Demo 4

This demo entry is used to test out further polynomials.

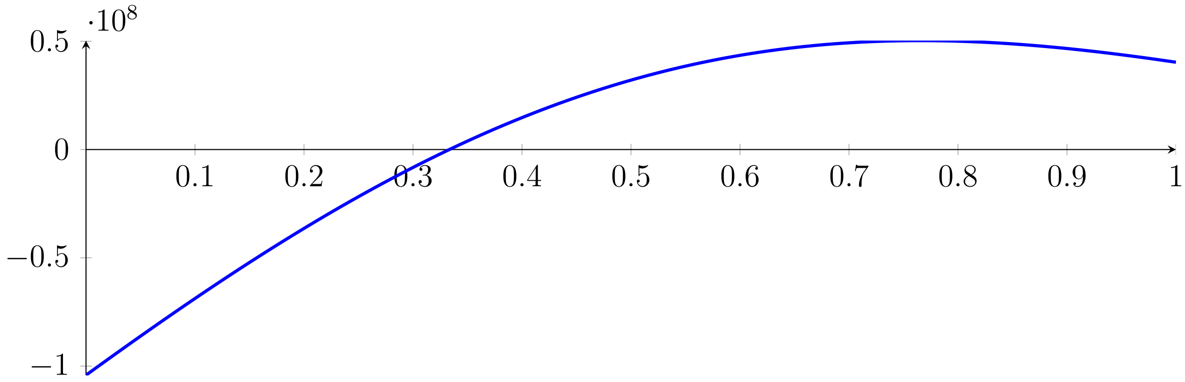
Contents

1 BezClip Applied to the Example Polynomial

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^6X^9 - 2.52262{\cdot}10^6X^8 - 1.43506{\cdot}10^7X^7 + 138542X^6 + 7.92823{\cdot}10^7X^5 + 3.97396{\cdot}10^7X^4 - 2.40625{\cdot}10^8X^3 - 8.33333{\cdot}10^7X^2 + 3.64583{\cdot}10^8X - 1.04167{\cdot}10^8$$

Called BezClip with input polynomial on interval $[0, 1]$:

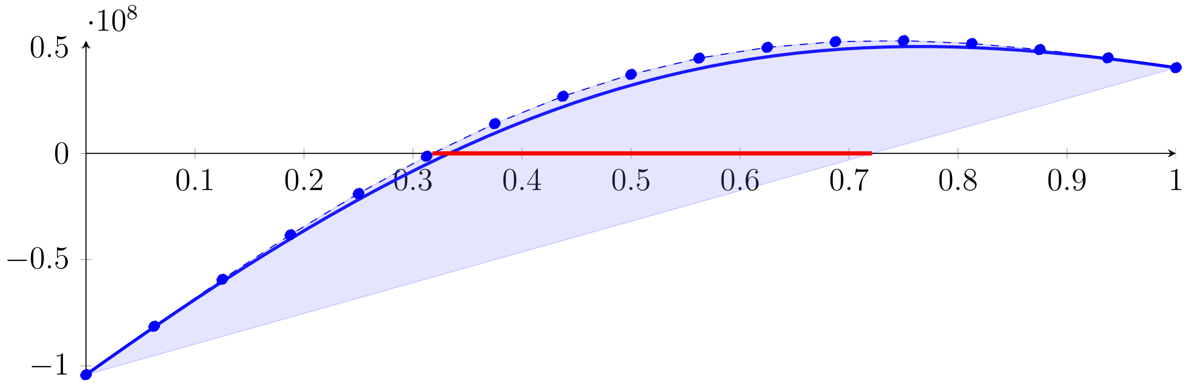
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^6X^9 - 2.52262{\cdot}10^6X^8 - 1.43506{\cdot}10^7X^7 + 138542X^6 + 7.92823{\cdot}10^7X^5 + 3.97396{\cdot}10^7X^4 - 2.40625{\cdot}10^8X^3 - 8.33333{\cdot}10^7X^2 + 3.64583{\cdot}10^8X - 1.04167{\cdot}10^8$$



1.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172{\cdot}10^6X^9 - 2.52262{\cdot}10^6X^8 - 1.43506{\cdot}10^7X^7 + 138542X^6 + 7.92823 \\ &\quad {\cdot}10^7X^5 + 3.97396{\cdot}10^7X^4 - 2.40625{\cdot}10^8X^3 - 8.33333{\cdot}10^7X^2 + 3.64583{\cdot}10^8X - 1.04167{\cdot}10^8 \\ &= -1.04167{\cdot}10^8B_{0,16}(X) - 8.13802{\cdot}10^7B_{1,16}(X) - 5.92882{\cdot}10^7B_{2,16}(X) - 3.83203 \\ &\quad {\cdot}10^7B_{3,16}(X) - 1.88844{\cdot}10^7B_{4,16}(X) - 1.34837{\cdot}10^6B_{5,16}(X) + 1.39781{\cdot}10^7B_{6,16}(X) \\ &\quad + 2.68604{\cdot}10^7B_{7,16}(X) + 3.71532{\cdot}10^7B_{8,16}(X) + 4.48105{\cdot}10^7B_{9,16}(X) + 4.98901 \\ &\quad {\cdot}10^7B_{10,16}(X) + 5.25504{\cdot}10^7B_{11,16}(X) + 5.30407{\cdot}10^7B_{12,16}(X) + 5.16832{\cdot}10^7B_{13,16}(X) \\ &\quad + 4.88488{\cdot}10^7B_{14,16}(X) + 4.49297{\cdot}10^7B_{15,16}(X) + 4.03108{\cdot}10^7B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.317999, 0.720989\}$$

Intersection intervals with the x axis:

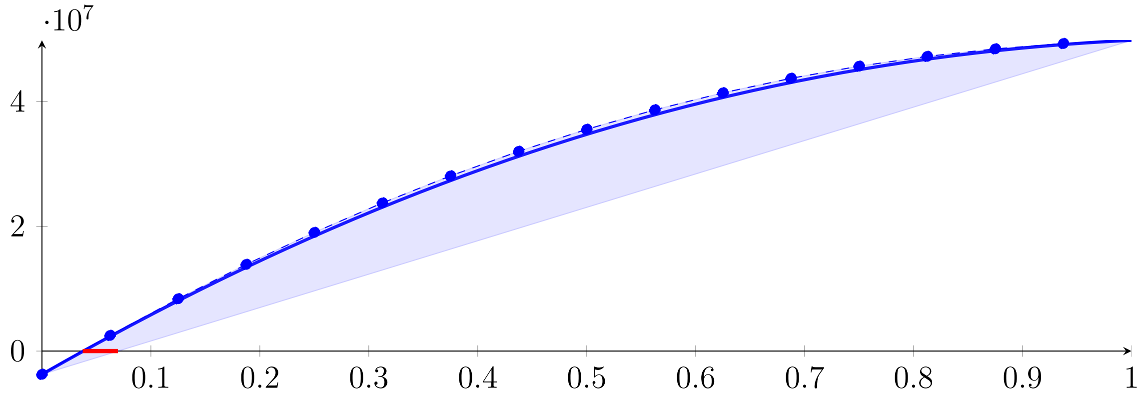
$$[0.317999, 0.720989]$$

Longest intersection interval: 0.402991
$\Longrightarrow$ Selective recursion: interval 1: $[0.317999, 0.720989]$,

1.2 Recursion Branch 1 1 in Interval 1: $[0.317999, 0.720989]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = {} & 1.59825 \cdot 10^{-06} X^{16} - 5.93153 \cdot 10^{-05} X^{15} - 0.00248867 X^{14} - 0.0650056 X^{13} - 0.909142 X^{12} \\ & - 5.03931 X^{11} + 36.4549 X^{10} + 692.921 X^{9} + 1886.97 X^{8} - 25792 X^{7} - 149671 X^{6} + 492605 X^{5} \\ & + 3.91945 \cdot 10^{6} X^{4} - 7.54561 \cdot 10^{6} X^{3} - 4.29225 \cdot 10^{7} X^{2} + 9.97982 \cdot 10^{7} X - 3.73501 \cdot 10^{6} \\ = {} & -3.73501 \cdot 10^{6} B_{0,16}(X) + 2.50237 \cdot 10^{6} B_{1,16}(X) + 8.38207 \cdot 10^{6} B_{2,16}(X) + 1.38906 \\ & \cdot 10^{7} B_{3,16}(X) + 1.90167 \cdot 10^{7} B_{4,16}(X) + 2.37512 \cdot 10^{7} B_{5,16}(X) + 2.80875 \cdot 10^{7} B_{6,16}(X) \\ & + 3.20213 \cdot 10^{7} B_{7,16}(X) + 3.55508 \cdot 10^{7} B_{8,16}(X) + 3.86768 \cdot 10^{7} B_{9,16}(X) + 4.14025 \\ & \cdot 10^{7} B_{10,16}(X) + 4.37336 \cdot 10^{7} B_{11,16}(X) + 4.56783 \cdot 10^{7} B_{12,16}(X) + 4.7247 \cdot 10^{7} B_{13,16}(X) \\ & + 4.84524 \cdot 10^{7} B_{14,16}(X) + 4.93093 \cdot 10^{7} B_{15,16}(X) + 4.98343 \cdot 10^{7} B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0374257, 0.069723\}$$

Intersection intervals with the x axis:

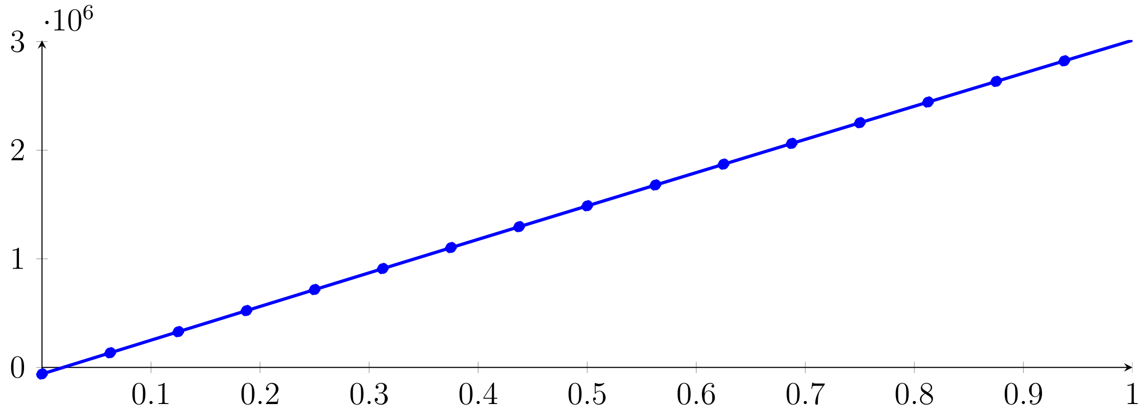
$$[0.0374257, 0.069723]$$

Longest intersection interval: 0.0322973
$\Longrightarrow$ Selective recursion: interval 1: $[0.333081, 0.346096]$,

1.3 Recursion Branch 1 1 1 in Interval 1: $[0.333081, 0.346096]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned} p = {} & 9.01396 \cdot 10^{-08} X^{16} - 2.65848 \cdot 10^{-07} X^{15} + 2.13948 \cdot 10^{-06} X^{14} - 1.33627 \cdot 10^{-06} X^{13} + 2.46973 \cdot 10^{-06} X^{12} \\ & - 2.45524 \cdot 10^{-06} X^{11} + 5.50112 \cdot 10^{-07} X^{10} - 1.64198 \cdot 10^{-07} X^{9} - 7.35598 \cdot 10^{-07} X^{8} - 1.00892 \cdot 10^{-06} X^{7} \\ & - 0.000177509 X^{6} + 0.0161038 X^{5} + 4.36155 X^{4} - 234.216 X^{3} - 45622.2 X^{2} + 3.11845 \cdot 10^{6} X - 60508.5 \\ = {} & -60508.5 B_{0,16}(X) + 134395 B_{1,16}(X) + 328918 B_{2,16}(X) + 523060 B_{3,16}(X) + 716822 B_{4,16}(X) \\ & + 910202 B_{5,16}(X) + 1.1032 \cdot 10^{6} B_{6,16}(X) + 1.29582 \cdot 10^{6} B_{7,16}(X) + 1.48805 \cdot 10^{6} B_{8,16}(X) \\ & + 1.6799 \cdot 10^{6} B_{9,16}(X) + 1.87137 \cdot 10^{6} B_{10,16}(X) + 2.06245 \cdot 10^{6} B_{11,16}(X) + 2.25315 \cdot 10^{6} B_{12,16}(X) \\ & + 2.44346 \cdot 10^{6} B_{13,16}(X) + 2.63339 \cdot 10^{6} B_{14,16}(X) + 2.82293 \cdot 10^{6} B_{15,16}(X) + 3.01209 \cdot 10^{6} B_{16,16}(X) \end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0194034, 0.0196929\}$$

Intersection intervals with the x axis:

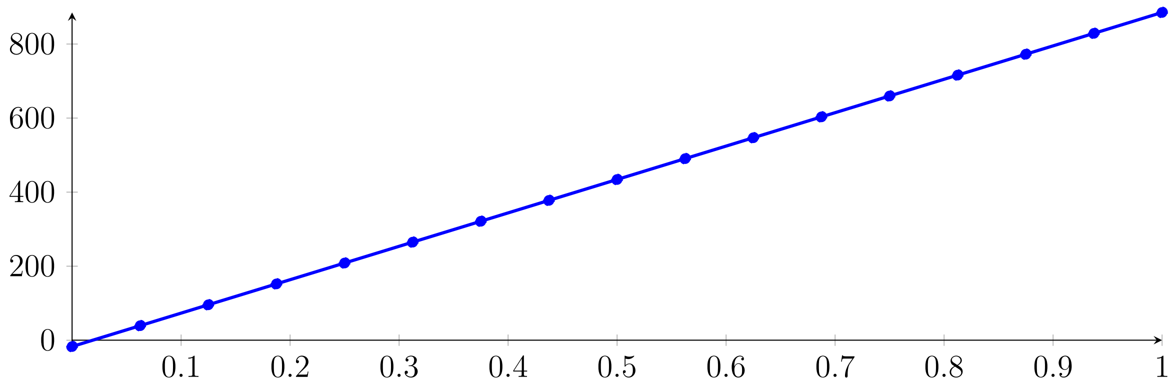
$$[0.0194034, 0.0196929]$$

Longest intersection interval: 0.000289554
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333337]$,

1.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333337]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$\begin{aligned}
p &= 2.55372 \cdot 10^{-11} X^{16} - 7.21263 \cdot 10^{-11} X^{15} + 6.24141 \cdot 10^{-10} X^{14} - 4.11162 \cdot 10^{-10} X^{13} \\
&\quad + 6.82359 \cdot 10^{-10} X^{12} - 7.09475 \cdot 10^{-10} X^{11} + 9.71305 \cdot 10^{-11} X^{10} - 3.46101 \cdot 10^{-11} X^{9} \\
&\quad - 2.13971 \cdot 10^{-10} X^{8} - 1.46061 \cdot 10^{-11} X^{7} - 1.63366 \cdot 10^{-11} X^{6} + 1.87916 \cdot 10^{-12} X^{5} \\
&\quad + 2.52576 \cdot 10^{-14} X^{4} - 5.67777 \cdot 10^{-09} X^{3} - 0.00382618 X^{2} + 902.448 X - 17.178 \\
&= -17.178 B_{0,16}(X) + 39.225 B_{1,16}(X) + 95.6279 B_{2,16}(X) + 152.031 B_{3,16}(X) + 208.434 B_{4,16}(X) \\
&\quad + 264.837 B_{5,16}(X) + 321.24 B_{6,16}(X) + 377.642 B_{7,16}(X) + 434.045 B_{8,16}(X) \\
&\quad + 490.448 B_{9,16}(X) + 546.851 B_{10,16}(X) + 603.253 B_{11,16}(X) + 659.656 B_{12,16}(X) \\
&\quad + 716.059 B_{13,16}(X) + 772.461 B_{14,16}(X) + 828.864 B_{15,16}(X) + 885.266 B_{16,16}(X)
\end{aligned}$$



Intersection of the convex hull with the x axis:

$$\{0.0190349, 0.019035\}$$

Intersection intervals with the x axis:

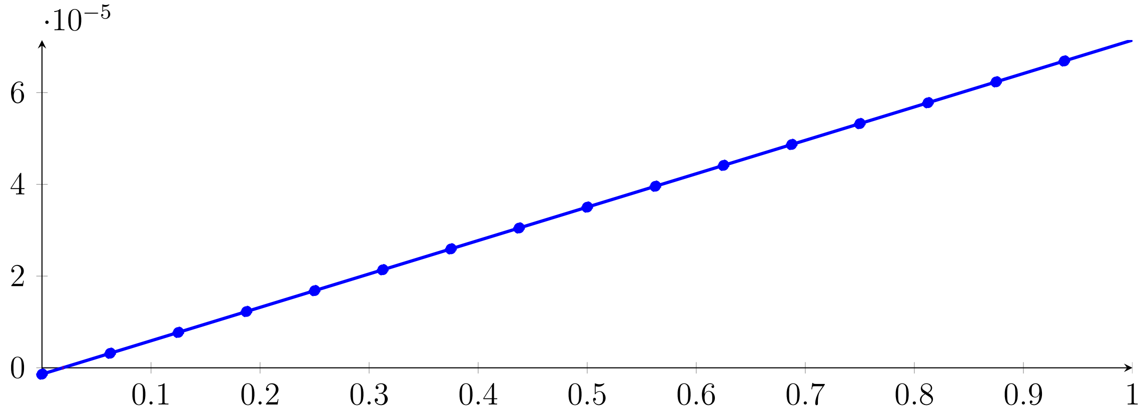
$$[0.0190349, 0.019035]$$

Longest intersection interval: $8.07045 \cdot 10^{-08}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

1.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

$$
\begin{aligned}
p = {} & 2.14261 \cdot 10^{-18} X^{16} - 6.28573 \cdot 10^{-18} X^{15} + 5.28612 \cdot 10^{-17} X^{14} - 3.67279 \cdot 10^{-17} X^{13} \\
& + 6.10136 \cdot 10^{-17} X^{12} - 6.60335 \cdot 10^{-17} X^{11} + 1.66661 \cdot 10^{-17} X^{10} - 8.36524 \cdot 10^{-18} X^{9} \\
& - 1.56919 \cdot 10^{-17} X^{8} - 1.85474 \cdot 10^{-18} X^{7} - 1.4308 \cdot 10^{-18} X^{6} + 1.1562 \cdot 10^{-19} X^{5} - 1.20437 \\
& \cdot 10^{-20} X^{4} - 4.63221 \cdot 10^{-22} X^{3} - 2.49207 \cdot 10^{-17} X^{2} + 7.28316 \cdot 10^{-05} X - 1.38634 \cdot 10^{-06} \\
= {} & -1.38634 \cdot 10^{-06} B_{0,16}(X) + 3.16564 \cdot 10^{-06} B_{1,16}(X) + 7.71761 \cdot 10^{-06} B_{2,16}(X) + 1.22696 \\
& \cdot 10^{-05} B_{3,16}(X) + 1.68216 \cdot 10^{-05} B_{4,16}(X) + 2.13735 \cdot 10^{-05} B_{5,16}(X) + 2.59255 \cdot 10^{-05} B_{6,16}(X) \\
& + 3.04775 \cdot 10^{-05} B_{7,16}(X) + 3.50295 \cdot 10^{-05} B_{8,16}(X) + 3.95814 \cdot 10^{-05} B_{9,16}(X) + 4.41334 \\
& \cdot 10^{-05} B_{10,16}(X) + 4.86854 \cdot 10^{-05} B_{11,16}(X) + 5.32374 \cdot 10^{-05} B_{12,16}(X) + 5.77893 \cdot 10^{-05} B_{13,16}(X) \\
& + 6.23413 \cdot 10^{-05} B_{14,16}(X) + 6.68933 \cdot 10^{-05} B_{15,16}(X) + 7.14453 \cdot 10^{-05} B_{16,16}(X)
\end{aligned}
$$



Intersection of the convex hull with the x axis:

$$\{0.0190348, 0.0190348\}$$

Intersection intervals with the x axis:

$$[0.0190348, 0.0190348]$$

Longest intersection interval: $6.51313 \cdot 10^{-15}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

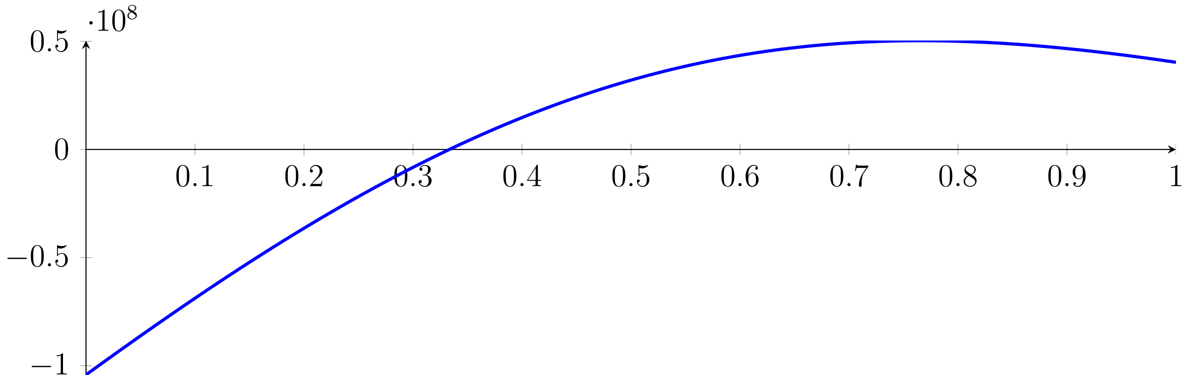
1.6 Recursion Branch 1 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 6!

1.7 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

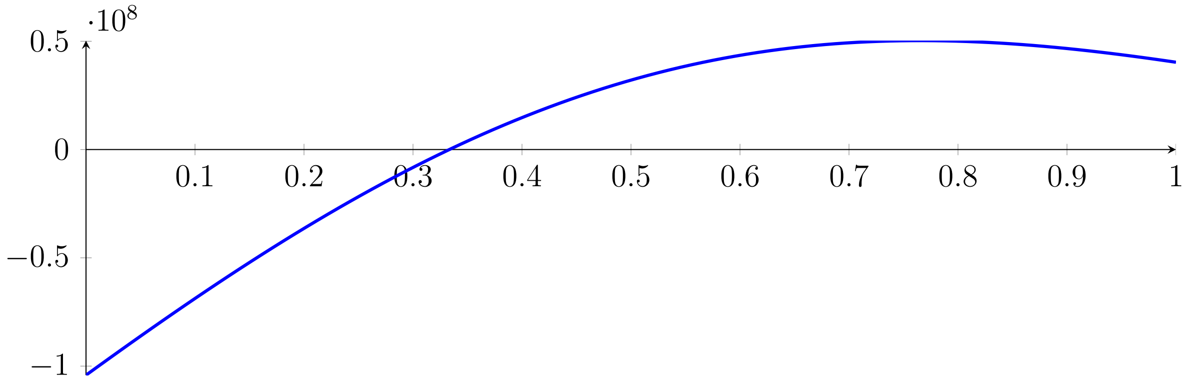
with precision $\varepsilon = 1\cdot10^{-32}$.

2 `QuadClip` Applied to the Example Polynomial

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called `QuadClip` with input polynomial on interval $[0, 1]$:

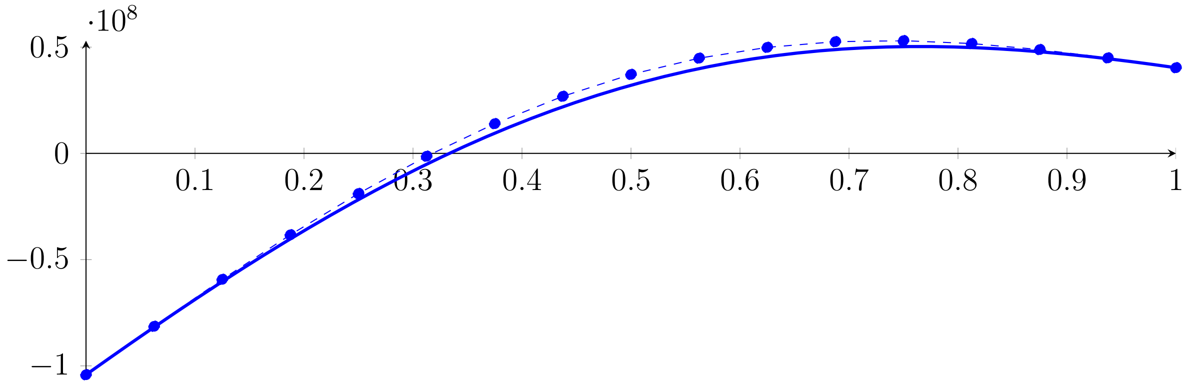
$$\begin{aligned} p = &-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\ &+ 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} \\ &+ 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \end{aligned}$$



2.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

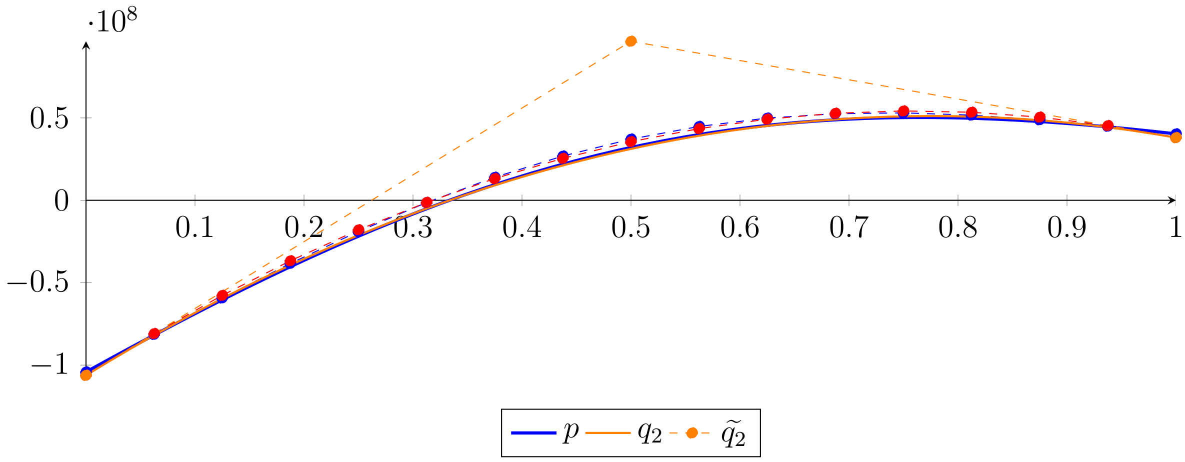
$$\begin{aligned} p = &-1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &+ 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \\ = &-1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203 \\ &\cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X) \\ &+ 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901 \\ &\cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X) \\ &+ 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.06197{\cdot}10^8 \\ &= -1.06197{\cdot}10^8 B_{0,2} + 9.65115{\cdot}10^7 B_{1,2} + 3.80384{\cdot}10^7 B_{2,2} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_2 &= 6049.18X^{16} - 48305.2X^{15} + 174971X^{14} - 380294X^{13} + 552846X^{12} - 567203X^{11} \\ &\quad + 422303X^{10} - 231038X^9 + 93003.6X^8 - 27320.1X^7 + 5752.57X^6 - 843.63X^5 \\ &\quad + 82.5145X^4 - 5.01388X^3 - 2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.06197{\cdot}10^8 \\ &= -1.06197{\cdot}10^8 B_{0,16} - 8.08583{\cdot}10^7 B_{1,16} - 5.76963{\cdot}10^7 B_{2,16} - 3.67107{\cdot}10^7 B_{3,16} - 1.79017 \\ &\quad {\cdot}10^7 B_{4,16} - 1.26924{\cdot}10^6 B_{5,16} + 1.31867{\cdot}10^7 B_{6,16} + 2.54662{\cdot}10^7 B_{7,16} + 3.55692{\cdot}10^7 B_{8,16} \\ &\quad + 4.34956{\cdot}10^7 B_{9,16} + 4.92456{\cdot}10^7 B_{10,16} + 5.2819{\cdot}10^7 B_{11,16} + 5.42159{\cdot}10^7 B_{12,16} \\ &\quad + 5.34363{\cdot}10^7 B_{13,16} + 5.04802{\cdot}10^7 B_{14,16} + 4.53476{\cdot}10^7 B_{15,16} + 3.80384{\cdot}10^7 B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.27233{\cdot}10^6$.

Bounding polynomials M and m:

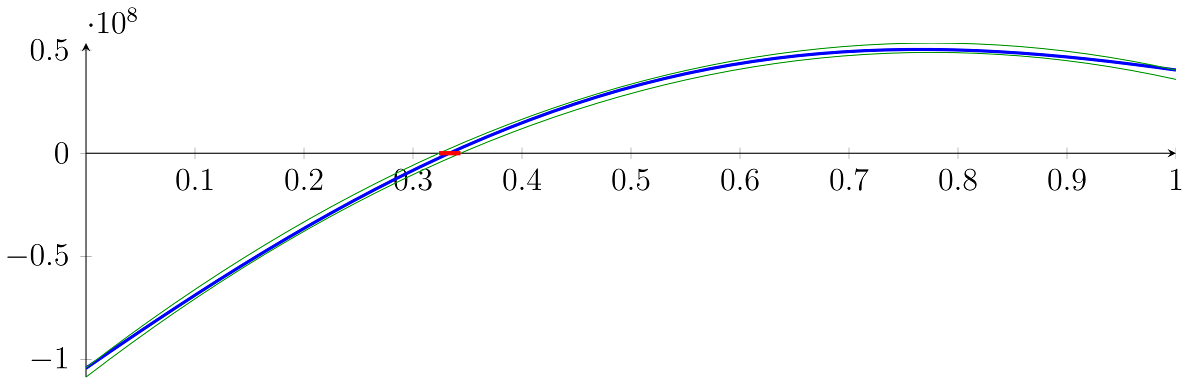
$$M = -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.03924{\cdot}10^8$$

$$m = -2.61181{\cdot}10^8 X^2 + 4.05417{\cdot}10^8 X - 1.08469{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.323946, 1.2283\} \qquad N(m) = \{0.343615, 1.20863\}$$

Intersection intervals:



$$[0.323946, 0.343615]$$

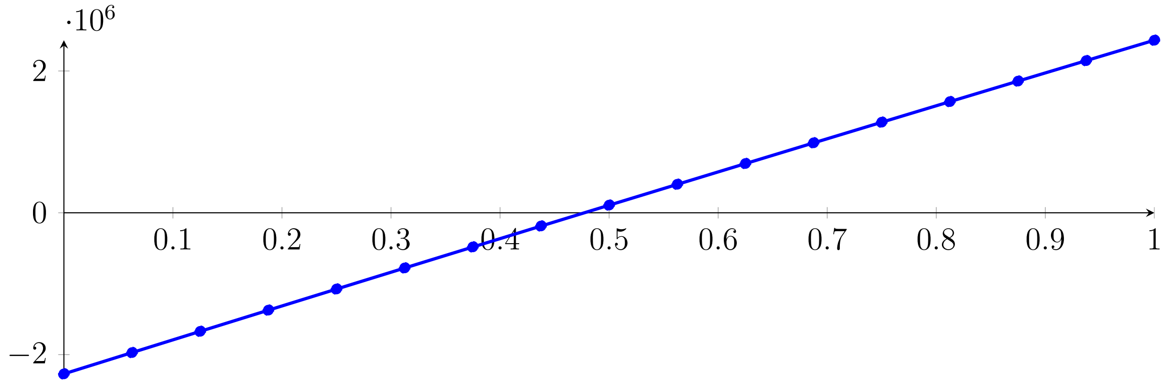
Longest intersection interval: 0.0196686

$\Longrightarrow$ Selective recursion: interval 1: $[0.323946, 0.343615]$,

2.2 Recursion Branch 1 1 in Interval 1: $[0.323946, 0.343615]$

Normalized monomial und Bézier representations and the Bézier polygon:

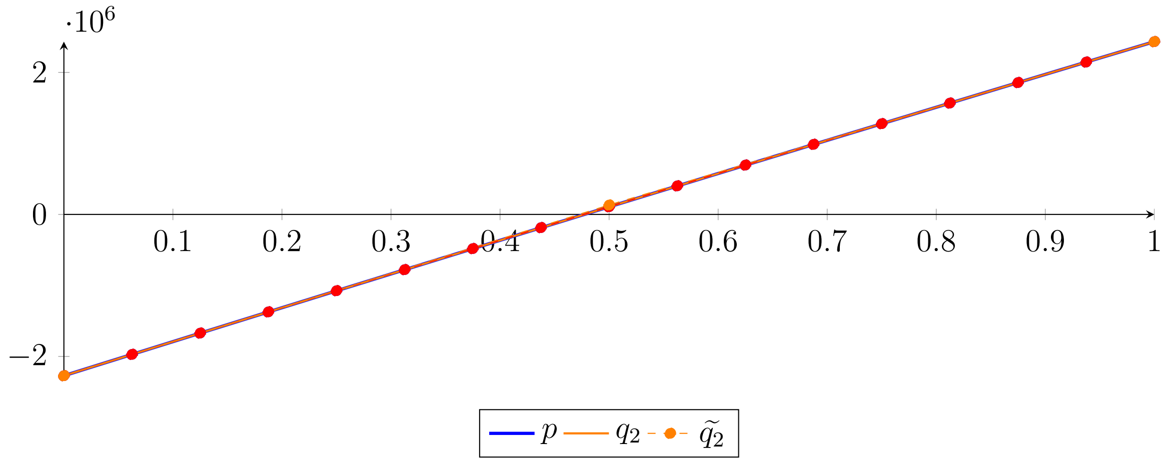
$$p = -1.54841{\cdot}10^{-10}X^{16} - 1.66965{\cdot}10^{-07}X^{15} - 2.92739{\cdot}10^{-07}X^{14} - 1.77943{\cdot}10^{-06}X^{13} - 1.17235{\cdot}10^{-06}X^{12} - 2.42234{\cdot}10^{-06}X^{11} - 6.86445{\cdot}10^{-07}X^{10} - 1.39162{\cdot}10^{-06}X^{9} + 1.07395{\cdot}10^{-06}X^{8} - 1.67072{\cdot}10^{-05}X^{7} - 0.00205879X^{6} + 0.132721X^{5} + 22.4437X^{4} - 850.239X^{3} - 103028X^{2} + 4.80874{\cdot}10^{6}X - 2.2715{\cdot}10^{6}$$
$$= -2.2715{\cdot}10^{6}B_{0,16}(X) - 1.97096{\cdot}10^{6}B_{1,16}(X) - 1.67127{\cdot}10^{6}B_{2,16}(X) - 1.37244{\cdot}10^{6}B_{3,16}(X) - 1.07448{\cdot}10^{6}B_{4,16}(X) - 777374B_{5,16}(X) - 481136B_{6,16}(X) - 185764B_{7,16}(X) + 108741B_{8,16}(X) + 402376B_{9,16}(X) + 695142B_{10,16}(X) + 987035B_{11,16}(X) + 1.27806{\cdot}10^{6}B_{12,16}(X) + 1.5682{\cdot}10^{6}B_{13,16}(X) + 1.85747{\cdot}10^{6}B_{14,16}(X) + 2.14586{\cdot}10^{6}B_{15,16}(X) + 2.43338{\cdot}10^{6}B_{16,16}(X)$$



Degree reduction and raising:

$$q_2 = -104265X^{2} + 4.80923{\cdot}10^{6}X - 2.27154{\cdot}10^{6}$$
$$= -2.27154{\cdot}10^{6}B_{0,2} + 133069B_{1,2} + 2.43342{\cdot}10^{6}B_{2,2}$$

$$\tilde{q}_2 = 22.7036X^{16} - 178.964X^{15} + 638.976X^{14} - 1366.64X^{13} + 1951.1X^{12} - 1960.92X^{11} + 1425.35X^{10} - 757.631X^{9} + 294.177X^{8} - 82.4368X^{7} + 16.2856X^{6} - 2.18949X^{5} + 0.191237X^{4} - 0.0101048X^{3} - 104265X^{2} + 4.80923{\cdot}10^{6}X - 2.27154{\cdot}10^{6}$$
$$= -2.27154{\cdot}10^{6}B_{0,16} - 1.97097{\cdot}10^{6}B_{1,16} - 1.67126{\cdot}10^{6}B_{2,16} - 1.37242{\cdot}10^{6}B_{3,16} - 1.07445{\cdot}10^{6}B_{4,16} - 777350B_{5,16} - 481118B_{6,16} - 185754B_{7,16} + 108740B_{8,16} + 402366B_{9,16} + 695123B_{10,16} + 987011B_{11,16} + 1.27803{\cdot}10^{6}B_{12,16} + 1.56818{\cdot}10^{6}B_{13,16} + 1.85746{\cdot}10^{6}B_{14,16} + 2.14587{\cdot}10^{6}B_{15,16} + 2.43342{\cdot}10^{6}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 40.5742$.

Bounding polynomials M and m:

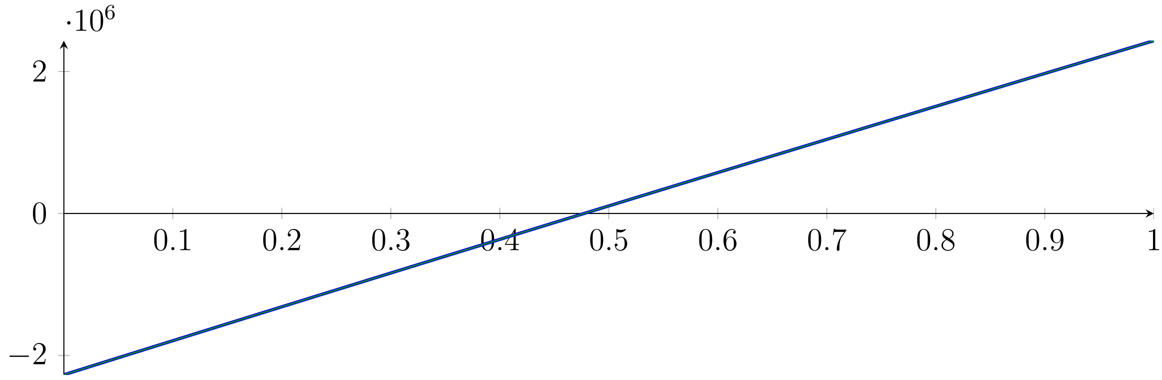
$$M = -104265X^{2} + 4.80923{\cdot}10^{6}X - 2.2715{\cdot}10^{6}$$

$$m = -104265X^2 + 4.80923\cdot 10^6 X - 2.27159\cdot 10^6$$

Root of M and m:

$$N(M) = \{0.47726, 45.6477\} \qquad N(m) = \{0.477278, 45.6477\}$$

Intersection intervals:



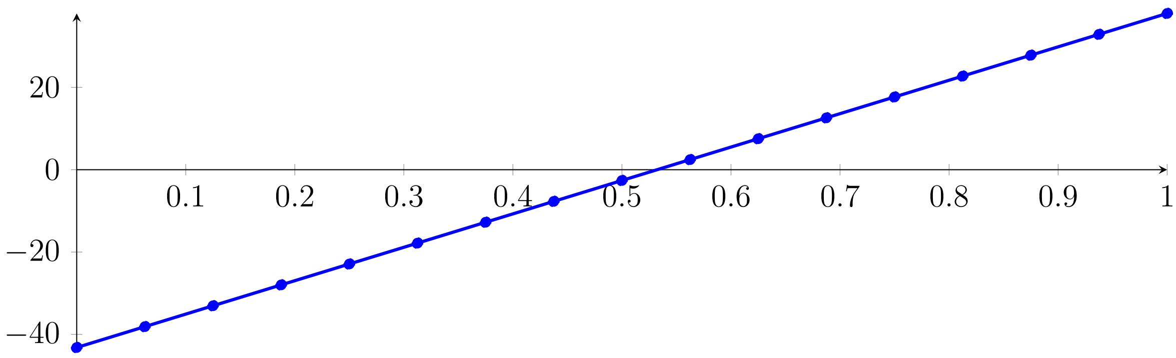
$$[0.47726, 0.477278]$$

Longest intersection interval: $1.72301\cdot 10^{-05}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

2.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

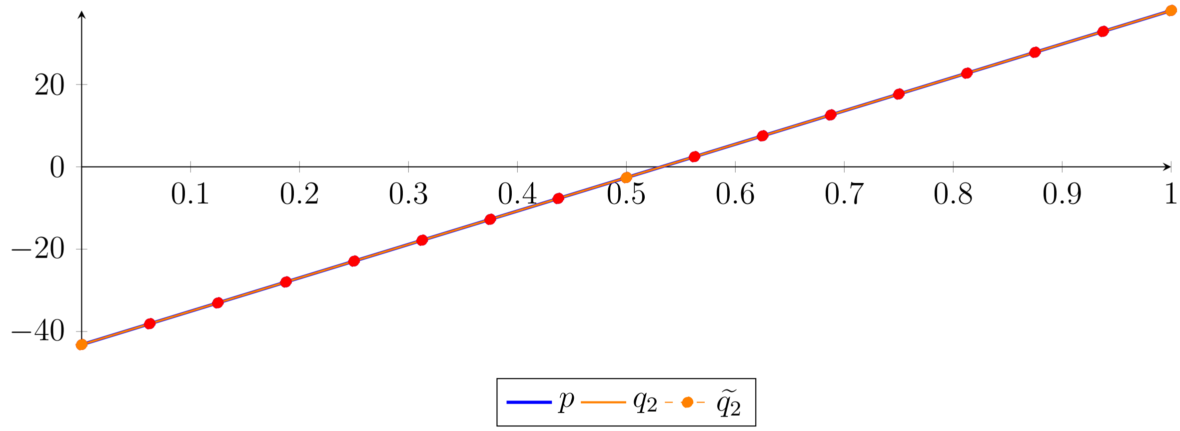
$$\begin{aligned} p = &-2.76723\cdot 10^{-13}X^{16} - 2.40874\cdot 10^{-12}X^{15} - 1.25233\cdot 10^{-11}X^{14} - 3.02935\cdot 10^{-11}X^{13} \\ &- 3.05617\cdot 10^{-11}X^{12} - 3.83107\cdot 10^{-11}X^{11} - 1.26692\cdot 10^{-11}X^{10} - 2.6672\cdot 10^{-11}X^{9} \\ &+ 2.0004\cdot 10^{-11}X^{8} + 4.12781\cdot 10^{-12}X^{7} + 2.44493\cdot 10^{-12}X^{6} - 1.21236\cdot 10^{-13}X^{5} \\ &+ 1.26288\cdot 10^{-14}X^{4} - 4.1267\cdot 10^{-12}X^{3} - 3.09388\cdot 10^{-05}X^{2} + 81.1506X - 43.1911 \\ = &-43.1911B_{0,16}(X) - 38.1192B_{1,16}(X) - 33.0473B_{2,16}(X) - 27.9754B_{3,16}(X) - 22.9035B_{4,16}(X) \\ &- 17.8316B_{5,16}(X) - 12.7597B_{6,16}(X) - 7.68778B_{7,16}(X) - 2.61587B_{8,16}(X) \\ &+ 2.45604B_{9,16}(X) + 7.52795B_{10,16}(X) + 12.5999B_{11,16}(X) + 17.6718B_{12,16}(X) \\ &+ 22.7437B_{13,16}(X) + 27.8156B_{14,16}(X) + 32.8875B_{15,16}(X) + 37.9594B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_2 &= -3.09389\cdot 10^{-05}X^2 + 81.1506X - 43.1911 \\ &= -43.1911B_{0,2} - 2.61586B_{1,2} + 37.9594B_{2,2} \end{aligned}$$

$$
\begin{aligned}
\widetilde{q}_2 &= 5.96265{\cdot}10^{-05}X^{16} - 0.000436042X^{15} + 0.00141812X^{14} - 0.00269475X^{13} \\
&\quad + 0.00329809X^{12} - 0.00268757X^{11} + 0.00143268X^{10} - 0.000439599X^{9} \\
&\quad + 1.98418{\cdot}10^{-05}X^{8} + 4.87608{\cdot}10^{-05}X^{7} - 2.46333{\cdot}10^{-05}X^{6} + 6.35808{\cdot}10^{-06}X^{5} \\
&\quad - 9.62755{\cdot}10^{-07}X^{4} + 8.21372{\cdot}10^{-08}X^{3} - 3.09429{\cdot}10^{-05}X^{2} + 81.1506X - 43.1911 \\
&= -43.1911B_{0,16} - 38.1192B_{1,16} - 33.0473B_{2,16} - 27.9754B_{3,16} - 22.9035B_{4,16} - 17.8316B_{5,16} \\
&\quad - 12.7597B_{6,16} - 7.68778B_{7,16} - 2.61587B_{8,16} + 2.45604B_{9,16} + 7.52795B_{10,16} + 12.5999B_{11,16} \\
&\quad + 17.6718B_{12,16} + 22.7437B_{13,16} + 27.8156B_{14,16} + 32.8875B_{15,16} + 37.9594B_{16,16}
\end{aligned}
$$



The maximum difference of the Bézier coefficients is $\delta = 1.5947{\cdot}10^{-09}$.

Bounding polynomials M and m:

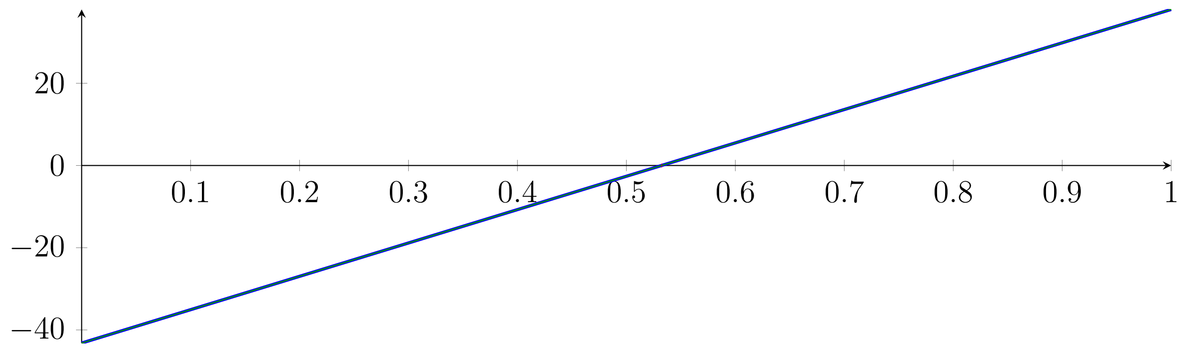
$$M = -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911$$

$$m = -3.09389{\cdot}10^{-05}X^2 + 81.1506X - 43.1911$$

Root of M and m:

$$N(M) = \{0.532235, 2.62293{\cdot}10^{6}\} \qquad N(m) = \{0.532235, 2.62293{\cdot}10^{6}\}$$

Intersection intervals:



$$[0.532235, 0.532235]$$

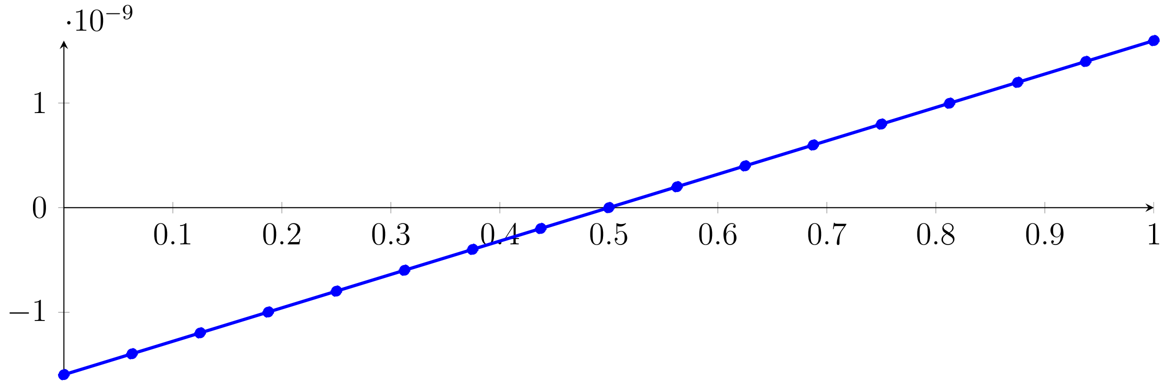
Longest intersection interval: $3.93535{\cdot}10^{-11}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

2.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

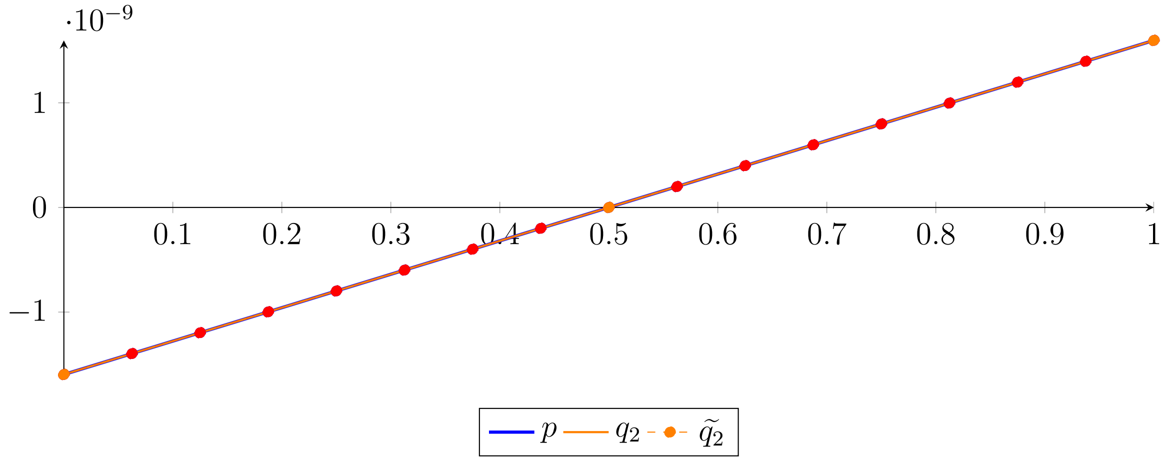
$$\begin{aligned}
p = &-4.89361\cdot 10^{-24}X^{16} - 1.05466\cdot 10^{-22}X^{15} - 3.09805\cdot 10^{-22}X^{14} - 1.16981\cdot 10^{-21}X^{13} \\
&- 9.76202\cdot 10^{-22}X^{12} - 1.69365\cdot 10^{-21}X^{11} - 5.95131\cdot 10^{-22}X^{10} - 1.05349\cdot 10^{-21}X^{9} \\
&+ 7.5893\cdot 10^{-22}X^{8} + 1.47859\cdot 10^{-22}X^{7} + 9.70322\cdot 10^{-23}X^{6} - 7.05688\cdot 10^{-24}X^{5} \\
&+ 1.47018\cdot 10^{-24}X^{4} - 4.84676\cdot 10^{-26}X^{2} + 3.19356\cdot 10^{-09}X - 1.59674\cdot 10^{-09} \\
= &-1.59674\cdot 10^{-09}B_{0,16}(X) - 1.39715\cdot 10^{-09}B_{1,16}(X) - 1.19755\cdot 10^{-09}B_{2,16}(X) - 9.97951 \\
&\cdot 10^{-10}B_{3,16}(X) - 7.98353\cdot 10^{-10}B_{4,16}(X) - 5.98756\cdot 10^{-10}B_{5,16}(X) - 3.99159\cdot 10^{-10}B_{6,16}(X) \\
&- 1.99561\cdot 10^{-10}B_{7,16}(X) + 3.6039\cdot 10^{-14}B_{8,16}(X) + 1.99633\cdot 10^{-10}B_{9,16}(X) + 3.99231 \\
&\cdot 10^{-10}B_{10,16}(X) + 5.98828\cdot 10^{-10}B_{11,16}(X) + 7.98425\cdot 10^{-10}B_{12,16}(X) + 9.98023\cdot 10^{-10}B_{13,16}(X) \\
&+ 1.19762\cdot 10^{-09}B_{14,16}(X) + 1.39722\cdot 10^{-09}B_{15,16}(X) + 1.59681\cdot 10^{-09}B_{16,16}(X)
\end{aligned}$$



Degree reduction and raising:

$$\begin{aligned}
q_2 &= -4.83666\cdot 10^{-26}X^{2} + 3.19356\cdot 10^{-09}X - 1.59674\cdot 10^{-09} \\
&= -1.59674\cdot 10^{-09}B_{0,2} + 3.6039\cdot 10^{-14}B_{1,2} + 1.59681\cdot 10^{-09}B_{2,2}
\end{aligned}$$

$$\begin{aligned}
\widetilde{q}_2 = &\ 9.45798\cdot 10^{-15}X^{16} - 7.39324\cdot 10^{-14}X^{15} + 2.61437\cdot 10^{-13}X^{14} - 5.52989\cdot 10^{-13}X^{13} \\
&+ 7.79462\cdot 10^{-13}X^{12} - 7.71893\cdot 10^{-13}X^{11} + 5.51461\cdot 10^{-13}X^{10} - 2.8712\cdot 10^{-13}X^{9} \\
&+ 1.08634\cdot 10^{-13}X^{8} - 2.94042\cdot 10^{-14}X^{7} + 5.52081\cdot 10^{-15}X^{6} - 6.84058\cdot 10^{-16}X^{5} + 5.20623 \\
&\cdot 10^{-17}X^{4} - 2.16513\cdot 10^{-18}X^{3} + 1.74369\cdot 10^{-20}X^{2} + 3.19356\cdot 10^{-09}X - 1.59674\cdot 10^{-09} \\
= &-1.59674\cdot 10^{-09}B_{0,16} - 1.39715\cdot 10^{-09}B_{1,16} - 1.19755\cdot 10^{-09}B_{2,16} - 9.97951\cdot 10^{-10}B_{3,16} - 7.98353 \\
&\cdot 10^{-10}B_{4,16} - 5.98756\cdot 10^{-10}B_{5,16} - 3.99159\cdot 10^{-10}B_{6,16} - 1.99561\cdot 10^{-10}B_{7,16} + 3.60393\cdot 10^{-14}B_{8,16} \\
&+ 1.99633\cdot 10^{-10}B_{9,16} + 3.99231\cdot 10^{-10}B_{10,16} + 5.98828\cdot 10^{-10}B_{11,16} + 7.98425\cdot 10^{-10}B_{12,16} \\
&+ 9.98023\cdot 10^{-10}B_{13,16} + 1.19762\cdot 10^{-09}B_{14,16} + 1.39722\cdot 10^{-09}B_{15,16} + 1.59681\cdot 10^{-09}B_{16,16}
\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 2.02367 \cdot 10^{-19}$.
Bounding polynomials M and m:

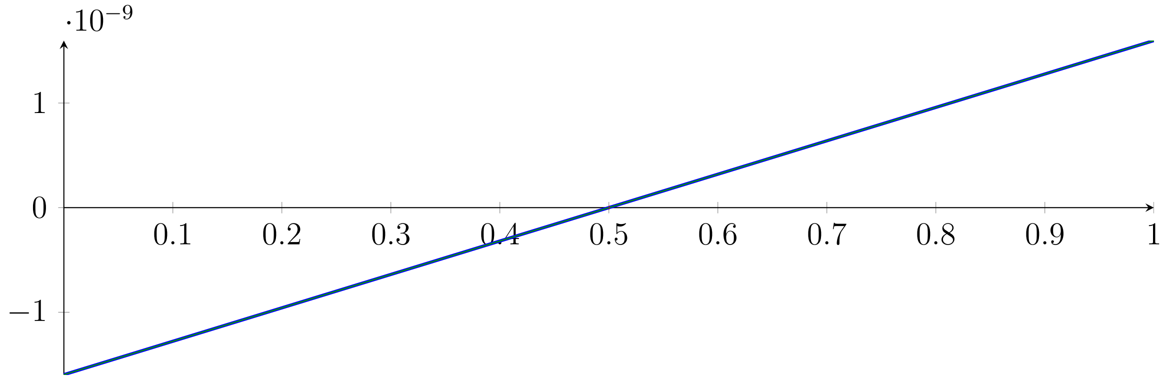
$$M = -4.82657 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$

$$m = -4.84676 \cdot 10^{-26} X^2 + 3.19356 \cdot 10^{-09} X - 1.59674 \cdot 10^{-09}$$

Root of M and m:

$$N(M) = \{0.5, 6.61662 \cdot 10^{16}\} \qquad N(m) = \{0.5, 6.58905 \cdot 10^{16}\}$$

Intersection intervals:



$$[0.5, 0.5]$$

Longest intersection interval: 0
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

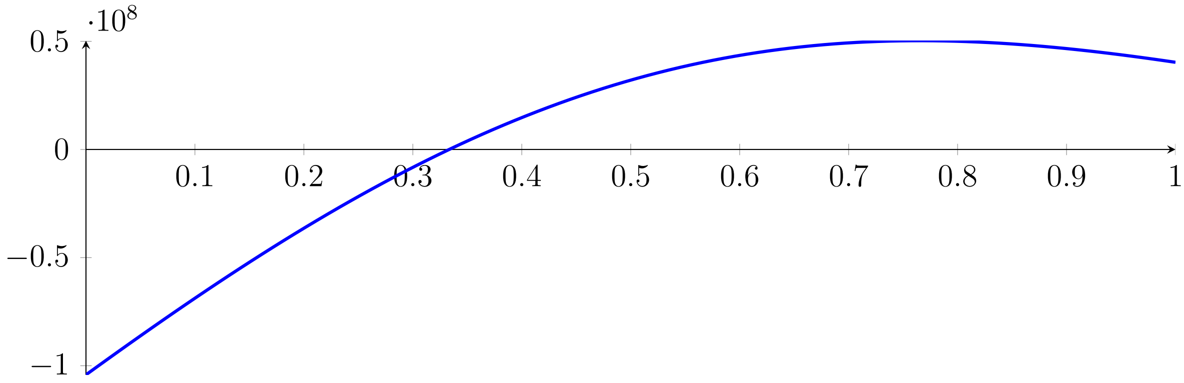
2.5 Recursion Branch 1 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Found root in interval $[0.333333, 0.333333]$ at recursion depth 5!

2.6 Result: 1 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$
\begin{aligned}
p = {} & -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} \\
& + 1.12172\cdot 10^{6}X^{9} - 2.52262\cdot 10^{6}X^{8} - 1.43506\cdot 10^{7}X^{7} + 138542X^{6} + 7.92823\cdot 10^{7}X^{5} \\
& + 3.97396\cdot 10^{7}X^{4} - 2.40625\cdot 10^{8}X^{3} - 8.33333\cdot 10^{7}X^{2} + 3.64583\cdot 10^{8}X - 1.04167\cdot 10^{8}
\end{aligned}
$$



Result: Root Intervals

$$[0.333333, 0.333333]$$

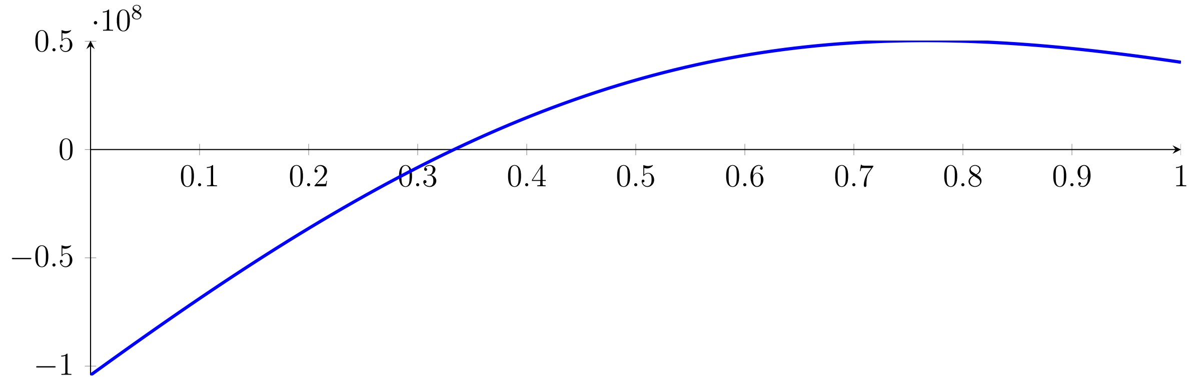
with precision $\varepsilon = 1\cdot 10^{-32}$.

3 CubeClip Applied to the Example Polynomial

$$-1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$

Called CubeClip with input polynomial on interval $[0, 1]$:

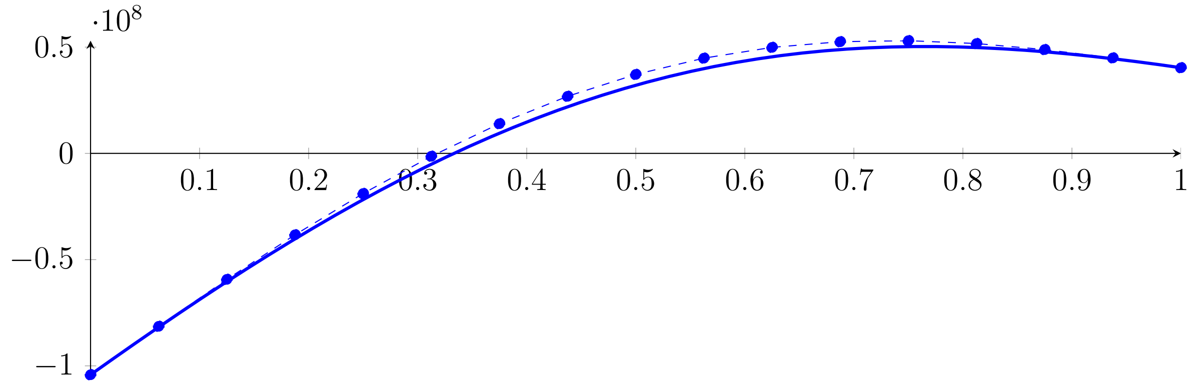
$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823\cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8}$$



3.1 Recursion Branch 1 for Input Interval $[0, 1]$

Normalized monomial und Bézier representations and the Bézier polygon:

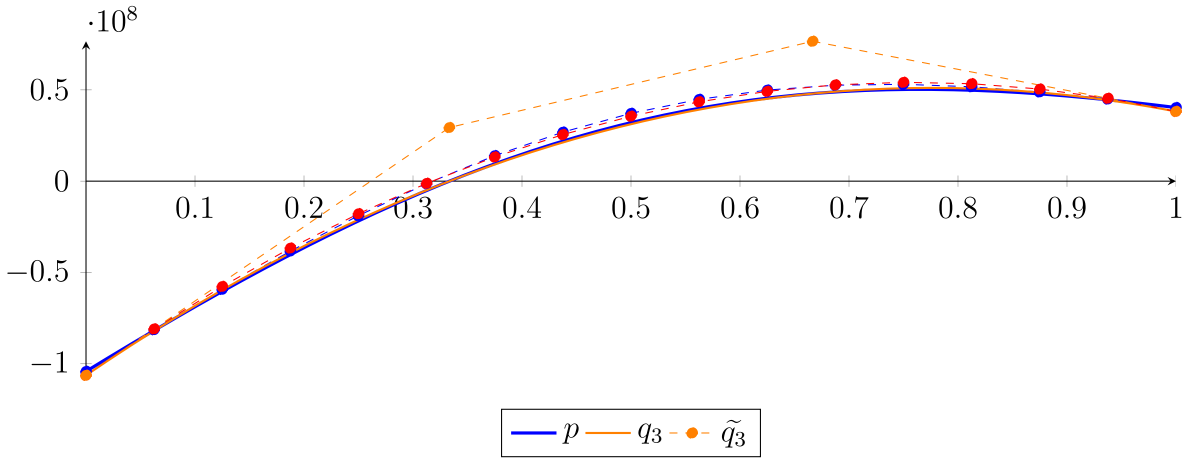
$$\begin{aligned} p &= -1.00001X^{16} - 39.6666X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} \\ &\quad + 451673X^{10} + 1.12172\cdot10^{6}X^{9} - 2.52262\cdot10^{6}X^{8} - 1.43506\cdot10^{7}X^{7} + 138542X^{6} + 7.92823 \\ &\quad \cdot10^{7}X^{5} + 3.97396\cdot10^{7}X^{4} - 2.40625\cdot10^{8}X^{3} - 8.33333\cdot10^{7}X^{2} + 3.64583\cdot10^{8}X - 1.04167\cdot10^{8} \\ &= -1.04167\cdot10^{8}B_{0,16}(X) - 8.13802\cdot10^{7}B_{1,16}(X) - 5.92882\cdot10^{7}B_{2,16}(X) - 3.83203 \\ &\quad \cdot10^{7}B_{3,16}(X) - 1.88844\cdot10^{7}B_{4,16}(X) - 1.34837\cdot10^{6}B_{5,16}(X) + 1.39781\cdot10^{7}B_{6,16}(X) \\ &\quad + 2.68604\cdot10^{7}B_{7,16}(X) + 3.71532\cdot10^{7}B_{8,16}(X) + 4.48105\cdot10^{7}B_{9,16}(X) + 4.98901 \\ &\quad \cdot10^{7}B_{10,16}(X) + 5.25504\cdot10^{7}B_{11,16}(X) + 5.30407\cdot10^{7}B_{12,16}(X) + 5.16832\cdot10^{7}B_{13,16}(X) \\ &\quad + 4.88488\cdot10^{7}B_{14,16}(X) + 4.49297\cdot10^{7}B_{15,16}(X) + 4.03108\cdot10^{7}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$q_3 = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.06335{\cdot}10^8$$
$$= -1.06335{\cdot}10^8 B_{0,3} + 2.93558{\cdot}10^7 B_{1,3} + 7.66068{\cdot}10^7 B_{2,3} + 3.81764{\cdot}10^7 B_{3,3}$$

$$\widetilde{q}_3 = 2461.93X^{16} - 19614.9X^{15} + 70879.5X^{14} - 153661X^{13} + 222746X^{12} - 227755X^{11} + 168826X^{10} - 91798.7X^9 + 36630.3X^8 - 10627.3X^7 + 2200.54X^6 - 316.059X^5 + 30.1958X^4 + 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.06335{\cdot}10^8$$
$$= -1.06335{\cdot}10^8 B_{0,16} - 8.08928{\cdot}10^7 B_{1,16} - 5.76618{\cdot}10^7 B_{2,16} - 3.66369{\cdot}10^7 B_{3,16} - 1.78131{\cdot}10^7 B_{4,16} - 1.18551{\cdot}10^6 B_{5,16} + 1.32508{\cdot}10^7 B_{6,16} + 2.55007{\cdot}10^7 B_{7,16} + 3.55692{\cdot}10^7 B_{8,16} + 4.34611{\cdot}10^7 B_{9,16} + 4.91815{\cdot}10^7 B_{10,16} + 5.27353{\cdot}10^7 B_{11,16} + 5.41273{\cdot}10^7 B_{12,16} + 5.33624{\cdot}10^7 B_{13,16} + 5.04457{\cdot}10^7 B_{14,16} + 4.53821{\cdot}10^7 B_{15,16} + 3.81764{\cdot}10^7 B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 2.16806{\cdot}10^6$.

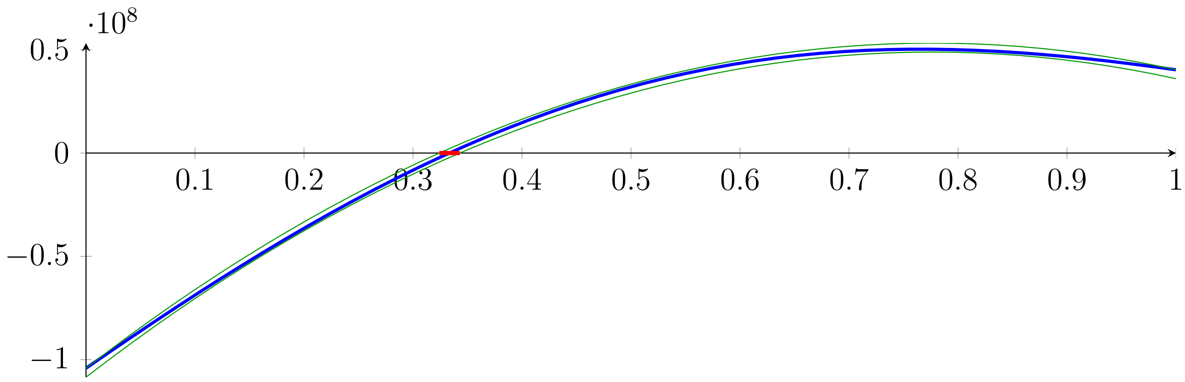
Bounding polynomials M and m:

$$M = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.04167{\cdot}10^8$$
$$m = 2.75812{\cdot}10^6 X^3 - 2.65319{\cdot}10^8 X^2 + 4.07072{\cdot}10^8 X - 1.08503{\cdot}10^8$$

Root of M and m:

$$N(M) = \{0.324143, 1.23113, 94.6401\} \qquad N(m) = \{0.342913, 1.21218, 94.6403\}$$

Intersection intervals:



$$[0.324143, 0.342913]$$

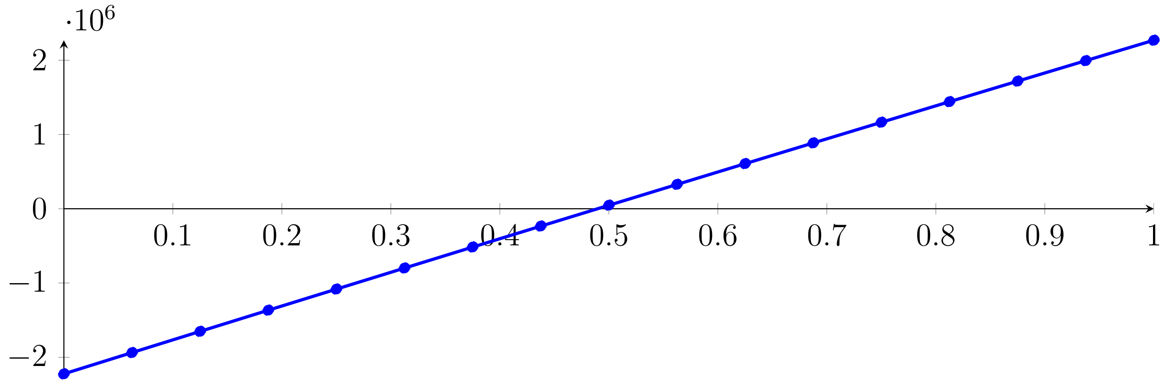
Longest intersection interval: 0.0187703

$\Longrightarrow$ Selective recursion: interval 1: $[0.324143, 0.342913]$,

3.2 Recursion Branch 1 1 in Interval 1: $[0.324143, 0.342913]$

Normalized monomial und Bézier representations and the Bézier polygon:

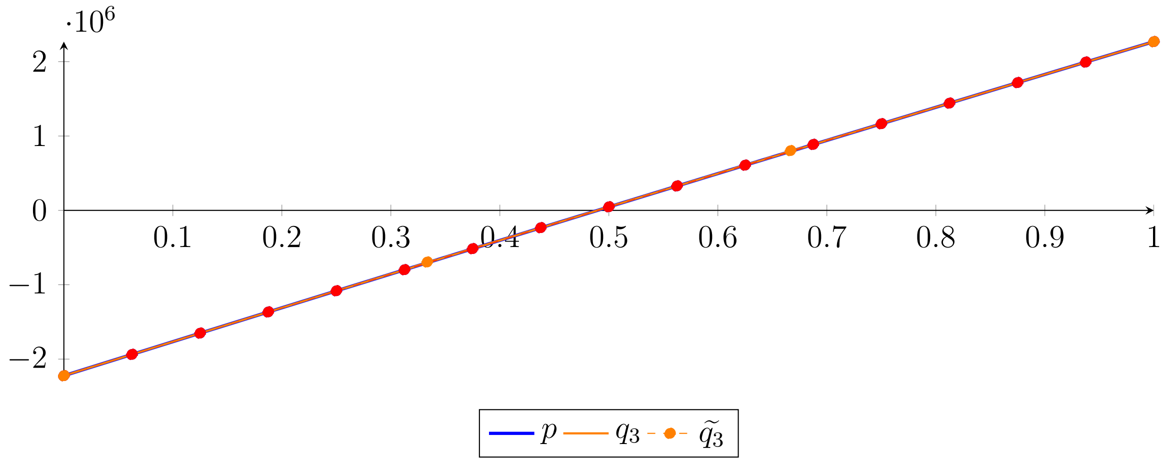
$$\begin{aligned} p = & -3.66617\cdot10^{-09}X^{16} - 1.53217\cdot10^{-07}X^{15} - 3.62234\cdot10^{-07}X^{14} - 1.65579\cdot10^{-06}X^{13} - 1.15373\cdot10^{-06}X^{12} \\ & - 2.3399\cdot10^{-06}X^{11} - 5.02543\cdot10^{-07}X^{10} - 1.38381\cdot10^{-06}X^{9} + 1.1237\cdot10^{-06}X^{8} - 1.19653\cdot10^{-05}X^{7} \\ & - 0.00155608X^{6} + 0.10496X^{5} + 18.6215X^{4} - 738.202X^{3} - 93855.7X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6} \\ = & -2.22335\cdot10^{6}B_{0,16}(X) - 1.93666\cdot10^{6}B_{1,16}(X) - 1.65074\cdot10^{6}B_{2,16}(X) - 1.36561\cdot10^{6}B_{3,16}(X) \\ & - 1.08127\cdot10^{6}B_{4,16}(X) - 797705B_{5,16}(X) - 514932B_{6,16}(X) - 232948B_{7,16}(X) + 48246.6B_{8,16}(X) \\ & + 328650B_{9,16}(X) + 608261B_{10,16}(X) + 887078B_{11,16}(X) + 1.1651\cdot10^{6}B_{12,16}(X) + 1.44233 \\ & \cdot10^{6}B_{13,16}(X) + 1.71876\cdot10^{6}B_{14,16}(X) + 1.99439\cdot10^{6}B_{15,16}(X) + 2.26922\cdot10^{6}B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6} \\ &= -2.22335\cdot10^{6}B_{0,3} - 694303B_{1,3} + 803454B_{2,3} + 2.26922\cdot10^{6}B_{3,3} \end{aligned}$$

$$\begin{aligned} \widetilde{q}_3 &= 16.4956X^{16} - 129.161X^{15} + 457.83X^{14} - 971.671X^{13} + 1375.95X^{12} - 1370.96X^{11} \\ &\quad + 987.265X^{10} - 519.476X^{9} + 199.587X^{8} - 55.434X^{7} + 10.9237X^{6} - 1.48019X^{5} \\ &\quad + 0.129516X^{4} - 700.679X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6} \\ &= -2.22335\cdot10^{6}B_{0,16} - 1.93666\cdot10^{6}B_{1,16} - 1.65074\cdot10^{6}B_{2,16} - 1.36561\cdot10^{6}B_{3,16} \\ &\quad - 1.08126\cdot10^{6}B_{4,16} - 797705B_{5,16} - 514932B_{6,16} - 232948B_{7,16} + 48246.4B_{8,16} \\ &\quad + 328650B_{9,16} + 608261B_{10,16} + 887078B_{11,16} + 1.1651\cdot10^{6}B_{12,16} + 1.44233 \\ &\quad \cdot10^{6}B_{13,16} + 1.71876\cdot10^{6}B_{14,16} + 1.99439\cdot10^{6}B_{15,16} + 2.26922\cdot10^{6}B_{16,16} \end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 0.270074$.

Bounding polynomials M and m:

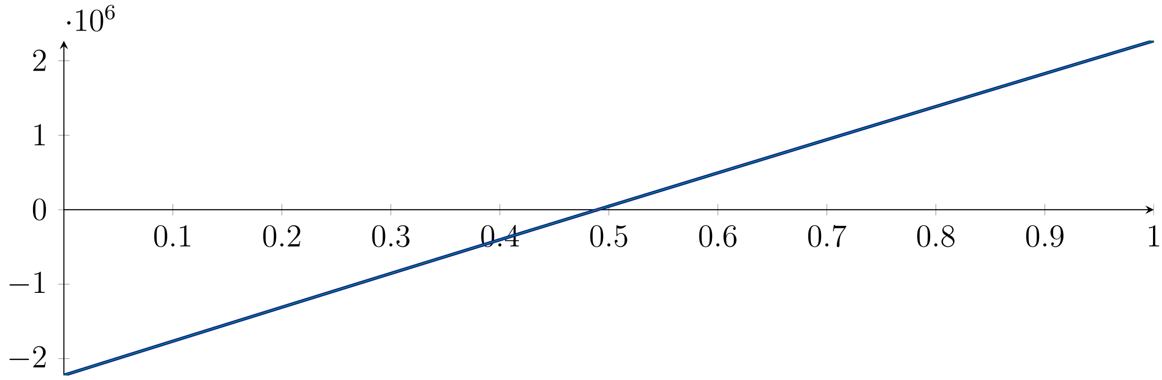
$$M = -700.673X^{3} - 93879.9X^{2} + 4.58715\cdot10^{6}X - 2.22335\cdot10^{6}$$

$$m = -700.673X^3 - 93879.9X^2 + 4.58715\cdot 10^6 X - 2.22335\cdot 10^6$$

Root of M and m:

$$N(M) = \{-172.127, 0.489616, 37.6521\} \qquad N(m) = \{-172.127, 0.489616, 37.6521\}$$

Intersection intervals:



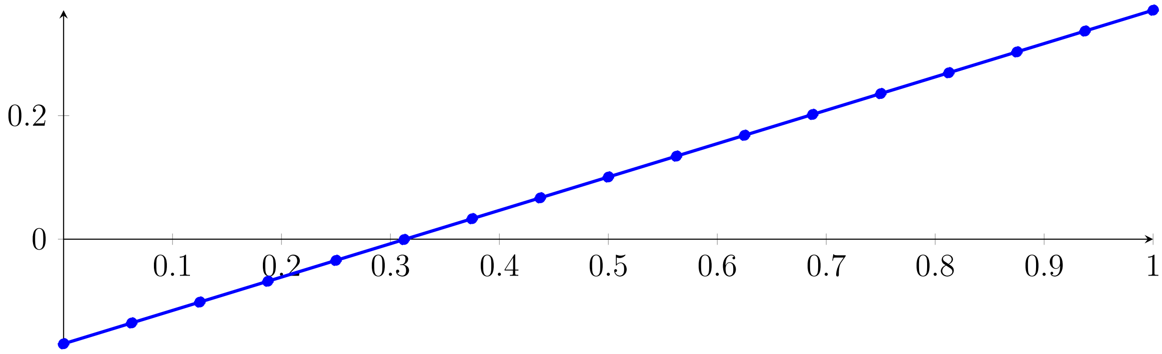
$$[0.489616, 0.489616]$$

Longest intersection interval: $1.20174\cdot 10^{-07}$
$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

3.3 Recursion Branch 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

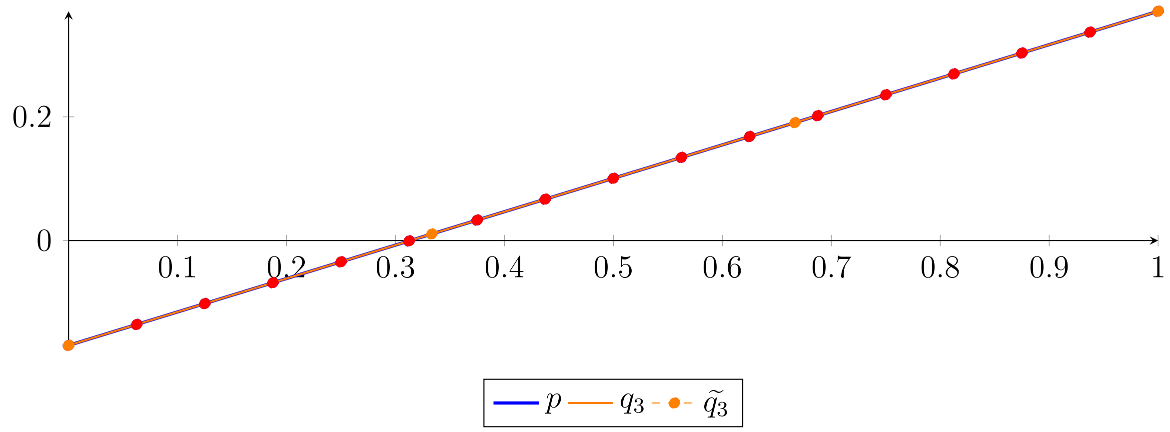
$$\begin{aligned} p = {} & 5.55524\cdot 10^{-15}X^{16} - 2.94313\cdot 10^{-14}X^{15} + 1.19384\cdot 10^{-13}X^{14} - 2.17482\cdot 10^{-13}X^{13} + 7.26155\cdot 10^{-14}X^{12} \\ & - 3.44766\cdot 10^{-13}X^{11} - 3.47292\cdot 10^{-15}X^{10} - 1.1287\cdot 10^{-13}X^{9} + 2.93027\cdot 10^{-14}X^{8} + 8.06213\cdot 10^{-15}X^{7} \\ & + 5.64349\cdot 10^{-15}X^{6} + 4.93312\cdot 10^{-17}X^{4} + 1.51788\cdot 10^{-17}X^{3} - 1.37072\cdot 10^{-09}X^{2} + 0.540148X - 0.169396 \\ = {} & -0.169396B_{0,16}(X) - 0.135637B_{1,16}(X) - 0.101877B_{2,16}(X) - 0.068118B_{3,16}(X) \\ & - 0.0343587B_{4,16}(X) - 0.000599476B_{5,16}(X) + 0.0331598B_{6,16}(X) \\ & + 0.0669191B_{7,16}(X) + 0.100678B_{8,16}(X) + 0.134438B_{9,16}(X) + 0.168197B_{10,16}(X) \\ & + 0.201956B_{11,16}(X) + 0.235715B_{12,16}(X) + 0.269475B_{13,16}(X) \\ & + 0.303234B_{14,16}(X) + 0.336993B_{15,16}(X) + 0.370752B_{16,16}(X) \end{aligned}$$



Degree reduction and raising:

$$\begin{aligned} q_3 &= -1.07065\cdot 10^{-17}X^3 - 1.37072\cdot 10^{-09}X^2 + 0.540148X - 0.169396 \\ &= -0.169396B_{0,3} + 0.0106536B_{1,3} + 0.190703B_{2,3} + 0.370752B_{3,3} \end{aligned}$$

$$\begin{aligned}\widetilde{q}_3 &= 8.59095\cdot 10^{-06}X^{16} - 6.82648\cdot 10^{-05}X^{15} + 0.000245968X^{14} - 0.000531568X^{13} \\ &+ 0.000767923X^{12} - 0.000782231X^{11} + 0.0005774X^{10} - 0.000312464X^{9} \\ &+ 0.000123994X^{8} - 3.57388\cdot 10^{-05}X^{7} + 7.34249\cdot 10^{-06}X^{6} - 1.04474\cdot 10^{-06}X^{5} \\ &+ 9.86739\cdot 10^{-08}X^{4} - 5.7553\cdot 10^{-09}X^{3} - 1.19186\cdot 10^{-09}X^{2} + 0.540148X - 0.169396 \\ &= -0.169396B_{0,16} - 0.135637B_{1,16} - 0.101877B_{2,16} - 0.068118B_{3,16} - 0.0343587B_{4,16} \\ &- 0.000599476B_{5,16} + 0.0331598B_{6,16} + 0.0669191B_{7,16} + 0.100678B_{8,16} \\ &+ 0.134438B_{9,16} + 0.168197B_{10,16} + 0.201956B_{11,16} + 0.235715B_{12,16} \\ &+ 0.269475B_{13,16} + 0.303234B_{14,16} + 0.336993B_{15,16} + 0.370752B_{16,16}\end{aligned}$$



The maximum difference of the Bézier coefficients is $\delta = 1.81206\cdot 10^{-10}$.

Bounding polynomials M and m:

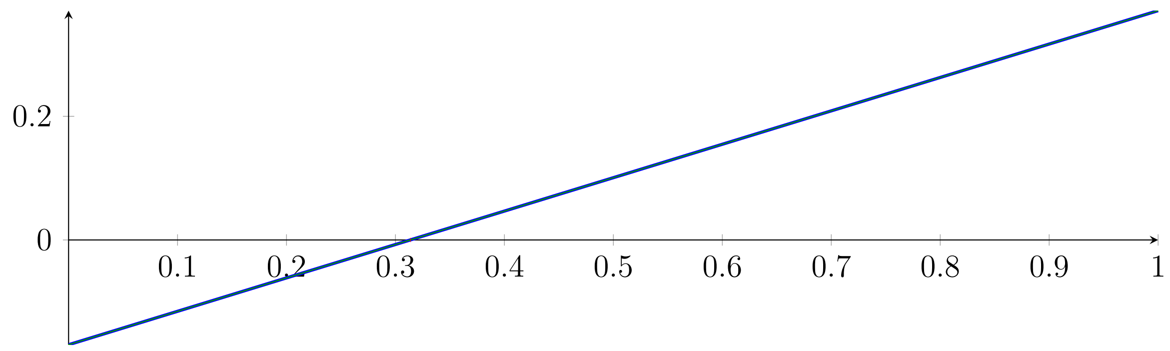
$$M = -1.07065\cdot 10^{-17}X^3 - 1.37072\cdot 10^{-09}X^2 + 0.540148X - 0.169396$$

$$m = -1.07065\cdot 10^{-17}X^3 - 1.37072\cdot 10^{-09}X^2 + 0.540148X - 0.169396$$

Root of M and m:

$$N(M) = \{-2.97569\cdot 10^{8}, 0.31361, 1.69542\cdot 10^{8}\} \quad N(m) = \{-2.97569\cdot 10^{8}, 0.31361, 1.69542\cdot 10^{8}\}$$

Intersection intervals:



$[0.31361, 0.31361]$

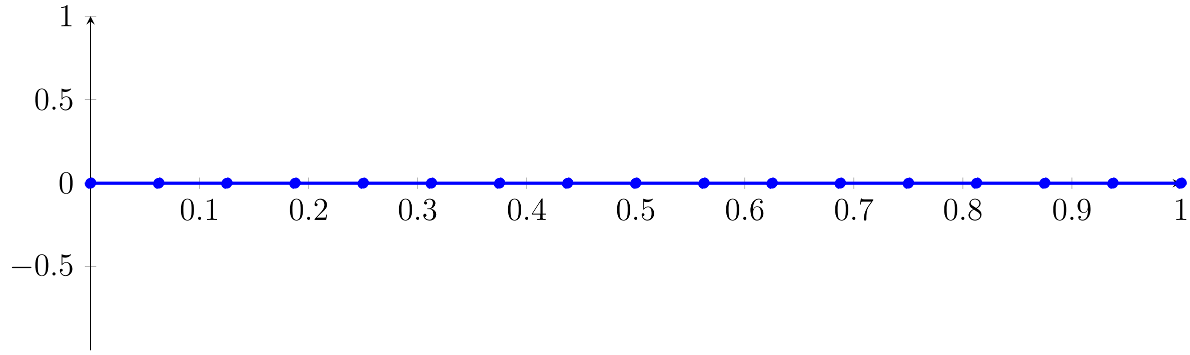
Longest intersection interval: $7.85803\cdot 10^{-10}$

$\Longrightarrow$ Selective recursion: interval 1: $[0.333333, 0.333333]$,

3.4 Recursion Branch 1 1 1 1 in Interval 1: $[0.333333, 0.333333]$

Normalized monomial und Bézier representations and the Bézier polygon:

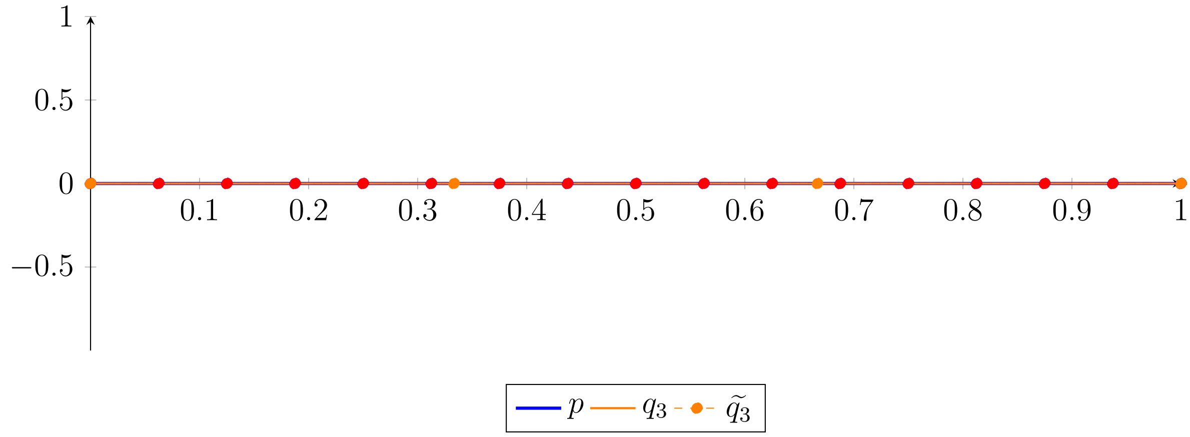
$$p = -1.51576\cdot10^{-21}X^{16} + 2.62009\cdot10^{-21}X^{15} - 3.98039\cdot10^{-20}X^{14} + 3.2136\cdot10^{-21}X^{13} - 5.16564\cdot10^{-20}X^{12} + 1.52429\cdot10^{-20}X^{11} - 1.44901\cdot10^{-20}X^{10} - 1.40466\cdot10^{-20}X^{9} + 2.34541\cdot10^{-20}X^{8} + 3.25289\cdot10^{-21}X^{7} + 2.38052\cdot10^{-21}X^{6} - 2.2582\cdot10^{-22}X^{5} + 3.52844\cdot10^{-23}X^{4} + 4.2445\cdot10^{-10}X - 2.39831\cdot10^{-08}$$
$$= -2.39831\cdot10^{-08}B_{0,16}(X) - 2.39566\cdot10^{-08}B_{1,16}(X) - 2.39301\cdot10^{-08}B_{2,16}(X) - 2.39036\cdot10^{-08}B_{3,16}(X) - 2.3877\cdot10^{-08}B_{4,16}(X) - 2.38505\cdot10^{-08}B_{5,16}(X) - 2.3824\cdot10^{-08}B_{6,16}(X) - 2.37974\cdot10^{-08}B_{7,16}(X) - 2.37709\cdot10^{-08}B_{8,16}(X) - 2.37444\cdot10^{-08}B_{9,16}(X) - 2.37179\cdot10^{-08}B_{10,16}(X) - 2.36913\cdot10^{-08}B_{11,16}(X) - 2.36648\cdot10^{-08}B_{12,16}(X) - 2.36383\cdot10^{-08}B_{13,16}(X) - 2.36118\cdot10^{-08}B_{14,16}(X) - 2.35852\cdot10^{-08}B_{15,16}(X) - 2.35587\cdot10^{-08}B_{16,16}(X)$$



Degree reduction and raising:

$$q_3 = -3.5866\cdot10^{-25}X^3 + 8.33643\cdot10^{-25}X^2 + 4.2445\cdot10^{-10}X - 2.39831\cdot10^{-08}$$
$$= -2.39831\cdot10^{-08}B_{0,3} - 2.38417\cdot10^{-08}B_{1,3} - 2.37002\cdot10^{-08}B_{2,3} - 2.35587\cdot10^{-08}B_{3,3}$$

$$\widetilde{q}_3 = -1.64958\cdot10^{-12}X^{16} + 1.3166\cdot10^{-11}X^{15} - 4.76688\cdot10^{-11}X^{14} + 1.03558\cdot10^{-10}X^{13} - 1.50448\cdot10^{-10}X^{12} + 1.54183\cdot10^{-10}X^{11} - 1.1456\cdot10^{-10}X^{10} + 6.24452\cdot10^{-11}X^{9} - 2.49793\cdot10^{-11}X^{8} + 7.26358\cdot10^{-12}X^{7} - 1.50649\cdot10^{-12}X^{6} + 2.16616\cdot10^{-13}X^{5} - 2.07725\cdot10^{-14}X^{4} + 1.24748\cdot10^{-15}X^{3} - 4.0727\cdot10^{-17}X^{2} + 4.2445\cdot10^{-10}X - 2.39831\cdot10^{-08}$$
$$= -2.39831\cdot10^{-08}B_{0,16} - 2.39566\cdot10^{-08}B_{1,16} - 2.39301\cdot10^{-08}B_{2,16} - 2.39036\cdot10^{-08}B_{3,16} - 2.3877\cdot10^{-08}B_{4,16} - 2.38505\cdot10^{-08}B_{5,16} - 2.3824\cdot10^{-08}B_{6,16} - 2.37974\cdot10^{-08}B_{7,16} - 2.37709\cdot10^{-08}B_{8,16} - 2.37444\cdot10^{-08}B_{9,16} - 2.37179\cdot10^{-08}B_{10,16} - 2.36913\cdot10^{-08}B_{11,16} - 2.36648\cdot10^{-08}B_{12,16} - 2.36383\cdot10^{-08}B_{13,16} - 2.36118\cdot10^{-08}B_{14,16} - 2.35852\cdot10^{-08}B_{15,16} - 2.35587\cdot10^{-08}B_{16,16}$$



The maximum difference of the Bézier coefficients is $\delta = 3.51589{\cdot}10^{-17}$.

Bounding polynomials M and m:

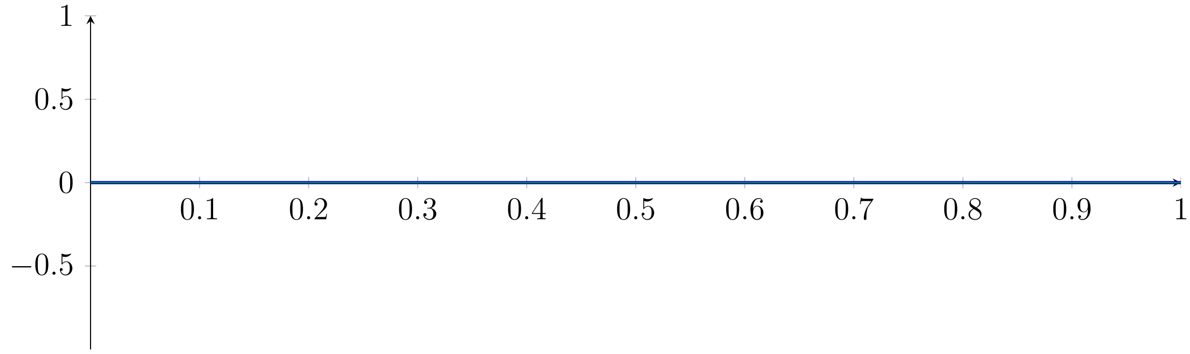
$$M = -3.5866{\cdot}10^{-25}X^3 + 8.33643{\cdot}10^{-25}X^2 + 4.2445{\cdot}10^{-10}X - 2.39831{\cdot}10^{-08}$$

$$m = -3.5866{\cdot}10^{-25}X^3 + 8.33643{\cdot}10^{-25}X^2 + 4.2445{\cdot}10^{-10}X - 2.39831{\cdot}10^{-08}$$

Root of M and m:

$$N(M) = \{-3.44011{\cdot}10^7, 56.504, 3.4401{\cdot}10^7\} \qquad N(m) = \{-3.44011{\cdot}10^7, 56.504, 3.4401{\cdot}10^7\}$$

Intersection intervals:

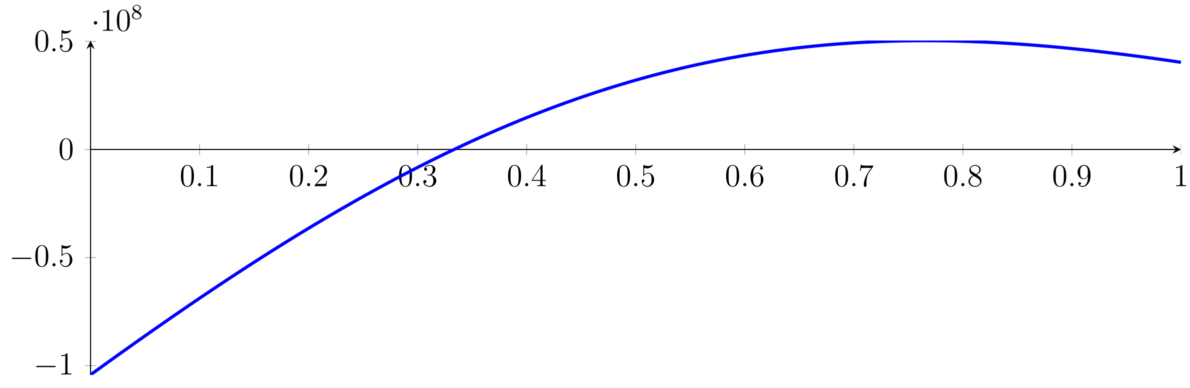


No intersection intervals with the x axis.

3.5 Result: 0 Root Intervals

Input Polynomial on Interval $[0, 1]$

$$p = -1X^{16} - 39.6667X^{15} - 651.667X^{14} - 5448.33X^{13} - 20440X^{12} + 18475.3X^{11} + 451673X^{10} + 1.12172{\cdot}10^{6}X^{9} - 2.52262{\cdot}10^{6}X^{8} - 1.43506{\cdot}10^{7}X^{7} + 138542X^{6} + 7.92823{\cdot}10^{7}X^{5} + 3.97396{\cdot}10^{7}X^{4} - 2.40625{\cdot}10^{8}X^{3} - 8.33333{\cdot}10^{7}X^{2} + 3.64583{\cdot}10^{8}X - 1.04167{\cdot}10^{8}$$



Result: Root Intervals

with precision $\varepsilon = 1{\cdot}10^{-32}$.