Robert Gallager's Minimum Delay Routing Algorithm Using Distributed Computation

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January 29, 2007

DSN

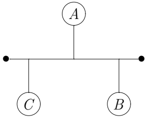
Road Map

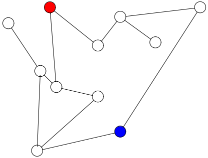
1 Introduction
2 Model
3 Algorithm
4 Conclusion

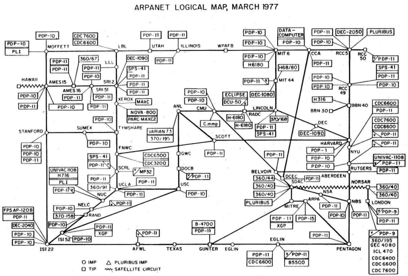
Introduction: Routing Algorithms

What are they?

Why do we need them?







Goals of Routing Algorithms

Primary Goal

Achieve "good" or even *optimal routing*.

- How to measure routing quality?
  $\longrightarrow$ Routing metrics

Other Aims

- little network overhead
- stability and reliablity
- adapt to changes
- quickly converge to optimal state
- scale well

Characteristics

Route Calculation Time

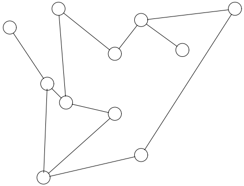
- Static routing algorithms
- Dynamic routing algorithms
- Quasi-static routing algorithms

Characteristics

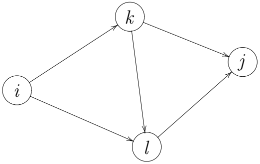
Other Characteristics

- Single-Path vs. Multi-Path Algorithms
- Centralized vs. Distributed Algorithms
- User vs. System Optimization

Model

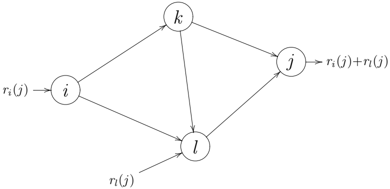


Model



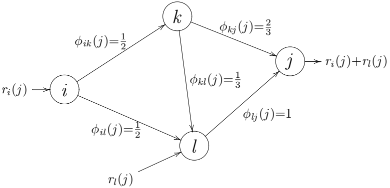
Set of n nodes enumerated by $\{1, 2, \dots, n\}$
Set of links: $\mathcal{L} := \{(i, j) \text{ is existing link}\}$

Model



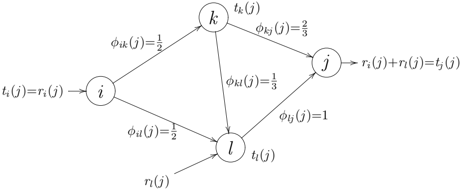
Input traffic entering at i and destined for j: $r_i(j)$.
e.g. in kbit/s

Model



Routing variables $\phi_{ik}(j)$:
Fraction of traffic destined for j travelling link (i, k).

Model



Sum over all traffic at node i destined for j: $t_i(j)$.

Constraints on ϕ

1. No traffic on non-existing links and no loopback traffic

$$\phi_{ik}(j) = 0 \quad \forall\,(i,j) \notin \mathcal{L} \text{ or } i = j$$

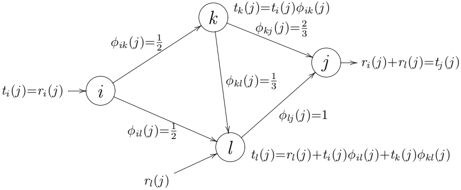
2. No loss of traffic is allowed.

$$\sum_{k=1}^{n} \phi_{ik}(j) = 1 \quad \forall\, i, j$$

3. All nodes are inter-connected.

$$\phi_{ik}(j) > 0, \phi_{kl}(j) > 0, \ldots, \phi_{mj}(j) > 0$$
$$\exists\, i, k, l, \ldots, m, j \;\forall\, i, j$$

Model



$$t_i(j) = r_i(j) + \sum_{l=1}^{n} t_l(j)\phi_{li}(j)$$

Variables

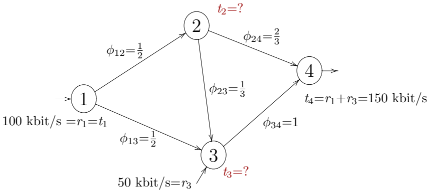
- Set of n nodes enumerate by $\{1, 2, \ldots, n\}$
- Set of links: $\mathcal{L} := \{(i,j) \text{ is existing link}\}$
- Input traffic set $\boldsymbol{r} := \{r_i(j)\}$
- Node flow set $\boldsymbol{t} := \{t_i(j)\}$
- Routing variable set $\boldsymbol{\phi} := \{\phi_{ik}(j)\}$.

Theorem 1

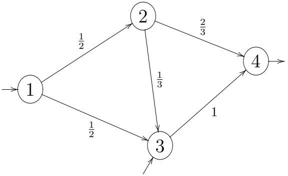
The routing variable set $\boldsymbol{\phi}$ will actually guide the network's flow.

Formally: An input set $\boldsymbol{r}$ and a routing variable set $\boldsymbol{\phi}$ uniquely define a network flow set $\boldsymbol{t}$.

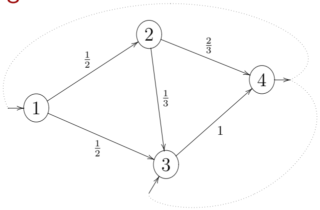
Routing Variables



Routing Variables

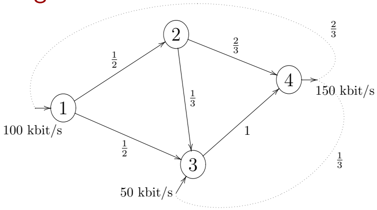


Routing Variables



Find steady state by introducing imaginary links which transfer traffic back to its source node.

Routing Variables



$$\phi_{ji}(j) := \frac{r_i(j)}{\sum_k r_k(j)}$$

Markov Transition Matrix

$$\Phi = (\phi_{ik}(j))_{i,k} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \end{pmatrix}$$

The second constraint on ϕ and $\phi_{ik}(j) \geq 0$ are the defining properties of a stochastic matrix.

Markov Equation

With $\phi_{ji}(j) := \frac{r_i(j)}{\sum_k r_k(j)}$ the aggregation equation

$$t_i(j) = r_i(j) + \sum_{l=1}^{n} t_l(j)\phi_{li}(j)$$

can be contracted to

$$t_i(j) = \sum_{l=1}^{n} t_l(j)\phi_{li}(j) \quad \Leftrightarrow \quad \bar{t} = \bar{t}\Phi$$

Equilibrium Distribution

$$\bar{t} = \bar{t}\Phi$$

Is the equation of a Markov chain in an equilibrium state.

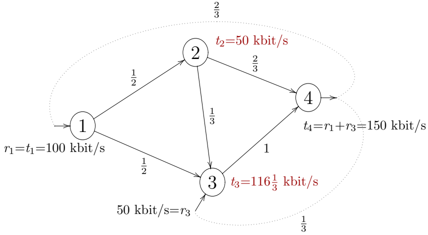
From Markov chain theory: If the transition matrix is irreducible, then exactly one equilibrium distribution $\bar{t}$ exists.

Equilibrium in the Example

$$\Phi = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 1 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \end{pmatrix} \qquad \lim_{n\to\infty} \Phi^n = \begin{pmatrix} \frac{6}{25} & \frac{3}{25} & \frac{7}{25} & \frac{9}{25} \\ \frac{6}{25} & \frac{3}{25} & \frac{7}{25} & \frac{9}{25} \\ \frac{6}{25} & \frac{3}{25} & \frac{7}{25} & \frac{9}{25} \\ \frac{6}{25} & \frac{3}{25} & \frac{7}{25} & \frac{9}{25} \end{pmatrix}$$

$$\Rightarrow \bar{t}' = \begin{pmatrix} \frac{6}{25} \\ \frac{3}{25} \\ \frac{7}{25} \\ \frac{9}{25} \end{pmatrix}^{\top} \quad \Rightarrow \quad \bar{t} = \begin{pmatrix} 100 \\ 50 \\ 116\frac{1}{3} \\ 150 \end{pmatrix}^{\top} \text{kbit/s}$$

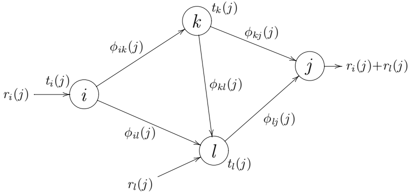
Equilibrium in the Example



Delay

Currently the model only describes traffic flow.

Now introduce delay.



Traffic and Delay

First define total traffic f_{ik} on a link (i, k)

$$f_{ik} = \sum_j t_i(j)\phi_{ik}(j)$$

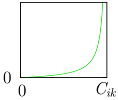
Traffic and Delay

Then calculate link delay $D_{ik}(f_{ik})$ from the traffic.

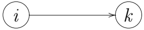
Only requirements of D_{ik}: convex and increasing.

For example

$$D_{ik}(f_{ik}) = \frac{f_{ik}}{C_{ik} - f_{ik}}$$



with link capacity C_{ik}.



Total delay

Finally define total delay D_T

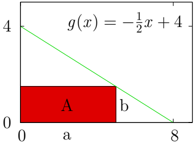
$$D_T = \sum_{i,k} D_{ik}(f_{ik})$$

Goal: Minimize D_T by setting optimal $\phi_{ik}(j)$.

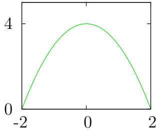
Use same general method as with maximizing rectangle area function in school.

General Method

Problem:
Find $a, b = g(a)$ with maximum area A.
Set first derivative to zero.



$$A(a) = a \cdot b = a \cdot g(a)$$
$$= -\tfrac{1}{2}a^2 + 4a$$
$$A'(a) = -a^2 + 4$$
$$A'(a) = 0 \text{ for } a = \pm\sqrt{4}$$
$$\Rightarrow b = 5$$



Derivative of D_T

Method: Determine the derivative of D_T and find a root.

But derive D_T by which parameter?

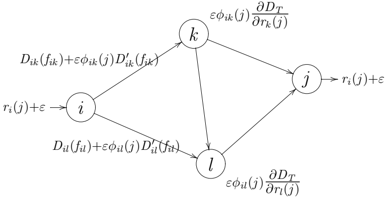
D_T is the sum of all delays D_{ik}.
Each D_{ik} is a function of the link traffic f_{ik}.
f_{ik} is somehow determined by $\boldsymbol{r}$, $\boldsymbol{t}$ and $\boldsymbol{\phi}$.

$$D'_{ik}(f_{ik}) = \frac{\mathrm{d}D_{ik}(f_{ik})}{\mathrm{d}f_{ik}}$$

Partial Derivatives of D_T

Easier: Determine partial derivative $\dfrac{\partial D_T}{\partial r_i(j)}$

How does more input traffic change total delay?

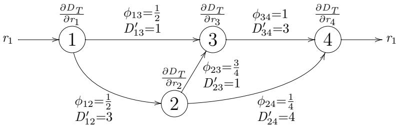


Partial Derivatives of D_T

Partial derivative regarding input traffic:

$$\frac{\partial D_T}{\partial r_i(j)} = \sum_k \phi_{ik}(j) \left(D'_{ik}(f_{ik}) + \frac{\partial D_T}{\partial r_k(j)} \right)$$

Calculate marginal (incremental) delay in this example:



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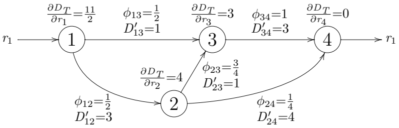
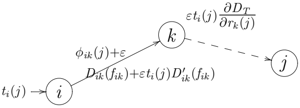


Partial Derivatives of D_T

However a future algorithm should change routing variables $\phi_{ik}(j)$.

So determine their change to delay: $\frac{\partial D_T}{\partial \phi_{ik}(j)}$



Finding a Root

$$\frac{\partial D_T}{\partial \phi_{ik}(j)} = t_i(j) \left(D'_{ik}(f_{ik}) + \frac{\partial D_T}{\partial r_k(j)} \right)$$

Find a stationary point of D_T regarded as a function of $\phi_{ik}(j)$ in which all $\frac{\partial D_T}{\partial \phi_{ik}(j)} = 0$ $(\nabla D_T(\boldsymbol{\phi}) = 0)$.

However $\boldsymbol{\phi}$ has the three constraints
$\Rightarrow$ Lagrange multipliers are required.

Lagrange Multipliers

Formalize the constraints into a function $g(\boldsymbol{\phi}) = 0$, with $\nabla g(\boldsymbol{\phi}) \neq 0$.

Introduce Lagrange multipliers λ and solve:

$$\nabla D_T(\boldsymbol{\phi}) = -\lambda g(\boldsymbol{\phi})$$
$$g(\boldsymbol{\phi}) = 0$$

Lagrange Multipliers

Result:

$$\frac{\partial D_T}{\partial \phi_{ik}(j)} \begin{cases} = \lambda_{ij}, & \phi_{ik}(j) > 0 \\ \geq \lambda_{ij}, & \phi_{ik}(j) = 0 \end{cases} \quad \forall\, i \neq j \;\forall\, (i,k) \in \mathcal{L}$$

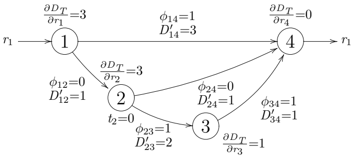
Note that the λ_{ij} do not depend on k.

$\Rightarrow$ All used links must have same marginal delay.
Unused must have greater marginal delay.

Only Necessary

However this condition is not sufficient.

Counter-example:

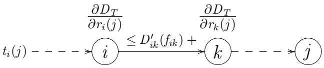


Sufficient Condition

Brilliant idea of Gallager: remove the factor $t_i(j)$

$$\frac{\partial D_T}{\partial r_i(j)} \leq D'_{ik}(f_{ik}) + \frac{\partial D_T}{\partial r_k(j)}$$

Intuitive reduction of delay:



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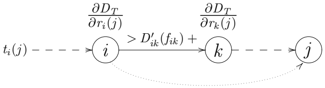
Intuitive reduction of delay (Contraposition):



Transformation into Algorithm

$$\frac{\partial D_T}{\partial r_i(j)} \leq D'_{ik}(f_{ik}) + \frac{\partial D_T}{\partial r_k(j)}$$

transformed into an iterative version useful for the future algorithm

$$D'_{ik}(f_{ik}) + \frac{\partial D_T}{\partial r_k(j)} \geq \min_{(i,m)\in\mathcal{L}} \left(D'_{im}(f_{im}) + \frac{\partial D_T}{\partial r_m(j)} \right)$$

The Algorithms Main Goal

- Calculate new routing variables (ϕ_{ik})
  - increase ϕ_{ik} on links with small marginal delay
  - decrease ϕ_{ik} on links with large marginal delay
- During iterative distributed computation:
  - stable state is reached
  - optimal solution is found
  - no deadlock occurs

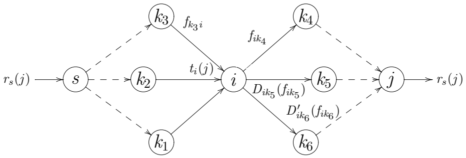
The Algorithm

1. Determine the necessary variables: $\frac{\partial D_{ik}}{\partial r_i(j)}$ and $D'_{ik}(f_{ik})$
2. Calculate new routing variables ϕ^1
   - main challenge: keep ϕ loop free

Variables Available to a Specific Node

- A node knows:
  - its incoming and outgoing links
  - its neighbors
  - the amount of traffic flow (can be measured)
  - its routing variables for all links and destinations

Variables Available to a Specific Node



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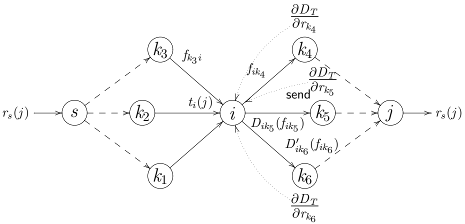
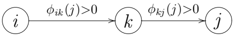


Determine Marginal Delay

- D_{ik} can be calculated or measured
- D'_{ik} can be calculated from D_{ik}
- D'_{ik} more often measured
- Still missing $\frac{\partial D_{ik}}{\partial r_i(j)}$

Downstream Concept

- Each node becomes $\frac{\partial D_{ik}}{\partial r_i(j)}$ from its downstream neighbors
- Node k is downstream from i with respect to destination j, if there is a path from i to j through k and all routing variables on the way down to j are positive (i.e. $\phi_{il_1}(j) > 0$ … $\phi_{l_n j}(j) > 0$)



Routing Variables Calculation

- Calculate new variables in three steps.
- Determine the best link (lowest marginal delay)
- Difference between each link k and the best link:

$$a_{ik}(j) = \underbrace{D'_{ik}(f_{ik}) + \frac{\partial D_T}{\partial r_k(j)}}_{\text{on link } k} - \underbrace{\left(D'_{ib}(f_{ib}) + \frac{\partial D_T}{\partial r_b(j)}\right)}_{\text{on the best link}}$$

Routing Variable Reduction

$\Delta_{ik}(j)$: the reduction of routing variable $\phi_{ik}(j)$

$$\Delta_{ik}(j) = \min\left\{\phi_{ik}(j),\ \frac{\eta}{t_i(j)} a_{ik}(j)\right\}$$

with a small scale factor η.

The New Routing Variables

$$\phi^1_{ik}(j) = \begin{cases} \phi_{ik}(j) - \Delta_{ik}(j), \\ \quad \text{if } (i,k) \text{ is not the best link} \\ \phi_{ib}(j) + \sum\limits_{\substack{(i,m)\in\mathcal{L} \\ m\neq b}} \Delta_{im}(j), \\ \quad \text{if } (i,k) \text{ is the best link} \\ \quad \text{and therefore } k = b \end{cases}$$

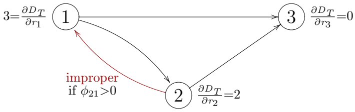
Blocked Set

- Blocked set $B_i(j)$: restrict flow from node i
  - require: $\phi_{ik}(j) = 0 \; \forall k \in B_i(j)$
- Nodes included in $B_i(j)$
  - nodes, which do not have link to node i
  - neighbors, which have downstream paths containing a loop

Improper Routing Variables

A routing variable $\phi_{ik}(j)$ is defined as improper if

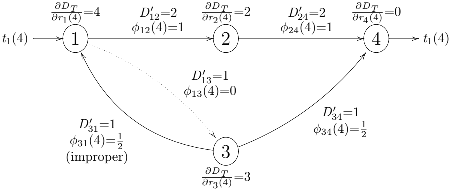
$$\phi_{ik}(j) > 0 \quad \text{and} \quad \frac{\partial D_T}{\partial r_i(j)} \leq \frac{\partial D_T}{\partial r_k(j)}$$



Blocked Set Definition

Formally $B_i(j)$ includes all nodes k, for which $\phi_{ik}(j) = 0$ and k can route packets to j over a path that contains some link (l, m) with *improper* $\phi_{lm}(j)$ and $\phi^1_{lm}(j) > 0$.

Example



Theorem 5

For every $D_0 > 0$
there exists a scale factor η for the algorithm A, such that if ϕ^0 satisfies $D_T(\phi^0) \leq D_0$, then

$$\lim_{m\to\infty} D_T(A^m(\phi)) = \min_{\phi} D_T(\phi)$$

Proof is done via seven lemmas over four pages (of twelve) in the paper.

Outline of Proof

Say $\phi^1 := A(\phi)$ and f^1 the new link flow.

First goal: calculate $D_T(\phi^1) - D_T(\phi)$.

Gallager uses auxiliary function ($0 \leq \lambda \leq 1$):
$D_T(\lambda) = \sum_{i,k} D_{ik}(f_{ik}^\lambda)$ with $f_{ik}^\lambda = f_{ik} + \lambda(f_{ik}^1 - f_{ik})$

and applies Taylor's remainder theorem in Lagrange form:

$$D_T(\phi^1) - D_T(\phi) = \left(\frac{\mathrm{d}D_T(\lambda)}{\mathrm{d}\lambda}\right)(0) + \frac{1}{2}\left(\frac{\mathrm{d}^2 D_T(\lambda)}{\mathrm{d}\lambda^2}\right)(\lambda^*)$$

Outline of Proof

Lemmas 1 to 4 are used to upper bound $\frac{\mathrm{d}D_T(\lambda)}{\mathrm{d}\lambda}$ and $\frac{\mathrm{d}^2 D_T(\lambda)}{\mathrm{d}\lambda^2}$.

Concluding in lemma 5:

For D_0 say $M := \max_{i,k} \max_{f:D_{ik}(f)\leq D_0} D''_{ik}(f)$

and let $\eta := \dfrac{1}{Mn^6}$, then

$$D_T(\phi^1) - D_T(\phi) \leq -\frac{1}{2\eta(n-1)^3} \sum_{i,j} \Delta_i^2(j)\, t_i^2(j)$$

Outline of Proof

In lemma 6 the last lemma is used to show a strict monotony criterion.

Let ϕ be routing variables with $D_T(\phi) < D_0$ but not the minimum.

Then $\exists\, \varepsilon > 0$ and m with $1 \leq m \leq n$:

$$\forall\, \phi^* : |\phi - \phi^*| < \varepsilon : D_T(A^m(\phi^*)) < D_T(\phi)$$

Proof includes a detailed analysis of the algorithm's steps for improper links and blocked nodes.

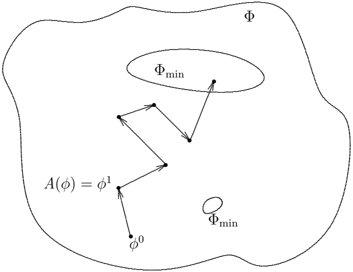
Outline of Proof

Let $\Phi \subseteq \mathbb{R}^n$ compact euclidean space of routing variables.

Then algorithm is a mapping $A : \Phi \to \Phi$, and $D_T : \Phi \to \mathbb{R}$ a real function.

Let $D_{\min}$ minimum of D_T over Φ and $\Phi_{\min}$ set of ϕ with $D_T(\phi) = D_{\min}$.

Outline of Proof



Outline of Proof

Because Φ is compact the sequence $\{A^m(\phi)\}$ has a convergent subsequence $\{\phi^l\}$.

Let $\phi' = \lim_{l\to\infty} \phi^l$, and since D_T is continuous $D_T(\phi') = \lim_{l\to\infty} D_T(\phi^l)$.

Left to prove: $D_T(\phi') = D_{\min}$.
Follows from $D_T(A^m(\phi)) < D_T(\phi)$.

Problems

- First drawback: required scale parameter η
- How can the start state be determined?
- What if links or nodes are dropped or added?
- Adapting to changing input traffic statistics.

Conclusion

- Rigorous mathematical approach
- Well designed mathematical model:
  - describe the minimum total delay problem
  - conditions for achieving global optimization
- Iterative, distributed routing algorithm
  - proved in detail that the algorithm will always progress into a network state with total minimum delay
- 209 citations on Google Scholar, 55 on Citeseer.