,16}(X) - 0.000599488B_{5,16}(X) + 0.0331598B_{6,16}(X) \\ &\quad + 0.066919B_{7,16}(X) + 0.100678B_{8,16}(X) + 0.134438B_{9,16}(X) + 0.168197B_{10,16}(X) \\ &\quad + 0.201956B_{11,16}(X) + 0.235715B_{12,16}(X) + 0.269475B_{13,16}(X) \\ &\quad + 0.303234B_{14,16}(X) + 0.336993B_{15,16}(X) + 0.370752B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

$$= -0.169396 B_{0,3} + 0.0106536 B_{1,3} + 0.190703 B_{2,3} + 0.370752 B_{3,3}$$

$$\tilde{q}_3 = 8.03185 \cdot 10^{-297} X^{16} - 6.20841 \cdot 10^{-296} X^{15} + 2.13274 \cdot 10^{-295} X^{14} - 4.26614 \cdot 10^{-295} X^{13}$$

$$+ 5.47461 \cdot 10^{-295} X^{12} - 4.70265 \cdot 10^{-295} X^{11} + 2.78551 \cdot 10^{-295} X^{10} - 1.22442 \cdot 10^{-295} X^9$$

$$+ 4.88954 \cdot 10^{-296} X^8 - 2.11494 \cdot 10^{-296} X^7 + 8.20665 \cdot 10^{-297} X^6 - 2.15458 \cdot 10^{-297} X^5$$

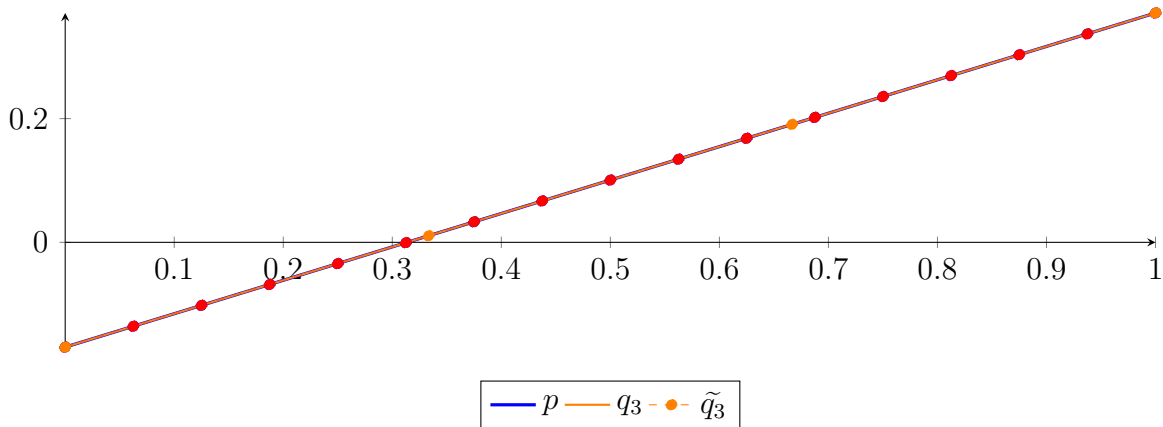
$$+ 3.01517 \cdot 10^{-298} X^4 - 1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

$$= -0.169396 B_{0,16} - 0.135637 B_{1,16} - 0.101877 B_{2,16} - 0.068118 B_{3,16} - 0.0343588 B_{4,16}$$

$$- 0.000599488 B_{5,16} + 0.0331598 B_{6,16} + 0.066919 B_{7,16} + 0.100678 B_{8,16}$$

$$+ 0.134438 B_{9,16} + 0.168197 B_{10,16} + 0.201956 B_{11,16} + 0.235715 B_{12,16}$$

$$+ 0.269475 B_{13,16} + 0.303234 B_{14,16} + 0.336993 B_{15,16} + 0.370752 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 5.62317 \cdot 10^{-29}$.

Bounding polynomials M and m :

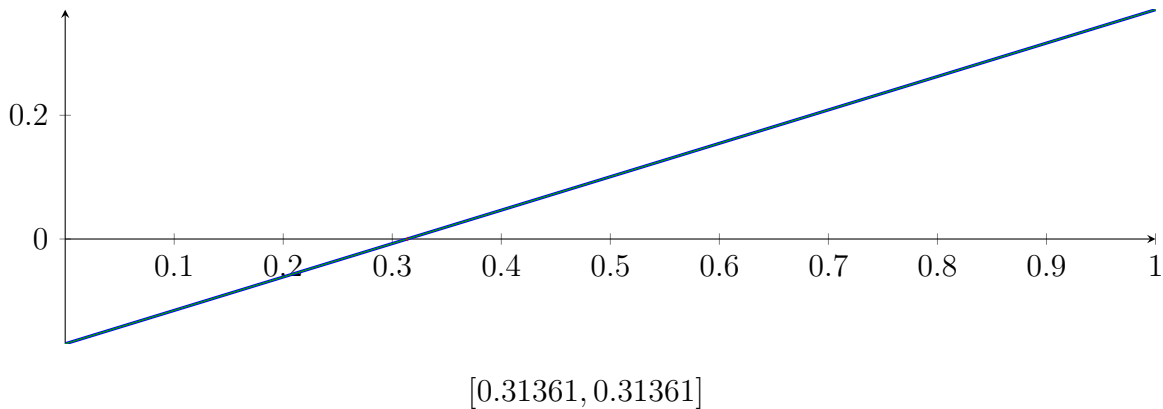
$$M = -1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

$$m = -1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

Root of M and m :

$$N(M) = \{-1.43506 \cdot 10^9, 0.31361, 3.09167 \cdot 10^8\} \quad N(m) = \{-1.43506 \cdot 10^9, 0.31361, 3.09167 \cdot 10^8\}$$

Intersection intervals:



Longest intersection interval: $2.08208 \cdot 10^{-28}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

243.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

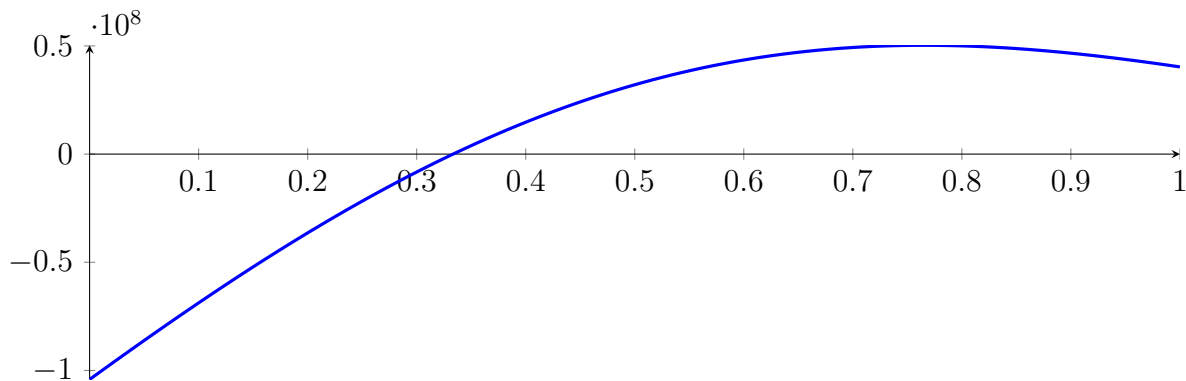
Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 4!

$$p(0) = -8.88188e-08 - p(1) -8.88188e-08$$

243.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

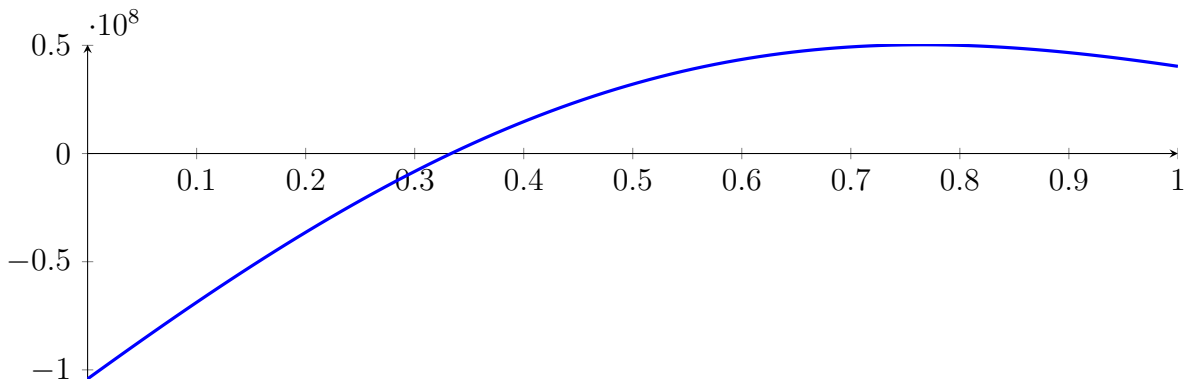
with precision $\varepsilon = 1 \cdot 10^{-16}$.

244 Running BezClip on f_{16} with epsilon 32

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

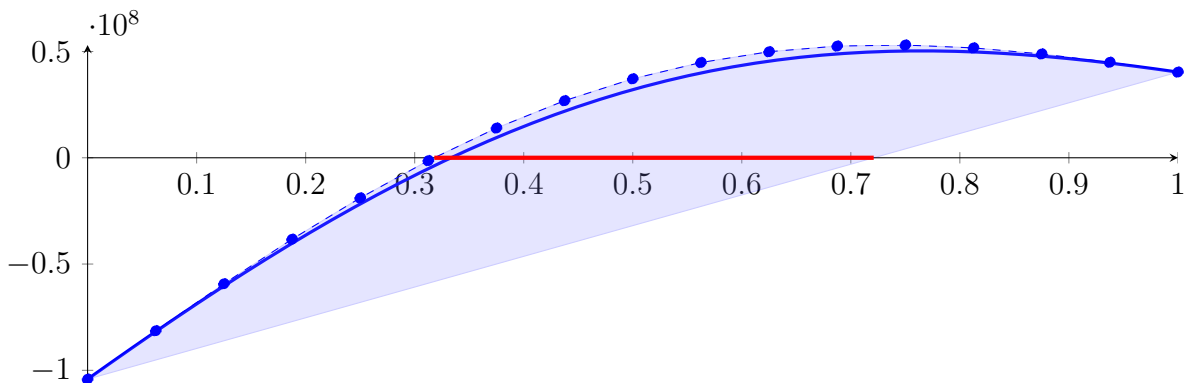
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



244.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

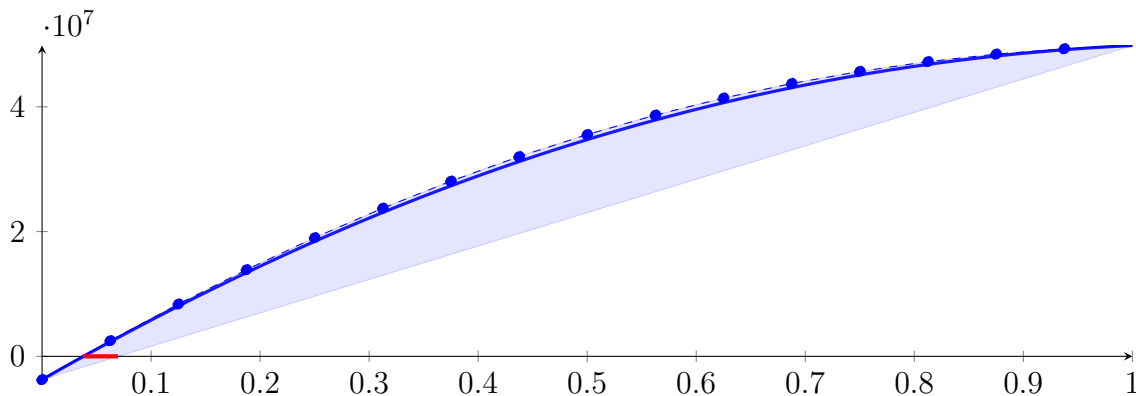
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

244.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.83858 \cdot 10^{-07} X^{16} - 5.37355 \cdot 10^{-05} X^{15} - 0.00254146 X^{14} - 0.064977 X^{13} - 0.909205 X^{12} \\ &\quad - 5.03924 X^{11} + 36.4549 X^{10} + 692.921 X^9 + 1886.97 X^8 - 25792 X^7 - 149671 X^6 + 492605 X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

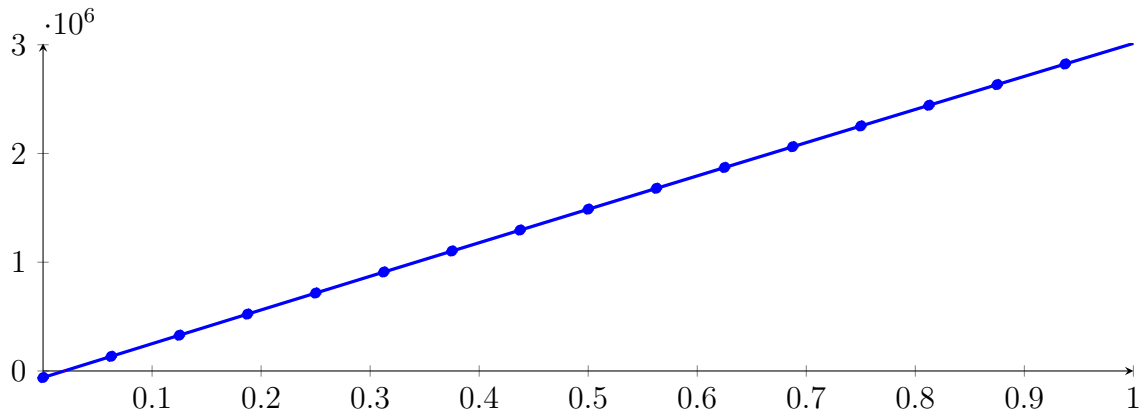
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

244.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -6.78234 \cdot 10^{-31} X^{16} - 2.34473 \cdot 10^{-27} X^{15} - 3.45581 \cdot 10^{-24} X^{14} - 2.75921 \cdot 10^{-21} X^{13} - 1.21242 \cdot 10^{-18} X^{12} \\ &\quad - 2.17572 \cdot 10^{-16} X^{11} + 4.23544 \cdot 10^{-14} X^{10} + 2.70025 \cdot 10^{-11} X^9 + 2.51306 \cdot 10^{-09} X^8 - 9.23474 \cdot 10^{-07} X^7 \\ &\quad - 0.000177459 X^6 + 0.0161038 X^5 + 4.36155 X^4 - 234.216 X^3 - 45622.2 X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

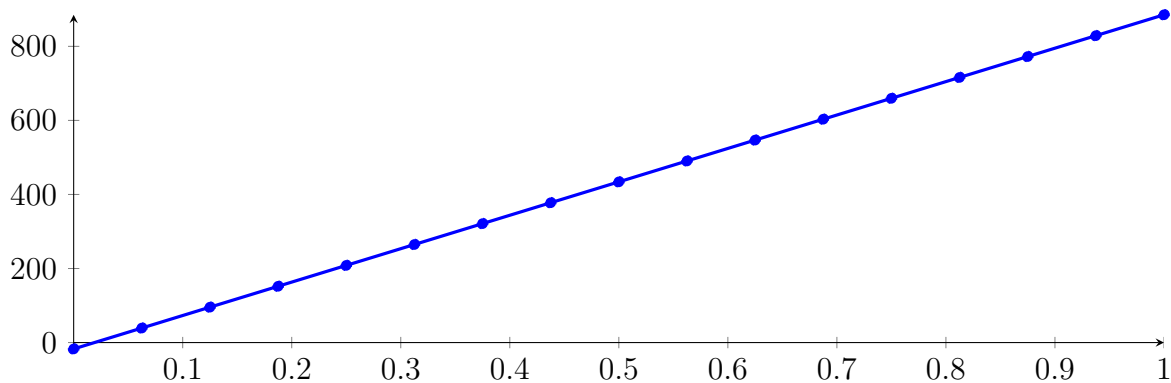
Longest intersection interval: 0.000289554

⇒ Selective recursion: interval 1: [\[0.333333, 0.333337\]](#),

244.4 Recursion Branch 1 1 1 1 in Interval 1: [\[0.333333, 0.333337\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.65599 \cdot 10^{-87} X^{16} - 1.97733 \cdot 10^{-80} X^{15} - 1.00659 \cdot 10^{-73} X^{14} - 2.77601 \cdot 10^{-67} X^{13} \\
 &\quad - 4.21367 \cdot 10^{-61} X^{12} - 2.61333 \cdot 10^{-55} X^{11} + 1.75275 \cdot 10^{-49} X^{10} + 3.8646 \cdot 10^{-43} X^9 \\
 &\quad + 1.2441 \cdot 10^{-37} X^8 - 1.57525 \cdot 10^{-31} X^7 - 1.04661 \cdot 10^{-25} X^6 + 3.27355 \cdot 10^{-20} X^5 \\
 &\quad + 3.06701 \cdot 10^{-14} X^4 - 5.67777 \cdot 10^{-09} X^3 - 0.00382618 X^2 + 902.448 X - 17.178 \\
 &= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
 &\quad + 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
 &\quad + 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
 &\quad + 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

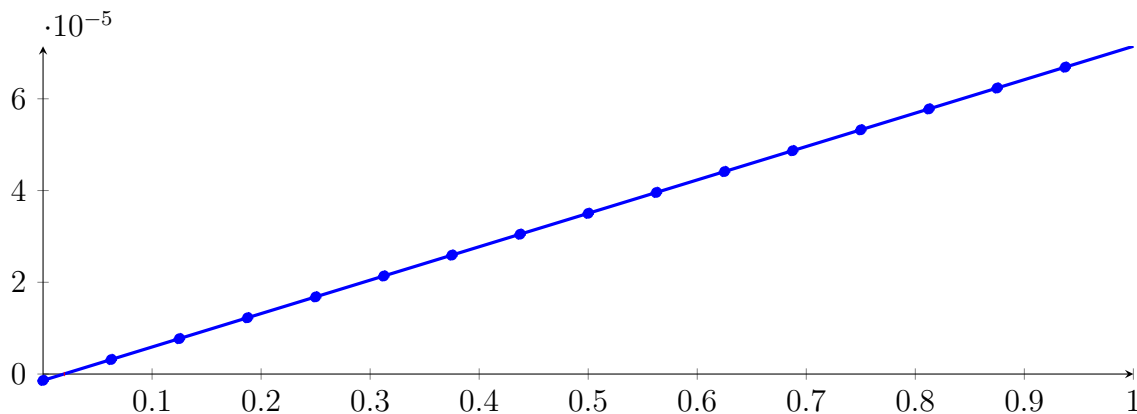
Longest intersection interval: $8.07045 \cdot 10^{-08}$

⇒ Selective recursion: interval 1: [\[0.333333, 0.333333\]](#),

244.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -5.36315 \cdot 10^{-201} X^{16} - 7.93495 \cdot 10^{-187} X^{15} - 5.0052 \cdot 10^{-173} X^{14} - 1.71037 \cdot 10^{-159} X^{13} \\
 &\quad - 3.21686 \cdot 10^{-146} X^{12} - 2.47211 \cdot 10^{-133} X^{11} + 2.05446 \cdot 10^{-120} X^{10} + 5.61285 \cdot 10^{-107} X^9 \\
 &\quad + 2.23891 \cdot 10^{-94} X^8 - 3.51264 \cdot 10^{-81} X^7 - 2.89181 \cdot 10^{-68} X^6 + 1.12075 \cdot 10^{-55} X^5 + 1.30109 \\
 &\quad \cdot 10^{-42} X^4 - 2.98449 \cdot 10^{-30} X^3 - 2.49207 \cdot 10^{-17} X^2 + 7.28316 \cdot 10^{-05} X - 1.38634 \cdot 10^{-06} \\
 &= -1.38634 \cdot 10^{-06} B_{0,16}(X) + 3.16564 \cdot 10^{-06} B_{1,16}(X) + 7.71761 \cdot 10^{-06} B_{2,16}(X) + 1.22696 \\
 &\quad \cdot 10^{-05} B_{3,16}(X) + 1.68216 \cdot 10^{-05} B_{4,16}(X) + 2.13735 \cdot 10^{-05} B_{5,16}(X) + 2.59255 \cdot 10^{-05} B_{6,16}(X) \\
 &\quad + 3.04775 \cdot 10^{-05} B_{7,16}(X) + 3.50295 \cdot 10^{-05} B_{8,16}(X) + 3.95814 \cdot 10^{-05} B_{9,16}(X) + 4.41334 \\
 &\quad \cdot 10^{-05} B_{10,16}(X) + 4.86854 \cdot 10^{-05} B_{11,16}(X) + 5.32374 \cdot 10^{-05} B_{12,16}(X) + 5.77893 \cdot 10^{-05} B_{13,16}(X) \\
 &\quad + 6.23413 \cdot 10^{-05} B_{14,16}(X) + 6.68933 \cdot 10^{-05} B_{15,16}(X) + 7.14453 \cdot 10^{-05} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

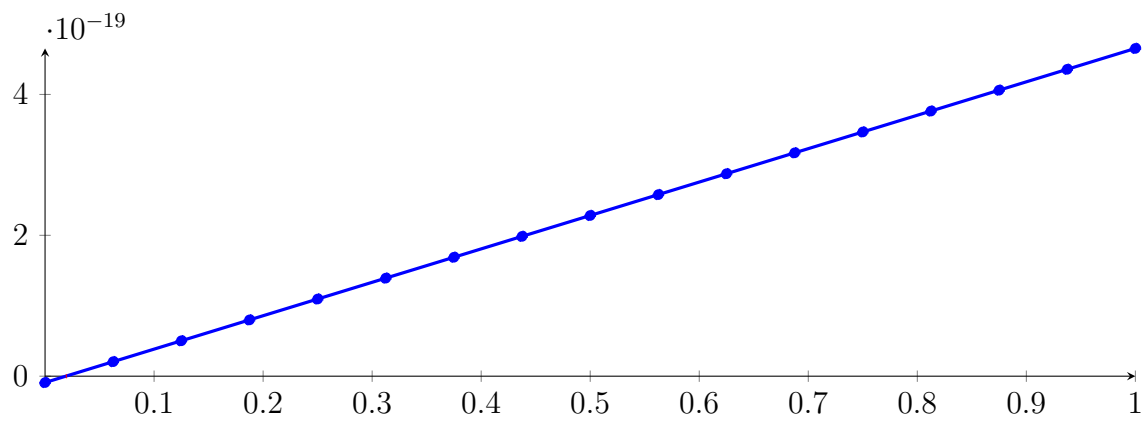
Longest intersection interval: $6.51314 \cdot 10^{-15}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

244.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.70149 \cdot 10^{-323} X^{16} + 1.97819 \cdot 10^{-322} X^{15} - 4.34527 \cdot 10^{-322} X^{14} + 3.97182 \cdot 10^{-322} X^{13} \\
 &\quad - 1.87464 \cdot 10^{-316} X^{12} - 2.21189 \cdot 10^{-289} X^{11} + 2.82229 \cdot 10^{-262} X^{10} + 1.18385 \cdot 10^{-234} X^9 \\
 &\quad + 7.25038 \cdot 10^{-208} X^8 - 1.74649 \cdot 10^{-180} X^7 - 2.20756 \cdot 10^{-153} X^6 + 1.31359 \cdot 10^{-126} X^5 + 2.34136 \\
 &\quad \cdot 10^{-99} X^4 - 8.24597 \cdot 10^{-73} X^3 - 1.05716 \cdot 10^{-45} X^2 + 4.74362 \cdot 10^{-19} X - 9.02941 \cdot 10^{-21} \\
 &= -9.02941 \cdot 10^{-21} B_{0,16}(X) + 2.06182 \cdot 10^{-20} B_{1,16}(X) + 5.02659 \cdot 10^{-20} B_{2,16}(X) + 7.99135 \\
 &\quad \cdot 10^{-20} B_{3,16}(X) + 1.09561 \cdot 10^{-19} B_{4,16}(X) + 1.39209 \cdot 10^{-19} B_{5,16}(X) + 1.68856 \cdot 10^{-19} B_{6,16}(X) \\
 &\quad + 1.98504 \cdot 10^{-19} B_{7,16}(X) + 2.28152 \cdot 10^{-19} B_{8,16}(X) + 2.57799 \cdot 10^{-19} B_{9,16}(X) + 2.87447 \\
 &\quad \cdot 10^{-19} B_{10,16}(X) + 3.17095 \cdot 10^{-19} B_{11,16}(X) + 3.46742 \cdot 10^{-19} B_{12,16}(X) + 3.7639 \cdot 10^{-19} B_{13,16}(X) \\
 &\quad + 4.06038 \cdot 10^{-19} B_{14,16}(X) + 4.35685 \cdot 10^{-19} B_{15,16}(X) + 4.65333 \cdot 10^{-19} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $4.2421 \cdot 10^{-29}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

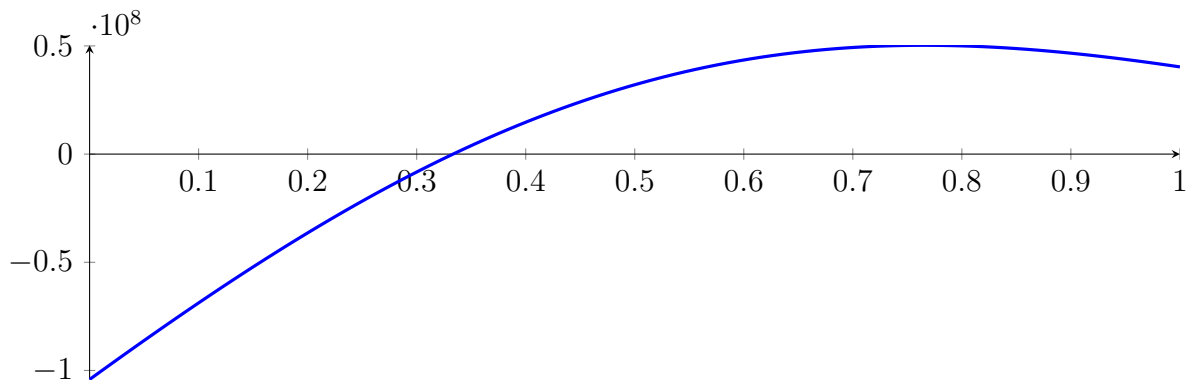
244.7 Recursion Branch 1 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 7!

244.8 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

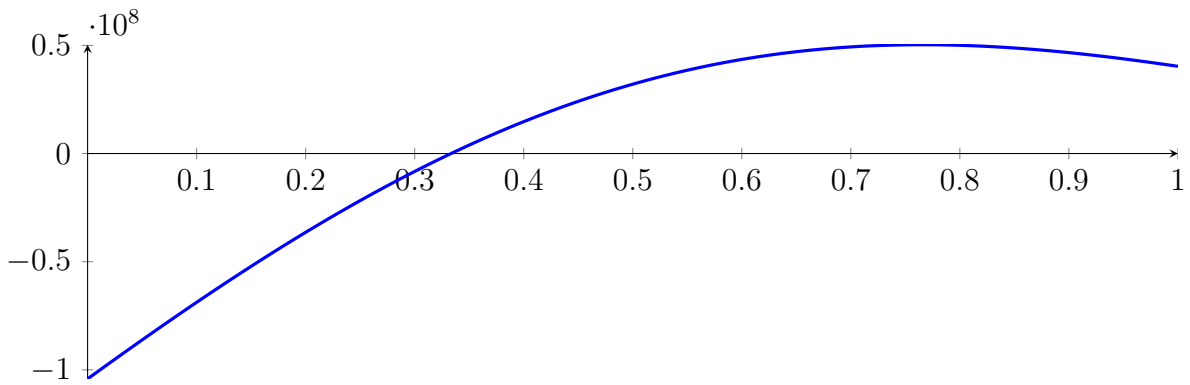
with precision $\varepsilon = 1 \cdot 10^{-32}$.

245 Running QuadClip on f_{16} with epsilon 32

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

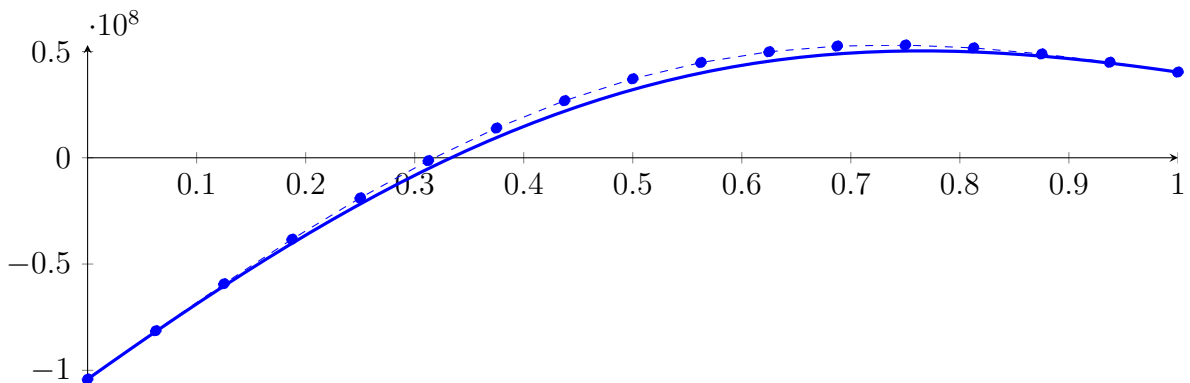
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



245.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = 3.74473 \cdot 10^{-288} X^{16} - 3.08889 \cdot 10^{-287} X^{15} + 1.17443 \cdot 10^{-286} X^{14} - 2.71603 \cdot 10^{-286} X^{13}$$

$$+ 4.23554 \cdot 10^{-286} X^{12} - 4.66107 \cdot 10^{-286} X^{11} + 3.6845 \cdot 10^{-286} X^{10} - 2.09811 \cdot 10^{-286} X^9$$

$$+ 8.58708 \cdot 10^{-287} X^8 - 2.54063 \cdot 10^{-287} X^7 + 5.57852 \cdot 10^{-288} X^6 - 9.21534 \cdot 10^{-289} X^5$$

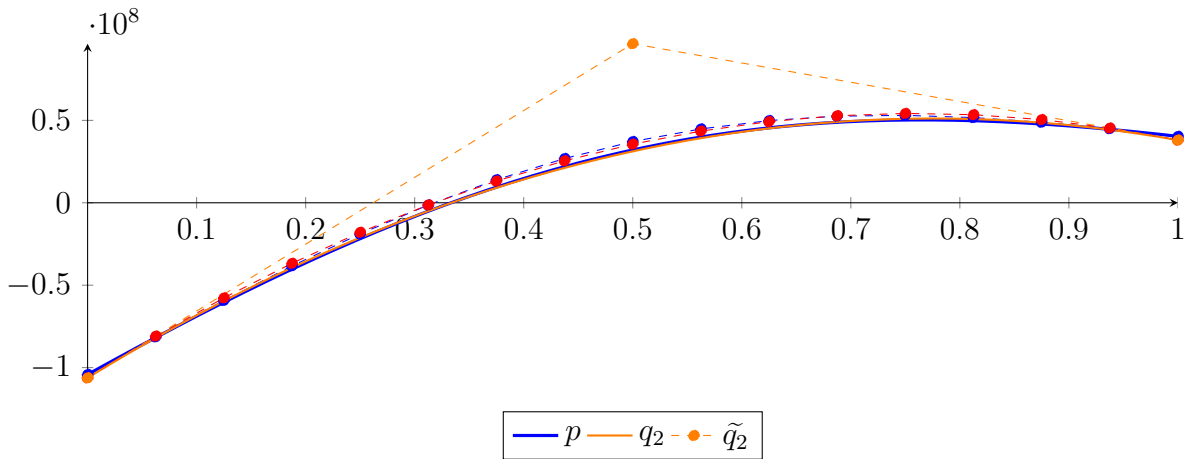
$$+ 1.02122 \cdot 10^{-289} X^4 - 5.95624 \cdot 10^{-291} X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

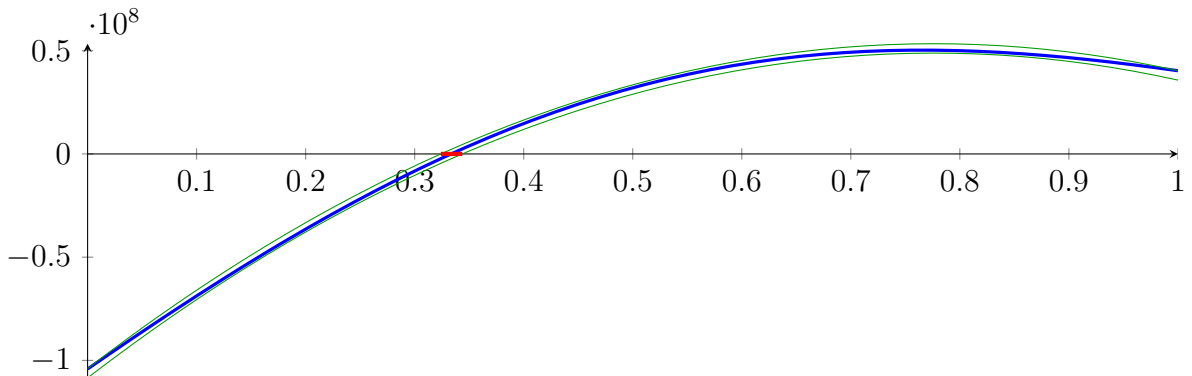
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

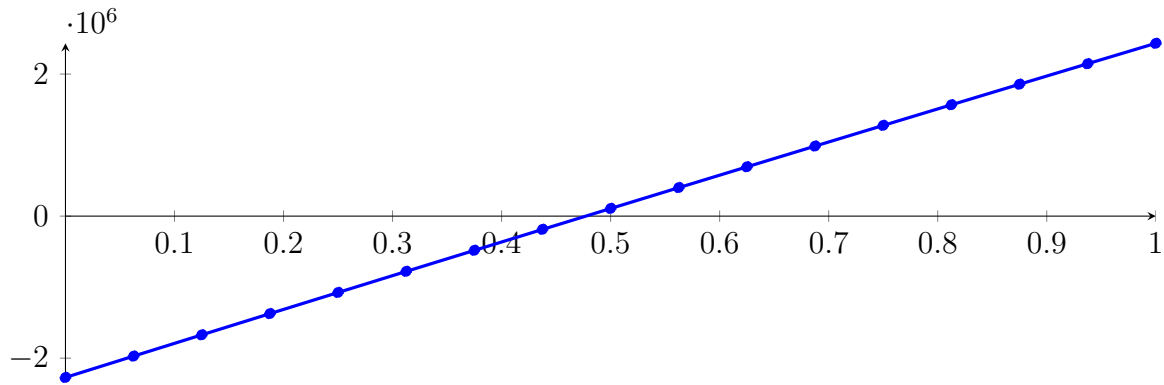
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

245.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

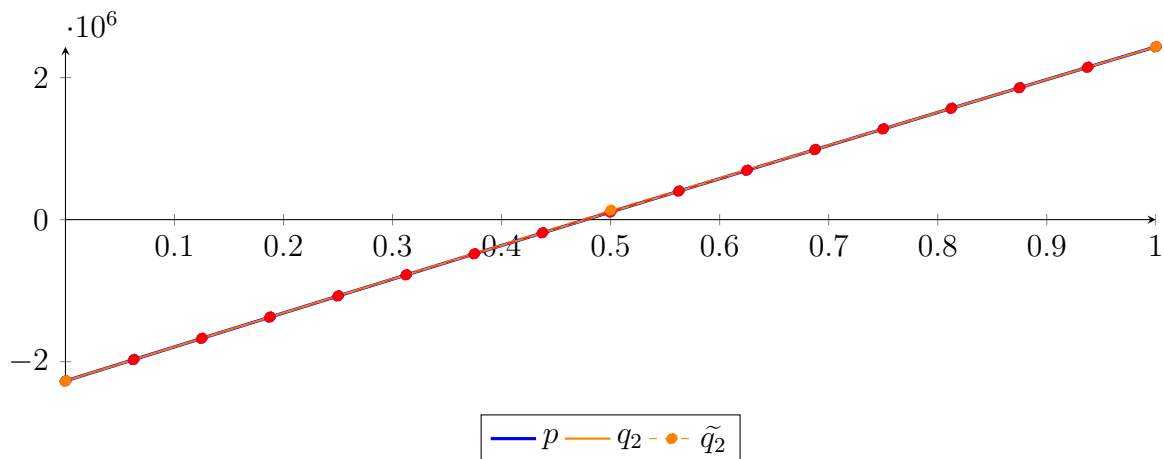
$$\begin{aligned}
 p &= -5.0162 \cdot 10^{-28} X^{16} - 1.14383 \cdot 10^{-24} X^{15} - 1.11125 \cdot 10^{-21} X^{14} - 5.84096 \cdot 10^{-19} X^{13} - 1.684 \cdot 10^{-16} X^{12} \\
 &\quad - 1.94707 \cdot 10^{-14} X^{11} + 2.73227 \cdot 10^{-12} X^{10} + 1.09727 \cdot 10^{-9} X^9 + 6.37314 \cdot 10^{-8} X^8 - 1.68645 \cdot 10^{-5} X^7 \\
 &\quad - 0.00205892 X^6 + 0.132721 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -8.18461 \cdot 10^{-291} X^{16} + 4.13745 \cdot 10^{-290} X^{15} - 5.8254 \cdot 10^{-290} X^{14} - 6.80919 \cdot 10^{-290} X^{13} \\
 &\quad + 3.48059 \cdot 10^{-289} X^{12} - 5.47245 \cdot 10^{-289} X^{11} + 4.8261 \cdot 10^{-289} X^{10} - 2.67867 \cdot 10^{-289} X^9 \\
 &\quad + 1.00314 \cdot 10^{-289} X^8 - 2.84766 \cdot 10^{-290} X^7 + 6.85072 \cdot 10^{-291} X^6 - 1.12501 \cdot 10^{-291} X^5 \\
 &\quad + 1.24731 \cdot 10^{-293} X^4 + 2.25944 \cdot 10^{-293} X^3 - 104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} \\
 &\quad + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

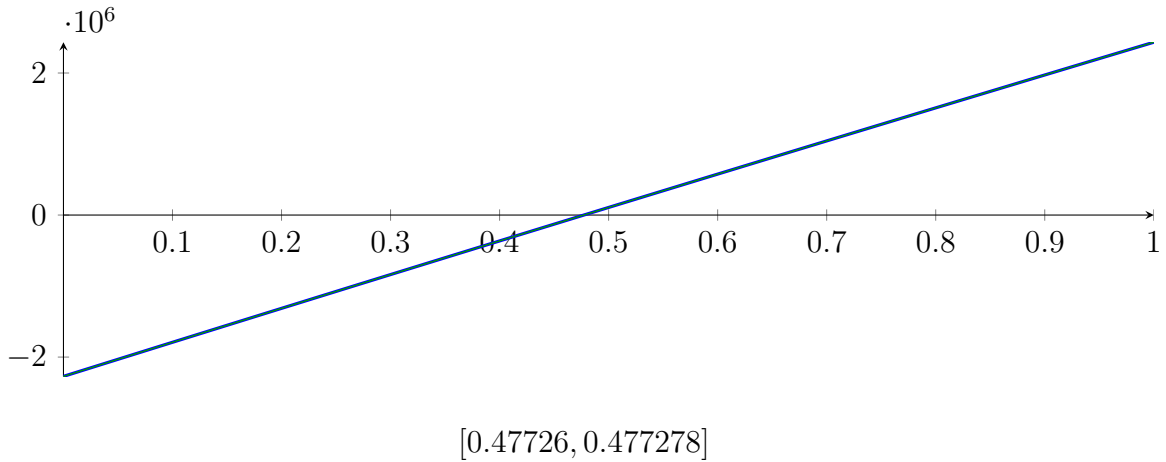
$$M = -104265X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



Longest intersection interval: $1.72301 \cdot 10^{-05}$
 \implies Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

245.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = -3.02667 \cdot 10^{-104} X^{16} - 4.019 \cdot 10^{-96} X^{15} - 2.27522 \cdot 10^{-88} X^{14} - 6.97783 \cdot 10^{-81} X^{13}$$

$$- 1.17785 \cdot 10^{-73} X^{12} - 8.12373 \cdot 10^{-67} X^{11} + 6.05916 \cdot 10^{-60} X^{10} + 1.48569 \cdot 10^{-52} X^9$$

$$+ 5.31875 \cdot 10^{-46} X^8 - 7.48919 \cdot 10^{-39} X^7 - 5.53349 \cdot 10^{-32} X^6 + 1.92471 \cdot 10^{-25} X^5$$

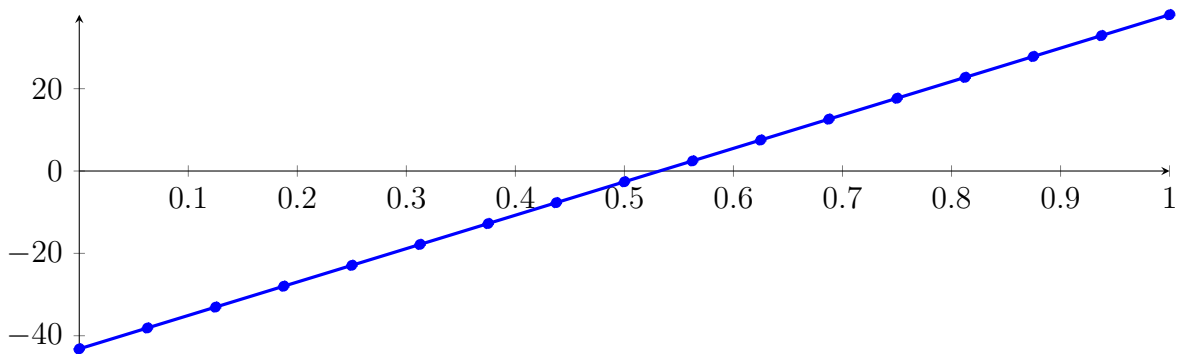
$$+ 2.00536 \cdot 10^{-18} X^4 - 4.12844 \cdot 10^{-12} X^3 - 3.09388 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$

$$= -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X)$$

$$- 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X)$$

$$+ 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X)$$

$$+ 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$

$$= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2}$$

$$\tilde{q}_2 = -1.88281 \cdot 10^{-295} X^{16} + 1.09893 \cdot 10^{-294} X^{15} - 2.26419 \cdot 10^{-294} X^{14} + 8.08223 \cdot 10^{-295} X^{13}$$

$$+ 4.92626 \cdot 10^{-294} X^{12} - 1.07605 \cdot 10^{-293} X^{11} + 1.11858 \cdot 10^{-293} X^{10} - 6.87288 \cdot 10^{-294} X^9$$

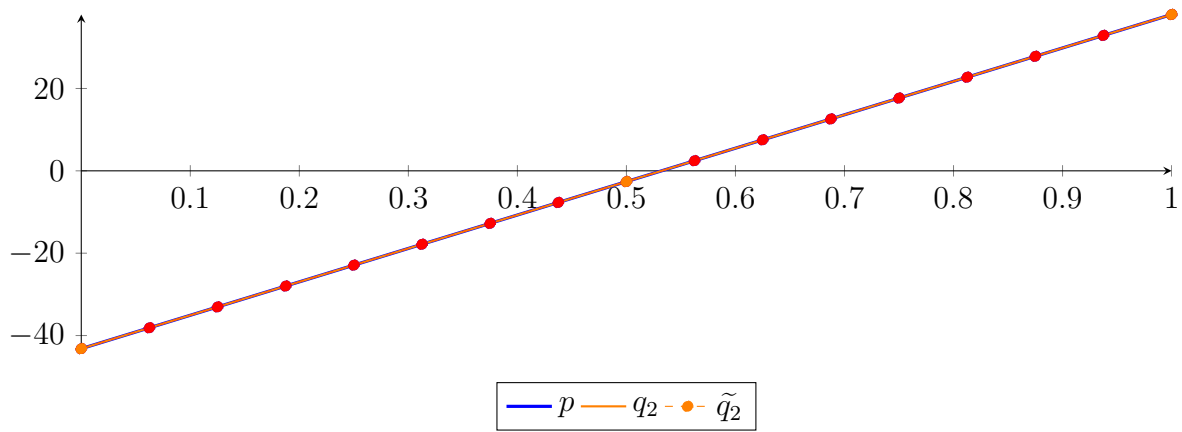
$$+ 2.54873 \cdot 10^{-294} X^8 - 5.2305 \cdot 10^{-295} X^7 + 3.18923 \cdot 10^{-296} X^6 + 1.34092 \cdot 10^{-296} X^5$$

$$- 4.89549 \cdot 10^{-297} X^4 + 5.89947 \cdot 10^{-298} X^3 - 3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$

$$= -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16}$$

$$- 12.7597B_{6,16} - 7.68778B_{7,16} - 2.61587B_{8,16} + 2.45604B_{9,16} + 7.52795B_{10,16} + 12.5999B_{11,16}$$

$$+ 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.06422 \cdot 10^{-13}$.

Bounding polynomials M and m :

$$M = -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$

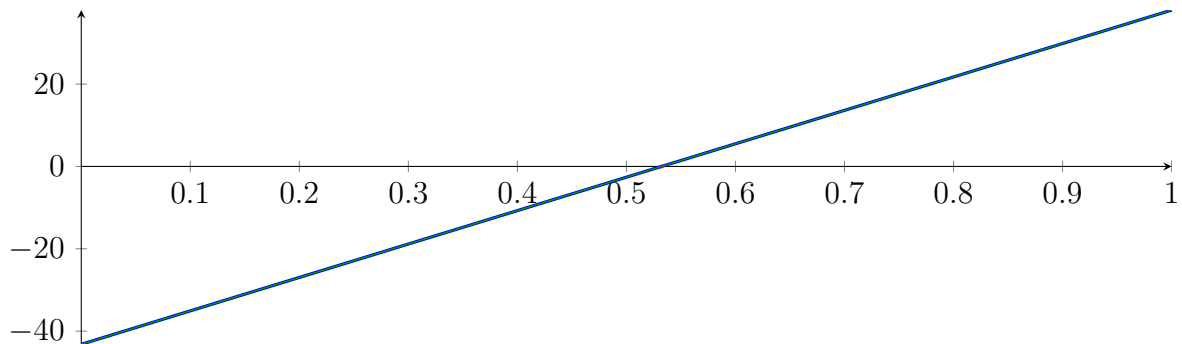
$$m = -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$

Root of M and m :

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\}$$

$$N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

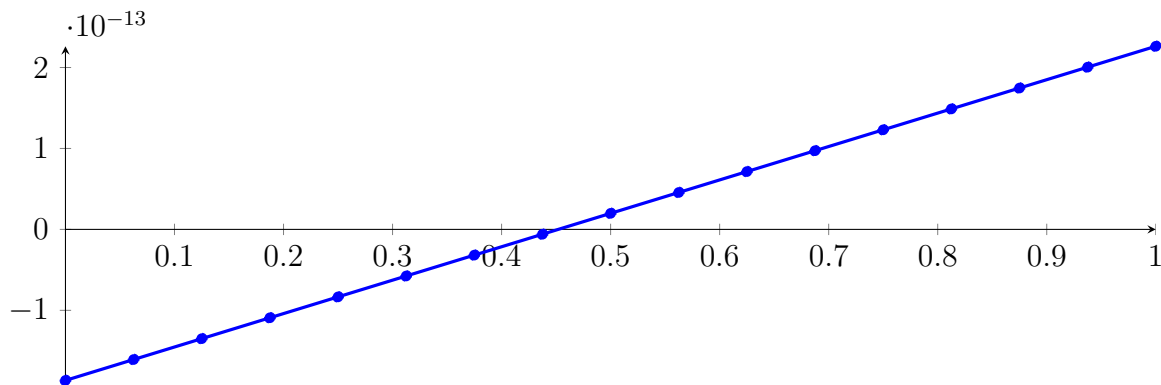
Longest intersection interval: $5.08738 \cdot 10^{-15}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

245.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

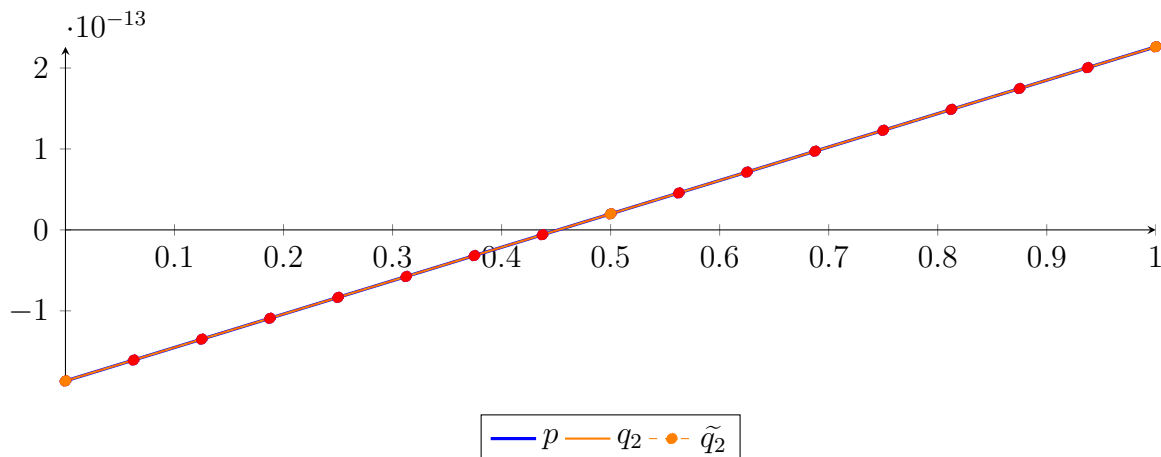
$$\begin{aligned}
 p &= -3.84502 \cdot 10^{-319} X^{16} - 1.59047 \cdot 10^{-310} X^{15} - 1.76985 \cdot 10^{-288} X^{14} - 1.06694 \cdot 10^{-266} X^{13} \\
 &\quad - 3.54011 \cdot 10^{-245} X^{12} - 4.79942 \cdot 10^{-224} X^{11} + 7.03641 \cdot 10^{-203} X^{10} + 3.39135 \cdot 10^{-181} X^9 \\
 &\quad + 2.3865 \cdot 10^{-160} X^8 - 6.60529 \cdot 10^{-139} X^7 - 9.59319 \cdot 10^{-118} X^6 + 6.55895 \cdot 10^{-97} X^5 + 1.34328 \\
 &\quad \cdot 10^{-75} X^4 - 5.43584 \cdot 10^{-55} X^3 - 8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13} \\
 &= -1.86598 \cdot 10^{-13} B_{0,16}(X) - 1.60795 \cdot 10^{-13} B_{1,16}(X) - 1.34993 \cdot 10^{-13} B_{2,16}(X) - 1.0919 \\
 &\quad \cdot 10^{-13} B_{3,16}(X) - 8.33872 \cdot 10^{-14} B_{4,16}(X) - 5.75845 \cdot 10^{-14} B_{5,16}(X) - 3.17818 \cdot 10^{-14} B_{6,16}(X) \\
 &\quad - 5.97912 \cdot 10^{-15} B_{7,16}(X) + 1.98236 \cdot 10^{-14} B_{8,16}(X) + 4.56263 \cdot 10^{-14} B_{9,16}(X) + 7.1429 \\
 &\quad \cdot 10^{-14} B_{10,16}(X) + 9.72317 \cdot 10^{-14} B_{11,16}(X) + 1.23034 \cdot 10^{-13} B_{12,16}(X) + 1.48837 \\
 &\quad \cdot 10^{-13} B_{13,16}(X) + 1.7464 \cdot 10^{-13} B_{14,16}(X) + 2.00443 \cdot 10^{-13} B_{15,16}(X) + 2.26245 \cdot 10^{-13} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13} \\
 &= -1.86598 \cdot 10^{-13} B_{0,2} + 1.98236 \cdot 10^{-14} B_{1,2} + 2.26245 \cdot 10^{-13} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -8.08289 \cdot 10^{-310} X^{16} + 4.33931 \cdot 10^{-309} X^{15} - 8.19282 \cdot 10^{-309} X^{14} + 3.30456 \cdot 10^{-309} X^{13} \\
 &\quad + 1.10762 \cdot 10^{-308} X^{12} - 2.01579 \cdot 10^{-308} X^{11} + 1.55412 \cdot 10^{-308} X^{10} - 6.55981 \cdot 10^{-309} X^9 \\
 &\quad + 2.12881 \cdot 10^{-309} X^8 - 1.08538 \cdot 10^{-309} X^7 + 5.4154 \cdot 10^{-310} X^6 - 1.38567 \cdot 10^{-310} X^5 + 9.27245 \\
 &\quad \cdot 10^{-312} X^4 + 1.80627 \cdot 10^{-312} X^3 - 8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13} \\
 &= -1.86598 \cdot 10^{-13} B_{0,16} - 1.60795 \cdot 10^{-13} B_{1,16} - 1.34993 \cdot 10^{-13} B_{2,16} - 1.0919 \cdot 10^{-13} B_{3,16} - 8.33872 \\
 &\quad \cdot 10^{-14} B_{4,16} - 5.75845 \cdot 10^{-14} B_{5,16} - 3.17818 \cdot 10^{-14} B_{6,16} - 5.97912 \cdot 10^{-15} B_{7,16} + 1.98236 \cdot 10^{-14} B_{8,16} \\
 &\quad + 4.56263 \cdot 10^{-14} B_{9,16} + 7.1429 \cdot 10^{-14} B_{10,16} + 9.72317 \cdot 10^{-14} B_{11,16} + 1.23034 \cdot 10^{-13} B_{12,16} \\
 &\quad + 1.48837 \cdot 10^{-13} B_{13,16} + 1.7464 \cdot 10^{-13} B_{14,16} + 2.00443 \cdot 10^{-13} B_{15,16} + 2.26245 \cdot 10^{-13} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.71792 \cdot 10^{-56}$.

Bounding polynomials M and m :

$$M = -8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13}$$

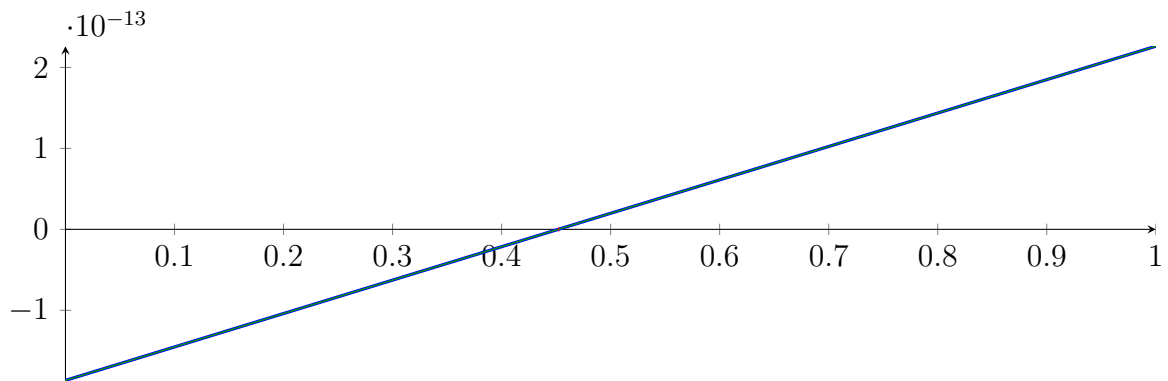
$$m = -8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13}$$

Root of M and m :

$$N(M) = \{0.451983, 5.15577 \cdot 10^{20}\}$$

$$N(m) = \{0.451983, 5.15577 \cdot 10^{20}\}$$

Intersection intervals:



$$[0.451983, 0.451983]$$

Longest intersection interval: $1.31668 \cdot 10^{-43}$

\implies Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

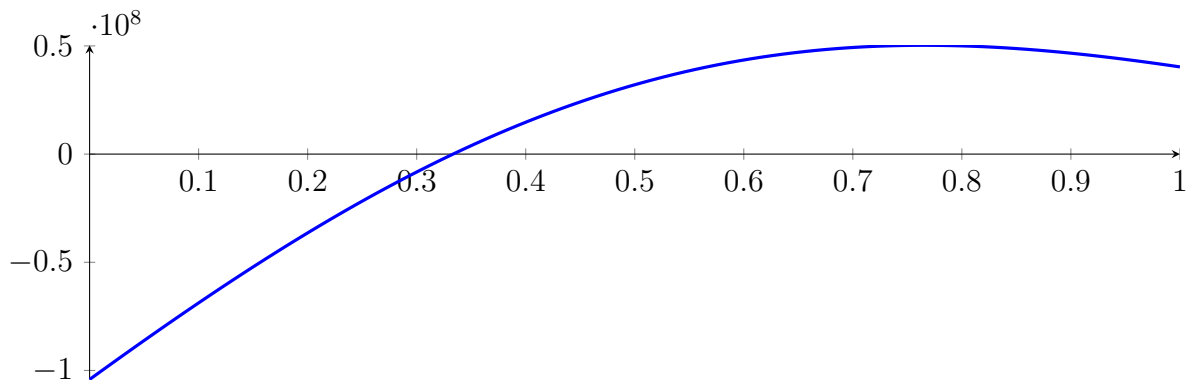
245.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 5!

245.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

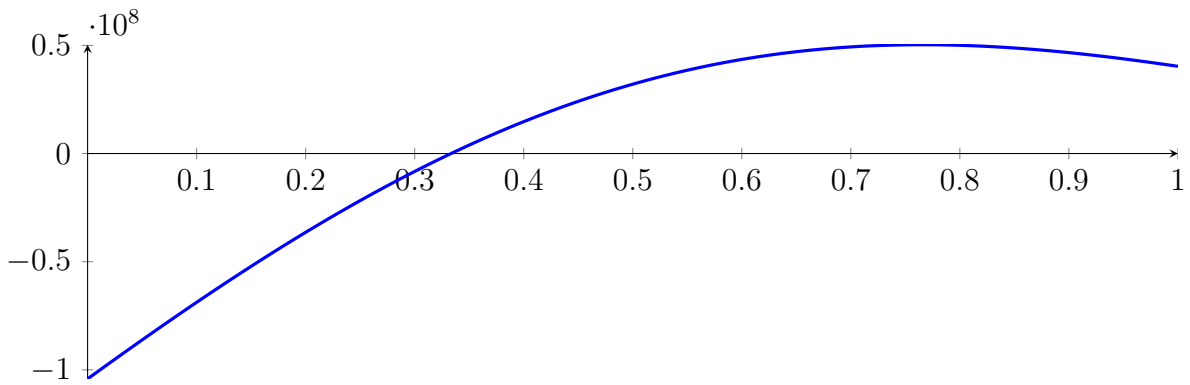
with precision $\varepsilon = 1 \cdot 10^{-32}$.

246 Running CubeClip on f_{16} with epsilon 32

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

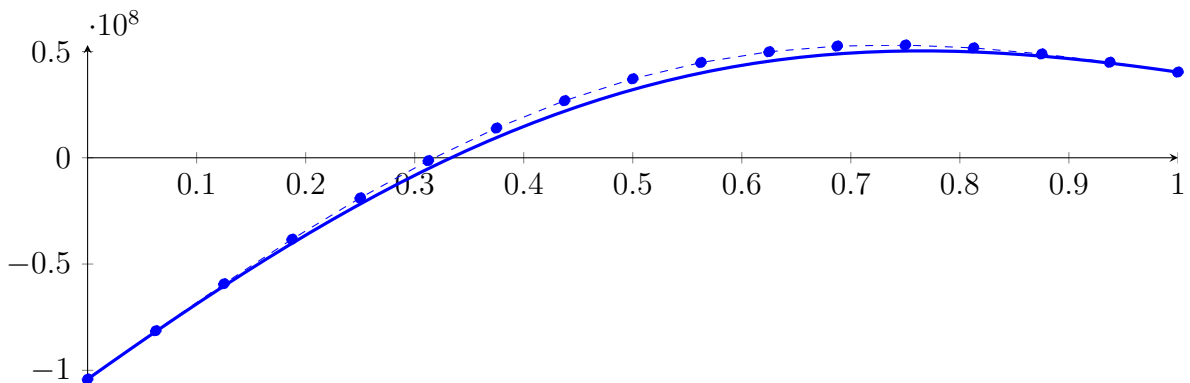
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



246.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 1.89955 \cdot 10^{-288} X^{16} - 1.4851 \cdot 10^{-287} X^{15} + 5.12896 \cdot 10^{-287} X^{14} - 1.02224 \cdot 10^{-286} X^{13}$$

$$+ 1.29027 \cdot 10^{-286} X^{12} - 1.07058 \cdot 10^{-286} X^{11} + 6.00739 \cdot 10^{-287} X^{10} - 2.54352 \cdot 10^{-287} X^9$$

$$+ 1.12089 \cdot 10^{-287} X^8 - 5.8673 \cdot 10^{-288} X^7 + 2.52637 \cdot 10^{-288} X^6 - 6.79624 \cdot 10^{-289} X^5$$

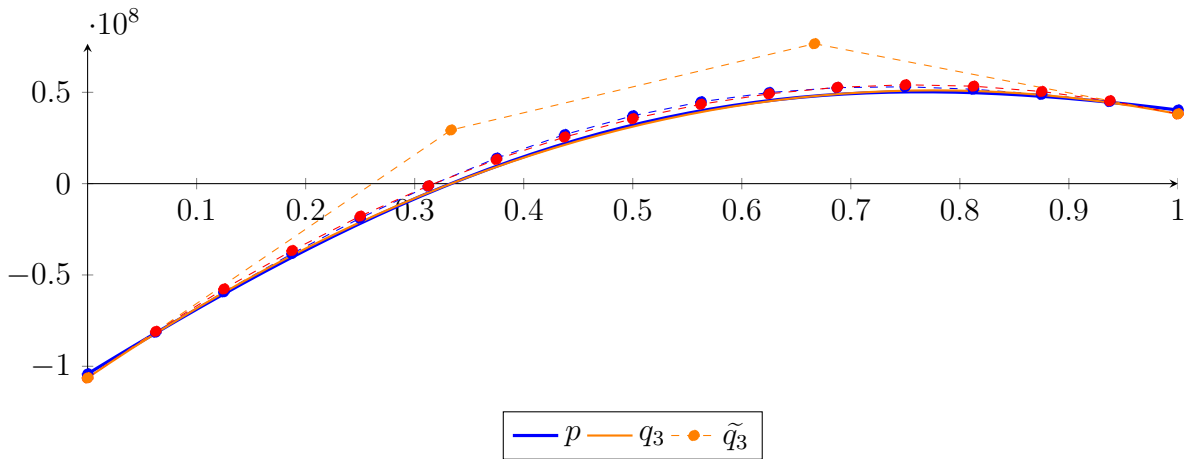
$$+ 9.36341 \cdot 10^{-290} X^4 + 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18551 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91815 \cdot 10^7 B_{10,16} + 5.27353 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

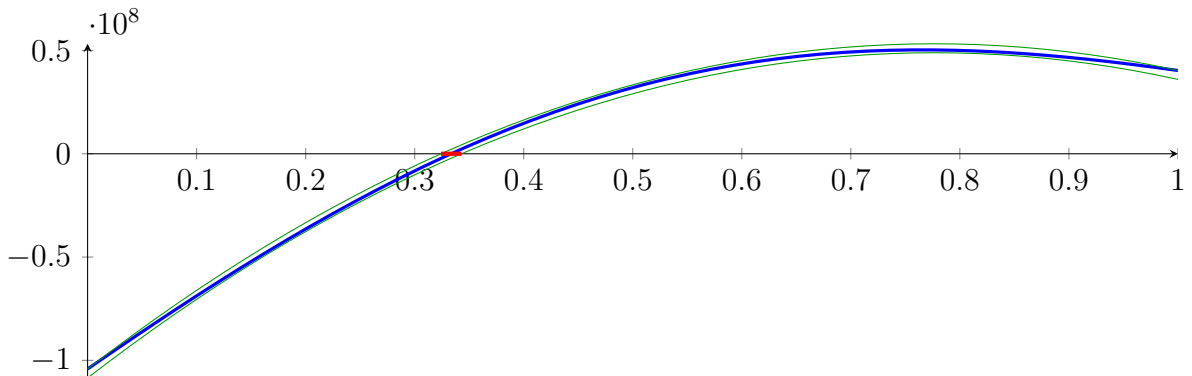
$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\}$$

$$N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

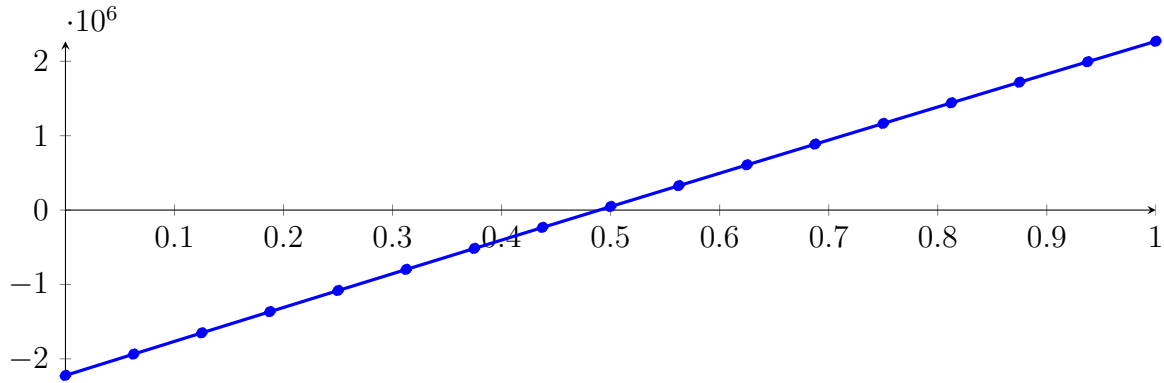
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

246.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

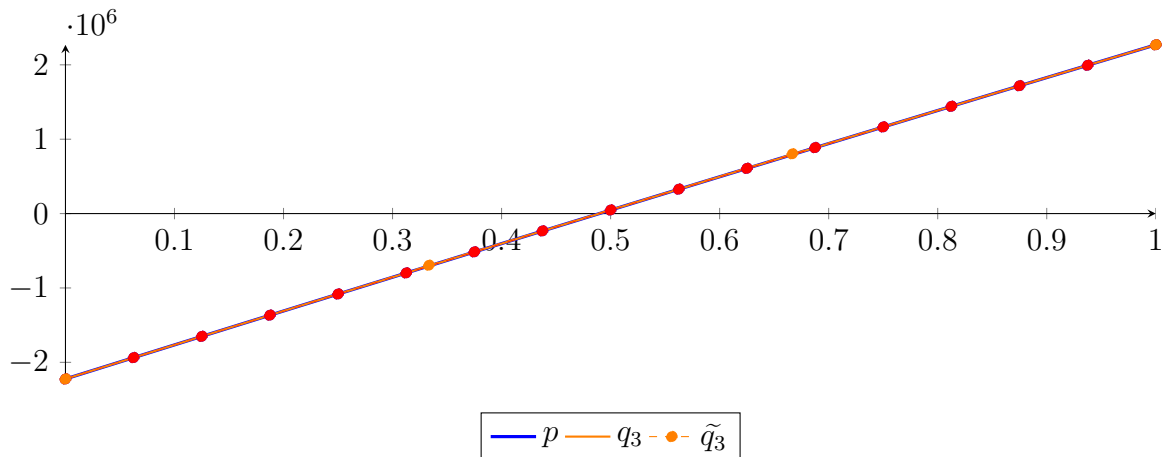
$$\begin{aligned}
 p &= -2.37433 \cdot 10^{-28} X^{16} - 5.67363 \cdot 10^{-25} X^{15} - 5.77631 \cdot 10^{-22} X^{14} - 3.1818 \cdot 10^{-19} X^{13} - 9.6142 \cdot 10^{-17} X^{12} \\
 &\quad - 1.16549 \cdot 10^{-14} X^{11} + 1.71065 \cdot 10^{-12} X^{10} + 7.20611 \cdot 10^{-10} X^9 + 4.39147 \cdot 10^{-08} X^8 - 1.21542 \cdot 10^{-05} X^7 \\
 &\quad - 0.00155624 X^6 + 0.10496 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 4.41595 \cdot 10^{-291} X^{16} - 1.48598 \cdot 10^{-290} X^{15} - 2.81102 \cdot 10^{-290} X^{14} + 2.65129 \cdot 10^{-289} X^{13} \\
 &\quad - 7.34593 \cdot 10^{-289} X^{12} + 1.17654 \cdot 10^{-288} X^{11} - 1.23119 \cdot 10^{-288} X^{10} + 8.63493 \cdot 10^{-289} X^9 \\
 &\quad - 3.92909 \cdot 10^{-289} X^8 + 1.01265 \cdot 10^{-289} X^7 - 6.13644 \cdot 10^{-291} X^6 - 3.92664 \cdot 10^{-291} X^5 \\
 &\quad + 9.04488 \cdot 10^{-292} X^4 - 700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} \\
 &\quad + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m :

$$M = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

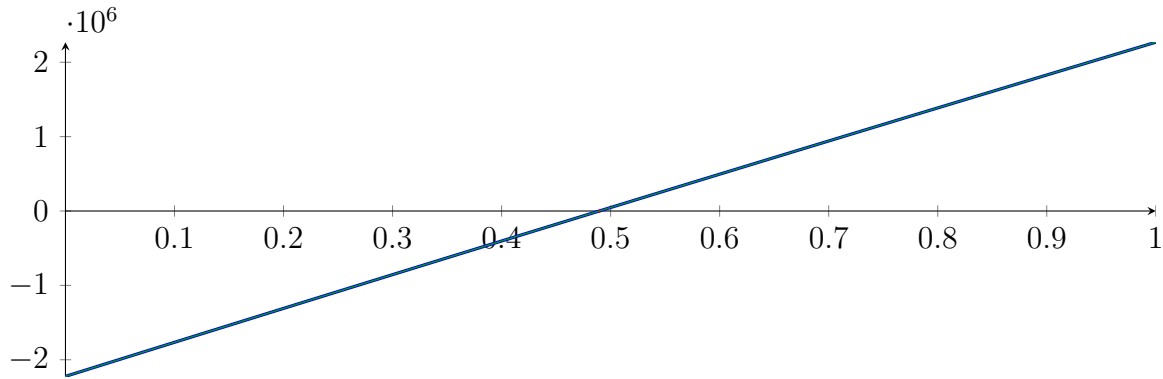
$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\}$$

$$N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

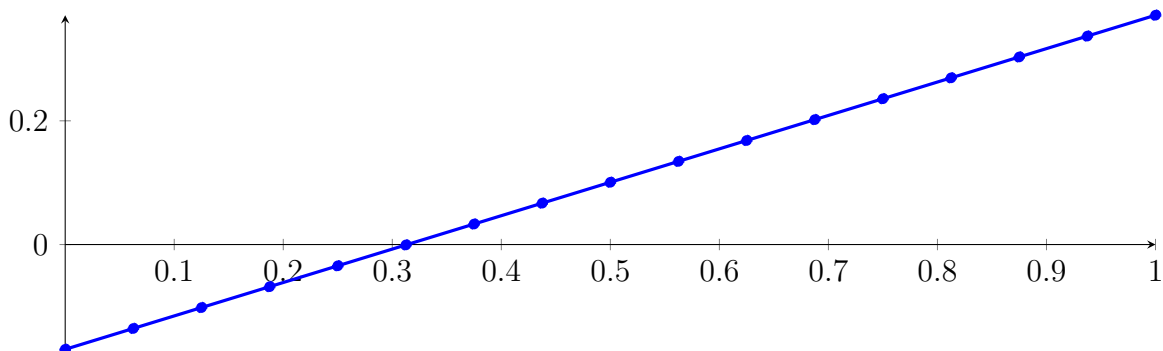
Longest intersection interval: $1.20174 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

246.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.49274 \cdot 10^{-139} X^{16} - 8.96277 \cdot 10^{-129} X^{15} - 7.623 \cdot 10^{-119} X^{14} - 3.51238 \cdot 10^{-109} X^{13} \\ &\quad - 8.90739 \cdot 10^{-100} X^{12} - 9.22984 \cdot 10^{-91} X^{11} + 1.03426 \cdot 10^{-81} X^{10} + 3.80998 \cdot 10^{-72} X^9 \\ &\quad + 2.04919 \cdot 10^{-63} X^8 - 4.33497 \cdot 10^{-54} X^7 - 4.81204 \cdot 10^{-45} X^6 + 2.51462 \cdot 10^{-36} X^5 \\ &\quad + 3.93622 \cdot 10^{-27} X^4 - 1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148X - 0.169396 \\ &= -0.169396B_{0,16}(X) - 0.135637B_{1,16}(X) - 0.101877B_{2,16}(X) - 0.068118B_{3,16}(X) \\ &\quad - 0.0343588B_{4,16}(X) - 0.000599488B_{5,16}(X) + 0.0331598B_{6,16}(X) \\ &\quad + 0.066919B_{7,16}(X) + 0.100678B_{8,16}(X) + 0.134438B_{9,16}(X) + 0.168197B_{10,16}(X) \\ &\quad + 0.201956B_{11,16}(X) + 0.235715B_{12,16}(X) + 0.269475B_{13,16}(X) \\ &\quad + 0.303234B_{14,16}(X) + 0.336993B_{15,16}(X) + 0.370752B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

$$= -0.169396 B_{0,3} + 0.0106536 B_{1,3} + 0.190703 B_{2,3} + 0.370752 B_{3,3}$$

$$\tilde{q}_3 = 8.03185 \cdot 10^{-297} X^{16} - 6.20841 \cdot 10^{-296} X^{15} + 2.13274 \cdot 10^{-295} X^{14} - 4.26614 \cdot 10^{-295} X^{13}$$

$$+ 5.47461 \cdot 10^{-295} X^{12} - 4.70265 \cdot 10^{-295} X^{11} + 2.78551 \cdot 10^{-295} X^{10} - 1.22442 \cdot 10^{-295} X^9$$

$$+ 4.88954 \cdot 10^{-296} X^8 - 2.11494 \cdot 10^{-296} X^7 + 8.20665 \cdot 10^{-297} X^6 - 2.15458 \cdot 10^{-297} X^5$$

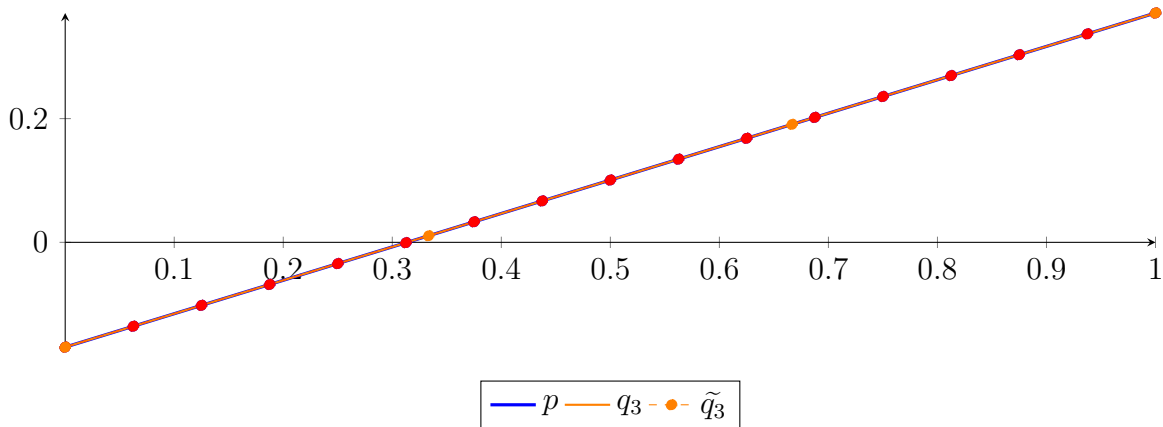
$$+ 3.01517 \cdot 10^{-298} X^4 - 1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

$$= -0.169396 B_{0,16} - 0.135637 B_{1,16} - 0.101877 B_{2,16} - 0.068118 B_{3,16} - 0.0343588 B_{4,16}$$

$$- 0.000599488 B_{5,16} + 0.0331598 B_{6,16} + 0.066919 B_{7,16} + 0.100678 B_{8,16}$$

$$+ 0.134438 B_{9,16} + 0.168197 B_{10,16} + 0.201956 B_{11,16} + 0.235715 B_{12,16}$$

$$+ 0.269475 B_{13,16} + 0.303234 B_{14,16} + 0.336993 B_{15,16} + 0.370752 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 5.62317 \cdot 10^{-29}$.

Bounding polynomials M and m :

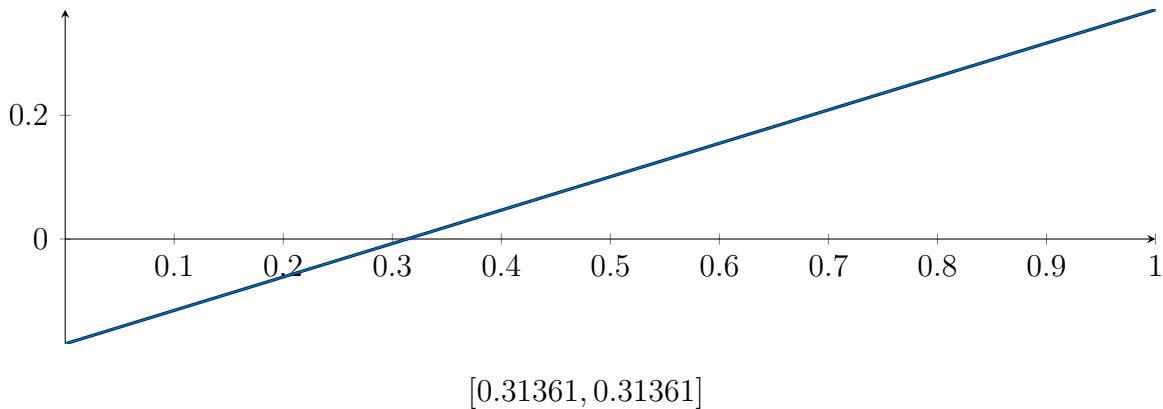
$$M = -1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

$$m = -1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

Root of M and m :

$$N(M) = \{-1.43506 \cdot 10^9, 0.31361, 3.09167 \cdot 10^8\} \quad N(m) = \{-1.43506 \cdot 10^9, 0.31361, 3.09167 \cdot 10^8\}$$

Intersection intervals:



Longest intersection interval: $2.08208 \cdot 10^{-28}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

246.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

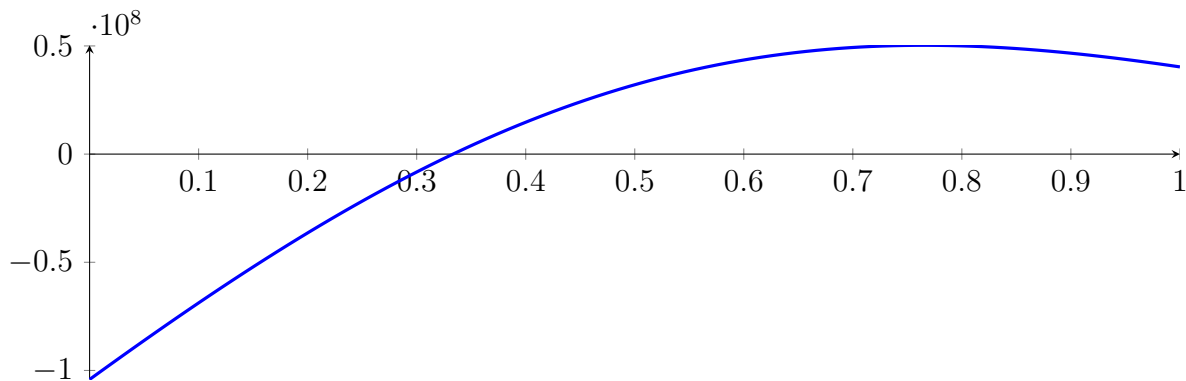
Reached interval $[0.333333, 0.333333]$ **without sign change** at depth 4!

$$p(0) = -8.88188e-08 - p(1) -8.88188e-08$$

246.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

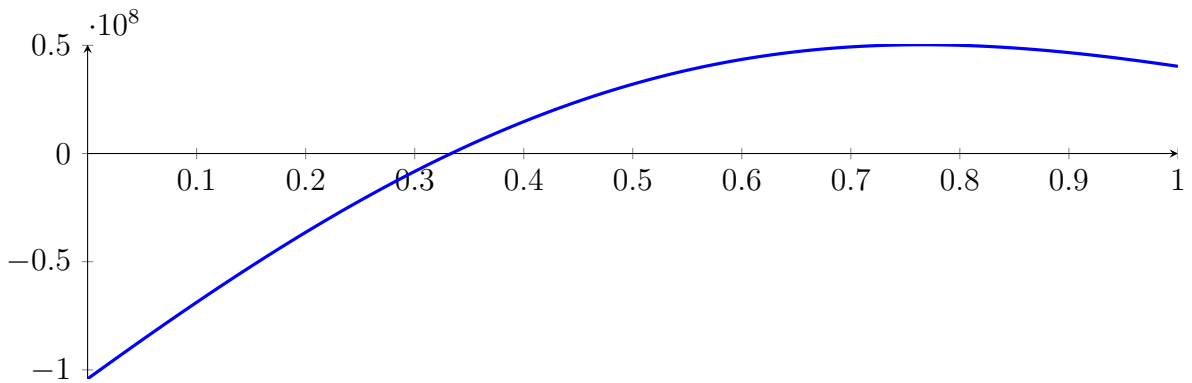
with precision $\varepsilon = 1 \cdot 10^{-32}$.

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$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

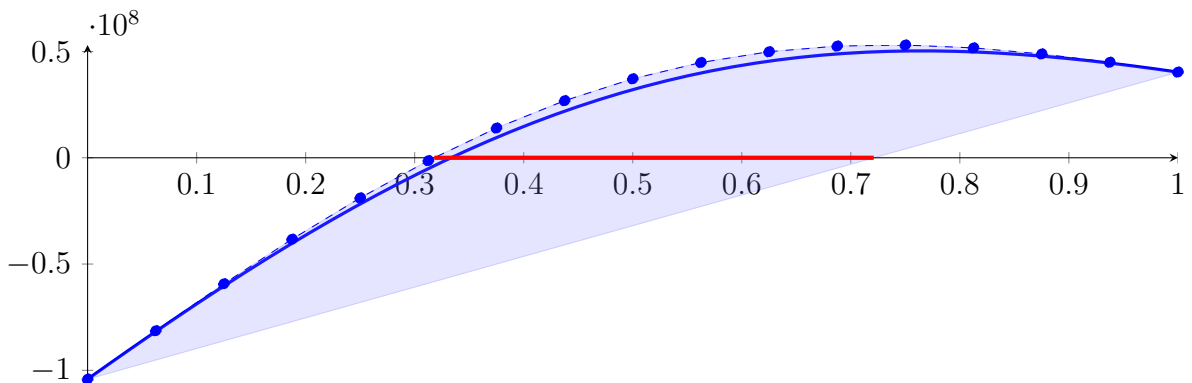
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



247.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

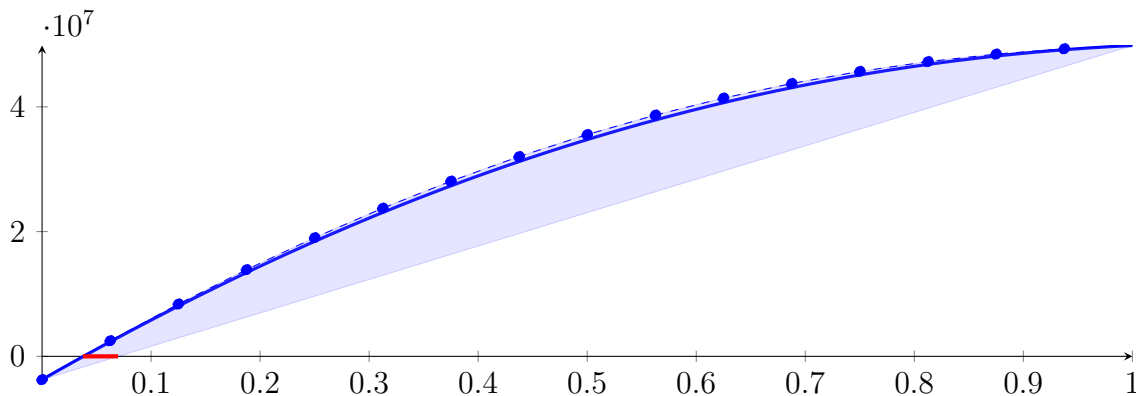
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

247.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.83858 \cdot 10^{-07} X^{16} - 5.37355 \cdot 10^{-05} X^{15} - 0.00254146 X^{14} - 0.064977 X^{13} - 0.909205 X^{12} \\ &\quad - 5.03924 X^{11} + 36.4549 X^{10} + 692.921 X^9 + 1886.97 X^8 - 25792 X^7 - 149671 X^6 + 492605 X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

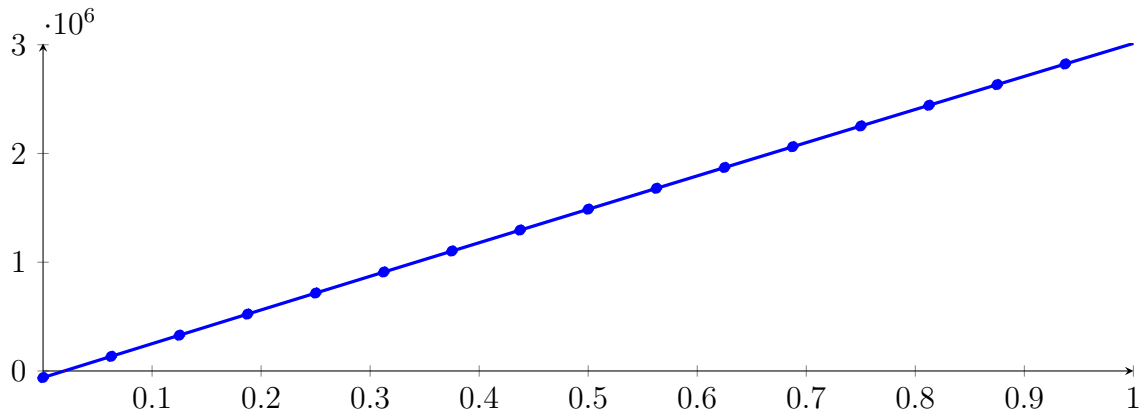
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

247.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -6.78234 \cdot 10^{-31} X^{16} - 2.34473 \cdot 10^{-27} X^{15} - 3.45581 \cdot 10^{-24} X^{14} - 2.75921 \cdot 10^{-21} X^{13} - 1.21242 \cdot 10^{-18} X^{12} \\ &\quad - 2.17572 \cdot 10^{-16} X^{11} + 4.23544 \cdot 10^{-14} X^{10} + 2.70025 \cdot 10^{-11} X^9 + 2.51306 \cdot 10^{-09} X^8 - 9.23474 \cdot 10^{-07} X^7 \\ &\quad - 0.000177459 X^6 + 0.0161038 X^5 + 4.36155 X^4 - 234.216 X^3 - 45622.2 X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

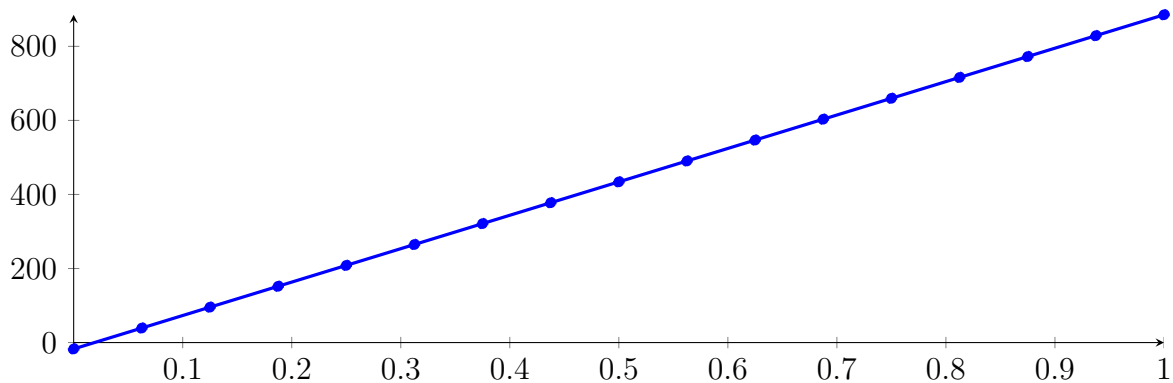
Longest intersection interval: 0.000289554

⇒ Selective recursion: interval 1: $[0.333333, 0.333337]$,

247.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.65599 \cdot 10^{-87} X^{16} - 1.97733 \cdot 10^{-80} X^{15} - 1.00659 \cdot 10^{-73} X^{14} - 2.77601 \cdot 10^{-67} X^{13} \\
 &\quad - 4.21367 \cdot 10^{-61} X^{12} - 2.61333 \cdot 10^{-55} X^{11} + 1.75275 \cdot 10^{-49} X^{10} + 3.8646 \cdot 10^{-43} X^9 \\
 &\quad + 1.2441 \cdot 10^{-37} X^8 - 1.57525 \cdot 10^{-31} X^7 - 1.04661 \cdot 10^{-25} X^6 + 3.27355 \cdot 10^{-20} X^5 \\
 &\quad + 3.06701 \cdot 10^{-14} X^4 - 5.67777 \cdot 10^{-09} X^3 - 0.00382618 X^2 + 902.448 X - 17.178 \\
 &= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
 &\quad + 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
 &\quad + 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
 &\quad + 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

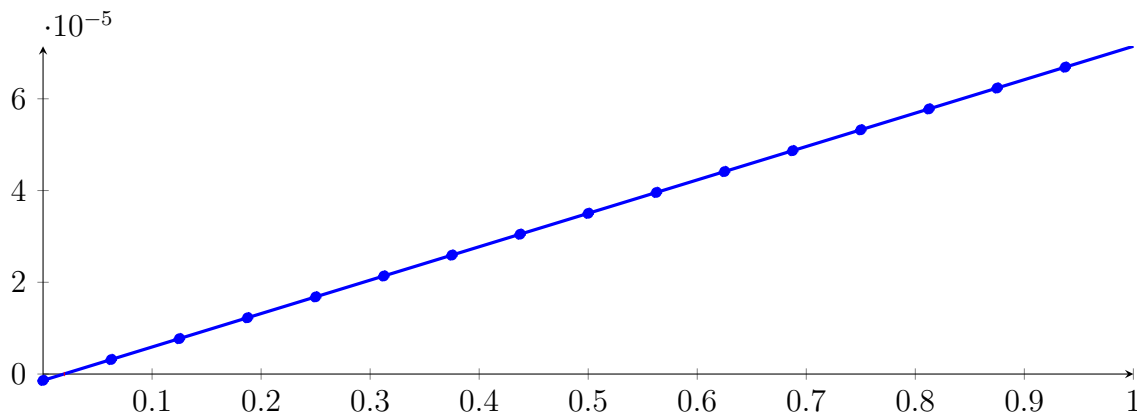
Longest intersection interval: $8.07045 \cdot 10^{-08}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

247.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -5.36315 \cdot 10^{-201} X^{16} - 7.93495 \cdot 10^{-187} X^{15} - 5.0052 \cdot 10^{-173} X^{14} - 1.71037 \cdot 10^{-159} X^{13} \\
 &\quad - 3.21686 \cdot 10^{-146} X^{12} - 2.47211 \cdot 10^{-133} X^{11} + 2.05446 \cdot 10^{-120} X^{10} + 5.61285 \cdot 10^{-107} X^9 \\
 &\quad + 2.23891 \cdot 10^{-94} X^8 - 3.51264 \cdot 10^{-81} X^7 - 2.89181 \cdot 10^{-68} X^6 + 1.12075 \cdot 10^{-55} X^5 + 1.30109 \\
 &\quad \cdot 10^{-42} X^4 - 2.98449 \cdot 10^{-30} X^3 - 2.49207 \cdot 10^{-17} X^2 + 7.28316 \cdot 10^{-05} X - 1.38634 \cdot 10^{-06} \\
 &= -1.38634 \cdot 10^{-06} B_{0,16}(X) + 3.16564 \cdot 10^{-06} B_{1,16}(X) + 7.71761 \cdot 10^{-06} B_{2,16}(X) + 1.22696 \\
 &\quad \cdot 10^{-05} B_{3,16}(X) + 1.68216 \cdot 10^{-05} B_{4,16}(X) + 2.13735 \cdot 10^{-05} B_{5,16}(X) + 2.59255 \cdot 10^{-05} B_{6,16}(X) \\
 &\quad + 3.04775 \cdot 10^{-05} B_{7,16}(X) + 3.50295 \cdot 10^{-05} B_{8,16}(X) + 3.95814 \cdot 10^{-05} B_{9,16}(X) + 4.41334 \\
 &\quad \cdot 10^{-05} B_{10,16}(X) + 4.86854 \cdot 10^{-05} B_{11,16}(X) + 5.32374 \cdot 10^{-05} B_{12,16}(X) + 5.77893 \cdot 10^{-05} B_{13,16}(X) \\
 &\quad + 6.23413 \cdot 10^{-05} B_{14,16}(X) + 6.68933 \cdot 10^{-05} B_{15,16}(X) + 7.14453 \cdot 10^{-05} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

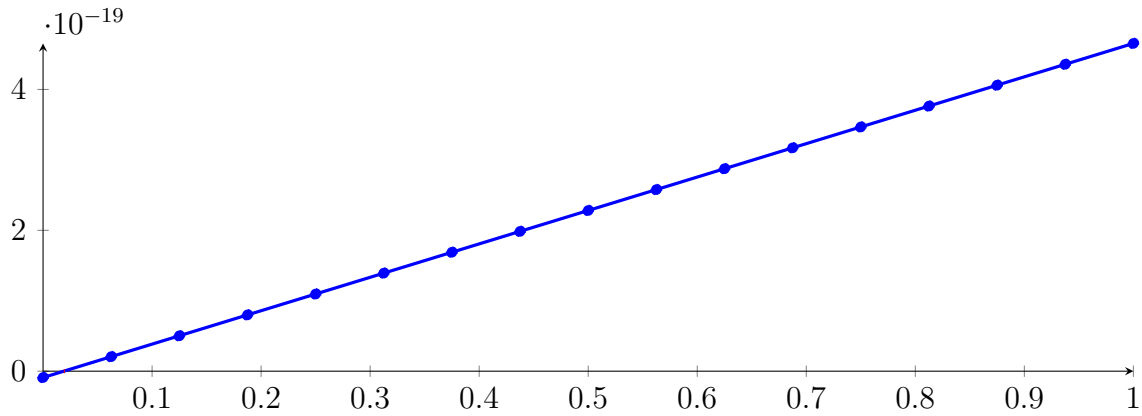
Longest intersection interval: $6.51314 \cdot 10^{-15}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

247.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.70149 \cdot 10^{-323} X^{16} + 1.97819 \cdot 10^{-322} X^{15} - 4.34527 \cdot 10^{-322} X^{14} + 3.97182 \cdot 10^{-322} X^{13} \\
 &\quad - 1.87464 \cdot 10^{-316} X^{12} - 2.21189 \cdot 10^{-289} X^{11} + 2.82229 \cdot 10^{-262} X^{10} + 1.18385 \cdot 10^{-234} X^9 \\
 &\quad + 7.25038 \cdot 10^{-208} X^8 - 1.74649 \cdot 10^{-180} X^7 - 2.20756 \cdot 10^{-153} X^6 + 1.31359 \cdot 10^{-126} X^5 + 2.34136 \\
 &\quad \cdot 10^{-99} X^4 - 8.24597 \cdot 10^{-73} X^3 - 1.05716 \cdot 10^{-45} X^2 + 4.74362 \cdot 10^{-19} X - 9.02941 \cdot 10^{-21} \\
 &= -9.02941 \cdot 10^{-21} B_{0,16}(X) + 2.06182 \cdot 10^{-20} B_{1,16}(X) + 5.02659 \cdot 10^{-20} B_{2,16}(X) + 7.99135 \\
 &\quad \cdot 10^{-20} B_{3,16}(X) + 1.09561 \cdot 10^{-19} B_{4,16}(X) + 1.39209 \cdot 10^{-19} B_{5,16}(X) + 1.68856 \cdot 10^{-19} B_{6,16}(X) \\
 &\quad + 1.98504 \cdot 10^{-19} B_{7,16}(X) + 2.28152 \cdot 10^{-19} B_{8,16}(X) + 2.57799 \cdot 10^{-19} B_{9,16}(X) + 2.87447 \\
 &\quad \cdot 10^{-19} B_{10,16}(X) + 3.17095 \cdot 10^{-19} B_{11,16}(X) + 3.46742 \cdot 10^{-19} B_{12,16}(X) + 3.7639 \cdot 10^{-19} B_{13,16}(X) \\
 &\quad + 4.06038 \cdot 10^{-19} B_{14,16}(X) + 4.35685 \cdot 10^{-19} B_{15,16}(X) + 4.65333 \cdot 10^{-19} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

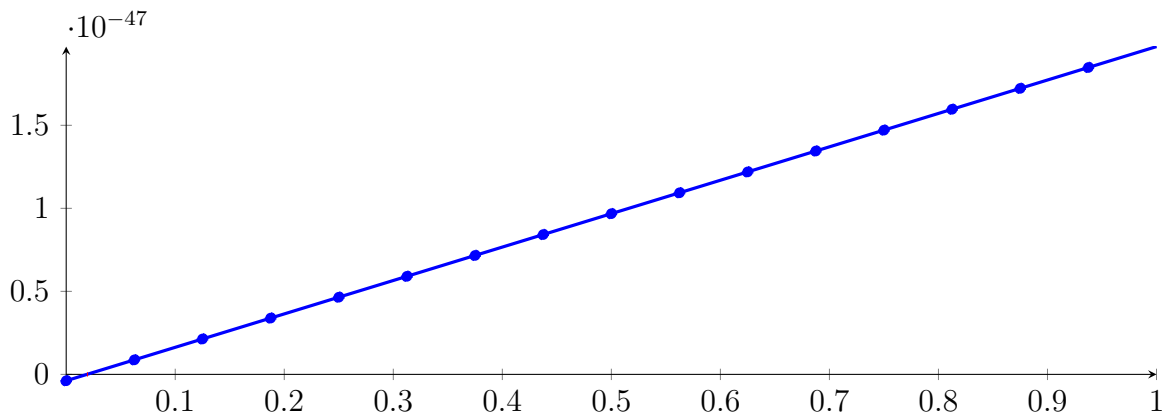
Longest intersection interval: $4.2421 \cdot 10^{-29}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

247.7 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 1.52489 \cdot 10^{-352} X^{16} + 4.56006 \cdot 10^{-351} X^{15} - 3.36159 \cdot 10^{-350} X^{14} + 2.87148 \cdot 10^{-350} X^{13} \\
 &\quad - 1.35884 \cdot 10^{-349} X^{12} + 1.13318 \cdot 10^{-349} X^{11} - 1.84828 \cdot 10^{-349} X^{10} + 7.52411 \cdot 10^{-350} X^9 \\
 &\quad - 2.19453 \cdot 10^{-350} X^8 + 2.78671 \cdot 10^{-351} X^7 - 1.28647 \cdot 10^{-323} X^6 + 1.80453 \cdot 10^{-268} X^5 + 7.58214 \\
 &\quad \cdot 10^{-213} X^4 - 6.29484 \cdot 10^{-158} X^3 - 1.90241 \cdot 10^{-102} X^2 + 2.01229 \cdot 10^{-47} X - 3.83037 \cdot 10^{-49} \\
 &= -3.83037 \cdot 10^{-49} B_{0,16}(X) + 8.74646 \cdot 10^{-49} B_{1,16}(X) + 2.13233 \cdot 10^{-48} B_{2,16}(X) + 3.39001 \\
 &\quad \cdot 10^{-48} B_{3,16}(X) + 4.6477 \cdot 10^{-48} B_{4,16}(X) + 5.90538 \cdot 10^{-48} B_{5,16}(X) + 7.16306 \cdot 10^{-48} B_{6,16}(X) \\
 &\quad + 8.42074 \cdot 10^{-48} B_{7,16}(X) + 9.67843 \cdot 10^{-48} B_{8,16}(X) + 1.09361 \cdot 10^{-47} B_{9,16}(X) + 1.21938 \\
 &\quad \cdot 10^{-47} B_{10,16}(X) + 1.34515 \cdot 10^{-47} B_{11,16}(X) + 1.47092 \cdot 10^{-47} B_{12,16}(X) + 1.59668 \cdot 10^{-47} B_{13,16}(X) \\
 &\quad + 1.72245 \cdot 10^{-47} B_{14,16}(X) + 1.84822 \cdot 10^{-47} B_{15,16}(X) + 1.97399 \cdot 10^{-47} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $1.79954 \cdot 10^{-57}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

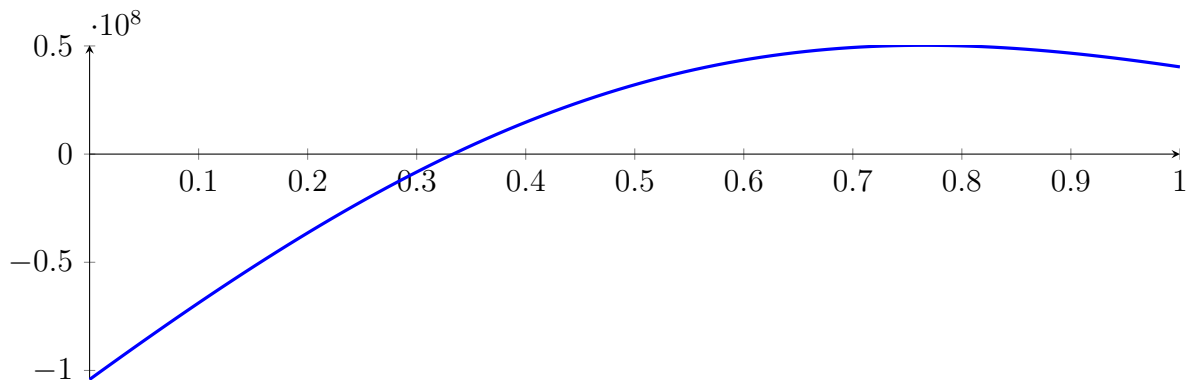
247.8 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 8!

247.9 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

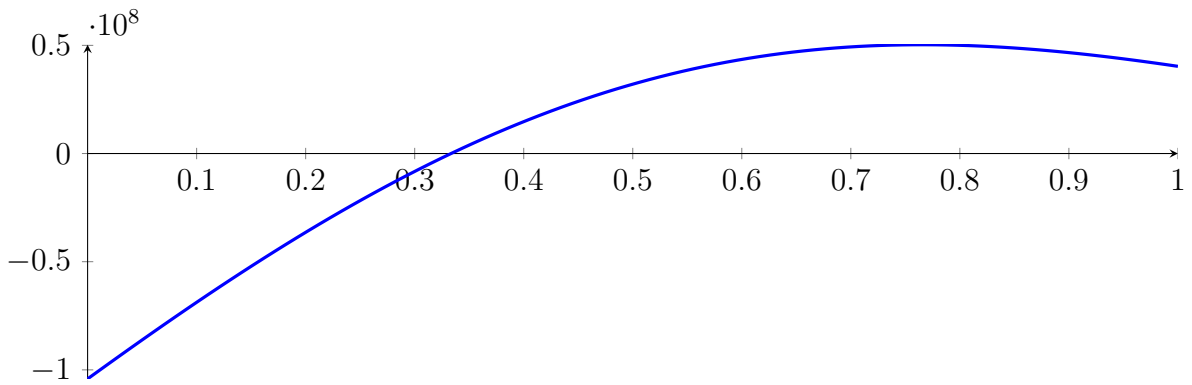
with precision $\varepsilon = 1 \cdot 10^{-64}$.

248 Running QuadClip on f_{16} with epsilon 64

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

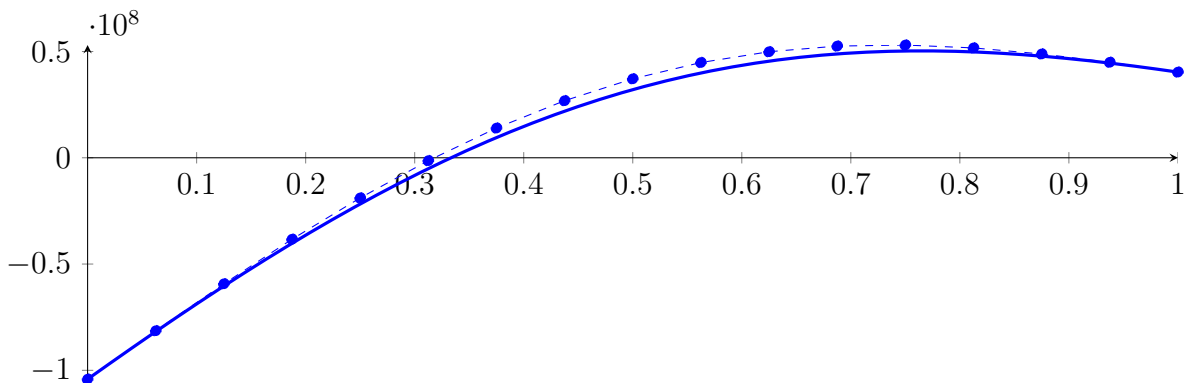
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



248.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = 3.74473 \cdot 10^{-288} X^{16} - 3.08889 \cdot 10^{-287} X^{15} + 1.17443 \cdot 10^{-286} X^{14} - 2.71603 \cdot 10^{-286} X^{13}$$

$$+ 4.23554 \cdot 10^{-286} X^{12} - 4.66107 \cdot 10^{-286} X^{11} + 3.6845 \cdot 10^{-286} X^{10} - 2.09811 \cdot 10^{-286} X^9$$

$$+ 8.58708 \cdot 10^{-287} X^8 - 2.54063 \cdot 10^{-287} X^7 + 5.57852 \cdot 10^{-288} X^6 - 9.21534 \cdot 10^{-289} X^5$$

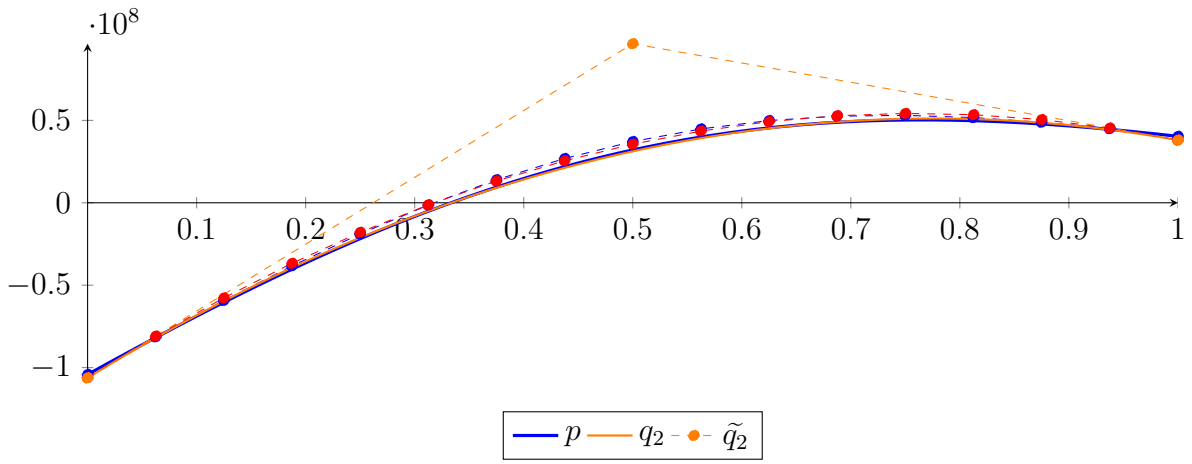
$$+ 1.02122 \cdot 10^{-289} X^4 - 5.95624 \cdot 10^{-291} X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

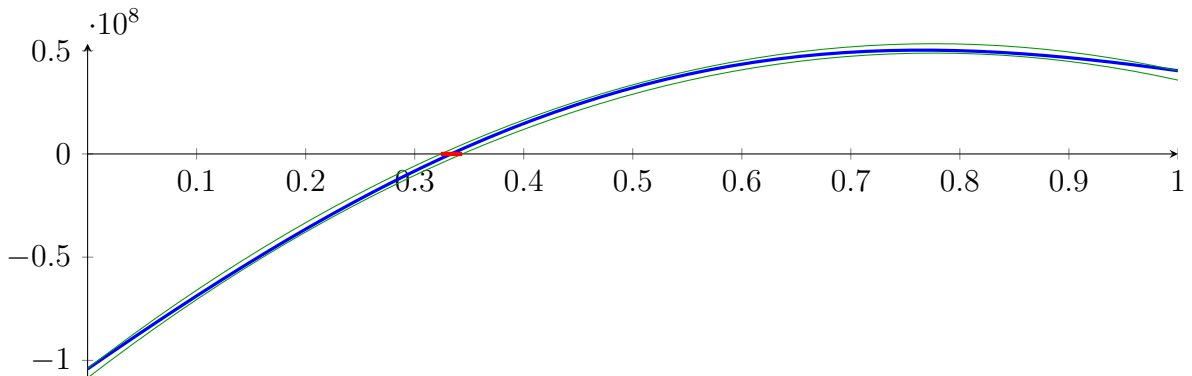
$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\}$$

$$N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

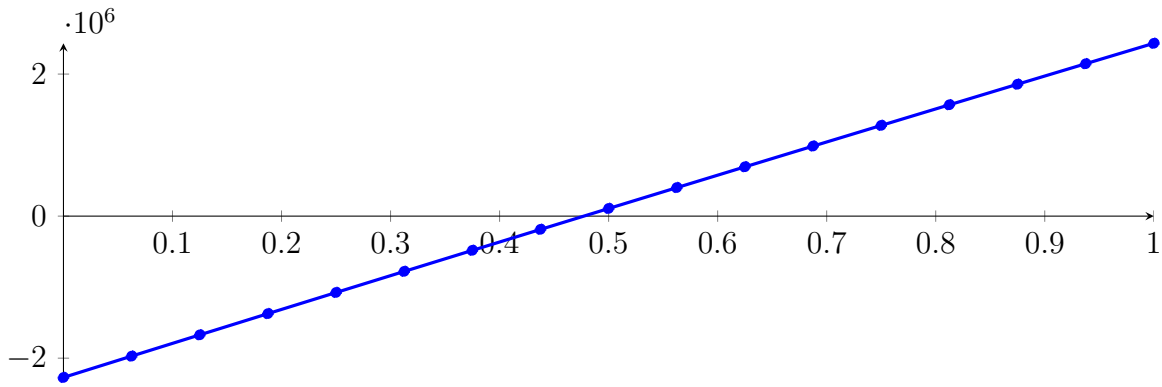
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

248.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

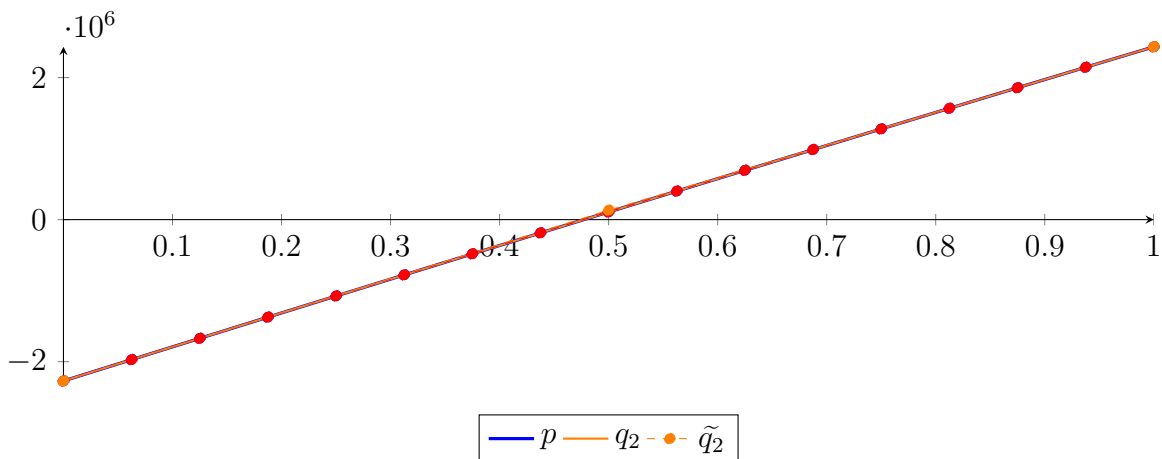
$$\begin{aligned}
 p &= -5.0162 \cdot 10^{-28} X^{16} - 1.14383 \cdot 10^{-24} X^{15} - 1.11125 \cdot 10^{-21} X^{14} - 5.84096 \cdot 10^{-19} X^{13} - 1.684 \cdot 10^{-16} X^{12} \\
 &\quad - 1.94707 \cdot 10^{-14} X^{11} + 2.73227 \cdot 10^{-12} X^{10} + 1.09727 \cdot 10^{-9} X^9 + 6.37314 \cdot 10^{-8} X^8 - 1.68645 \cdot 10^{-5} X^7 \\
 &\quad - 0.00205892 X^6 + 0.132721 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -8.18461 \cdot 10^{-291} X^{16} + 4.13745 \cdot 10^{-290} X^{15} - 5.8254 \cdot 10^{-290} X^{14} - 6.80919 \cdot 10^{-290} X^{13} \\
 &\quad + 3.48059 \cdot 10^{-289} X^{12} - 5.47245 \cdot 10^{-289} X^{11} + 4.8261 \cdot 10^{-289} X^{10} - 2.67867 \cdot 10^{-289} X^9 \\
 &\quad + 1.00314 \cdot 10^{-289} X^8 - 2.84766 \cdot 10^{-290} X^7 + 6.85072 \cdot 10^{-291} X^6 - 1.12501 \cdot 10^{-291} X^5 \\
 &\quad + 1.24731 \cdot 10^{-293} X^4 + 2.25944 \cdot 10^{-293} X^3 - 104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} \\
 &\quad + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

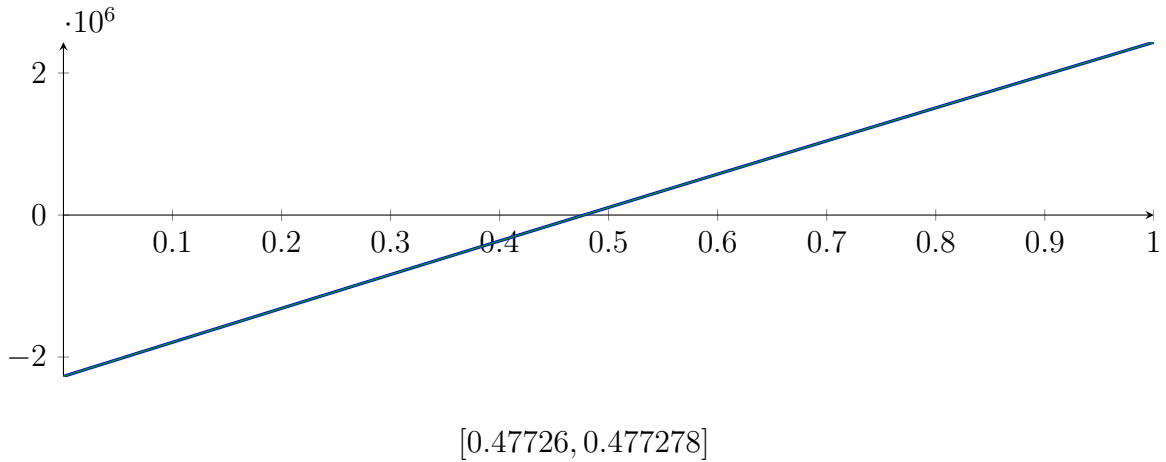
$$M = -104265X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:

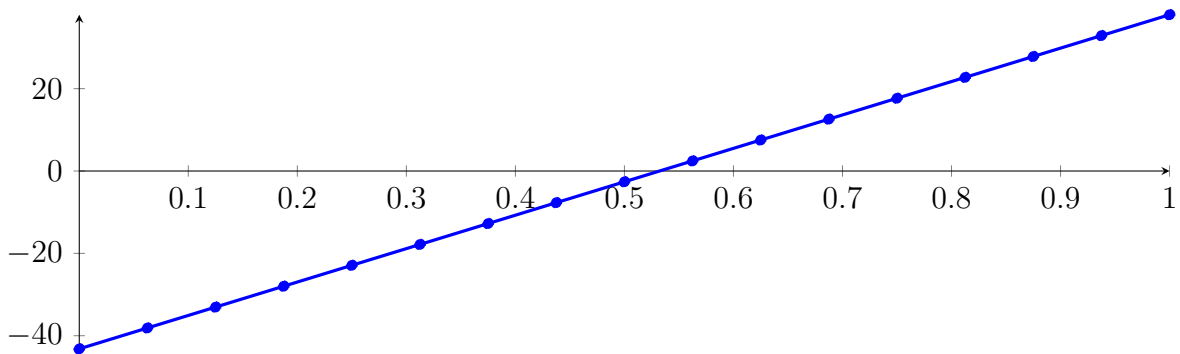


Longest intersection interval: $1.72301 \cdot 10^{-05}$
 \implies Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

248.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.02667 \cdot 10^{-104} X^{16} - 4.019 \cdot 10^{-96} X^{15} - 2.27522 \cdot 10^{-88} X^{14} - 6.97783 \cdot 10^{-81} X^{13} \\
 &\quad - 1.17785 \cdot 10^{-73} X^{12} - 8.12373 \cdot 10^{-67} X^{11} + 6.05916 \cdot 10^{-60} X^{10} + 1.48569 \cdot 10^{-52} X^9 \\
 &\quad + 5.31875 \cdot 10^{-46} X^8 - 7.48919 \cdot 10^{-39} X^7 - 5.53349 \cdot 10^{-32} X^6 + 1.92471 \cdot 10^{-25} X^5 \\
 &\quad + 2.00536 \cdot 10^{-18} X^4 - 4.12844 \cdot 10^{-12} X^3 - 3.09388 \cdot 10^{-05} X^2 + 81.1506X - 43.1911 \\
 &= -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\
 &\quad - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X) \\
 &\quad + 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X) \\
 &\quad + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$

$$= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2}$$

$$\tilde{q}_2 = -1.88281 \cdot 10^{-295} X^{16} + 1.09893 \cdot 10^{-294} X^{15} - 2.26419 \cdot 10^{-294} X^{14} + 8.08223 \cdot 10^{-295} X^{13}$$

$$+ 4.92626 \cdot 10^{-294} X^{12} - 1.07605 \cdot 10^{-293} X^{11} + 1.11858 \cdot 10^{-293} X^{10} - 6.87288 \cdot 10^{-294} X^9$$

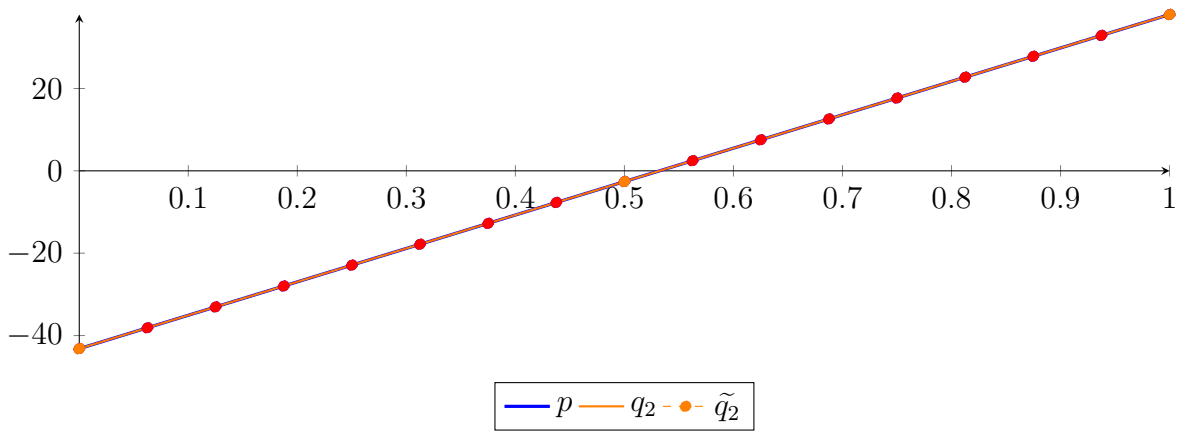
$$+ 2.54873 \cdot 10^{-294} X^8 - 5.2305 \cdot 10^{-295} X^7 + 3.18923 \cdot 10^{-296} X^6 + 1.34092 \cdot 10^{-296} X^5$$

$$- 4.89549 \cdot 10^{-297} X^4 + 5.89947 \cdot 10^{-298} X^3 - 3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$

$$= -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16}$$

$$- 12.7597B_{6,16} - 7.68778B_{7,16} - 2.61587B_{8,16} + 2.45604B_{9,16} + 7.52795B_{10,16} + 12.5999B_{11,16}$$

$$+ 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.06422 \cdot 10^{-13}$.

Bounding polynomials M and m :

$$M = -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$

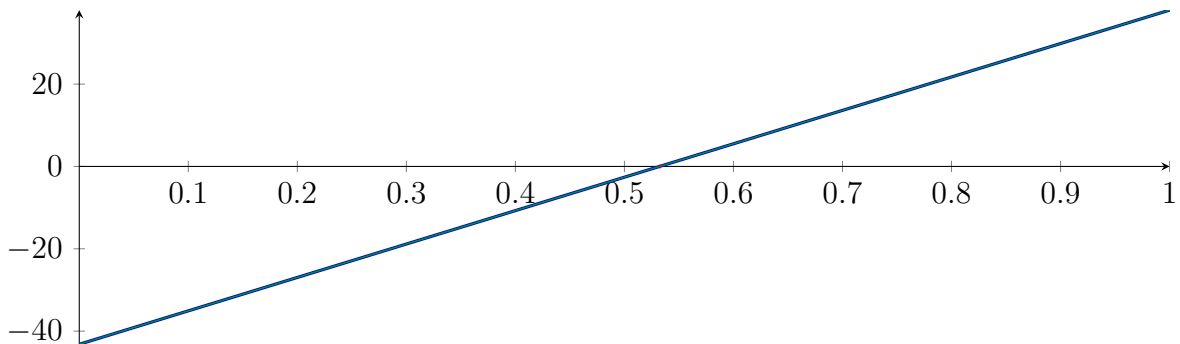
$$m = -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$

Root of M and m :

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\}$$

$$N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

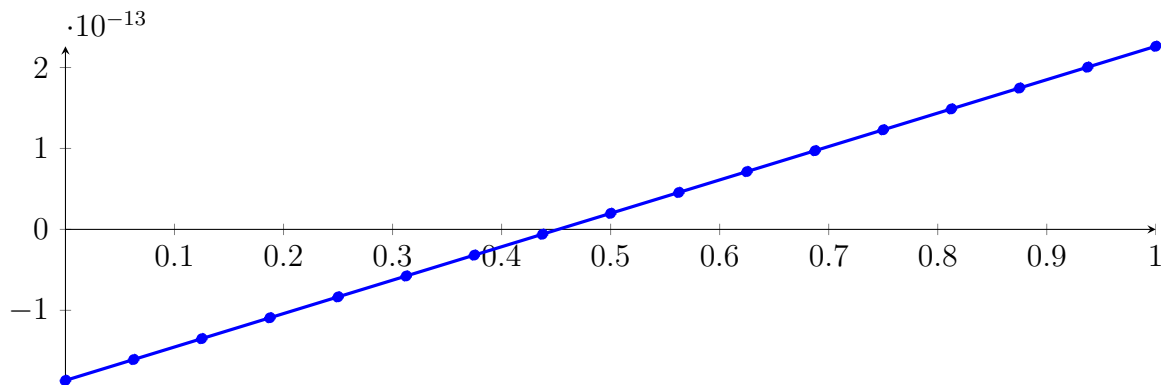
Longest intersection interval: $5.08738 \cdot 10^{-15}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

248.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

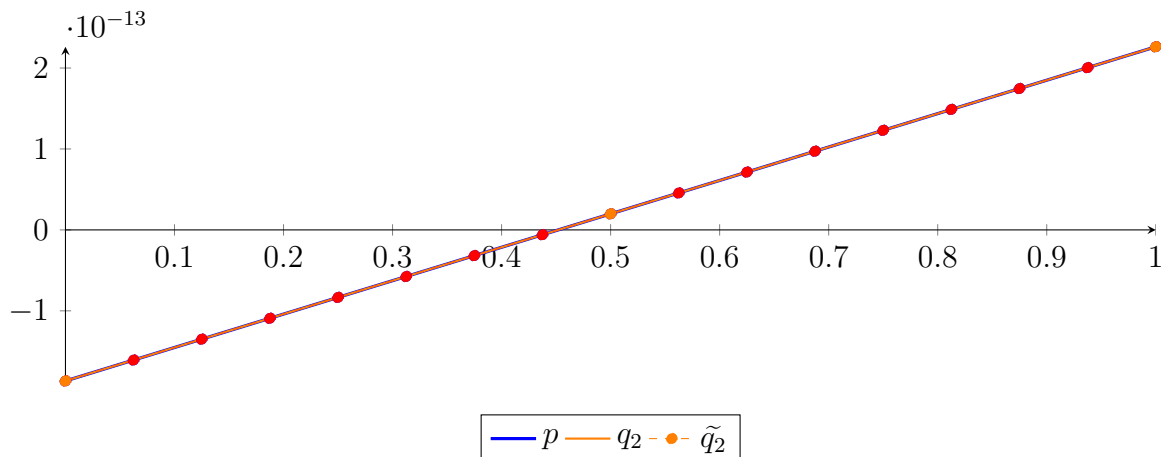
$$\begin{aligned}
 p &= -3.84502 \cdot 10^{-319} X^{16} - 1.59047 \cdot 10^{-310} X^{15} - 1.76985 \cdot 10^{-288} X^{14} - 1.06694 \cdot 10^{-266} X^{13} \\
 &\quad - 3.54011 \cdot 10^{-245} X^{12} - 4.79942 \cdot 10^{-224} X^{11} + 7.03641 \cdot 10^{-203} X^{10} + 3.39135 \cdot 10^{-181} X^9 \\
 &\quad + 2.3865 \cdot 10^{-160} X^8 - 6.60529 \cdot 10^{-139} X^7 - 9.59319 \cdot 10^{-118} X^6 + 6.55895 \cdot 10^{-97} X^5 + 1.34328 \\
 &\quad \cdot 10^{-75} X^4 - 5.43584 \cdot 10^{-55} X^3 - 8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13} \\
 &= -1.86598 \cdot 10^{-13} B_{0,16}(X) - 1.60795 \cdot 10^{-13} B_{1,16}(X) - 1.34993 \cdot 10^{-13} B_{2,16}(X) - 1.0919 \\
 &\quad \cdot 10^{-13} B_{3,16}(X) - 8.33872 \cdot 10^{-14} B_{4,16}(X) - 5.75845 \cdot 10^{-14} B_{5,16}(X) - 3.17818 \cdot 10^{-14} B_{6,16}(X) \\
 &\quad - 5.97912 \cdot 10^{-15} B_{7,16}(X) + 1.98236 \cdot 10^{-14} B_{8,16}(X) + 4.56263 \cdot 10^{-14} B_{9,16}(X) + 7.1429 \\
 &\quad \cdot 10^{-14} B_{10,16}(X) + 9.72317 \cdot 10^{-14} B_{11,16}(X) + 1.23034 \cdot 10^{-13} B_{12,16}(X) + 1.48837 \\
 &\quad \cdot 10^{-13} B_{13,16}(X) + 1.7464 \cdot 10^{-13} B_{14,16}(X) + 2.00443 \cdot 10^{-13} B_{15,16}(X) + 2.26245 \cdot 10^{-13} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13} \\
 &= -1.86598 \cdot 10^{-13} B_{0,2} + 1.98236 \cdot 10^{-14} B_{1,2} + 2.26245 \cdot 10^{-13} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -8.08289 \cdot 10^{-310} X^{16} + 4.33931 \cdot 10^{-309} X^{15} - 8.19282 \cdot 10^{-309} X^{14} + 3.30456 \cdot 10^{-309} X^{13} \\
 &\quad + 1.10762 \cdot 10^{-308} X^{12} - 2.01579 \cdot 10^{-308} X^{11} + 1.55412 \cdot 10^{-308} X^{10} - 6.55981 \cdot 10^{-309} X^9 \\
 &\quad + 2.12881 \cdot 10^{-309} X^8 - 1.08538 \cdot 10^{-309} X^7 + 5.4154 \cdot 10^{-310} X^6 - 1.38567 \cdot 10^{-310} X^5 + 9.27245 \\
 &\quad \cdot 10^{-312} X^4 + 1.80627 \cdot 10^{-312} X^3 - 8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13} \\
 &= -1.86598 \cdot 10^{-13} B_{0,16} - 1.60795 \cdot 10^{-13} B_{1,16} - 1.34993 \cdot 10^{-13} B_{2,16} - 1.0919 \cdot 10^{-13} B_{3,16} - 8.33872 \\
 &\quad \cdot 10^{-14} B_{4,16} - 5.75845 \cdot 10^{-14} B_{5,16} - 3.17818 \cdot 10^{-14} B_{6,16} - 5.97912 \cdot 10^{-15} B_{7,16} + 1.98236 \cdot 10^{-14} B_{8,16} \\
 &\quad + 4.56263 \cdot 10^{-14} B_{9,16} + 7.1429 \cdot 10^{-14} B_{10,16} + 9.72317 \cdot 10^{-14} B_{11,16} + 1.23034 \cdot 10^{-13} B_{12,16} \\
 &\quad + 1.48837 \cdot 10^{-13} B_{13,16} + 1.7464 \cdot 10^{-13} B_{14,16} + 2.00443 \cdot 10^{-13} B_{15,16} + 2.26245 \cdot 10^{-13} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.71792 \cdot 10^{-56}$.

Bounding polynomials M and m :

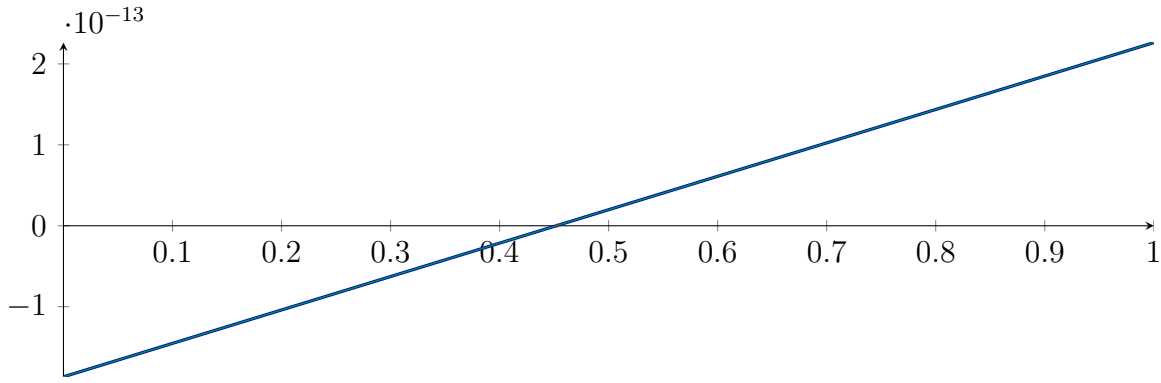
$$M = -8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13}$$

$$m = -8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13}$$

Root of M and m :

$$N(M) = \{0.451983, 5.15577 \cdot 10^{20}\} \quad N(m) = \{0.451983, 5.15577 \cdot 10^{20}\}$$

Intersection intervals:



$$[0.451983, 0.451983]$$

Longest intersection interval: $1.31668 \cdot 10^{-43}$

\implies Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

248.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 5.66252 \cdot 10^{-361} X^{16} + 8.30503 \cdot 10^{-360} X^{15} - 8.71157 \cdot 10^{-361} X^{14} + 3.25232 \cdot 10^{-359} X^{13}$$

$$+ 1.65157 \cdot 10^{-358} X^{12} - 1.58551 \cdot 10^{-358} X^{11} - 7.2669 \cdot 10^{-359} X^{10} - 4.15252 \cdot 10^{-359} X^9$$

$$+ 9.34316 \cdot 10^{-359} X^8 + 1.45338 \cdot 10^{-359} X^6 + 2.59562 \cdot 10^{-311} X^5 + 4.03733 \cdot 10^{-247} X^4$$

$$- 1.24083 \cdot 10^{-183} X^3 - 1.38821 \cdot 10^{-119} X^2 + 5.43584 \cdot 10^{-56} X - 3.10342 \cdot 10^{-56}$$

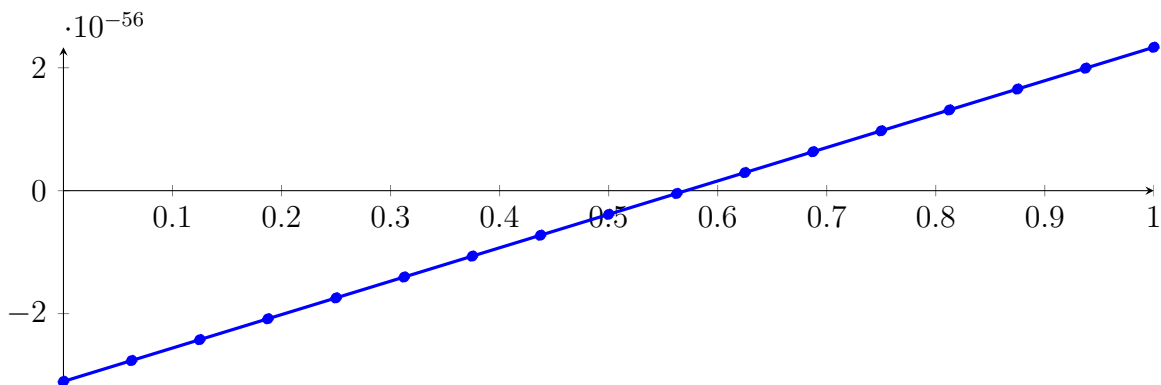
$$= -3.10342 \cdot 10^{-56} B_{0,16}(X) - 2.76368 \cdot 10^{-56} B_{1,16}(X) - 2.42394 \cdot 10^{-56} B_{2,16}(X) - 2.0842$$

$$\cdot 10^{-56} B_{3,16}(X) - 1.74446 \cdot 10^{-56} B_{4,16}(X) - 1.40472 \cdot 10^{-56} B_{5,16}(X) - 1.06498 \cdot 10^{-56} B_{6,16}(X)$$

$$- 7.25243 \cdot 10^{-57} B_{7,16}(X) - 3.85503 \cdot 10^{-57} B_{8,16}(X) - 4.57628 \cdot 10^{-58} B_{9,16}(X) + 2.93977$$

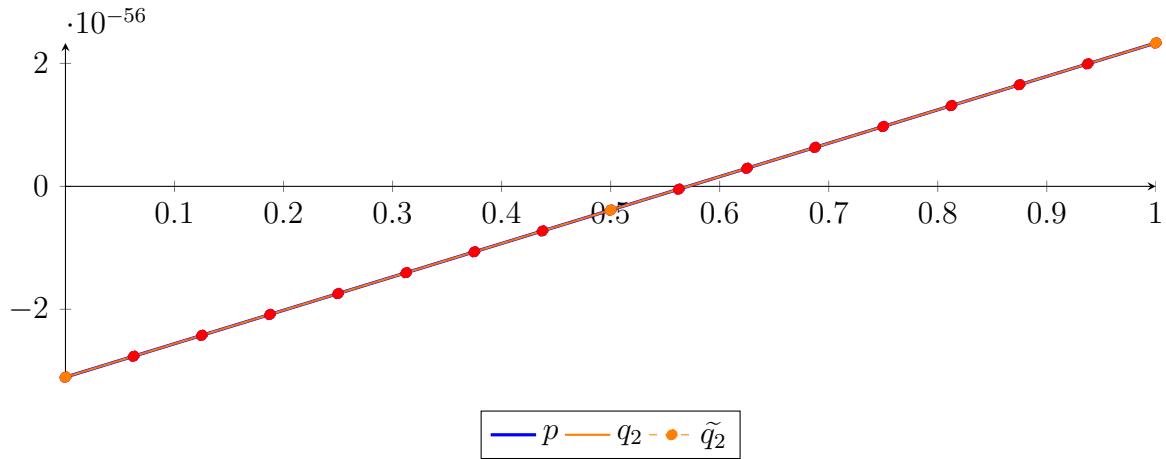
$$\cdot 10^{-57} B_{10,16}(X) + 6.33717 \cdot 10^{-57} B_{11,16}(X) + 9.73457 \cdot 10^{-57} B_{12,16}(X) + 1.3132 \cdot 10^{-56} B_{13,16}(X)$$

$$+ 1.65294 \cdot 10^{-56} B_{14,16}(X) + 1.99268 \cdot 10^{-56} B_{15,16}(X) + 2.33242 \cdot 10^{-56} B_{16,16}(X)$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -1.38821 \cdot 10^{-119} X^2 + 5.43584 \cdot 10^{-56} X - 3.10342 \cdot 10^{-56} \\
 &= -3.10342 \cdot 10^{-56} B_{0,2} - 3.85503 \cdot 10^{-57} B_{1,2} + 2.33242 \cdot 10^{-56} B_{2,2} \\
 \tilde{q}_2 &= -1.35612 \cdot 10^{-352} X^{16} + 8.15544 \cdot 10^{-352} X^{15} - 1.72777 \cdot 10^{-351} X^{14} + 5.92647 \cdot 10^{-352} X^{13} \\
 &\quad + 4.18743 \cdot 10^{-351} X^{12} - 9.40291 \cdot 10^{-351} X^{11} + 1.01181 \cdot 10^{-350} X^{10} - 6.40657 \cdot 10^{-351} X^9 \\
 &\quad + 2.39508 \cdot 10^{-351} X^8 - 4.50359 \cdot 10^{-352} X^7 - 2.71062 \cdot 10^{-354} X^6 + 2.21064 \cdot 10^{-353} X^5 - 5.44842 \\
 &\quad \cdot 10^{-354} X^4 + 4.71019 \cdot 10^{-355} X^3 - 1.38821 \cdot 10^{-119} X^2 + 5.43584 \cdot 10^{-56} X - 3.10342 \cdot 10^{-56} \\
 &= -3.10342 \cdot 10^{-56} B_{0,16} - 2.76368 \cdot 10^{-56} B_{1,16} - 2.42394 \cdot 10^{-56} B_{2,16} - 2.0842 \cdot 10^{-56} B_{3,16} - 1.74446 \\
 &\quad \cdot 10^{-56} B_{4,16} - 1.40472 \cdot 10^{-56} B_{5,16} - 1.06498 \cdot 10^{-56} B_{6,16} - 7.25243 \cdot 10^{-57} B_{7,16} - 3.85503 \cdot 10^{-57} B_{8,16} \\
 &\quad - 4.57628 \cdot 10^{-58} B_{9,16} + 2.93977 \cdot 10^{-57} B_{10,16} + 6.33717 \cdot 10^{-57} B_{11,16} + 9.73457 \cdot 10^{-57} B_{12,16} \\
 &\quad + 1.3132 \cdot 10^{-56} B_{13,16} + 1.65294 \cdot 10^{-56} B_{14,16} + 1.99268 \cdot 10^{-56} B_{15,16} + 2.33242 \cdot 10^{-56} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.20413 \cdot 10^{-185}$.

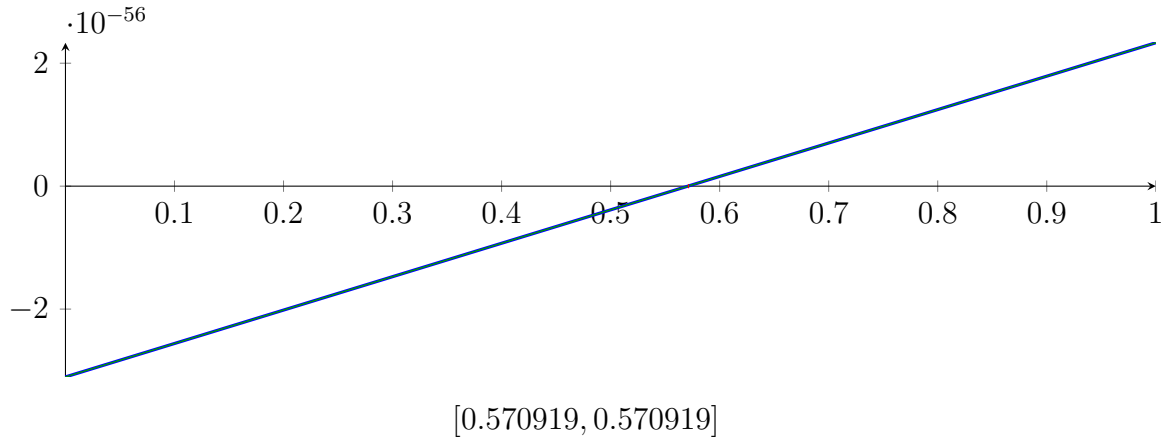
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -1.38821 \cdot 10^{-119} X^2 + 5.43584 \cdot 10^{-56} X - 3.10342 \cdot 10^{-56} \\
 m &= -1.38821 \cdot 10^{-119} X^2 + 5.43584 \cdot 10^{-56} X - 3.10342 \cdot 10^{-56}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.570919, 3.91572 \cdot 10^{63}\} \qquad N(m) = \{0.570919, 3.91572 \cdot 10^{63}\}$$

Intersection intervals:



Longest intersection interval: $2.28268 \cdot 10^{-129}$
 \implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

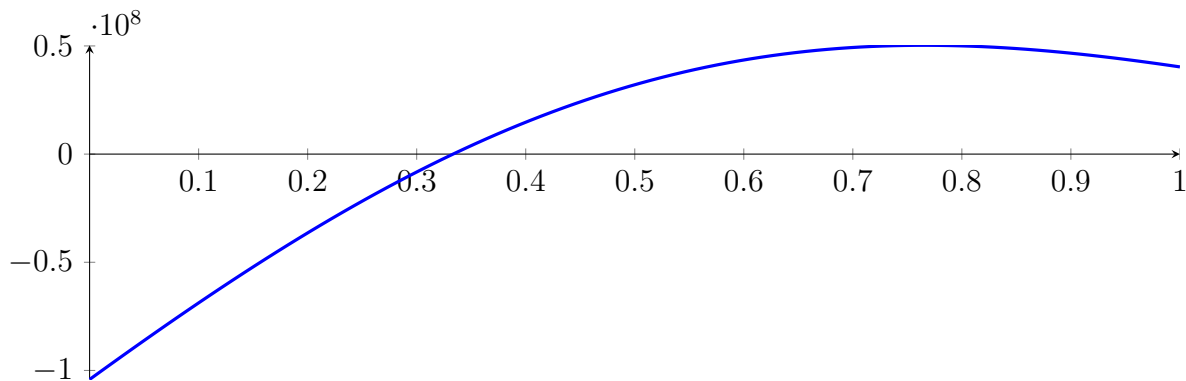
248.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

248.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

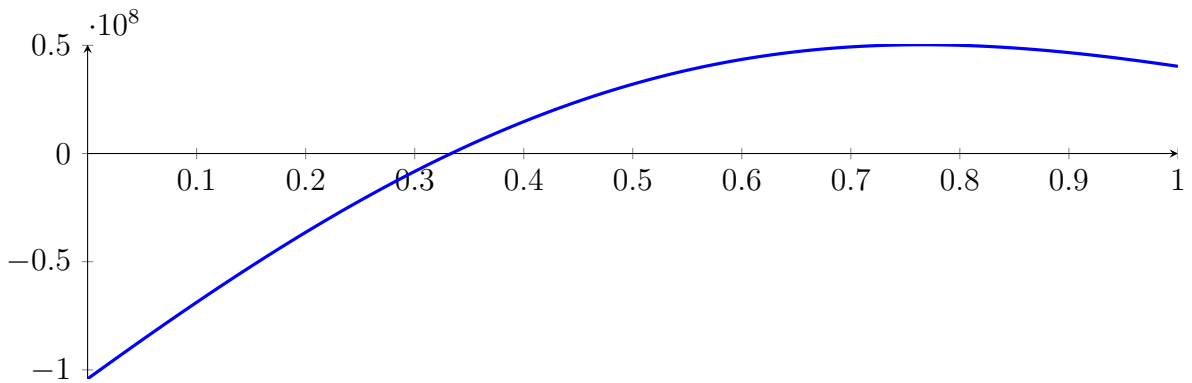
with precision $\varepsilon = 1 \cdot 10^{-64}$.

249 Running CubeClip on f_{16} with epsilon 64

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

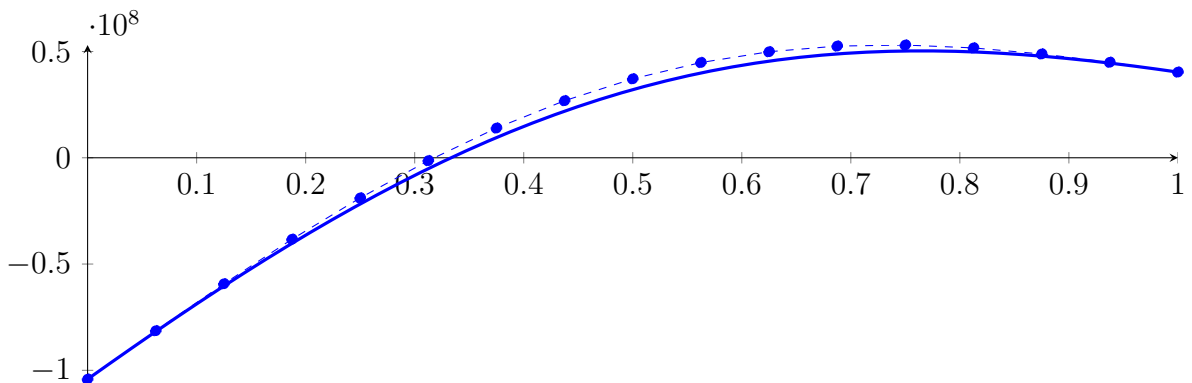
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



249.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 1.89955 \cdot 10^{-288} X^{16} - 1.4851 \cdot 10^{-287} X^{15} + 5.12896 \cdot 10^{-287} X^{14} - 1.02224 \cdot 10^{-286} X^{13}$$

$$+ 1.29027 \cdot 10^{-286} X^{12} - 1.07058 \cdot 10^{-286} X^{11} + 6.00739 \cdot 10^{-287} X^{10} - 2.54352 \cdot 10^{-287} X^9$$

$$+ 1.12089 \cdot 10^{-287} X^8 - 5.8673 \cdot 10^{-288} X^7 + 2.52637 \cdot 10^{-288} X^6 - 6.79624 \cdot 10^{-289} X^5$$

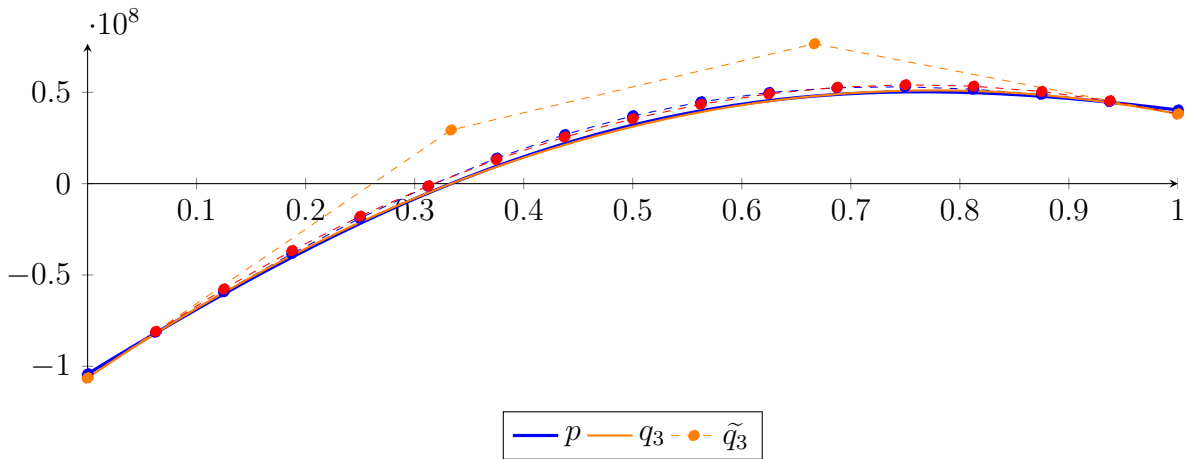
$$+ 9.36341 \cdot 10^{-290} X^4 + 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18551 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91815 \cdot 10^7 B_{10,16} + 5.27353 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

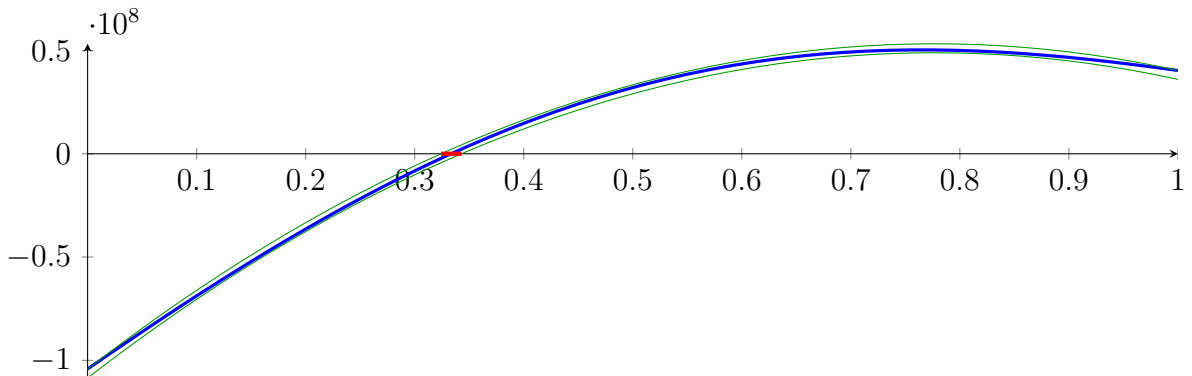
$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\}$$

$$N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

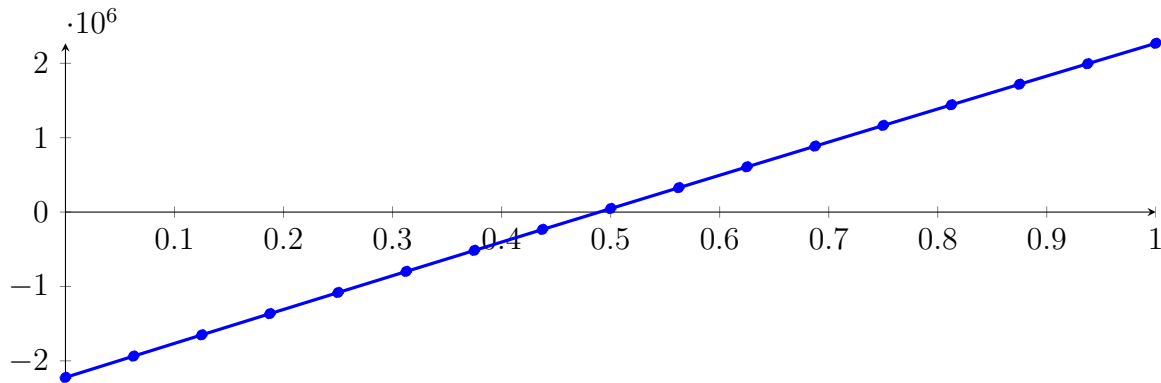
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

249.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

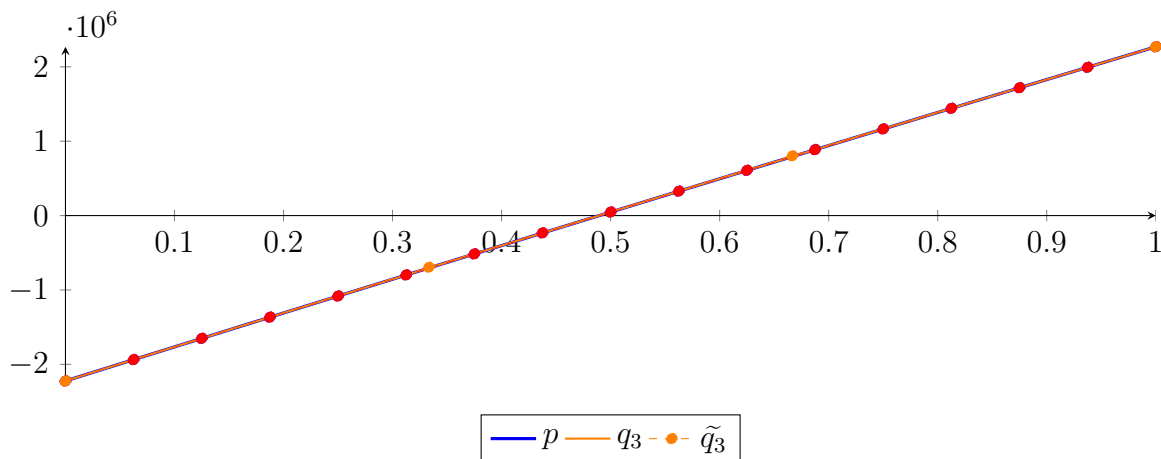
$$\begin{aligned}
 p &= -2.37433 \cdot 10^{-28} X^{16} - 5.67363 \cdot 10^{-25} X^{15} - 5.77631 \cdot 10^{-22} X^{14} - 3.1818 \cdot 10^{-19} X^{13} - 9.6142 \cdot 10^{-17} X^{12} \\
 &\quad - 1.16549 \cdot 10^{-14} X^{11} + 1.71065 \cdot 10^{-12} X^{10} + 7.20611 \cdot 10^{-10} X^9 + 4.39147 \cdot 10^{-08} X^8 - 1.21542 \cdot 10^{-05} X^7 \\
 &\quad - 0.00155624 X^6 + 0.10496 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 4.41595 \cdot 10^{-291} X^{16} - 1.48598 \cdot 10^{-290} X^{15} - 2.81102 \cdot 10^{-290} X^{14} + 2.65129 \cdot 10^{-289} X^{13} \\
 &\quad - 7.34593 \cdot 10^{-289} X^{12} + 1.17654 \cdot 10^{-288} X^{11} - 1.23119 \cdot 10^{-288} X^{10} + 8.63493 \cdot 10^{-289} X^9 \\
 &\quad - 3.92909 \cdot 10^{-289} X^8 + 1.01265 \cdot 10^{-289} X^7 - 6.13644 \cdot 10^{-291} X^6 - 3.92664 \cdot 10^{-291} X^5 \\
 &\quad + 9.04488 \cdot 10^{-292} X^4 - 700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} \\
 &\quad + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m :

$$M = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

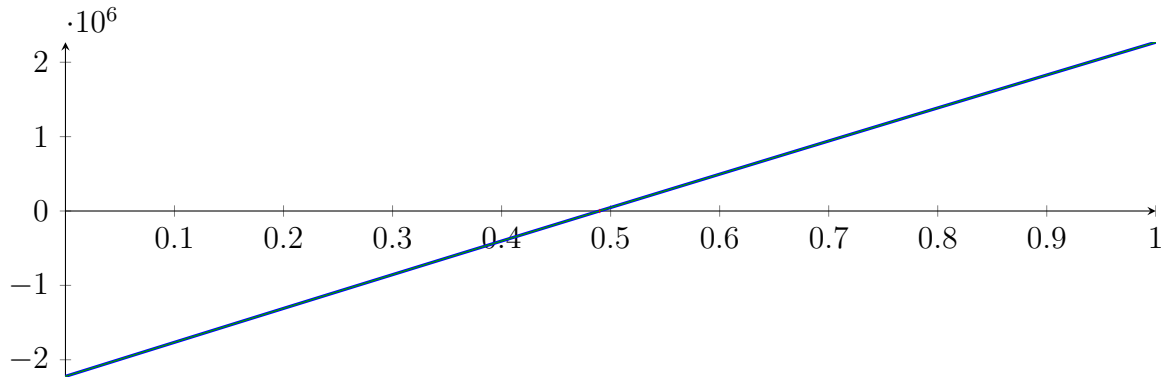
$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\}$$

$$N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

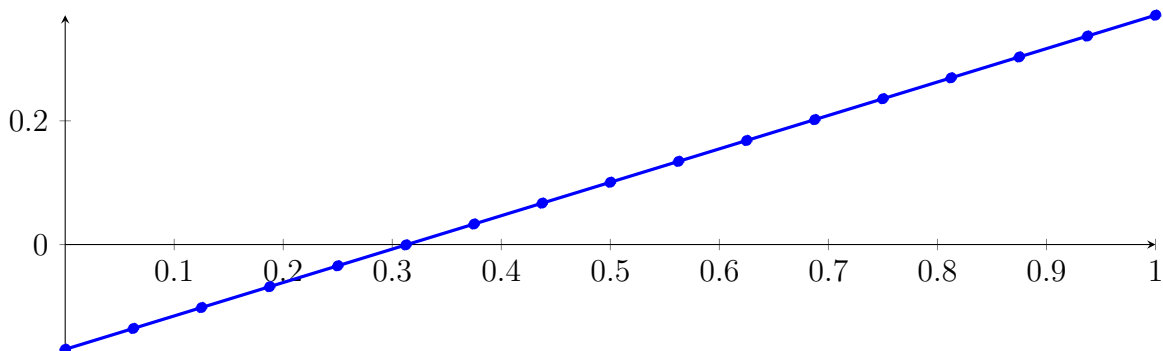
Longest intersection interval: $1.20174 \cdot 10^{-07}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

249.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.49274 \cdot 10^{-139} X^{16} - 8.96277 \cdot 10^{-129} X^{15} - 7.623 \cdot 10^{-119} X^{14} - 3.51238 \cdot 10^{-109} X^{13} \\ &\quad - 8.90739 \cdot 10^{-100} X^{12} - 9.22984 \cdot 10^{-91} X^{11} + 1.03426 \cdot 10^{-81} X^{10} + 3.80998 \cdot 10^{-72} X^9 \\ &\quad + 2.04919 \cdot 10^{-63} X^8 - 4.33497 \cdot 10^{-54} X^7 - 4.81204 \cdot 10^{-45} X^6 + 2.51462 \cdot 10^{-36} X^5 \\ &\quad + 3.93622 \cdot 10^{-27} X^4 - 1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396 \\ &= -0.169396 B_{0,16}(X) - 0.135637 B_{1,16}(X) - 0.101877 B_{2,16}(X) - 0.068118 B_{3,16}(X) \\ &\quad - 0.0343588 B_{4,16}(X) - 0.000599488 B_{5,16}(X) + 0.0331598 B_{6,16}(X) \\ &\quad + 0.066919 B_{7,16}(X) + 0.100678 B_{8,16}(X) + 0.134438 B_{9,16}(X) + 0.168197 B_{10,16}(X) \\ &\quad + 0.201956 B_{11,16}(X) + 0.235715 B_{12,16}(X) + 0.269475 B_{13,16}(X) \\ &\quad + 0.303234 B_{14,16}(X) + 0.336993 B_{15,16}(X) + 0.370752 B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

$$= -0.169396 B_{0,3} + 0.0106536 B_{1,3} + 0.190703 B_{2,3} + 0.370752 B_{3,3}$$

$$\tilde{q}_3 = 8.03185 \cdot 10^{-297} X^{16} - 6.20841 \cdot 10^{-296} X^{15} + 2.13274 \cdot 10^{-295} X^{14} - 4.26614 \cdot 10^{-295} X^{13}$$

$$+ 5.47461 \cdot 10^{-295} X^{12} - 4.70265 \cdot 10^{-295} X^{11} + 2.78551 \cdot 10^{-295} X^{10} - 1.22442 \cdot 10^{-295} X^9$$

$$+ 4.88954 \cdot 10^{-296} X^8 - 2.11494 \cdot 10^{-296} X^7 + 8.20665 \cdot 10^{-297} X^6 - 2.15458 \cdot 10^{-297} X^5$$

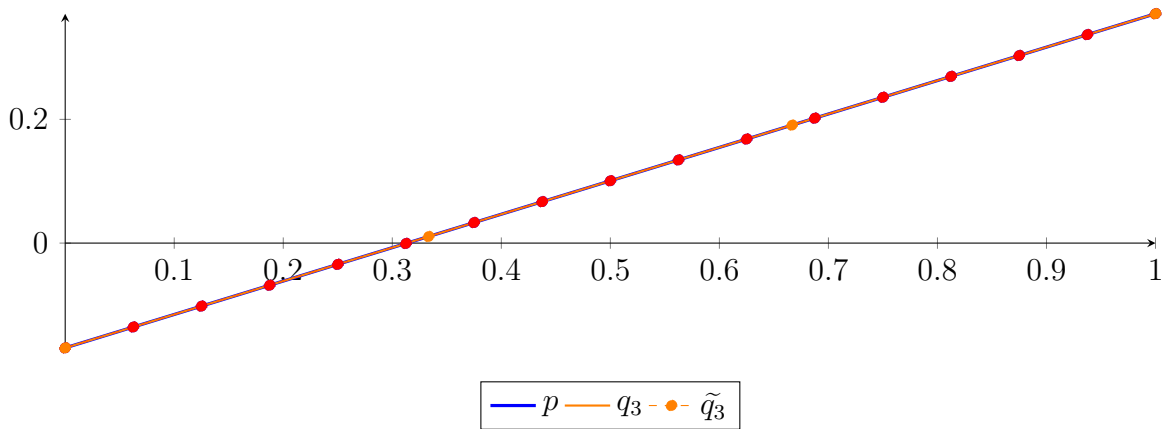
$$+ 3.01517 \cdot 10^{-298} X^4 - 1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

$$= -0.169396 B_{0,16} - 0.135637 B_{1,16} - 0.101877 B_{2,16} - 0.068118 B_{3,16} - 0.0343588 B_{4,16}$$

$$- 0.000599488 B_{5,16} + 0.0331598 B_{6,16} + 0.066919 B_{7,16} + 0.100678 B_{8,16}$$

$$+ 0.134438 B_{9,16} + 0.168197 B_{10,16} + 0.201956 B_{11,16} + 0.235715 B_{12,16}$$

$$+ 0.269475 B_{13,16} + 0.303234 B_{14,16} + 0.336993 B_{15,16} + 0.370752 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 5.62317 \cdot 10^{-29}$.

Bounding polynomials M and m :

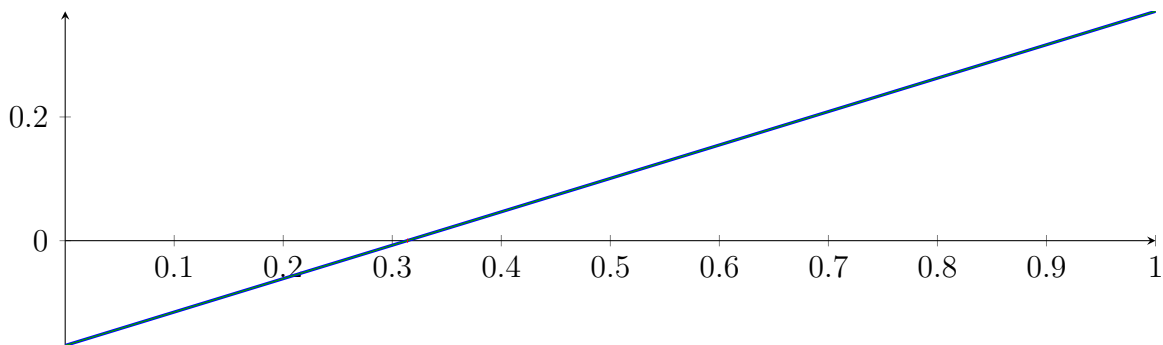
$$M = -1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

$$m = -1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

Root of M and m :

$$N(M) = \{-1.43506 \cdot 10^9, 0.31361, 3.09167 \cdot 10^8\} \quad N(m) = \{-1.43506 \cdot 10^9, 0.31361, 3.09167 \cdot 10^8\}$$

Intersection intervals:



$$[0.31361, 0.31361]$$

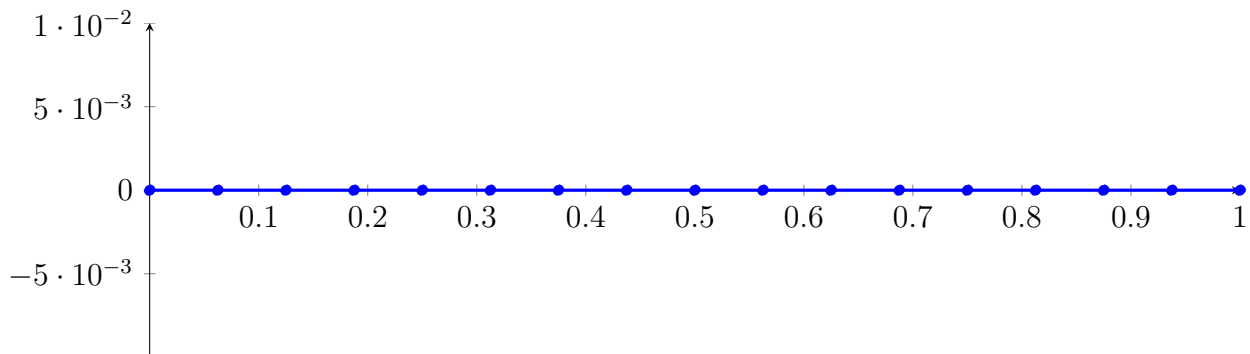
Longest intersection interval: $2.08208 \cdot 10^{-28}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

249.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

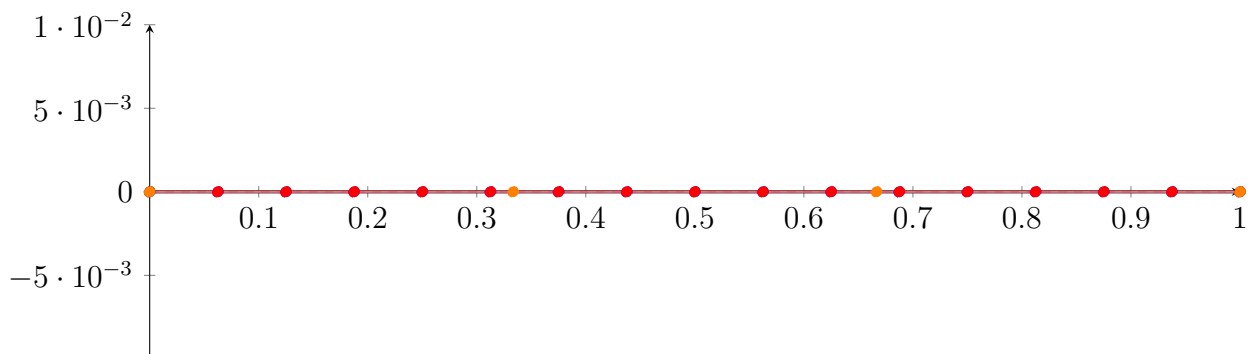
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 3.09811 \cdot 10^{-312} X^{16} + 1.13739 \cdot 10^{-311} X^{15} + 1.09495 \cdot 10^{-310} X^{14} - 3.56495 \cdot 10^{-311} X^{13} \\
 &\quad - 1.48688 \cdot 10^{-309} X^{12} + 1.48302 \cdot 10^{-309} X^{11} + 7.222 \cdot 10^{-310} X^{10} + 1.21378 \cdot 10^{-310} X^9 \\
 &\quad + 7.23716 \cdot 10^{-285} X^8 - 7.35315 \cdot 10^{-248} X^7 - 3.92029 \cdot 10^{-211} X^6 + 9.83929 \cdot 10^{-175} X^5 + 7.39728 \\
 &\quad \cdot 10^{-138} X^4 - 1.09887 \cdot 10^{-101} X^3 - 5.94215 \cdot 10^{-65} X^2 + 1.12463 \cdot 10^{-28} X - 8.88188 \cdot 10^{-08} \\
 &= -8.88188 \cdot 10^{-08} B_{0,16}(X) - 8.88188 \cdot 10^{-08} B_{1,16}(X) - 8.88188 \cdot 10^{-08} B_{2,16}(X) - 8.88188 \\
 &\quad \cdot 10^{-08} B_{3,16}(X) - 8.88188 \cdot 10^{-08} B_{4,16}(X) - 8.88188 \cdot 10^{-08} B_{5,16}(X) - 8.88188 \cdot 10^{-08} B_{6,16}(X) \\
 &\quad - 8.88188 \cdot 10^{-08} B_{7,16}(X) - 8.88188 \cdot 10^{-08} B_{8,16}(X) - 8.88188 \cdot 10^{-08} B_{9,16}(X) - 8.88188 \\
 &\quad \cdot 10^{-08} B_{10,16}(X) - 8.88188 \cdot 10^{-08} B_{11,16}(X) - 8.88188 \cdot 10^{-08} B_{12,16}(X) - 8.88188 \cdot 10^{-08} B_{13,16}(X) \\
 &\quad - 8.88188 \cdot 10^{-08} B_{14,16}(X) - 8.88188 \cdot 10^{-08} B_{15,16}(X) - 8.88188 \cdot 10^{-08} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -1.09887 \cdot 10^{-101} X^3 - 5.94215 \cdot 10^{-65} X^2 + 1.12463 \cdot 10^{-28} X - 8.88188 \cdot 10^{-08} \\
 &= -8.88188 \cdot 10^{-08} B_{0,3} - 8.88188 \cdot 10^{-08} B_{1,3} - 8.88188 \cdot 10^{-08} B_{2,3} - 8.88188 \cdot 10^{-08} B_{3,3} \\
 \tilde{q}_3 &= -6.98397 \cdot 10^{-303} X^{16} + 5.61515 \cdot 10^{-302} X^{15} - 2.01778 \cdot 10^{-301} X^{14} + 4.27019 \cdot 10^{-301} X^{13} \\
 &\quad - 5.92096 \cdot 10^{-301} X^{12} + 5.69601 \cdot 10^{-301} X^{11} - 3.9714 \cdot 10^{-301} X^{10} + 2.10656 \cdot 10^{-301} X^9 \\
 &\quad - 8.95545 \cdot 10^{-302} X^8 + 3.10786 \cdot 10^{-302} X^7 - 8.35303 \cdot 10^{-303} X^6 + 1.57296 \cdot 10^{-303} X^5 - 1.80277 \\
 &\quad \cdot 10^{-304} X^4 - 1.09887 \cdot 10^{-101} X^3 - 5.94215 \cdot 10^{-65} X^2 + 1.12463 \cdot 10^{-28} X - 8.88188 \cdot 10^{-08} \\
 &= -8.88188 \cdot 10^{-08} B_{0,16} - 8.88188 \cdot 10^{-08} B_{1,16} - 8.88188 \cdot 10^{-08} B_{2,16} - 8.88188 \cdot 10^{-08} B_{3,16} - 8.88188 \\
 &\quad \cdot 10^{-08} B_{4,16} - 8.88188 \cdot 10^{-08} B_{5,16} - 8.88188 \cdot 10^{-08} B_{6,16} - 8.88188 \cdot 10^{-08} B_{7,16} - 8.88188 \cdot 10^{-08} B_{8,16} \\
 &\quad - 8.88188 \cdot 10^{-08} B_{9,16} - 8.88188 \cdot 10^{-08} B_{10,16} - 8.88188 \cdot 10^{-08} B_{11,16} - 8.88188 \cdot 10^{-08} B_{12,16} \\
 &\quad - 8.88188 \cdot 10^{-08} B_{13,16} - 8.88188 \cdot 10^{-08} B_{14,16} - 8.88188 \cdot 10^{-08} B_{15,16} - 8.88188 \cdot 10^{-08} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.05675 \cdot 10^{-139}$.

Bounding polynomials M and m :

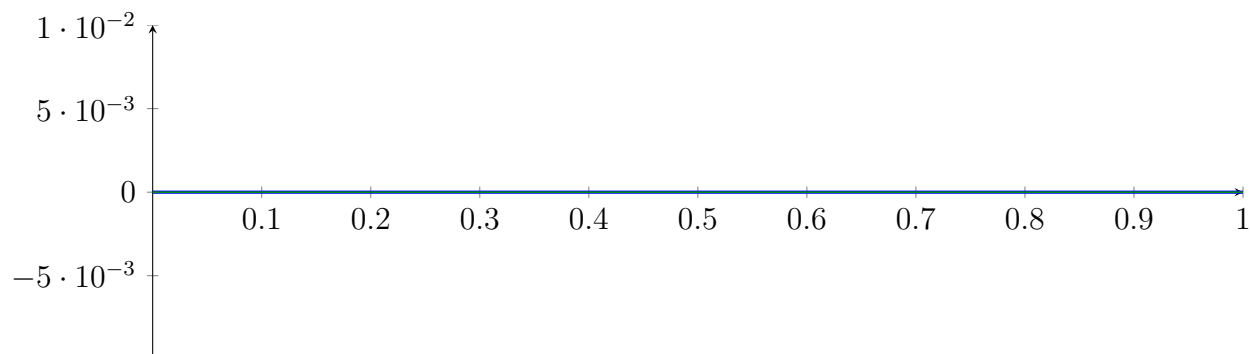
$$M = -1.09887 \cdot 10^{-101} X^3 - 5.94215 \cdot 10^{-65} X^2 + 1.12463 \cdot 10^{-28} X - 8.88188 \cdot 10^{-08}$$

$$m = -1.09887 \cdot 10^{-101} X^3 - 5.94215 \cdot 10^{-65} X^2 + 1.12463 \cdot 10^{-28} X - 8.88188 \cdot 10^{-08}$$

Root of M and m :

$$N(M) = \{-6.89243 \cdot 10^{36}, 1.34848 \cdot 10^{12}, 1.48489 \cdot 10^{36}\} \quad N(m) = \{-6.89243 \cdot 10^{36}, 1.34848 \cdot 10^{12}, 1.48489 \cdot 10^{36}\}$$

Intersection intervals:

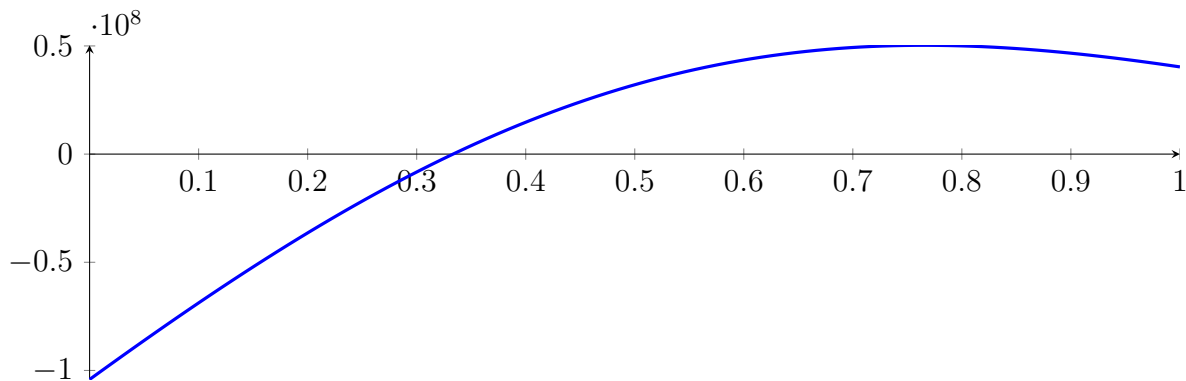


No intersection intervals with the x axis.

249.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

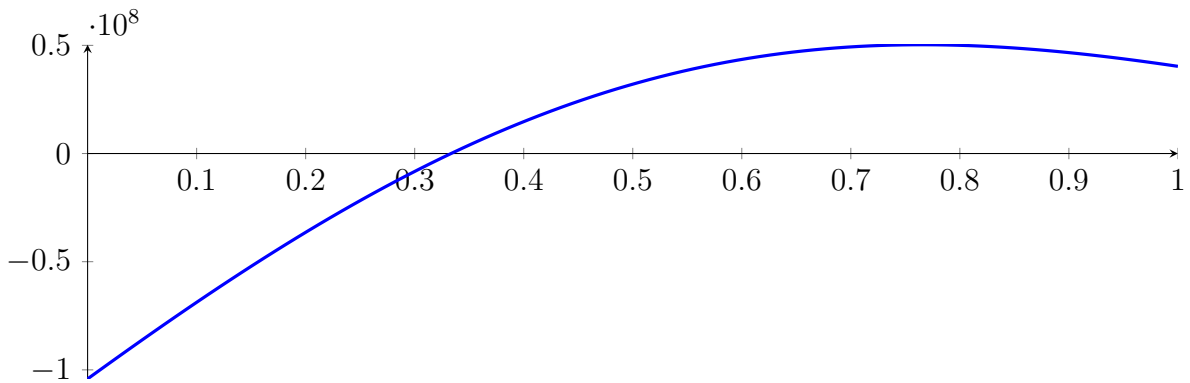
with precision $\varepsilon = 1 \cdot 10^{-64}$.

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$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called BezClip with input polynomial on interval $[0, 1]$:

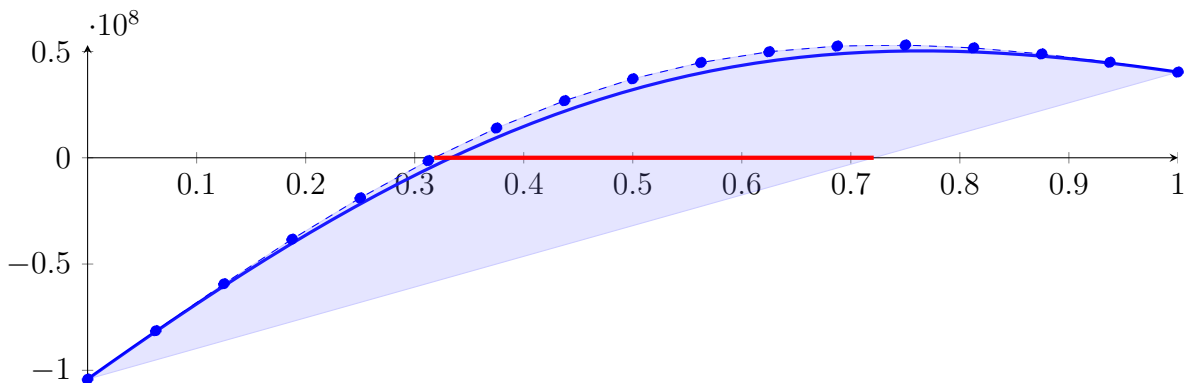
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



250.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

$$[0.317999, 0.720989]$$

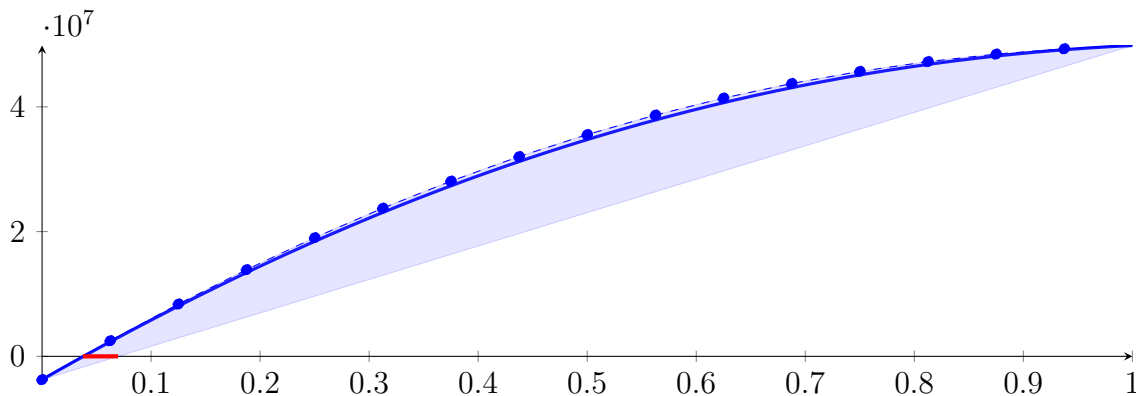
Longest intersection interval: 0.402991

⇒ Selective recursion: interval 1: [\[0.317999, 0.720989\]](#),

250.2 Recursion Branch 1 1 in Interval 1: [\[0.317999, 0.720989\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.83858 \cdot 10^{-07} X^{16} - 5.37355 \cdot 10^{-05} X^{15} - 0.00254146 X^{14} - 0.064977 X^{13} - 0.909205 X^{12} \\ &\quad - 5.03924 X^{11} + 36.4549 X^{10} + 692.921 X^9 + 1886.97 X^8 - 25792 X^7 - 149671 X^6 + 492605 X^5 \\ &\quad + 3.91945 \cdot 10^6 X^4 - 7.54561 \cdot 10^6 X^3 - 4.29225 \cdot 10^7 X^2 + 9.97982 \cdot 10^7 X - 3.73501 \cdot 10^6 \\ &= -3.73501 \cdot 10^6 B_{0,16}(X) + 2.50237 \cdot 10^6 B_{1,16}(X) + 8.38207 \cdot 10^6 B_{2,16}(X) + 1.38906 \\ &\quad \cdot 10^7 B_{3,16}(X) + 1.90167 \cdot 10^7 B_{4,16}(X) + 2.37512 \cdot 10^7 B_{5,16}(X) + 2.80875 \cdot 10^7 B_{6,16}(X) \\ &\quad + 3.20213 \cdot 10^7 B_{7,16}(X) + 3.55508 \cdot 10^7 B_{8,16}(X) + 3.86768 \cdot 10^7 B_{9,16}(X) + 4.14025 \\ &\quad \cdot 10^7 B_{10,16}(X) + 4.37336 \cdot 10^7 B_{11,16}(X) + 4.56783 \cdot 10^7 B_{12,16}(X) + 4.7247 \cdot 10^7 B_{13,16}(X) \\ &\quad + 4.84524 \cdot 10^7 B_{14,16}(X) + 4.93093 \cdot 10^7 B_{15,16}(X) + 4.98343 \cdot 10^7 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

$$[0.0374257, 0.069723]$$

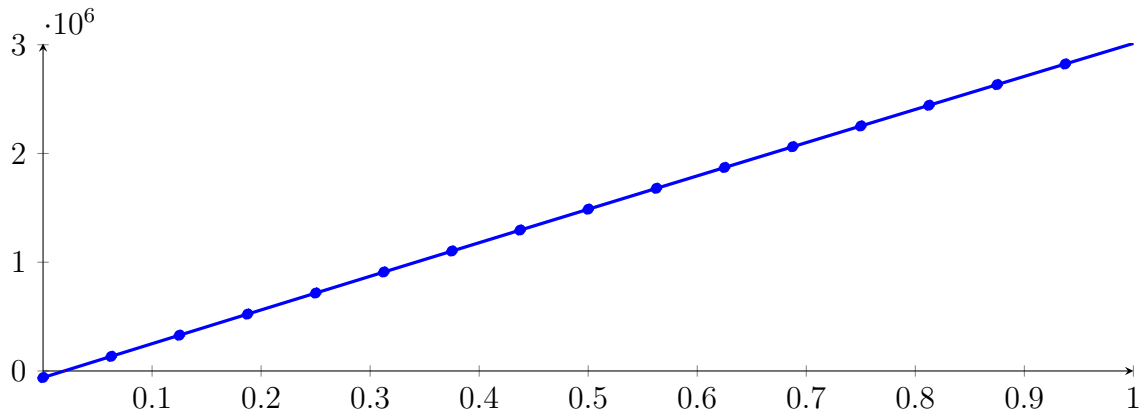
Longest intersection interval: 0.0322973

⇒ Selective recursion: interval 1: [\[0.333081, 0.346096\]](#),

250.3 Recursion Branch 1 1 1 in Interval 1: [\[0.333081, 0.346096\]](#)

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -6.78234 \cdot 10^{-31} X^{16} - 2.34473 \cdot 10^{-27} X^{15} - 3.45581 \cdot 10^{-24} X^{14} - 2.75921 \cdot 10^{-21} X^{13} - 1.21242 \cdot 10^{-18} X^{12} \\ &\quad - 2.17572 \cdot 10^{-16} X^{11} + 4.23544 \cdot 10^{-14} X^{10} + 2.70025 \cdot 10^{-11} X^9 + 2.51306 \cdot 10^{-09} X^8 - 9.23474 \cdot 10^{-07} X^7 \\ &\quad - 0.000177459 X^6 + 0.0161038 X^5 + 4.36155 X^4 - 234.216 X^3 - 45622.2 X^2 + 3.11845 \cdot 10^6 X - 60508.5 \\ &= -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ &\quad + 910202 B_{5,16}(X) + 1.1032 \cdot 10^6 B_{6,16}(X) + 1.29582 \cdot 10^6 B_{7,16}(X) + 1.48805 \cdot 10^6 B_{8,16}(X) \\ &\quad + 1.6799 \cdot 10^6 B_{9,16}(X) + 1.87137 \cdot 10^6 B_{10,16}(X) + 2.06245 \cdot 10^6 B_{11,16}(X) + 2.25315 \cdot 10^6 B_{12,16}(X) \\ &\quad + 2.44346 \cdot 10^6 B_{13,16}(X) + 2.63339 \cdot 10^6 B_{14,16}(X) + 2.82293 \cdot 10^6 B_{15,16}(X) + 3.01209 \cdot 10^6 B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

$$[0.0194034, 0.0196929]$$

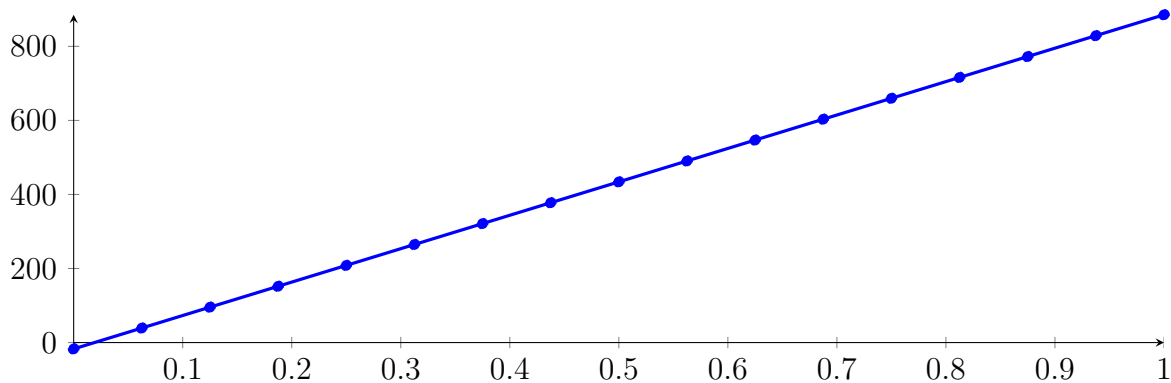
Longest intersection interval: 0.000289554

⇒ Selective recursion: interval 1: $[0.333333, 0.333337]$,

250.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.65599 \cdot 10^{-87} X^{16} - 1.97733 \cdot 10^{-80} X^{15} - 1.00659 \cdot 10^{-73} X^{14} - 2.77601 \cdot 10^{-67} X^{13} \\
 &\quad - 4.21367 \cdot 10^{-61} X^{12} - 2.61333 \cdot 10^{-55} X^{11} + 1.75275 \cdot 10^{-49} X^{10} + 3.8646 \cdot 10^{-43} X^9 \\
 &\quad + 1.2441 \cdot 10^{-37} X^8 - 1.57525 \cdot 10^{-31} X^7 - 1.04661 \cdot 10^{-25} X^6 + 3.27355 \cdot 10^{-20} X^5 \\
 &\quad + 3.06701 \cdot 10^{-14} X^4 - 5.67777 \cdot 10^{-09} X^3 - 0.00382618 X^2 + 902.448 X - 17.178 \\
 &= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
 &\quad + 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
 &\quad + 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
 &\quad + 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

$$[0.0190349, 0.019035]$$

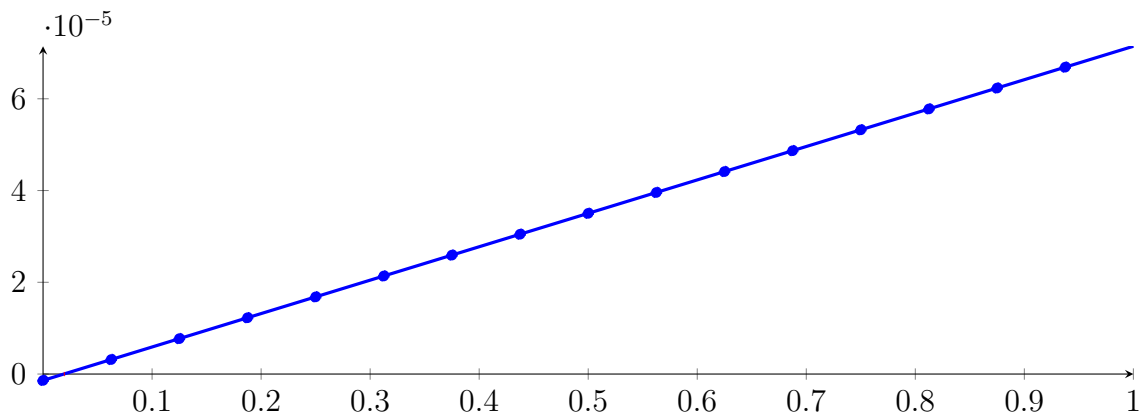
Longest intersection interval: $8.07045 \cdot 10^{-08}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

250.5 Recursion Branch 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -5.36315 \cdot 10^{-201} X^{16} - 7.93495 \cdot 10^{-187} X^{15} - 5.0052 \cdot 10^{-173} X^{14} - 1.71037 \cdot 10^{-159} X^{13} \\
 &\quad - 3.21686 \cdot 10^{-146} X^{12} - 2.47211 \cdot 10^{-133} X^{11} + 2.05446 \cdot 10^{-120} X^{10} + 5.61285 \cdot 10^{-107} X^9 \\
 &\quad + 2.23891 \cdot 10^{-94} X^8 - 3.51264 \cdot 10^{-81} X^7 - 2.89181 \cdot 10^{-68} X^6 + 1.12075 \cdot 10^{-55} X^5 + 1.30109 \\
 &\quad \cdot 10^{-42} X^4 - 2.98449 \cdot 10^{-30} X^3 - 2.49207 \cdot 10^{-17} X^2 + 7.28316 \cdot 10^{-05} X - 1.38634 \cdot 10^{-06} \\
 &= -1.38634 \cdot 10^{-06} B_{0,16}(X) + 3.16564 \cdot 10^{-06} B_{1,16}(X) + 7.71761 \cdot 10^{-06} B_{2,16}(X) + 1.22696 \\
 &\quad \cdot 10^{-05} B_{3,16}(X) + 1.68216 \cdot 10^{-05} B_{4,16}(X) + 2.13735 \cdot 10^{-05} B_{5,16}(X) + 2.59255 \cdot 10^{-05} B_{6,16}(X) \\
 &\quad + 3.04775 \cdot 10^{-05} B_{7,16}(X) + 3.50295 \cdot 10^{-05} B_{8,16}(X) + 3.95814 \cdot 10^{-05} B_{9,16}(X) + 4.41334 \\
 &\quad \cdot 10^{-05} B_{10,16}(X) + 4.86854 \cdot 10^{-05} B_{11,16}(X) + 5.32374 \cdot 10^{-05} B_{12,16}(X) + 5.77893 \cdot 10^{-05} B_{13,16}(X) \\
 &\quad + 6.23413 \cdot 10^{-05} B_{14,16}(X) + 6.68933 \cdot 10^{-05} B_{15,16}(X) + 7.14453 \cdot 10^{-05} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

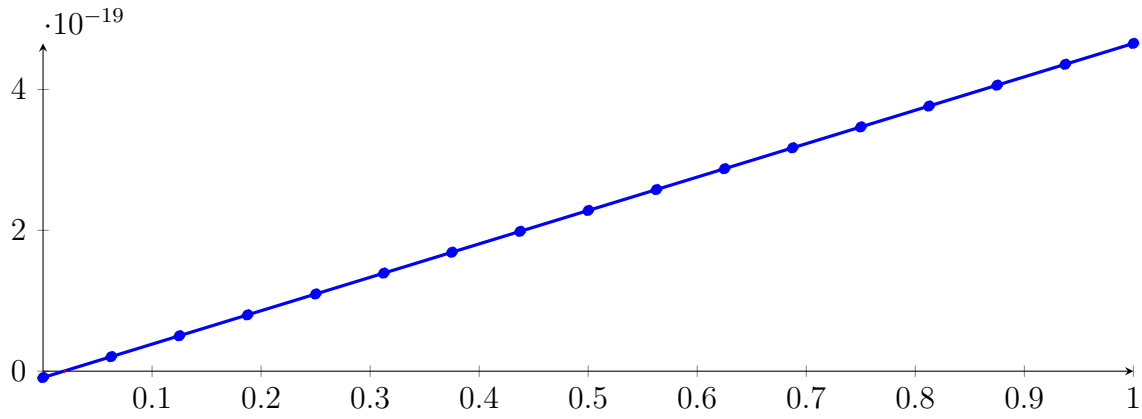
Longest intersection interval: $6.51314 \cdot 10^{-15}$

⇒ Selective recursion: interval 1: [0.333333, 0.333333],

250.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -1.70149 \cdot 10^{-323} X^{16} + 1.97819 \cdot 10^{-322} X^{15} - 4.34527 \cdot 10^{-322} X^{14} + 3.97182 \cdot 10^{-322} X^{13} \\
 &\quad - 1.87464 \cdot 10^{-316} X^{12} - 2.21189 \cdot 10^{-289} X^{11} + 2.82229 \cdot 10^{-262} X^{10} + 1.18385 \cdot 10^{-234} X^9 \\
 &\quad + 7.25038 \cdot 10^{-208} X^8 - 1.74649 \cdot 10^{-180} X^7 - 2.20756 \cdot 10^{-153} X^6 + 1.31359 \cdot 10^{-126} X^5 + 2.34136 \\
 &\quad \cdot 10^{-99} X^4 - 8.24597 \cdot 10^{-73} X^3 - 1.05716 \cdot 10^{-45} X^2 + 4.74362 \cdot 10^{-19} X - 9.02941 \cdot 10^{-21} \\
 &= -9.02941 \cdot 10^{-21} B_{0,16}(X) + 2.06182 \cdot 10^{-20} B_{1,16}(X) + 5.02659 \cdot 10^{-20} B_{2,16}(X) + 7.99135 \\
 &\quad \cdot 10^{-20} B_{3,16}(X) + 1.09561 \cdot 10^{-19} B_{4,16}(X) + 1.39209 \cdot 10^{-19} B_{5,16}(X) + 1.68856 \cdot 10^{-19} B_{6,16}(X) \\
 &\quad + 1.98504 \cdot 10^{-19} B_{7,16}(X) + 2.28152 \cdot 10^{-19} B_{8,16}(X) + 2.57799 \cdot 10^{-19} B_{9,16}(X) + 2.87447 \\
 &\quad \cdot 10^{-19} B_{10,16}(X) + 3.17095 \cdot 10^{-19} B_{11,16}(X) + 3.46742 \cdot 10^{-19} B_{12,16}(X) + 3.7639 \cdot 10^{-19} B_{13,16}(X) \\
 &\quad + 4.06038 \cdot 10^{-19} B_{14,16}(X) + 4.35685 \cdot 10^{-19} B_{15,16}(X) + 4.65333 \cdot 10^{-19} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

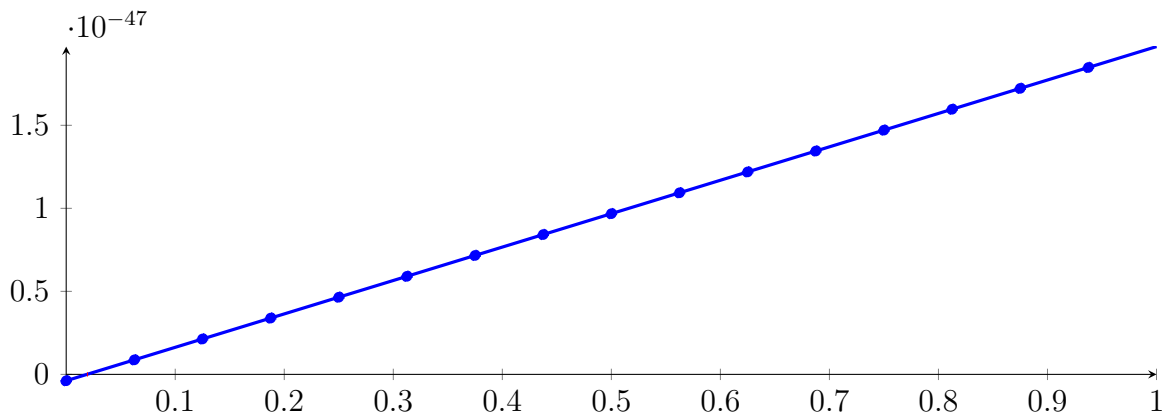
Longest intersection interval: $4.2421 \cdot 10^{-29}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

250.7 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 1.52489 \cdot 10^{-352} X^{16} + 4.56006 \cdot 10^{-351} X^{15} - 3.36159 \cdot 10^{-350} X^{14} + 2.87148 \cdot 10^{-350} X^{13} \\
 &\quad - 1.35884 \cdot 10^{-349} X^{12} + 1.13318 \cdot 10^{-349} X^{11} - 1.84828 \cdot 10^{-349} X^{10} + 7.52411 \cdot 10^{-350} X^9 \\
 &\quad - 2.19453 \cdot 10^{-350} X^8 + 2.78671 \cdot 10^{-351} X^7 - 1.28647 \cdot 10^{-323} X^6 + 1.80453 \cdot 10^{-268} X^5 + 7.58214 \\
 &\quad \cdot 10^{-213} X^4 - 6.29484 \cdot 10^{-158} X^3 - 1.90241 \cdot 10^{-102} X^2 + 2.01229 \cdot 10^{-47} X - 3.83037 \cdot 10^{-49} \\
 &= -3.83037 \cdot 10^{-49} B_{0,16}(X) + 8.74646 \cdot 10^{-49} B_{1,16}(X) + 2.13233 \cdot 10^{-48} B_{2,16}(X) + 3.39001 \\
 &\quad \cdot 10^{-48} B_{3,16}(X) + 4.6477 \cdot 10^{-48} B_{4,16}(X) + 5.90538 \cdot 10^{-48} B_{5,16}(X) + 7.16306 \cdot 10^{-48} B_{6,16}(X) \\
 &\quad + 8.42074 \cdot 10^{-48} B_{7,16}(X) + 9.67843 \cdot 10^{-48} B_{8,16}(X) + 1.09361 \cdot 10^{-47} B_{9,16}(X) + 1.21938 \\
 &\quad \cdot 10^{-47} B_{10,16}(X) + 1.34515 \cdot 10^{-47} B_{11,16}(X) + 1.47092 \cdot 10^{-47} B_{12,16}(X) + 1.59668 \cdot 10^{-47} B_{13,16}(X) \\
 &\quad + 1.72245 \cdot 10^{-47} B_{14,16}(X) + 1.84822 \cdot 10^{-47} B_{15,16}(X) + 1.97399 \cdot 10^{-47} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

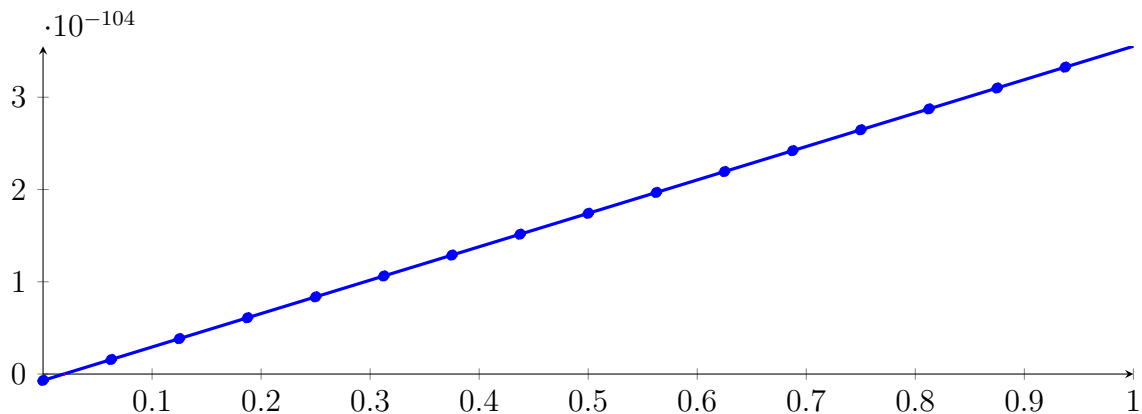
Longest intersection interval: $1.79954 \cdot 10^{-57}$

⇒ Selective recursion: interval 1: $[0.333333, 0.333333]$,

250.8 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 1.13315 \cdot 10^{-409} X^{16} + 3.67576 \cdot 10^{-408} X^{15} + 3.40877 \cdot 10^{-408} X^{14} - 2.82512 \cdot 10^{-407} X^{13} \\
 &+ 5.53724 \cdot 10^{-407} X^{12} - 2.84772 \cdot 10^{-407} X^{11} + 1.69055 \cdot 10^{-406} X^{10} + 8.70138 \cdot 10^{-407} X^9 \\
 &- 4.59486 \cdot 10^{-407} X^8 + 2.21974 \cdot 10^{-407} X^7 - 1.24305 \cdot 10^{-407} X^6 + 1.41256 \cdot 10^{-409} X^4 \\
 &- 3.66835 \cdot 10^{-328} X^3 - 6.16067 \cdot 10^{-216} X^2 + 3.6212 \cdot 10^{-104} X - 6.8929 \cdot 10^{-106} \\
 &= -6.8929 \cdot 10^{-106} B_{0,16}(X) + 1.57396 \cdot 10^{-105} B_{1,16}(X) + 3.83722 \cdot 10^{-105} B_{2,16}(X) + 6.10047 \\
 &\cdot 10^{-105} B_{3,16}(X) + 8.36372 \cdot 10^{-105} B_{4,16}(X) + 1.0627 \cdot 10^{-104} B_{5,16}(X) + 1.28902 \cdot 10^{-104} B_{6,16}(X) \\
 &+ 1.51535 \cdot 10^{-104} B_{7,16}(X) + 1.74167 \cdot 10^{-104} B_{8,16}(X) + 1.968 \cdot 10^{-104} B_{9,16}(X) + 2.19432 \\
 &\cdot 10^{-104} B_{10,16}(X) + 2.42065 \cdot 10^{-104} B_{11,16}(X) + 2.64697 \cdot 10^{-104} B_{12,16}(X) + 2.8733 \cdot 10^{-104} B_{13,16}(X) \\
 &+ 3.09963 \cdot 10^{-104} B_{14,16}(X) + 3.32595 \cdot 10^{-104} B_{15,16}(X) + 3.55228 \cdot 10^{-104} B_{16,16}(X)
 \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $3.23835 \cdot 10^{-114}$

\implies Selective recursion: interval 1: [0.333333, 0.333333],

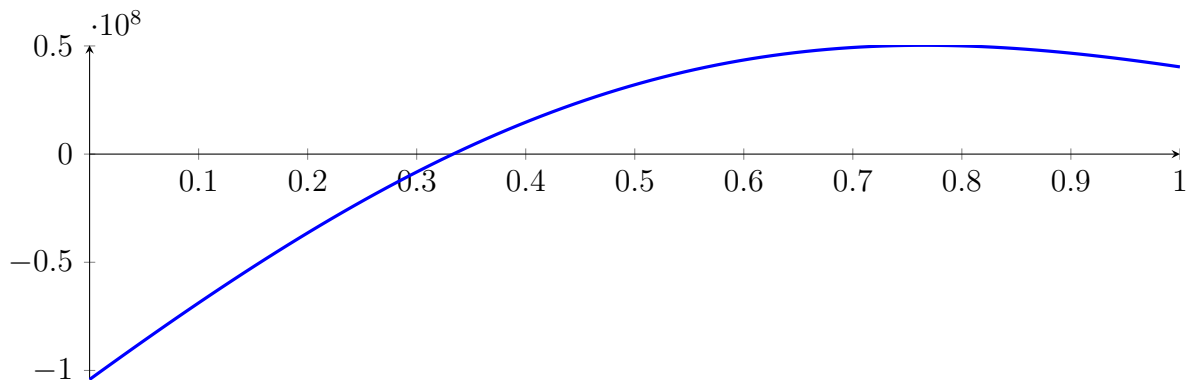
250.9 Recursion Branch 1 1 1 1 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Found root in interval [0.333333, 0.333333] at recursion depth 9!

250.10 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

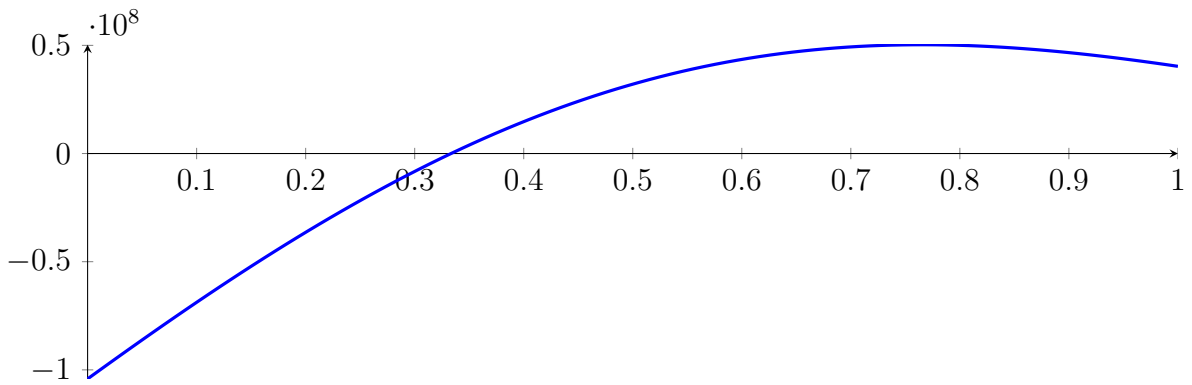
with precision $\varepsilon = 1 \cdot 10^{-128}$.

251 Running QuadClip on f_{16} with epsilon 128

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called QuadClip with input polynomial on interval $[0, 1]$:

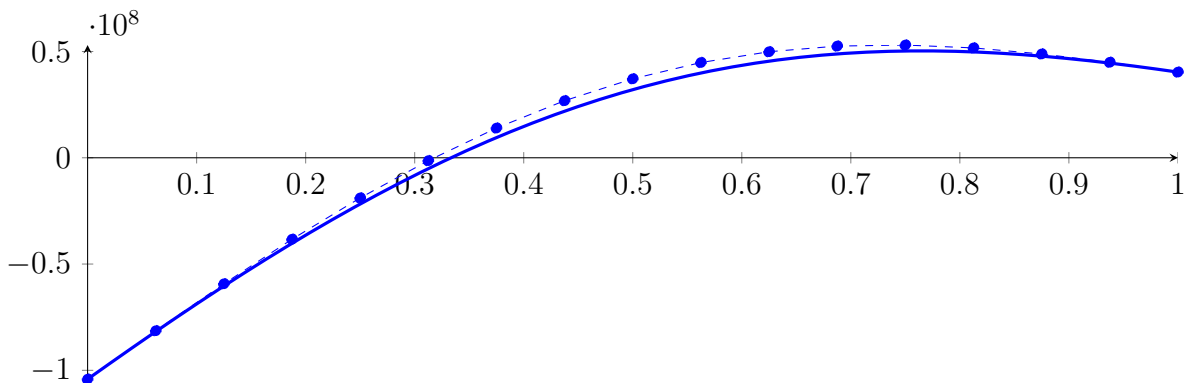
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



251.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,2} + 9.65115 \cdot 10^7 B_{1,2} + 3.80384 \cdot 10^7 B_{2,2}$$

$$\tilde{q}_2 = 3.74473 \cdot 10^{-288} X^{16} - 3.08889 \cdot 10^{-287} X^{15} + 1.17443 \cdot 10^{-286} X^{14} - 2.71603 \cdot 10^{-286} X^{13}$$

$$+ 4.23554 \cdot 10^{-286} X^{12} - 4.66107 \cdot 10^{-286} X^{11} + 3.6845 \cdot 10^{-286} X^{10} - 2.09811 \cdot 10^{-286} X^9$$

$$+ 8.58708 \cdot 10^{-287} X^8 - 2.54063 \cdot 10^{-287} X^7 + 5.57852 \cdot 10^{-288} X^6 - 9.21534 \cdot 10^{-289} X^5$$

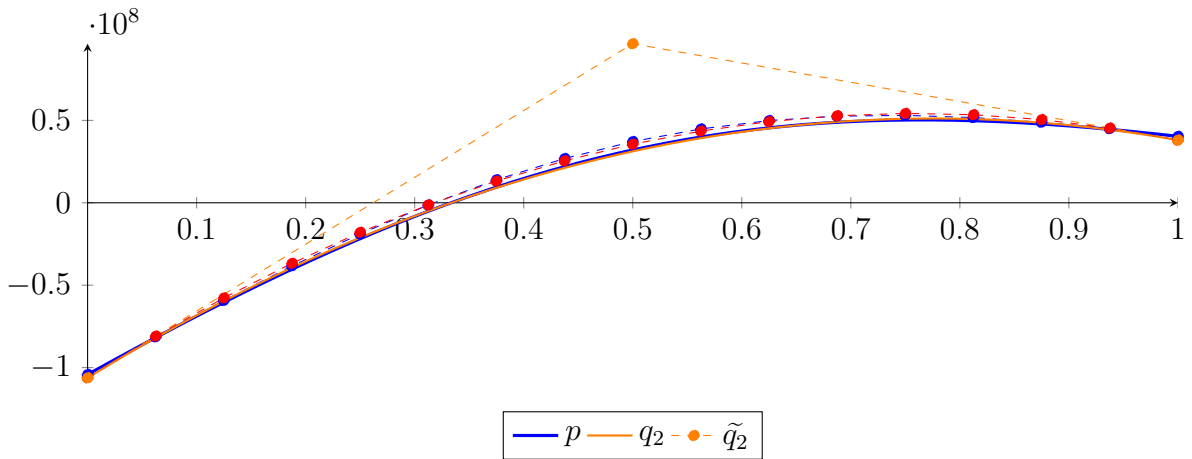
$$+ 1.02122 \cdot 10^{-289} X^4 - 5.95624 \cdot 10^{-291} X^3 - 2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.06197 \cdot 10^8$$

$$= -1.06197 \cdot 10^8 B_{0,16} - 8.08583 \cdot 10^7 B_{1,16} - 5.76963 \cdot 10^7 B_{2,16} - 3.67107 \cdot 10^7 B_{3,16} - 1.79017$$

$$\cdot 10^7 B_{4,16} - 1.26924 \cdot 10^6 B_{5,16} + 1.31867 \cdot 10^7 B_{6,16} + 2.54662 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34956 \cdot 10^7 B_{9,16} + 4.92456 \cdot 10^7 B_{10,16} + 5.2819 \cdot 10^7 B_{11,16} + 5.42159 \cdot 10^7 B_{12,16}$$

$$+ 5.34363 \cdot 10^7 B_{13,16} + 5.04802 \cdot 10^7 B_{14,16} + 4.53476 \cdot 10^7 B_{15,16} + 3.80384 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233 \cdot 10^6$.

Bounding polynomials M and m :

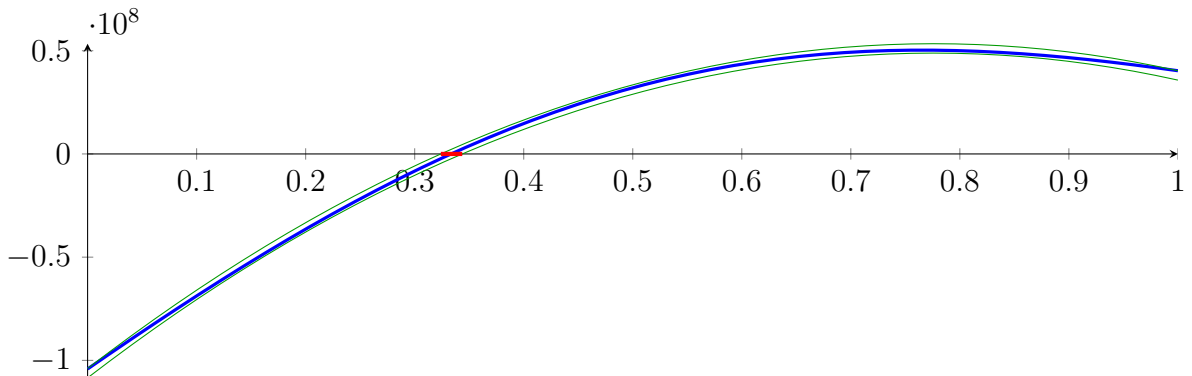
$$M = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.03924 \cdot 10^8$$

$$m = -2.61181 \cdot 10^8 X^2 + 4.05417 \cdot 10^8 X - 1.08469 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

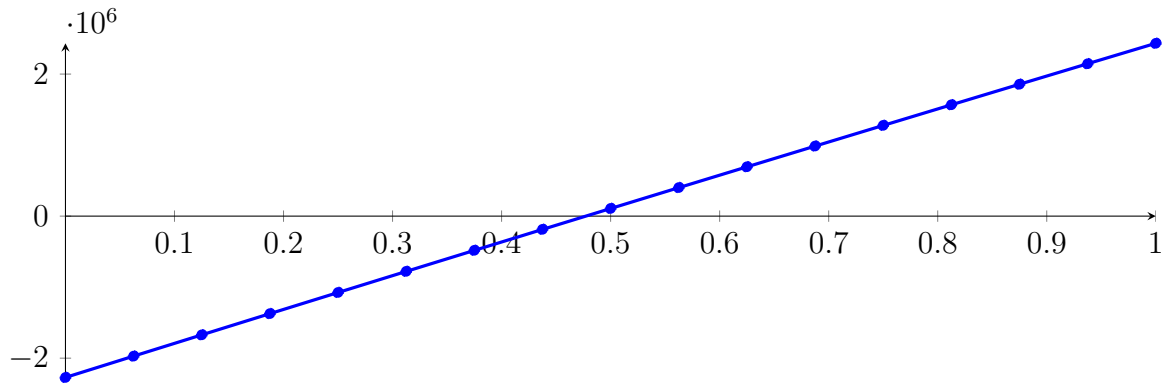
Longest intersection interval: 0.0196686

\implies Selective recursion: interval 1: $[0.323946, 0.343615]$,

251.2 Recursion Branch 1 1 in Interval 1: [0.323946, 0.343615]

Normalized monomial und Bézier representations and the Bézier polygon:

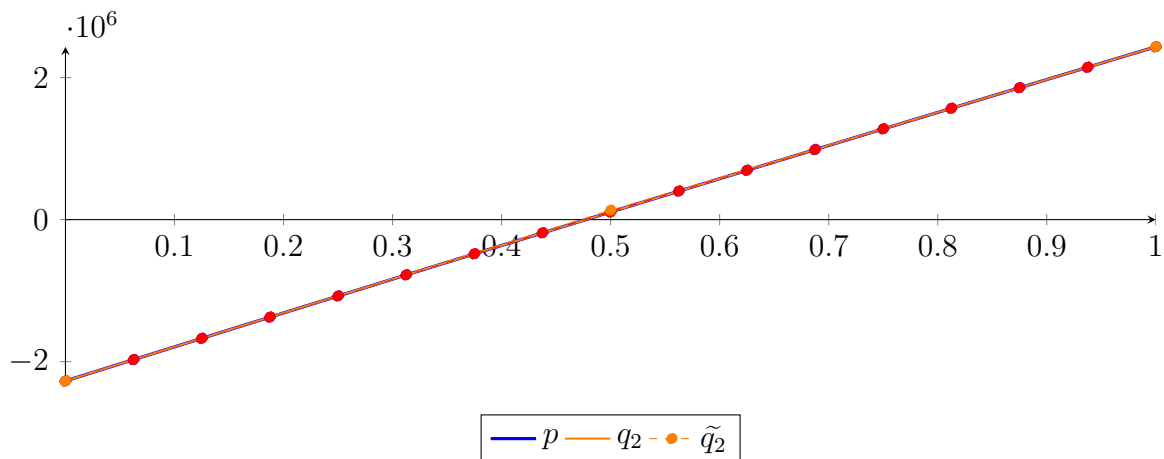
$$\begin{aligned}
 p &= -5.0162 \cdot 10^{-28} X^{16} - 1.14383 \cdot 10^{-24} X^{15} - 1.11125 \cdot 10^{-21} X^{14} - 5.84096 \cdot 10^{-19} X^{13} - 1.684 \cdot 10^{-16} X^{12} \\
 &\quad - 1.94707 \cdot 10^{-14} X^{11} + 2.73227 \cdot 10^{-12} X^{10} + 1.09727 \cdot 10^{-9} X^9 + 6.37314 \cdot 10^{-8} X^8 - 1.68645 \cdot 10^{-5} X^7 \\
 &\quad - 0.00205892 X^6 + 0.132721 X^5 + 22.4437 X^4 - 850.239 X^3 - 103028 X^2 + 4.80874 \cdot 10^6 X - 2.2715 \cdot 10^6 \\
 &= -2.2715 \cdot 10^6 B_{0,16}(X) - 1.97096 \cdot 10^6 B_{1,16}(X) - 1.67127 \cdot 10^6 B_{2,16}(X) - 1.37244 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.07448 \cdot 10^6 B_{4,16}(X) - 777374 B_{5,16}(X) - 481136 B_{6,16}(X) - 185764 B_{7,16}(X) + 108741 B_{8,16}(X) \\
 &\quad + 402376 B_{9,16}(X) + 695142 B_{10,16}(X) + 987035 B_{11,16}(X) + 1.27806 \cdot 10^6 B_{12,16}(X) + 1.5682 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.85747 \cdot 10^6 B_{14,16}(X) + 2.14586 \cdot 10^6 B_{15,16}(X) + 2.43338 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,2} + 133069 B_{1,2} + 2.43342 \cdot 10^6 B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -8.18461 \cdot 10^{-291} X^{16} + 4.13745 \cdot 10^{-290} X^{15} - 5.8254 \cdot 10^{-290} X^{14} - 6.80919 \cdot 10^{-290} X^{13} \\
 &\quad + 3.48059 \cdot 10^{-289} X^{12} - 5.47245 \cdot 10^{-289} X^{11} + 4.8261 \cdot 10^{-289} X^{10} - 2.67867 \cdot 10^{-289} X^9 \\
 &\quad + 1.00314 \cdot 10^{-289} X^8 - 2.84766 \cdot 10^{-290} X^7 + 6.85072 \cdot 10^{-291} X^6 - 1.12501 \cdot 10^{-291} X^5 \\
 &\quad + 1.24731 \cdot 10^{-293} X^4 + 2.25944 \cdot 10^{-293} X^3 - 104265 X^2 + 4.80923 \cdot 10^6 X - 2.27154 \cdot 10^6 \\
 &= -2.27154 \cdot 10^6 B_{0,16} - 1.97097 \cdot 10^6 B_{1,16} - 1.67126 \cdot 10^6 B_{2,16} - 1.37242 \cdot 10^6 B_{3,16} \\
 &\quad - 1.07445 \cdot 10^6 B_{4,16} - 777350 B_{5,16} - 481118 B_{6,16} - 185754 B_{7,16} + 108740 B_{8,16} \\
 &\quad + 402366 B_{9,16} + 695123 B_{10,16} + 987011 B_{11,16} + 1.27803 \cdot 10^6 B_{12,16} + 1.56818 \\
 &\quad \cdot 10^6 B_{13,16} + 1.85746 \cdot 10^6 B_{14,16} + 2.14587 \cdot 10^6 B_{15,16} + 2.43342 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m :

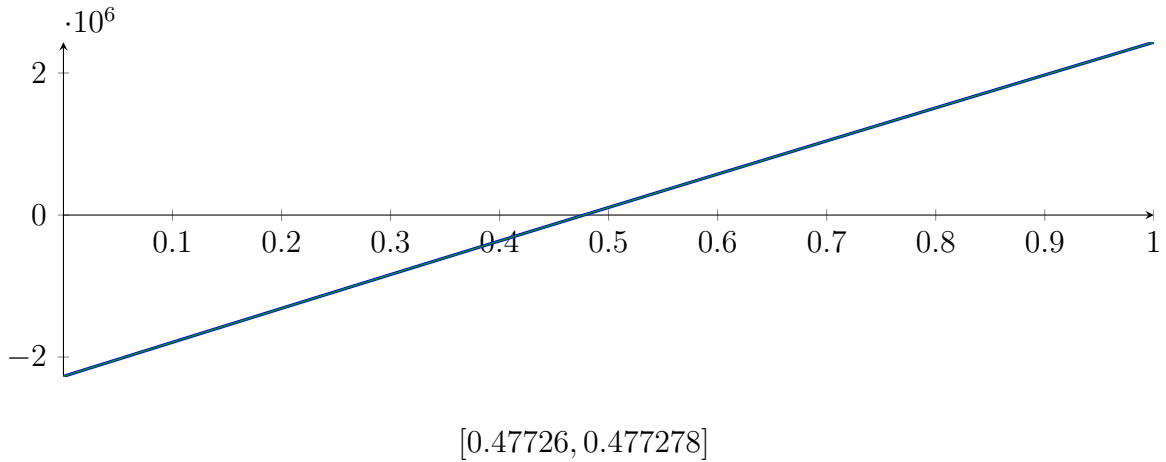
$$M = -104265X^2 + 4.80923 \cdot 10^6 X - 2.2715 \cdot 10^6$$

$$m = -104265X^2 + 4.80923 \cdot 10^6 X - 2.27159 \cdot 10^6$$

Root of M and m :

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:

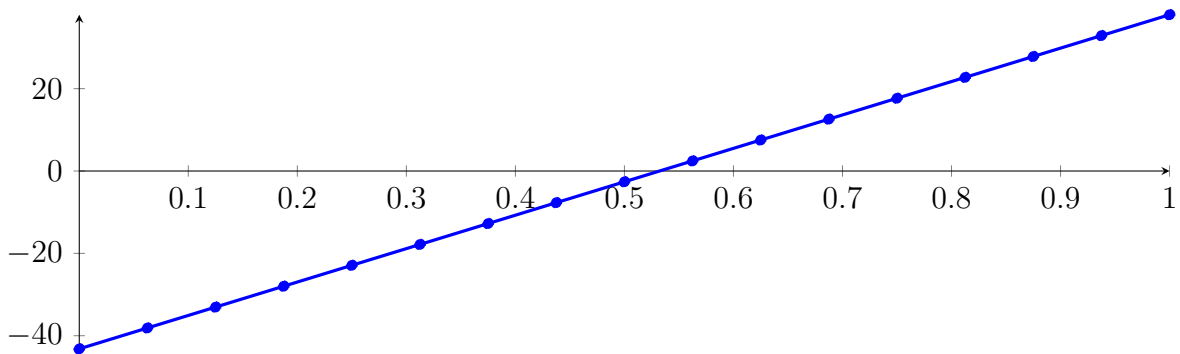


Longest intersection interval: $1.72301 \cdot 10^{-05}$
 \implies Selective recursion: [interval 1: \[0.333333, 0.333333\]](#),

251.3 Recursion Branch 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= -3.02667 \cdot 10^{-104} X^{16} - 4.019 \cdot 10^{-96} X^{15} - 2.27522 \cdot 10^{-88} X^{14} - 6.97783 \cdot 10^{-81} X^{13} \\
 &\quad - 1.17785 \cdot 10^{-73} X^{12} - 8.12373 \cdot 10^{-67} X^{11} + 6.05916 \cdot 10^{-60} X^{10} + 1.48569 \cdot 10^{-52} X^9 \\
 &\quad + 5.31875 \cdot 10^{-46} X^8 - 7.48919 \cdot 10^{-39} X^7 - 5.53349 \cdot 10^{-32} X^6 + 1.92471 \cdot 10^{-25} X^5 \\
 &\quad + 2.00536 \cdot 10^{-18} X^4 - 4.12844 \cdot 10^{-12} X^3 - 3.09388 \cdot 10^{-05} X^2 + 81.1506X - 43.1911 \\
 &= -43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\
 &\quad - 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X) \\
 &\quad + 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X) \\
 &\quad + 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_2 = -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$

$$= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2}$$

$$\tilde{q}_2 = -1.88281 \cdot 10^{-295} X^{16} + 1.09893 \cdot 10^{-294} X^{15} - 2.26419 \cdot 10^{-294} X^{14} + 8.08223 \cdot 10^{-295} X^{13}$$

$$+ 4.92626 \cdot 10^{-294} X^{12} - 1.07605 \cdot 10^{-293} X^{11} + 1.11858 \cdot 10^{-293} X^{10} - 6.87288 \cdot 10^{-294} X^9$$

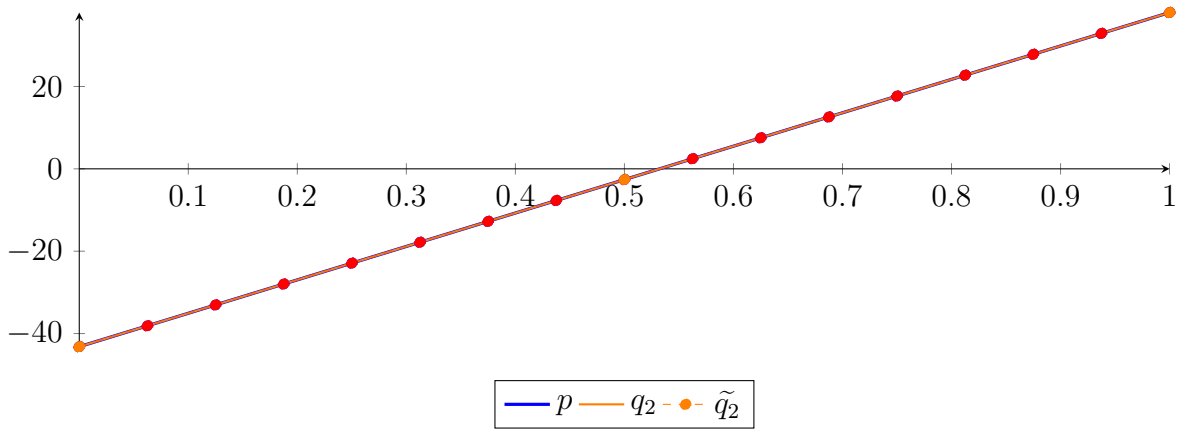
$$+ 2.54873 \cdot 10^{-294} X^8 - 5.2305 \cdot 10^{-295} X^7 + 3.18923 \cdot 10^{-296} X^6 + 1.34092 \cdot 10^{-296} X^5$$

$$- 4.89549 \cdot 10^{-297} X^4 + 5.89947 \cdot 10^{-298} X^3 - 3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$

$$= -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16}$$

$$- 12.7597B_{6,16} - 7.68778B_{7,16} - 2.61587B_{8,16} + 2.45604B_{9,16} + 7.52795B_{10,16} + 12.5999B_{11,16}$$

$$+ 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.06422 \cdot 10^{-13}$.

Bounding polynomials M and m :

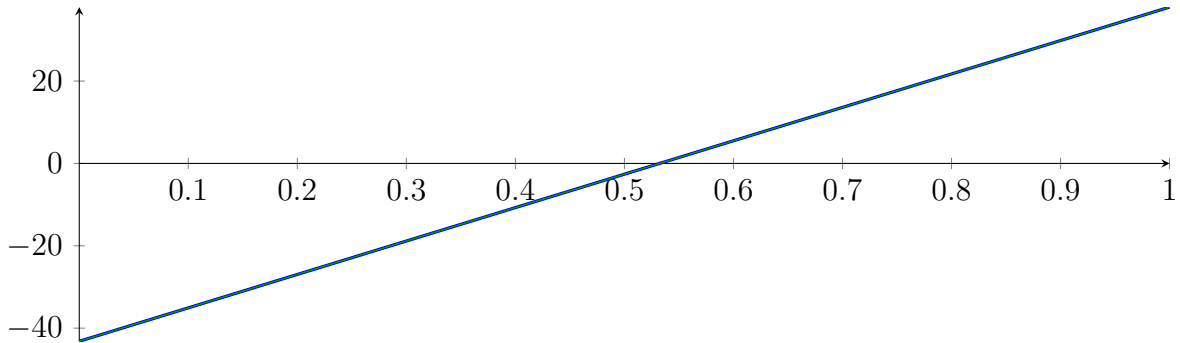
$$M = -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$

$$m = -3.09389 \cdot 10^{-05} X^2 + 81.1506X - 43.1911$$

Root of M and m :

$$N(M) = \{0.532235, 2.62293 \cdot 10^6\} \quad N(m) = \{0.532235, 2.62293 \cdot 10^6\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

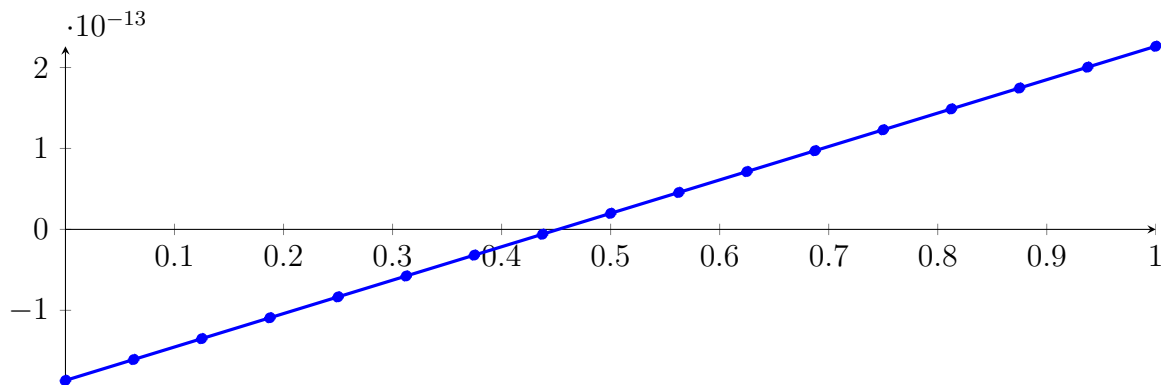
Longest intersection interval: $5.08738 \cdot 10^{-15}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

251.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

Normalized monomial und Bézier representations and the Bézier polygon:

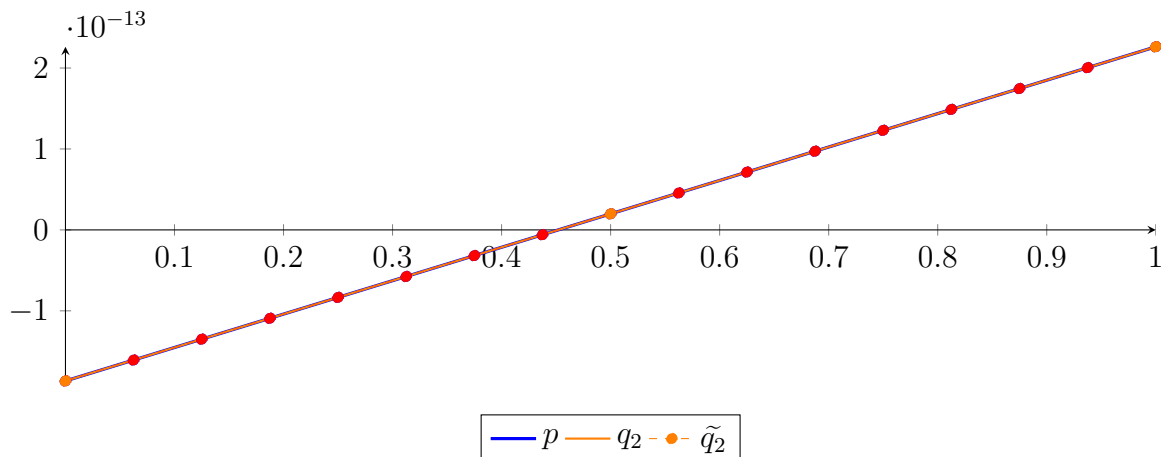
$$\begin{aligned}
 p &= -3.84502 \cdot 10^{-319} X^{16} - 1.59047 \cdot 10^{-310} X^{15} - 1.76985 \cdot 10^{-288} X^{14} - 1.06694 \cdot 10^{-266} X^{13} \\
 &\quad - 3.54011 \cdot 10^{-245} X^{12} - 4.79942 \cdot 10^{-224} X^{11} + 7.03641 \cdot 10^{-203} X^{10} + 3.39135 \cdot 10^{-181} X^9 \\
 &\quad + 2.3865 \cdot 10^{-160} X^8 - 6.60529 \cdot 10^{-139} X^7 - 9.59319 \cdot 10^{-118} X^6 + 6.55895 \cdot 10^{-97} X^5 + 1.34328 \\
 &\quad \cdot 10^{-75} X^4 - 5.43584 \cdot 10^{-55} X^3 - 8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13} \\
 &= -1.86598 \cdot 10^{-13} B_{0,16}(X) - 1.60795 \cdot 10^{-13} B_{1,16}(X) - 1.34993 \cdot 10^{-13} B_{2,16}(X) - 1.0919 \\
 &\quad \cdot 10^{-13} B_{3,16}(X) - 8.33872 \cdot 10^{-14} B_{4,16}(X) - 5.75845 \cdot 10^{-14} B_{5,16}(X) - 3.17818 \cdot 10^{-14} B_{6,16}(X) \\
 &\quad - 5.97912 \cdot 10^{-15} B_{7,16}(X) + 1.98236 \cdot 10^{-14} B_{8,16}(X) + 4.56263 \cdot 10^{-14} B_{9,16}(X) + 7.1429 \\
 &\quad \cdot 10^{-14} B_{10,16}(X) + 9.72317 \cdot 10^{-14} B_{11,16}(X) + 1.23034 \cdot 10^{-13} B_{12,16}(X) + 1.48837 \\
 &\quad \cdot 10^{-13} B_{13,16}(X) + 1.7464 \cdot 10^{-13} B_{14,16}(X) + 2.00443 \cdot 10^{-13} B_{15,16}(X) + 2.26245 \cdot 10^{-13} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13} \\
 &= -1.86598 \cdot 10^{-13} B_{0,2} + 1.98236 \cdot 10^{-14} B_{1,2} + 2.26245 \cdot 10^{-13} B_{2,2}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_2 &= -8.08289 \cdot 10^{-310} X^{16} + 4.33931 \cdot 10^{-309} X^{15} - 8.19282 \cdot 10^{-309} X^{14} + 3.30456 \cdot 10^{-309} X^{13} \\
 &\quad + 1.10762 \cdot 10^{-308} X^{12} - 2.01579 \cdot 10^{-308} X^{11} + 1.55412 \cdot 10^{-308} X^{10} - 6.55981 \cdot 10^{-309} X^9 \\
 &\quad + 2.12881 \cdot 10^{-309} X^8 - 1.08538 \cdot 10^{-309} X^7 + 5.4154 \cdot 10^{-310} X^6 - 1.38567 \cdot 10^{-310} X^5 + 9.27245 \\
 &\quad \cdot 10^{-312} X^4 + 1.80627 \cdot 10^{-312} X^3 - 8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13} \\
 &= -1.86598 \cdot 10^{-13} B_{0,16} - 1.60795 \cdot 10^{-13} B_{1,16} - 1.34993 \cdot 10^{-13} B_{2,16} - 1.0919 \cdot 10^{-13} B_{3,16} - 8.33872 \\
 &\quad \cdot 10^{-14} B_{4,16} - 5.75845 \cdot 10^{-14} B_{5,16} - 3.17818 \cdot 10^{-14} B_{6,16} - 5.97912 \cdot 10^{-15} B_{7,16} + 1.98236 \cdot 10^{-14} B_{8,16} \\
 &\quad + 4.56263 \cdot 10^{-14} B_{9,16} + 7.1429 \cdot 10^{-14} B_{10,16} + 9.72317 \cdot 10^{-14} B_{11,16} + 1.23034 \cdot 10^{-13} B_{12,16} \\
 &\quad + 1.48837 \cdot 10^{-13} B_{13,16} + 1.7464 \cdot 10^{-13} B_{14,16} + 2.00443 \cdot 10^{-13} B_{15,16} + 2.26245 \cdot 10^{-13} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.71792 \cdot 10^{-56}$.

Bounding polynomials M and m :

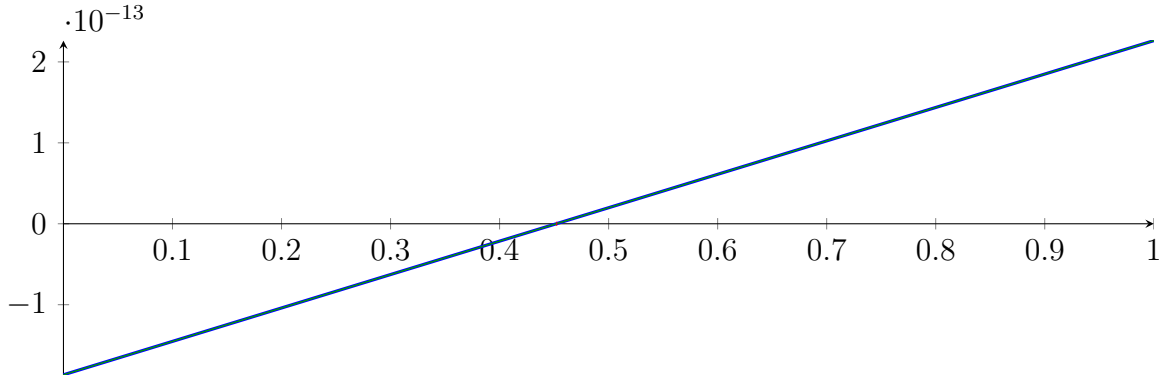
$$M = -8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13}$$

$$m = -8.00741 \cdot 10^{-34} X^2 + 4.12843 \cdot 10^{-13} X - 1.86598 \cdot 10^{-13}$$

Root of M and m :

$$N(M) = \{0.451983, 5.15577 \cdot 10^{20}\} \quad N(m) = \{0.451983, 5.15577 \cdot 10^{20}\}$$

Intersection intervals:



$$[0.451983, 0.451983]$$

Longest intersection interval: $1.31668 \cdot 10^{-43}$

⇒ Selective recursion: **interval 1:** $[0.333333, 0.333333]$,

251.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$p = 5.66252 \cdot 10^{-361} X^{16} + 8.30503 \cdot 10^{-360} X^{15} - 8.71157 \cdot 10^{-361} X^{14} + 3.25232 \cdot 10^{-359} X^{13}$$

$$+ 1.65157 \cdot 10^{-358} X^{12} - 1.58551 \cdot 10^{-358} X^{11} - 7.2669 \cdot 10^{-359} X^{10} - 4.15252 \cdot 10^{-359} X^9$$

$$+ 9.34316 \cdot 10^{-359} X^8 + 1.45338 \cdot 10^{-359} X^6 + 2.59562 \cdot 10^{-311} X^5 + 4.03733 \cdot 10^{-247} X^4$$

$$- 1.24083 \cdot 10^{-183} X^3 - 1.38821 \cdot 10^{-119} X^2 + 5.43584 \cdot 10^{-56} X - 3.10342 \cdot 10^{-56}$$

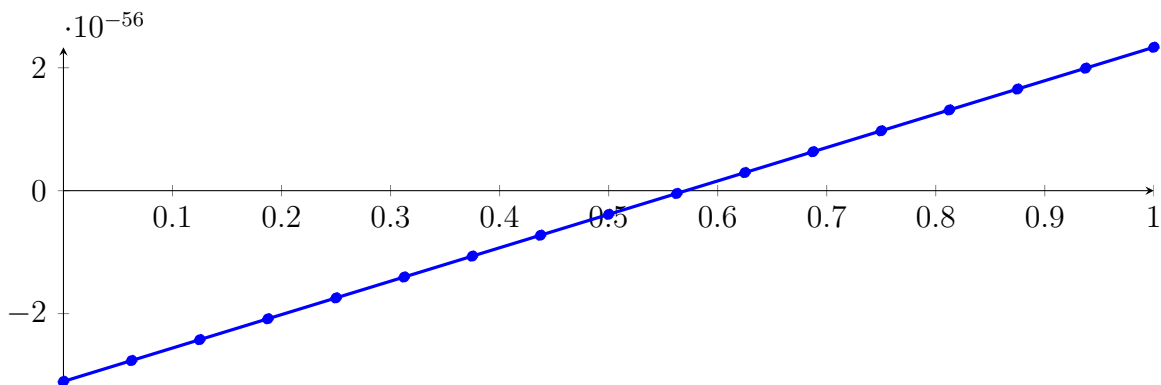
$$= -3.10342 \cdot 10^{-56} B_{0,16}(X) - 2.76368 \cdot 10^{-56} B_{1,16}(X) - 2.42394 \cdot 10^{-56} B_{2,16}(X) - 2.0842$$

$$\cdot 10^{-56} B_{3,16}(X) - 1.74446 \cdot 10^{-56} B_{4,16}(X) - 1.40472 \cdot 10^{-56} B_{5,16}(X) - 1.06498 \cdot 10^{-56} B_{6,16}(X)$$

$$- 7.25243 \cdot 10^{-57} B_{7,16}(X) - 3.85503 \cdot 10^{-57} B_{8,16}(X) - 4.57628 \cdot 10^{-58} B_{9,16}(X) + 2.93977$$

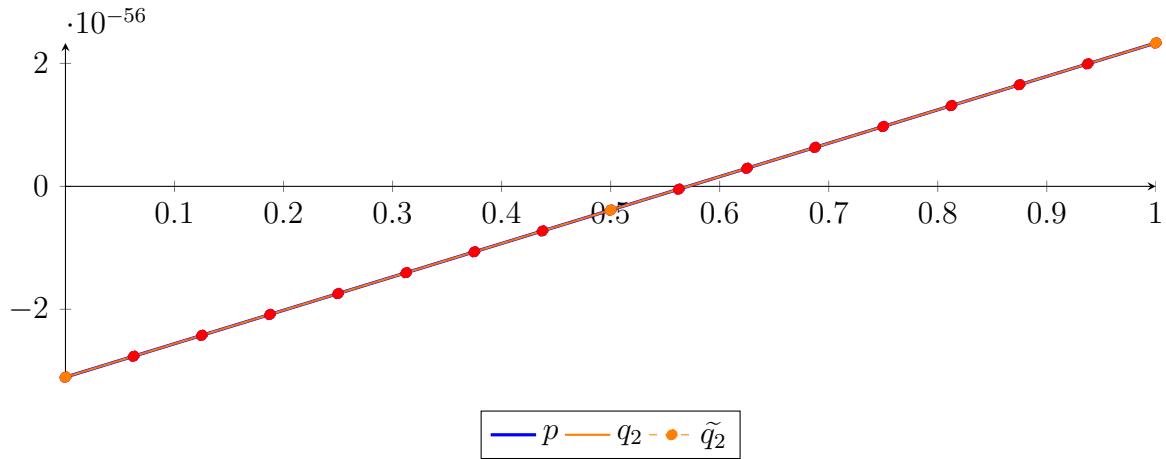
$$\cdot 10^{-57} B_{10,16}(X) + 6.33717 \cdot 10^{-57} B_{11,16}(X) + 9.73457 \cdot 10^{-57} B_{12,16}(X) + 1.3132 \cdot 10^{-56} B_{13,16}(X)$$

$$+ 1.65294 \cdot 10^{-56} B_{14,16}(X) + 1.99268 \cdot 10^{-56} B_{15,16}(X) + 2.33242 \cdot 10^{-56} B_{16,16}(X)$$



Degree reduction and raising:

$$\begin{aligned}
 q_2 &= -1.38821 \cdot 10^{-119} X^2 + 5.43584 \cdot 10^{-56} X - 3.10342 \cdot 10^{-56} \\
 &= -3.10342 \cdot 10^{-56} B_{0,2} - 3.85503 \cdot 10^{-57} B_{1,2} + 2.33242 \cdot 10^{-56} B_{2,2} \\
 \tilde{q}_2 &= -1.35612 \cdot 10^{-352} X^{16} + 8.15544 \cdot 10^{-352} X^{15} - 1.72777 \cdot 10^{-351} X^{14} + 5.92647 \cdot 10^{-352} X^{13} \\
 &\quad + 4.18743 \cdot 10^{-351} X^{12} - 9.40291 \cdot 10^{-351} X^{11} + 1.01181 \cdot 10^{-350} X^{10} - 6.40657 \cdot 10^{-351} X^9 \\
 &\quad + 2.39508 \cdot 10^{-351} X^8 - 4.50359 \cdot 10^{-352} X^7 - 2.71062 \cdot 10^{-354} X^6 + 2.21064 \cdot 10^{-353} X^5 - 5.44842 \\
 &\quad \cdot 10^{-354} X^4 + 4.71019 \cdot 10^{-355} X^3 - 1.38821 \cdot 10^{-119} X^2 + 5.43584 \cdot 10^{-56} X - 3.10342 \cdot 10^{-56} \\
 &= -3.10342 \cdot 10^{-56} B_{0,16} - 2.76368 \cdot 10^{-56} B_{1,16} - 2.42394 \cdot 10^{-56} B_{2,16} - 2.0842 \cdot 10^{-56} B_{3,16} - 1.74446 \\
 &\quad \cdot 10^{-56} B_{4,16} - 1.40472 \cdot 10^{-56} B_{5,16} - 1.06498 \cdot 10^{-56} B_{6,16} - 7.25243 \cdot 10^{-57} B_{7,16} - 3.85503 \cdot 10^{-57} B_{8,16} \\
 &\quad - 4.57628 \cdot 10^{-58} B_{9,16} + 2.93977 \cdot 10^{-57} B_{10,16} + 6.33717 \cdot 10^{-57} B_{11,16} + 9.73457 \cdot 10^{-57} B_{12,16} \\
 &\quad + 1.3132 \cdot 10^{-56} B_{13,16} + 1.65294 \cdot 10^{-56} B_{14,16} + 1.99268 \cdot 10^{-56} B_{15,16} + 2.33242 \cdot 10^{-56} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 6.20413 \cdot 10^{-185}$.

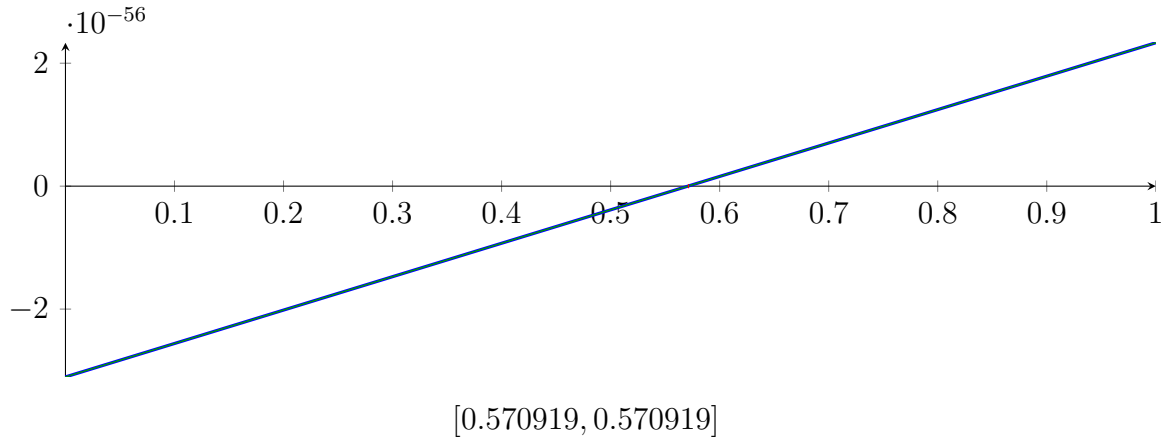
Bounding polynomials M and m :

$$\begin{aligned}
 M &= -1.38821 \cdot 10^{-119} X^2 + 5.43584 \cdot 10^{-56} X - 3.10342 \cdot 10^{-56} \\
 m &= -1.38821 \cdot 10^{-119} X^2 + 5.43584 \cdot 10^{-56} X - 3.10342 \cdot 10^{-56}
 \end{aligned}$$

Root of M and m :

$$N(M) = \{0.570919, 3.91572 \cdot 10^{63}\} \qquad N(m) = \{0.570919, 3.91572 \cdot 10^{63}\}$$

Intersection intervals:



Longest intersection interval: $2.28268 \cdot 10^{-129}$
 \implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

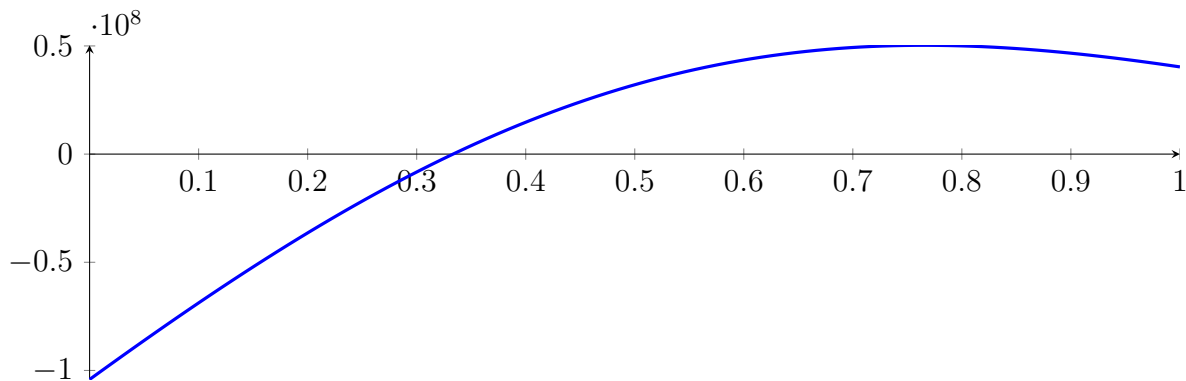
251.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

251.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

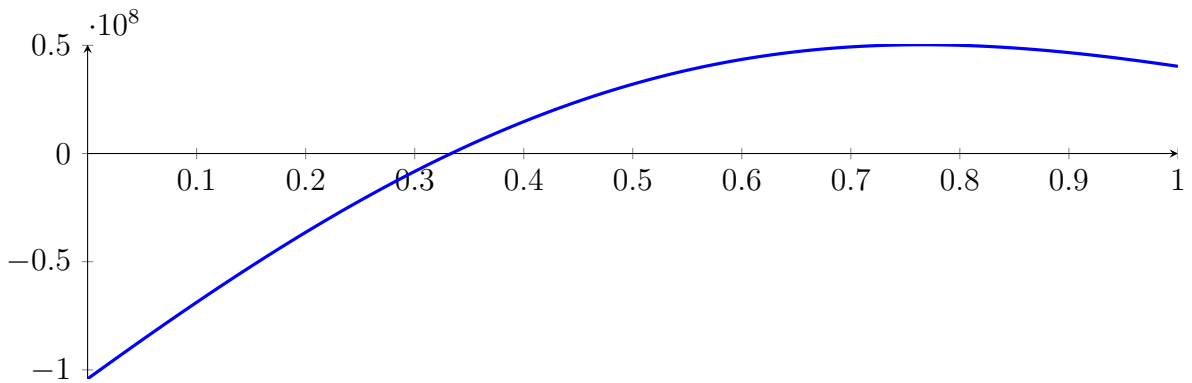
with precision $\varepsilon = 1 \cdot 10^{-128}$.

252 Running CubeClip on f_{16} with epsilon 128

$$\begin{aligned}
 & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + \\
 & 18475.3X^{11} + 451673X^{10} + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot \\
 & 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - \\
 & 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

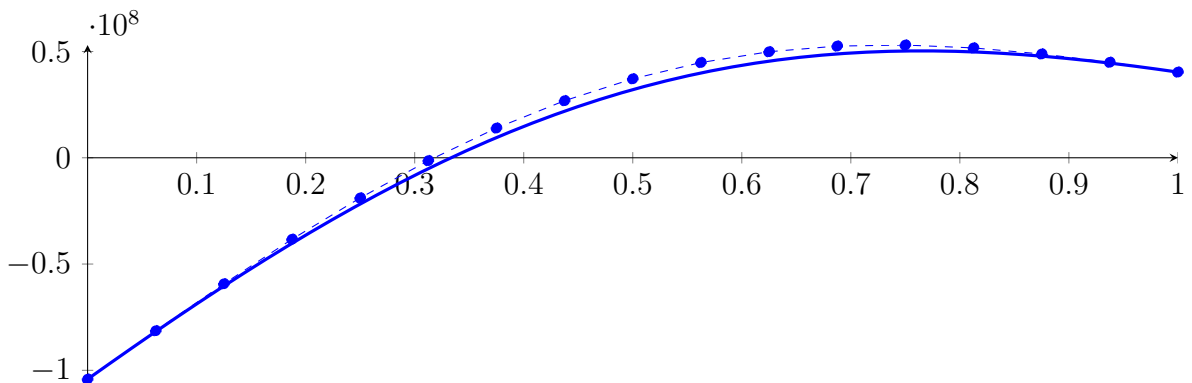
$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8
 \end{aligned}$$



252.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
 & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\
 & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \\
 = & -1.04167 \cdot 10^8 B_{0,16}(X) - 8.13802 \cdot 10^7 B_{1,16}(X) - 5.92882 \cdot 10^7 B_{2,16}(X) - 3.83203 \\
 & \cdot 10^7 B_{3,16}(X) - 1.88844 \cdot 10^7 B_{4,16}(X) - 1.34837 \cdot 10^6 B_{5,16}(X) + 1.39781 \cdot 10^7 B_{6,16}(X) \\
 & + 2.68604 \cdot 10^7 B_{7,16}(X) + 3.71532 \cdot 10^7 B_{8,16}(X) + 4.48105 \cdot 10^7 B_{9,16}(X) + 4.98901 \\
 & \cdot 10^7 B_{10,16}(X) + 5.25504 \cdot 10^7 B_{11,16}(X) + 5.30407 \cdot 10^7 B_{12,16}(X) + 5.16832 \cdot 10^7 B_{13,16}(X) \\
 & + 4.88488 \cdot 10^7 B_{14,16}(X) + 4.49297 \cdot 10^7 B_{15,16}(X) + 4.03108 \cdot 10^7 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,3} + 2.93558 \cdot 10^7 B_{1,3} + 7.66068 \cdot 10^7 B_{2,3} + 3.81764 \cdot 10^7 B_{3,3}$$

$$\tilde{q}_3 = 1.89955 \cdot 10^{-288} X^{16} - 1.4851 \cdot 10^{-287} X^{15} + 5.12896 \cdot 10^{-287} X^{14} - 1.02224 \cdot 10^{-286} X^{13}$$

$$+ 1.29027 \cdot 10^{-286} X^{12} - 1.07058 \cdot 10^{-286} X^{11} + 6.00739 \cdot 10^{-287} X^{10} - 2.54352 \cdot 10^{-287} X^9$$

$$+ 1.12089 \cdot 10^{-287} X^8 - 5.8673 \cdot 10^{-288} X^7 + 2.52637 \cdot 10^{-288} X^6 - 6.79624 \cdot 10^{-289} X^5$$

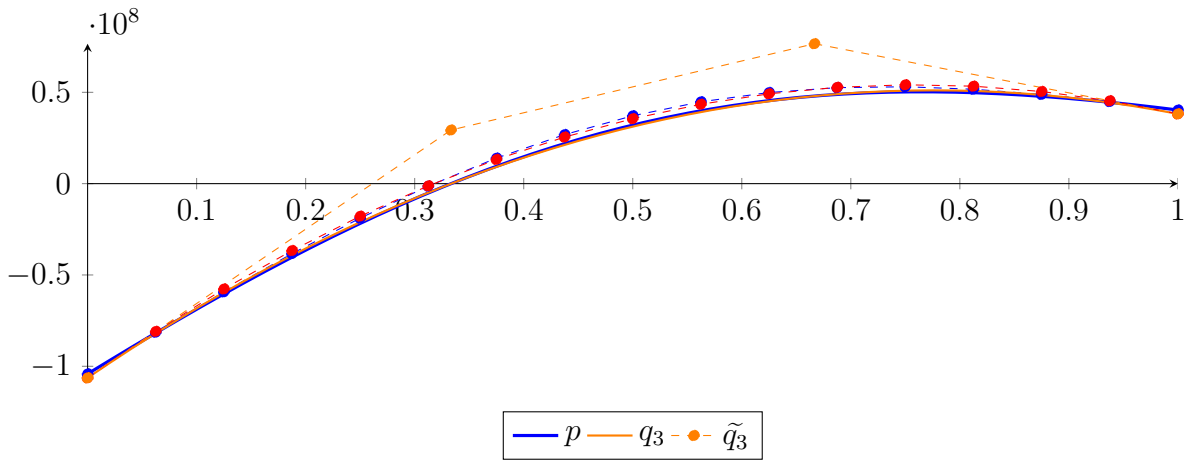
$$+ 9.36341 \cdot 10^{-290} X^4 + 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.06335 \cdot 10^8$$

$$= -1.06335 \cdot 10^8 B_{0,16} - 8.08928 \cdot 10^7 B_{1,16} - 5.76618 \cdot 10^7 B_{2,16} - 3.66369 \cdot 10^7 B_{3,16} - 1.78131$$

$$\cdot 10^7 B_{4,16} - 1.18551 \cdot 10^6 B_{5,16} + 1.32508 \cdot 10^7 B_{6,16} + 2.55007 \cdot 10^7 B_{7,16} + 3.55692 \cdot 10^7 B_{8,16}$$

$$+ 4.34611 \cdot 10^7 B_{9,16} + 4.91815 \cdot 10^7 B_{10,16} + 5.27353 \cdot 10^7 B_{11,16} + 5.41273 \cdot 10^7 B_{12,16}$$

$$+ 5.33624 \cdot 10^7 B_{13,16} + 5.04457 \cdot 10^7 B_{14,16} + 4.53821 \cdot 10^7 B_{15,16} + 3.81764 \cdot 10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806 \cdot 10^6$.

Bounding polynomials M and m :

$$M = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.04167 \cdot 10^8$$

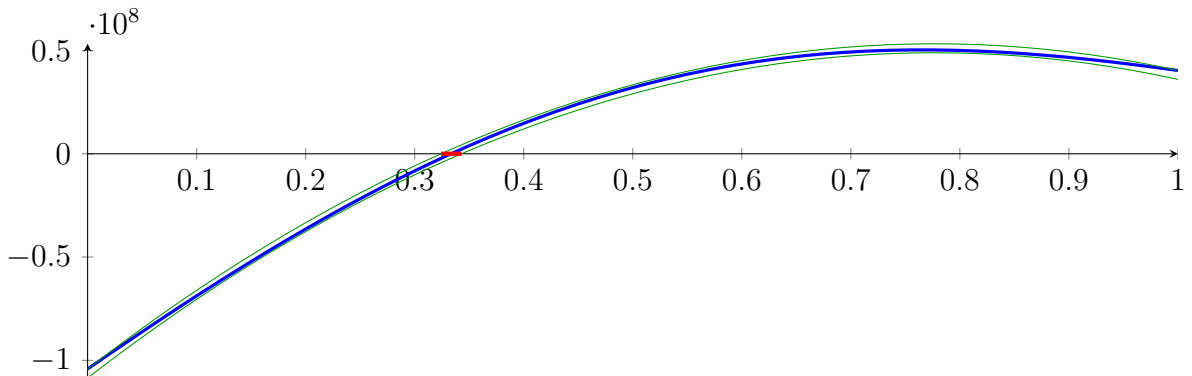
$$m = 2.75812 \cdot 10^6 X^3 - 2.65319 \cdot 10^8 X^2 + 4.07072 \cdot 10^8 X - 1.08503 \cdot 10^8$$

Root of M and m :

$$N(M) = \{0.324143, 1.23113, 94.6401\}$$

$$N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

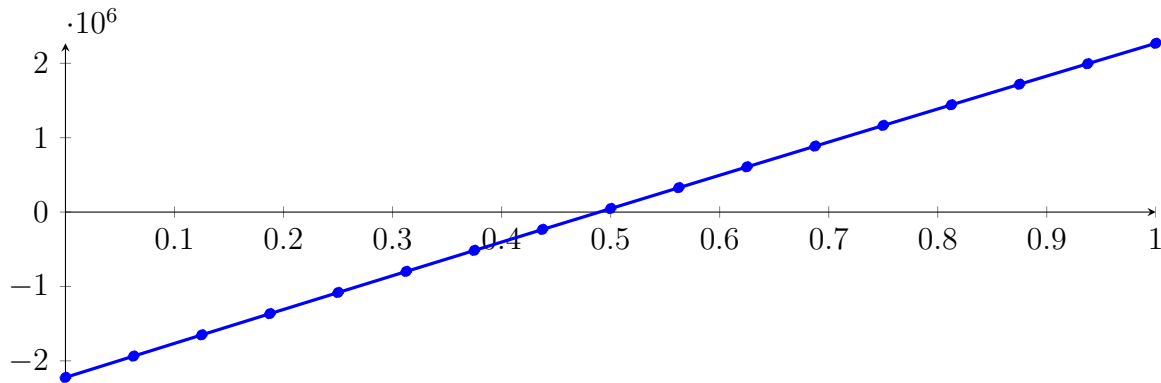
Longest intersection interval: 0.0187703

\implies Selective recursion: interval 1: $[0.324143, 0.342913]$,

252.2 Recursion Branch 1 1 in Interval 1: [0.324143, 0.342913]

Normalized monomial und Bézier representations and the Bézier polygon:

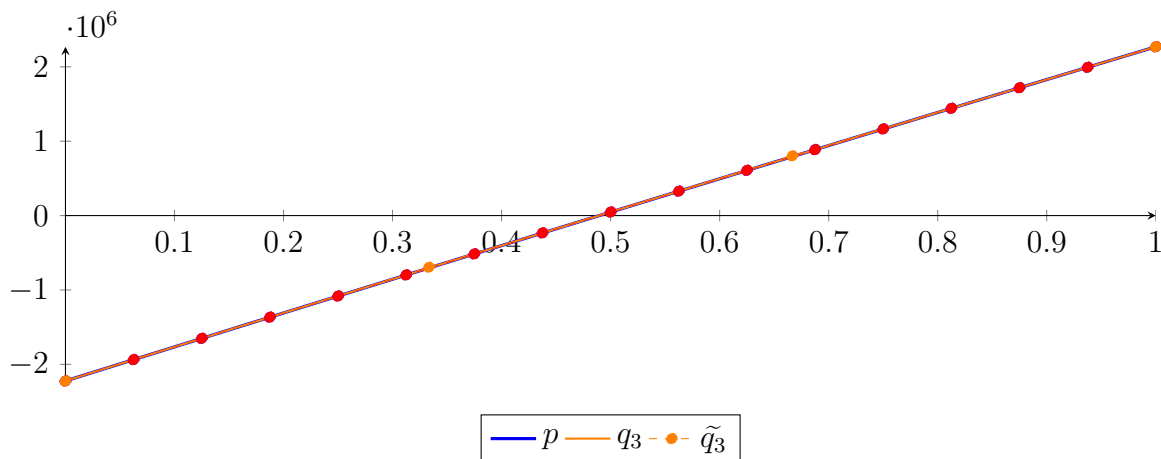
$$\begin{aligned}
 p &= -2.37433 \cdot 10^{-28} X^{16} - 5.67363 \cdot 10^{-25} X^{15} - 5.77631 \cdot 10^{-22} X^{14} - 3.1818 \cdot 10^{-19} X^{13} - 9.6142 \cdot 10^{-17} X^{12} \\
 &\quad - 1.16549 \cdot 10^{-14} X^{11} + 1.71065 \cdot 10^{-12} X^{10} + 7.20611 \cdot 10^{-10} X^9 + 4.39147 \cdot 10^{-08} X^8 - 1.21542 \cdot 10^{-05} X^7 \\
 &\quad - 0.00155624 X^6 + 0.10496 X^5 + 18.6215 X^4 - 738.202 X^3 - 93855.7 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16}(X) - 1.93666 \cdot 10^6 B_{1,16}(X) - 1.65074 \cdot 10^6 B_{2,16}(X) - 1.36561 \cdot 10^6 B_{3,16}(X) \\
 &\quad - 1.08127 \cdot 10^6 B_{4,16}(X) - 797705 B_{5,16}(X) - 514932 B_{6,16}(X) - 232948 B_{7,16}(X) + 48246.6 B_{8,16}(X) \\
 &\quad + 328650 B_{9,16}(X) + 608261 B_{10,16}(X) + 887078 B_{11,16}(X) + 1.1651 \cdot 10^6 B_{12,16}(X) + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16}(X) + 1.71876 \cdot 10^6 B_{14,16}(X) + 1.99439 \cdot 10^6 B_{15,16}(X) + 2.26922 \cdot 10^6 B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,3} - 694303 B_{1,3} + 803454 B_{2,3} + 2.26922 \cdot 10^6 B_{3,3}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{q}_3 &= 4.41595 \cdot 10^{-291} X^{16} - 1.48598 \cdot 10^{-290} X^{15} - 2.81102 \cdot 10^{-290} X^{14} + 2.65129 \cdot 10^{-289} X^{13} \\
 &\quad - 7.34593 \cdot 10^{-289} X^{12} + 1.17654 \cdot 10^{-288} X^{11} - 1.23119 \cdot 10^{-288} X^{10} + 8.63493 \cdot 10^{-289} X^9 \\
 &\quad - 3.92909 \cdot 10^{-289} X^8 + 1.01265 \cdot 10^{-289} X^7 - 6.13644 \cdot 10^{-291} X^6 - 3.92664 \cdot 10^{-291} X^5 \\
 &\quad + 9.04488 \cdot 10^{-292} X^4 - 700.673 X^3 - 93879.9 X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6 \\
 &= -2.22335 \cdot 10^6 B_{0,16} - 1.93666 \cdot 10^6 B_{1,16} - 1.65074 \cdot 10^6 B_{2,16} - 1.36561 \cdot 10^6 B_{3,16} \\
 &\quad - 1.08126 \cdot 10^6 B_{4,16} - 797705 B_{5,16} - 514932 B_{6,16} - 232948 B_{7,16} + 48246.4 B_{8,16} \\
 &\quad + 328650 B_{9,16} + 608261 B_{10,16} + 887078 B_{11,16} + 1.1651 \cdot 10^6 B_{12,16} + 1.44233 \\
 &\quad \cdot 10^6 B_{13,16} + 1.71876 \cdot 10^6 B_{14,16} + 1.99439 \cdot 10^6 B_{15,16} + 2.26922 \cdot 10^6 B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m :

$$M = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

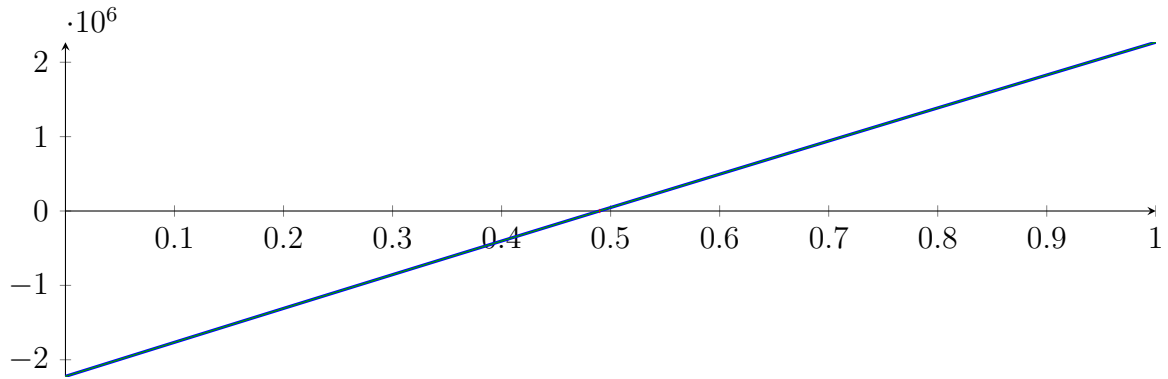
$$m = -700.673X^3 - 93879.9X^2 + 4.58715 \cdot 10^6 X - 2.22335 \cdot 10^6$$

Root of M and m :

$$N(M) = \{-172.127, 0.489616, 37.6521\}$$

$$N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



$$[0.489616, 0.489616]$$

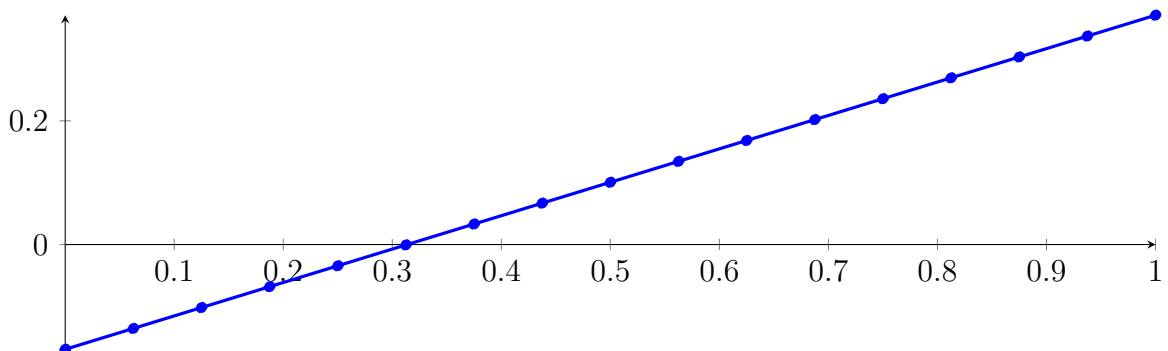
Longest intersection interval: $1.20174 \cdot 10^{-07}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

252.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -4.49274 \cdot 10^{-139} X^{16} - 8.96277 \cdot 10^{-129} X^{15} - 7.623 \cdot 10^{-119} X^{14} - 3.51238 \cdot 10^{-109} X^{13} \\ &\quad - 8.90739 \cdot 10^{-100} X^{12} - 9.22984 \cdot 10^{-91} X^{11} + 1.03426 \cdot 10^{-81} X^{10} + 3.80998 \cdot 10^{-72} X^9 \\ &\quad + 2.04919 \cdot 10^{-63} X^8 - 4.33497 \cdot 10^{-54} X^7 - 4.81204 \cdot 10^{-45} X^6 + 2.51462 \cdot 10^{-36} X^5 \\ &\quad + 3.93622 \cdot 10^{-27} X^4 - 1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148X - 0.169396 \\ &= -0.169396B_{0,16}(X) - 0.135637B_{1,16}(X) - 0.101877B_{2,16}(X) - 0.068118B_{3,16}(X) \\ &\quad - 0.0343588B_{4,16}(X) - 0.000599488B_{5,16}(X) + 0.0331598B_{6,16}(X) \\ &\quad + 0.066919B_{7,16}(X) + 0.100678B_{8,16}(X) + 0.134438B_{9,16}(X) + 0.168197B_{10,16}(X) \\ &\quad + 0.201956B_{11,16}(X) + 0.235715B_{12,16}(X) + 0.269475B_{13,16}(X) \\ &\quad + 0.303234B_{14,16}(X) + 0.336993B_{15,16}(X) + 0.370752B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = -1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

$$= -0.169396 B_{0,3} + 0.0106536 B_{1,3} + 0.190703 B_{2,3} + 0.370752 B_{3,3}$$

$$\tilde{q}_3 = 8.03185 \cdot 10^{-297} X^{16} - 6.20841 \cdot 10^{-296} X^{15} + 2.13274 \cdot 10^{-295} X^{14} - 4.26614 \cdot 10^{-295} X^{13}$$

$$+ 5.47461 \cdot 10^{-295} X^{12} - 4.70265 \cdot 10^{-295} X^{11} + 2.78551 \cdot 10^{-295} X^{10} - 1.22442 \cdot 10^{-295} X^9$$

$$+ 4.88954 \cdot 10^{-296} X^8 - 2.11494 \cdot 10^{-296} X^7 + 8.20665 \cdot 10^{-297} X^6 - 2.15458 \cdot 10^{-297} X^5$$

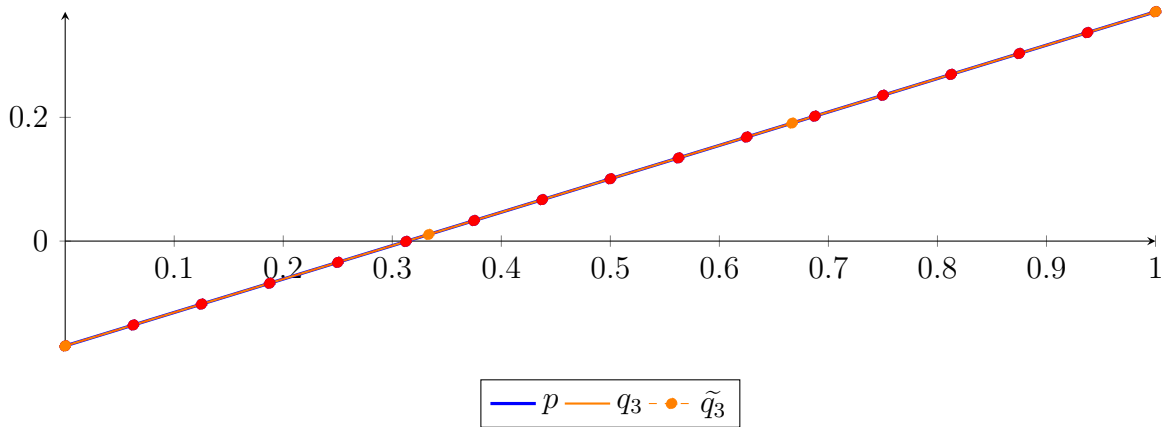
$$+ 3.01517 \cdot 10^{-298} X^4 - 1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

$$= -0.169396 B_{0,16} - 0.135637 B_{1,16} - 0.101877 B_{2,16} - 0.068118 B_{3,16} - 0.0343588 B_{4,16}$$

$$- 0.000599488 B_{5,16} + 0.0331598 B_{6,16} + 0.066919 B_{7,16} + 0.100678 B_{8,16}$$

$$+ 0.134438 B_{9,16} + 0.168197 B_{10,16} + 0.201956 B_{11,16} + 0.235715 B_{12,16}$$

$$+ 0.269475 B_{13,16} + 0.303234 B_{14,16} + 0.336993 B_{15,16} + 0.370752 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 5.62317 \cdot 10^{-29}$.

Bounding polynomials M and m :

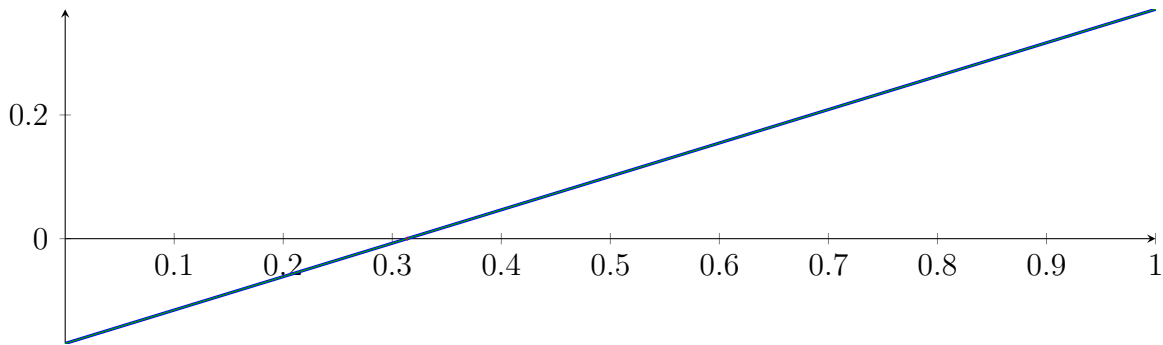
$$M = -1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

$$m = -1.21745 \cdot 10^{-18} X^3 - 1.37072 \cdot 10^{-09} X^2 + 0.540148 X - 0.169396$$

Root of M and m :

$$N(M) = \{-1.43506 \cdot 10^9, 0.31361, 3.09167 \cdot 10^8\} \quad N(m) = \{-1.43506 \cdot 10^9, 0.31361, 3.09167 \cdot 10^8\}$$

Intersection intervals:



$$[0.31361, 0.31361]$$

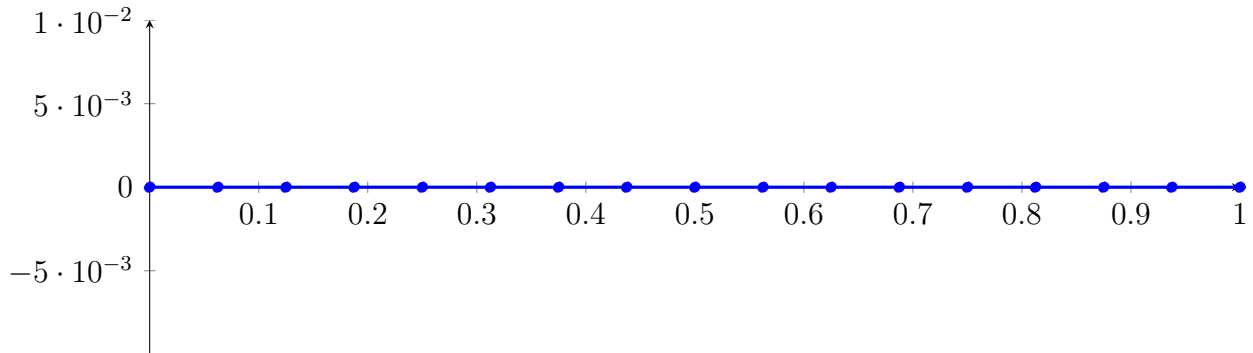
Longest intersection interval: $2.08208 \cdot 10^{-28}$

\implies Selective recursion: interval 1: $[0.333333, 0.333333]$,

252.4 Recursion Branch 1 1 1 1 in Interval 1: [0.333333, 0.333333]

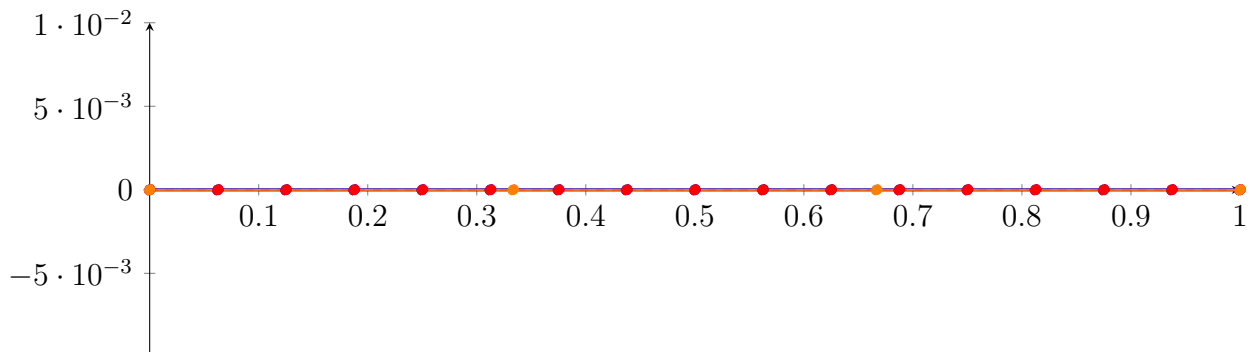
Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
 p &= 3.09811 \cdot 10^{-312} X^{16} + 1.13739 \cdot 10^{-311} X^{15} + 1.09495 \cdot 10^{-310} X^{14} - 3.56495 \cdot 10^{-311} X^{13} \\
 &\quad - 1.48688 \cdot 10^{-309} X^{12} + 1.48302 \cdot 10^{-309} X^{11} + 7.222 \cdot 10^{-310} X^{10} + 1.21378 \cdot 10^{-310} X^9 \\
 &\quad + 7.23716 \cdot 10^{-285} X^8 - 7.35315 \cdot 10^{-248} X^7 - 3.92029 \cdot 10^{-211} X^6 + 9.83929 \cdot 10^{-175} X^5 + 7.39728 \\
 &\quad \cdot 10^{-138} X^4 - 1.09887 \cdot 10^{-101} X^3 - 5.94215 \cdot 10^{-65} X^2 + 1.12463 \cdot 10^{-28} X - 8.88188 \cdot 10^{-08} \\
 &= -8.88188 \cdot 10^{-08} B_{0,16}(X) - 8.88188 \cdot 10^{-08} B_{1,16}(X) - 8.88188 \cdot 10^{-08} B_{2,16}(X) - 8.88188 \\
 &\quad \cdot 10^{-08} B_{3,16}(X) - 8.88188 \cdot 10^{-08} B_{4,16}(X) - 8.88188 \cdot 10^{-08} B_{5,16}(X) - 8.88188 \cdot 10^{-08} B_{6,16}(X) \\
 &\quad - 8.88188 \cdot 10^{-08} B_{7,16}(X) - 8.88188 \cdot 10^{-08} B_{8,16}(X) - 8.88188 \cdot 10^{-08} B_{9,16}(X) - 8.88188 \\
 &\quad \cdot 10^{-08} B_{10,16}(X) - 8.88188 \cdot 10^{-08} B_{11,16}(X) - 8.88188 \cdot 10^{-08} B_{12,16}(X) - 8.88188 \cdot 10^{-08} B_{13,16}(X) \\
 &\quad - 8.88188 \cdot 10^{-08} B_{14,16}(X) - 8.88188 \cdot 10^{-08} B_{15,16}(X) - 8.88188 \cdot 10^{-08} B_{16,16}(X)
 \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
 q_3 &= -1.09887 \cdot 10^{-101} X^3 - 5.94215 \cdot 10^{-65} X^2 + 1.12463 \cdot 10^{-28} X - 8.88188 \cdot 10^{-08} \\
 &= -8.88188 \cdot 10^{-08} B_{0,3} - 8.88188 \cdot 10^{-08} B_{1,3} - 8.88188 \cdot 10^{-08} B_{2,3} - 8.88188 \cdot 10^{-08} B_{3,3} \\
 \tilde{q}_3 &= -6.98397 \cdot 10^{-303} X^{16} + 5.61515 \cdot 10^{-302} X^{15} - 2.01778 \cdot 10^{-301} X^{14} + 4.27019 \cdot 10^{-301} X^{13} \\
 &\quad - 5.92096 \cdot 10^{-301} X^{12} + 5.69601 \cdot 10^{-301} X^{11} - 3.9714 \cdot 10^{-301} X^{10} + 2.10656 \cdot 10^{-301} X^9 \\
 &\quad - 8.95545 \cdot 10^{-302} X^8 + 3.10786 \cdot 10^{-302} X^7 - 8.35303 \cdot 10^{-303} X^6 + 1.57296 \cdot 10^{-303} X^5 - 1.80277 \\
 &\quad \cdot 10^{-304} X^4 - 1.09887 \cdot 10^{-101} X^3 - 5.94215 \cdot 10^{-65} X^2 + 1.12463 \cdot 10^{-28} X - 8.88188 \cdot 10^{-08} \\
 &= -8.88188 \cdot 10^{-08} B_{0,16} - 8.88188 \cdot 10^{-08} B_{1,16} - 8.88188 \cdot 10^{-08} B_{2,16} - 8.88188 \cdot 10^{-08} B_{3,16} - 8.88188 \\
 &\quad \cdot 10^{-08} B_{4,16} - 8.88188 \cdot 10^{-08} B_{5,16} - 8.88188 \cdot 10^{-08} B_{6,16} - 8.88188 \cdot 10^{-08} B_{7,16} - 8.88188 \cdot 10^{-08} B_{8,16} \\
 &\quad - 8.88188 \cdot 10^{-08} B_{9,16} - 8.88188 \cdot 10^{-08} B_{10,16} - 8.88188 \cdot 10^{-08} B_{11,16} - 8.88188 \cdot 10^{-08} B_{12,16} \\
 &\quad - 8.88188 \cdot 10^{-08} B_{13,16} - 8.88188 \cdot 10^{-08} B_{14,16} - 8.88188 \cdot 10^{-08} B_{15,16} - 8.88188 \cdot 10^{-08} B_{16,16}
 \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.05675 \cdot 10^{-139}$.

Bounding polynomials M and m :

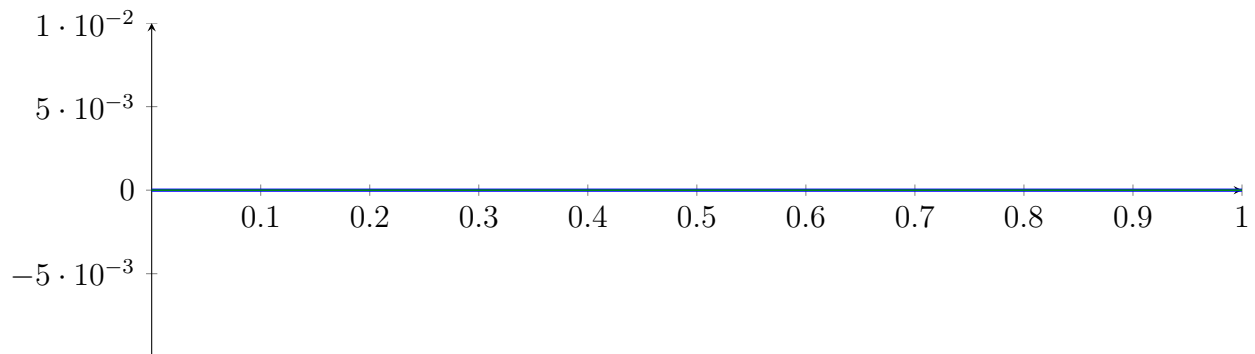
$$M = -1.09887 \cdot 10^{-101} X^3 - 5.94215 \cdot 10^{-65} X^2 + 1.12463 \cdot 10^{-28} X - 8.88188 \cdot 10^{-08}$$

$$m = -1.09887 \cdot 10^{-101} X^3 - 5.94215 \cdot 10^{-65} X^2 + 1.12463 \cdot 10^{-28} X - 8.88188 \cdot 10^{-08}$$

Root of M and m :

$$N(M) = \{-6.89243 \cdot 10^{36}, 1.34848 \cdot 10^{12}, 1.48489 \cdot 10^{36}\} \quad N(m) = \{-6.89243 \cdot 10^{36}, 1.34848 \cdot 10^{12}, 1.48489 \cdot 10^{36}\}$$

Intersection intervals:

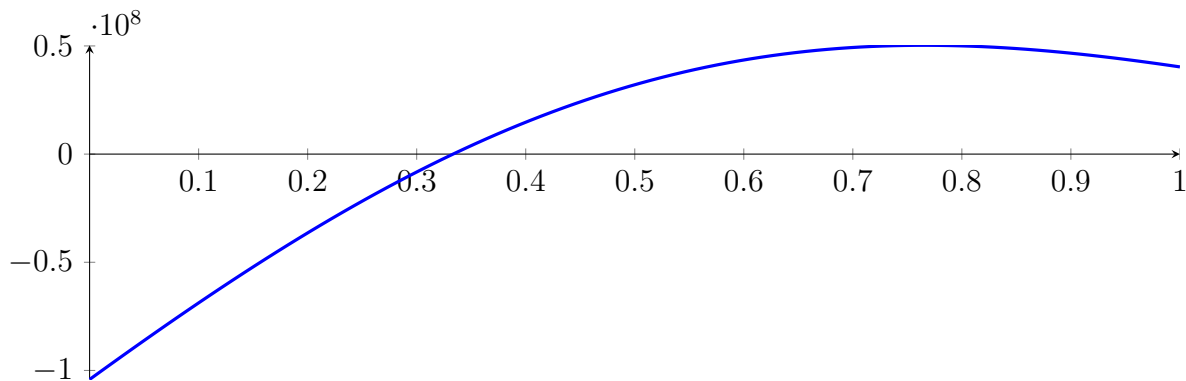


No intersection intervals with the x axis.

252.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$\begin{aligned} p = & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ & + 1.12172 \cdot 10^6 X^9 - 2.52262 \cdot 10^6 X^8 - 1.43506 \cdot 10^7 X^7 + 138542X^6 + 7.92823 \cdot 10^7 X^5 \\ & + 3.97396 \cdot 10^7 X^4 - 2.40625 \cdot 10^8 X^3 - 8.33333 \cdot 10^7 X^2 + 3.64583 \cdot 10^8 X - 1.04167 \cdot 10^8 \end{aligned}$$



Result: Root Intervals

with precision $\varepsilon = 1 \cdot 10^{-128}$.